THE STRUCTURE OF FROBENIUS KERNELS FOR AUTOMORPHISM GROUP SCHEMES

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Abstract. We establish structure results for Frobenius kernels of automorphism group schemes for surfaces of general type in positive characteristics. It turns out that there are surprisingly few possibilities. This relies on properties of the famous Witt algebra, which is a simple Lie algebra without finite-dimensional counterpart over the complex numbers, together with is twisted forms. The result actually holds true for arbitrary proper integral schemes under the assumption that the Frobenius kernel has large isotropy group at the generic point. This property is measured by a new numerical invariant called the foliation rank.

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Introduction

Let k be an algebraically closed ground field of characteristic $p \geq 0$ and X be a proper scheme. Then the *automorphism group scheme* $\mathrm{Aut}_{X/k}$ is locally of finite type, and the connected component $\mathrm{Aut}^0_{X/k}$ is of finite type. The corresponding Lie algebra $\mathfrak{h} = H^0(X, \Theta_{X/k})$ is the space of global vector fields. If X is smooth and of

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general type, then the group $\mathrm{Aut}(X)$ is actually finite, according to a general result of Martin-Deschamps and Lewin-Ménégaux [40].

Throughout this paper we are mainly interested in characteristic $p > 0$. Then the group scheme $\mathrm{Aut}_{X/k}$ comes with a relative Frobenius map, and the resulting Frobenius kernel $H = \mathrm{Aut}_{X/k}[F]$ is a *height-one group scheme*. The group of rational points is trivial, but the coordinate ring may contain nilpotent elements. The Lie algebra $\mathfrak{h} = H^0(X, \Theta_{X/k})$ remains the space of global vector fields, or equivalently the space of k-linear derivations $D : \mathscr{O}_X \to \mathscr{O}_X$. The p-fold composition in the associative ring of k-linear differential operators endows the Lie algebra with an additional structure, the so-called *p-map* $D \mapsto D^{[p]}$, which turns $\mathfrak{h}$ into a *restricted Lie algebra*. By the *Demazure–Gabriel Correspondence*, height-one group schemes and restricted Lie algebras determine each other.

Our goal is to uncover the *structural properties* of the height-one group scheme $H = \mathrm{Aut}_{X/k}[F]$, or equivalently the restricted Lie algebra $\mathfrak{h} = H^0(X, \Theta_{X/k})$, and our initial motivation was to understand the case of surfaces of general type. Such surfaces with $\mathfrak{h} \neq 0$ where first constructed Russell [46] and Lang [33]. These constructions rely on Tango curves [55], and come with a purely inseparable covering by a ruled surface. By a similar construction with abelian surfaces, Shepherd-Barron produced examples in characteristic $p = 2$ that are non-uniruled ([51], Theorem 5.3). Ekedahl already had examples with rational double points for arbitrary $p > 0$ ([13], pages 145–146); the vector fields, however, do not extend to a resolution of singularities. Recently, Martin studied infinitesimal automorphism group schemes of elliptic and quasielliptic surfaces ([38], [39]).

However, almost nothing seems to be known about the general structure of the height-one group schemes $H = \mathrm{Aut}_{X/k}[F]$, and one would expect little restrictions in this respect. The main result of this paper asserts that in some sense, quite the opposite is true:

Theorem. *(See Cor. 12.2) Let X be a proper normal surface with $h^0(\omega_X^\vee) = 0$. Then the Frobenius kernel H for the automorphism group scheme $\mathrm{Aut}_{X/k}$ is isomorphic to the Frobenius kernel of one of the following three basic types of group schemes:*

$$\mathrm{SL}_2 \quad \textit{and} \quad \mathbb{G}_a^{\oplus n} \quad \textit{and} \quad \mathbb{G}_a^{\oplus n} \rtimes \mathbb{G}_m,$$

for some integer $n \geq 0$.

In the latter two cases, the respective Frobenius kernels are $\alpha_p^{\oplus n}$ and the semidirect product $\alpha_p^{\oplus n} \rtimes \mu_p$. The key idea in the proof for the above result is to relate our problem to the the famous *Witt algebra* $\mathfrak{g}_0 = \mathrm{Der}_E(F_0)$ formed with the truncated polynomial ring $F_0 = E[t]/(t^p)$ over certain function fields E. This algebra was indeed introduced by Ernst Witt, compare the discussion in [53]. Note that it has nothing to do with the ring of Witt vectors, or Witt groups for quadratic forms.

Our result applies, in particular, to surfaces of general type, and properly elliptic surfaces. It actually holds holds true for any proper integral scheme X whose *foliation rank* is at most one (see Theorem 12.1). This is a new invariant that can be defined as follows: Forming the quotient $Y = X/H$ by the Frobenius kernel of the automorphism group scheme, the canonical map $X \to Y$ induces a height-one extension $E = k(Y) \subset k(X) = F$ of function fields, and the foliation rank $r \geq 0$ is given by $[F : E] = p^r$. Via the Jacobson Correspondence, this can also be expressed in terms of the *inertia subgroup scheme* for the induced action of the base-change H_F on $F\otimes_E F$. This geometric interpretation of the Jacobson Correspondence seems to be of independent interest (see Section 5).

The condition $r = 1$ means that $\deg(X/Y) = p$. This situation is paradoxical, because it may hold even with large Frobenius kernels H. We now compare $H \subset \operatorname{Aut}_{X/k}$ with the generic fiber of the relative group scheme $G = \operatorname{Aut}_{X/Y}$. In other words, we relate the restricted Lie algebra $\mathfrak{h} = H^0(X, \Theta_{X/k})$ over k with the restricted Lie algebra $\mathfrak{g} = \operatorname{Der}_E(F)$ over the function field $E = k(Y)$.

This $\mathfrak{g}$ is a a twisted form of the *Witt algebra* $\mathfrak{g}_0 = \operatorname{Der}_E(F_0)$ formed with the truncated polynomial ring $F_0 = E[t]/(t^p)$. This $\mathfrak{g}_0$ is one of the simple algebras in characteristic $p > 0$ having no finite-dimensional counterpart over the complex numbers. The classification of its subalgebras due to Premet and Steward [44] is one key ingredient for our proof. Among other surprising features, $\mathfrak{g}_0$ contains Cartan algebras of different dimensions. A key observations is that *the bigger Cartan algebras disappear after passing to twisted forms like* $\mathfrak{g}$*, leaving few possibilities for subalgebras.* This is an algebraic incarnation for the fact that the reduced part of a group scheme may not be a subgroup scheme, and if it is, it may not be normal.

The semidirect products $\alpha_p^{\oplus n} \rtimes \mu_p$ indeed occur as Frobenius kernels of automorphism group schemes. In Section 14, we construct examples of surfaces as coverings $X \to \mathbb{P}^2$ of degree p or divisors $X \subset \mathbb{P}^3$ of degree $2p+1$, such that $\mathfrak{h} = k^n \rtimes \mathfrak{gl}_1(k)$, for certain integers $n \geq 0$. So far, we do not know if $\mathfrak{h} = \mathfrak{sl}_2(k)$ may also occur. In our examples, the minimal resolutions are surfaces S of general type, and X are their canonical models.

Such X are also called *canonically polarized surface*. They come with two *Chern numbers* $c_1^2 = c_1^2(L^\bullet_{X/k}) = K_X^2$ and $c_2 = c_2(L^\bullet_{X/k})$. This was introduced by Ekedahl, Hyland and Shepherd-Barron [15] for general proper surfaces whose local rings are complete intersections, such that the cotangent complex is perfect. Using Noether's inequality and results from Ekedahl [14], we show with more classical methods:

Theorem. *(see Thm. 13.2) Let X be canonically polarized surface, with Chern numbers c_1^2 and c_2. Then the Lie algebra $\mathfrak{h} = H^0(X, \Theta_{X/k})$ for the Frobenius kernel $H = \operatorname{Aut}_{X/k}[F]$ has the property $\dim(\mathfrak{h}) \leq \Phi(c_1^2, c_2)$ for the polynomial*

$$\Phi(x, y) = \frac{1}{144}(14641x^2 + 242xy + 60x + y^2 - 12y + 288).$$

The paper is organized as follows: Section 1 contains general facts on restricted Lie algebras and their semidirect products. In Section 2 we examine multiplicative and additive vectors, and the toral rank. In Section 3 we collect general facts on automorphism group schemes for proper schemes, the quotient by height-one group schemes, and discuss twisted forms of some relevant restricted Lie algebras. Section 5 contains a geometric interpretation of the Jacobson correspondence, in terms of inertia group schemes at generic points. We introduce the foliation rank and establish its basic properties in Section 6. In Section 7 we analyze the removal of subvector spaces under certain twists. Then we make a detailed analysis of the automorphism group scheme for radical extensions of prime degree in Section 8, followed by an examination of the corresponding Witt algebras in Section 9. In Section 10 we show how structural properties of restricted Lie algebras over different fields are inherited. Our main result on the structure of the Frobenius kernel for automorphism groups is contained in Section 11. Section 13 contains the bound for surfaces of general type. In the final Section 14, we construct examples.

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1. Restricted Lie algebras

In this section we review some standard results on restricted Lie algebras and height-one group schemes that are relevant for the applications we have in mind. Let k be a ground field of characteristic $p > 0$. For each ring R, not necessarily commutative or associative, the vector space $\mathrm{Der}_k(R)$ of k-derivations $D : R \to R$ is closed under forming commutators $[D, D']$ and p-fold compositions D^p in the associative ring $\mathrm{End}_k(R)$. One now views $\mathrm{Der}_k(R)$ as a *Lie algebra*, endowed the map $D \mapsto D^p$ as an additional structure.

This leads to the following abstraction: A *restricted Lie algebra* is a Lie algebra $\mathfrak{g}$, together with a map $\mathfrak{g} \to \mathfrak{g}$, $x \mapsto x^{[p]}$ called the *p-map*, subject to the following three axioms:

(R 1) We have $\mathrm{ad}_{x^{[p]}} = (\mathrm{ad}_x)^p$ for all vectors $x \in \mathfrak{g}$.

(R 2) Moreover $(\lambda \cdot x)^{[p]} = \lambda^p \cdot x^{[p]}$ for all vectors $x \in \mathfrak{g}$ and scalars $\lambda \in k$.

(R 3) The formula $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{r=1}^{p-1} s_r(x, y)$ holds for all $x, y \in \mathfrak{g}$.

Here the summands $s_r(x, y)$ are universal expressions defined by

$$s_r(t_0, t_1) = -\frac{1}{r}\sum_u (\mathrm{ad}_{t_{u(1)}} \circ \mathrm{ad}_{t_{u(2)}} \circ \ldots \circ \mathrm{ad}_{t_{u(p-1)}})(t_1),$$

where $\mathrm{ad}_a(x) = [a, x]$ denotes the *adjoint representation*, and the the index runs over all maps $u : \{1, \ldots, p-1\} \to \{0, 1\}$ taking the value zero exactly r times. For $p = 2$ the expression simplifies to $s_1 = [t_0, t_1]$, whereas $p = 3$ gives $s_1 = [t_1, [t_0, t_1]]$ and $s_2 = [t_0, [t_0, t_1]]$. Restricted Lie algebras were introduced and studied by Jacobson [25], and also go under the name *p-Lie algebras*. We refer to the monographs of Demazure and Gabriel [11], in particular Chapter II, §7, or Strade and Farnsteiner [54] for more details.

Throughout the paper, terms like homomorphisms, subalgebras, ideals, extensions etc. are understood in the *restricted sense*, if not said otherwise. For example, an *ideal* $\mathfrak{a} \subset \mathfrak{g}$ is a vector subspace such that $[x, y], x^{[p]} \in \mathfrak{a}$ whenever $x \in \mathfrak{a}$ and $y \in \mathfrak{g}$. Note that this holds for the *center* $\mathfrak{C}(\mathfrak{g}) = \{a \in \mathfrak{g} \mid [a, x] = 0 \text{ for all } x \in \mathfrak{g}\}$, because $[a^{[p]}, x] = (\mathrm{ad}_a)^p(x) = (\mathrm{ad}_a)^{p-1}([a, x]) = 0$.

For abelian $\mathfrak{g}$, the p-map becomes *semi-linear*, which means that it corresponds to a linear map $\mathfrak{g} \to \mathfrak{g}$ when the scalar multiplication in the range is redefined via Frobenius. In turn, those $\mathfrak{g}$ correspond to modules over the associative polynomial ring $k[F]$, in which the relation $F\lambda = \lambda^p F$ holds. Every right ideal is principal; this also holds for left ideals, provided that k is perfect, and then the structure theory developed by Jacobson applies ([28], Chapter 3).

In contrast, for non-abelian $\mathfrak{g}$ the p-map *fails to be additive*, and it is challenging to understand its structure. However, by axiom (R 1) it is determined by the bracket up to central elements, because $[a^{[p]},x] = (\mathrm{ad}_a)^p(x)$. In particular, if the center is trivial, the p-map is unique, once it exists. This also explains the terminology *restricted*.

Recall that for each group scheme G, the *Lie algebra* $\mathfrak{g} = \mathrm{Lie}(G)$ is defined by the short exact sequence

$$0 \longrightarrow \mathrm{Lie}(G) \longrightarrow G(k[\epsilon]) \longrightarrow G(k) \longrightarrow 0,$$

where $k[\epsilon]$ is the ring of dual numbers, and the map is the restriction with respect to the inclusion $k \subset k[\epsilon]$. As explained in [11], Chapter II, §7 it carries the structure of a restricted Lie algebra, in a functorial way. Also recall that the relative Frobenius $F : G \to G^{(p)}$ is a homomorphism. The resulting *Frobenius kernel* $G[F]$ is a group scheme whose underlying topological space is a singleton.

Let us call G *of height one* if it is of finite type and annihilated by the relative Frobenius map. We then also say that G is a *height-one group scheme*. According to [11], Chapter II, §7, Theorem 3.5 the canonical map

$$\mathrm{Hom}(G,H) \longrightarrow \mathrm{Hom}(\mathrm{Lie}(G),\mathrm{Lie}(H))$$

is bijective whenever G has height one. In particular, the functor $G \mapsto \mathrm{Lie}(G)$ is an equivalence between the category of height-one group schemes and the category of finite-dimensional restricted Lie algebras. We call this the *Demazure–Gabriel Correspondence*. The inverse functor sends $\mathfrak{g}$ to the spectrum of the dual for the Hopf algebra $U^{[p]}(\mathfrak{g})$, which is the universal enveloping algebra $U(\mathfrak{g})$ modulo the ideal generated by the elements $x^p - x^{[p]}$, for $x \in \mathfrak{g}$. From this one deduces the formulas

$$|G| = h^0(\mathscr{O}_G) = p^{\dim(\mathfrak{g})} \quad \text{and} \quad \mathrm{edim}(\mathscr{O}_{G,e}) = \dim(\mathfrak{g}).$$

As customary, we write $\mathfrak{gl}_n(k)$ for the restricted Lie algebra of $n\times n$-matrices, where bracket and p-map are given by commutators and p-powers, and $\mathfrak{sl}_n(k)$ for the ideal of trace zero matrices. Furthermore, k^n denotes the standard vector space, endowed with trivial bracket and p-map.

Let $\mathfrak{a} \subset \mathfrak{g}$ be an ideal, and consider the vector space $\mathrm{Der}_k(\mathfrak{a})$ of all k-linear derivations. Then $\mathrm{Der}_k(\mathfrak{a}) \subset \mathfrak{gl}(\mathfrak{a})$ is a subalgebra. Derivations $D : \mathfrak{g} \to \mathfrak{g}$ satisfying the additional condition $D(a^{[p]}) = (\mathrm{ad}_a)^{p-1}(D(a))$ for all $a \in \mathfrak{a}$ are called *restricted derivations*. Write $\mathrm{Der}'_k(\mathfrak{a})$ for the vector space of all restricted k-derivations. According to [27], Theorem 4 the inclusion $\mathrm{Der}'_k(\mathfrak{a}) \subset \mathrm{Der}_k(\mathfrak{a})$ is a subalgebra. By the Jacobi identity and axiom (R 1), the adjoint map defines a homomorphism

$$\mathfrak{g} \longrightarrow \mathrm{Der}'_k(\mathfrak{a}), \quad x \longmapsto (a \mapsto [x,a]). \tag{1}$$

Given restricted Lie algebras $\mathfrak{h}$ and $\mathfrak{a}$, we are are now interested in *extensions*

$$0 \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{h} \longrightarrow 0,$$

such that $\mathfrak{a} \subset \mathfrak{g}$ becomes an ideal with quotient $\mathfrak{g}/\mathfrak{a} = \mathfrak{h}$. The extension *splits* if the ideal $\mathfrak{a} \subset \mathfrak{g}$ admits a complementary subalgebra $\mathfrak{h}' \subset \mathfrak{g}$. Composing the inverse for the projection $\mathfrak{h}' \to \mathfrak{h}$ with (1), we obtain a homomorphism $\varphi : \mathfrak{h} \to \mathrm{Der}'_k(\mathfrak{a})$.

Conversely, suppose we have such a homomorphism, written as $h \mapsto (a \mapsto \varphi_h(a))$. On the vector space sum $\mathfrak{a} \oplus \mathfrak{h}$, we now define bracket and p-map by

$$\begin{aligned} [a+h, a'+h'] &= [a,a'] + [h,h'] + \varphi_h(a') - \varphi_{h'}(a), \\ (a+h)^{[p]} &= a^{[p]} + h^{[p]} + \sum_{r=1}^{p-1} s_r(a,h). \end{aligned} \tag{2}$$

Lemma 1.1. *The above endows the vector space $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{h}$ with the structure of a restricted Lie algebra, such that $\mathfrak{a}$ and $\mathfrak{h}$ is an ideal and subalgebra, respectively.*

Proof. As explained in [7], Chapter I, §1.8 the bracket turns $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{h}$ into a Lie algebra, having $\mathfrak{a}$ as an ideal and $\mathfrak{h}$ as a subalgebra. Now choose bases $a_i \in \mathfrak{a}$ and $h_j \in \mathfrak{h}$, such that a_i, h_j form a basis for $\mathfrak{g}$. We claim that

$$(\mathrm{ad}_{a_i})^p = \mathrm{ad}_{(a_i)^{[p]}} \quad \text{and} \quad (\mathrm{ad}_{h_j})^p = \mathrm{ad}_{(h_j)^{[p]}} \tag{3}$$

as k-linear endomorphisms of $\mathfrak{g}$. Indeed: Since $\mathfrak{a}$ and $\mathfrak{h}$ are restricted, and by the definition of the bracket in $\mathfrak{g}$, it is enough to verify $(\mathrm{ad}_{a_i})^p(h) = -\varphi_h((a_i)^{[p]})$ for every vector $h \in \mathfrak{h}$, and $(\mathrm{ad}_{h_j})^p(a) = \varphi_{(h_j)^{[p]}}(a)$ for every $a \in \mathfrak{a}$. Since the derivations φ_h are restricted, we have

$$-\varphi_h((a_i)^{[p]}) = -\mathrm{ad}_{a_i}^{p-1}(\varphi_h(a_i)) = -\mathrm{ad}_{a_i}^{p-1}([h,a_i]) = (\mathrm{ad}_{a_i})^p(h).$$

The argument for $(\mathrm{ad}_{h_j})^p(a)$ is similar. Thus (3) holds. According to [54], Theorem 2.3 there is a unique p-map satisfying (3) and the axioms (R 1)–(R 3). By construction, this p-map on $\mathfrak{g}$ coincides with the given p-map on $\mathfrak{a}$ and $\mathfrak{h}$. It thus coincides with (2), in light of the third axiom. □

In the above situation, the restricted Lie algebras $\mathfrak{g} = \mathfrak{a} \rtimes_\varphi \mathfrak{h}$ are called *semidirect products*. Obviously, every split extension of $\mathfrak{h}$ by $\mathfrak{a}$ is of this form. Of particular importance for us is the case $\mathfrak{a} = k^n$ and $\mathfrak{b} = \mathfrak{gl}_1(k)$, where the homomorphism $\varphi : \mathfrak{gl}_1(k) \to \mathfrak{gl}(k^n) = \mathrm{Der}_k(k^n)$ sends scalars to scalar matrices. The resulting restricted Lie algebra is written as $k^n \rtimes \mathfrak{gl}_1(k)$. Here bracket and p-map are given by the formulas

$$[v+\lambda e, v'+\lambda' e] = \lambda v' - \lambda' v \quad \text{and} \quad (v+\lambda e)^{[p]} = \lambda^{p-1}(v+\lambda e), \tag{4}$$

where $e \in \mathfrak{gl}_1(k)$ is the unit element, and $v, v' \in k^n$ are vectors, and $\lambda, \lambda' \in k$ are scalars.

2. Toral rank and p-closed vectors

Let $\mathfrak{g}$ be a finite-dimensional restricted Lie algebra over a ground field k of characteristic $p > 0$, and G be the corresponding height-one group scheme, such that $\mathrm{Lie}(G) = \mathfrak{g}$. Recall that $x \in \mathfrak{g}$ is called *p-closed* if $x^{[p]} \in kx$. Such vectors are called *multiplicative* if $x^{[p]} \neq 0$, and *additive* if $x^{[p]} = 0$. If the vector is non-zero, $\mathfrak{h} = kx$ is a one-dimensional subalgebra, hence corresponds to a subgroup scheme $H \subset G$ of order p. For multiplicative vectors, this is a twisted form of the diagonalizable group scheme $\mu_p = \mathbb{G}_m[F]$. In the additive case, it is isomorphic to the unipotent group scheme $\alpha_p = \mathbb{G}_a[F]$. This basic fact has many geometric applications: For results concerning K3 surfaces, Enriques surfaces and Kummer surfaces, see [48], [50] and [32].

Proposition 2.1. *Every vector in $\mathfrak{g} = k^n \rtimes \mathfrak{gl}_1(k)$ is p-closed. The same holds for $\mathfrak{g} = \mathfrak{sl}_2(k)$ in characteristic $p \geq 3$.*

Proof. The first assertion immediately follows from (4). Recall that $\mathfrak{sl}_2(k)$ is the restricted Lie algebra comprising the traceless matrices $A = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \in \mathrm{Mat}_2(k)$. The characteristic polynomial $\chi_A(T) = T^2 + d$ depends only on the determinant $d = -a^2 - bc$, so the possible Jordan normal forms are

$$\begin{pmatrix} \sqrt{d} & 0 \\ 0 & -\sqrt{d} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Computing p-powers via the above normal forms, we see that $A^{[p]} = d^{(p-1)/2}A$. $\square$

The traceless matrices $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $x = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $y = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ form a basis of $\mathfrak{sl}_2(k)$, and the structural constants are given by

$$[h, x] = 2x, \quad [h, y] = 2y, \quad [x, y] = h^{[p]} = h, \quad x^{[p]} = y^{[p]} = 0.$$

One also says that (h, x, y) is an $\mathfrak{sl}_2(k)$-*triple*. For $p \geq 3$, it follows that for each non-zero $a \in \mathfrak{sl}_2(k)$, the adjoint map ad_a is bijective, hence $\mathfrak{sl}_2(k)$ is simple. In contrast, for $p = 2$ we have a central extension $0 \to \mathfrak{gl}_1(k) \to \mathfrak{sl}_2(k) \to k^2 \to 0$, where the kernel corresponds to scalar matrices. The extension does not split, because $A^{[2]} \neq 0$ for all matrices not contained in the kernel.

If k is algebraically closed, the *toral rank* for a restricted Lie algebra $\mathfrak{g}$ is the maximal integer $r \geq 0$ for which there is an embedding $\mathfrak{gl}_1(k)^{\oplus r} \subset \mathfrak{g}$. In terms of vectors, the condition means that there are linearly independent $x_1, \ldots, x_r \in \mathfrak{g}$ with $[x_i, x_j] = 0$ and $x_i^{[p]} = x_i$. For general fields k, we define the toral rank as the toral rank of the base-change $\mathfrak{g} \otimes_k k^{\mathrm{alg}}$. Following the notation in [12], Exposé XII, Section 2 we denote this integer by $\rho_t(\mathfrak{g}) \geq 0$. By Hilbert's Nullstellensatz the toral rank does not change under field extensions. According to [6], Lemma 1.7.2 it satisfies $\rho_t(\mathfrak{g}) = \rho_t(\mathfrak{n}) + \rho_t(\mathfrak{g}/\mathfrak{n})$ for each ideal $\mathfrak{a} \subset \mathfrak{g}$. In other words, it is additive in extensions. Obviously $0 \leq \rho_t(\mathfrak{g}) \leq \dim(\mathfrak{g})$.

Proposition 2.2. *The following are equivalent:*

(i) *The restricted Lie algebra $\mathfrak{g}$ has maximal toral rank $\rho_t(\mathfrak{g}) = \dim(\mathfrak{g})$.*
(ii) *The group scheme G is a twisted form of some $\mu_p^{\oplus r}$.*
(iii) *The group scheme G is multiplicative.*

Proof. It suffices to treat the case that k is algebraically closed. The implications (i)$\Leftrightarrow$(ii)$\Rightarrow$(iii) are obvious. Now suppose that (iii) holds. Since $k = k^{\mathrm{alg}}$ the group scheme G is diagonalizable, whence the spectrum of the Hopf algebra $k[\Lambda]$ for some finitely generated abelian group Λ. We have $p\Lambda = 0$ because G has height one. Choosing an $\mathbb{F}_p$-basis for Λ gives $G = \mu_p^{\oplus r}$, thus (ii) holds. $\square$

The other extreme is somewhat more involved:

Proposition 2.3. *The following are equivalent:*

(i) *The restricted Lie algebra $\mathfrak{g}$ has minimal toral rank $\rho_t(\mathfrak{g}) = 0$.*
(ii) *There is some exponent $\nu \geq 0$ with $x^{[p^\nu]} = 0$ for all vectors $x \in \mathfrak{g}$.*
(iii) *There are ideals $0 = \mathfrak{a}_0 \subset \ldots \subset \mathfrak{a}_r = \mathfrak{g}$ inside $\mathfrak{g}$ with quotients $\mathfrak{a}_i/\mathfrak{a}_{i-1} \simeq k$.*
(iv) *There are normal subgroup schemes $0 = N_0 \subset \ldots \subset N_r = G$ inside G with quotients $N_i/N_{i-1} \simeq \alpha_p$.*

(v) *The group scheme G is unipotent.*

Proof. The implications (iv)⇒(v) and (iii)⇒(ii) and (ii)⇒(i) are trivial, whereas (iv)⇔(iii) follows from the Demazure–Gabriel Correspondence.

We next verify (v)⇒(i). Without loss of generality we may assume that k is algebraically closed. Then there is a composition series G_j inside G such that G_j/G_{j-1} is isomorphic to a subgroup scheme of the additive group $\mathbb{G}_a$. This already lies in $\alpha_p = \mathbb{G}_a[F]$, because G has height one. For the corresponding subalgebras $\mathfrak{b}_j$ inside $\mathfrak{g}$ this means $\mathfrak{b}_j/\mathfrak{b}_{j-1} \subset k$. The additivity of toral rank implies $\rho_t(\mathfrak{g}) = 0$.

To see (i)⇒(ii) we may assume that k is algebraically closed, and then the implication follows from [43], Corollary 2. For (ii)⇒(iii) we use $(\mathrm{ad}_a)^{p^\nu} = \mathrm{ad}_{a^{[p^\nu]}} = 0$, and conclude with Engel's Theorem ([7], Chapter I, §4.2) that the underlying Lie algebra $\mathfrak{g}$ is nilpotent. Now recall that the center $\mathfrak{C}(\mathfrak{g})$ is invariant under the p-map. In turn, the upper central series, which is recursively defined by $\mathfrak{g}_{i+1}/\mathfrak{g}_i = \mathfrak{C}(\mathfrak{g}/\mathfrak{g}_i)$, yields a sequence of ideals $0 = \mathfrak{g}_0 \subset \ldots \subset \mathfrak{g}_s = \mathfrak{g}$ having abelian quotients. This reduces our problem to the case that $\mathfrak{g}$ itself is abelian. We proceed by induction on $n = \dim(\mathfrak{g})$. The case $n = 0$ is trivial. Suppose now that $n > 0$, and that (iii) holds for $n-1$. Fix some $x \neq 0$, and consider the largest exponent $d \geq 1$ such that $x^{[p^d]} \neq 0$. Replacing x by $x^{[p^d]}$, we may assume that $x^{[p]} = 0$. Then $\mathfrak{a}_1 = kx$ is a one-dimensional ideal. The quotient $\mathfrak{g}' = \mathfrak{g}/\mathfrak{a}$ has dimension $n' = n-1$, and furthermore $\rho_t(\mathfrak{g}') = 0$ by additivity of toral rank. To the induction hypothesis applies to $\mathfrak{g}' = \mathfrak{g}/\mathfrak{a}$, and the Isomorphism Theorem gives the desired ideals in $\mathfrak{g}$. □

3. Automorphism group schemes

Let k be a ground field. Write (Aff/k) for the category of affine k-schemes, which we usually write as $T = \mathrm{Spec}(R)$. Recall that an *algebraic space* is a contravariant functor $X : (\mathrm{Aff}/k) \to (\mathrm{Set})$ satisfying the sheaf axiom with respect to the étale topology, such that the diagonal $X \to X \times X$ is relatively representable by schemes, and that there is an étale surjection $U \to X$ from some scheme U. According to [52], Lemma 076M, the sheaf axiom already holds with respect to the fppf topology. Throughout, we use the fppf topology if not stated otherwise. Algebraic spaces are important generalizations of schemes, because modifications, quotients, families, or moduli spaces of schemes are frequently algebraic spaces rather than schemes. We refer to the monographs of Olsson [42], Laumon and Moret-Bailly [34], Artin [2], Knutson [31], and to the stacks project [52], Part 4.

Let X be a scheme, or more generally an algebraic space, that is separated and of finite type. Recall that the R-valued points of the Hilbert functor $\mathrm{Hilb}_{X/k}$ are the closed subschemes $Z \subset X \otimes R$ such that the projection $Z \to \mathrm{Spec}(R)$ is proper and flat. Regarding automorphisms $f : X \otimes R \to X \otimes R$ as graphs, we see that $\mathrm{Aut}_{X/k}$ is an open subfunctor. According to [1], Theorem 6.1 the Hilbert functor is representable by an algebraic space that is separated and locally of finite type. In turn, the same holds for $\mathrm{Aut}_{X/k}$, which additionally carries a group structure. Using descent and translations, one sees that it must be schematic. The Lie algebra for the automorphism group scheme is given by

$$\mathrm{Lie}(\mathrm{Aut}_{X/k}) = H^0(X, \Theta_{X/k}),$$

where $\Theta_{X/k} = \underline{\mathrm{Hom}}(\Omega^1_{X/k}, \mathscr{O}_X)$ is the coherent sheaf dual to the sheaf of Kähler differentials.

We now assume that X is proper, and that the ground field has characteristic $p > 0$. Then $\mathfrak{g} = H^0(X, \Theta_X)$ is a restricted Lie algebra of finite dimension, which corresponds to the Frobenius kernel $G[F]$ for the automorphism group scheme $G = \mathrm{Aut}_{X/k}$. Note that $G[F]$ is a height-one group scheme, of order p^n, where $n = h^0(\Theta_{X/k})$.

Let H be a group scheme that is separated and locally of finite type, $f : H \to G$ be a homomorphism, and P be a H-torsor. The latter is an algebraic space, endowed with a free and transitive H-action. The set of isomorphism classes comprise the non-abelian cohomology $H^1(k, H)$, formed with respect to the fppf topology. On the product $P \times X$ we get a diagonal action. This action is free, because it is free on the first factor. It follows that the quotient ${}^PX = H\backslash(P \times X)$ exists as an algebraic space (see for example [35], Lemma 1.1). We have ${}^PX \simeq X$ provided that P is trivial, that is, contains a rational point. In any case, there is an étale surjection $U \to P$ from some scheme U. According to Hilbert's Nullstellensatz, every closed point $a \in U$ defines a finite field extension $k' = \kappa(a)$, and we see that ${}^PX \otimes k' \simeq X \otimes k'$. We therefore say that ${}^PX$ is a *twisted form* of X. Indeed, every algebraic space Y that becomes isomorphic to X after some field extension is of this form, with $H = \mathrm{Aut}_{X/k}$.

Our $f : H \to G = \mathrm{Aut}_{X/k}$ induces a homomorphism $c : H \to \mathrm{Aut}_{G/k}$, which sends $h \in H(R)$ to the inner automorphism $g \mapsto f(h)gf(h)^{-1}$. This gives a twisted form ${}^PG$ of G, and its Lie algebra ${}^P\mathfrak{g}$ is a twisted form of $\mathfrak{g}$. In fact, one may view $\mathfrak{g}$ as a vector scheme as in Section 7, regard bracket and p-map as morphisms of schemes, and obtains ${}^P\mathfrak{g}$ by taking the rational points on the twisted form of the vector scheme, formed via the derivative $c' : H \to \mathrm{Aut}_{\mathfrak{g}/k}$.

Lemma 3.1. *There is a canonical identification* ${}^P\mathrm{Aut}_{X/k} = \mathrm{Aut}_{{}^PX/k}$*, where on the left we take twist with respect to* $c : H \to \mathrm{Aut}_{G/k}$*. The restricted Lie algebra for this group scheme is* ${}^P\mathfrak{g}$*, where we twist with respect to* $c' : H \to \mathrm{Aut}_{\mathfrak{g}/k}$*.*

Proof. This follows from very general considerations in [18], Chapter III, which can be made explicit as follows: Consider the canonical morphism

$$P \times \mathrm{Aut}_{X/k} \longrightarrow \mathrm{Aut}_{{}^PX}, \quad (p, \psi) \longmapsto (H \cdot (p, x) \mapsto H \cdot (p, \psi(x)),$$

where the description on the right is viewed as a natural transformation for R-valued points. This is well-defined, because in presence of $p \in P(R)$ the projection $\{p\} \times X(R) \to ({}^PX)(R)$ is bijective. For each $h \in H(R)$, the element $(hp, h\psi h^{-1})$ sends the orbit $H \cdot (p, x) = H \cdot (hp, hx)$ to the orbit $H \cdot (hp, h\psi(x)) = H \cdot (p, \psi(x))$. Thus the above transformation descends to a morphism ${}^P\mathrm{Aut}_{X/k} \to \mathrm{Aut}_{{}^PX}$, where H acts via conjugacy on $\mathrm{Aut}_{X/k}$. The same argument applies for the Frobenius kernel, and equivalently to the restricted Lie algebra. $\square$

We now change notation, and suppose that $G = X$ is a height-one group scheme, and write $\mathfrak{g} = \mathrm{Lie}(G)$. One easily checks that $\mathrm{Aut}_{G/k}$ is a closed subgroup scheme of the general linear group $\mathrm{GL}_{V/k}$, where $V = H^0(G, \mathscr{O}_G)$. By the Demazure–Gabriel Correspondence, used in the relative form, we get an identification $\mathrm{Aut}_{G/k} = \mathrm{Aut}_{\mathfrak{g}/k}$. The latter can be constructed directly: Choose a basis $e_1, \ldots, e_n \in \mathfrak{g}$. Then $\mathrm{Aut}_{\mathfrak{g}/k}$ is the closed subgroup scheme inside $\mathrm{GL}_{k,n}$ respecting the structural equations $[e_r, e_s] = \sum \lambda_{r,s,i} e_i$ and $e_r^{[p]} = \sum \mu_{r,j} e_j$. For later use, we compute some automorphism group schemes $\mathrm{Aut}_{\mathfrak{g}/k}$:

Proposition 3.2. *The following table gives automorphism group scheme and resulting cohomology sets for the restricted Lie algebras k, $\mathfrak{gl}_1(k)$, $k \rtimes \mathfrak{gl}_1(k)$ and $\mathfrak{sl}_2(k)$, where $\star$ denotes a singleton, and the last column is only valid for $p \geq 3$:*

$\mathfrak{g}$	k	$\mathfrak{gl}_1(k)$	$k \rtimes \mathfrak{gl}_1(k)$	$\mathfrak{sl}_2(k)$
$\mathrm{Aut}_{\mathfrak{g}/k}$	$\mathbb{G}_m$	μ_{p-1}	$\mathbb{G}_a \rtimes \mathbb{G}_m$	PGL_2
$H^1(k, \mathrm{Aut}_{\mathfrak{g}/k})$	$\{1\}$	$k^\times / k^{\times(p-1)}$	$\star$	$k^\times / k^{\times 2}$

Proof. For the first case $\mathfrak{g} = k$ we immediately get $\mathrm{Aut}_{\mathfrak{g}/k} = \mathrm{GL}_1 = \mathbb{G}_m$, and Hilbert 90 gives $H^1(k, \mathbb{G}_m) = 0$.

In the second case, the restricted Lie algebra $\mathfrak{g} = \mathfrak{gl}_1(k)$ is generated by one element A_1, which gives an embedding $\mathrm{Aut}_{\mathfrak{g}/k} \subset \mathbb{G}_m$. The structure for $\mathfrak{g}$ is given by $A_1^{[p]} = A_1$. For each k-algebra R and each invertible scalar $\lambda \in R^\times$ we thus have $\lambda^p A_1^{[p]} = \lambda A_1$, and thus $\lambda^{p-1} = 1$. Conversely, each such λ gives an automorphism, hence $\mathrm{Aut}_{\mathfrak{g}/k} = \mu_{p-1}$. The Kummer sequence yields $H^1(k, \mathrm{Aut}_{\mathfrak{g}/k}) = k^\times / k^{\times p-1}$.

The restricted Lie algebra $\mathfrak{g} = k \rtimes \mathfrak{gl}_1(k)$ is generated inside $\mathfrak{gl}_2(k)$ by the matrices $A_1 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $A_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, which gives an embedding $\mathrm{Aut}_{\mathfrak{g}/k} \subset \mathrm{GL}_2$. For each R-valued point $\varphi = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ from the automorphism group scheme, the condition $[\varphi(A_1), \varphi(A_2)] = \varphi(A_1)$ implies $c = 0$ and $a = ad$. It follows $a \in R^\times$ and $d = 1$, and we obtain $\mathrm{Aut}_{\mathfrak{g}/k} \subset \mathbb{G}_a \rtimes \mathbb{G}_m$. Conversely, one easily sees that each matrix with $c = 0$ and $d = 1$ yields an automorphism of the restricted Lie algebra. Now let T be a torsor over k with respect to $\mathbb{G}_a \rtimes \mathbb{G}_m$. The induced $\mathbb{G}_m$-torsor has a rational point, by Hilbert 90. Its preimage $T' \subset T$ is a torsor for $\mathbb{G}_a$. Over any affine scheme, the higher cohomology of $\mathbb{G}_a$ vanishes, so T' also contains a rational point, and the torsor T is trivial.

We come to the last case $\mathfrak{g} = \mathfrak{sl}_2(k)$, which is freely generated by the matrices $A_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $A_2 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $A_3 = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. This gives an inclusion $\mathrm{Aut}_{\mathfrak{g}/k} \subset \mathrm{GL}_3$. We already saw in the proof for Proposition 2.1 that $A^{[p]} = \det(A)^{(p-1)/2} A$ for all $A = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)$. Moreover, $\det(A) = -a^2 - bc$ defines, up to sign, the standard smooth quadratic form on $\mathfrak{sl}_2(k)$ viewed as the affine space $\mathbb{A}^3$, and one obtains an inclusion $\mathrm{Aut}_{\mathfrak{g}/k} \subset O(3)$. Combining Jacobson's arguments in [30], page 283 and [26], page 493 we conclude every automorphism of $\mathfrak{sl}_2(k)$ is given by $A \mapsto SAS^{-1}$, and it follows that that $\mathrm{Aut}_{\mathfrak{g}/k} \subset \mathrm{SO}(3)$. The latter is smooth, connected and three-dimensional. On the other hand, the conjugacy map defines an inclusion $\mathrm{PGL}_2 \to \mathrm{Aut}_{\mathfrak{g}/k}$. Since PGL_2 is smooth, connected and three-dimensional, we see that the inclusions $\mathrm{PGL}_2 \subset \mathrm{Aut}_{\mathfrak{g}/k} \subset \mathrm{SO}(3)$ are qualities.

Finally, we have a central extension $0 \to \mathbb{G}_m \to \mathrm{GL}_2 \to \mathrm{PGL}_2 \to 1$, and get maps in non-abelian cohomology

$$H^1(k, \mathrm{GL}_2) \longrightarrow H^1(k, \mathrm{PGL}_2) \longrightarrow H^2(k, \mathbb{G}_m).$$

The term on the left is a singleton, by Hilbert 90, whereas the term on the right equals the Brauer group $\mathrm{Br}(k)$. It follows that the coboundary map is injective ([18], Chapter IV, Proposition 4.2.8)), and is contained in the 2-torsion of the Brauer group ([19], Proposition 1.4). From the Kummer sequence one gets $\mathrm{Br}(k)[2] = H^1(k, \mu_2) = k^\times/k^{\times 2}$. Each member of this group can be realized by some smooth conic curve in $\mathbb{P}^2$. Summing up, we have an identification $H^1(k, \mathrm{Aut}_{\mathfrak{g}/k}) = k^\times/k^{\times 2}$. □

4. Quotients by height-one group schemes

Let k be a ground field of characteristic $p > 0$, and G a height-one group scheme, with restricted Lie algebra $\mathfrak{g}$. Suppose X is an scheme endowed with a G-action. Taking derivatives, we obtain a homomorphism $\mathfrak{g} \to H^0(X, \Theta_{X/k})$ of restricted Lie algebras. According to [11], Chapter II, §7, Proposition 3.10 any such homomorphism comes from a unique G-action. Note that this does not require any finiteness assumption for the scheme X.

We now show that such actions admits a *categorical quotient* in the category (Sch/k) ([41], Definition 0.5). To this end we temporarily change notation and write the schemes in question as pairs, comprising a topological space and a structure sheaf. Our task is to construct the categorical quotient $(Y, \mathscr{O}_Y)$ for the action on $(X, \mathscr{O}_X)$. First recall that the image $\mathscr{O}_X^p$ of the homomorphism $\mathscr{O}_X \to \mathscr{O}_X$, $f \mapsto f^p$ is a quasicoherent $\mathscr{O}_X$-algebra, with algebra structure $f \cdot g^p = (fg)^p$. In turn, the ringed space $(X, \mathscr{O}_X^p)$ is a $\mathbb{F}_p$-scheme. Choose a vector space basis $D_1, \ldots, D_n \in \mathfrak{g}$. The canonical inclusion $\mathscr{O}_X^p \subset \mathscr{O}_X$ turns $\mathscr{O}_X$ into a quasicoherent $\mathscr{O}_X^p$-algebra, and yields the absolute Frobenius morphism $(X, \mathscr{O}_X) \to (X, \mathscr{O}_X^p)$. The derivations $D_i : \mathscr{O}_X \to \mathscr{O}_X$ are $\mathscr{O}_X^p$-linear, and we write $\mathscr{O}_X^{\mathfrak{g}} = \bigcap_{i=1}^n \mathrm{Ker}(D_i)$ for the intersection of kernels. This is another quasicoherent $\mathscr{O}_X^p$-algebra. Setting $Y = X$ and $\mathscr{O}_Y = \mathscr{O}_X^{\mathfrak{g}}$, we obtain a scheme $(Y, \mathscr{O}_Y)$ that is affine over $(X, \mathscr{O}_X^p)$. The identity $\mathrm{id} : X \to Y$ and the canonical inclusion $\iota : \mathscr{O}_Y \subset \mathscr{O}_X$ define a morphism of $\mathbb{F}_p$-schemes

$$(\mathrm{id}, \iota) : (X, \mathscr{O}_X) \longrightarrow (Y, \mathscr{O}_Y).$$

The following should be well-known:

Lemma 4.1. *The above morphism of schemes is a categorical quotient in* (Sch/k)*. Moreover, the formation of the quotient is compatible with flat base-change in the scheme* $(Y, \mathscr{O}_Y)$*.*

Proof. First note that the $\mathscr{O}_Y \subset \mathscr{O}_X$ is invariant with respect to multiplication of scalars $\lambda \in k$, so the morphism belongs to the category (Sch/k). Furthermore, the formation of kernels and finite intersections for maps between quasicoherent sheaves on schemes is compatible with flat base-change, and in particular the formation of $(Y, \mathscr{O}_Y)$ is compatible with flat base-change.

We now verify the universal property. Let $(T, \mathscr{O}_T)$ be scheme endowed with the trivial G-action, and $(f, \varphi) : (X, \mathscr{O}_X) \to (T, \mathscr{O}_T)$ be an equivariant morphism. Obviously, there is a unique continuous map $g : Y \to T$ with $f = g \circ \mathrm{id}$. The trivial G-action on $(T, \mathscr{O}_T)$ corresponds to the zero map $\mathfrak{g} \to H^0(T, \mathscr{O}_T)$, and equivariance ensures that $f^{-1}(\mathscr{O}_T) \to \mathscr{O}_X$ factors over the injection $\mathscr{O}_Y \subset \mathscr{O}_X$. This gives a unique morphism $(g, \psi) : (Y, \mathscr{O}_Y) \to (T, \mathscr{O}_T)$ of ringed spaces that factors (f, φ). For each point $a \in X$, the local map $\mathscr{O}_{T,f(a)} \to \mathscr{O}_{X,a}$ factors over $\mathscr{O}_{Y,a}$, and it follows that $\psi : \mathscr{O}_{T,g(a)} \to \mathscr{O}_{Y,a}$ is local. Thus (g, ψ) is a morphism in the category (Sch/k), which shows the universal property. □

We now revert back to the usual notation, and write $Y = X/G$ for the quotient of the action $\mu : G \times X \to X$, with quotient map $q : X \to Y$. Clearly, this map is surjective, Y carries the quotient topology, and the set-theoretical image of $\mu \times \mathrm{pr}_2 : G \times X \to X \times X$ equals the fiber product $X \times_Y X$. By construction, for each open set $U \subset Y$ and each local section $f \in \Gamma(U, q_*(\mathscr{O}_X))$, we have $f \in \Gamma(U, \mathscr{O}_Y)$ if and only if $f \circ \mu = f \circ \mathrm{pr}_2$ as morphisms $G \times q^{-1}(U) \to \mathbb{A}^1$. Summing up, our categorical quotient is also a *uniform geometric quotient*, in the sense of [41], Definition 0.7. The following observation will be useful:

Proposition 4.2. *Suppose that X is integral, with function field $F = \mathscr{O}_{X,\eta}$. Then $\mathscr{O}_{Y,a} = \mathscr{O}_{X,a} \cap (F^{\mathfrak{g}})$ for each point $a \in X$. Moreover, the scheme Y is normal provided this holds for Y.*

Proof. Set $R = \mathscr{O}_{X,a}$, such that $R^{\mathfrak{g}} = \mathscr{O}_{Y,a}$. Choose a basis $D_1, \ldots, D_n \in \mathfrak{g}$, and consider the resulting commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & R^{\mathfrak{g}} & \longrightarrow & R & \longrightarrow & R^{\oplus n} \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F^{\mathfrak{g}} & \longrightarrow & F & \longrightarrow & F^{\oplus n} \end{array}$$

where the horizontal maps on the right are given by $s \mapsto (D_1(s), \ldots, D_n(s))$. The commutativity of the left square gives $R^{\mathfrak{g}} \subset R \cap F^{\mathfrak{g}}$, and the injectivity of the vertical map on the right ensures the reverse inclusion, by a diagram chase. Now suppose that R is normal, and $f \in F^{\mathfrak{g}}$ satisfies an integral equation over the subring $R^{\mathfrak{g}}$. This is also an integral equation over R, hence $f \in R \cap F^{\mathfrak{g}} = R^{\mathfrak{g}}$. □

5. Inertia and Jacobson correspondence

The goal of this section is provide a new, more geometric interpretation of the *Jacobson Correspondence* ([25] and [29]). We start by recalling this correspondence, which relates certain subfields and restricted Lie algebras, in Bourbaki's formulation ([8], Chapter V, §13, No. 3, Theorem 3):

Let F be a field of characteristic $p > 0$. It comes with a subfield F^p and a restricted Lie algebra $\mathfrak{g} = \mathrm{Der}(F)$ over F^p that is also endowed with the structure of an F-vector space. Note that the bracket is F^p-linear but in general not F-bilinear. Rather, we have the formula

$$[\lambda D, \lambda' D'] = \lambda\lambda' \cdot [D, D'] + \lambda D(\lambda') \cdot D' - \lambda' D'(\lambda) \cdot D. \tag{5}$$

Throughout, a subgroup $\mathfrak{h} \subset \mathfrak{g}$ is called an *F^p-subalgebra with F-multiplication* if it is stable under bracket, p-map, and multiplication by scalars $\lambda \in F$. It is thus a restricted Lie algebra over F^p, endowed with the F-multiplication as *additional structure.* Consider the ordered sets

$$\Phi = \{E \mid F^p \subset E \subset F \text{ is an intermediate field}\},$$
$$\Psi = \{\mathfrak{h} \mid \mathfrak{h} \subset \mathfrak{g} \text{ is an } F^p\text{-subalgebra with } F\text{-multiplication}\}.$$

Similar to classical Galois theory for separable algebraic extensions, one has inclusion-reversing maps $\Phi \to \Psi$ and $\Psi \to \Phi$ given by

$$E \longmapsto \mathrm{Der}_E(F) \quad \text{and} \quad \mathfrak{h} \longmapsto F^{\mathfrak{h}},$$

respectively. Here $F^{\mathfrak{h}}$ denotes the intersection of the kernels for $D : F \to F$, where $D \in \mathfrak{h}$ runs over all elements. Then the Jacobson Correspondence asserts that the above maps induces a bijection between the intermediate fields $F^p \subset E \subset F$ having $[F : E] < \infty$ and the F^p-subalgebras with F-multiplication $\mathfrak{h} \subset \mathfrak{g}$ having $\dim_F(\mathfrak{h}) < \infty$. Moreover, under this bijection $[F : E] = p^{\dim_F(\mathfrak{h})}$ holds.

In particular, if F has *finite p-degree*, which means that $F^p \subset F$ is finite, we get an unconditional identification

$$\{\text{intermediate fields } E\} = \{F^p\text{-subalgebras } \mathfrak{h} \text{ with } F\text{-multiplication}\}.$$

Forgetting the F-multiplication, the restricted Lie algebra $\mathfrak{h} = \mathrm{Der}_E(F)$ corresponds to a height-one group scheme H, with $\mathfrak{h} = \mathrm{Lie}(H)$. By construction, this coincides with the Frobenius kernel of the affine group scheme $\mathrm{Aut}_{F/E}$.

We now consider the following set-up geared towards geometric applications: Let k be a ground field of characteristic $p > 0$, and F be some extension field; one should think of the function field of some proper integral scheme. Let H be a height-one group scheme over k, with corresponding restricted Lie algebra $\mathfrak{h} = \mathrm{Lie}(H)$. Suppose we have a faithful action of the group scheme H on the scheme $\mathrm{Spec}(F)$, in other words, a homomorphism $\mathfrak{h} \to \mathrm{Der}_k(F)$ that is k-linear and injective. Throughout, we regard this homomorphisms also as an inclusion.

Let $E = F^{\mathfrak{h}}$, such that $\mathfrak{h} \subset \mathrm{Der}_E(F)$. Then the field E contains the composite $k \cdot F^p$, and its spectrum is the categorical quotient $\mathrm{Spec}(F)/H$, according to Proposition 4.1. Moreover, we obtain subspace $\mathfrak{h} \subset \mathfrak{h} \cdot E \subset \mathfrak{h} \cdot F$ inside $\mathrm{Der}_E(F)$. This are subvector spaces over k and E and F, respectively. Obviously we have

$$\dim_F(\mathfrak{h} \cdot F) \leq \dim_E(\mathfrak{h} \cdot E) \leq \dim_k(\mathfrak{h}).$$

Let us unravel how these various fields and vector spaces are related:

Proposition 5.1. *In the above situation, the following holds:*

(i) *The subspace $\mathfrak{h} \subset \mathrm{Der}_E(F)$ contains an F-basis, such that $\mathfrak{h} \cdot F = \mathrm{Der}_E(F)$.*
(ii) *The canonical inclusions $E = F^{\mathfrak{h}} \subset F^{\mathfrak{h} \cdot E} \subset F^{\mathfrak{h} \cdot F}$ are equalities.*
(iii) *The subspace $\mathfrak{h} \cdot E \subset \mathrm{Der}_E(F)$ is stable with respect to bracket and p-map.*
(iv) *The extension $E \subset F$ is finite, of degree $[E : F] = p^{\dim_F(\mathfrak{h} \cdot F)}$.*

Proof. To see (ii), choose a k-generating set $D_1, \ldots, D_n \in \mathfrak{h}$. Clearly, $F^{\mathfrak{h}}$ coincides with the intersection of the $\mathrm{Ker}(D_i : F \to F)$. Since $D_1, \ldots, D_n \in \mathfrak{h} \cdot F$ is an F-generating set as well, this intersection coincides with $F^{\mathfrak{h} \cdot F}$, and the equalities $F^{\mathfrak{h}} = F^{\mathfrak{h} \cdot E} = F^{\mathfrak{h} \cdot F}$ follow.

We next verify that the F-vector subspace $\mathfrak{h} \cdot F \subset \mathrm{Der}_E(F)$ is stable under bracket and p-map. The former follows from (5), the latter from the axioms (R 2) and (R 3) for restricted Lie algebras in Section 1. In turn, $\mathfrak{h} \cdot F \subset \mathrm{Der}_E(F)$ is an F^p-subalgebra with F-multiplication, obviously of finite F-dimension. Now the Jacobson Correspondence applied to $E = F^{\mathfrak{h} \cdot F}$ shows (iv). Applying the correspondence once more reveals $\mathfrak{h} \cdot F = \mathrm{Der}_E(F)$, and (i) follows. The above reasoning likewise shows that the E-vector subspace $\mathfrak{h} \cdot E \subset \mathrm{Der}_E(F)$ is stable under bracket and p-map, which reveals (iii). □

We now seek a more geometric understanding of the above facts. Set $\mathfrak{h}_E = \mathfrak{h} \otimes_k E$, and consider the E-linearization $\mathfrak{h}_E \to \mathrm{Der}_E(F)$ of our inclusion $\mathfrak{h} \subset \mathrm{Der}_E(F)$. Write

$\mathfrak{h}_E^{\text{triv}} \subset \mathfrak{h}_E$ for the kernel. This is an ideal, giving an inclusion $\mathfrak{h}_E/\mathfrak{h}_E^{\text{triv}} \subset \mathrm{Der}_E(F)$. Now recall that H denotes the height-one group scheme with $\mathrm{Lie}(H) = \mathfrak{h}$. Write $H_E = H \otimes_k E$ for its base-change, and $H_E^{\text{triv}} \subset H_E$ for the normal subgroup scheme corresponding to $\mathfrak{h}_E^{\text{triv}}$. This acts trivially on $\mathrm{Spec}(F)$, whereas the quotient H_E/H_E^{triv} acts faithfully.

Proposition 5.2. *The action of the group scheme H_E on $\mathrm{Spec}(F)$ is transitive.*

Proof. Recall that for any site $\mathcal{C}$, the action of a group-valued sheaf G on a sheaf Z is called *transitive* if the morphism $\mu \times \mathrm{pr}_2 : G \times Z \to Z \times Z$ is an epimorphism, where $\mu : G \times Z \to Z$ denotes the action.

In our situation the site is (Aff/E), endowed with the fppf topology. Set $G = H_E/H_E^{\text{triv}}$ and $Z = \mathrm{Spec}(F)$. We have to check that for any R-valued points $a, b \in Z(R)$, there is an fppf extension $R \subset R'$ and some $\sigma \in G(R')$ that sends the base-change $a \otimes R'$ to $b \otimes R'$. Replacing R by $R \otimes_E F$, we may assume that R is an F-algebra. Choose a p-basis for the extension $E \subset F$, such that $F = E[T_1, \ldots, T_r]/(T_1^p - \mu_1, \ldots, T_r^p - \mu_r)$ for some scalars $\mu_i \in E$. Then

$$F \otimes_E R = R[s_1, \ldots, s_r]/(s_1^p, \ldots, s_r^p)$$

for the elements $s_i = T_1 \otimes 1 - 1 \otimes T_1$. The R-valued points of Z thus correspond to $s_i \mapsto \lambda_i$, where $\lambda_i \in R$ satisfy $\lambda_i^p = 0$. It suffices to treat the case that $a, b \in Z(R)$ is given by by $s_i \mapsto 0$ and $s_i \mapsto \lambda_i$, respectively.

The differentials $dT_i \in \Omega^1_{F/E}$ form an F-basis. The dual basis inside $\mathrm{Der}_E(F) = \mathrm{Hom}(\Omega^1_{F/E}, F)$ are the partial derivatives $\partial/\partial T_i$. Clearly we have $[\partial/\partial T_i, \partial/\partial T_j] = (\partial/\partial T_i)^{[p]} = 0$. Consequently, the linear combination $D = \sum \lambda_i \partial/\partial T_i$ satisfies $D^{[p]} = 0$, thus D is an *additive* element inside $\mathrm{Der}_R(F \otimes_E R)$. Note that this would fail with coefficients from $F \otimes_E R$ rather then R. By the Demazure–Gabriel Correspondence, it yields a homomorphism of group schemes $\alpha_{p,R} \to \mathrm{Aut}_{F/E} \otimes_E R$.

According to Proposition 5.1 we have $\mathrm{Der}_E(F) = \mathfrak{h} \cdot F$, so there are elements $D_1, \ldots, D_r \in \mathfrak{h} \cdot E = \mathfrak{h}_E/\mathfrak{h}_E^{\text{triv}}$ that form an F-basis of $\mathrm{Der}_E(F)$. In particular, we may write $\sum \lambda_i \partial/\partial T_i = \sum \alpha_i D_i$ for some $\alpha_i \in R$. In turn, we get an additive element $D \in (\mathfrak{h}_E/\mathfrak{h}_E^{\text{triv}}) \otimes_E R$, so our homomorphism of group schemes has a factorization $\alpha_{p,R} \to G_R$. For $R' = R[\sigma]/(\sigma^p)$ we get a canonical element $\sigma \in \alpha_{p,R'}$, whose image is likewise denoted by $\sigma \in G(R')$. By construction, we have

$$\sigma^*(s_j) = D(s_j) = \sum_i \lambda_i \partial T_j/\partial T_i = \lambda_j,$$

for all $1 \leq j \leq n$, and the desired property $\sigma \cdot a = b$ follows. $\square$

Note that the E-scheme $Z = \mathrm{Spec}(F)$ does not contain a rational point, except for $\mathfrak{h} = 0$. The existence of such a point would allow us to form the inertia subgroup scheme and view Z as a *homogeneous space*. However, we can achieve this after further base-change:

Regard $A = F \otimes_E F$ as an F-algebra via $\lambda \mapsto 1 \otimes \lambda$. Then the multiplication map $\lambda \otimes \mu \mapsto \lambda\mu$ yields a canonical retraction. Indeed, A is a local Artin ring with residue field $A/\mathfrak{m}_A = F$. In turn, $Z_F = Z \otimes F$ has a unique rational point $z_0 \in Z_F$. Write $H_F^{\text{inert}} = I(z_0)$ for the resulting *inertia subgroup scheme* inside $H_F = H \otimes_k F$. By the Demazure–Gabriel Correspondence, it is given by a Lie subalgebra $\mathfrak{h}_F^{\text{inert}}$ inside

$\mathfrak{h}_F = \mathfrak{h} \otimes_k F$, which we call the *inertia Lie algebra*. We now interpret the base change Z_F as a homogeneous space:

Proposition 5.3. *The orbit morphism $H_F \cdot \{z_0\} \to Z_F$ induces an identification $H_F/H_F^{\text{inert}} = \operatorname{Spec}(F \otimes_E F)$. Moreover, the inertia Lie algebra $\mathfrak{h}_F^{\text{inert}}$ is the kernel for the canonical surjection*

$$\mathfrak{h} \otimes_k F \longrightarrow \mathfrak{h} \cdot F = \operatorname{Der}_E(F).$$

Finally, the degree of the field extension $E \subset F$ can be expressed as $[F : E] = p^c$, where $c \geq 0$ is the codimension of the inertia Lie algebra $\mathfrak{h}_F^{\text{inert}} \subset \mathfrak{h}_F$.

Proof. According to Proposition 5.2, the H_F-action on Z_F is transitive, and it follows that the orbit $H_F \cdot \{z_0\} \to Z_F$ is an epimorphism. By definition of the inertia subgroup scheme, the induced morphism $H_F/H_F^{\text{inert}} \to Z_F$ is a monomorphism. Hence the latter is an isomorphism. This is a finite scheme, and the F-dimension for the ring of global sections for the homogeneous space is given by p^c. It follows $[F : E] = p^c$.

It remains to see that the inertia Lie algebra $\mathfrak{h}_F^{\text{inert}}$ coincides with the kernel K of the canonical surjection $\mathfrak{h}_F \to \mathfrak{h} \cdot F$. We saw in Proposition 5.1 and the preceding paragraph that

$$p^{\dim_F(\mathfrak{h}\cdot F)} = [F : E] = h^0(\mathscr{O}_Z \otimes_E F) = p^{\dim(\mathfrak{h}_F/\mathfrak{h}_F^{\text{inert}})}.$$

It thus suffices to verify that the canonical map $\mathfrak{h}_F^{\text{inert}} \to \operatorname{Der}_E(F)$ is zero. Suppose this is not the case, and fix some non-zero $D \in \mathfrak{h}_F^{\text{inert}}$ from the kernel. Choose a p-basis for $E \subset F$ and write

$$F = E[T_1, \ldots, T_r]/(T_1^p - \mu_1, \ldots, T_r^p - \mu_r)$$

for some scalars $\mu_i \in E^\times$. The partial derivatives $\partial/\partial T_i \in \operatorname{Der}_E(F)$ form another F-basis, and $D = \sum \lambda_i \partial/\partial T_i$. Without restriction, we may assume $\lambda_1 \neq 0$. Now make a base-change to $R = F$, such that

$$A = F \otimes_E F = R[s_1, \ldots, s_r]/(s_1^p, \ldots, s_r^p)$$

as in the proof for Proposition 5.2. Then $D(s_1) = \lambda_1 \otimes 1 \notin \mathfrak{m}_A$. But this implies that H_F^{inert} does not fix the closed point $z_0 \in Z_F = \operatorname{Spec}(A)$, contradiction. $\square$

6. The foliation rank

Throughout this section, k is a ground field of characteristic $p > 0$, and X is a proper scheme. Note that everything carries over verbatim to proper algebraic spaces. Let $H = \operatorname{Aut}_{X/k}[F]$ be the resulting height-one group scheme, whose restricted Lie algebra is $\mathfrak{h} = H^0(X, \Theta_{X/k})$. To simplify exposition, we also assume that X is integral. Let $\eta \in X$ be the generic point and $F = k(X)$ be the function field. The quotient $Y = X/H$ is integral as well, and we denote its function field by $E = k(Y)$. This field extension $E \subset F$ is finite and purely inseparable. This yields a numerical invariant:

Definition 6.1. The *foliation rank* of the proper integral scheme X is the integer $r \geq 0$ defined by the formula $\deg(X/Y) = p^r$.

In other words, we have $[F:E]=p^r$. Since the field extension $E\subset F$ has height one, the foliation rank $r\geq 0$ is also given by $r=\dim_F(\Omega^1_{F/E})$, which can also be seen as the rank of the coherent sheaf $\Omega^1_{X/Y}$. Dualizing the surjection $\Omega^1_{X/k}\rightarrow\Omega^1_{X/Y}$ gives an inclusion $\mathscr{F}=\Theta_{X/Y}\subset\Theta_{X/k}$. The subsheaf $\mathscr{F}$ is closed under Lie brackets and p-maps, hence constitutes a *foliation*, where the integer $\operatorname{rank}(\mathscr{F})=\operatorname{rank}(\Omega^1_{X/Y})$ coincides with our foliation rank $r\geq 0$.

To obtain an interpretations of the foliation rank in terms of group schemes, consider the restricted Lie algebras

$$\mathfrak{h}=\operatorname{Lie}(H)=H^0(X,\Theta_{X/k})\quad\text{and}\quad\mathfrak{g}=\operatorname{Der}_E(F)=\Theta_{X/Y,\eta}.$$

The former is a finite-dimensional over the ground field k. The latter a finite-dimensional over the function field E, and can be seen as the Lie algebra for the automorphism group scheme for $\operatorname{Spec}(F)$ viewed as a finite E-scheme. The localization map $\mathfrak{h}=H^0(X,\Theta_{X/k})\rightarrow\Theta_{X/k,\eta}$ respects brackets and p-powers, and factors over the subalgebra $\mathfrak{g}=\Theta_{X/Y,\eta}$. This gives a k-linear map $\mathfrak{h}\rightarrow\mathfrak{g}$, together with its E-linearization

$$\mathfrak{h}\otimes_k E\longrightarrow\mathfrak{g},\quad \delta\otimes\lambda\longmapsto(f\mapsto\lambda\delta_\eta(f)).$$

The latter is a homomorphism of restricted Lie algebras over E. The map $\mathfrak{h}\rightarrow\mathfrak{g}$ is injective, because the coherent sheaf $\Theta_{X/k}$ is torsion free, and we often view it as an inclusion $\mathfrak{h}\subset\mathfrak{g}$. Note, however, that its E-linearization in general it is *neither injective nor surjective*. This is perhaps the main difference to the classical situation of group actions rather than group scheme actions.

We are now in the situation studied in Section 5. Let $\mathfrak{h}_F^{\text{inert}}$ be the inertia Lie algebra inside the base-change $\mathfrak{h}_F=\mathfrak{h}\otimes_k F$, corresponding to the inertia group scheme with respect to the F-rational point in $\operatorname{Spec}(F\otimes_E F)$. From Proposition 5.3 we obtain:

Proposition 6.2. *The foliation rank $r\geq 0$ of the scheme X coincides with the codimension of $\mathfrak{h}_F^{\text{inert}}\subset\mathfrak{h}_F$.*

In some sense, this measures how free the Frobenius kernel of the automorphism group scheme acts generically:

Proposition 6.3. *The foliation rank of the scheme X satisfies $0\leq r\leq h^0(\Theta_{X/k})$. We have $r=0$ if and only if the Frobenius kernel $H=\operatorname{Aut}_{X/k}[F]$ vanishes. The condition $r=h^0(\Theta_{X/k})$ holds if and only if H acts freely on some dense open set $U\subset X$.*

Proof. The inequality $r\leq h^0(\Theta_{X/k})$ follows from Proposition 6.2. If the group scheme H is trivial we have $h^0(\Theta_{X/k})=0$ and hence $r=0$. Conversely, if H is non-trivial there is a non-zero derivation $D:\mathscr{O}_X\rightarrow\mathscr{O}_X$. Since the structure sheaf is torsion-free, the derivation remains non-zero at the generic point, which implies that $E=F^{\mathfrak{h}}$ dose not coincide with F, and thus $r>0$.

If H acts freely on some dense open set, the projection $\epsilon:X\rightarrow Y$ to the quotient $Y=X/H$ is a principal homogeneous H-space over the dense open set $V\subset Y$ corresponding to U. In turn $[F:E]=h^0(\mathscr{O}_H)$, and thus $r=\dim(\mathfrak{h})=h^0(\Theta_{X/k})$.

Finally, suppose that the foliation rank takes the maximal possible value $r=h^0(\Theta_{X/k})$. Then the inertia Lie algebra $\mathfrak{h}_F^{\text{inert}}\subset\mathfrak{h}_F$ has codimension $r=\dim(\mathfrak{h}_F)$, thus is trivial. It follows that the group scheme H_E acts freely on $\mathrm{Spec}(F)$ viewed as an E-scheme. Thus there is an open dense set $V \subset Y$ over which the projection $\epsilon : X \to Y$ becomes a principal homogeneous H-space, and the H-action on $U = \epsilon^{-1}(V)$ is free. □

We next describe how the foliation rank behaves under birational maps:

Proposition 6.4. *Let $f : X \to X'$ be a birational morphism to another proper integral scheme X', with the property $\mathscr{O}_{Y'} = f_*(\mathscr{O}_X)$. Then the respective foliation ranks satisfy $r \leq r'$.*

Proof. According to Blanchard's Lemma, there is a unique homomorphism $f_* : \mathrm{Aut}^0_{X/k} \to \mathrm{Aut}^0_{X'/k}$ of group schemes making the morphism $f : X \to X'$ equivariant. Indeed, the original form of the lemma for complex-analytic spaces ([5], Proposition I.1) was extended to schemes by Brion, Samuel and Uma ([9], Proposition 4.2.1).

The homomorphism of group schemes is a monomorphism, because f is birational, and the schemes in question are integral. In particular, the induced homomorphism on Frobenius kernel gives a closed embedding $H \subset H'$, and an injection $\mathfrak{h} \subset \mathfrak{h}'$ of restricted Lie algebras. For the common function field $F = k(X) = k(X')$, we get $F^{\mathfrak{h}} \supset F^{\mathfrak{h}'}$, and $r \leq r'$ follows. □

The following gives an upper bound on the foliation rank:

Proposition 6.5. *Let $i \geq 0$ be some integer, and suppose that the coherent sheaf $\mathscr{F} = \underline{\mathrm{Hom}}(\Omega^i_{X/k}, \mathscr{O}_X)$ satisfies $h^0(\mathscr{F}) = 0$. Then X has foliation rank $r < i$.*

Proof. We have to show that the vector space $\mathfrak{h} \cdot F = \mathrm{Der}_E(F)$ has dimension at most $i - 1$. Seeking a contradiction, we suppose that there are k-derivations $D_1, \ldots, D_i : \mathscr{O}_X \to \mathscr{O}_X$ that are F-linearly independent. Then the same holds for the corresponding $\mathscr{O}_X$-linear maps $s_1, \ldots, s_i : \mathscr{O}_X \to \Theta_{X/k}$. Consequently their wedge product $s_1 \wedge \ldots \wedge s_i : \mathscr{O}_X \to \Lambda^i(\Theta_{X/k})$ is generically non-zero. The universal property of exteriour powers gives a canonical map $\Lambda^i(\Theta_{X/k}) \to \underline{\mathrm{Hom}}(\Omega^i_{X/k}, \mathscr{O}_X) = \mathscr{F}$, which is generically bijective. Thus $s_1 \wedge \ldots \wedge s_i$ yield a non-zero global section of $\mathscr{F}$, contradiction. □

Corollary 6.6. *Let X be a geometrically normal surface with $h^0(\omega_X^\vee) = 0$. Then the foliation rank is $r \leq 1$.*

Proof. Replacing the ground field k by the field $H^0(X, \mathscr{O}_X)$, it suffices to treat the case $h^0(\mathscr{O}_X) = 1$. By Serre's Criterion, the scheme X is regular in codimension one, so the locus of non-smoothness $\mathrm{Sing}(X/k)$ is finite. Let $f : S \to X$ be a resolution of singularities. Suppose for the moment that the regular surface S is smooth. Then $\omega_S = \Omega^2_{S/k}$. Consider the following chain of canonical maps

$$\Omega^2_{X/k} \longrightarrow f_*f^*(\Omega^2_{X/k}) \longrightarrow f_*(\Omega^2_{S/k}) \longrightarrow f_*(\omega_S) \longrightarrow \omega_X,$$

where to the right is the trace map. All these maps are bijective on the complement $U = X \smallsetminus \mathrm{Sing}(X/k)$, so the same holds for the dual map

$$\varphi : \omega_X^\vee \longrightarrow \underline{\mathrm{Hom}}(\Omega^2_{X/k}, \mathscr{O}_X) = \mathscr{F}.$$

According to [23], Corollary 1.8 and Theorem 1.9, these rank-one sheaves are reflexive and satisfy the Serre Condition (S_2). Since $\varphi|U$ is bijective, already φ is bijective, by loc. cit. Theorem 1.12. The assertion thus follows from the theorem.

It remains to treat the case that the ground field k is imperfect. Choose a perfect closure k'. The base-change $X' = X \otimes_k k'$ is normal, and the above reasoning applies to any resolution of singularities $S' \to X'$. It follows that $\omega_X^\vee$ and $\mathscr{F}$ become isomorphic after base-changing to k'. If follows that $\operatorname{Hom}(\omega_X, \mathscr{F})$ is one-dimensional. Choose a non-zero element $\varphi : \omega_X \to \mathscr{F}$. Then $\varphi \otimes k'$ must be bijective, and by descent the same holds for φ. $\square$

This applies in particular to smooth surfaces S of Kodaira dimension $\operatorname{kod}(S) \geq 1$, which comprise surfaces of general type, and the properly elliptic surfaces, including those with quasi-elliptic fibration. It also applies to surfaces S with Kodaira dimension zero, provided that the dualizing sheaf of the minimal model X is non-trivial.

Let S be smooth surface of general type, and X be its canonical model. This is the homogeneous spectrum $P(S, \omega_S)$ of the graded ring $R(S, \omega_S) = \bigoplus H^0(S, \omega_S^{\otimes t})$. Then X is normal, the singularities are at most rational double points, and the dualizing sheaf ω_X is ample. We also say that X is a *canonically polarized surfaces*. Obviously $h^0(\omega_X^{\otimes -1}) = 0$, and X has foliation rank $r \leq 1$. According to Proposition 6.4, the same holds for S.

Proposition 6.7. *Suppose that X has foliation rank $r = 1$, and let $D \in H^0(X, \Theta_{X/k})$ be any non-zero global section. Then for each point $x \in X$, the local ring $\mathscr{O}_{Y,\epsilon(x)}$ is the kernel for the additive map $D : \mathscr{O}_{X,x} \to \mathscr{O}_{X,x}$.*

Proof. Set $y = \epsilon(x)$. The local ring is given by $\mathscr{O}_{Y,y} = \mathscr{O}_{X,x}^{\mathfrak{h}}$, which is contained in the kernel $\mathscr{O}_{X,x}^D$ of the derivation D. Let $f \in \mathscr{O}_{X,x}^D$, and $D' \in \mathfrak{h}$ be another derivation. Then $D' = \lambda D$ for some element λ from the function field $F = \operatorname{Frac}(\mathscr{O}_{X,x})$, and thus $D'(f) = \lambda D(f) = 0$ inside F. Since the localization map $\mathscr{O}_{X,x} \to F$ is injective, we already have $D'(f) = 0$ inside $\mathscr{O}_{X,x}$. This shows $f \in \mathscr{O}_{Y,y}$. In turn, the inclusion $\mathscr{O}_{Y,y} \subset \mathscr{O}_{X,x}^D$ is an equality. $\square$

We will later see that for $r = 1$ each vector in $\mathfrak{g}$ is p-closed. Thus the non-zero elements $D \in \mathfrak{h}$ indeed yield height-one group schemes $N \subset H$ of order $|N| = p$, such that $Y = X/N$.

7. Invariant subspaces

Let k be a ground field of characteristic $p \geq 0$ and V be a finite-dimensional vector space of dimension $n \geq 0$. Let us write $\operatorname{GL}_{V/k}$ for the group-valued functor on the category (Aff/k) of affine k-schemes $T = \operatorname{Spec}(R)$ defined by

$$\operatorname{GL}_{V/k}(R) = \operatorname{Aut}_R(V \otimes_k R).$$

This satisfies the sheaf axiom with respect to the fppf topology. In fact, it is representable by an affine group scheme, and the choice of a basis $e_1, \ldots, e_n \in V$ yields $\operatorname{GL}_{V/k} \simeq \operatorname{GL}_{n,k}$.

Let us write $\underline{V}$ for the abelian functor whose R-valued points are $\underline{V}(R) = V \otimes_k R$. As explained in [21], Chapter I, Section 9.6 this is an affine scheme represented by the spectrum of the symmetric algebra on the dual vector space V^*. Moreover, the structure morphism $\underline{V} \to \mathrm{Spec}(k)$ carries the structure of a *vector bundle* of rank n with $\underline{V}(k) = V$, and the canonical homomorphism $\mathrm{GL}_{V/k} \to \mathrm{Aut}_{\underline{V}/k}$ of group schemes is bijective. Combining [20], Theorem 11.7 with [4], Exposé VIII, Corollary 2.3 and [8], Chapter V, §10, No. 5, Proposition 9 one sees that each $\mathrm{GL}_{V/k}$-torsor is trivial, that is, the non-abelian cohomology set $H^1(k, \mathrm{GL}_{V/k})$ with respect to the fppf topology is a singleton. In other words, all vector bundles $E \to \mathrm{Spec}(k)$ of rank n are isomorphic to $\underline{V}$.

Now let $H \subset \mathrm{GL}_{V/k}$ be a subgroup scheme, and $T \to \mathrm{Spec}(k)$ be a H-torsor. Then the quotient

$$^{T}\underline{V} = T \wedge^H \underline{V} = H\backslash(T \times \underline{V})$$

with respect to the diagonal action $\sigma \cdot (t, v) = (t\sigma^{-1}, \sigma v) = (\sigma t, \sigma v)$ is another vector bundle called the *T-twist*. *We now consider the following general problem: What subbundles exist in the T-twist whose pull-back to T are contained in the pullback of a fixed subbundle $\underline{V}' \subset \underline{V}$?* By fppf descent, these pullbacks correspond to subbundles inside the induced bundle $\underline{V} \times T \to T$ whose total space is invariant with respect to the diagonal H-action.

The n-dimensional vector space $^{T}V = (^{T}\underline{V})(k)$ of k-rational points is likewise called the *T-twist* of V. If H is finite and $T = \mathrm{Spec}(L)$ is the spectrum of a field, we are thus looking for k-vector subspaces $U \subset {}^{T}V$ such that $U \otimes_k L$ is contained in the base-change $V' \otimes_k L$, or equivalently to L-vector subspaces in $V' \otimes_k L$ that are invariant for the diagonal H-action.

Suppose now that $p > 0$, and that $H = \alpha_p$ is the *infinitesimal group scheme* defined by $H(R) = \{\alpha \in R \mid \alpha^p = 0\}$, where the group law is given by addition. Recall that the Lie algebra of $\mathrm{GL}_{V/k}$ is the vector space $\mathfrak{gl}(V) = \mathrm{End}_k(V)$, where the Lie bracket is given by commutators $[f, g] = fg - gf$, and the p-map $f^{[p]} = f^p$ is the p-fold composition. The inclusion homomorphism $H \to \mathrm{GL}_{V/k}$ corresponds to a vector $f \in \mathfrak{gl}(V)$ that is nilpotent, with all Jordan blocks of size $\leq p$. On R-valued points, the map becomes

$$H(R) \longrightarrow \mathrm{GL}_{V/k}(R), \quad \alpha \longmapsto \sum_{i=0}^{p-1} \frac{(\alpha f)^i}{i!}.$$

Set $e^{\alpha f} = \sum_{i=0}^{p-1} (\alpha f)^i / i!$ to simplify notation. By naturality, the above maps are determined by the single matrix e^{tf} with entries in the truncated polynomial ring $R = k[t]/(t^p)$. The following is well-known:

Lemma 7.1. *Each torsor T for the infinitesimal group scheme $H = \alpha_p$ is isomorphic to the spectrum of $L = k[s]/(s^p - \omega)$ for some $\omega \in k$, where the group elements $\alpha \in H(R)$ act via $s \mapsto s + \alpha$. The torsor T is non-trivial if and only if L is a field. Moreover, each purely inseparable field extension $k \subset L$ of degree p, the spectrum* $\mathrm{Spec}(L)$ *admits the structure of a H-torsor.*

Proof. Consider the relative Frobenius map $F : \mathbb{G}_a \to \mathbb{G}_a$ on the additive group, which comes from the k-linear map $k[t] \to k[t]$ given by $t \mapsto t^p$. Then $H = \alpha_p$ is the kernel. The short exact sequence $0 \to H \to \mathbb{G}_a \xrightarrow{F} \mathbb{G}_a \to 0$ yields a long exact sequence

$$k \longrightarrow k \longrightarrow H^1(k, H) \longrightarrow H^1(k, \mathbb{G}_a) \longrightarrow H^1(k, \mathbb{G}_a).$$

The terms on the right vanish. It follows that each H-torsor T arises as the fiber for $F : \mathbb{G}_a \to \mathbb{G}_a$ over some rational point $\omega \in \mathbb{G}_a(k)$ Thus T is equivariantly isomorphic to the spectrum of $k[s]/(s^p - \omega)$, where the group elements $\alpha \in H(R)$ act via $s \mapsto s + \alpha$. If T is non-trivial, the polynomial $s^p - \omega \in k[s]$ has no root in k. We infer that it is irreducible, because the algebra $L = k[s]/(s^p - \omega)$ has prime degree p. Thus L is a field, which is purely inseparable over k. Conversely, if L is a field, then T has no rational point, and the torsor is trivial.

Finally, let $k \subset L$ be a purely inseparable extension of degree p. For each element in L not contained in k we get an identification $L = k[s]/(s^p - \omega)$. Thus $\operatorname{Spec}(L)$ arises as fiber of the relative Frobenius map, thus admits the structure of a H-torsor. □

For the applications we have in mind, we now consider the particular situation that $V = k[t]/(t^p)$ is the underlying vector space of dimension $n = p$ coming from the truncated polynomial ring, and $V' = tk[t]/(t^p)$ is given by the maximal ideal. Each vector can be uniquely written as a polynomial $f(t) = \sum_{i=0}^{p-1} \lambda_i t^i$, with coefficients $\lambda_i \in k$. This vector space comes with a canonical action of the additive group $\mathbb{G}_a$, where the elements $\alpha \in \mathbb{G}_a(R) = R$ act via $f(t) \mapsto f(t + \alpha)$. With respect to the canonical basis $t^0, \ldots, t^{p-1} \in V \otimes_k R$, this automorphism is given by $\alpha \mapsto (\alpha_{ij})$, where the matrix entries are $\alpha_{ij} = \binom{j}{j-i}\alpha^{j-i}$. In turn, we get an induced action of the Frobenius kernel $H = \alpha_p$. Note that $V' \subset V$ is not H-invariant, because some $\alpha_{0j} = \alpha^j$ are non-zero for $\alpha \neq 0$.

Now let $T = \operatorname{Spec}(L)$ be a H-torsor. The resulting twist ${}^T V$, which is another vector space of dimension $n = p$. Note that both V and ${}^T V$ are isomorphic to $k^{\oplus p}$, but there is no canonical isomorphism. The following observation will be crucial for later applications:

Proposition 7.2. *In the above situation, there is no vector $x \neq 0$ inside the twist ${}^T V$ such that the induced element $x \otimes 1$ inside ${}^T V \otimes_k L = V \otimes_k L$ is contained in the base change $V' \otimes_k L$.*

Proof. Seeking a contradiction, we assume that such an element exists. Its image $x \otimes 1$ inside ${}^T V \otimes_k L = V \otimes_k L$ takes the form $f(t) = \sum_{i=1}^{p-1} \lambda_i t^i$ with coefficients $\lambda_i \in L$. According to Lemma 7.1, we have $L = k[s]/(s^p - \omega)$ for some $\omega \in k$, and the group elements $\alpha \in H(R)$ act via $s \mapsto s + \alpha$. Write $\lambda_i = \sum_{j=0}^{p-1} \lambda_{ij} \omega^j$ with coefficients $\lambda_{ij} \in k$. The H-invariance of the vector $f(t) \in V \otimes_k L$ with respect to the diagonal H-action means

$$\sum_{i=1}^{p-1} \sum_{j=0}^{p-1} \lambda_{ij} (s + \alpha)^j (t + \alpha)^i = \sum_{i=1}^{p-1} \sum_{j=0}^{p-1} \lambda_{ij} s^j t^i \tag{6}$$

for each $\alpha \in H(R)$. Our task is to infer $\lambda_{ij} = 0$. We now consider the universal situation, where α is the class of the indeterminate in the truncated polynomial ring $R = k[u]/(u^p)$. Then (6) becomes an equation in the residue class ring $k[t, s, u]/(t^p, s^p - \omega, u^p)$. Writing

$$(s + \alpha)^j (t + \alpha)^i = s^j t^i + \alpha(j s^{j-1} t^i + i s^j t^{i-1}) + \alpha^2(\ldots)$$

as a polynomial in α and comparing coefficients in (6) at the linear terms we get the equation

$$\sum_{i=1}^{p-1}\sum_{j=0}^{p-1}\lambda_{ij}(js^{j-1}t^i + is^jt^{i-1}) = 0.$$

It thus suffices to verify that the $p(p-1)$ elements $a_{ij} = js^{j-1}t^i + is^jt^{i-1}$ are k-linearly independent in the ring $k[s,t]/(t^p, s^p-\omega)$. We do so by comparing them with the standard basis vectors s^jt^i, and use the lexicographic order for the exponents $(i,j)\in\mathbb{N}^2$, where $(u,v)<(i,j)$ means that $u<i$, or $u=i$ and $v<j$. Obviously

$$a_{ij} = \sum_{(u,v)\leq(i-1,j)} \mu_{(u,v),(i,j)}s^vt^u$$

for some coefficients $\mu_{(u,v),(i,j)}\in k$, with $\mu_{(i-1,j),(i,j)} = i$. The resulting matrix $(\mu_{(u,v),(i,j)})$ has p^2 rows and $p(p-1)$ columns, and we see that the last $p(p-1)$ rows form a square matrix in row-echelon form, whose diagonal entries $\mu_{(i-1,j),(i,j)} = i$ lie in the unit group $\mathbb{F}_p^\times$. Consequently, the whole matrix has maximal rank, thus the vectors a_{ij} are linearly independent. □

8. Automorphisms for prime-degree radical extensions

Let k be a ground field of characteristic $p>0$. For each scalar $\omega\in k$, write

$$L = L_\omega = k[t]/(t^p-\omega)$$

for the resulting finite algebra of rank p. Each element can be uniquely written as $\sum_{i=0}^{p-1}\lambda_it^i$, and we call such expressions *truncated polynomials*. Write $\mathrm{Aut}_{L/k}$ for the group-valued functor on the category (Aff/k) whose R-valued points are the R-linear automorphisms of $L\otimes R$. This functor is representable by an affine group scheme. In this section we make a detailed study of the opposite group scheme

$$G = G_\omega = \mathrm{Aut}_{\mathrm{Spec}(L_\omega)/k} = (\mathrm{Aut}_{L_\omega/k})^{\mathrm{op}},$$

which comprises the automorphisms of the affine scheme $\mathrm{Spec}(L)$. We shall see that G is geometrically non-smooth, so understanding the scheme structure is of paramount importance. Note that the Lie algebras $\mathfrak{g} = \mathrm{Lie}(G)$ where discovered by Witt, compare the discussions in [10], Introduction and also [58], footnote on page 3. These so-called *Witt algebras* will be studied in the next section.

Any automorphism $g: L\otimes_k R\to L\otimes_k R$ is determined by the image of the generator t, which is some truncated polynomial $\varphi_g(t) = \sum_{i=0}^{p-1}\alpha_it^i$. The multiplication $gh\in G(R)$ of group elements corresponds to the substitution $\varphi_h(\varphi_g(t))$ of truncated polynomials.

The inverse group element g^{-1} defines another truncated polynomial $\varphi_{g^{-1}} = \sum_{i=0}^{p-1}\beta_it^i$, such that $\sum\alpha_i(\sum\beta_jt^j)^i = t = \sum\beta_i(\sum\alpha_jt^j)^i$. Note, however, that the truncated polynomials attached to group elements are never units in the ring $L\otimes_k R$, unless $R=0$. To avoid this ambiguity in notation we use the additional symbol $\varphi_g(t)$ to denote the image of the indeterminate under $g\in G(R)$.

The coefficients in the truncated polynomials $\varphi_g(t) = \sum_{i=0}^{p-1}\lambda_it^i$ for the group elements $g\in G(R)$ define a monomorphism $G\to\mathbb{A}^p$.

Proposition 8.1. *The monomorphism $G \to \mathbb{A}^p$ is is an embedding, and its image is the intersection of the closed set defined by the Fermat equation*

$$\lambda_0^p + (\lambda_1 - 1)^p\omega + \lambda_2^p\omega^2 + \ldots + \lambda_{p-1}^p\omega^{p-1} = 0 \tag{7}$$

with the open set given by $\det(\alpha_{ij}) \neq 0$. *Here the matrix entries come from the truncated polynomials* $(\sum \lambda_i t^i)^j = \sum \alpha_{ij} t^i$, *with* $0 \leq i, j \leq p-1$.

Proof. For each $g \in G(R)$, with truncated polynomial $\varphi_g(t) = \sum \lambda_i t^i$, the images $(\sum \lambda_i t^i)^j$ of the basis vectors t^j form a R-basis of $L \otimes R$, thus $G \to \mathbb{A}^p$ factors over the open set $U \subset \mathbb{A}^p$ given by $\det(\alpha_{ij}) \neq 0$. Since $t^p = \omega$, we also have $(\sum \lambda_i t^i)^p = \omega$, so the monomorphism also factors over the closed set $Z \subset \mathbb{A}^p$ defined by (7). Any tuple $(\lambda_0, \ldots, \lambda_{p-1}) \in \mathbb{A}^p(R)$ lying in $U \cap Z$ gives via the truncated polynomial $\sum \lambda_i t^i$ some group element $g \in G(R)$. It follows that the monomorphism $G \to A^p$ is an embedding, with image $U \cap Z$. $\square$

Note that throughout, we regard the coefficients λ_i either as scalars or as indeterminates, depending on the context. This abuse of notation simplifies exposition and should not cause confusion.

Proposition 8.2. *The neutral element* $e \in G$ *has coordinates* $(0, 1, 0, \ldots, 0)$ *with respect to the embedding* $G \subset \mathbb{A}^p$. *If the scalar* $\omega \in k$ *is not a* p*-power, then the group of rational points is* $G(k) = \{e\}$.

Proof. The truncated polynomial of the neutral element is $\varphi_e(t) = t$, which gives the coordinates of $e \in G$. Now suppose that $\omega \notin k^p$. By Proposition 8.1, it suffices to verify that the polynomial equation $\omega^0 T_0^p + \omega^1 T_1^p + \ldots + \omega^{p-1} T_{p-1}^p = 0$ has no non-trivial solution. The latter means that $1, \omega, \ldots, \omega^{p-1} \in k$ are linearly independent over k^p. This indeed holds, because $k^p \subset k$ is an extension of height ≤ 1, hence the minimal polynomial of any $\lambda \in k$ not contained in k^p is of the form $T^p - \lambda^p$. $\square$

We now consider the Frobenius pullback $G^{(p)}$ and its reduced part $G^{(p)}_{\text{red}} = (G^{(p)})_{\text{red}}$. Note that over imperfect fields, reduced parts of group schemes may fail to be subgroup schemes, see [17], Proposition 1.6 for an example. The following shows that even if it is a subgroup scheme, it might be non-normal. Note that this phenomenon seems to be the crucial ingredient for the main results of this paper.

Proposition 8.3. *The reduced part* $G^{(p)}_{\text{red}} \subset G^{(p)}$ *is non-normal subgroup scheme. Moreover,* $G^{(p)}_{\text{red}} \simeq U \rtimes \mathbb{G}_m$, *where* U *has a composition series of length* $p-2$ *whose quotients are isomorphic to the additive group* $\mathbb{G}_a$. *In particular,* G *is affine, irreducible, and of dimension* $p-1$.

Proof. Recall that $G^{(p)}$ is defined as the base-change of G with respect to the absolute Frobenius map on $\operatorname{Spec}(k)$. Clearly, the Frobenius pullback of $L_\omega = k[t]/(t^p - \omega)$ is isomorphic to $L_0 = k[t]/(t^p)$, and for our automorphism group schemes this means $G^{(p)} \simeq G_0$. Thus we may assume $\omega = 0$, and work with $G = G^{(p)}$.

The embedding $G \subset \mathbb{A}^p$ in Proposition 8.1 is now given by the conditions $\lambda_0^p = 0$ and $\det(\alpha_{ij}) \neq 0$. View the entries of the matrix (α_{ij}) as elements from the ring $A = k[\lambda_0, \ldots, \lambda_{p-1}]/(\lambda_0^p)$. Taken modulo the radical $\operatorname{Rad}(A) = (\lambda_0)$, the matrix takes lower triangular form, with diagonal entries $1, \lambda_1, \ldots, \lambda_1^{p-1}$. In turn, the embedding $G \subset \mathbb{A}^p$ is given by $\lambda_0^p = 0$ and $\lambda_1 \neq 0$. Consequently, the reduced part G_{red} is given by $\lambda_0 = 0$ and $\lambda_1 \neq 0$, which is smooth. Moreover, we see that G is affine, irreducible, and of dimension $p-1$.

Given two truncated polynomials $\varphi_g = \sum \alpha_i t^i$ and $\varphi_h = \sum \beta_i t^i$ with constant term $\alpha_0 = \beta_0 = 0$, the substitution $\varphi_g(\varphi_h(t))$ has constant term $\alpha_0 + \beta_0 = 0$, so the subsets $G_{\mathrm{red}}(R) \subset G(R)$ are subgroups. Over $R = k[u, v, \epsilon]/(uv-1, \epsilon^2)$, the truncated polynomials $\varphi_g = \epsilon + t$ and $\varphi_h = ut$ have $\varphi_{g^{-1}}(\varphi_h(\varphi_g))) = u\epsilon + t$, so the subgroup $G_{\mathrm{red}}(R) \subset G(R)$ fails to be normal.

Summing up, $G_{\mathrm{red}} \subset G$ is a smooth non-normal subgroup scheme. The map $\sum_{i=1}^{p-1} \lambda_i t^i \mapsto \lambda_1$ defines a short exact sequence

$$0 \longrightarrow U \longrightarrow G_{\mathrm{red}} \longrightarrow \mathbb{G}_m \longrightarrow 0.$$

The inclusion $U \subset \mathbb{A}^p$ is given by $\lambda_0 = 0$ and $\lambda_1 = 1$, hence the underlying scheme of U is a copy of the affine space $\mathbb{A}^{p-2}$. By Lazard's Theorem ([11], Chapter IV, 4.1), the group scheme U admits a composition series whose quotients are isomorphic to the additive group $\mathbb{G}_a$. Moreover, the projection $G_{\mathrm{red}} \to \mathbb{G}_m$ has a section via $\lambda_1 \mapsto \lambda_1 t$. This is a homomorphism, hence G_{red} is a semidirect product. $\square$

Clearly, the group elements $g \in G(R)$ with linear truncated polynomial $\varphi_g = \lambda_0 + \lambda_1 t$ form a closed subgroup scheme $B \subset G$. It sits in a short exact sequence

$$0 \longrightarrow \alpha_p \longrightarrow B \longrightarrow \mathbb{G}_m \longrightarrow 0, \tag{8}$$

where the map on the left is given by $\lambda_0 \mapsto \lambda_0 + t$, and the map on the right comes from $\lambda_0 + \lambda_1 t \to \lambda_1$. In particular, B is a connected solvable group scheme. We see later that B is maximal with respect to this property, so one may regard it as a Borel group. However, we want to stress that this lies in a non-smooth group scheme, and B itself is non-reduced. We therefore call $B \subset G$ a *non-reduced Borel group*.

An element $\lambda_1 \in \mathbb{G}_m(R)$ lies in the image if and only if $\lambda_0 = (1-\lambda_1)\omega^{1/p}$ exists in R. It follows that the extension (8) splits if and only if $\omega \in k$ is a p-power. Moreover, we see that the canonical map $\mathbb{G}_m \to \mathrm{Aut}_{\alpha_p/k} = \mathbb{G}_m$ is the identity. Note that the group of all such extension of $\mathbb{G}_m$ by α_p, with non-trivial $\mathbb{G}_m$-action, is identified with k/k^p, according to [11], Chapter III, Corollary 6.4. Note that for $p = 2$ the inclusion $B \subset G$ is an equality. In any case, the pullback of the extension (8) along the inclusion $\mu_p \subset \mathbb{G}_m$ admits a splitting given by $\lambda_1 \mapsto \lambda_1 t$, and one sees $B \times_{\mathbb{G}_m} \mu_p = \alpha_p \rtimes \mu_p$.

Write $G[F]$ for the kernel of the relative Frobenius map $G \to G^{(p)}$, which is a normal subgroup scheme of height one. We now consider the resulting $G/G[F] \subset G^{(p)}$.

Proposition 8.4. *The group scheme $G/G[F]$ is smooth, and coincides with the reduced part $G^{(p)}_{\mathrm{red}}$ inside the Frobenius pullback $G^{(p)}$.*

Proof. We may assume that k is algebraically closed. We first verify that $G/G[F]$ is reduced. The short exact sequence (8) yields an inclusion $\alpha_p \subset G$. This is not normal, but contained in the Frobenius kernel $G[F]$. The resulting projection $G/\alpha_p \to G/G[F]$ is faithfully flat, and it suffices to check that the homogeneous space G/α_p is reduced. Since G acts transitively, it is enough to verify that the the local ring at the image in G/α_p of the origin $e \in G$ is regular. According to

[48], Proposition 2.2 it is enough to check that in the local ring $\mathscr{O}_{G,e}$, the ideal $\mathfrak{a}$ corresponding to the subgroup scheme $\alpha_p \subset G$ has finite projective dimension. But this is clear, because it is given by the complete intersection $\lambda_1 = \ldots = \lambda_{p-1} = 0$.

Thus $G/G[F]$ is reduced. The reduced closed subschemes $G/G[F]$ and $G^{(p)}_{\mathrm{red}}$ inside the Frobenius pullback have the same underlying set, whence $G/G[F] = G^{(p)}_{\mathrm{red}}$. The latter is smooth by Proposition 8.3, whence the same holds for the former. ☐

Now consider the conjugacy map $c : G \to \mathrm{Aut}_{G/k}$, sending $g \in G(R)$ to the automorphism $x \mapsto gxg^{-1}$. In terms of truncated polynomials, gxg^{-1} is given by the triple substitution $\varphi_{g^{-1}}(\varphi_x(\varphi_g(t)))$. According to [47], Theorem 4.13 we have:

Proposition 8.5. *The conjugacy map $c : G \to \mathrm{Aut}_{G/k}$ is an isomorphism.*

In other words, the center and the scheme of outer automorphisms are trivial. One also says the the group scheme G is *complete*. Now recall that $G = G_\omega$ depends on a scalar $\omega \in k$.

Proposition 8.6. *For each pair of scalars $\omega, \omega' \in k^\times$, the following are equivalent:*

(i) *The k-algebras L_ω and $L_{\omega'}$ are isomorphic.*
(ii) *The group schemes G_ω and $G_{\omega'}$ are isomorphic.*
(iii) *We have $k^p(\omega) = k^p(\omega')$ as subfields inside k.*

Proof. According to [18], Chapter III, Corollary 2.5.2 the category of twisted forms for L_ω, and the category of twisted forms of G_ω are both equivalent to the category of G_ω-torsors. This implies the equivalence of (i) and (ii).

It remains to check (i)⇔(iii). Write $L_\omega = k[t]/(t^p - \omega)$ and $L_{\omega'} = k[t']/(t'^p - \omega')$. Suppose first that these algebras are isomorphic. Choose an isomorphism and regard it as an identification $L_\omega = L_{\omega'}$. Then $t' = \sum_{i=0}^{p-1} \lambda_i t^i$, consequently $\omega' = \sum \lambda_i^p \omega^i$, and hence $k^p(\omega') \subset k^p(\omega)$. By symmetry, the reverse implication holds as well.

Conversely, suppose that $k^p(\omega) = k^p(\omega')$. If this subfield coincides with k^p, then both $\omega, \omega' \in k$ are p-powers, hence both algebras L_ω, $L_{\omega'}$ are isomorphic to $k[t]/(t^p)$. Suppose now that the subfield is different from k^p. Taking p-th roots we get $k(\omega^{1/p}) = k(\omega'^{1/p})$ inside some perfect closure k^{perf}. These fields are isomorphic to L_ω and $L_{\omega'}$, because both scalars $\omega, \omega' \in k$ are not p-powers. ☐

In particular, each $L = L_\omega$ is a twisted form of L_0, and each $G = G_\omega$ is a twisted form of G_0. Up to isomorphism, these twisted forms correspond to classes in non-abelian cohomology set $H^1(k, G_0)$. We will use this throughout to gain insight into G, by using facts on G_0. For example, from Proposition 8.1 we see that the *locus of non-smoothness* $\mathrm{Sing}(G_0/k)$, defined as in [17], Section 2, equals the whole scheme G_0. Hence the same holds for G, because it is a twisted form of G_0.

We now write $\mathrm{Sing}(G)$ for the *singular locus* of G, which comprise all points $a \in G$ where the local ring $\mathscr{O}_{G,a}$ is singular. Note that the formation of such loci commutes with base-changes along separable extension, but usually not with inseparable extensions.

Proposition 8.7. *The local ring at the origin is singular, with embedding dimension $\mathrm{edim}(\mathscr{O}_{G,e}) = p$. Moreover, the inclusion $\mathrm{Sing}(G) \subset G$ is not an equality if and only if $\omega \in k$ is not a p-power. In this case, the singular locus has codimension one in G.*

Proof. Since $e \in G$ is a rational point, the embedding dimension of $\mathscr{O}_{G,a}$ does not change under ground field extensions. If $\omega \in k^p$ we have $G \simeq G_0$ and thus every point $a \in G$ the local ring $\mathscr{O}_{G,a}$ is singular. Now suppose that ω is not a p-power, and consider the p-Fermat hypersurface $X \subset \mathbb{P}^{p-1}$ defined by the homogeneous polynomial $\lambda_0 T_0^p + \ldots + \lambda_{p-1} T_{p-1}^p$, with coefficient $\lambda_i = \omega^i$. The field extension $k^p \subset E$ generated by $\lambda_i/\lambda_0 = \omega^i$ is nothing but $k^p(\omega)$. It has degree $[E : k] = p$, hence its p-degree is $d = 1$. According to [49], Theorem 3.3 the singular locus $\mathrm{Sing}(X) \subset X$ has codimension $d = 1$. It follows that $\mathrm{Sing}(G) \subset G$ is not an equality.

Seeking a contradiction, we now assume that $Z = \mathrm{Sing}(G)$ has codimension ≥ 2. Then the scheme G is normal, by Serre's Criterion. Choose a normal compactification $Y = \bar{G}$. The canonical map $k \to H^0(Y, \mathscr{O}_Y)$ is bijective, because we have the rational point $e \in G$. According to [49], Lemma 1.3 the base-change $Y \otimes_k k(\omega^{1/p})$ remains integral. On the other hand, we just saw that $G \otimes k(\omega^{1/p})$ is non-reduced, contradiction. □

9. Witt algebras

We keep the notation from the previous section. Our goal now is to understand the restricted Lie algebra $\mathfrak{g} = \mathfrak{g}_\omega$, or equivalently the Frobenius kernel, attached to the automorphism group scheme $G = G_\omega$ of the spectrum of the ring $L = L_\omega = k[t]/(t^p - \omega)$.

From $G = \mathrm{Aut}_{\mathrm{Spec}(L)/k}$ we get an identification $\mathfrak{g} = \mathrm{Der}_k(L)$. Any k-derivation $\delta : L \to L$ can be seen as an L-linear map $\Omega^1_{L/k} \to L_\omega$. The module of Kähler differentials is a free L-module of rank one, generated by dt. Let $\partial \in \mathfrak{g}$ be the dual basis vector. In turn we get the canonical k-basis $t^i\partial$, with $0 \leq i \leq p-1$, and the Lie bracket is given by

$$[t^i\partial, t^j\partial] = (j-i)t^{i+j-1}\partial.$$

The p-map $(f\partial)^{[p]} = (f\partial)^p$ is the p-fold composition in $\mathrm{End}_k(L)$. It can be made explicit as follows: For each truncated polynomial $f = \sum_{i=0}^{p-1} \lambda_i t^i$, we write $f^{p-1} = \sum_{i=0}^{p-1} C_i t^i$, where the $C_i \in k$ are certain polynomial expressions in the coefficients $\lambda_0, \ldots, \lambda_{p-1}$ and ω, which also depend on the prime $p > 0$. Set $C = C_{p-1}$.

Proposition 9.1. *We have $(f\partial)^{[p]} = C \cdot f\partial$ for every element $f\partial \in \mathfrak{g}$. Moreover, the factor C is homogeneous of degree $p-1$ in the coefficients of $f = \sum \lambda_i t^i$.*

Proof. Clearly we have $\partial^p = 0$. According to Hochschild's Formula ([24], Lemma 1) the p-fold composition of $f\partial$ is given by

$$(f\partial)^p = f^p\partial^p + g\partial = g\partial,$$

where $g = (f\partial)^{p-1}(f)$. Consider the differential operator $D = \partial f\partial \ldots f\partial$, where the number of ∂-factors is $p-1$, such that $g = fD(f)$. According to [16], Theorem 2 we have $D(f) = -\partial^{p-1}(f^{p-1})$. Clearly, $\partial^{p-1}(t^i) = 0$ for $0 \leq i \leq p-2$, whereas $\partial^{p-1}(t^{p-1}) = (p-1)! = -1$. Summing up, we have $D(f) = C$, hence $g = fC = Cf$, and the statement on the p-map follows. From $(\sum \lambda_i t^i)^{p-1} = \sum C_i t^i$ one immediately sees that each $C_i = C_i(\lambda_0, \ldots, \lambda_{p-1})$ is homogeneous of degree $p-1$. □

This has a remarkable consequence:

Corollary 9.2. *Every vector in the restricted Lie algebra $\mathfrak{g}$ is p-closed.*

So each non-zero vector $f\partial \in \mathfrak{g}$ defines a subgroup scheme $H \subset G$ of order p. Note that the additive vectors might be viewed as rational points on the hypersurface of degree $p-1$ defined by the homogeneous equation $C(\lambda_0, \ldots, \lambda_{p-1}) = 0$. For example, with $p = 5$ the polynomial becomes

$$(\lambda_0^3\lambda_4 + 2\lambda_0^2\lambda_1\lambda_3 + \lambda_0^2\lambda_2^2 + 2\lambda_0\lambda_1^2\lambda_2 + \lambda_1^4) + \omega(2\lambda_0\lambda_1\lambda_4^2 + 4\lambda_0\lambda_2\lambda_3\lambda_4 + 4\lambda_0\lambda_3^3 + \\ 2\lambda_1^2\lambda_3\lambda_4 + 2\lambda_1\lambda_2^2\lambda_4 + 2\lambda_1\lambda_2\lambda_3^2 + 4\lambda_2^3\lambda_3) + \omega^2(4\lambda_2\lambda_4^3 + \lambda_3^2\lambda_4^2).$$

For the primes $p = 2$ and $p = 3$ we get $C(\lambda_0 + \lambda_1) = \lambda_1$ and $C(\lambda_0, \lambda_1, \lambda_2) = \lambda_1^2 - \lambda_0\lambda_2$, respectively, which then reveals the structure of $\mathfrak{g}$:

Corollary 9.3. *For $p = 2$ we have $\mathfrak{g} \simeq k \rtimes \mathfrak{gl}_1(k)$. For $p = 3$ we get $\mathfrak{g} \simeq \mathfrak{sl}_2(k)$, provided that $-1 \in k^\times$ is a square.*

Proof. In the first case, one easily checks that the linear bijection $\mathfrak{g} \to k \rtimes \mathfrak{gl}_1(k)$ given by $(a + bt) \mapsto (a, b)$ respects bracket and p-map. In the second case we set $i = \sqrt{-1}$, and see that the linear bijection

$$\mathfrak{g} \longrightarrow \mathfrak{sl}_2(k), \quad (a + bt + ct^2) \longmapsto \begin{pmatrix} ia & b \\ c & -ia \end{pmatrix}$$

likewise respects bracket and p-map. $\square$

Consider the *adjoint representation* $\mathrm{Ad} : G \to \mathrm{Aut}_{\mathfrak{g}/k}$, which sends each $g \in G(R)$ to the derivative of c_g. From [57], Theorem A we get:

Proposition 9.4. *For $p \geq 5$ the adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}_{\mathfrak{g}/k}$ is an isomorphism of group schemes.*

So for $p \geq 5$ our G can be seen as the automorphism group scheme for the ring L, the group scheme G, and the restricted Lie algebra $\mathfrak{g}$. Consequently, the three conditions in Proposition 8.6 are also equivalent to $\mathfrak{g}_\omega \simeq \mathfrak{g}_{\omega'}$. For $p = 2, 3$ the adjoint representation $G \to \mathrm{Aut}_{\mathfrak{g}/k}$ is not bijective, according to Proposition 3.2.

We now come to the crucial result of this paper:

Theorem 9.5. *Suppose that the scalar $\omega \in k$ is not a p-power, and that $k^\times = k^{\times(p-1)}$. Then each subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ of dimension $1 \leq n \leq p-1$ is isomorphic to either k or $\mathfrak{gl}_1(k)$ or $k \rtimes \mathfrak{gl}_1(k)$ or $\mathfrak{sl}_2(k)$.*

Proof. In the special case $p = 2$ the dimension of $\mathfrak{g}'$ must be $n = 1$, and it follows that $\mathfrak{g}'$ is a twisted form of k or $\mathfrak{gl}_1(k)$. According to Proposition 3.2, all such twisted forms are trivial, so our assertion indeed holds.

From now on we assume $p \geq 3$. Recall that $\mathfrak{g} = \mathrm{Lie}(G)$ is a twisted form of $\mathfrak{g}_0 = \mathrm{Lie}(G_0)$, where G_0 is the automorphism group scheme for the spectrum of $L_0 = k[t]/(t^p)$. Let $\mathfrak{g}_{0,\mathrm{red}}$ be the subalgebra corresponding to the reduced part $G_{0,\mathrm{red}}$. According to Proposition 10.3 below, there is no vector $x \neq 0$ in $\mathfrak{g}$ such that $x \otimes 1 \in \mathfrak{g} \otimes k(\omega^{1/p})$ is contained in $\mathfrak{g}_{0,\mathrm{red}} \otimes k(\omega^{1/p})$. In particular, the latter does not contain the base-change $\mathfrak{g}' \otimes k(\omega^{1/p})$.

It follows that the further base-change $\mathfrak{g}' \otimes k^{\mathrm{alg}}$ is not contained in $\mathfrak{g}_{0,\mathrm{red}} \otimes k^{\mathrm{alg}}$. Such subalgebras where studied by Premet and Stewart in [44], Section 2.2: They remark on page 971 that a subalgebra in $\mathfrak{g}_0 \otimes k^{\mathrm{alg}}$ is not contained in $\mathfrak{g}_{0,\mathrm{red}} \otimes k^{\mathrm{alg}}$ if and only if it does not preserve any proper non-zero ideal, and call such subalgebras *transitive*. We thus may apply loc. cit., Lemma 2.2 and infer that $\mathfrak{g}'$ is a twisted form of k or $\mathfrak{gl}_1(k)$ or $k \rtimes \mathfrak{gl}_1(k)$ or $\mathfrak{sl}_2(k)$. By assumption, the group $k^\times/k^{\times(p-1)}$ vanishes. Since p is odd, this ensures that $k^\times/k^{\times 2}$ vanishes as well. According to Lemma 3.2, the four restricted Lie algebras in question have no twisted forms over our field k, thus $\mathfrak{g}'$ is isomorphic to one of them. $\square$

Note that for $p = 2$ the assumption $k^\times = k^{\times(p-1)}$ is vacuous, and for $p = 3$ means that the field k is *quadratically closed*.

10. Twisting adjoint representations

We keep the assumption of the preceding section, and establish the crucial ingredients for the proof of Theorem 9.5. Recall that we are in characteristic $p > 0$, that $G = G_\omega$ is the automorphism group scheme of the spectrum of $L = L_\omega = k[t]/(t^p - \omega)$, for some scalar $\omega \in k$. The resulting restricted Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ is the p-dimensional vector space $\mathrm{Der}_k(L)$, which comprises the derivations $f(t)\partial$, where $\partial = \sum_{i=0}^{p-1} \lambda_i t^i$ is a truncated polynomial. Moreover, that the group elements $g \in G(R)$ act on $L \otimes_k R$ via the substitution $t \mapsto \varphi_g(t)$, for the corresponding truncated polynomial $\varphi_g(t) = \sum \lambda_i t^i$, and that its coefficients define an embedding $G \subset \mathbb{A}^p$ of the underlying scheme. For each $g \in G(R)$, write c_g for the induced inner automorphism $x \mapsto gxg^{-1}$. The resulting conjugacy map $c : G \to \mathrm{Aut}_{G/k}$ is given in terms of truncated polynomials by the formula

$$\varphi_{gxg^{-1}}(t) = \varphi_g(\varphi_x(\varphi_{g^{-1}}(t))).$$

By functoriality, the elements $c_g \in \mathrm{Aut}_{G/k}(R)$ induce an automorphism $\mathrm{Ad}_g = \mathrm{Lie}(c_g)$ of $\mathfrak{g} \otimes_k R$, which defines the adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}_{\mathfrak{g}/k}$.

Proposition 10.1. *Let $g \in G(R)$, and write $\varphi(t) = \varphi_{g^{-1}}(t)$ for the truncated polynomial of the inverse g^{-1}. Then the formal derivative $\varphi'(t)$ is a unit in the ring $L \otimes_k R$, and for each $f(t)\partial \in \mathfrak{g} \otimes_k R$ we have*

$$\mathrm{Ad}_g(f(t)\partial) = \frac{f(\varphi(t))}{\varphi'(t)}\partial.$$

Proof. By definition, the element $f(t)\partial \in \mathfrak{g} \otimes_k R \subset G(R[\epsilon])$ acts on the algebra $L \otimes R[\epsilon]$ via

$$h(t) \longmapsto h(t) + \epsilon f(t)\partial(h) = h(t) + \epsilon f(t) h'(t).$$

Thus the adjoint $\mathrm{Ad}_g(f(t)\partial) = g \circ f(t)\partial \circ g^{-1}$ is given by the following composition:

$$t \longmapsto \varphi_g(t) \longmapsto \varphi_g(t) + \epsilon f(t)\varphi_g'(t) \longmapsto \varphi_g(\varphi_{g^{-1}}(t)) + \epsilon f(\varphi_{g^{-1}}(t))\varphi_g'(\varphi_{g^{-1}}(t)). \tag{9}$$

For a moment, let us regard the truncated polynomials $\varphi_g(t)$ and $\varphi_{g^{-1}}(t)$ as elements in the polynomial ring $R[t]$. Then $t = \varphi_g(\varphi_{g^{-1}}(t)) + (t^p - \omega)h(t)$ for some polynomial $h(t)$. Taking formal derivatives and applying the chain rule, we obtain

$$1 = \varphi_g'(\varphi_{g^{-1}}(t)) \cdot \varphi_{g^{-1}}(t) + (t^p - \omega)h'(t).$$

This gives $1 = \varphi_g'(\varphi_{g^{-1}}(t)) \cdot \varphi_{g^{-1}}(t)$ in the truncated polynomial ring $L \otimes_k R = R[t]/(t^p)$. It follows that $\varphi(t) = \varphi_{g^{-1}}(t)$ is a unit, with inverse $\varphi_g'(\varphi_{g^{-1}}(t))$. Substituting for the term on the right in (9) gives the desired formula for $\mathrm{Ad}_g(f(t)\partial)$. $\square$

Now consider the additive vector $\partial \in \mathfrak{g}$, which corresponds to an inclusion of the infinitesimal group scheme $H = \alpha_p$ into the group scheme G. The R-valued points $h \in H(R) = \{\lambda_0 \in R \mid \lambda^p = 0\}$ correspond to truncated polynomials $\varphi_h(t) = t + \lambda$. The inverse R-valued point has $\varphi_{h^{-1}} = t - \lambda$, with formal derivative $\varphi'_{h^{-1}}(t) = 1$. This immediately gives:

Corollary 10.2. *With the above notation, we have* $\mathrm{Ad}_h(f(t)\partial) = f(t-\lambda)\partial$ *for every element* $f(t)\partial \in \mathfrak{g} \otimes_k R$.

Recall that $G = G_\omega$ depends on some scalar $\omega \in k$, and is a twisted form of G_0. The latter coincides with its own Frobenius pullback. By Proposition 8.3, the reduced part $G_{0,\mathrm{red}}$ is a non-normal subgroup scheme. Recall that the embedding $G_0 \subset \mathbb{A}^p$ is given by $\lambda_0^p = 0$ and $\lambda_1 \neq 0$, such that $G_{0,\mathrm{red}}$ is defined by $\lambda_0 = 0$ and $\lambda_1 \neq 0$. Write $\mathfrak{g}_{0,\mathrm{red}} \subset \mathfrak{g}_0$ for the resulting subalgebra, which comprises the derivations $f\partial$ where the truncated polynomial $f = \sum_{i=0}^{p-1} \lambda_i t^i$ has $\lambda_0 = 0$. Write $H_0 \subset G_0$ for the copy of α_p given by the additive vector $\partial \in \mathfrak{g}_0$.

Now suppose that our ground field k is imperfect, that our scalar $\omega \in k$ is not a p-power, and consider the resulting field extension $k(\omega^{1/p})$. In light of Lemma 7.1, we may endow its spectrum T with the structure of an H_0-torsor. Lemma 3.1 gives an identification ${}^T\mathfrak{g}_0 = \mathfrak{g}_\omega$, and thus an identification $\mathfrak{g}_0 \otimes_k k(\omega^{1/p}) = \mathfrak{g}_\omega \otimes_k k(\omega^{1/p})$ The following fact was a crucial ingredient for the proof of Theorem 9.5:

Proposition 10.3. *The twisted form* $\mathfrak{g}_\omega$ *contains no vector* $x \neq 0$ *such that the induced vector* $x \otimes 1$ *inside* $\mathfrak{g}_0 \otimes_k k(\omega^{1/p}) = \mathfrak{g}_\omega \otimes_k k(\omega^{1/p})$ *is contained in the base-change* $\mathfrak{g}_{0,\mathrm{red}} \otimes_k k(\omega^{1/p})$.

Proof. Setting $V = \mathfrak{g}_0$ and $V' = \mathfrak{g}_{0,\mathrm{red}}$, we see that that action of $H_0 = \alpha_p$ via the adjoint representation $G_0 \to \mathrm{Aut}_{\mathfrak{g}_0/k}$ is exactly as described in Proposition 7.2, and the assertion follows. □

11. Subalgebras

Throughout this section, k is a field of characteristic $p > 0$, and $k \subset E$ is a field extension. Suppose we have a group scheme H of finite type over k, a group scheme G of finite type over E, and a homomorphism $f : H \otimes_k E \to G$. We shall see that in certain circumstances, important structural properties of the Frobenius kernel $G[F]$ are inherited to $H[F]$.

Consider the finite-dimensional restricted Lie algebra $\mathfrak{h} = \mathrm{Lie}(H)$ over k and $\mathfrak{g} = \mathrm{Lie}(G)$ over E. Our homomorphism of group schemes induces an E-linear homomorphism

$$\mathrm{Lie}(f) : \mathfrak{h} \otimes_k E \to \mathfrak{g}, \quad x \otimes \alpha \longmapsto \alpha x,$$

of restricted Lie algebras which corresponds to a k-linear homomorphism $\mathfrak{h} \to \mathfrak{g}$ of restricted Lie algebras. We are mainly interested in the case that E is the function field of an integral k-scheme X of finite type, such that $\mathfrak{g}$ is an infinite-dimensional k-vector space. Set $N = \mathrm{Ker}(f)$, with Lie algebra $\mathfrak{n} = \mathrm{Ker}(\mathrm{Lie}(f))$.

Proposition 11.1. *The following are equivalent:*

(i) *The k-linear homomorphism $\mathfrak{h} \to \mathfrak{g}$ is injective.*

(ii) *For every non-trivial subgroup scheme $H' \subset H$ that is minimal with respect to inclusion, the base-change $H' \otimes_k E$ is not contained in the kernel $N \subset H \otimes_k E$.*

Proof. We prove the contrapositive: Suppose $\mathfrak{h} \to \mathfrak{g}$ is not injective. Inside the kernel, choose a subalgebra $\mathfrak{h}' \neq 0$ that is minimal with respect to inclusion. Then the induced E-linear map $\mathfrak{h}' \otimes_k E \to \mathfrak{g}$ is zero. By the Demazure–Gabriel Correspondence, the corresponding subgroup scheme $H' \subset H$ of height one is minimal with respect to inclusion, and $H' \otimes_k E \subset N$. Conversely, suppose $H' \otimes_k E \subset N$ for some H' as in (ii). Choose some non-zero vector x from $\mathfrak{h}' = \mathrm{Lie}(H')$. By construction, it lies in the kernel for $\mathfrak{h} \to \mathfrak{g}$. $\square$

We now suppose that the above equivalent conditions hold, and regard the injective map as an inclusion $\mathfrak{h} \subset \mathfrak{g}$. To simplify exposition, we also assume that k is algebraically closed, and that $\mathfrak{h}$ contains an E-basis for $\mathfrak{g}$. In other words, the induced linear map $\mathfrak{h} \otimes_k E \to \mathfrak{g}$ is surjective. Note that this E-linear map is usually *not injective*. However, we shall see that important structural properties of $\mathfrak{g}$ transfer to $\mathfrak{h}$. We start with a series of three elementary but useful observations:

Lemma 11.2. *If every vector in $\mathfrak{g}$ is p-closed, the same holds for every vector in $\mathfrak{h}$.*

Proof. Fix some non-zero $x \in \mathfrak{h}$. By assumption we have $x^{[p]} = \alpha x$ for some $\alpha \in E$, and our task is to verify that this scalar already lies in k. Since the latter is algebraically closed, it is enough to verify that α is algebraic over k. By induction on $i \geq 0$ we get $x^{[p^i]} = \alpha^{n_i} x$ for some strictly increasing sequence $0 = n_0 < n_1 < \ldots$ of integers. Since $\dim_k(\mathfrak{h}) < \infty$ there is a non-trivial relation $\sum_{i=0}^r \lambda_i x^{[p^i]}$ for some $r \geq 0$ and some coefficients $\lambda_i \in k$. This gives $\sum \lambda_i \alpha^{n_i} x = 0$. Since $x \neq 0$ we must have $\sum \lambda_i \alpha^{n_i} = 0$, hence $\alpha \in E$ is algebraic over k. $\square$

Lemma 11.3. *The restricted Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ have the same toral rank, and the kernel $\mathfrak{n}$ for $\mathfrak{h} \otimes_k E \to \mathfrak{g}$ has toral rank $\rho_t(\mathfrak{n}) = 0$.*

Proof. It follows from [6], Lemma 1.7.2 that $\rho_t(\mathfrak{h}) = \rho_t(\mathfrak{g}) + \rho_t(\mathfrak{n})$, and in particular $\rho_t(\mathfrak{g}) \geq \rho_t(\mathfrak{h})$. For the reverse inequality, suppose there are k-linearly independent vectors $x_1, \ldots, x_r \in \mathfrak{h}$ with $[x_i, x_j] = 0$ and $x_i^{[p]} = x_i$. We have to check that the vectors are E-linearly independent. Suppose there is a non-trivial relation. Without loss of generality, we may assume that $x_1, \ldots, x_{r-1}$ are E-linearly independent, and that $x_r = \sum_{i=1}^{r-1} \lambda_i x_i$ for some coefficients $\lambda_i \in E$. From the axioms of the p-map we get

$$\sum \lambda_i x_i = x_r = x_r^{[p]} = (\sum \lambda_i x_i)^{[p]} = \sum \lambda_i^p x_i^{[p]} = \sum \lambda_i^p x_i.$$

Comparing coefficients gives $\lambda_i^p = \lambda_i$. Thus λ_i lie in the prime field, in particular in k. In turn, the vectors are k-linearly dependent, contradiction. $\square$

Let us call the restricted Lie algebra $\mathfrak{h}$ *simple* if it is non-zero, and contains no ideal besides $\mathfrak{a} = 0$ and $\mathfrak{a} = \mathfrak{h}$.

Lemma 11.4. *Suppose there is a restricted Lie algebra $\mathfrak{h}'$ over k such that $\mathfrak{g}$ is a twisted form of the base-change $\mathfrak{h}' \otimes_k E$. If $\mathfrak{h}'$ is simple of dimension $n' \geq 2$, we must have $\mathfrak{h} \simeq \mathfrak{h}'$ and $\mathfrak{g} \simeq \mathfrak{h}' \otimes_k E$.*

Proof. Let H and H' be the finite group schemes of height one corresponding to the restricted Lie algebras $\mathfrak{h}$ and $\mathfrak{h}'$, respectively. Consider the Hom scheme $X \subset \mathrm{Hilb}_{H\times H'}$ of surjective homomorphisms $H \to H'$. By assumption, this scheme contains a point with values in the algebraic closure E^{alg}. By Hilbert's Nullstellensatz, there must be a point with values in k, hence there there is a surjective homomorphism $H \to H'$. It corresponds to a short exact sequence of restricted Lie algebras

$$0 \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{h}' \longrightarrow 0.$$

We claim that the ideal $\mathfrak{a}$ vanishes. Suppose this is not the case. Clearly, $\mathfrak{h}'$ and $\mathfrak{g}$ have the same toral rank. By Lemma 11.3, also $\mathfrak{g}$ and $\mathfrak{h}$ have the same toral rank. According to [6], Lemma 1.7.2 we have $\rho_t(\mathfrak{n}) = 0$, so the p-map on $\mathfrak{a}$ is nilpotent. On the other hand, the p-map on $\mathfrak{h}'$ is not nilpotent, because the Lie algebra is simple of dimension $\dim(\mathfrak{h}') \geq 2$. The same holds for $\mathfrak{g}$, and we infer that the induced map $\mathfrak{a} \otimes_k E \to \mathfrak{g}$ is not surjective. Its image $\mathfrak{b} \subsetneq \mathfrak{g}$ is non-zero, because $\mathfrak{h} \subset \mathfrak{g}$. Since $\mathfrak{g}$ is simple, there is are elements $x \in \mathfrak{b}$ and $y \in \mathfrak{g}$ with $[x,y] \notin \mathfrak{b}$. Such vectors may be chosen with $x \in \mathfrak{a}$ and $y \in \mathfrak{h}$. Consequently, $\mathfrak{a} \subset \mathfrak{h}$ is not an ideal, contradiction.

This shows that $\mathfrak{h} = \mathfrak{h}'$. In particular, $\mathfrak{h}$ and $\mathfrak{g}$ have the same vector space dimension, so our surjection $\mathfrak{h}\otimes_k E \to \mathfrak{g}$ must be bijective. Our assertions follow. ☐

For each $a \in \mathfrak{h}$, the Lie bracket $\mathrm{ad}_a(x) = [a,x]$ defines a k-linear endomorphism of $\mathfrak{h}$, but also an E-linear endomorphism of $\mathfrak{g}$. Write $\mathrm{ad}_{\mathfrak{h},a}$ and $\mathrm{ad}_{\mathfrak{g},a}$ for the respective maps, and $\mu_{\mathfrak{h},a}(t) \in k[t]$ and $\mu_{\mathfrak{g},a}(t) \in E[t]$ for the resulting minimal polynomials.

Lemma 11.5. *We have $\mu_{\mathfrak{h},a}(t) = \mu_{\mathfrak{g},a}(t)$. In particular, the endomorphism $\mathrm{ad}_{\mathfrak{g},a}$ is trigonalizable, and its eigenvalues coincide with those of $\mathrm{ad}_{\mathfrak{h},a}$. Moreover, the former is diagonalizable if and only if this holds for the latter.*

Proof. The surjection $\mathfrak{h} \otimes_k E \to \mathfrak{g}$ already reveals that $\mu_{\mathfrak{g},a}(t)$ divides $\mu_{\mathfrak{h},a}(t)$. The latter decomposes into linear factors over k, because this field is algebraically closed. We conclude that $\mu_{\mathfrak{g},a}(t) = \sum \lambda_i t^i$ actually lies in $k[t]$, and decomposes into linear factors over k. Moreover, for each vector x from $\mathfrak{h} \subset \mathfrak{g}$ we have $\sum \lambda_i \mathrm{ad}^i_{\mathfrak{h},a}(x) = 0$, hence $\mu_{\mathfrak{h},a}(t)$ divides $\mu_{\mathfrak{g},a}(t)$. In turn, the two minimal polynomials coincide. The remaining assertions follow immediately. ☐

We now consider some special cases for $\mathfrak{g}$, and deduce structure results for $\mathfrak{h}$. Recall that k^n denotes the n-dimensional restricted Lie algebra over k with trivial bracket and p-map. The following fact is obvious:

Proposition 11.6. *If $\mathfrak{g}$ is isomorphic to E^m then the restricted Lie algebra $\mathfrak{h}$ is isomorphic to k^n for some integer $n \geq m$.*

Recall that $k^n \rtimes_\varphi \mathfrak{gl}_1(k)$ denotes the semidirect product formed with respect to the homomorphism $\varphi : \mathfrak{gl}_1(k) \to \mathfrak{gl}(k^n) = \mathrm{Der}'_k(k^n)$ that sends scalars to scalar matrices.

Proposition 11.7. *If $\mathfrak{g}$ is isomorphic to $E^m \rtimes \mathfrak{gl}_1(E)$, then the restricted Lie algebra $\mathfrak{h}$ is isomorphic to $k^n \rtimes \mathfrak{gl}_1(k)$ for some $n \geq m$.*

Proof. Without loss of generality we may assume $\mathfrak{g} = E^m \rtimes \mathfrak{gl}_1(E)$. First recall that bracket and p-map are given by the formulas

$$(10) \qquad [v + \lambda e, v' + \lambda' e] = \lambda v' - \lambda' v \quad \text{and} \quad (v + \lambda e)^{[p]} = \lambda^{p-1}(v + \lambda e),$$

where $v \in E^n$, and $e \in \mathfrak{gl}_1(E)$ denotes the unit. In particular, each vector is p-closed. Moreover, $a = v + \lambda e$ is multiplicative if and only if $\lambda \neq 0$, and $a^{[p]} = a$ if and only if $\lambda \in \mu_{p-1}(E)$. For any such vector, we see that the endomorphism $\mathrm{ad}_a(x) = [a, x]$ is diagonalizable, and $E^n \subset \mathfrak{g}$ is the eigenspace with respect to the eigenvalue $\alpha = \lambda$, whereas the line $Ea \subset \mathfrak{g}$ is the eigenspace for $\alpha = 0$.

From the extension $0 \to E^m \to \mathfrak{g} \to \mathfrak{gl}_1(E) \to 0$ one sees that $\mathfrak{g}$ has toral rank one. According to Lemma 11.3, the same holds for $\mathfrak{h}$. Choose some non-zero vector $a \in \mathfrak{h}$ with $a^{[p]} = a$. Then $a = v + \lambda e$ for some $\lambda \in \mu_{p-1}(E) \subset k^\times$. Replacing a by $\lambda^{-1} a$ we may assume $\lambda = 1$. By Lemma 11.5, the adjoint representation $\mathrm{ad}_{\mathfrak{h},a}$ is diagonalizable, with eigenvalues $\alpha = 0$ and $\alpha = 1$. Let $\mathfrak{h} = U_0 \oplus U_1$ be the corresponding eigenspace decomposition. Then U_0 lies in the corresponding eigenspace for $\mathrm{ad}_{\mathfrak{g},a}$, which is $E^n \subset \mathfrak{g}$. It follows that U_0 has trivial Lie bracket and p-map. The choice of a k-basis gives $U_0 = k^n$ for some $n \geq 0$. Likewise, U_1 is contained in Ea. Thus the bracket vanishes on U_1, and the p-map is injective. Using Lemma 11.3, we infer that $U_1 = ka$. The vector space decomposition $\mathfrak{h} = U_0 \oplus U_1$ thus becomes a semidirect product $\mathfrak{h} = k^n \rtimes \mathfrak{gl}_1(k)$. We must have $m \geq n$ because the map $\mathfrak{h} \otimes_k E \to \mathfrak{g}$ is surjective. □

Recall that $\mathfrak{sl}_2(E)$ is simple for $p \geq 3$. Using Lemma 11.4, we immediately obtain:

Proposition 11.8. *Suppose $p \geq 3$. If $\mathfrak{g}$ is isomorphic to a twisted form of $\mathfrak{sl}_2(E)$, then the restricted Lie algebra $\mathfrak{h}$ is isomorphic to $\mathfrak{sl}_2(k)$.*

12. Structure results for Frobenius kernels

We now come to our main result. Let k be an algebraically closed field of characteristic $p > 0$, and X be a proper integral scheme or more generally a proper integral algebraic space, $H = \mathrm{Aut}_{X/k}[F]$ be the Frobenius kernel for the automorphism group scheme, and $\mathfrak{h} = H^0(X, \Theta_{X/k})$ the corresponding restricted Lie algebra over k. Let $H_F = H \otimes_k F$ be the base-change to the function field $F = k(X)$, and $H_F^{\mathrm{inert}} \subset H_F$ the inertia subgroup scheme for the rational point in the spectrum of $F \otimes_E F$, with corresponding restricted Lie algebra $\mathfrak{h}_F^{\mathrm{inert}} \subset \mathfrak{h}_F$. Recall that the foliation rank $r \geq 0$ is given by

$$r = \dim(\mathfrak{h}_F / \mathfrak{h}_F^{\mathrm{inert}}) = \dim(\Omega^1_{F/E}) \quad \text{and} \quad [H_F : H_F^{\mathrm{inert}}] = [F : E] = p^r,$$

where $E = F^{\mathfrak{h}}$ is the kernel for all derivations $D : F \to F$ from the Lie algebra $\mathfrak{h}$. This is nothing but the function field $E = k(Y)$ of the quotient $Y = X/H$.

Theorem 12.1. *Suppose that the proper integral scheme X has foliation rank $r \leq 1$. Then the Frobenius kernel $H = \mathrm{Aut}_{X/k}[F]$ is isomorphic to the Frobenius kernel of one of the following three basic types of group schemes:*

$$\mathrm{SL}_2 \quad \textit{and} \quad \mathbb{G}_a^{\oplus n} \quad \textit{and} \quad \mathbb{G}_a^{\oplus n} \rtimes \mathbb{G}_m,$$

for some integer $n \geq 0$.

Proof. The case $r = 0$ is trivial, so we assume $r = 1$, such that $\mathfrak{h} \neq 0$. By assumption the subfield $E = F^{\mathfrak{h}}$ has $[F : E] = p$, and we thus have $F = E[T]/(T^p - \omega)$ for some $\omega \in E$. Thus the restricted Lie algebra $\mathfrak{g} = \mathrm{Der}_E(F)$ is a twisted form of the Witt algebra $\mathfrak{g}_0$ over E. By construction, we have an inclusion $\mathfrak{h} \subset \mathfrak{g}$. It generates a restricted subalgebra $\mathfrak{g}' = \mathfrak{h} \cdot E$.

We now replace the field E by some separable closure E^{sep}, and likewise F by $F \otimes_E E^{\text{sep}}$. According to Theorem 9.5 the restricted Lie algebra $\mathfrak{g}'$ is isomorphic to either $\mathfrak{sl}_2(E)$ or E or $\mathfrak{g}_1(E)$ or $E \rtimes \mathfrak{g}_1(E)$. By the results in Section 11, this ensures that $\mathfrak{h}$ is isomorphic to $\mathfrak{sl}_2(k)$ or k^n or $k^n \rtimes \mathfrak{g}_1(k)$ for some $n \geq 0$. This are the Frobenius kernels for the group schemes in question, and our assertion follows. $\square$

In the latter two cases, the respective Frobenius kernels are $\alpha_p^{\oplus n}$ and $\alpha_p^{\oplus n} \rtimes \mu_p$. With Proposition 6.6, we immediately get the following consequence:

Corollary 12.2. *Suppose that X is a proper normal surface with $h^0(\omega_X^\vee) = 0$. Then $H = \operatorname{Aut}_{X/k}[F]$ is isomorphic to the Frobenius kernel of one of the three basic types of group schemes in the Theorem.*

This applies in particular to smooth surface S of Kodaira dimension $\operatorname{kod}(S) \geq 1$, to surfaces of general type S and their minimal models X, or normal surfaces with $c_1 = 0$ and $\omega_X \neq \mathscr{O}_S$ having at most rational double points.

13. Canonically polarized surfaces

Let k be a ground field of characteristic $p > 0$, and X be a proper normal surface with $h^0(\mathscr{O}_X) = 1$ whose complete local rings $\mathscr{O}_{X,a}^\wedge$ are complete intersections. Then the cotangent complex $L^\bullet_{X/k}$ is perfect, and we obtain two *Chern numbers*

$$c_1^2 = c_1^2(L^\bullet_{X/k}) \quad \text{and} \quad c_2 = c_2(L^\bullet_{X/k}),$$

as explained by Ekedahl, Hyland and Shepherd–Barron [15], Section 3. In some sense, these integers are the most fundamental numerical invariants of the surface X. Note that c_1^2 is nothing but the self-intersection number $K_X^2 = (\omega_X \cdot \omega_X)$ of the dualizing sheaf. If the singularities are also rational, hence rational double points, the Chern numbers of X coincide with the Chern numbers of the minimal resolution of singularities S, according to loc. cit. Proposition 3.12 and Corollary 3.13. For more details on rational double points, we refer to [37] and [3].

Recall that a *canonically polarized surface* is the canonical model X of a smooth surface S of general type. Then ω_X is ample, all local rings $\mathscr{O}_{X,a}$ are either regular or rational double points, and the above applies. Let us record the following facts:

Lemma 13.1. *Suppose that X is canonically polarized. Then*

$$\chi(\mathscr{O}_X) = \frac{1}{12}(c_1^2 + c_2) \quad \textit{and} \quad h^0(\omega_X) \leq \frac{1}{2}(c_1^2 + 4) \quad \textit{and} \quad c_2 \leq 5c_1^2 + 36.$$

Proof. Let $f : S \to X$ be the minimal resolution of singularities. Then S is a smooth minimal surface of general type, and the first formula holds for S instead of X by Hirzebruch–Riemann–Roch. We already observed that the surfaces X and S have the same Chern numbers, and the structure sheaves have the same cohomology. Thus the formula also holds for X. In particular we have $h^0(\omega_X) = h^2(\mathscr{O}_X) = h^2(\mathscr{O}_S) = h^0(\omega_S)$, and Noether's Inequality (for example [36], Section 8.3) for the minimal surface of general type S gives the second formula. This ensures

$$\chi(\mathscr{O}_X) = 1 - h^1(\omega_X) + h^0(\omega_X) \leq 1 + h^0(\omega_X) = 1 + h^0(\omega_S) \leq (c_1^2 + 6)/2.$$

Combining with $\chi(\mathscr{O}_X) = (c_1^2 + c_2)/12$ we get the third inequality. $\square$

Theorem 13.2. *Let X be canonically polarized surface, with Chern numbers c_1^2, c_2. Then the Lie algebra $\mathfrak{h} = H^0(X, \Theta_{X/k})$ for the Frobenius kernel $H = \operatorname{Aut}_{X/k}[F]$ has the property $\dim(\mathfrak{h}) \leq \Phi(c_1^2, c_2)$ for the polynomials*

$$\Phi(x,y) = \frac{1}{144}(14641x^2 + 242xy + 60x + y^2 - 12y + 288).$$

Moreover, we also have the weaker bound $\dim(\mathfrak{h}) \leq \Psi(c_1^2)$ with the polynomial

$$\Psi(x) = \frac{441}{4}x^2 + 63x + 8.$$

Proof. According to [14], Theorem 1.20 the invertible sheaf $\omega_X^{\otimes 5}$ is very ample, and defines a closed embedding $X \subset \mathbb{P}^n$, such that $n+1 = h^0(\omega_X^{\otimes 5})$ and $\mathscr{O}_X(1) = \omega_X^{\otimes 5}$. Serre Duality gives $h^i(\omega_X^{\otimes 5}) = h^{2-i}(\omega_X^{\otimes -4})$. This vanishes for $i = 2$, because ω_X is ample, and also for $i = 1$ by [14], Theorem 1.7. Thus we have $h^0(\omega_X^{\otimes 5}) = \chi(\omega_X^{\otimes 5})$, and Riemann–Roch gives

$$n + 1 = h^0(\omega_X^{\otimes 5}) = \chi(\mathscr{O}_X) + \frac{1}{2}\left((5K_X)^2 - (5K_X \cdot K_X)\right) = \chi(\mathscr{O}_X) + 10c_1^2. \tag{11}$$

Applying the functor $\operatorname{Hom}(\cdot, \mathscr{O}_X)$ to the canonical surjection $\Omega^1_{\mathbb{P}^n/k} \otimes \mathscr{O}_X \to \Omega^1_{X/k}$ gives an inclusion

$$H^0(X, \Theta_{X/k}) = \operatorname{Hom}(\Omega^1_{X/k}, \mathscr{O}_X) \subset \operatorname{Hom}(\Omega^1_{\mathbb{P}^n/k} \otimes \mathscr{O}_X, \mathscr{O}_X) = \operatorname{Hom}(\Omega^1_{\mathbb{P}^n}, \mathscr{O}_X). \tag{12}$$

Likewise, the Euler Sequence $0 \to \Omega^1_{\mathbb{P}^n/k} \to \bigoplus_{i=0}^n \mathscr{O}_{\mathbb{P}^n}(-1) \to \mathscr{O}_{\mathbb{P}^n} \to 0$ induces the exact sequence

$$0 \longrightarrow H^0(\mathscr{O}_X) \longrightarrow \bigoplus_{i=0}^n H^0(\mathscr{O}_X(1)) \longrightarrow \operatorname{Hom}(\Omega^1_{\mathbb{P}^n/k}, \mathscr{O}_X) \longrightarrow H^1(\mathscr{O}_X),$$

where we have used the identification $\operatorname{Ext}^i(\mathscr{E}^\vee, \mathscr{O}_X) = H^i(X, \mathscr{E}|X)$ for locally free sheaves $\mathscr{E}$ on $\mathbb{P}^n$. Combining the above sequence with (11) and (12) gives the estimate

$$h^0(\Theta_X) \leq \dim \operatorname{Hom}(\Omega^1_{\mathbb{P}^n/k}, \mathscr{O}_X) \leq (n+1) \cdot h^0(\omega_X^{\otimes 5}) - 1 + h^1(\mathscr{O}_X).$$

With (11) and Serre duality, the right-hand side can be rewritten as

$$(\chi(\mathscr{O}_X) + 10c_1^2) \cdot (\chi(\mathscr{O}_X) + 10c_1^2) - \chi(\mathscr{O}_X) + h^0(\omega_X).$$

Using $\chi(\mathscr{O}_X) = (c_1^2 + c_2)/12$ and $h^0(\omega_X) \leq (c_1^2 - 4)/2$ from Lemma 13.1, we get the desired bound $\dim(\mathfrak{h}) \leq \Phi(c_1^2, c_2)$. Finally $c_2 \leq 5c_1^2 + 36$ yields the weaker bound $\dim(\mathfrak{h}) \leq \Psi(c_1^2)$. □

14. Examples

Let k be a ground field of characteristic $p > 0$. In this section we give examples of canonically polarized surfaces X with $\operatorname{Aut}_{X/k}[F] = \alpha_p^{\oplus n} \rtimes \mu_p$ for certain $n \geq 0$. In other words, the corresponding restricted Lie algebra is $\mathfrak{h} = H^0(X, \Theta_{X/k}) = k^n \rtimes \mathfrak{gl}_1(k)$. The examples are simply-connected, with c_1^2 being a cubic polynomials in p. Note that the full automorphism group scheme $\operatorname{Aut}_{X/k}$ for canonically polarized surface with fixed c_1^2 is bounded, according to [56], Theorem 3.1. Also note that we do not have examples with $\mathfrak{h} = \mathfrak{sl}_2(k)$.

We start by constructing examples with $n \geq 1$. View $\mathbb{P}^2$ as the homogeneous spectrum of $k[T_0, T_1, T_2]$. Fix some $d \geq 1$, set $\mathscr{L} = \mathscr{O}_{\mathbb{P}^2}(d)$ and consider the section

$$s = T_0 T_1 T_2^{pd-2} + T_1 T_2 T_0^{pd-2} + T_2 T_0 T_1^{pd-2} \in \Gamma(\mathbb{P}^2, \mathscr{L}^{\otimes pd}). \tag{13}$$

Regarded as $\mathscr{L}^{\otimes -p} \to \mathscr{O}_{\mathbb{P}^2}$, this endows the coherent sheaf $\mathscr{A} = \bigoplus_{i=0}^{p-1} \mathscr{L}^{\otimes -i}$ with the structure of a $\mathbb{Z}/p\mathbb{Z}$-graded $\mathscr{O}_{\mathbb{P}^2}$-algebra, and we define $X = \operatorname{Spec}(\mathscr{A})$ as the relative spectrum.

Proposition 14.1. *In the above setting, suppose $p \neq 3$ and $d \geq 4$. Then*

$$\mathfrak{h} = k^n \rtimes \mathfrak{gl}_1(k) \quad \textit{and} \quad \operatorname{Aut}_{X/k}[F] = \alpha_p^{\oplus n} \rtimes \mu_p,$$

where $n = (d+1)(d+2)/2$. Moreover, X is a canonically polarized surface with Chern invariants $c_1^2 = p(pd-d-3)^2$ and $c_2 = 3p + dp(p-1)(pd-3)$.

Proof. Being locally a hypersurface in affine three-space, the scheme X is Gorenstein. According to [14], Proposition 1.7 the dualizing sheaf is $\omega_X = \pi^*(\omega_{\mathbb{P}^2} \otimes \mathscr{L}^{p-1})$, which equals the pullback of $\mathscr{O}_{\mathbb{P}^2}(pd-d-3)$. The statement on c_1^2 follows. Using $d(p-1) - 3 \geq d - 3 \geq 1$ we see that ω_X is ample. Since $\pi : X \to \mathbb{P}^2$ is finite, the Euler characteristic $\chi(\mathscr{O}_X)$ equals

$$\sum_{i=0}^{p-1} \chi(\mathscr{O}_{\mathbb{P}^2}(-id)) = \sum_{i=0}^{p-1} \binom{2-id}{2} = \frac{12p - 9d(p-1)p + d^2(p-1)p(2p-1)}{12}.$$

Now suppose for the moment that we already know that X is geometrically normal, with only rational double points. Then X is a canonically polarized surface, and Lemma 13.1 yields the statement on c_2.

We proceed by computing $\mathfrak{h} = H^0(X, \Theta_{X/k})$ as a vector space. The grading of the structure sheaf $\mathscr{A} = \bigoplus_{i=0}^{p-1} \mathscr{L}^{\otimes -i}$ corresponds to an action of $G = \mu_p$ on the scheme X, with quotient $\mathbb{P}^2$. Let $D : \mathscr{O}_X \to \Theta_{X/k}$ be the corresponding multiplicative vector field, and $\mathscr{O}_X(\Delta) \subset \Theta_{X/k}$ the saturation of the image, for some effective Weil divisor $\Delta \subset X$. Lemma 14.5 below gives an exact sequence

$$0 \longrightarrow \mathscr{O}_X(\Delta) \xrightarrow{D} \Theta_{X/k} \longrightarrow \omega_X^{\otimes -1}(-\Delta).$$

The term on the right has no non-zero global sections, because ω_X is ample, and consequently $H^0(X, \mathscr{O}_X(\Delta)) = H^0(X, \Theta_{X/k})$. We have $\omega_X = \pi^*(\omega_{\mathbb{P}^2}) \otimes \mathscr{O}_X((p-1)\Delta)$ by [45], Proposition 2 combined with Proposition 3, which gives $\mathscr{O}_X(\Delta) = \pi^*(\mathscr{L})$. Consequently

$$H^0(X, \mathscr{O}_X(\Delta)) = H^0(\mathbb{P}^2, \mathscr{A} \otimes \mathscr{L}) = H^0(\mathbb{P}^2, \mathscr{L}) \oplus H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}) = k^n \times k,$$

with the integer $n = (d+1)(d+2)/2$, as desired.

It is not difficult to compute bracket and p-map for $\mathfrak{h} = H^0(X, \Theta_{X/k})$. The coordinate rings for the affine open sets $U_i = D_+(T_i)$ of $\mathbb{P}^2$ are the homogeneous localizations $R_i = k[T_0, T_1, T_2]_{(T_i)}$, and the preimages $\pi^{-1}(U_i)$ are the spectra of $A_i = R_i[t_i]/(t_i^p - s_i)$. Here s_i denotes the dehomogenization of (13) with respect to T_i. We have $\Theta_{A_i/R_i} = A_i \partial/\partial t_i$, and our multiplicative vector field restricts becomes $D = t_i \partial/\partial t_i$. For any $b_i, b_i' \in R_i$, one immediately calculates

$$[b_i \partial/\partial t_i, t_i \partial/\partial t_i] = b_i \partial/\partial t_i, \quad [b_i \partial/\partial t_i, b_i' \partial/\partial t_i] = 0 \quad \text{and} \quad (b_i \partial/\partial t_i)^{[p]} = 0.$$

Choosing a basis for $H^0(\mathbb{P}^2,\mathscr{L})$, we infer that the vector space decomposition $\mathfrak{h} = H^0(\mathbb{P}^2,\mathscr{L})\oplus H^0(\mathbb{P}^2,\mathscr{O}_{\mathbb{P}^2})$ becomes a semi-direct product structure $\mathfrak{h}=k^n\rtimes\mathfrak{gl}_1(k)$ for the restricted Lie algebra.

It remains to check that $\operatorname{Sing}(X/k)$ is finite, and that all singularities are rational double points. For this we may assume that k is algebraically closed. In light of the symmetry in (13), it suffices to verify this on the preimage $V=\pi^{-1}(U)$ of the open set $U=D_+(T_0)$. Setting $x=T_1/T_0$ and $y=T_2/T_0$, we see that V has coordinate ring $A=k[x,y,t]/(f)$ with

$$f=t^p-xy-x^{pd-2}y-xy^{pd-2}.$$

The singular locus comprises the common zeros of f and the partial derivatives $\partial f\partial x=-y+2x^{pd-3}y-y^{pd-2}=0$ and $\partial f/\partial y=-x-x^{pd-2}+2xy^{pd-3}=0$. Clearly there are only finitely many singularities with $x=0$ or $y=0$. For the remaining part of $\operatorname{Sing}(X)$, it suffices to examine the system of polynomial equations

$$-1+2x^{pd-3}-y^{pd-3}=0\quad\text{and}\quad -1-x^{pd-3}+2y^{pd-3}=0. \tag{14}$$

This actually is a system of linear equations in the powers x^{pd-3} and y^{pd-3}. Using $p\neq 3$ we conclude that it has only finitely many solutions.

It remains to verify that all singularities are rational double points. It would be tedious and cumbersome to do this explicitly. We resort to a trick of independent interest, where we actually show that there are only rational double points of A-type: By Proposition 14.3 and Lemma 14.4, it suffices to verify that no singular point is a zero for the polynomial $\frac{\partial^2 f}{\partial x\partial y}\cdot\frac{\partial^2 f}{\partial x\partial y}-\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}$, which gives the additional equation

$$1+4x^{2pd-6}+4y^{2pd-6}-4x^{pd-3}-4y^{pd-3}-28(xy)^{pd-3}=0. \tag{15}$$

Assume some singular point also satisfies the above. Clearly $x\neq 0$ or $y\neq 0$. If both inequalities hold we easily get from (14) that $x^{pd-3}=y^{pd-3}=1$, which in light of the above gives $1-28\equiv 0$ modulo p, contradiction. We thus may assume $x\neq 0$ and $y=0$. Now (14) ensures $-1+x^{pd-3}=0$, and (15) yields $1+4+4\equiv 0$, again contradiction. $\square$

Note that $\pi : X\to\mathbb{P}^2$ is a universal homeomorphism, so the geometric fundamental group of X is trivial, and $b_1=0$ and $b_2=1$. Let $x_1,\ldots,x_r$ be the geometric singularities on X. We saw above that they are rational double points of certain types $A_{n_1},\ldots,A_{n_r}$. As discussed in Section 13, the Chern number c_2 is the alternating sum of the Betti numbers on the minimal resolution of X, which yields the formula $c_2-3=n_1+\ldots+n_r$.

We now describe a different construction that gives surfaces with $\mathfrak{h}=\mathfrak{gl}_1(k)$. Regard $\mathbb{P}^3$ as the homogeneous spectrum of $k[T_0,\ldots,T_3]$. Consider the homogeneous polynomial

$$s=T_0T_1T_2^{2p-1}-T_0T_1T_3^{2p-1}+T_0^{2p}T_2+T_1^{2p}T_3+T_2^{2p+1}+T_3^{2p+1} \tag{16}$$

and the resulting surface $X\subset\mathbb{P}^3$ of degree $2p+1$.

Proposition 14.2. *In the above setting, the surface X has*

$$\mathfrak{h}=\mathfrak{gl}_1(k)\quad\textit{and}\quad \operatorname{Aut}_{X/k}[F]=\mu_p.$$

Moreover, X is a canonically polarized surface with $c_1^2 = (2p-3)^2(2p+1)$ and $c_2 = 8p^3 - 4p^2 + 2p + 3$.

Proof. The Adjunction Formula gives $\omega_X = \mathscr{O}_X(2p-3)$. This sheaf is ample, and the statement on the Chern number c_1^2 is immediate. The short exact sequence $0 \to \mathscr{O}_{\mathbb{P}^3}(-2p-1) \to \mathscr{O}_{\mathbb{P}^3} \to \mathscr{O}_X \to 0$ yields

$$\chi(\mathscr{O}_X) = \chi(\mathscr{O}_{\mathbb{P}^3}) - \chi(\mathscr{O}_{\mathbb{P}^3}(-2p-1)) = 1 - (2-2p)(1-2p)(-2p)/6.$$

Suppose for the moment that X is normal, and the singularities are rational double points. Then X is a canonically polarized surfaces, and Lemma 13.1 yields the statement on the Chern number c_2.

We proceed to compute $\mathfrak{g} = H^0(X, \Theta_X)$. Consider the global vector field

$$D = T_0\partial/\partial T_0 - T_1\partial/\partial T_1$$

on $\mathbb{P}^3$. From $D(T_0) = T_0$, $D(T_1) = -T_1$ and $D(T_2) = D(T_3) = 0$ we see $D^p = D$. One easily checks that $D(s) = 0$, so one gets an induced multiplicative vector field on X, which we likewise denote by D. Let $\mathscr{O}_X(\Delta) \subset \Theta_{X/k}$ be the saturation of the mapping $D : \mathscr{O}_X \to \Theta_{X/k}$ With Lemma 14.5 below we get $H^0(X, \mathscr{O}_X(\Delta)) = H^0(X, \Theta_{X/k})$. We claim that this vector space is one-dimensional.

To see this, let us first examine the fixed scheme for the action of $G = \mu_p$ on $\mathbb{P}^3$ corresponding to the multiplicative vector field D. Using the quotient rule, we compute

$$D(T_1/T_0) = -2T_1/T_0, \quad D(T_2/T_0) = -T_2/T_0, \quad D(T_3/T_0) = -T_3/T_0.$$

With $x_i = T_i/T_0$ one gets $D|U_0 = -2x_1\partial/\partial x_1 - x_2\partial/\partial x_2 - x_3\partial/\partial x_3$ on the affine open set $U_0 = D_+(T_0)$. So on this chart, the fixed scheme is given by the ideal $\mathfrak{a} = (2x_1, x_2, x_3)$. Note the different behavior for $p \neq 2$ and $p = 2$. With $y_j = T_j/T_3$, one gets $D = y_0\partial/\partial y_0 - y_1\partial/\partial y_1$ on $U_3 = D_+(T_3)$. Making similar computations on the remaining charts, we infer that the fixed scheme in $\mathbb{P}^3$ is contained in the union of the disjoint lines $L = V_+(T_2, T_3)$ and $L' = V_+(T_0, T_1)$, regardless of $p > 0$. Clearly $L' \cap X$ is finite, whereas $L \subset X$. Thus $\Delta \subset L$, and hence $\mathscr{O}_X(\Delta) \subset \mathscr{O}_X(L)$.

Serre Duality yields $h^0(\mathscr{O}_X(L)) = h^2(\omega_X(-L))$. From the short exact sequence $0 \to \omega_X(-L) \to \omega_X \to \omega_X|_L \to 0$ we get an exact sequence

$$H^1(L, \omega_X|_L) \longrightarrow H^2(X, \omega_X(-L)) \longrightarrow H^2(X, \omega_X) \longrightarrow 0.$$

The term on the left vanishes, because $\omega_X|_L = \mathscr{O}_{\mathbb{P}^1}(2p-3)$, and $h^0(\omega_X) = 1$, therefore also $h^2(\omega_X(-L)) = 1$. Combining the above facts we infer that $H^0(X, \mathscr{O}_X(\Delta))$ indeed is one-dimensional.

It remains to verify that $\mathrm{Sing}(X/k)$ is finite, and that the singularities are rational double points. For this we may assume that k is algebraically closed. We compute partial derivatives:

$$\partial f/\partial T_0 = T_1 \cdot \prod (T_2 - \zeta T_3) \quad \text{and} \quad \partial f/\partial T_1 = T_0 \cdot \prod (T_2 - \zeta T_3),$$

where the products run over the $(2p-1)$-th roots of unities $\zeta \in k^\times$. From the defining equation (16) we see that there are only finitely many points on X satisfying $T_1 = T_0 = 0$. All other singularities lie on a plane given by some $T_2 - \zeta T_3 = 0$. Now consider

$$\partial f/\partial T_2 = -T_0T_1T_2^{2p-2} + (T_0^2 + T_2^2)^p \quad \text{and} \quad \partial f/\partial T_3 = T_0T_1T_3^{2p-2} + (T_1^2 + T_3^2)^p.$$

Substituting $T_3 = \zeta^{-1}T_2$ and the first term in $\partial f/\partial T_3$, we see that the other singularities are common solutions of $-T_0T_1T_2^{2p-2} + T_0^{2p} + T_2^{2p} = 0$ and an equation $T_0^2 + \alpha T_1^2 + \beta T_2^2 = 0$, hence finite in number. Summing up, $\mathrm{Sing}(X)$ is finite.

It remains to verify that every singularity is a rational double point. Again we show that that only A-types occurs. We do the computation on the open set $U_0 = D_+(T_0)$, the other coordinate charts can be handled in a similar way. Set $x_i = T_i/T_0$. The dehomogenization of (16) is $f = x_1x_2^{2p-1} - x_1x_3^{2p-1} + x_2 + x_1^{2p}x_3 + x_2^{2p+1} + x_3^{2p+1}$. Suppose there is a singularity that is not a rational double point of A-type. By Proposition 14.4 combined with Lemma 14.3 below it must be a common zero of

$$\frac{\partial f}{\partial x_2} = -x_1x_2^{2p-2} + 1 + x_2^{2p} \quad \text{and} \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} \cdot \frac{\partial^2 f}{\partial x_2 \partial x_1} - \frac{\partial^2 f}{\partial x_1^2} \cdot \frac{\partial^2 f}{\partial x_2^2} = x_2^{4p-4},$$

giving the desired contradiction. □

Note that the inclusion of the surface $X \subset \mathbb{P}^3$ induces a bijection on fundamental groups ([22], Exposé X, Theorem 3.10), so again the geometric fundamental group of X is trivial.

It remains to establish the technical results used in the preceding arguments. Let A be a complete local k-algebra that is regular of dimension three, with maximal ideal $\mathfrak{m}_A$ and residue field $k = A/\mathfrak{m}_A$. Note that for each choice of regular system of parameters $x, y, z \in A$ one obtains an identification $A = k[[x, y, z]]$.

Lemma 14.3. *Let $f \in A$ be an irreducible element such that $f \equiv x_0y_0 + \lambda z_0^2$ modulo $\mathfrak{m}_A^3$ for some regular system of parameters x_0, y_0, z_0 and some $\lambda \in k$. Then there exists another regular system of parameters x, y, z such that $(f) = (xy + z^n)$, for some $n \geq 2$. Hence $B = A/(f)$ is a rational double point of type A_{n-1}.*

Proof. We construct by induction on $n \geq 0$ certain regular system of parameters $x_n, y_n, z_n \in A$ such that $x_n \equiv x_{n-1}$ and $y_n \equiv y_{n-1}$ modulo $\mathfrak{m}_A^n$, and $z_n = z_0$, and

$$f = x_ny_n + x_n\phi_n + y_n\psi_n + h_n \tag{17}$$

for some $\phi_n, \psi_n \in \mathfrak{m}_A^{n+2}$ and $h_n \in z_n^2 k[[z_n]]$. For $n = 0$ we take x_0, y_0, z_0 as in our assumptions. If we already have defined $x_n, y_n, z_n \in A$, we set

$$x_{n+1} = x_n + \psi_n, \quad y_{n+1} = y_n + \phi_n \quad \text{and} \quad z_{n+1} = z_n. \tag{18}$$

Clearly $x_{n+1} \equiv x_n$ and $y_{n+1} \equiv y_n$ modulo $\mathfrak{m}_A^{n+1}$, and $z_{n+1} = z_0$. In particular the above is a regular system of parameters. Since $\phi_n\psi_n \in \mathfrak{m}_A^{2n+4}$, we may write

$$-\phi_n\psi_n = x_{n+1}\phi_{n+1} + y_{n+1}\psi_{n+1} + \sigma_n$$

with $\phi_{n+1}, \psi_{n+1} \in \mathfrak{m}_A^{2n+3}$ and $\sigma_n \in z_{n+1}^{2n+4} k[[z_{n+1}]]$. Combining (17) and (18), we get

$$f = x_{n+1}y_{n+1} + x_{n+1}\phi_{n+1} + y_{n+1}\psi_{n+1} + h_{n+1}$$

where $h_{n+1} = h_n + \sigma_n$ belongs to $z_{n+1}^2 k[[z_{n+1}]]$. This completes our inductive definition. Note that $h_{n+1} \equiv h_n$ modulo $\mathfrak{m}_A^{2n+4}$.

By construction, the x_n, y_n, z_n are convergent sequences in A with respect to the $\mathfrak{m}_A$-adic topology. The limits $x, y, z \in A$ give the desired regular system of parameters: Since the ϕ_n, ψ_n converge to zero, we have $f = xy + h(z)$, where h is the limit of the $h_n \in k[[z]]$. We must have $h \neq 0$, because f is irreducible. Hence $h = uz^n$ with $u \in k[[z]]^\times$ and $n \geq 2$. Replacing x by $u^{-1}x$, we finally get $(f) = (xy - z^n)$. Summing up, $B = A/(f)$ is a rational double point of type A_{n-1}. ☐

The condition in the proposition can be checked with partial derivatives, at least if k is algebraically closed. This makes the criterion applicable for computations:

Proposition 14.4. *Let $f \in \mathfrak{m}_A^2$. Suppose that k is algebraically closed, and that*

$$\left(\frac{\partial^2 f}{\partial u_1 \partial u_2}\right) \cdot \left(\frac{\partial^2 f}{\partial u_1 \partial u_2}\right) - \left(\frac{\partial^2 f}{\partial u_1^2}\right) \cdot \left(\frac{\partial^2 f}{\partial u_2^2}\right) \notin \mathfrak{m}_A \tag{19}$$

for some system of parameters $u_1, u_2, u_3 \in A$. Then there exists another system of parameters x, y, z such that $f \equiv xy + \lambda z^2$ modulo $\mathfrak{m}_A^3$, for some $\lambda \in k$.

Proof. Write $f = q + g$, where $q = q(u_1, u_2, u_3)$ is a homogeneous polynomial of degree two and $g \in \mathfrak{m}_A^3$. Write $q = q_1 + u_3 l$, where $l = l(u_1, u_2, u_3)$ is homogeneous of degree one and $q_1 = q_1(u_1, u_2)$. If q_1 is a square, a straightforward computation with partial derivatives produces a contradiction to (19). Since k is algebraically closed we have a factorization $q_1 = L_1 \cdot L_2$ where $L_1 = L_1(u_1, u_2)$ and $L_2 = L_2(u_1, u_2)$ are independent homogeneous polynomials of degree one. Then $w_1 = L_1$, $w_2 = L_2$ and w_3 form another regular system of parameters of A, and we have

$$q = w_1 w_2 + w_3 l = w_1 w_2 + a w_3 w_1 + b w_3 w_2 + c w_3^2,$$

with $a, b, c \in k$. We finally set $x = w_1 + b w_3$, $y = w_2 + a w_3$ and $z = w_3$. This is is a further regular system of parameters, with $q = xy + \lambda z^2$, where $\lambda = c - ab$. Therefore, $f \equiv xy + \lambda z^2$ modulo $\mathfrak{m}_A^3$, as claimed. ☐

We also used a general fact on coherent sheaves: Let X be a noetherian scheme that is integral and normal, $\mathscr{E}$ be a coherent sheaf of rank two, $s : \mathscr{O}_X \to \mathscr{E}^\vee$ a non-zero global section. The double dual $\mathscr{O}_X(-\Delta)$ for the image of the dual map $s^\vee : \mathscr{E}^{\vee\vee} \to \mathscr{O}_X$ defines an effective Weil divisor $\Delta \subset X$. By [23], Corollary 1.8 the duals of coherent sheaves on X are reflexive. Dualizing $\mathscr{E}^{\vee\vee} \to \mathscr{O}_X(-\Delta) \subset \mathscr{O}_X$ we see that the homomorphism s factors over an inclusion $\mathscr{O}_X(\Delta) \subset \mathscr{E}^\vee$. The latter is called the *saturation* of the section $s \in \Gamma(X, \mathscr{E}^\vee)$.

Lemma 14.5. *In the above setting there is a four-term exact sequence*

$$0 \longrightarrow \mathscr{O}_X(\Delta) \longrightarrow \mathscr{E}^\vee \longrightarrow \mathscr{L}(-\Delta) \longrightarrow \mathscr{N} \longrightarrow 0,$$

where $\mathscr{L} = \underline{\mathrm{Hom}}(\Lambda^2(\mathscr{E}), \mathscr{O}_X)$, and $\mathscr{N}$ is a coherent sheaf whose support has codimension at least two.

Proof. According to [23], Theorem 1.12 it suffices to construct a short exact sequence $0 \to \mathscr{O}_X(\Delta) \to \mathscr{E}^\vee \to \mathscr{L}(-\Delta) \to 0$ on the complement of some closed set $Z \subset X$ of codimension at least two. Let $\mathscr{E}_0$ be the quotient of $\mathscr{E}$ by its torsion subsheaf. The surjection $\mathscr{E} \to \mathscr{E}_0$ induces an equality $\mathscr{E}_0^\vee = \mathscr{E}^\vee$, so we may assume that $\mathscr{E}$ is torsion free. It is then locally free in codimension one, so it suffices to treat the case that $\mathscr{E}$ is locally free. By construction, the cokernel $\mathscr{F}$ for $\mathscr{O}_X(\Delta) \subset \mathscr{E}^\vee$ is torsion-free of rank one, so we may assume that it is invertible. Taking determinants shows $\mathscr{F} \simeq \mathscr{L}(-\Delta)$. ☐

References

[1] M. Artin: Algebraization of formal moduli I. In: D. Spencer, S. Iyanaga (eds.), Global Analysis, pp. 21–71. Univ. Tokyo Press, Tokyo, 1969.

[2] M. Artin: Algebraic spaces. Yale University Press, New Haven, Conn.-London, 1971.

[3] M. Artin: Coverings of the rational double points in characteristic p. In: W. Baily, T. Shioda (eds.), Complex analysis and algebraic geometry, pp. 11–22. Iwanami Shoten, Tokyo, 1977.

[4] M. Artin, A. Grothendieck, J.-L. Verdier (eds.): Théorie des topos et cohomologie étale des schémas (SGA 4) Tome 2. Springer, Berlin, 1973.

[5] A. Blanchard: Sur les variétés analytiques complexes. Ann. Sci. Ecole Norm. Sup. 73 (1956), 157–202.

[6] R. Block, R. Wilson: Classification of the restricted simple Lie algebras. J. Algebra 114 (1988), 115–259.

[7] N. Bourbaki: Lie groups and Lie algebras. Chapters 1–3. Springer, Berlin, 1989.

[8] N. Bourbaki: Algebra II. Chapters 4–7. Springer, Berlin, 1990.

[9] M. Brion, P. Samuel, V. Uma: Lectures on the structure of algebraic groups and geometric applications. Hindustan Book Agency, New Delhi, 2013.

[10] H.-J. Chang: Über Wittsche Lie-Ringe. Abh. Math. Semin. Univ. Hambg. 14 (1941), 151–184.

[11] M. Demazure, P. Gabriel: Groupes algébriques. Masson, Paris, 1970.

[12] M. Demazure, A. Grothendieck (eds.): Schémas en groupes II (SGA 3 Tome 2). Springer, Berlin, 1970.

[13] T. Ekedahl: Foliations and inseparable morphisms. In: S. Bloch (ed.), Algebraic geometry, Bowdoin, 1985, pp. 139–149. Amer. Math. Soc., Providence, RI, 1987.

[14] T. Ekedahl: Canonical models of surfaces of general type in positive characteristic. Inst. Hautes Études Sci. Publ. Math. 67 (1988), 97–144.

[15] T. Ekedahl, J. Hyland, N. Shepherd-Barron: Moduli and periods of simply connected Enriques surfaces. Preprint, math.AG/1210.0342.

[16] T. Evans, D. Fuchs: On the structure of the restricted Lie algebra for the Witt algebra in a finite characteristic. Funct. Anal. Appl. 36 (2002), 140–144.

[17] A. Fanelli, S. Schröer: Del Pezzo surfaces and Mori fiber spaces in positive characteristic. Trans. Amer. Math. Soc. 373 (2020), 1775–1843.

[18] J. Giraud: Cohomologie non abélienne. Springer, Berlin, 1971.

[19] A. Grothendieck: Le groupe de Brauer I. In: J. Giraud (ed.) et al.: Dix exposés sur la cohomologie des schémas, pp. 46–66. North-Holland, Amsterdam, 1968.

[20] A. Grothendieck: Le groupe de Brauer III. In: J. Giraud (ed.) et al.: Dix exposés sur la cohomologie des schémas, pp. 88–189. North-Holland, Amsterdam, 1968.

[21] A. Grothendieck, J. Dieudonné: Éléments de géométrie algébrique I: Le langage des schémas. Springer, Berlin, 1970.

[22] A. Grothendieck: Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2). North-Holland Publishing Company, Amsterdam, 1968.

[23] R. Hartshorne: Generalised divisors on Gorenstein schemes. K-Theory 8 (1994), 287–339.

[24] G. Hochschild: Simple algebras with purely inseparable splitting fields of exponent 1. Trans. AmerṀath. Soc. 79 (1955), 477–489.

[25] N. Jacobson: Abstract derivation and Lie algebras. Trans. Amer. math. Soc. 42 (1937), 206–224.

[26] N. Jacobson: Classes of restricted Lie algebras of characteristic p. I. Amer. J. Math. 63 (1941), 481–515.

[27] N. Jacobson: Restricted Lie algebras of characteristic p. Trans. Amer. Math. Soc. 50 (1941), 15–25.

[28] N. Jacobson: The Theory of Rings. American Mathematical Society, New York, 1943.

[29] N. Jacobson: Galois theory of purely inseparable fields of exponent one. Amer. J. Math. 66 (1944), 645–648.

[30] N. Jacobson: Lie algebras. Interscience Publishers, New York–London, 1962.

[31] D. Knutson: Algebraic spaces. Springer, Berlin, 1971.

[32] S. Kondo, S. Schröer: Kummer surfaces associated with group schemes. Preprint, arXiv: 1912.12015, to appear in Manuscr. Math.

[33] W. Lang: Examples of surfaces of general type with vector fields. In: M. Artin, J. Tate (eds.), Arithmetic and geometry II, pp. 167–173. Birkhäuser, Boston, MA, 1983.

[34] G. Laumon, L. Moret-Bailly: Champs algebriques. Springer, Berlin, 2000.

[35] B. Laurent, S. Schröer: Para-abelian varieties and Albanese maps. Preprint, arXiv:2101.10829.

[36] C. Liedtke: Algebraic Surfaces in Positive Characteristic. In Birational Geometry, Rational Curves and Arithmetic. Springer 2013.

[37] J. Lipman: Rational singularities, with applications to algebraic surfaces and unique factorization. Inst. Hautes Études Sci. Publ. Math. 36 (1969), 195–279.

[38] G. Martin: Infinitesimal automorphisms of algebraic varieties and vector fields on elliptic surfaces. Preprint, arXiv:2004.07227.

[39] G. Martin: Automorphism schemes of (quasi-)bielliptic surfaces Preprint, arXiv:2008. 11679.

[40] M. Martin-Deschamps, R. Lewin-Ménégaux: Applications rationnelles séparables dominantes sur une variété de type général. Bull. Soc. Math. France 106 (1978), 279–287.

[41] D. Mumford, J. Fogarty, F. Kirwan: Geometric invariant theory. Third edition. Springer, Berlin, 1993.

[42] M. Olsson: Algebraic spaces and stacks. American Mathematical Society, Providence, RI, 2016.

[43] A. Premet: Regular Cartan subalgebras and nilpotent elements in restricted Lie algebras. Math. USSR-Sb. 66 (1990), 555–570.

[44] A. Premet, D. Stewart: Classification of the maximal subalgebras of exceptional Lie algebras over fields of good characteristic. J. Amer. Math. Soc. 32 (2019), 965–1008.

[45] A. Rudakov, I. Safarevic: Inseparable morphisms of algebraic surfaces. Math. USSR, Izv. 10 (1976), 1205–1237.

[46] P. Russell: Factoring the Frobenius morphism of an algebraic surface. In: L. Bădescu and D. Popescu (eds.), Algebraic geometry, pp. 366–380. Springer, Berlin, 1984.

[47] P. Sancho de Salas: Automorphism scheme of a finite field extension. Trans. Amer. Math. Soc. 352 (2000), 595–608.

[48] S. Schröer: Kummer surfaces for the selfproduct of the cuspidal rational curve. J. Algebraic Geom. 16 (2007), 305–346.

[49] S. Schröer: On fibrations whose geometric fibers are nonreduced. Nagoya Math. J. 200 (2010), 35–57.

[50] S. Schröer: Enriques surfaces with normal K3-like coverings. J. Math. Soc. Japan. 73 (2021), 433–496.

[51] N. Shepherd-Barron: Some foliations on surfaces in characteristic 2. J. Algebraic Geom. 5 (1996), 521–535.

[52] Stacks project authors: Stacks project. https://stacks.math.columbia.edu, 2018.

[53] H. Strade: Die Klassifikation der einfachen Lie-Algebren über Körpern mit positiver Charakteristik: Methoden und Resultate. Jahresber. Deutsch. Math.-Verein. 95 (1993), 28–46.

[54] H. Strade, R. Farnsteiner: Modular Lie algebras and their representations. Marcel Dekker, New York, 1988.

[55] H. Tango: On the behavior of extensions of vector bundles under the Frobenius map. Nagoya Math. J. 48 (1972), 73–89.

[56] N. Tziolas: Automorphisms of smooth canonically polarized surfaces in positive characteristic. Adv. Math. 310 (017), 235–289.

[57] W. Waterhouse: Automorphisms and twisted forms of generalized Witt Lie algebras. Trans. Amer. Math. Soc. 327 (1991), 185–200.

[58] H. Zassenhaus: Über Lie'sche Ringe mit Primzahlcharakteristik. Abh. Math. Sem. Univ. Hamburg 13 (1939), 1–100.

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