DEL PEZZO SURFACES AND MORI FIBER SPACES
IN POSITIVE CHARACTERISTIC

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ABSTRACT. We settle a question that originates from results and remarks by
Kollár on extremal rays in the minimal model program: In positive characteris-
tics, there are no Mori fibrations on threefolds with only terminal singularities
whose generic fibers are geometrically non-normal surfaces. To show this we
establish some general structure results for del Pezzo surfaces over imperfect
ground fields. This relies on Reid's classification of non-normal del Pezzo
surfaces over algebraically closed fields, combined with a detailed analysis of
geometrical non-reducedness over imperfect fields of p-degree one.

CONTENTS

INTRODUCTION

A smooth proper scheme V of dimension two whose dualizing sheaf ω_V is anti-
ample is called a *del Pezzo surface*. This notion immediately generalizes from the

smooth case to the Gorenstein case. Smooth del Pezzo surfaces were first studied by del Pezzo [20] in the nineteenth century. Over algebraically closed ground fields, they are either $\mathbb{P}^1 \times \mathbb{P}^1$ or the blowing-up of $\mathbb{P}^2$ in at most eight points in general position (see [19] or [22] for this classification in modern language). In some sense, del Pezzo surfaces are the two-dimensional analogs of the projective line. The higher-dimensional generalizations are called *Fano varieties*.

Del Pezzo surfaces and Fano varieties play an important role in the *minimal model program*, which is a tremendously successful approach to achieve a classification of higher-dimensional algebraic schemes over algebraically closed ground fields k. If $f : X \to B$ is a *Mori fibration*, that is, the contraction of an extremal ray of fiber-type, then the generic fiber $V = X_\eta$ is a Fano variety over the function field

$$F = \mathscr{O}_{B,\eta} = \kappa(\eta) = k(B).$$

Furthermore, $\dim(V) = \dim(X) - \dim(B)$ and the Picard number is $\rho(V) = 1$. In particular, Mori fibrations on *threefolds* X yield del Pezzo surfaces V over function fields F of algebraic curves B. Of course, one may also study more general fibrations $f : X \to B$ having del Pezzo surfaces or Fano varieties as generic fiber. One may call them *del Pezzo fibrations* or *Fano fibrations*. In any case, to understand the geometry of X, it is imperative to understand the geometry of $V = X_\eta$ over the non-closed field F.

If the total space X is smooth, that is, the sheaf of Kähler differentials $\Omega^1_{X/k}$ is locally free of rank $\dim(X)$, the generic fiber $V = X_\eta$ is regular, in the sense that all local rings $\mathscr{O}_{V,a}$ are regular. In characteristic zero, this ensures that V is smooth as well, and one may understand it in terms of the base-change $V \otimes_F F^{\mathrm{alg}}$, together with the action of the Galois group $\mathrm{Gal}(F^{\mathrm{alg}}/F)$. This was exploited, for example, in [46], [15], and [16].

In positive characteristics $p > 0$, however, this is no longer true, and it may easily happen that the *geometric generic fiber* $V \otimes_F F^{\mathrm{alg}}$ is non-normal, or even non-reduced. The former already plays an important role in the Enriques classification of surfaces: In characteristic $p = 2$ and $p = 3$, there are *quasielliptic fibrations*, where the generic fiber is a regular genus-one curve and the geometric generic fiber is the rational cuspidal curve [9]. The latter easily happens in higher-dimensions: The hypersurface $X \subset \mathbb{P}^n \times \mathbb{P}^n$ given by the bihomogeneous equation

$$X : \quad S_0 T_0^p + S_1 T_1^p + \ldots + S_n T_n^p = 0$$

is smooth, whereas the geometric generic fiber for the projection $\mathrm{pr}_1 : X \to \mathbb{P}^n$ is a p-fold hyperplane.

Del Pezzo surfaces in positive characteristics $p > 0$ and their log-generalizations have been studied, among others, by Reid [51], Schröer [58], Maddock [43], Cascini, Tanaka [13], Cascini, Tanaka, and Witaszek [14], Bernasconi [7], and Das [18]. The minimal model program (MMP) was introduced by Mori [46], and originally focused on the situation over the complex numbers, although Shepherd-Barron analyzed Fano threefolds in positive characteristics [65]. Recently, MMP made tremendous advances in positive characteristic, for example by the work of Hacon and Xu [33] and Tanaka [66].

However, many foundational issues remained open. About 25 years ago, Kollár's analysis of extremal rays on threefolds raised the question whether geometric non-normality may appear on Mori fibrations on threefolds ([39, Remarks in 1.2]), as it

does in the Enriques classification of surfaces. Our main result is that this–perhaps surprisingly–does not happen.

Theorem (See Theorem 15.2). *Suppose k is an algebraically closed ground field of characteristic $p > 0$. Let X be a threefold with only terminal singularities, and let $f : X \to B$ be a Mori fibration of relative dimension two. Then the generic fiber $V = X_\eta$ is geometrically normal.*

This generalizes results of Saito [55], who treated the case where X is a Fano threefold with $\rho(X) = 2$, and Patakfalvi and Waldron [49], who ruled out the cases $p \geq 5$. Besides the above non-existence result concerning Mori fibrations, we also show that del Pezzo fibration with unusual properties actually do exist.

Theorem (See Theorem 14.8). *Let k be an algebraically closed ground field of characteristic $p = 2$.*

(i) *There is a Mori fibration $Y \to \mathbb{P}^1$ whose generic fiber is a non-smooth del Pezzo surface that is geometrically normal.*
(ii) *There is a del Pezzo fibration $X \to \mathbb{P}^1$ whose generic fiber is geometrically non-normal with Picard number two.*

Here X arises from Y by some blowing-up whose center $Z \subset Y$ is a horizontal curve.

Situations like in (i) are probably well known, and we list it here because it leads to case (ii).

The preceding results are special cases of our analysis of del Pezzo surfaces V over arbitrary imperfect fields F, by relating the geometry of the surface to the arithmetic of the ground field. In fact, we develop several techniques that work for general algebraic schemes over imperfect ground fields, which should be useful in many other situations.

The key idea is to use the *p-degree* $\mathrm{pdeg}(F) \geq 0$ of the ground field systematically. This notion was introduced by Teichmüller [67] under the name *degree of imperfection*, and was further studied by Becker and MacLane [6]. It can be seen as the dimension of the vector space of absolute Kähler differentials $\Omega^1_{F/\mathbb{Z}}$. If F is the function field of some integral algebraic scheme B over a perfect field, we just have $\mathrm{pdeg}(F) = \dim(B)$. The non-existence result concerning Mori fibrations on threefolds comes from the following statement.

Theorem (See Theorem 14.1). *Let V be a regular del Pezzo surface with Picard number one over a ground field F of p-degree one. Then V is geometrically normal.*

As a rule of thumb, the higher the p-degree, the more unusual the geometry of algebraic F-schemes may become. Some scholars may regard the ensuing possibilities as "pathological" or "psychedelic", but we believe that these effects are rather natural and deserve systematic further study.

The crucial idea for our results is to study the *locus of non-smoothness* $\mathrm{Sing}(V/F)$ for regular but geometrically non-normal del Pezzo surfaces. It carries a natural scheme structure via Fitting ideals, and we look at its divisorial part $N \subset \mathrm{Sing}(V/F)$ and its reduction $D = N_{\mathrm{red}}$. This is an effective Cartier divisor $D \subset V$, and we analyze how it could fit in with *Reid's Classification* [51] of non-normal del Pezzo surface $Y = V_K$ obtained by base-changing along sufficiently large finite purely inseparable field extensions $F \subset K$. This non-normal del Pezzo surface will

be analyzed in terms of its *conductor square*

$$
\begin{CD}
R @>>> X \\
@VVV @VV{\nu}V \\
C @>>> Y,
\end{CD}
$$

where $\nu : X \to Y$ is the normalization, $R \subset X$ is the ramification divisor, and $C \subset Y$ is the conductor curve. The latter is not a Cartier divisor, but has the same support of the Cartier divisor $D_K \subset Y$.

Another important input comes from Serre's *characterization of one-dimension Gorenstein rings* in terms of length conditions [63], which was extended by Reid in the language of schemes [51]. These Gorenstein conditions can be used to obtain information about the local rings of $\mathscr{O}_{Y,y}$ and the structure of N and its reduction $D = N_{\mathrm{red}}$. Finally, we combine the geometry of our schemes with the arithmetic of ground fields by introducing the *geometric generic embedding dimension*

$$
\mathrm{edim}(\mathscr{O}_{N,\eta}/F) = \mathrm{edim}(\mathscr{O}_{N \otimes_F F^{\mathrm{perf}}})
$$

for arbitrary integral algebraic schemes N. Another key observation is the following, which strengthens some general bound of the second author [60].

Theorem (See Theorem 1.4). *Let F be a field with $\mathrm{pdeg}(F) \leq 1$. Then for each proper integral scheme N, we have $\mathrm{edim}(\mathscr{O}_{N,\eta}/F) \leq 1$.*

Another important ingredient is *Maddock's bound* [43]: If V is a normal del Pezzo surface in characteristic $p > 0$ with irregularity $h^1(\mathscr{O}_V) > 0$, then this irregularity is actually bounded from below by $h^1(\mathscr{O}_V) \geq \frac{p^2-1}{6} K_V^2$.

The paper is organized as follows: In the first two sections we establish various foundational facts about generic geometric embedding dimension $\mathrm{edim}(\mathscr{O}_{N,\eta}/F)$ and the locus of non-smoothness $\mathrm{Sing}(V/F)$. Section 3 contains some computations with complete local rings for conductor squares, mainly in codimension one and two, which gives some information on the behavior of the locus of non-smoothness. In Section 4 we collect some general facts on proper schemes that are geometrically non-normal, and give some results on their Picard schemes. This is applied in Section 5 to del Pezzo surfaces that are geometrically non-normal, where we also tabulate the possibilities according to Reid's classification. Section 6 treats the case that the ramification divisor $R \subset X$ is smooth: It turns out that only one case of Reid's classification is possible, which has Picard number $\rho(V) = 2$. The cases that the ramification divisor is non-smooth is much more challenging. Here we start with a preliminary investigation, excluding further possibilities in Section 7 for characteristic two, and Section 8 for characteristic three. In the remaining cases, the reduced locus of non-smoothness $D = \mathrm{Sing}(V/F)$ are curves of low genera with rather peculiar properties. We investigate the structure of such curves in Sections 9 and 12, where non-uniqueness of coefficient fields plays a decisive role. This is applied in Sections 10 and 11, where we rule out the remaining cases in characteristic two. Section 13 contains a similar analysis for characteristic three. The paper closes with an appendix, where we give a general treatment of conductor squares and Gorenstein conditions, with some slight generalizations of results of Serre, Reid, and many others.

1. GEOMETRIC GENERIC EMBEDDING DIMENSION AND p-DEGREE

We start by collecting some notions and results pertaining to local rings at generic points for algebraic schemes. Let A be a local Artin ring, with maximal ideal $\mathfrak{m}_A \subset A$ and residue field $\kappa = A/\mathfrak{m}_A$. The two basic numerical invariants are *embedding dimension* and *Hilbert–Samuel multiplicity*

$$\operatorname{edim}(A) = \dim_\kappa \mathfrak{m}_A/\mathfrak{m}_A^2 \geq 0 \quad \text{and} \quad e(A) = \operatorname{length}(A) \geq 1.$$

For the general theory of Hilbert–Samuel multiplicities, we refer to [12, Chapter VIII, §7]. The following immediate observations show that both integers give a measure for non-regularity.

Proposition 1.1. *The following are equivalent:*

    (i) *The local Artin ring A is regular.*
    (ii) *The projection $A \to \kappa$ is bijective.*
    (iii) *The embedding dimension is $\operatorname{edim}(A) = 0$.*
    (iv) *The Hilbert–Samuel multiplicity is $e(A) = 1$.*

Now suppose F is a ground field of characteristic $p > 0$, and let A be a local ring that is essentially of finite-type as an F-algebra, with Krull dimension $\dim(A) = 0$. These are precisely the local Artin rings where the field extension $F \subset \kappa(A)$ is finitely generated. They can also be regarded as stalks $\mathscr{O}_{Y,\eta}$ at the generic point η of irreducible schemes Y that are separated and of finite-type. If the latter holds, we also say that Y is an *algebraic scheme*.

For each algebraic field extension $F \subset K$, the ring $A_K = A \otimes_F K$ is essentially of finite-type over K, hence noetherian, and integral over A and therefore $\dim(A_K) = 0$. It follows that A_K is an Artin ring. If $F \subset K$ is purely inseparable, the rings stay local, and we can consider the integers $\operatorname{edim}(A_K)$ and $e(A_K)$ as above. Of particular interest is the situation where $K = F^{\mathrm{perf}}$ is the *perfect closure*. Let us call the integer

$$\operatorname{edim}(A/F) = \operatorname{edim}(A \otimes_F F^{\mathrm{perf}}) \geq 0$$

the *geometric embedding dimension*. Similarly, define

$$e(A/F) = e(A \otimes_F F^{\mathrm{perf}}) \geq 1$$

as the *geometric Hilbert–Samuel multiplicity*. Furthermore, if Y is an irreducible algebraic scheme with generic point $\eta \in Y$, we call the integers

$$\operatorname{edim}(\mathscr{O}_{Y,\eta}/F) = \operatorname{edim}(\mathscr{O}_{Y,\eta} \otimes_F F^{\mathrm{perf}}) \quad \text{and} \quad e(\mathscr{O}_{Y,\eta}/F) = e(\mathscr{O}_{Y,\eta} \otimes_F F^{\mathrm{perf}})$$

the *geometric generic embedding dimension* and the *geometric generic Hilbert–Samuel multiplicity* of the scheme Y. We thus get the following.

Proposition 1.2. *For an irreducible algebraic scheme Y without embedded components, the following are equivalent:*

    (i) *The scheme Y is geometrically reduced.*
    (ii) *The geometric generic embedding dimension is $\operatorname{edim}(\mathscr{O}_{Y,\eta}/F) = 0$.*
    (iii) *The geometric generic Hilbert–Samuel multiplicity is $e(\mathscr{O}_{Y,\eta}/F) = 1$.*

Proof. Conditions (ii) and (iii) are equivalent by Proposition 1.1. The assumption that Y has no embedded components means that the scheme satisfies Serre's Condition (S_1). This also holds for each base-change $Y_K = Y \otimes_F K$, according to [28, Proposition 6.7.1]. It follows that Y is geometrically reduced if and only if the local Artin ring $A = \mathscr{O}_{Y,\eta}$ is geometrically reduced.

Each of the Conditions (i)–(iii) imply that the scheme Y and the local Artin ring A are reduced, so it suffices to treat this situation. Then the field extension $F \subset A$ is geometrically reduced if and only if it is separable. The latter holds if and only if $A \otimes_F F^{\mathrm{perf}}$ is reduced, according to MacLane's Criterion [11, Chapter V, §2, No. 4, Theorem 2]. $\square$

The following notion goes back to Teichmüller [67], under the name of *degree of imperfection*, and was further studied by Becker and MacLane [6].

Definition 1.3. The dimension of the vector space of absolute Kähler differentials $\Omega^1_{F/\mathbb{Z}} = \Omega^1_{F/F^p}$ is called the *p-degree* $\mathrm{pdeg}(F)$ of the field F.

It could be seen as the cardinality of a p-basis for $F \subset F^{1/p}$, or $F^p \subset F$. If these algebraic extensions are finite, the degree is of the form $[F^{1/p} : F] = p^n$ with exponent $n = \mathrm{pdeg}(F)$.

The second author established in [60, Theorem 2.3] the relation

$$\mathrm{edim}(\mathscr{O}_{Y,\eta}/F) < \mathrm{pdeg}(F)$$

for every proper normal scheme Y with $h^0(\mathscr{O}_Y) = 1$ that is not geometrically reduced. As a consequence, if the ground field has $\mathrm{pdeg}(F) \leq 1$, then every proper normal scheme Y with $h^0(\mathscr{O}_Y) = 1$ is geometrically reduced. The following extension to *arbitrary* proper integral schemes, which could be non-normal or could have $h^0(\mathscr{O}_Y) \geq 2$, will be crucial for our applications.

Theorem 1.4. *Suppose the ground field F has p-degree $\mathrm{pdeg}(F) \leq 1$. Then for each proper integral scheme Y, we have $\mathrm{edim}(\mathscr{O}_{Y,\eta}/F) \leq 1$.*

Proof. The conclusion is trivial if Y is geometrically reduced. So we assume that Y is not geometrically reduced. Let $X \to Y$ be the normalization. Then X is a proper scheme that is integral but not geometrically reduced. The ring of global sections $K = H^0(X, \mathscr{O}_X)$ is integral with $[K : F] < \infty$, whence $F \subset K$ is a finite field extension. According to [60, Theorem 2.3] we have $\mathrm{edim}(\mathscr{O}_{X,\eta}/K) < \mathrm{pdeg}(K)$. But $\mathrm{pdeg}(K) = \mathrm{pdeg}(F) = 1$ according to [6, Theorem 3]. So by Proposition 1.2, the algebraic scheme X is geometrically reduced over the field K.

Now choose a perfect closure $E = K^{\mathrm{perf}}$. The composite extension $F \subset K \subset E$ is a perfect closure for K. Write $A = \mathscr{O}_{X,\eta} = \mathscr{O}_{Y,\eta}$ for the common function field of the integral schemes X and Y. Then

$$A \otimes_F E = A \otimes_K (K \otimes_F E).$$

Since $\mathrm{pdeg}(F) \leq 1$, the finite field extension $F \subset K$ is obtained by adjoining a single element $\alpha \in K$, such that $K = F(\alpha)$, according to [6, Theorem 1]. Let $f \in F[T]$ be the minimal polynomial of this generator $\alpha \in K$. Then $K \otimes_F E = E[T]/(f)$, thus

$$A \otimes_F E = A \otimes_K E[T]/(f) = (A \otimes_K E)[T]/(f).$$

Since $A \otimes_K E$ is a field, it follows that the residue class of the indeterminate T generates the maximal ideal of the local Artin ring $A \otimes_F E$, and consequently $\mathrm{edim}(\mathscr{O}_{Y,\eta}/F) \leq 1$. $\square$

2. Singular loci on algebraic schemes

In this section we establish some useful facts on closed subschemes contained in singular loci, and introduce some general notation along the way. Let F be a

ground field and let V be an algebraic F-scheme, which means a separated scheme of finite-type. As customary, we write $\mathrm{Sing}(V)$ for the set of points $a \in V$ where the local ring $\mathscr{O}_{V,a}$ is not regular, and call it the *singular locus*. Note that $\mathrm{Sing}(V) \subset V$ is a closed set, according to [28, Corollary 6.12.5]. One may regard it as a closed subscheme, endowed with reduced scheme structure. For every field extension $F \subset K$, we have $\mathrm{Sing}(V) \otimes_F K \subset \mathrm{Sing}(V \otimes_F K)$ by [28, Proposition 6.5.1]. Note that this inclusion is an equality provided that $F \subset K$ is separable, but in general one has a strict inclusion. The following relative version is a remedy for this defect.

Definition 2.1. The *locus of non-smoothness* $\mathrm{Sing}(V/F) \subset V$ is the set of points $a \in V$ where the local ring $\mathscr{O}_{V,a}$ is not geometrically regular as an F-algebra.

According to [28, Definition 6.7.6], this means that for some finite field extension $F \subset K$, the resulting semilocal noetherian ring $\mathscr{O}_{V,a} \otimes_F K$ becomes non-regular. By [28, Proposition 6.7.7] this already appears for some purely inseparable field extension $F \subset K$. The relative and absolute notions are related as follows.

Proposition 2.2. *Let $K = F^{\mathrm{perf}} = F^{1/p^\infty}$ be the perfect closure. Then $\mathrm{Sing}(V/F)$ is the image of $\mathrm{Sing}(V \otimes_F K)$ under the universal homeomorphism $V \otimes_F K \to V$.*

Proof. By [28, Corollary 6.7.8], the locus of non-smoothness $\mathrm{Sing}(V/F)$ coincides with the image of $\mathrm{Sing}(V \otimes_F K/K)$ under the projection $V \otimes_F K \to V$. But over a perfect field, the singular locus coincides with the locus of non-smoothness. $\square$

In particular, we see that $\mathrm{Sing}(V/F) \subset V$ is a closed set, and it commutes with ground field extensions. In contrast to the singular locus, it comes with a canonical scheme structure that is usually non-reduced. This relies on Fitting ideals for coherent sheaves $\mathscr{F}$. If $V = \mathrm{Spec}(R)$ is affine, we may choose a finite presentation

$$
(1) \qquad\qquad R^{\oplus s} \xrightarrow{\ A\ } R^{\oplus r} \longrightarrow M \longrightarrow 0
$$

for the R-module $M = \Gamma(V, \mathscr{F})$ with some matrix $A \in \mathrm{Mat}_{r \times s}(R)$. For each integer $0 \leq n \leq r$, the ideal $\mathfrak{a}_n \subset R$ generated by the $(r - n)$-minors of the matrix A is called the *nth Fitting ideal* of the module M. For $n > r$, we set $\mathfrak{a}_n = R$. Indeed, this depends only on the module and not on the chosen presentation; see the discussion in [23, Section 20.2]. By gluing, we thus get Fitting ideals as coherent ideal sheaves $\mathscr{I}_n \subset \mathscr{O}_V$ for $\mathscr{F}$ on arbitrary algebraic schemes V. The corresponding closed set is the locus of points $a \in V$ where some presentation (1) with $n \leq r$ exists on some affine open neighborhood such that the matrix $A \otimes \kappa(a)$ has rank $< r - n$, in other words $\dim_{\kappa(a)}(M \otimes_R \kappa(a)) > n$. According to Nakayama's Lemma, the latter means that $\mathscr{F}$ needs at least $n + 1$ generators on each affine open neighborhood of the point a.

Proposition 2.3. *Suppose that the algebraic scheme V is equidimensional of dimension $n = \dim(V)$. Then the closed set defined by the nth Fitting ideal for the sheaf of Kähler differentials $\Omega^1_{V/F}$ coincides with the locus of non-smoothness $\mathrm{Sing}(V/F)$.*

Proof. Let $U \subset V$ be the complementary open set. As discussed above, this is the set of all points $a \in V$ where the stalk $\Omega^1_{V/F,a}$ can be generated by n elements.

Likewise, let $U' \subset V$ be the complement of the locus of non-smoothness. Recall that the *local dimension* $\dim_a(V)$ is the limit of the dimensions of open neighborhoods of $a \in V$. It takes constant value n because V is equidimensional, and

coincides with the *relative dimension* $\dim_a(f) = \dim_a f^{-1}f(a)$ of the structure morphism $f : V \to \mathrm{Spec}(F)$. According to [30, Proposition 17.15.15] the open set U' is the set of points where $\Omega^1_{V/F}$ is locally free of rank $n = \dim_a(f)$.

We thus have $U' \subset U$. Seeking a contradiction, we assume that the inclusion is strict. Making a base-change, we may assume that F is algebraically closed. By Hilbert's Nullstellensatz there is a rational point $a \in U \smallsetminus U'$, and thus we get $\Omega^1_{V/F} \otimes \kappa(a) = \mathfrak{m}_a/\mathfrak{m}_a^2$. The former vector space has dimension $\leq n$ because $a \in U$. The latter vector space has dimension $\mathrm{edim}(\mathscr{O}_{V,a}) > \dim(\mathscr{O}_{V,a}) = n$ because $a \notin U'$, a contradiction. $\square$

For equidimensional algebraic schemes V, we usually regard the locus of non-smoothness $\mathrm{Sing}(V/F)$ as a closed subscheme, endowed with the scheme structure coming from the nth Fitting ideal for the coherent sheaf $\Omega^1_{V/F}$, where $n = \dim(V)$. This subscheme is stable under ground field extension, and has the following strange property.

Proposition 2.4. *Let $Z \subset \mathrm{Sing}(V/F)$ be a reduced closed subscheme and let $\eta \in Z$ be a generic point. Suppose that $\mathscr{O}_{Z,\eta} = \mathscr{O}_{V,\eta}/(f_1,\ldots,f_r)$ for some regular sequence $f_1,\ldots,f_r \in \mathscr{O}_{V,\eta}$. Then the scheme Z is geometrically non-reduced.*

Proof. Suppose that Z is geometrically reduced. Base-changing to the algebraic closure, we may assume that $F = F^{\mathrm{alg}}$. Then $\eta \in \mathrm{Sing}(V/F) = \mathrm{Sing}(V)$, such that the local ring $\mathscr{O}_{V,\eta}$ becomes singular. On the other hand, the local Artin ring $\mathscr{O}_{Z,\eta}$ is regular. Since the sequence $f_1,\ldots,f_r$ is regular, the local $\mathscr{O}_{V,\eta}$ must be regular, a contradiction. $\square$

Since regular subschemes in regular schemes are locally given by regular sequences, we obtain the following.

Corollary 2.5. *Suppose that V is regular and let $Z \subset V$ be some regular subscheme contained in $\mathrm{Sing}(V/F)$. Then Z is geometrically non-reduced.*

The following special case, which already appears in [30, Proposition 17.15.1], will play an important role throughout.

Corollary 2.6. *Let $a \in V$ be a closed point contained in $\mathrm{Sing}(V/F)$ whose local ring $\mathscr{O}_{V,a}$ is regular. Then the finite field extension $F \subset \kappa(a)$ is not separable. In particular, the closed point $a \in V$ is not rational.*

Since prime divisors in normal schemes are generically defined by a single equation, we also have the following consequence.

Corollary 2.7. *Suppose that V is normal and let $Z \subset V$ be some prime divisor contained in $\mathrm{Sing}(V/F)$. Then Z is geometrically non-reduced.*

Assume now that V is geometrically integral. Let $\mathscr{I} \subset \mathscr{O}_V$ be the Fitting ideal corresponding to the locus of non-smoothness. There are only finitely many points $a_1,\ldots,a_r \in V$ of codimension one with $\mathscr{I}_{a_i} \neq 0$. They admit a common affine open neighborhood $U = \mathrm{Spec}(R_0)$, even if there is no ample sheaf ([26, proof for Theorem 1.5]). Let $R = S^{-1}R_0$ be the ensuing semilocal ring with Fitting ideal $\mathfrak{a} \subset R$. The schematic image of the morphism $\mathrm{Spec}(R/\mathfrak{a}) \to V$ is called the *divisorial part* $N \subset \mathrm{Sing}(V/F)$ for the locus of non-smoothness. If non-empty, the subscheme $N \subset V$ is purely one-codimensional and without embedded component. Moreover, its formation commutes with ground field extensions.

Throughout the paper, one main idea is to analyze the divisorial part $N \subset \mathrm{Sing}(V/F)$ and the resulting reduction $D = N_{\mathrm{red}}$, which is also purely one-co-dimensional and without embedded component. Note, however, that $D = N_{\mathrm{red}}$ does not necessarily commute with inseparable ground field extensions. This phenomenon will play a crucial role.

If V is, moreover, normal, such that the one-dimensional local rings $\mathscr{O}_{V,a_i}$ are discrete valuation rings, the Fitting ideals take the form $\mathscr{I}_{a_i} = \mathfrak{m}_{a_i}^{m_i}$, and one may regard the subscheme $N \subset V$ as the effective *Weil divisor* $N = \sum m_i D_i$, where $D_i = \overline{\{a_i\}}$. Its reduction is $D = \sum D_i$. If, furthermore, V is locally factorial, the Weil divisors N and $D = N_{\mathrm{red}}$ are Cartier divisors. For each base-change $Y = V \otimes_F K$, we get an induced *Cartier divisor* $D \otimes_F K$, supported by the locus of non-smoothness.

3. Local computations

Fix a ground field K of characteristic $p > 0$. Let X and Y be algebraic schemes and let $\nu : X \to Y$ be a finite modification. We then have the *conductor square*

$$
(2) \qquad
\begin{array}{ccc}
R & \longrightarrow & X \\
\downarrow & & \downarrow{\nu} \\
C & \longrightarrow & Y,
\end{array}
$$

where $R \subset X$ denotes the *ramification locus*, $C \subset Y$ is the *conductor scheme*, and $\nu : R \to C$ is the *gluing map*, as discussed in detail in Appendix A. Throughout, we also assume that X is normal, that Y satisfies conditions (S_2) and (G_1), and that $\nu : X \to Y$ is a universal homeomorphism. Here (G_1) means that the local rings $\mathscr{O}_{Y,y}$ are Gorenstein for all points $y \in Y$ of codimension one.

The goal of this section is to study the complete local rings $\mathscr{O}_{Y,y}^{\wedge}$ at points $y \in Y$ of codimension ≤ 2 contained in the conductor scheme C, and the schematic structure of the locus of non-smoothness $N = \mathrm{Sing}(Y/K)$. To avoid technical problems with respect to Kähler differentials of complete local rings, we also assume that the ground field K has finite p-degree; compare to the discussion in [61, Section 1]. Let us start with the case that the ramification locus is regular.

Proposition 3.1. *If the ramification locus $R \subset X$ is regular, then the conductor scheme $C \subset Y$ is normal, and for each point $x \in X$ of codimension one whose image $y \in Y$ is contained in C, the field extension $\mathscr{O}_{C,y} \subset \mathscr{O}_{R,x}$ is purely inseparable of degree two. In particular, the characteristic must be $p = 2$.*

Proof. By assumption, we have $\mathscr{O}_{R,x} = \kappa(x)$. It follows that the local Artin ring $\mathscr{O}_{C,y} \subset \mathscr{O}_{R,x}$ is integral, whence also $\mathscr{O}_{C,y} = \kappa(y)$. According to Proposition A.2, the Gorenstein assumption ensures that the field extension $\kappa(y) \subset \kappa(x)$ has degree two. It is purely inseparable because the gluing map $R \to C$ is a universal homeomorphism. $\qquad \square$

Theorem 3.2. *Assumptions are as in Proposition 3.1. Let $x \in X$ be a point of codimension two whose image $y \in Y$ is contained in the conductor scheme C. Suppose that the local ring $\mathscr{O}_{X,x}$ is regular, and that the residue field extension $\kappa = \kappa(y) \subset \kappa(x)$ is trivial. Then we have an isomorphism of complete local rings*

$$
\mathscr{O}_{Y,y}^{\wedge} \simeq \kappa[[a, b, c]]/(a^2 - b^2 c).
$$

If, furthermore, the field extension $K \subset \kappa$ is separable, the locus of non-smoothness $N = \mathrm{Sing}(Y/K)$ corresponds to the subscheme defined by the equation $b^2 = 0$, and N is formally isomorphic to the spectrum of $\kappa[[a, b, c]]/(a^2, b^2)$. The preimage $\nu^{-1}(N) \subset X$ coincides with $2R \subset X$ in an open neighborhood of $x \in X$.

Proof. Since the local ring $\mathscr{O}_{Y,y}^{\wedge}$ is complete, there exists a *coefficient field* $\kappa \subset \mathscr{O}_{Y,y}^{\wedge}$ ([12, Chapter IX, §3, No. 3, Theorem 1]). Recall that this is a subfield that bijects onto the residue field. By assumption, the schemes R and X are regular at $x \in X$, so the corresponding ideal is generated by a single element $u \in \mathscr{O}_{X,x}$, and we may extend it to a regular system of parameters $u, v \in \mathscr{O}_{X,x}$. Then we get an identification $\mathscr{O}_{R,x}^{\wedge} = \kappa[[u, v]]/(u) = \kappa[[v]]$ with field of fractions $\kappa((v))$. The subring $\mathscr{O}_{C,y}^{\wedge}$ is normal by Proposition A.1. It contains the coefficient field κ by construction, and also the square v^2 because the field extension $\mathrm{Frac}(\mathscr{O}_{C,y}^{\wedge}) \subset \mathrm{Frac}(\mathscr{O}_{R,x}^{\wedge})$ is purely inseparable of degree $p = 2$. This gives $\kappa[[v^2]] \subset \mathscr{O}_{C,y}^{\wedge} \subset \mathscr{O}_{R,x}^{\wedge} = \kappa[[v]]$. The composite extension and the extension on the right both have degree two, whence the left extension has degree one. Thus $\mathscr{O}_{C,y}^{\wedge} = \kappa[[v^2]]$ because the former is normal. A computation with formal power series immediately shows that the diagram

$$
\begin{array}{ccc}
\kappa[[v]] & \longleftarrow & \kappa[[u, v]] \\
\uparrow & & \uparrow \\
\kappa[[v^2]] & \longleftarrow & \kappa[[u, uv, v^2]]
\end{array}
$$

is cartesian, where the horizontal maps send u and uv to zero. The three generators $a = uv$, $b = u$, and $c = v^2$ satisfy the relation $a^2 = b^2 c$. With the ring $B = \kappa[[a, b, c]]/(a^2 - b^2 c)$, we obtain a surjective homomorphism $B \to \kappa[[u, uv, v^2]]$. This map is actually bijective by Krull's Principal Ideal Theorem, because the $a^2 - b^2 c$ is irreducible, viewed as a polynomial in a. This establishes $\mathscr{O}_{Y,y}^{\wedge} = B$.

Now suppose that the field extension $K \subset \kappa$ is separable. After shrinking Y we find a closed embedding into an affine space $\mathbb{A}_K^r$ for some $r \geq 0$. This gives a surjection of K-algebras $\mathscr{O}_{\mathbb{A}_K^r, y}^{\wedge} \to \mathscr{O}_{Y,y}^{\wedge}$. According to [27, Theorem 19.6.4] there is a coefficient field in $\mathscr{O}_{\mathbb{A}_K^r, y}^{\wedge}$ containing K. This induces a new coefficient field $\kappa \subset \mathscr{O}_{Y,y}^{\wedge}$ containing the ground field K. The ring extensions $K \subset \kappa \subset B$ yield an exact sequence

$$
(3) \qquad\qquad \Omega_{\kappa/K}^1 \otimes_\kappa B \longrightarrow \Omega_{B/K}^1 \longrightarrow \Omega_{B/\kappa}^1 \longrightarrow 0
$$

of Kähler differentials. The map on the left is injective. To see this, consider the exact sequence $\Omega_{\kappa/K}^1 \otimes_\kappa \mathrm{Frac}(B) \to \Omega_{\mathrm{Frac}(B)/K}^1 \to \Omega_{\mathrm{Frac}(B)/\kappa}^1 \to 0$. Here the map on the left is injective, in fact a direct summand, according to [27, Theorem 20.5.7], because the extension $\kappa \subset \mathrm{Frac}(B)$ is separable. Using the injectivity of $\Omega_{\kappa/K}^1 \otimes_\kappa B \to \Omega_{\kappa/K}^1 \otimes_\kappa \mathrm{Frac}(B)$ and the functoriality of the sequences of Kähler differentials, we infer that the map on the left in (3) is injective.

Set $n = \dim(Y)$. Obviously, $\kappa = \kappa(y)$ is the field of fractions of the integral closed subscheme $\overline{\{y\}} \subset Y$ of dimension $n - 2$. In turn, the extension $K \subset \kappa$ has transcendence degree $\mathrm{trdeg}(\kappa/K) = n - 2$. According to [11, Chapter V, §16, No. 7, Theorem 5] the vector space $\Omega_{\kappa/K}^1$ is free of rank $n - 2$, thus the B-module $\Omega_{\kappa/K}^1 \otimes_\kappa B$ is free of the same rank. Since K has finite p-degree, so does κ, which ensures that $\Omega_{B/\kappa}^1$ is generated by the differentials da, db, dc modulo the relation $b^2 dc = 0$ (compare to the discussion in [61, Section 1]). Thus $\Omega_{B/K}^1$ has $n + 1$

generators and one relation, and we may use them to compute Fitting ideals. By Proposition 2.3, the locus of non-smoothness $N = \mathrm{Sing}(Y/K)$ is formally defined by the coefficient $b^2 \in B$, and the description of $\mathscr{O}^\wedge_{N,y}$ follows. Since $b^2 = u^2$ in $\mathscr{O}^\wedge_{X,x} = k[[u, v]]$, the statement on $\nu^{-1}(N)$ is also clear. $\qquad \square$

Next, we examine the case that the ramification locus is non-reduced.

Proposition 3.3. *If the ramification locus takes the form $R = 2R_{\mathrm{red}}$, then for each point $x \in X$ of codimension one whose image $y \in Y$ is contained in the conductor scheme C, the extension of local rings $\mathscr{O}_{C,y} \subset \mathscr{O}_{R,x}$ is isomorphic to*

$$\kappa(y) \subset \kappa(x)[\epsilon] \quad or \quad \kappa(y)[\epsilon] \subset \kappa(x)[\epsilon],$$

where ϵ is an indeterminate subject to $\epsilon^2 = 0$. In the first case, we have $\kappa(y) = \kappa(x)$. In the second case, the residue field extension $\kappa(y) \subset \kappa(x)$ is purely inseparable of degree two and we are in characteristic $p = 2$.

Proof. Write $\kappa = \kappa(x)$, and choose a coefficient field so that the local ring $\mathscr{O}_{R,x}$ becomes isomorphic to $\bar{A} = \kappa[\epsilon]$. According to Proposition A.2, we have to understand the K-subalgebras $\bar{B} \subset \bar{A}$ so that $\bar{A}$ is finite over $\bar{B}$ with $\mathrm{length}_{\bar{B}}(\bar{A}) = 2\,\mathrm{length}_{\bar{B}}(\bar{B})$. Setting $\kappa' = \kappa(y)$, we see that $[\kappa : \kappa']$ is either one or two.

In the former case, $\kappa' = \kappa$ and the length of $\bar{A}$ as a module over itself coincides with its length as a module over $\bar{B}$. It follows that $\bar{B} = \kappa'$. Replacing the coefficient field in $\bar{A}$ by the image of $\bar{B}$, we reach the first alternative.

In the second case, the length of $\bar{A}$ as a module over itself is twice its length as a module over $\bar{B}$, and we infer that both Artin rings have length two. Hence $\bar{B} = \kappa'[\epsilon']$. Inside the overring $\bar{A} = \kappa[\epsilon]$, the element ϵ' is non-zero and lies in the maximal ideal, hence generates the maximal ideal. Replacing ϵ by ϵ', we arrive at the second alternative. $\qquad \square$

Theorem 3.4. *Assumptions are as in Proposition 3.3. Let $x \in X$ be a point of codimension two whose image $y \in Y$ is contained in the conductor scheme C. Suppose that the local rings $\mathscr{O}_{X,x}$ and $\mathscr{O}_{C,y}$ are regular. Then*

$$\mathscr{O}^\wedge_{Y,y} \simeq \kappa[[a, b, c]]/(a^2 - b^3),$$

where $\kappa = \kappa(y)$ is the residue field. If, furthermore, the field extension $K \subset \kappa$ is separable, the following holds:

  (i) *In characteristic $p = 3$, the locus of non-smoothness $N = \mathrm{Sing}(Y/K)$ is defined by the equation $a = 0$, such that $\mathscr{O}^\wedge_{N,y} \simeq \kappa[[a, b, c]]/(a, b^3)$, and the preimage $\nu^{-1}(N) \subset X$ coincides with $\frac{3}{2}R = 3R_{\mathrm{red}}$ in a neighborhood of $x \in X$.*

  (ii) *In characteristic $p = 2$, the subscheme $N = \mathrm{Sing}(Y/K)$ is defined by $b^2 = 0$, such that $\mathscr{O}^\wedge_{N,y} \simeq \kappa[[a, b, c]]/(a^2, b^2)$. The preimage $\nu^{-1}(N) \subset X$ coincides with $2R = 4R_{\mathrm{red}}$ in a neighborhood of $x \in X$.*

Proof. Choose a coefficient field $\kappa \subset \mathscr{O}^\wedge_{X,x}$. Since $\mathscr{O}_{C,y}$ is regular, we are in the first alternative of Proposition 3.3, hence $R_{\mathrm{red}} \to C$ is birational. The inclusion $\mathscr{O}_{C,y} \subset \mathscr{O}_{R_{\mathrm{red}},x}$ is an equality, because $\mathscr{O}_{C,y}$ is normal, hence the local ring $\mathscr{O}_{R_{\mathrm{red}},x}$ is regular. Therefore the ideal for the reduced ramification locus is generated by a member u of some regular system of parameters $u, v \in \mathscr{O}^\wedge_{X,x}$. Whence $\mathscr{O}^\wedge_{X,x} = \kappa[[u, v]]$ and $\mathscr{O}^\wedge_{R,x} = k[[u, v]]/(u^2)$. The rings $\mathscr{O}_{C,y}$ and $\mathscr{O}_{R,x}$ are regular and Cohen–Macaulay, respectively, whence the finite extension $\mathscr{O}_{C,y} \subset \mathscr{O}_{R,x}$ of degree two is

flat ([28, Proposition 6.1.5]). Thus we may write $\mathscr{O}_{R,x} = \mathscr{O}_{C,y}[U]/(U^2 + \mu U + \xi)$ for some generator U. By adding an element from $\mathscr{O}_{C,y} = \mathscr{O}_{R_{\mathrm{red}},x}$, we may assume that U is nilpotent. Then $\mu = \xi = 0$, because polynomial rings over discrete valuation rings are factorial. Choose a uniformizer $V \in \mathscr{O}_{C,y}$. Replacing $u, v \in \mathscr{O}_{X,x}^{\wedge}$ by representatives of the classes of U and V, we may assume that $\mathscr{O}_{C,y}^{\wedge} \subset \mathscr{O}_{R,x}^{\wedge}$ is given by $k[[v]] \subset \kappa[[u,v]]/(u^2)$. A computation with formal power series shows that the diagram

$$
\begin{array}{ccc}
\kappa[[u,v]]/(u^2) & \longleftarrow & \kappa[[u,v]] \\
\uparrow & & \uparrow \\
\kappa[[v]] & \longleftarrow & \kappa[[u^2, u^3, v]]
\end{array}
$$

is cartesian. The generators $a = u^3$, $b = u^2$, and $c = v$ satisfy the relations $a^2 = b^3$, and as in the proof for Theorem 3.2 we infer $\mathscr{O}_{Y,y}^{\wedge} = \kappa[[a,b,c]]/(a^2 - b^3)$. Likewise, we get the statements on the locus of non-smoothness $N = \mathrm{Sing}(Y/K)$ and its preimage $\nu^{-1}(N) \subset X$. □

For the applications we have in mind, we do not have to bother for primes $p \geq 5$, due to the following observation on the tangent sheaf.

Proposition 3.5. *Assumptions are as in Theorem 3.4, with $K \subset \kappa$ separable. The stalk of the tangent sheaf $\Theta_{Y/K,y}$ is free if and only if we are in characteristic $p \leq 3$.*

Proof. Set $n = \dim(Y)$, choose a transcendence basis $\xi_1, \ldots, \xi_{n-2} \in \kappa$ for the field extension $K \subset \kappa$, and write $B = \mathscr{O}_{Y,y}^{\wedge} = \kappa[[a,b,c]]/(a^2 - b^3)$. As in the last paragraph of the proof for Theorem 3.2, the B-module $\Omega^1_{B/K}$ is generated by da, db, dc together with $d\xi_1, \ldots, d\xi_{n-2}$, modulo the relation $2a\,da - 3b^2\,db = 0$. Thus we have an exact sequence

$$
B \longrightarrow B^{\oplus n+1} \longrightarrow \Omega^1_{B/K} \longrightarrow 0,
$$

where the map on the right sends the standard basis vectors to the differentials da, db, dc, and the map on the left is given by the $(n+1) \times 1$-matrix $(2a, -3b^2, 0, \ldots, 0)^t$. Dualizing gives an exact sequence

$$
0 \longrightarrow \Theta_{B/K} \longrightarrow B^{\oplus n+1} \longrightarrow B \longrightarrow B/(2a, -3b^2) \longrightarrow 0.
$$

In characteristic $p = 2$ and $p = 3$, the ideal on the right becomes principal, whence the residue class module $M = B/(2a, -3b^2)$ has finite projective dimension. The Auslander–Buchsbaum Formula gives $\mathrm{pd}(M) \leq \dim(B) = 2$, so the syzygy $\Theta_{B/K}$ must be free.

Conversely, suppose that $\Theta_{B/K}$ is free. Then the B-module $B/(2a, -3b^2)$ has finite projective dimension. Seeking a contradiction, we assume that $p \geq 5$. It follows that B/I has finite projective dimension as well, where $I = (a, b^2, c)$. The latter ideal has finite colength $l = 2$. Now consider its Frobenius power $I' = (a^p, b^{2p}, c^p) \subset B$. According to a result of Miller [45, Corollary 5.2.3] such a Frobenius power $I' \subset B$ has colength $l' = lp^2 = 2p^2$. On the other hand, we may identify B with the subring of $\kappa[[t^2, t^3, c]] \subset \kappa[[t, c]]$ via the identification $a = t^3$ and $b = t^2$. Then $a^p = t^{3p}$ and $b^{2p} = t^{4p}$. From this we infer that the $3p^2$ elements

$$
t^i c^j \in B/I', \quad i = 0, 2, 3, \ldots, 3p-1, 3p+1, \quad \text{and} \quad 0 \leq j \leq p-1
$$

form a κ-basis. Thus $l' = 3p^2$, a contradiction. □

It remains to treat the case where the conductor scheme is non-reduced.

Theorem 3.6. *Assumptions are as in Proposition 3.3. Let $x \in X$ be a point of codimension two whose image $y \in Y$ is contained in the conductor scheme C. Suppose that the local rings $\mathscr{O}_{X,x}$ and $\mathscr{O}_{R_{\mathrm{red}},x}$ and $\mathscr{O}_{C_{\mathrm{red}},y}$ are regular, that $\mathscr{O}_{C,y}$ is non-reduced, and that the ring extension $\mathscr{O}_{C,y} \subset \mathscr{O}_{R,x}$ is flat, with trivial residue field extension $\kappa = \kappa(y) \subset \kappa(x)$. Then*

$$\mathscr{O}_{Y,y}^{\wedge} \simeq \kappa[[a,b,c]]/(a^2 - cb^4).$$

If, furthermore, the field extension $K \subset \kappa$ is separable, then the locus of non-smoothness $N = \mathrm{Sing}(Y/K)$ is defined by the equation $b^4 = 0$, becomes formally isomorphic to $\kappa[[a,b,c]]/(a^2, b^4)$, and the preimage $\nu^{-1}(N) \subset X$ coincides with $2R = 4R_{\mathrm{red}}$ in a neighborhood of $x \in X$.

Proof. Choose a coefficient field $\kappa \subset \mathscr{O}_{Y,y}^{\wedge}$. Since $\mathscr{O}_{C,y}$ is non-reduced, so is the overring $\mathscr{O}_{R,x}$, and we are in the second alternative of Proposition 3.3, and in particular $p = 2$. Thus we may choose a regular system of parameters $u, v \in \mathscr{O}_{X,x}^{\wedge}$ so that $\mathscr{O}_{R,x}^{\wedge} = \kappa[[u,v]]/(u^2)$. Since $\mathscr{O}_{C_{\mathrm{red}},y}$ and $\mathscr{O}_{R_{\mathrm{red}},x}$ are regular and Cohen–Macaulay, respectively, the degree two extension $\mathscr{O}_{C_{\mathrm{red}},y} \subset \mathscr{O}_{R_{\mathrm{red}},x}$ is flat ([28, Proposition 6.1.5]). Choose a uniformizer $\pi \in \mathscr{O}_{C_{\mathrm{red}},y}^{\wedge}$, such that $\mathscr{O}_{C_{\mathrm{red}},y}^{\wedge} = \kappa[[\pi]]$, and write $\mathscr{O}_{R_{\mathrm{red}},x}^{\wedge} = \kappa[[\pi, V]]/(V^2 - \varphi)$ for some formal power series $\varphi \in \kappa[[\pi]]$. The latter is not a square, because the ring is reduced, and becomes a square in κ, because the residue field extension $\kappa = \kappa(y) \subset \kappa(x)$ is trivial. Replacing V, we may assume that φ lies in the maximal ideal. Since $\mathscr{O}_{R_{\mathrm{red}},x}^{\wedge}$ is regular, the element $\varphi \in \kappa[[\pi]]$ is a uniformizer, and we may assume $\pi = \varphi$. Clearly, the image $V \in \mathscr{O}_{R_{\mathrm{red}},x}^{\wedge}$ is a uniformizer.

The nilradical $\mathfrak{n} \subset \mathscr{O}_{C,y}$ is a torsion-free module of rank one over $\mathscr{O}_{C_{\mathrm{red}},y}$, whence free. Let $U \in \mathfrak{n}$ be a basis. By assumption the ring extension $\mathscr{O}_{C,y} \subset \mathscr{O}_{R,x}$ is flat, hence the ring $\mathscr{O}_{R,x}/(U)$ is torsion-free. It is generically reduced by the local description in Proposition 3.3, hence reduced. Replacing u, v by representatives of U, V we may assume that $\mathscr{O}_{R,x}^{\wedge} = \kappa[[u,v]]/(u^2)$ and $\mathscr{O}_{C,y}^{\wedge} = \kappa[[u,v^2]]/(u^2)$. A computation with formal power series shows that the diagram

$$
\begin{array}{ccc}
\kappa[[u,v]]/(u^2) & \longleftarrow & \kappa[[u,v]] \\
\uparrow & & \uparrow \\
\kappa[[u,v^2]]/(u^2) & \longleftarrow & \kappa[[u,u^2v,v^2]]
\end{array}
$$

is cartesian. The generators $a = u^2 v$ and $b = u$ and $c = v^2$ satisfy the relation $a^2 = b^4 c$. The statement on the locus of non-smoothness $N = \mathrm{Sing}(Y/K)$ and its preimage on X follows as in Theorem 3.4. $\qquad\square$

4. Geometrically non-normal schemes

Let F be a ground field of characteristic $p > 0$, and let V be an algebraic scheme, that is, the structure morphism $V \to \mathrm{Spec}(F)$ is separated and of finite-type. We use the letters F and V because in applications, the former frequently arises as a *function field* and the latter becomes the *generic fiber* of some fibration. Recall that V is called *geometrically normal* if $V \otimes_F K$ is normal for all field extensions $F \subset K$, with similar locution for other scheme-theoretic properties of V.

Given a field extension $F \subset K$, we set $Y = (V \otimes_F K)_{\mathrm{red}}$ and write $\nu : X \to Y$ for the normalization. Note that for the schemes X and Y, we regard K rather

than F as the ground field. In our applications, we are mainly interested in the situation where V is geometrically integral but not geometrically normal. In this section, however, we make some fairly general observations. Let us start with the following well-known fact.

Lemma 4.1. *There is a finite purely inseparable field extension $F \subset K$ so that X is geometrically normal and Y is geometrically reduced.*

Proof. Choose a perfect closure $F \subset F^{\mathrm{perf}}$. Then $Y_\infty = (V \otimes_F F^{\mathrm{perf}})_{\mathrm{red}}$ is geometrically reduced, and its normalization X_∞ is geometrically normal. Let $F \subset F_\lambda \subset F^{\mathrm{perf}}$, $\lambda \in L$ be the filtered ordered set of finite subextensions. According to [29, Theorem 8.8.2], for some index λ there is a closed subscheme $Y_\lambda \subset V \otimes_F F_\lambda$ and a finite morphism $X_\lambda \to Y_\lambda$ inducing $X_\infty \to Y_\infty \subset V \otimes_F F^{\mathrm{perf}}$. It follows that X_λ is geometrically normal and Y_λ is geometrically reduced. This yields the desired finite purely inseparable field extension $K = F_\lambda$. $\square$

Now suppose that $F \subset K$ is a field extension so that the X is geometrically normal and Y is geometrically reduced. As explained in Appendix A, the finite birational morphism $\nu : X \to Y$ comes with the conductor square

$$\begin{array}{ccc} R & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \nu} \\ C & \longrightarrow & Y, \end{array}$$

where $C \subset Y$ is the conductor scheme and $R \subset X$ is the ramification locus. Both correspond to the coherent sheaf $\mathfrak{C}$ defined as the annihilator of $(\nu_* \mathscr{O}_X)/\mathscr{O}_Y$, which is an ideal sheaf in both $\mathscr{O}_Y$ and $\nu_*(\mathscr{O}_X)$. For each further field extension $K \subset K'$, the scheme $Y' = Y \otimes_K K'$ is reduced, and $X' = X \otimes_K K'$ is normal, whence the induced map $X' \to Y'$ is the normalization. Since kernels for homomorphisms between quasicoherent sheaves, and in particular annihilator ideals for coherent sheaves, commute with ground field extensions, we see that the base-change of the above diagram along $K \subset K'$ is the conductor square for the normalization $X' \to Y'$. Applying Lemma 4.1 to the conductor scheme and the ramification locus, we obtain the following.

Proposition 4.2. *There is a finite purely inseparable field extension $F \subset K$ such that X is geometrically normal, and the schemes Y, C_{red}, R_{red} are geometrically reduced.*

Now suppose additionally that the algebraic scheme V is proper, and, furthermore, satisfies the condition $h^0(\mathscr{O}_V) = 1$. Then we can form the *Picard scheme* $\mathrm{Pic}_{V/F}$, its connected component of the origin $\mathrm{Pic}^0_{V/F}$ and the resulting *Néron–Severi scheme* $\mathrm{NS}_{V/F}$. These are commutative *group schemes* sitting in a short exact sequence

$$0 \longrightarrow \mathrm{Pic}^0_{V/F} \longrightarrow \mathrm{Pic}_{V/F} \longrightarrow \mathrm{NS}_{V/F} \longrightarrow 0.$$

The cokernel is a smooth zero-dimensional group scheme, whence is determined by the group $\mathrm{NS}_{V/F}(F^{\mathrm{sep}})$, where F^{sep} is some separable closure, together with the action of the Galois group $\mathrm{Gal}(F^{\mathrm{sep}}/F)$. The group of rational points is denoted by $\mathrm{NS}(V/F) = \mathrm{NS}_{V/F}(F)$. Inside, we have the usually smaller subgroup $\mathrm{NS}(V)$ of rational points coming from invertible sheaves on V.

We say that V *has completely constant Néron–Severi group scheme* if the inclusion $\mathrm{NS}(V) \subset \mathrm{NS}_{V/F}(F^{\mathrm{sep}})$ is an equality. In other words, the group scheme is constant, and each point actually corresponds to an invertible sheaf. In turn, the canonical map $\mathrm{Pic}(V) \to \mathrm{NS}_{V/F}(F^{\mathrm{sep}})$ is surjective. The kernel is $\mathrm{Pic}^0(V) = \mathrm{Pic}(V) \cap \mathrm{Pic}^0_{V/F}(F)$, and we get identifications

$$\mathrm{NS}(V) = \mathrm{NS}(V/F) = \mathrm{NS}_{V/F}(F^{\mathrm{sep}}) = \mathrm{Pic}(V)/\mathrm{Pic}^0(V),$$

as customary over algebraically closed ground fields. The following observation will be useful.

Lemma 4.3. *There is a finite separable field extension $F \subset F'$ so that the base-change $V' = V \otimes_F F'$ has completely constant Néron–Severi group scheme $\mathrm{NS}_{V'/F'}$.*

Proof. By finiteness of the base number, the underlying set of the group scheme $\mathrm{NS}_{V/F}$ is countable. For each point $l \in \mathrm{NS}_{V/F}$ the residue field extension $F \subset \kappa(l)$ is finite and separable. Choose an enumeration $l_n \in \mathrm{NS}_{V/F}$, $n \geq 0$ of them. The resulting étale algebras $A_n = \kappa(l_0) \otimes_F \ldots \otimes_F \kappa(l_n)$ form an increasing sequence $A_0 \subset A_1 \subset \ldots$, and we write $A = \varinjlim_{n \geq 0} A_n$ for their direct limit. Choose a residue field $F' = A/\mathfrak{m}$. Clearly, the field extension $F \subset F'$ is separable algebraic. The images $F_n \subset F'$ of the étale algebras A_n form an increasing sequence $F_0 \subset F_1 \subset \ldots$ of finite separable field extensions. If this sequence is stationary, the field $F' = F_n$, $n \gg 0$ solves our problem. Seeking a contradiction, we assume that the sequence is not stationary. Choose an embedding of F' into some separable closure F^{sep}. Then the images $H_n \subset \mathrm{NS}_{V/F}(F^{\mathrm{sep}})$ of the groups $\mathrm{NS}_{V/F}(F_n)$ form a sequence $H_0 \subset H_1 \subset \ldots$ that is not stationary, but lies inside the finitely generated abelian group $\mathrm{NS}_{V/F}(F^{\mathrm{sep}})$, a contradiction. $\square$

Note that the corresponding statement for the Picard scheme does not hold, for example if F is finite and $\dim \mathrm{Pic}^0_{V/F} \geq 1$, or if F is imperfect and the Picard scheme contains a copy of $\mathbb{G}_a$ or $\mathbb{G}_m$.

The ramification locus $R \subset X$ has in general $h^0(\mathscr{O}_R) \neq 1$, so forming the Picard scheme is problematic. The same difficulty occurs for the conductor scheme $C \subset Y$. To avoid cumbersome statements, we say that a proper K-scheme Z has *completely constant Néron–Severi group scheme* if the connected components $Z_1, \ldots, Z_r \subset Z$ satisfy $h^0(\mathscr{O}_{Z_i}) = 1$ and have completely constant $\mathrm{NS}_{Z_i/K}$. Note that the condition $h^0(\mathscr{O}_{Z_i}) = 1$ automatically holds if Z_i is geometrically connected and geometrically reduced.

Proposition 4.4. *There is a finite separable field extension $F \subset F'$ and a finite purely inseparable field extension $F \subset K'$ so that, for the resulting finite field extension $K = F' \otimes_F K'$, the following holds:*

(i) *The normalization X of $Y = (V \otimes_F K)_{\mathrm{red}}$ is geometrically normal.*
(ii) *The inclusion $\mathrm{Sing}(Y) \subset \mathrm{Sing}(Y/K)$ is an equality of closed sets.*
(iii) *The schemes Y, C_{red}, and R_{red} are geometrically reduced.*
(iv) *The schemes X, Y, C_{red}, and R_{red} have completely constant Néron–Severi group scheme.*

Proof. According to Proposition 4.2, there is a finite purely inseparable field extension $F \subset K'$ so that the conditions (i) and (iii) hold for any separable field extension $F \subset F'$. In light of Proposition 2.2, we may also achieve (ii).

For the last property, choose a separable closure $F \subset F^{\mathrm{sep}}$, and consider the filtered order set of finite subextensions $F \subset F_\lambda \subset F^{\mathrm{sep}}$, $\lambda \in L$. Write $Z_1, \dots, Z_r \subset X \otimes_F F^{\mathrm{sep}}$ for the connected components. According to [29, Theorem 8.8.2] there are closed subschemes $Z_{i,\lambda} \subset X \otimes_F F_\lambda$ for some index $\lambda \in L$ inducing the $Z_i \subset X \otimes_F F^{\mathrm{sep}}$. Clearly, the $Z_{i,\lambda}$ are geometrically connected and geometrically reduced, thus $h^0(\mathscr{O}_{Z_{i,\lambda}}) = 1$. According to Lemma 4.3, we may enlarge the index $\lambda \in L$ to achieve that the $Z_{i,\lambda}$ have completely constant Néron–Severi group scheme. The same arguments apply for Y, C_{red}, and R_{red}. Summing up, we may choose the index $\lambda \in L$ so large that, for $F' = F_\lambda$ also condition (iv) holds.   □

The following notion turns out to be useful.

Definition 4.5. A proper scheme V, with structure morphism $V \to \mathrm{Spec}(F)$, and a purely inseparable field extension $F \subset K$ are called *adapted* if conditions (i)–(iv) of Proposition 4.4 hold.

According to Proposition 4.4, for any proper F-scheme V there is a finite separable extension $F \subset F'$ and a finite purely inseparable extension $F \subset K'$ such that $V \otimes_F F' \to \mathrm{Spec}(F')$ and $F' \subset F' \otimes_F K'$ are adapted. We usually reduce to this situation when analyzing the existence or non-existence of certain proper F-schemes V. The following observation will be useful.

Proposition 4.6. *Suppose that the proper scheme V and the finite purely inseparable extension $F \subset K$ are adapted, and that the ramification locus $R \subset X$ satisfies $h^0(\mathscr{O}_R) = 1$ and $h^1(\mathscr{O}_R) = 0$. Then*

$$h^0(\mathscr{O}_X) = h^0(\mathscr{O}_Y) = h^0(\mathscr{O}_C) = 1 \quad \text{and} \quad h^1(\mathscr{O}_Y) = h^1(\mathscr{O}_X) + h^1(\mathscr{O}_C).$$

If, moreover, $h^2(\mathscr{O}_Y) = 0$, the canonical sequence

$$0 \longrightarrow \mathrm{NS}(Y) \longrightarrow \mathrm{NS}(X) \oplus \mathrm{NS}(C) \longrightarrow \mathrm{NS}(R)$$

of Néron–Severi groups is exact.

Proof. The gluing map $R \to C$ is schematically dominant, which implies that the homomorphism $H^0(C, \mathscr{O}_C) \to H^0(R, \mathscr{O}_R)$ is injective, so $h^0(\mathscr{O}_C) = 1$. The short exact sequence (31) yields an exact sequence

$$0 \longrightarrow H^0(Y, \mathscr{O}_Y) \longrightarrow H^0(X, \mathscr{O}_X) \oplus H^0(C, \mathscr{O}_C) \longrightarrow H^0(R, \mathscr{O}_R) \longrightarrow 0.$$

The map on the right is indeed surjective, because $h^0(\mathscr{O}_R) = 1$. Since the normalization $\nu : X \to Y$ is schematically dominant, the map $H^0(Y, \mathscr{O}_Y) \to H^0(X, \mathscr{O}_X)$ is injective. This map is actually bijective, by the preceding exact sequence and $h^0(\mathscr{O}_C) = h^0(\mathscr{O}_R)$. We thus have an identification $H^0(X, \mathscr{O}_X) = H^0(Y, \mathscr{O}_Y)$. The scheme V is geometrically connected, by our overall assumption $h^0(\mathscr{O}_V) = 1$, so the reduction $Y = (V \otimes_F K)_{\mathrm{red}}$ is geometrically connected as well. It is also geometrically reduced, because V and $F \subset K$ are adapted. Consequently, $h^0(\mathscr{O}_X) = h^0(\mathscr{O}_Y) = 1$. The short exact sequence (31) also yields an exact sequence

$$0 \longrightarrow H^1(Y, \mathscr{O}_Y) \longrightarrow H^1(X, \mathscr{O}_X) \oplus H^1(C, \mathscr{O}_C) \longrightarrow 0,$$

whence $h^1(\mathscr{O}_Y) = h^1(\mathscr{O}_X) + h^1(\mathscr{O}_C)$. The above cohomology groups are the Lie algebras for the respective Picard groups, and we also infer that the restriction map

$$(4) \qquad\qquad \mathrm{Pic}^0_{Y/K} \longrightarrow \mathrm{Pic}^0_{X/K} \oplus \mathrm{Pic}^0_{C/K}$$

of group schemes has finite étale kernel. Now suppose that also $h^2(\mathscr{O}_Y) = 0$. Then the group scheme $\mathrm{Pic}^0_{Y/K}$ is smooth ([47, Lecture 27]), of dimension $h^1(\mathscr{O}_Y) = h^1(\mathscr{O}_X) + h^1(\mathscr{O}_C)$, so the restriction map is surjective. The short exact sequence (33) gives an exact sequence

$$0 \longrightarrow \mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}(X) \oplus \mathrm{Pic}(C) \longrightarrow \mathrm{Pic}(R)$$

of Picard groups. The map on the left is indeed injective because the mapping $K^\times = H^0(X, \mathscr{O}_X^\times) \to H^0(R, \mathscr{O}_R^\times) = K^\times$ is surjective. This also holds for all extension fields, and so (4) is actually an isomorphism of group schemes. The assumption $H^1(R, \mathscr{O}_R) = 0$ ensures that $\mathrm{Pic}^0_{R/K} = 0$, in particular $\mathrm{NS}(R) = \mathrm{Pic}(R)$. In turn, we obtain a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Pic}^0(Y) & \longrightarrow & \mathrm{Pic}^0(X) \oplus \mathrm{Pic}^0(C) & \longrightarrow & 0 & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i''} & & \\
0 & \longrightarrow & \mathrm{Pic}(Y) & \longrightarrow & \mathrm{Pic}(X) \oplus \mathrm{Pic}(C) & \longrightarrow & \mathrm{Pic}(R) & &
\end{array}
$$

with exact rows. Consequently, the desired exact sequence of Néron–Severi groups $0 \to \mathrm{Coker}(i') \to \mathrm{Coker}(i) \to \mathrm{Coker}(i'')$ follows from the Snake Lemma. $\qquad\square$

In the above situation, we can regard $\mathrm{NS}(Y)$ as the kernel of some homomorphism between finitely generated abelian groups. If, moreover, the Néron–Severi groups of X, C, R are torsion-free, we thus have

$$\mathrm{NS}(Y) = \mathrm{Ker}(\Psi) \subset \mathbb{Z}^{\oplus n}$$

for some *integral matrix* $\Psi \in \mathrm{Mat}_{m \times n}(\mathbb{Z})$, where the size of the matrix is given by $m = \rho(R)$ and $n = \rho(X) + \rho(C)$. Here $\rho(X) = \mathrm{rank}\,\mathrm{NS}(X)$, etc., denotes *Picard numbers*.

In the next sections, we shall exploit this in the following way: every Cartier divisor $D \subset V$ induces a Cartier divisor $D_K \subset Y$, which represents an integral vector in the kernel of the matrix Ψ. This applies in particular for the canonical divisor $D = K_V$, if the scheme V is Gorenstein, or the reduction $D = N_{\mathrm{red}}$ of the divisorial part $N \subset \mathrm{Sing}(V/F)$ for the locus of non-smoothness, if the scheme V is locally factorial. As we shall see, sometimes geometric reasons preclude the existence of such integral solutions for the system of linear equations

$$\Psi \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We close this section with a well-known useful observation.

Proposition 4.7. *Suppose that the local rings $\mathscr{O}_{V,a}$, $a \in V$ are geometrically unibranch. Then the normalization map $\nu : X \to Y = (V \otimes_F K)_{\mathrm{red}}$ is a universal homeomorphism.*

Proof. The conditions on the local rings mean that the spectrum of the strict henselization $\mathscr{O}^s_{V,a}$ is irreducible. If $F \subset K$ is a finite purely inseparable field extension, then the ring $\mathscr{O}^s_{V,a} \otimes_F K$ remains local, indeed strictly local, and coincides with the strictly local ring at the point $b \in V \otimes_F K$ corresponding to $a \in V$, according to [30, Proposition 18.6.8]. In particular, the normalization A' of the strictly local integral domain $A = (\mathscr{O}^s_{V \otimes_F K, b})_{\mathrm{red}}$ yields a universal homeomorphism

$\mathrm{Spec}(A') \to \mathrm{Spec}(A)$. It follows that the normalization $X \to Y = (V \otimes_F K)_{\mathrm{red}}$ is a universal homeomorphism.   □

Note that the conclusion holds, in particular, when V is normal.

5. Regular del Pezzo surfaces

Let F be a ground field. Throughout the paper, we use the following general notion of del Pezzo surfaces.

Definition 5.1. A *del Pezzo surface* is a proper two-dimensional scheme V with $h^0(\mathscr{O}_V) = 1$ that is Gorenstein, and whose dualizing sheaf ω_V is antiample.

The most immediate examples are surfaces of degree two or three in $\mathbb{P}^3$, or complete intersections of two quadrics in $\mathbb{P}^4$. The antiample sheaf ω_V has no sections, at least if V is integral, hence $h^2(\mathscr{O}_V) = 0$. The selfintersection number $(\omega_V \cdot \omega_V) > 0$ is called the *degree* of the del Pezzo surface, and the integer $h^1(\mathscr{O}_V) \geq 0$ is commonly referred to as the *irregularity*. Examples of del Pezzo surfaces with irregularity $h^1(\mathscr{O}_V) > 0$ were constructed by Reid [51], the second author [58], and Maddock [43].

In what follows, we suppose that the ground field F has characteristic $p > 0$, and assume that the del Pezzo surface V is normal, locally factorial, and geometrically integral, but geometrically non-normal. We are mainly interested in the case that V is even regular, but the weaker assumptions of local factoriality lies at the core for most our arguments. The locus of non-smoothness $\mathrm{Sing}(V/F)$ contains a non-empty divisorial part $N \subset \mathrm{Sing}(V/F)$, which is an effective Cartier divisor. Let $D = N_{\mathrm{red}}$ be its reduction. The recurrent idea of this paper is to study the behavior of the effective Cartier divisor $D \subset V$ under base-change, in particular if the ground field has $\mathrm{pdeg}(F) = 1$.

After replacing the ground field F by some finite separable extension, we may assume that there is some finite purely inseparable field extension $F \subset K$ so that $V \to \mathrm{Spec}(F)$ and $F \subset K$ are adapted, according to Proposition 4.4. As in the preceding section, we write $\nu : X \to Y$ for the normalization of $Y = V \otimes_F K$. Note that X is Cohen–Macaulay, but not necessarily Gorenstein. Let

$$\text{(5)} \qquad \begin{array}{ccc} R & \longrightarrow & X \\ \downarrow & & \downarrow \nu \\ C & \longrightarrow & Y \end{array}$$

be the conductor square, where $C \subset Y$ is the *conductor curve* and $R \subset X$ is the *ramification divisor*, as discussed in Appendix A. Both are Weil divisors that are not necessarily Cartier. Moreover, the vertical maps are universal homeomorphisms. Note that we regard K as a new ground field for Y and X.

Recall that the *Hirzebruch surface with numerical invariant* $e \geq 0$ is the smooth surface $S = \mathrm{Proj}(\mathrm{Sym}\,\mathscr{E})$ for the locally free sheaf $\mathscr{E} = \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(e)$. It comes with a ruling $r : S \to \mathbb{P}^1$ and a section $E = \mathrm{Proj}(\mathrm{Sym}\,\mathscr{L})$ with $\mathscr{L} = \mathscr{O}_{\mathbb{P}^1}(e)$, which has selfintersection number $E^2 = -e$. If $e > 0$ the curve $E \subset S$ is negative-definite and contracts to a rational singularity. The resulting *contracted Hirzebruch surface* can

be regarded as the weighted projective space $\mathbb{P}(1,1,e) = \operatorname{Proj} K[T_0, T_1, T_2]$, with grading $\deg(T_0) = \deg(T_1) = 1$ and $\deg(T_2) = e$. One may view it also as a toric variety.

By abuse of notation, we say that a proper curve Z is a *split conic* if it is isomorphic to a divisor of degree two inside $\mathbb{P}^2$ given by one of the following three homogeneous equations:

$$T_0^2 + T_1 T_2 = 0, \quad \text{or} \quad T_0 T_1 = 0, \quad \text{or} \quad (T_0 + T_1 + T_2)^2 = 0.$$

In the first case Z is isomorphic to the *projective line* $\mathbb{P}^1$. In the second case, we say that Z is a *pair of lines* $\mathbb{P}^1 \cup \mathbb{P}^1$. In the last case, Z is isomorphic to the *split ribbon* $\mathbb{P}^1 \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$ on the projective line $\mathbb{P}^1$ with ideal sheaf $\mathscr{O}_{\mathbb{P}^1}(-1)$. Any such ribbon is split, because $\operatorname{Ext}^1(\mathscr{O}_{\mathbb{P}^1}(-1), \mathscr{O}_{\mathbb{P}^1}) = 0$. We refer to [5] for the general theory of ribbons.

Proposition 5.2. *The normal surface X is either the projective plane $\mathbb{P}^2$, a contracted Hirzebruch surface $\mathbb{P}(1,1,e)$ with numerical invariant $e \geq 2$, or a Hirzebruch surface S with $e \geq 0$. The ramification curve R is a split conic, in particular $h^0(\mathscr{O}_R) = 1$ and $h^1(\mathscr{O}_R) = 0$.*

Proof. By assumption, V is geometrically integral and X is geometrically normal, respectively. In light of Reid's Classification ([51, Theorem 1.1]), the assertion holds if we pass to some finite field extension $K \subset K'$. It remains to check that the assertion already holds over K. For this we use the assumption that V and $F \subset K$ are adapted.

We first treat the ramification curve R and set $R' = R \otimes_K K'$. If R' is a projective line, we choose an invertible sheaf $\mathscr{L}$ on R of degree one, which defines an isomorphism $R \to \mathbb{P}^1$. If $R' = R_1' \cup R_2'$ is a pair of lines, there are invertible sheaves $\mathscr{L}_1$ and $\mathscr{L}_2$ on R with $\deg(\mathscr{L}_i | R_j') = \delta_{ij}$. The resulting morphisms $f_i : R \to \mathbb{P}^1$ reveal that R is a pair of lines. Now suppose that R' is a split ribbon. The scheme R_{red} is geometrically reduced, and its ideal sheaf $\mathscr{I} \subset \mathscr{O}_R$ is invertible of degree one on R_{red}. It follows $R_{\mathrm{red}} = \mathbb{P}^1$. Such ribbons split, so $R = \mathbb{P}^1 \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$.

In all cases, we may regard $R \subset \mathbb{P}^2$ as a divisor of degree two. The resulting short exact sequence $0 \to \mathscr{O}_{\mathbb{P}^2}(-2) \to \mathscr{O}_{\mathbb{P}^2} \to \mathscr{O}_R \to 0$ induces a long exact sequence

$$H^0(\mathscr{O}_{\mathbb{P}^2}) \to H^0(\mathscr{O}_R) \to H^1(\mathscr{O}_{\mathbb{P}^2}(-2)) \to H^1(\mathscr{O}_{\mathbb{P}^2}) \to H^1(\mathscr{O}_R) \to H^2(\mathscr{O}_{\mathbb{P}^2}(-2)).$$

The cohomology groups $H^i(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(-2))$ vanish for all $i \geq 1$, and it follows that $h^0(\mathscr{O}_R) = 1$ and $h^1(\mathscr{O}_R) = 0$.

We now turn to the normal surface X. Suppose that $X' = X \otimes F$ is isomorphic to a Hirzebruch surface. Fix a ruling $r' : X' \to \mathbb{P}^1_{K'}$, and let $\mathscr{L}'$ be the preimage of $\mathscr{O}_{\mathbb{P}^1_{K'}}(1)$. This invertible sheaf descends to an invertible sheaf $\mathscr{L}$ on X. The latter is semiample, and defines a ruling $r : X \to \mathbb{P}^1$. Now choose an invertible sheaf $\mathscr{N}'$ that has degree one on the fibers of $r' : X' \to \mathbb{P}^1_{K'}$. This also descends, and it follows that X is a Hirzebruch surface. The case that X' is the projective plane is treated in a similar way.

Now suppose that X' is a contracted Hirzebruch surface, and let $S' \to X'$ be the minimal resolution of singularities. Let $a' \in X'$ be the singularity and let $a \in X$ be its image. Using [3, Theorem 4], we infer that $S' \to X'$ is the blowing-up of the

reduced center $a' \in X'$, and that the exceptional divisor $E' \subset S'$ is a projective line. According to Reid's Classification, the ramification divisor $R' \subset X'$ is linearly equivalent to $2H'$, where $H' \subset X'$ is the image of any fiber $F' \subset S'$ from the ruling. Note that eH' generates the Picard group, but that H' is not Cartier.

Suppose first that there is a Weil divisor $H \subset X$ whose base-change is linearly equivalent to $H' \subset X'$. Since $h^0(\mathcal{O}_X(H)) = 2$, there are two such Weil divisors $H_1 \neq H_2$. Their intersection $H_1 \cap H_2$ has length one and contains $a \in X$. So its blowing-up $S \to X$ yields a twisted form of the Hirzebruch surface, and the exceptional divisor $E \subset S$ is a twisted form of the projective line. The latter intersects the strict transform of H in a rational point, so $E = \mathbb{P}^1$. Arguing as above, one can easily see that S is a Hirzebruch surface and hence $X = \mathbb{P}(1, 1, e)$.

Seeking a contradiction, we now assume that there is no Weil divisor on X inducing $H' \subset X'$. Recall that the ramification divisor R is a split conic. It must be a projective line if $a \in R$. This yields a contradiction, because on X' there are no lines linearly equivalent to $2H'$ passing through the singularity. Thus $R \subset X$ lies in the smooth locus. Hence R and the linearly equivalent $2H$ are Cartier, and we must have $e = 2$. It also follows that the split conic R is a projective line, with selfintersection $R^2 = 2$, and one easily computes $h^0(\mathcal{O}_X(R)) = 4$. Since $\mathrm{Sing}(Y) = \mathrm{Sing}(Y/K)$, the local ring $\mathcal{O}_{X,a} = \mathcal{O}_{Y,a}$ is singular. It must be a twisted form of the rational double point of type A_1 given by the equation $z^2 - xy = 0$. We see that the local Artin scheme $\mathrm{Sing}(X/K)$ has length two, and we conclude with Lemma 14.2 that $a \in X$ is a rational point. To proceed, choose two further rational points $b, c \in R$ and consider the reduced closed subscheme $Z = \{a, b, c\}$. The invertible sheaf $\mathcal{L} = \mathcal{O}_X(R)$ has $h^0(\mathcal{L}) = 4$. For dimension reasons, the restriction map $H^0(X, \mathcal{L}) \to H^0(Z, \mathcal{L}|Z)$ is not injective, so there is an effective Cartier divisor $A \subset X$ linearly equivalent to R and containing Z. Examining its base-change to X', we easily infer that A is reducible. Consequently, there is a Weil divisor on X inducing $H' \subset X'$, a contradiction.   □

In what follows, we shall use the following notation: If $X = \mathbb{P}^2$ is the projective plane, write $H \subset X$ for a hyperplane. Then $\mathrm{Pic}(X)$ is freely generated by the class of H, the intersection pairing is given by $H^2 = 1$, and the canonical class is $K_X = -3H$.

If $X = \mathbb{P}(1, 1, e)$ is the contraction of a Hirzebruch surface S with numerical invariant $e \geq 2$, we write $E \subset S$ for the section with $E^2 = -e$ and $F \subset S$ for a fiber of the ruling $r : S \to \mathbb{P}^1$. (It should be clear from the context whether the symbol F means a fiber for the ruling or a ground field.) The Picard group $\mathrm{Pic}(S)$ is freely generated by the classes of E and F, the intersection form has Gram matrix $\left(\begin{smallmatrix} -e & 1 \\ 1 & 0 \end{smallmatrix} \right)$, and the canonical class is $K_S = -2E - (e+2)F$. Let $f : S \to X$ be the contraction of the negative-definite curve $E \subset S$ and write $H = f(F)$ for the image of the fiber. The Weil divisor $H \subset X$ is not Cartier, but eH is Cartier, with $f^*(eH) = eF + E$, and freely generates $\mathrm{Pic}(X)$. The intersection pairing is given by $(eH)^2 = e$, and the canonical class is $K_X = -(e+2)H$. Note that X is Gorenstein if and only if $e = 2$.

If $X = S$ is a Hirzebruch surface with numerical invariant $e \geq 0$, we likewise write $E, F \subset S$ for the section with $E^2 = -e$ and the fiber for a ruling. Here the canonical class is given by $K_X = -2E - (e+2)F$.

We now tabulate the five possibilities that follow from Reid's Classification of non-normal del Pezzo surfaces over algebraically closed ground fields ([51, Theorem 1.1]).

Theorem 5.3. *If the del Pezzo surface V and the field extension $F \subset K$ are adapted, then the possibilities for the normalization $X \to Y = V_K$ are as follows:*

Case	(i)	(ii)	(iii)	(iv)	(v)
X	$\mathbb{P}^2$	$\mathbb{P}^2$	$\mathbb{P}(1,1,e)$	S	S
e	1	1	≥ 2	≥ 0	≥ 0
$\mathscr{O}_X(R)$	$2H$	H	$2H$	$E+F$	E
ω_X	$-3H$	$-3H$	$-(e+2)H$	$-2E-(e+2)F$	$-2E-(e+2)F$
$\nu^*(\omega_Y)$	$-H$	$-2H$	$-eH$	$-E-(e+1)F$	$-E-(e+2)F$
K_Y^2	1	4	e	$e+2$	$e+4$

We refer to this table as *Reid's Classification*. Note that Cases (i) and (iii) may be treated on the same footing because $\mathbb{P}^2 = \mathbb{P}(1,1,1)$. It is remarkable that in all cases we have $\rho(V) \leq 2$. In particular, the dualizing sheaf ceases to be antiample if we blow-up V in two closed points. For later use, we record the following fact.

Proposition 5.4. *With the exception of Case* (v) *in Reid's Classification, the pullback map* $\mathrm{Pic}(V) \to \mathrm{Pic}(C)$, $\mathscr{L} \mapsto \mathscr{L}_C$ *is injective.*

Proof. In any case, the projection $Y = V \otimes_F K \to V$ induces an inclusion $\mathrm{Pic}(V) \subset \mathrm{Pic}(Y)$, and the short exact sequence (33) yields an exact sequence

$$0 \longrightarrow \mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}(X) \oplus \mathrm{Pic}(C) \longrightarrow \mathrm{Pic}(R).$$

The map on the left is indeed injective because $h^0(\mathscr{O}_R) = 1$. For the cases in question, the restriction map $\mathrm{Pic}(X) \to \mathrm{Pic}(R)$ is injective, and it follows that the restriction map $\mathrm{Pic}(Y) \to \mathrm{Pic}(C)$ is injective as well. $\square$

Since $h^0(\mathscr{O}_R) = 1$ and $h^1(\mathscr{O}_R) = 0$ and $h^2(\mathscr{O}_Y) = 0$, we can apply Proposition 4.6 and conclude that we also have an exact sequence

$$(6) \qquad 0 \longrightarrow \mathrm{NS}(Y) \longrightarrow \mathrm{NS}(X) \oplus \mathrm{NS}(C) \xrightarrow{\Psi} \mathrm{NS}(R).$$

Note that all terms in this sequence are finitely generated free abelian groups, so we may regard $\mathrm{NS}(Y)$ as the kernel of some matrix $\Psi \in \mathrm{Mat}_{s \times (r+s)}(\mathbb{Z})$, where $r = \rho(X) = \rho(Y)$ and $s = \rho(C) = \rho(R)$. Since Y is Gorenstein, the class of the dualizing sheaf ω_Y defines an element in this kernel. This already gives an important restriction.

Proposition 5.5. *The ramification divisor $R \subset X$ is either the projective line $\mathbb{P}^1$ or the split ribbon $\mathbb{P}^1 \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$.*

Proof. Seeking a contradiction, we assume that $R = R_1 \cup R_2$ is a pair of lines. By Proposition 4.7, the map $R \to C$ is a universal homeomorphism. Setting $C_i = \nu(R_i)$ we get $C = C_1 \cup C_2$. According to Proposition A.2 the induced morphisms $R_i \to C_i$ have degree two, and the characteristic must be $p = 2$.

We now go through the five cases of Reid's Classification: In Case (i), the matrix Ψ whose kernel gives $\mathrm{NS}(Y)$ takes the form

$$\Psi = \begin{pmatrix} 1 & -2 & 0 \\ 1 & 0 & -2 \end{pmatrix} \in \mathrm{Mat}_{2 \times 3}(\mathbb{Z}).$$

The entries in the first column are the intersection numbers $(H \cdot R_i) = 1$, whereas the other non-zero entries are the negative degrees of the gluing maps $\nu\colon R_i \to C_i$. We have $\nu^*(K_Y) = -H$, so the class of $K_Y \in \mathrm{Ker}(\Psi)$ is a column vector of the form $-(1, m, n)^t$ for some integers $m, n \in \mathbb{Z}$. The only solution is $m, n = 1/2$, which is not integral, a contradiction.

In Case (iii), the Néron–Severi group $\mathrm{NS}(X)$ is generated by eH, which has intersection numbers $(eH \cdot R_i) = 1$. Consequently, the matrix describing $\mathrm{NS}(Y)$ is as in the previous paragraph, and we get a contradiction again. In the Cases (ii) and (v) the ramification divisor R is not linearly equivalent to the sum of two effective divisors, so these cases are impossible as well.

It remains to deal with Case (iv), where $X = S$ is a Hirzebruch surface with numerical invariant $e \geq 0$. Then $\mathrm{Pic}(X)$ is freely generated by $E, F \subset X$. The ramification divisor is $R = E \cup F$, say with $R_1 = E$ and $R_2 = F$. Now the matrix describing $\mathrm{NS}(Y)$ takes the form

$$\Psi = \begin{pmatrix} -e & 1 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{pmatrix} \in \mathrm{Mat}_{2 \times 4}(\mathbb{Z}),$$

where the entries on the left half are given by the intersection numbers

$$(E \cdot R_1) = -e, \quad (E \cdot R_2) = 1, \quad (F \cdot R_1) = 1, \quad \text{and} \quad (F \cdot R_2) = 0.$$

We have $\nu^*(K_Y) = -E - (e+1)F$, whence $K_Y \in \mathrm{Ker}(\Psi)$ corresponds to a column vector of the form $-(1, e+1, m, n)^t$ for some integers $m, n \in \mathbb{Z}$. However, the only solution has $n = 1/2$, again a contradiction. $\qquad\square$

Recall that $N \subset \mathrm{Sing}(V/F)$ denotes the divisorial part of the locus of non-smoothness. Then $N \subset V$ is a Cartier divisor, since our normal surface V is assumed to be locally factorial. In turn, we get a Cartier divisor $N_K \subset V_K = Y$ whose support coincides with the conductor curve $C \subset Y$. Another consequence of the Gorenstein condition is the following.

Proposition 5.6. *The conductor curve C is geometrically integral.*

Proof. Proposition 5.5 ensures that C is geometrically irreducible. Since the del Pezzo surface V and the field extension $F \subset K$ are adapted, the scheme C_{red} is geometrically reduced. It thus suffices to verify that C is reduced. This is obvious if the ramification curve R is a line. Assume now that R is a split ribbon, so we are in Case (i) or Case (iii) in Reid's Classification. Let us treat both at the same time, allowing $e = 1$ for $X = \mathbb{P}^2$. Seeking a contradiction, we now assume that C is not reduced. According to Proposition 3.3, the morphism $R_{\mathrm{red}} \to C_{\mathrm{red}}$ has degree two. To proceed, consider again the exact sequence of Néron–Severi groups

$$0 \longrightarrow \mathrm{NS}(Y) \longrightarrow \mathrm{NS}(X) \oplus \mathrm{NS}(C) \xrightarrow{\Psi} \mathrm{NS}(R).$$

According to [10, Proposition 9.1.5] for each proper geometrically irreducible curve Z the restriction map $\mathrm{NS}(Z) \to \mathbb{Z}$ given by $\mathscr{L} \mapsto \deg(\mathscr{L}|Z_{\mathrm{red}})$ is injective. Choosing identifications $\mathrm{NS}(C) = \mathbb{Z}$ and $\mathrm{NS}(R) = \mathbb{Z}$, we see that the linear map in the above exact sequence becomes the matrix $\Psi = (1, -2)$, because $(eH \cdot R_{\mathrm{red}}) = 1$ and $R_{\mathrm{red}} \to C_{\mathrm{red}}$ has degree two. By Reid's Classification, the invertible sheaf $\nu^{-1}(\omega_Y)$ is given by $-eH$, so the corresponding element in $\mathrm{NS}(X) \oplus \mathrm{NS}(C)$ is of the form $-(1, m)^t$. Since this lies in $\mathrm{Ker}(\Psi)$, we must have $m = 1/2$, a contradiction. $\qquad\square$

Next, we restrict the possible characteristics.

Proposition 5.7. *If the ramification divisor R is a split ribbon, then the characteristic must be $p \leq 3$ and the induced morphism $\nu : R_{\mathrm{red}} \to C$ is birational. If $p = 3$, the normal surface X is either $\mathbb{P}^2$ or $\mathbb{P}(1,1,3)$. In case $p = 2$, the only possibilities are $\mathbb{P}^2$, $\mathbb{P}(1,1,2)$, or $\mathbb{P}(1,1,4)$.*

Proof. Clearly, the tangent sheaf $\Theta_{V/F} = \underline{\mathrm{Hom}}(\Omega^1_{V/F}, \mathscr{O}_V)$ on the normal surface V is locally free in codimension one, whence locally free at almost all points $a \in V$. In turn, $\Theta_{Y/K}$ is locally free at almost all points $y \in Y$. Since C_{red} is geometrically reduced, we can use Bertini Theorems to find a closed point $y \in C_{\mathrm{red}}$ in the regular locus so that the field extension $K \subset \kappa(y)$ is separable (for example [31, Proposition 4.3] or [37, Chapter I, Theorem 6.3]). Now we can apply Proposition 3.5 and deduce $p \leq 3$.

In Reid's Classification, the ramification divisor $R \subset X$ is linearly equivalent to a multiple of the same effective divisor only in Case (i) and (iii). In other words, $X = \mathbb{P}(1,1,e)$ is a contracted Hirzebruch surface with numerical invariant $e \geq 1$, and ramification divisor $R = 2H$. Consider the divisorial part $N \subset \mathrm{Sing}(V/F)$ and its preimage $N_K \subset Y$. If $p = 3$, according to Theorem 3.4, the resulting Cartier divisor on X is $\nu^{-1}(N_K) = 3H$. Since $\mathrm{Pic}(X)$ is freely generated by eH, we must have $e|3$. If $p = 2$, Theorem 3.4 implies that $\nu^{-1}(N_K) = 4H$, therefore $e|4$. The assertion follows. $\square$

Proposition 5.8. *If the ramification divisor R is the projective line, then the characteristic must be $p = 2$, and the gluing map $\nu : R \to C$ is isomorphic to a finite flat morphism $\mathbb{P}^1 \to \mathbb{P}^1$ that is radical of degree two. Furthermore, the inclusion $\mathscr{O}_C \subset \nu_*(\mathscr{O}_R)$ is a direct summand of $\mathscr{O}_C$-modules. The normal surface X is either $\mathbb{P}^2$, $\mathbb{P}(1,1,2)$, or $\mathbb{P}(1,1,4)$.*

Proof. From Proposition 3.1 we deduce $p = 2$, that the ramification curve C is regular, and that $\nu : R \to C$ is finite, flat, and radical. Since this holds true if we replace K by any finite purely inseparable field extension, the curve C must be smooth. By Lüroth's Theorem, it is a projective line $\mathbb{P}^1$.

The cokernel for the inclusion $\mathscr{O}_C \subset \nu_*(\mathscr{O}_R)$ is an invertible sheaf $\mathscr{O}_{\mathbb{P}^1}(n)$ for some integer n, with Euler characteristic $\chi(\mathscr{O}_R) - \chi(\mathscr{O}_C) = 0$. Thus $n = -1$. Using $\mathrm{Ext}^1(\mathscr{O}_{\mathbb{P}^1}(-1), \mathscr{O}_{\mathbb{P}^1}) = H^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(1)) = 0$, we infer that the inclusion $\mathscr{O}_C \subset \nu_*(\mathscr{O}_R)$ is a direct summand.

According to Reid's Classification, $X = \mathbb{P}(1,1,e)$ is a contracted Hirzebruch surface with invariant $e \geq 1$, and $\mathscr{O}_X(R) = \mathscr{O}_X(2H)$. According to Theorem 3.2, we have $\nu^{-1}(N_K) = 2R$, and thus $e|4$. $\square$

Note that the gluing map $\nu : R \to C$ coincides with the Frobenius map on $\mathbb{P}^1$ if the ground field F is perfect, such that the function field $k(C)$ has p-degree one. Combining the preceding results, we already obtain a non-existence result, which generalizes [49, Corollary 1.4].

Theorem 5.9. *Normal del Pezzo surfaces that are locally factorial, geometrically integral but not geometrically normal do not exist in characteristic $p \geq 5$.*

Recall that $N \subset \mathrm{Sing}(V/F)$ denotes the divisorial part of the locus of non-smoothness. Its preimage $N_K \subset Y$ has the same support as the conductor curve $C \subset Y$. According to Proposition 5.6, we actually have $(N_K)_{\mathrm{red}} = C$. It follows that N is irreducible, and that $D = N_{\mathrm{red}}$ is integral.

Proposition 5.10. *Suppose $p = 2$. Then the geometric generic embedding dimension and geometric generic Hilbert–Samuel multiplicity for the curve N are*

$$\mathrm{edim}(\mathscr{O}_{N,\eta}/F) = 2 \quad and \quad e(\mathscr{O}_{N,\eta}/F) = 4.$$

The preimage on normal surface X is given by $\nu^{-1}(N_K) = 2R$. If N is not integral, then $N = 2D$, and $D = N_{\mathrm{red}}$ satisfies

$$\mathrm{edim}(\mathscr{O}_{D,\eta}/F) = 1 \quad and \quad e(\mathscr{O}_{D,\eta}/F) = 2.$$

Proof. Suppose that the ramification divisor $R \subset X$ is a line. By Proposition 5.8, the same holds for the conductor curve $C \subset Y$. Choose a rational point $x \in R$. Its image $y \in C$ is rational as well. Thus we may apply Theorem 3.2, and the values for $\mathrm{edim}(\mathscr{O}_{N,\eta}/F)$ and $e(\mathscr{O}_{N,\eta}/F)$ and the multiplicity in $\nu^{-1}(N_K) = 2R$ follow. If the ramification locus R is a double line, we argue analogously with Theorem 3.4. The case described in Theorem 3.6 does not occur, thanks to Proposition 5.6.

Now suppose that the curve N is non-reduced with $N = nD$. The multiplicities satisfy $e(\mathscr{O}_{N,\eta}/F) = n \cdot e(\mathscr{O}_{D,\eta}/F)$, hence $n|4$. If $n = 4$, the integral curve D would be generically smooth, hence V would be smooth at some point contained in D. This is a contradiction. Thus we have $n = 2$ and $e(\mathscr{O}_{D,\eta}/F) = 2$. The value $\mathrm{edim}(\mathscr{O}_{D,\eta}/F) = 1$ again follows from the descriptions in Theorems 3.2 and 3.4. $\square$

In the same way, one verifies the case of characteristic three.

Proposition 5.11. *Suppose $p = 3$. Then the geometric generic embedding dimension and geometric generic Hilbert–Samuel multiplicity for the curve N are*

$$\mathrm{edim}(\mathscr{O}_{N,\eta}/F) = 1 \quad and \quad e(\mathscr{O}_{N,\eta}/F) = 3.$$

The preimage on the normal surface X is given by $\nu^{-1}(N_K) = 3R_{\mathrm{red}}$, and the curve N is integral.

According to Proposition 5.6, the conductor curve $C \subset Y$ coincides with the reduced divisorial part of $\mathrm{Sing}(Y)$. It will be crucial to understand the effective Cartier divisors on the non-normal surface Y supported by C. For every invertible sheaf $\mathscr{L}$ on Y, the short exact sequence $0 \to \mathscr{L} \to \mathscr{L}_X \oplus \mathscr{L}_C \to \mathscr{L}_R \to 0$ of coherent sheaves on Y yields a long exact sequence

$$(7) \qquad 0 \longrightarrow H^0(Y, \mathscr{L}) \longrightarrow H^0(X, \mathscr{L}_X) \oplus H^0(C, \mathscr{L}_C) \longrightarrow H^0(R, \mathscr{L}_R).$$

In other words, the global sections of $\mathscr{L}$ can be regarded as pairs $s = (s_X, s_C)$ with $s_X|R = s_C|R$.

Proposition 5.12. *The mapping*

$$H^0(X, \mathscr{L}_X(-R)) \longrightarrow H^0(Y, \mathscr{L}), \quad s_X \longmapsto (s_X, 0),$$

is a bijection between the global section of the reflexive sheaf $\mathscr{L}_X(-R)$ and the global sections of the invertible sheaf $\mathscr{L}$ that vanish along $C = N_{\mathrm{red}}$.

Proof. The global sections $s = (s_X, s_C)$ that vanish along the conductor curve have $s_C = 0$, and hence $s_X|R = 0$. The assertion follows using the exact sequence $0 \to \mathscr{L}(-R) \to \mathscr{L}_X \to \mathscr{L}_R \to 0$. $\square$

6. Smooth ramification loci

Let F be a ground field of characteristic $p > 0$, and let V be a normal del Pezzo surface that is locally factorial, geometrically integral but geometrically non-normal. After making a finite separable ground field extension, we may assume that there is a finite purely inseparable field extension $F \subset K$ so that $V \to \mathrm{Spec}(F)$ and $F \subset K$ are adapted. Let $\nu : X \to Y$ be the normalization, let $C \subset Y$ be the conductor curve, and let $R \subset X$ be the ramification divisor. We then have a commutative diagram

$$
\begin{array}{ccc}
R & \longrightarrow & X \\
\downarrow & & \downarrow{\scriptstyle \nu} \\
C & \longrightarrow & Y.
\end{array}
$$

In this section, we treat the case that the ramification locus R is smooth. By Proposition 5.8, we have $p = 2$ and $R = \mathbb{P}^1$, and the gluing map $R \to C$ is isomorphic to a finite flat morphism $\mathbb{P}^1 \to \mathbb{P}^1$ that is radical of degree two. The possibilities for the embedding $R \subset X$ are given by Reid's Classification in Theorem 5.3. Furthermore, we have a proper birational morphism $S \to X$ from a Hirzebruch surface S with numerical invariant $e \geq 0$. Let us start with a vanishing result.

Proposition 6.1. *We have $H^1(V, \omega_V^{\otimes n}) = 0$ for all integers n.*

Proof. By Serre duality, it suffices to check this for $n \geq 1$. Set $\mathscr{L} = \omega_V^{\otimes n}$. The short exact sequence $0 \to \mathscr{L}_Y \to \mathscr{L}_X \oplus \mathscr{L}_C \to \mathscr{L}_R \to 0$ yields an exact sequence

$$
H^0(R, \mathscr{L}_R) \longrightarrow H^1(Y, \mathscr{L}_Y) \longrightarrow H^1(X, \mathscr{L}_X) \oplus H^1(C, \mathscr{L}_C) \longrightarrow H^1(R, \mathscr{L}_R).
$$

The term on the left vanishes because $\mathscr{L}_R = \mathscr{O}_{\mathbb{P}^1}(-m)$ for some integer $m \geq 1$. According to Proposition 5.8, the inclusion $\mathscr{O}_C \subset \nu_*(\mathscr{O}_R)$ is a direct summand, and it follows that $H^1(C, \mathscr{L}_C) \to H^1(R, \mathscr{L}_R)$ is injective. In turn, the canonical mapping $H^1(Y, \mathscr{L}_Y) \to H^1(X, \mathscr{L}_X)$ is injective as well. From Reid's Classification in Theorem 5.3, one sees that $\mathscr{L}_X \simeq \mathscr{O}_X(-A)$ for some effective Cartier divisor $A \subset X$ that is geometrically connected and geometrically reduced, which ensures $h^0(\mathscr{O}_A) = 1$. The normal surface X is either the projective plane, a contracted Hirzebruch surface, or a Hirzebruch surface, which all have $H^1(X, \mathscr{O}_X) = 0$. The exact sequence $0 \to \mathscr{L}_X \to \mathscr{O}_X \to \mathscr{O}_A \to 0$ yields an exact sequence

$$
H^0(X, \mathscr{O}_X) \longrightarrow H^0(A, \mathscr{O}_A) \longrightarrow H^1(X, \mathscr{L}_X) \longrightarrow H^1(X, \mathscr{O}_X),
$$

hence the term $H^1(X, \mathscr{L}_X)$ vanishes. $\qquad\square$

Let $N \subset \mathrm{Sing}(V/F)$ be the divisorial part of the locus of non-smoothness. This irreducible curve has geometric generic embedding dimension $\mathrm{edim}(\mathscr{O}_{N,\eta}/F) = 2$, and the preimage on the normal surface X is given by $\nu^{-1}(N_K) = 2R$, by Proposition 5.10. The idea now is to study the integral curve $D = N_{\mathrm{red}}$.

Proposition 6.2. *The possibilities for the integral curve $D = N_{\mathrm{red}}$ and the numerical invariant $e \geq 0$ of the Hirzebruch surface S are given by the following table, according to Cases* (i)–(v) *in Reid's Classification:*

Case	(i)		(ii)	(iii)			(iv)			(v)		
X	$\mathbb{P}^2$		$\mathbb{P}^2$	$\mathbb{P}(1,1,e)$			S			S		
e	1		1	2		4	0		1	0	1	2
$\mathscr{O}_X(R)$	$2H$		H	$2H$		$2H$	$E+F$		$E+F$	E	E	E
$h^0(\mathscr{O}_D)$	1	1	1	1	1	1	1	1	1	1	4	2
$h^1(\mathscr{O}_D)$	2	7	1	1	3	1	1	3	1	0	0	0
$\mathrm{edim}(\mathscr{O}_{D,\eta}/F)$	1	2	2	1	2	2	1	2	2	1	2	1
D	$\frac{1}{2}N$	N	N	$\frac{1}{2}N$	N	N	$\frac{1}{2}N$	N	N	$\frac{1}{2}N$	N	$\frac{1}{2}N$
$\nu^{-1}(D_K)$	R	$2R$	$2R$	R	$2R$	$2R$	R	$2R$	$2R$	R	$2R$	R

Proof. According to Proposition 5.10, we either have $D = \frac{1}{2}N$ or $D = N$, and the respective values for the geometric generic embedding dimension $d = \mathrm{edim}(\mathscr{O}_{D,\eta}/F)$ are $d = 1$ or $d = 2$. We first treat the Cases (i)–(iii) from Reid's Classification. According to Proposition 5.8, we have $e = 2^\nu$ with $0 \leq \nu \leq 2$. Moreover, $\mathrm{Pic}(V)$ is cyclic, and generated by the dualizing sheaf, so we have $\mathscr{O}_V(-D) = \omega_V^{\otimes m}$ for some integer $m \geq 1$. The short exact sequence $0 \to \omega_V^{\otimes m} \to \mathscr{O}_V \to \mathscr{O}_D \to 0$ yields an exact sequence

$$H^0(V, \mathscr{O}_V) \longrightarrow H^0(D, \mathscr{O}_D) \longrightarrow H^1(V, \omega_V^{\otimes m}).$$

The term on the right vanishes, by Proposition 6.1, thus $h^0(\mathscr{O}_D) = 1$. Riemann–Roch gives $2h^1(\mathscr{O}_D) = 2 - 2\chi(\mathscr{O}_D) = 2 + \deg(\omega_D)$. By the Adjunction Formula, this integer equals

$$2 + (K_V + D) \cdot D = 2 + (\nu^{-1}(K_Y) + \nu^{-1}(D_K)) \cdot \nu^{-1}(D_K),$$

and the tabulated values for $h^1(\mathscr{O}_D)$ easily follow in each of the three Cases (i)–(iii).

In Case (ii), $D = \frac{1}{2}N$ does not occur. Here $X = \mathbb{P}^2$, and the ramification locus $R \subset X$ is a line. In light of the exact sequence (6), we may regard $\mathrm{Pic}(Y)$ as the kernel of the matrix $\Psi = (1, -2)$. The first matrix entry is the intersection number $(H \cdot R) = 1$, and the second entry is the negative degree for the gluing map $\nu: R \to C$. Since $\nu^{-1}(N_K) = 2R$, we may regard N_K as the vector $\binom{2}{1}$, which is primitive. Thus $N_K \in \mathrm{Pic}(Y)$ and hence $N \in \mathrm{Pic}(V)$ are generators, and therefore $N = D$. In Case (iii), $D = \frac{1}{2}N$ with $X = \mathbb{P}(1,1,4)$ does not occur for similiar reasons.

Next, we analyze Case (iv). Here $X = S$ is a Hirzebruch surface with numerical invariant $e \geq 0$, and the ramification curve $R \subset X$ is linearly equivalent to $E + F$. Seeking a contradiction, we assume $e \geq 2$. Then $E \cdot (E + F) = -e + 1 < 0$. The exact sequence $0 \to \mathscr{O}_X(F) \to \mathscr{O}_X(E+F) \to \mathscr{O}_E(E+F) \to 0$ gives an exact sequence

$$0 \longrightarrow H^0(X, \mathscr{O}_X(F)) \longrightarrow H^0(X, \mathscr{O}_X(E+F)) \longrightarrow H^0(E, \mathscr{O}_E(E+F)).$$

The term on the right vanishes, and the Projection Formula applied to the ruling $r : X \to \mathbb{P}^1$ gives $H^0(X, \mathscr{O}_X(F)) = H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(1))$, whence R must be a pair of lines, a contradiction. Thus $e = 0$ or $e = 1$.

If $e = 0$, then $X = S = \mathbb{P}^1 \times \mathbb{P}^1$, and the Picard group $\mathrm{Pic}(Y)$ can be regarded as the kernel of the matrix $\Psi = (1, 1, -2)$. The first two entries are the intersection numbers $(E \cdot R) = E \cdot (E + F) = 1$ and $(F \cdot R) = F \cdot (E + F) = 1$, and the last entry

is the negative degree of the gluing map $R \to C$. Since $\nu^*(\omega_Y) = \mathscr{O}_X(-E-F)$ and $\nu^{-1}(N_K) = 2R$, the dualizing sheaf ω_Y is given by the column vector $-(1,1,1)^t$, whereas N_K corresponds to the column vector $(2,2,2)^t$. Thus $\mathscr{O}_V(N)$ is a multiple of ω_V, and we compute the possible cohomological invariants for D as above.

Now suppose the numerical invariant is $e = 1$, that is, $X = S$ is the projective plane blown-up in a rational point. Then $\mathrm{Pic}(Y)$ is the kernel of the matrix $\Psi = (0, 1, -2)$. The first two entries are the intersection numbers $(E \cdot R) = E \cdot (E+F) = 0$ and $(F \cdot R) = F \cdot (E+F) = 1$, and the last entry is the negative degree of the gluing map $R \to C$. Now $\nu^*(\omega_Y) = \mathscr{O}_X(-E-2F)$ and $\nu^{-1}(N_K) = 2R$, so the dualizing sheaf ω_Y corresponds to the column vector $-(1,2,1)^t$, whereas the locus of non-smoothness N_K gives $(2,2,1)^t$. The latter is primitive, so $D = N$. To proceed, consider the invertible sheaf $\mathscr{L} = \omega_V(N)$. Then $\mathscr{L}_Y$ corresponds to the column vector $(1,0,0)^t$, thus $\mathscr{L}_X = \mathscr{O}_X(E)$. The Adjunction Formula gives

$$\deg(\omega_D) = (K_V + D) \cdot D = E \cdot (2E + 2F) = 0.$$

Therefore $h^0(\mathscr{O}_D) = h^1(\mathscr{O}_D)$. To compute this number, consider the short exact sequence $0 \to \mathscr{O}_Y(-N_K) \to \mathscr{O}_Y \to \mathscr{O}_{N_K} \to 0$. It gives a long exact sequence

$$H^1(Y, \mathscr{O}_Y) \longrightarrow H^1(N_K, \mathscr{O}_{N_K}) \longrightarrow H^2(Y, \mathscr{O}_Y(-N_K)) \longrightarrow H^2(Y, \mathscr{O}_Y),$$

in which the outer terms vanish, by Proposition 4.6. Moreover, $h^2(\mathscr{O}_Y(-N_K)) = h^0(\mathscr{L}_Y)$ by Serre Duality. Since $\mathscr{L}_X = \mathscr{O}_X(E)$ we have $\mathscr{L}_R = \mathscr{O}_R$ and $\mathscr{L}_C = \mathscr{O}_C$. As $E \subset X$ is negative-definite, so $h^0(\mathscr{O}_X(E)) = 1$. Using the exact sequence (7), we infer $h^0(\mathscr{L}_Y) = 1$, therefore $h^1(\mathscr{O}_{N_K}) = 1$ and $h^1(\mathscr{O}_D) = 1$.

We come to the final Case (v). Here $X = S$ is a Hirzebruch surface with numerical invariant $e \geq 0$, and the ramification locus is $R = E$. Now we may regard $\mathrm{Pic}(Y)$ as the kernel of the matrix $(-e, 1, -2)$. The first two entries are the intersection numbers $(E \cdot E) = -e$ and $(F \cdot E) = 1$, and the last entry is the negative degree of the gluing map $R \to C$. The dualizing sheaf ω_Y is given by the column vector $-(1, e+2, 1)^t$, whereas $N_K \subset Y$ corresponds to $(2, 0, -e)^t$. Note that we have $N^2 = (2E)^2 = -4e$.

First suppose that the integer $e \geq 0$ is odd. Then the column vector $(2, 0, -e)^t$ is primitive, and it follows that $D = N$. We compute

$$\deg(\omega_D) = (K_Y + N_K) \cdot N_K = (E - (e+2)F) \cdot 2E = -2(2e+2).$$

Then $h^1(\mathscr{O}_D) = 0$ and $h^0(\mathscr{O}_D) = 2e + 2$, by Riemann–Roch. Hence the purely inseparable field extension $F \subset H^0(D, \mathscr{O}_D)$ has degree $2e + 2$. Proposition 5.10 gives $4 = e(\mathscr{O}_{D,\eta}/F) \geq h^0(\mathscr{O}_D) = 2e + 2$, hence $e = 1$ and $h^0(\mathscr{O}_D) = 4$.

Second we assume that $e \geq 0$ is even: the column vector $(2, 0, -e)^t$ attached to the Cartier divisor $N_Y \subset Y$ is therefore divisible. Let $\mathscr{L}$ be the invertible sheaf on V so that $\mathscr{L}_Y$ corresponds to the primitive vector $(1, 0, -e/2)^t$. Then we have $\mathscr{L}_X = \mathscr{O}_X(E)$ and $\mathscr{L}_R = \mathscr{O}_{\mathbb{P}^1}(-e)$ and $\mathscr{L}_C = \mathscr{O}_{\mathbb{P}^1}(-e/2)$. Using the exact sequence (7) we infer that $h^0(\mathscr{L}) = 1$, and any non-zero section $s \in H^0(V, \mathscr{L})$ defines the unique effective Cartier divisor $D = \frac{1}{2}N$. It follows $D^2 = -e$, and the Adjunction Formula gives

$$\deg(\omega_D) = (K_Y + D_K) \cdot D_K = -(e+2)F \cdot E = -2(e/2 + 1) < 0,$$

thus $h^1(\mathscr{O}_D) = 0$ and $h^0(\mathscr{O}_D) = e/2 + 1$. On the other hand, Proposition 5.10 gives $2 = e(\mathscr{O}_{D,\eta}/F) \geq h^0(\mathscr{O}_D) = e/2 + 1$. The only solutions are $e = 0$ or $e = 2$, such that $h^0(\mathscr{O}_D) = 1$ and $h^0(\mathscr{O}_D) = 2$, respectively. $\square$

So far, we have exploited that the local rings $\mathscr{O}_{V,a}$ are factorial. If they are regular, one can say even more.

Proposition 6.3. *Suppose $D = \frac{1}{2}N$. If the del Pezzo surface V is regular, the curve D is regular as well.*

Proof. Consider the finite morphism $\varphi : X \to V$ obtained by composing the normalization $X \to Y = V_K$ with the projection $V_K \to V$. It is flat, because V is regular and X is Cohen–Macaulay (for example [28, Proposition 6.1.5]). We have $\nu^{-1}(N) = 2R$, thus $\nu^{-1}(D) = R$. Since here the scheme R is regular, the scheme D must be regular as well, by [28, Proposition 2.1.13]. □

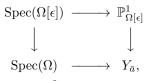Now we relate the geometric information provided by Proposition 6.2 to the arithmetic of the ground field F.

Theorem 6.4. *Suppose the del Pezzo surface V is regular and the ground field F has $\mathrm{pdeg}(F) = 1$. Then V has Picard number $\rho(V) = 2$ and belongs to Case (v) from Reid's Classification, with $e = 2$. In particular, $X = S$ must be a Hirzebruch surface with invariant $e = 2$, and the ramification divisor is $R = E$.*

Proof. We go through the table in Proposition 6.2. Since $\mathrm{pdeg}(F) \leq 1$ the integral curve D has $\mathrm{edim}(\mathscr{O}_{D,\eta}/F) \leq 1$, according to Theorem 1.4. This rules out all cases with $D = N$. So $D = \frac{1}{2}N$, and the integral curve D is regular by Proposition 6.3. If, furthermore, $h^0(\mathscr{O}_D) = 1$, then D is geometrically reduced, by [60, Theorem 2.3]. But this contradicts Corollary 2.7 (compare also Proposition 5.10). The only possibility left is Case (v), with $e = 2$. From Proposition 4.6 we infer that $\rho(Y) = 2$. Since $Y \to V$ is a universal homeomorphism, we also have $\rho(V) = 2$. □

Now suppose we are in Case (v) of Reid's Classification, which is the only possibility if the ground field has $\mathrm{pdeg}(F) = 1$. The Hirzebruch surface $X = S$ comes with a ruling $r : X \to \mathbb{P}^1$, together with the contraction $X \to \mathbb{P}(1,1,2)$ with exceptional divisor $E = R$. The latter factors over the non-normal surface Y, but the former does not. However, if we write $\psi : \mathbb{P}^1 \to \mathbb{P}^1$ for the composition of the gluing map $\nu : R \to C$ and the inverse of $R = E \subset S = X \to \mathbb{P}^1$, we get a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ \nu\ } & Y \\
\downarrow & & \downarrow \\
\mathbb{P}^1 & \xrightarrow[\ \psi\ ]{} & \mathbb{P}^1.
\end{array}
$$

Over each geometric point $\bar{a} : \mathrm{Spec}(\Omega) \to \mathbb{P}^1$, the fibers, $X_{\bar{a}} = \mathbb{P}^1_{\Omega[\epsilon]}$ and $Y_{\bar{a}}$ sit in the conductor square

$$
\begin{array}{ccc}
\mathrm{Spec}(\Omega[\epsilon]) & \longrightarrow & \mathbb{P}^1_{\Omega[\epsilon]} \\
\downarrow & & \downarrow \\
\mathrm{Spec}(\Omega) & \longrightarrow & Y_{\bar{a}},
\end{array}
$$

where ϵ is an indeterminate modulo $\epsilon^2 = 0$. In other words, all geometric fibers $Y_{\bar{a}}$ are split ribbons $\mathbb{P}^1 \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$. Let us call such morphisms *quasirulings*. In other words, these are conic bundles where all fibers are geometrically non-reduced.

Since the projection $Y = V \otimes_F K \to V$ is a universal homeomorphism, the two morphisms $Y \to \mathbb{P}(1, 1, 2)$ and $Y \to \mathbb{P}^1$ are induced from morphisms

$$c : V \longrightarrow W \quad \text{and} \quad f : V \longrightarrow \mathbb{P}^1.$$

The latter indeed goes to the projective line rather than a twisted form because $\mathrm{Pic}(V) = \mathrm{Pic}(Y)$. The image W of the former is a twisted form of the contracted Hirzebruch surface with $W \otimes_F K = \mathbb{P}(1, 1, 2)$. In particular, W is a del Pezzo surface.

Proposition 6.5. *The del Pezzo surface W is regular, and the exceptional divisor for the contraction $c : V \to W$ is $D = \frac{1}{2}N$, which is a projective line $\mathbb{P}^1_{F'}$ over some purely inseparable quadratic field extension $F \subset F'$.*

Proof. According to Proposition 6.2, the exceptional divisor must be the integral curve $D = \frac{1}{2}N$, which has $h^0(\mathscr{O}_D) = 2$ and $h^1(\mathscr{O}_D) = 0$. In turn, $F' = H^0(D, \mathscr{O}_D)$ is a purely inseparable field extension of degree two. Moreover, the curve D is regular, by Proposition 6.3. Since D^2 coincides with $R^2 = -e = -2$, the scheme D contains an F'-rational point, and it follows that $D = \mathbb{P}^1_{F'}$. According to Castelnuovo's Contraction Theorem, [64, Chapter 6, page 102], the curve $D \subset V$ is an exceptional divisor of the first kind, and W is regular. $\square$

We shall see in Section 14 that such regular del Pezzo surfaces W and V actually do exist over any imperfect ground field F of characteristic $p = 2$.

Note that the quasiruling $f : V \to \mathbb{P}^1$ and the contraction $c : V \to W$ exists even without the assumption that $V \to \mathrm{Spec}(F)$ and $F \subset K$ are adapted: If $F \subset F'$ is a Galois extension, the Galois action on $\mathrm{Pic}(V \otimes_F F')$ must be trivial, because it respects the cone of curves, and the two extremal rays have different selfintersection. The only difference is that the range of the quasiruling may be a twisted form of the projective line, rather than the projective line. Such twisted forms correspond to elements of order two in the Brauer group $\mathrm{Br}(F)$.

7. Non-reduced ramification locus in characteristic two

In this section, we study the situation over ground fields F of characteristic $p = 2$ where the ramification divisor $R \subset X$ is non-smooth, hence a split ribbon $R = \mathbb{P}^1 \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$. Notation is as in Section 6: The normal del Pezzo surface V is locally factorial, geometrically integral but geometrically non-normal. We assume that the structure morphism $V \to \mathrm{Spec}(F)$ and the field extension $F \subset K$ are adapted, and consider the normalization $\nu : X \to Y = V_K$ and the ensuing conductor square

$$
\begin{array}{ccc}
R & \longrightarrow & X \\
\downarrow & & \downarrow \nu \\
C & \longrightarrow & Y.
\end{array}
$$

Recall that we have a proper birational morphism $S \to X$ from a Hirzebruch surface S with numerical invariant $e \geq 0$. Furthermore, $N \subset \mathrm{Sing}(V/F)$ denotes the divisorial part of the locus of non-smoothness, and we write $D = N_{\mathrm{red}}$ for its reduction.

Proposition 7.1. *The possible numerical invariants $h^i(\mathscr{O}_D) \geq 0$ of the integral curve D, together with its geometric generic embedding dimension $\mathrm{edim}(\mathscr{O}_{D,\eta}/F)$,*

the relation to the divisor $N \subset V$, and the preimage $\nu^{-1}(D_K) \subset X$ are given by the following table:

Case	(i)			(iii)						
X	$\mathbb{P}^2$			$\mathbb{P}(1,1,e)$						
e	1			2				4		
$\mathscr{O}_X(R)$	$2H$			$2H$				$2H$		
$h^0(\mathscr{O}_D)$	1	2	1	1	2	1	2	1	2	4
$h^1(\mathscr{O}_D)$	7	8	2	3	4	1	2	1	2	4
$\mathrm{edim}(\mathscr{O}_{D,\eta}/F)$	2		1	2		1		2		
D	N		$\frac{1}{2}N$	N		$\frac{1}{2}N$		N		
$\nu^{-1}(D_K)$	$2R$		R	$2R$		R		$2R$		

Proof. According to Proposition 5.7, only Case (i), and Case (iii) with $e = 2$ or $e = 4$ from Reid's Classification in Theorem 5.3 are possible. We may treat them simultaneously, by regarding the normalization as $X = \mathbb{P}(1,1,e)$ with $e = 2^\nu$ with $0 \le \nu \le 2$. In any case, we have $\nu^*(K_Y) = -eH$. According to Theorems 3.4 and 3.6, the preimage $\nu^{-1}(N_K) = 2R$ is linearly equivalent to $4H$. Without restriction, we may assume that $R = 2H$, such that $R_{\mathrm{red}} = H$. Using (6), we have an exact sequence of Néron–Severi groups

$$0 \longrightarrow \mathrm{NS}(Y) \longrightarrow \mathrm{NS}(X) \oplus \mathrm{NS}(C) \xrightarrow{\Psi} \mathrm{NS}(R_{\mathrm{red}}).$$

The induced morphism $\nu : R_{\mathrm{red}} \to C_{\mathrm{red}}$ has degree $d \le 2$, and the group $\mathrm{Pic}(X)$ is freely generated by $eH \subset X$, which has $(eH \cdot R_{\mathrm{red}}) = (eH \cdot H) = 1$. Thus $\mathrm{NS}(Y)$ can be regarded as the kernel of the matrix $\Psi = (1, -d) \in \mathrm{Mat}_{1 \times 2}(\mathbb{Z})$. The numerical class of K_Y is of the form $-\binom{a}{b}$ with $a = 1$ and $-a + db = 0$. The existence of an integral solution forces $d = 1$, so the morphism $\nu : R_{\mathrm{red}} \to C_{\mathrm{red}}$ is birational. Furthermore, the numerical class of K_Y is given by $-\binom{1}{1}$.

The numerical class of N_K is given by the vector $\frac{4}{e}\binom{1}{1}$. We have $H \subset \nu^{-1}(D_K)$, but equality is impossible, because otherwise D_K and whence D would be geometrically reduced. This yields, in each of the three cases $e = 2^\nu$, the indicated two possibilities $D = \frac{1}{d}N$ with $d = 1$ and $d = 2$. Consequently, we have

$$(8) \qquad \nu^{-1}(D_K) = \frac{4}{de}eH \quad \text{and} \quad \nu^{-1}(K_Y) = -eH.$$

Since the coefficient $4/de$ is an integer, the case with $d = 2$ and $e = 4$ is impossible. The Adjunction Formula, base-change, and the Projection Formula yield

$$\deg(\omega_D) = (K_V + D) \cdot D = (K_Y + D_K) \cdot D_K = (\nu^{-1}(K_Y) + \nu^{-1}(D_K)) \cdot \nu^{-1}(D_K).$$

Together with (8) we infer

$$\deg(\omega_D) = \left(-1 + \frac{4}{de}\right)\frac{4}{de}(eH)^2 = \frac{4(4 - de)}{d^2 e}.$$

This yields

$$\chi(\mathscr{O}_D) = -\frac{1}{2}\deg(\omega_D) = \begin{cases} -6 & \text{if } e = 1,\, d = 1; \\ -1 & \text{if } e = 1,\, d = 2; \\ -2 & \text{if } e = 2,\, d = 1; \\ 0 & \text{else.} \end{cases}$$

Using that $h^0(\mathscr{O}_D)$ is a p-power and divides $\chi(\mathscr{O}_D)$, we obtain the possibilities indicated in the table. $\square$

This narrows down further if we take the arithmetic of the ground field F into account.

Proposition 7.2. *If the ground field has* $\mathrm{pdeg}(F) \leq 1$, *then the del Pezzo surface V belongs to Case* (iii) *of Proposition* 7.1, *and the curve $D \subset V$ has numerical invariants* $h^0(\mathscr{O}_D) = h^1(\mathscr{O}_D) = 1$ *or* $h^0(\mathscr{O}_D) = h^1(\mathscr{O}_D) = 2$.

Proof. Without restriction, we may assume that the ground field F is separably closed. Then the length of any finite irreducible F-scheme is a 2-power. According to Theorem 1.4, we must have $\mathrm{edim}(\mathscr{O}_{D,\eta}/F) \leq 1$, which rules out all but three subcases in Proposition 7.1. In these remaining cases, $D = \frac{1}{2}N$ is a proper integral curve inside the del Pezzo surface with geometric generic embedding dimension $\mathrm{edim}(\mathscr{O}_{D,\eta}/F) = 1$. Furthermore, the curve D contains no F-rational points.

In the first subcase, the curve D has numerical invariants $h^0(\mathscr{O}_D) = 1$ and $h^1(\mathscr{O}_D) = 2$. We show that such a curve does not exist: According to [60, Theorem 2.3], the curve D is non-normal. Let $\tilde{D} \to D$ be the normalization map. Since $\tilde{D}$ is normal and $\mathrm{edim}(\mathscr{O}_{\tilde{D},\eta}/F) = 1$, the same result ensures that $h^0(\mathscr{O}_{\tilde{D}}) = 2^\nu$ for some exponent $\nu \geq 1$. In particular, $h^0(\mathscr{O}_{\tilde{D}})$ is an even number. Now consider the conductor square

$$
\begin{array}{ccc}
A & \longrightarrow & \tilde{D} \\
\downarrow & & \downarrow \\
B & \longrightarrow & D.
\end{array}
$$

From this we obtain a long exact sequence

$$0 \to H^0(\mathscr{O}_D) \to H^0(\mathscr{O}_{\tilde{D}}) \oplus H^0(\mathscr{O}_B) \to H^0(\mathscr{O}_A) \to H^1(\mathscr{O}_D) \to H^1(\mathscr{O}_{\tilde{D}}) \to 0.$$

The numbers $h^1(\mathscr{O}_{\tilde{D}})$ and $h^0(\mathscr{O}_A)$ are even because $h^0(\mathscr{O}_{\tilde{D}})$ is even. Since $D(F) = \varnothing$ and F is separably closed, the number $h^0(\mathscr{O}_B)$ is even as well. Thus all terms in the exact sequence have even vector space dimension over the ground field F, except $H^0(\mathscr{O}_D)$, which is one-dimensional. On the other hand, the alternating sum of vector space dimensions in the exact sequence vanishes, a contradiction. $\square$

8. NON-REDUCED RAMIFICATION LOCUS IN CHARACTERISTIC THREE

In this section, we study the situation over ground fields F of characteristic $p = 3$. Notation is as in Section 6: The normal del Pezzo surface V is locally factorial, geometrically integral but geometrically non-normal. We assume that the structure morphism $V \to \mathrm{Spec}(F)$ and the field extension $F \subset K$ are adapted, and consider the normalization $\nu : X \to Y = V_K$ and the ensuing conductor square

$$
\begin{array}{ccc}
R & \longrightarrow & X \\
\downarrow & & \downarrow{\scriptstyle \nu} \\
C & \longrightarrow & Y.
\end{array}
$$

We saw in Propositions 5.5 and 5.8 that the ramification divisor is a split ribbon $R = \mathbb{P}^1 \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$. Recall that we have a proper birational morphism $S \to X$ from a Hirzebruch surface S with numerical invariant $e \geq 0$. Furthermore, $N \subset \mathrm{Sing}(V/F)$

denotes the divisorial part of the locus of non-smoothness, and we write $D = N_{\mathrm{red}}$ for its reduction.

Proposition 8.1. *Assumptions are as above. Then N is reduced, and the curve $D = N$ has $\mathrm{edim}(\mathscr{O}_{D,\eta}/F) = 1$ and $\nu^{-1}(D_K) = \frac{3}{2}R$. Only two cases in Reid's Classification are possible, and the numerical invariants are given by the following table:*

Case	(i)		(iii)	
X	$\mathbb{P}^2$		$\mathbb{P}(1,1,3)$	
e	1		3	
$\mathscr{O}_X(R)$	$2H$		$2H$	
$h^0(\mathscr{O}_D)$	1	3	1	3
$h^1(\mathscr{O}_D)$	4	6	1	3

Proof. The curve N is reduced by Proposition 5.11, with $\mathrm{edim}(\mathscr{O}_{N,\eta}/F) = 1$. We proceed by going through the cases in Reid's Classification. Since the ramification divisor R is the split ribbon, only Case (iii) with $e = 3$ and Case (i) from Theorem 5.3 are possible. We may treat them simultaneously, by regarding the normalization as $X = \mathbb{P}(1,1,e)$ with $e = 3$ and $e = 1$, respectively.

The split ribbon has $h^0(\mathscr{O}_R) = 1$ and $h^1(\mathscr{O}_R) = 0$, and the del Pezzo surface has $h^2(\mathscr{O}_Y) = 0$. Proposition 4.6 gives an exact sequence

$$0 \longrightarrow \mathrm{NS}(Y) \longrightarrow \mathrm{NS}(X) \oplus \mathrm{NS}(C) \xrightarrow{\Psi} \mathrm{NS}(R).$$

Furthermore, the Weil divisor $R_{\mathrm{red}} \subset X$ is linearly equivalent to the line $H \subset X$, and the induced morphism $\nu : R_{\mathrm{red}} \to C$ is birational. Furthermore, the restriction map $\mathrm{Pic}(R) \to \mathrm{Pic}(R_{\mathrm{red}})$ is bijective. The group $\mathrm{Pic}(X)$ is freely generated by $eH \subset X$, which has $(eH \cdot R_{\mathrm{red}}) = (eH \cdot H) = 1$. Thus $\mathrm{NS}(Y)$ can be regarded as the kernel of the matrix $\Psi = (1, -1) \in \mathrm{Mat}_{1 \times 2}(\mathbb{Z})$. Then $\nu^*(K_Y) = -eH$ generates the Picard group $\mathrm{Pic}(X)$, and $\nu^{-1}(N_K)$ is linearly equivalent to $3H = \frac{3}{e} \cdot eH$. The numerical class of ω_Y corresponds to the vector $-\binom{1}{1}$, whereas the numerical class of N_K is given by the vector $\frac{3}{e}\binom{1}{1}$. The Adjunction Formula for $D = N$ yields

$$\deg(\omega_D) = (K_V + D) \cdot D = (K_Y + N_K) \cdot N_K = \left(-eH + \frac{3}{e}eH\right) \cdot \frac{3}{e}eH = 3(3/e - 1).$$

In Case (i) we have $e = 1$ and thus $\deg(\omega_D) = 6$. Now we use that $h^0(\mathscr{O}_D)$ is a p-power that divides $\chi(\mathscr{O}_D) = -\frac{1}{2}\deg(\omega_D) = -3$, and get the indicated values for $h^i(\mathscr{O}_D)$. In Case (iii) we have $e = 3$, thus $\deg(\omega_D) = 0$ and $h^0(\mathscr{O}_D) = h^1(\mathscr{O}_D)$. Here $h^0(\mathscr{O}_D)$ is a p-power that divides the selfintersection number $D^2 = (eH)^2 = 3$, which again gives the possible values for $h^i(\mathscr{O}_D)$. $\square$

We proceed to rule out all the two subcases with $h^0(\mathscr{O}_D) > 1$. This relies on a general result of Maddock on normal del Pezzo surfaces with irregularity $h^1(\mathscr{O}_V) > 0$: According to [43, Corollary 1.4] the bound

$$(9) \qquad\qquad h^1(\mathscr{O}_V) \geq \frac{p^2 - 1}{6} K_V^2$$

then holds. Let us check that this result indeed applies.

Proposition 8.2. *Our del Pezzo surface V has irregularity $h^1(\mathscr{O}_V) > 0$ provided the curve $D = N$ has $h^0(\mathscr{O}_D) > 1$.*

Proof. Seeking a contradiction we assume $h^1(\mathscr{O}_V) = 0$ and thus $h^1(\mathscr{O}_C) = 0$. The short exact sequence $0 \to \mathscr{O}_V(-D) \to \mathscr{O}_V \to \mathscr{O}_D \to 0$ yields a long exact sequence

$$(10) \qquad H^1(V, \mathscr{O}_V) \longrightarrow H^1(D, \mathscr{O}_D) \longrightarrow H^2(V, \mathscr{O}_V(-D)) \longrightarrow H^2(V, \mathscr{O}_V),$$

where the outer terms vanish. Together with Serre duality we get $h^1(\mathscr{O}_D) = h^0(\mathscr{L})$ for the invertible sheaf $\mathscr{L} = \omega_V(D)$. In case $X = \mathbb{P}(1,1,3)$ we have $h^1(\mathscr{O}_D) = 3$ and $\mathscr{L}_X = \mathscr{O}_X$, whence $\mathscr{L}$ is numerically trivial and $h^0(\mathscr{L}) \le 1$, a contradiction.

Now suppose we are in case $X = \mathbb{P}^2$. Then $\mathscr{L}_X = \mathscr{O}_{\mathbb{P}^2}(2)$. Since the morphism $R_{\mathrm{red}} \to C$ is birational, we have $\deg(\mathscr{L}_C) = 2$ and with Riemann–Roch $h^0(\mathscr{L}_C) = 3$. The short exact sequence $0 \to \mathscr{L}_Y \to \mathscr{L}_X \oplus \mathscr{L}_C \to \mathscr{L}_R \to 0$ yields an exact sequence

$$0 \longrightarrow H^0(Y, \mathscr{L}_Y) \longrightarrow H^0(X, \mathscr{L}_X) \oplus H^0(C, \mathscr{L}_C) \longrightarrow H^0(R, \mathscr{L}_R).$$

Since $\mathscr{L}_X(-R) = \mathscr{O}_{\mathbb{P}^2}$ we see that the restriction map $H^0(X, \mathscr{L}_X) \to H^0(R, \mathscr{L}_R)$ is surjective, with one-dimensional kernel. So $h^1(\mathscr{O}_D) = h^0(\mathscr{L}) = 1 + h^0(\mathscr{L}_C) = 4$. Hence we are in the case of Proposition 8.1 where $h^0(\mathscr{O}_D) = 1$, a contradiction. $\square$

Proposition 8.3. *The two subcases in Proposition 8.1 with $h^0(\mathscr{O}_D) > 1$ do not exist.*

Proof. The inclusion $C \subset D_K \subset Y$ yields a commutative diagram

$$(11)$$

$$\begin{array}{ccccccc} H^1(Y, \mathscr{O}_Y) & \longrightarrow & H^1(D_K, \mathscr{O}_{D_K}) & \longrightarrow & H^2(Y, \mathscr{O}_Y(-D_K)) & \longrightarrow & 0 \\ & \searrow^{\simeq} & \downarrow & & & & \\ & & H^1(C, \mathscr{O}_C) & & & & \end{array}$$

with exact row. The vertical map is surjective, and the diagonal map is bijective. Serre duality gives $h^2(\mathscr{O}_Y(-D_K)) = h^0(\mathscr{L}_Y)$ for the invertible sheaf $\mathscr{L} = \omega_V(D)$.

First suppose that we are in Case (iii) of Proposition 8.1. Then $K_V^2 = 3$. Moreover, $\mathscr{L}$ is numerically trivial, thus $h^0(\mathscr{L}) \le 1$, and $h^1(\mathscr{O}_D) = 3$. It follows that $h^1(\mathscr{O}_V) = h^1(\mathscr{O}_C) \le 3$. On the other hand, Maddock's bound (9) gives $h^1(\mathscr{O}_V) \ge 8/6 \cdot K_V^2 = 4$, a contradiction.

Finally suppose we are in Case (i) of Proposition 8.1. Then $K_V^2 = 1$, so Maddock's bound ensures $h^1(\mathscr{O}_V) \ge 2$. On the other hand, the ring of global sections $F' = H^0(D, \mathscr{O}_D)$ is a purely inseparable field extension $F \subset F'$ of degree $[F' : F] = 3$, and the cohomology group $H^1(D, \mathscr{O}_D)$ is a two-dimension vector space over F'. In turn, $H^1(D_K, \mathscr{O}_{D_K})$ is a free module of rank two over the local Artin ring $A' = F' \otimes_F K$, which has length two. The inclusion $C \subset D_K$ of curves induces a surjection $A' = H^0(D_K, \mathscr{O}_{D_K}) \to H^0(C, \mathscr{O}_C)$ whose kernel is an ideal $\mathfrak{a} \subset A'$ of length one. This ideal annihilates the image of the surjection $A' \oplus A' \simeq H^1(D_K, \mathscr{O}_{D_K}) \to H^1(C, \mathscr{O}_C)$, and we conclude that $\dim_K H^1(C, \mathscr{O}_C) \le 2$. In turn, our del Pezzo surface V has irregularity $h^1(\mathscr{O}_V) \le 2$. Summing up, we have $h^1(\mathscr{O}_C) = h^1(\mathscr{O}_V) = 2$.

Using the exact sequence in (11), we conclude that $h^0(\mathscr{L}) \ge 6 - 2 = 4$. On the other hand, the arguments in the proof for Proposition 8.2 show that $h^0(\mathscr{L}) = 1 + h^0(\mathscr{L}_C)$. According to Proposition 3.3, the conductor curve C is integral. In our situation we have $h^0(\mathscr{O}_C) = 1$ and $h^1(\mathscr{O}_C) = 2$, and $\deg(\mathscr{L}_C) = 2$. Riemann–Roch gives $\chi(\mathscr{L}_C) = 1$. Furthermore, $\deg(\omega_C) = 2$, hence $\omega_C \otimes \mathscr{L}_C^\vee$ is numerically trivial, and $h^1(\mathscr{L}_C) \le 1$. Combining these observations we get $h^0(\mathscr{L}_C) = \chi(\mathscr{L}_C) + h^1(\mathscr{L}_C) \le 2$, and therefore $h^0(\mathscr{L}) \le 3$, a contradiction. $\square$

Proposition 8.4. *In Case* (iii) *of Proposition* 8.1 *with* $h^1(\mathscr{O}_D) = 1$, *the del Pezzo surface has* $H^1(V, \mathscr{O}_V) = 0$.

Proof. Seeking a contradiction, we assume that $h^1(\mathscr{O}_V) > 0$. According to Maddock's bound (9), the irregularity $h^1(\mathscr{O}_V)$ is at least two. Now recall that the restriction map $H^1(Y, \mathscr{O}_Y) \to H^1(C, \mathscr{O}_C)$ is bijective. The inclusion of curves $C \subset D_K$ induces a surjection $H^1(D_K, \mathscr{O}_{D_K}) \to H^1(C, \mathscr{O}_C)$ on cohomology groups, and we thus have $h^1(\mathscr{O}_D) \geq 2$, a contradiction. $\square$

9. Some peculiar genus-one curves

In this section we study some rather peculiar algebraic curves of arithmetic genus one. These curves showed up on certain geometrically non-normal del Pezzo surfaces for $p = 2$ and $p = 3$, and their structure will lead to non-existence results. These genus-one curves, however, are of interest in all characteristics.

Let F be an imperfect ground field of characteristic $p > 0$, and let $F \subset \tilde{F}$ be a purely inseparable extension of degree $[\tilde{F} : F] = p$. Choose an element $\beta \in F^\times$ so that $\tilde{F} = F(\beta^{1/p})$. Set $\tilde{D} = \mathbb{P}^1_{\tilde{F}}$, write $a = (0 : 1)$ for the origin, and let $A \subset \mathbb{P}^1_{\tilde{F}}$ be its first infinitesimal neighborhood. The structure sheaf $\mathscr{O}_A$ is a skyscraper sheaf supported by $a \in \tilde{D}$. By abuse of notation, we also write $\mathscr{O}_A$ when we mean the stalk $\mathscr{O}_{A,a}$ or the ring of global sections $H^0(A, \mathscr{O}_A)$. This said, we may write

$$\mathscr{O}_A = \tilde{F}[\epsilon] = \tilde{F} \oplus \tilde{F}\epsilon.$$

As customary, ϵ denotes an indeterminate subject to $\epsilon^2 = 0$. Consider the F-subalgebra $\mathscr{O}_B$ generated by $\beta^{1/p} + \epsilon \in \mathscr{O}_A$. Using $(\beta^{1/p} + \epsilon)^p = \beta + \epsilon^p = \beta$, we see that the resulting map $\tilde{F} \to \mathscr{O}_B$ given by $\beta^{1/p} \mapsto \beta^{1/p} + \epsilon$ is bijective. The crucial point here is that the images of $H^0(\mathscr{O}_{\tilde{D}})$ and $H^0(\mathscr{O}_B)$ define two different copies of the field $\tilde{F}$ inside the F-algebra $H^0(\mathscr{O}_A)$, which become the same after projecting onto the residue field $\kappa(a) = \tilde{F}$. In other words, the construction relies on the non-uniqueness of coefficient fields. The inclusion $\mathscr{O}_B \subset \mathscr{O}_A$ yields a finite morphism $A \to B$, and the resulting cocartesian square

$$\begin{array}{ccc}
A & \longrightarrow & \tilde{D} \\
\downarrow & & \downarrow{\scriptstyle \nu} \\
B & \longrightarrow & D
\end{array}$$

(12)

defines a proper integral curve D containing a unique singularity $b = \nu(a) \in D$ with residue field $\kappa(b) = \tilde{F}$. The normalization map $\nu : \tilde{D} \to D$ is a universal homeomorphism, and the cocartesian square yields a long exact sequence

(13) $0 \to H^0(\mathscr{O}_D) \to H^0(\mathscr{O}_{\tilde{D}}) \oplus H^0(\mathscr{O}_B) \to H^0(\mathscr{O}_A) \to H^1(\mathscr{O}_D) \to 0,$

from which one can deduce the structure of the curve.

Proposition 9.1. *The curve D is Gorenstein and has cohomological invariants* $h^0(\mathscr{O}_D) = 1$ *and* $h^1(\mathscr{O}_D) = 1$. *Moreover, the reduced base-change* $(D \otimes_F \tilde{F})_{\mathrm{red}}$ *is isomorphic to the projective line* $\mathbb{P}^1_{\tilde{F}}$. *For $p = 2$, the base-change $D \otimes_F \tilde{F}$ is the split ribbon on $\mathbb{P}^1_{\tilde{F}}$ with respect to the ideal sheaf $\mathscr{O}_{\mathbb{P}^1_{\tilde{F}}}(-2)$.*

Proof. The length of $\mathscr{O}_A$ as an $\mathscr{O}_B$-module is two, hence D is Gorenstein, by Proposition A.3. The conductor square (12) gives

$$\chi(\mathscr{O}_D) = \chi(\mathscr{O}_{\tilde{D}}) + \chi(\mathscr{O}_B) - \chi(\mathscr{O}_A) = p + p - 2p = 0,$$

and consequently $h^1(\mathscr{O}_D) = h^0(\mathscr{O}_D)$. Consider the field extensions $F \subset H^0(\mathscr{O}_D) \subset H^0(\mathscr{O}_{\tilde{D}}) = \tilde{F}$, which has degree p. Suppose the first inclusion is not an equality. Then the second inclusion must be an equality, and this also holds for the inclusion $H^0(\mathscr{O}_D) \subset H^0(\mathscr{O}_B)$. By the exactness of (13), the two maps

$$H^0(\mathscr{O}_D) = H^0(\mathscr{O}_{\tilde{D}}) \longrightarrow H^0(\mathscr{O}_A) \longleftarrow H^0(\mathscr{O}_B)$$

must have the same image. But the former map factors over the first summand in $\mathscr{O}_A = \tilde{F} \oplus \tilde{F}\epsilon$, whereas the latter does not, a contradiction. Thus we have $H^0(\mathscr{O}_D) = F$, and hence $h^1(\mathscr{O}_D) = h^0(\mathscr{O}_D) = 1$.

To understand the base-change, write $\mathbb{P}^1_{\tilde{F}} = \operatorname{Proj} \tilde{F}[U, V]$, and look at the affine open subset $D_+(V) = \operatorname{Spec} \tilde{F}[u]$ with $u = U/V$. The conductor square (12) yields the cartesian diagram

(14)
$$
\begin{array}{ccc}
\tilde{F}[u]/(u^2) & \longleftarrow & \tilde{F}[u] \\
\uparrow & & \uparrow \\
F[\beta^{1/p} + u] & \longleftarrow & R,
\end{array}
$$

where $\operatorname{Spec}(R)$ defines an affine open neighborhood of the singularity $b \in D$. Base-changing the diagram along $F \subset \tilde{F}$, we get a new cartesian diagram

$$
\begin{array}{ccc}
\tilde{F} \otimes_F \tilde{F}[u]/(u^2) & \longleftarrow & \tilde{F} \otimes_F \tilde{F}[u] \\
\uparrow & & \uparrow \\
\tilde{F} \otimes_F F[\beta^{1/p} + u] & \longleftarrow & \tilde{F} \otimes_F R.
\end{array}
$$

Clearly, the birational morphism $\varphi : \mathbb{P}^1_{\tilde{F}} = (\tilde{D}_{\tilde{F}})_{\mathrm{red}} \to (D_{\tilde{F}})_{\mathrm{red}}$ is an isomorphism over the complement of the the point $\tilde{b} \in D_{\tilde{F}}$ corresponding to the singularity $b \in D$. The schematic fiber $\varphi^{-1}(\tilde{b}) \subset \mathbb{P}^1_{\tilde{F}}$ is given by the spectrum $\tilde{F} \otimes_F \tilde{F}[u]/(u^2)$ modulo the ideal generated by

$$\beta^{1/p} \otimes 1 - 1 \otimes \beta^{1/p} \quad \text{and} \quad \beta^{1/p} \otimes 1 - 1 \otimes (\beta^{1/p} + \epsilon).$$

Clearly, they generate the maximal ideal of the local Artin ring $\tilde{F} \otimes_F \tilde{F}[u]/(u^2)$, and we conclude that $(D_{\tilde{F}})_{\mathrm{red}} = \mathbb{P}^1_{\tilde{F}}$.

Finally, suppose $p = 2$. Then the nilradical $\mathscr{I}$ for the scheme $D_{\tilde{F}}$ is torsion-free of rank one, with $\mathscr{I}^2 = 0$, whence $\mathscr{I} = \mathscr{O}_{\mathbb{P}^1}(n)$ for some integer n. We have $n = -2$ because $\chi(\mathscr{I}) = -1$. Such a ribbon is necessarily split, because $\operatorname{Ext}^1(\Omega^1_{\mathbb{P}^1}, \mathscr{O}_{\mathbb{P}^1}(-2)) = 0$. $\qquad\square$

The curve D does not seem to have a nice description in terms of equations. However, the situation improves when we pass to more singular models.

Proposition 9.2. *There is a finite birational universal homeomorphism $D \to D'$ to the effective Cartier divisor $D' \subset \mathbb{P}(1, 1, p)$ in weighted projective space defined by the homogeneous equation $t^p - \beta(x^p - \beta y^p)^p = 0$ of degree p^2. In characteristic two, this birational morphism is actually an isomorphism.*

Proof. Let us continue to use the notation from the preceding proof. From the cartesian square (14) we see that the subring $R \subset \tilde{F}[u]$ comprises all polynomials of the form

$$\lambda(\beta^{1/p} + u) + \lambda_2 u^2 + \ldots + \lambda_n u^n$$

with coefficients $\lambda \in F$ and $\lambda_i \in \tilde{F}$. Consider the two particular polynomials

(15) $$x = \beta^{1/p} + u \quad \text{and} \quad t = \beta^{1/p} u^p \in R,$$

which satisfy the relation $\beta x^{p^2} = \beta^{p+1} + t^p$, hence

(16) $$t^p - \beta(x^p - \beta)^p = 0.$$

Regarding x, t as indeterminates and using the Reduction Criterion for the field of fraction $F[x, t] \subset F(x, t)$, we see that the polynomial $t^p - \beta(x^p - \beta)^p$ is irreducible. It follows that the canonical map

$$R' = F[x, t]/(t^p - \beta(x^p - \beta)^p) \longrightarrow R \subset \tilde{F}[u]$$

is injective. To see that the resulting inclusion of function fields is bijective, consider as denominator $s = x^p - \beta$. Then $(t/s)^p = \beta$, such that $(R')_s = \tilde{F}[x] = \tilde{F}[u]$. We conclude that the morphism $\mathrm{Spec}(R) \to \mathrm{Spec}(R')$ is a universal homeomorphism, and becomes an isomorphism upon removal of the singular point $b \in \mathrm{Spec}(R)$ and its image $b' \in \mathrm{Spec}(R')$.

Homogenization of (16) yields the homogeneous equation $T^p - \beta(X^p - \beta Y^p)^p = 0$, where the degrees of the generators are $\deg(X) = \deg(Y) = 1$ and $\deg(T) = p$. Inside the weighted projective space $\mathbb{P} = \mathbb{P}(1, 1, p)$, it defines a closed subscheme D'. This is a Cartier divisor, because the coherent sheaf $\mathscr{O}_{\mathbb{P}}(p^2)$ is invertible. It lies in the smooth locus of $\mathbb{P}(1, 1, p)$, because $(0 : 0 : 1)$ is the only singular point of the weighted projective space. By construction, we have an identification $D' \cap D_+(Y) = \mathrm{Spec}(R')$. In particular, the scheme D' is regular outside the singular point b'. The situation is similar on $D' \cap D_+(X)$, and we infer $\mathrm{Sing}(D') = \{b'\}$. Summing up, the morphism $D \to D'$ yields an isomorphism between the regular loci.

Now consider the case $p = 2$. Here we claim that the inclusion $R' \subset R$ is actually an equality. Indeed, we have

$$u^2 = x^2 - \beta, \quad \beta^{1/2} u^2 = t, \quad \text{and} \quad \beta^{1/2} u^3 = xt + \beta u^2,$$

which implies $\beta^{1/2} u^n \in R'$ for all $n \geq 2$, and thus $R' = R$. Thus our birational morphism $D \to D'$ is an isomorphism. $\qquad\square$

Conversely, given some $\beta \in F^\times$ that is not a p-power, we can consider the curve D' defined by the homogeneous equation $T^p - \beta(X^p - \beta Y^p)^p = 0$ inside the weighted projective space $\mathbb{P} = \mathbb{P}(1, 1, p)$. We already observed that the homogeneous polynomial is irreducible, such that the scheme D' is integral.

Proposition 9.3. *The curve D' has cohomological invariants*

$$h^0(\mathscr{O}_{D'}) = 1 \quad \text{and} \quad h^1(\mathscr{O}_{D'}) = (p^3 - p^2 - 2p + 2)/2.$$

In characteristic $p = 2$, this means $h^0(\mathscr{O}_{D'}) = h^1(\mathscr{O}_{D'}) = 1$.

Proof. Consider the long exact cohomology sequence coming from the short exact sequence $0 \to \mathscr{O}_{\mathbb{P}}(-p^2) \to \mathscr{O}_{\mathbb{P}} \to \mathscr{O}_{D'} \to 0$. It starts with

$$H^0(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(-p^2)) \longrightarrow H^0(\mathbb{P}, \mathscr{O}_{\mathbb{P}}) \longrightarrow H^0(D', \mathscr{O}_{D'}) \longrightarrow H^1(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(-p^2)).$$

The outer terms vanish, and we get $h^0(\mathscr{O}_{D'}) = 1$. The long exact sequence continues with

$$H^1(\mathbb{P}, \mathscr{O}_{\mathbb{P}}) \longrightarrow H^1(D', \mathscr{O}_{D'}) \longrightarrow H^2(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(-p^2)) \longrightarrow H^2(\mathbb{P}, \mathscr{O}_{\mathbb{P}}).$$

Again, the outer terms vanish, according to [21, Theorem in Section 1.4], and Serre duality gives $h^2(\mathscr{O}_{\mathbb{P}}(-p^2)) = h^0(\mathscr{O}_{\mathbb{P}}(n))$ with $n = p^2 - p - 2 = (p+1)(p-2)$. The

vector space $H^0(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(n))$ equals the homogeneous component of degree n in the graded ring $S = F[X, Y, T]$. A basis for this vector space is given by the monomials $X^i Y^j T^k$ with exponents satisfying

$$i + j + pk = (p+1)(p-2).$$

This equation has solutions only for $0 \leq k \leq p-2$, and for fixed such k there are $s_k = (p+1)(p-2) - pk + 1$ solutions. It follows that $h^1(\mathscr{O}_{D'})$ equals

$$\sum_{k=0}^{p-2} s_k = (p-1)(p+1)(p-2) - p\frac{(p-1)(p-2)}{2} + (p-1) = (p^3 - p^2 - 2p + 2)/2.$$

Setting $p = 2$ gives the value $h^1(\mathscr{O}_{D'}) = 1$. □

The Jacobian Criterion immediately reveals that D' contains no smooth point. The subscheme $D' \subset \mathbb{P}(1, 1, p)$ lies in $D_+(X) \cup D_+(Y)$, thus is contained in the smooth locus of the weighted projective space. The intersections $D' \cap V_+(Y)$ and $D' \cap V_+(X)$ are the two non-rational points $(1 : 0 : \beta^{1/p})$ and $(0 : 1 : -\beta^{1+1/p})$, respectively. Both residue fields are isomorphic to $\tilde{F}$, and the subscheme is reduced, hence the corresponding local rings are regular. To understand the singular locus of D', it suffices to look at the affine chart $D_+(Y)$, which is the spectrum of $k[t, x]$, with $t = T/Y^p$ and $x = X/Y$, and our homogeneous equation becomes

$$t^p - \beta(x^p - \beta)^p = 0.$$

Let u be new indeterminate, and consider the polynomial ring $\tilde{F}[u]$ over the field extension $\tilde{F} = F(\beta^{1/p})$. Then (15) defines an injective homomorphism of F-algebras

(17) $$R' = F[x, t]/(t^p - \beta(x^p - \beta)^p) \longrightarrow \tilde{F}[u].$$

Localizing with denominator $s = x^p - \beta$, one sees that

$$t/s \longmapsto \beta^{1/p} \quad \text{and} \quad x - t/s \longmapsto u,$$

such that $(R')_s = \tilde{F}[u]_s$. It follows that the normalization of $D' \subset \mathbb{P}(1, 1, p)$ is given by (17). In particular, $b' = (\beta^{1/p} : 1 : 0)$ is the only singular point, and its preimage $a \in \tilde{D}$ is given by the origin $a = (0 : 1) \in \mathbb{P}^1_{\tilde{F}}$. We now consider the normalization $\tilde{D} \to D'$ and the resulting cartesian square

$$
\begin{array}{ccc}
A' & \longrightarrow & \tilde{D} \\
\downarrow & & \downarrow \\
B' & \longrightarrow & D',
\end{array}
$$

where $B' \subset D'$ is defined by the conductor ideal, and $A' \subset \tilde{D}$ is the preimage.

Proposition 9.4. *The ramification locus $A' \subset \tilde{D}$ is given by $\mathscr{O}_{A'} = \tilde{F}[u]/(u^{p(p-1)})$, the F-subalgebra $\mathscr{O}_{B'} \subset \mathscr{O}_{A'}$ is generated by $\beta^{1/p} + u$ and $\beta^{1/p} u^p$, and the morphisms $\nu' : \tilde{D} \to D'$ factor uniquely over $\nu : \tilde{D} \to D$.*

Proof. The conductor square gives a long exact cohomology sequence

$$0 \to H^0(\mathscr{O}_{D'}) \to H^0(\mathscr{O}_{\tilde{D}}) \oplus H^0(\mathscr{O}_{B'}) \to H^0(\mathscr{O}_{A'}) \to H^1(\mathscr{O}_{D'}) \to 0.$$

We have $h^0(\mathscr{O}_{\tilde{D}}) = p$ and $h^0(\mathscr{O}_{A'}) = pl$ for some integer $l \geq 1$, and our task is to verify $l = p^2 - p$. Being a complete intersection, the curve D' is Gorenstein, thus

$h^0(\mathscr{O}_{B'}) = pl/2$. Taking alternating vector space dimension, the above sequence yields

$$1 - (p + pl/2) + pl - (p^3 - p^2 - 2p + 2)/2 = 0,$$

and the first assertion follows. Since $\mathscr{O}_{D'} \to \mathscr{O}_{B'}$ is surjective and $\mathscr{O}_{B'} \to \mathscr{O}_{A'}$ is injective, the second assertion follows from the description (15) of the morphism (17). For the last assertion, we use the universal properties of cocartesian squares of schemes and the corresponding cartesian squares of rings: The image of the composite map $\mathscr{O}_{D'} \subset \mathscr{O}_{\tilde{D}} \to \mathscr{O}_{A'} \to \mathscr{O}_A$ is generated by $\beta^{1/p} + u$, and thus factors over the subring $\mathscr{O}_B \subset \mathscr{O}_A$. □

We now establish a structure result on certain algebraic curves of arithmetic genus one. Recall that for ground fields with $\mathrm{pdeg}(F) = 1$, there is precisely one purely inseparable field extension $\tilde{F}$ with $[\tilde{F} : F] = p$, namely $\tilde{F} = F^{1/p}$.

Theorem 9.5. *Let D be a proper integral curve with cohomological invariants $h^0(\mathscr{O}_D) = h^1(\mathscr{O}_D) = 1$. Assume D in not regular, contains no rational points, and the local rings $\mathscr{O}_{D,a}$ are geometrically unibranch. Assume further that $\mathrm{Pic}^p_{D/F}$ contains a rational point, and that the ground field has $\mathrm{pdeg}(F) = 1$. Then the curve D is given by a cocartesian square as in (12), for some non-zero scalar $\beta \in \tilde{F} = F^{1/p}$. For $p = 2$, this D is isomorphic to the curve inside $\mathbb{P}(1,1,2)$ given by the homogeneous equation $t^2 - \beta(x^2 - \beta y^2)^2 = 0$.*

Proof. Let $\nu : \tilde{D} \to D$ be the normalization map, with conductor square

(18)
$$\begin{array}{ccc} A & \longrightarrow & \tilde{D} \\ \downarrow & & \downarrow{\scriptstyle\nu} \\ B & \longrightarrow & D. \end{array}$$

Then $\tilde{D}$ is another proper integral curve, and we write $\tilde{F} = H^0(\tilde{D}, \mathscr{O}_{\tilde{D}})$ for the field of global sections. This gives a finite field extension $F \subset \tilde{F}$. Let

(19) $0 \to H^0(\mathscr{O}_D) \to H^0(\mathscr{O}_{\tilde{D}}) \oplus H^0(\mathscr{O}_B) \to H^0(\mathscr{O}_A) \to H^1(\mathscr{O}_D) \to H^1(\mathscr{O}_{\tilde{D}}) \to 0$

be the exact sequence resulting from the conductor square. Our proof proceeds in a sequence of little steps.

Step 1. We have $h^0(\mathscr{O}_{\tilde{D}}) = p$, $h^1(\mathscr{O}_{\tilde{D}}) = 0$, $h^0(\mathscr{O}_B) = p$, and $h^0(\mathscr{O}_A) = 2p$. To see this, set $d = h^0(\mathscr{O}_{\tilde{D}}) = [\tilde{F} : F]$ for the degree of the field extension $F \subset \tilde{F}$, and write $h^0(\mathscr{O}_A) = dl$ for some integer $l \geq 1$. Since the curve D is Gorenstein, we then have $h^0(\mathscr{O}_B) = dl/2$. Clearly, $h^1(\mathscr{O}_{\tilde{D}}) \leq 1$. If equality holds, then $d = 1$. Using that the alternating sum of vector space dimensions in (19) vanishes, we get $l = 0$, a contradiction. Therefore $h^1(\mathscr{O}_{\tilde{D}}) = 0$. In turn, we have $(d + dl/2) - dl = 0$, hence $l = 2$. If $d = 1$ we get $h^0(\mathscr{O}_B) = 1$, which implies that D contains a rational point, a contradiction. This shows $d > 1$. In order to check $d = p$, we may base-change to the separable closure of the ground field F, and assume that F is separably closed. Since $h^0(\mathscr{O}_D) = 1$ we have an exact sequence

$$\mathrm{Pic}(D) \longrightarrow \mathrm{Pic}_{D/F}(F) \longrightarrow \mathrm{Br}(F).$$

Since the Brauer group vanishes, the rational point on $\mathrm{Pic}^p_{D/F}$ comes from an invertible sheaf on D of degree p, and its preimage $\mathscr{L}$ on $\tilde{D}$ likewise has $\chi(\mathscr{L}) - \chi(\mathscr{O}_{\tilde{D}}) = \deg(\mathscr{L}) = p$. This degree is a multiple of $d > 1$, and we conclude $d = p$.

Step 2. *The field extension* $F \subset \tilde{F}$ *is purely inseparable.* Suppose this does not hold. Choose some $\alpha \in \tilde{F}$ whose minimal polynomial $f \in F[T]$ has at least two roots $\omega_1 \neq \omega_2$ in some algebraic closure Ω. This gives a subfield $F[T]/(f) \subset \tilde{F}$ such that the spectrum of $\tilde{F} \otimes_F \Omega$ is disconnected. It follows that the curve $\tilde{D}$ is geometrically reducible, whence the same holds for D. On the other hand, D is geometrically connected because $h^0(\mathscr{O}_D) = 1$. In turn, there must be a closed point on $D \otimes_F \Omega$ whose local ring has reducible spectrum. Its image $a \in D$ is a closed point where the local ring $\mathscr{O}_{D,a}$ is not geometrically unibranch, a contradiction.

Step 3. *The scheme* A *contains only one point.* Suppose this is not the case. Then the condition $h^0(\mathscr{O}_A) = 2p$ implies that $\mathscr{O}_A = \tilde{F} \times \tilde{F}$, hence the subring $\mathscr{O}_B \subset \mathscr{O}_A$ is reduced and we have $A = \{a_1, a_2\}$. The scheme B contains at least one and at most two points. The case $B = \{b\}$ is impossible, because then the local ring $\mathscr{O}_{D,b}$ is not unibranch. Thus $B = \{b_1, b_2\}$, such that $\mathscr{O}_B = F_1 \times F_2$ for some intermediate fields $F \subset F_i \subset \tilde{F}$. Since the local rings $\mathscr{O}_{D,b_i}$ are Gorenstein, we have $[\tilde{F} : F_i] = [F_i : F] = p/2$, which forces $p = 2$ and $[F_i : F] = 1$. The latter implies that D contains rational points, a contradiction.

Step 4. *We have* $\mathscr{O}_A = \tilde{F}[\epsilon]$ *for some indeterminate* ϵ *subject to* $\epsilon^2 = 0$. The task is to show that A is non-reduced. Seeking a contradiction, we assume that A is integral. Since $\mathrm{pdeg}(F) = 1$, we must have $\mathscr{O}_A = F^{1/p^2}$, and there is but one intermediate field between $F \subset \mathscr{O}_A$, namely $F^{1/p}$. The difference map $H^0(\mathscr{O}_{\tilde{D}}) \oplus H^0(\mathscr{O}_B) \to H^0(\mathscr{O}_A)$ is not surjective, by the exact sequence (19), hence its image is the intermediate field, and the cokernel $H^1(\mathscr{O}_D)$ has dimension p, a contradiction.

Step 5. *The composite map* $\mathscr{O}_B \subset \mathscr{O}_A \to \mathscr{O}_A/\mathfrak{m}_A$ *is bijective.* Consider the inclusions of fields $F \subset \mathscr{O}_B/\mathfrak{m}_B \subset \mathscr{O}_A/\mathfrak{m}_A = \tilde{F}$. The first inclusion is strict because D contains no rational points. The composite extension has degree p, hence $\mathscr{O}_B/\mathfrak{m}_B = \tilde{F}$. Since $h^0(\mathscr{O}_B) = p$, we must have $\mathfrak{m}_B = 0$.

Step 6. *The normal curve* $\tilde{D}$ *is isomorphic to* $\mathbb{P}^1_{\tilde{F}}$. We saw in Step 4 that there is an invertible sheaf $\mathscr{L}$ on $\tilde{D}$ of degree $\deg(\mathscr{L}) = p$. Let us temporarily regard $\tilde{F}$ as the ground field for $\tilde{D}$, such that $\deg(\mathscr{L}) = p/[\tilde{F} : F] = 1$ and $\deg(\omega_{\tilde{D}}) = -2\chi(\mathscr{O}_{\tilde{D}}) = -2$. It follows that $h^1(\mathscr{L}) = h^0(\mathscr{L}^\vee \otimes \omega_{\tilde{D}}) = 0$, and Riemann–Roch gives $h^0(\mathscr{L}) = \chi(\mathscr{L}) = \deg(\mathscr{L}) + \chi(\mathscr{O}_{\tilde{D}}) = 2$. Each non-zero section $s \in H^0(\tilde{D}, \mathscr{L})$ vanishes at a unique $\tilde{F}$-rational point, and two linearly independent sections vanish at different points. It follows that $\mathscr{L}$ is globally generated, thus defines a morphism $\tilde{D} \to \mathbb{P}^1_{\tilde{F}}$ of degree one, which thus must be an isomorphism.

The proof concludes as follows: Write $\tilde{F} = F(\beta^{1/p})$ for some $\beta \in F^\times$ that is not a p-power. The conductor scheme B is canonically isomorphic to $\mathrm{Spec}(\tilde{F})$, via the composite map in Step 6, but the inclusion $\mathscr{O}_B \subset \mathscr{O}_A$ is given by

$$\beta^{1/p} \mapsto \beta^{1/p} + \gamma\epsilon$$

for some $\epsilon \in F^\times$. Replacing the generator $\beta^{1/p} \in \mathscr{O}_B$ by $\gamma^{-1}\beta^{1/p}$, we may assume $\gamma = 1$; thus the scheme D obtained from $\tilde{D} = \mathbb{P}^1_{\tilde{F}}$ by the denormalization in (12). According to Proposition 9.2, there exists a finite birational universal homeomorphism $D \to D'$, where $D' \subset \mathbb{P}(1,1,p)$ is given by the equation $t^p - \beta(x^p - \beta y^p)^p = 0$. Moreover, if $p = 2$, the finite birational universal homeomorphism is actually an isomorphism. This means that D is isomorphic to the curve in weighted projective space given by $t^2 - \beta(x^2 - \beta y^2)^2 = 0$. $\qquad\square$

10. NON-EXISTENCE WITHOUT IRREGULARITY IN CHARACTERISTIC TWO

Let F be a ground field of characteristic $p = 2$. Throughout, we use the notation from Section 5: Let V be a regular del Pezzo surface that is geometrically integral but geometrically non-normal. Let $N \subset \mathrm{Sing}(V/F)$ be the divisorial part of the locus of non-smoothness, and let $D = N_{\mathrm{red}}$ be the underlying reduced scheme. We now settle the subcase from Proposition 7.1 occurring in Proposition 7.2:

Theorem 10.1. *Assume that $h^0(\mathscr{O}_D) = h^1(\mathscr{O}_D) = 1$, and that over the algebraic closure $\Omega = \bar{F}$, the normalization of V_Ω is the contracted Hirzebruch surface $\mathbb{P}(1, 1, 2)$, with non-reduced ramification locus $R = 2H$. Then the ground field F must have $\mathrm{pdeg}(F) \geq 2$.*

Proof. By assumption, we are in case (iii) of Reid's Classification. After making a finite separable extension of F, we may assume that there is a finite purely inseparable extension $F \subset K$ so that the normalization $\nu : X \to Y = V_K$ is given by $X = \mathbb{P}(1, 1, 2)$, with non-reduced ramification divisor $R = 2H$. The latter has selfintersection number $R^2 = 2$. According to Proposition 7.1, we have $\nu^{-1}(D_K) = R$, thus $D^2 = 2$, in particular there is an invertible sheaf of degree two on D. Moreover, the curve D is geometrically irreducible. Being the divisorial part of non-smoothness on some regular scheme, the integral curve D is geometrically non-reduced and contains no rational points, by Corollaries 2.7 and 2.6.

Seeking a contradiction, we now assume that $\mathrm{pdeg}(F) \leq 1$. In light of [60, Theorem 2.3] the curve D is not regular. Thus Theorem 9.5 applies, hence D is given by the homogeneous equation $t^2 - \beta(x^2 - \beta y^2)^2 = 0$ inside the two-dimensional weighted projective space $\mathbb{P}(1, 1, 2)$ for some non-square $\beta \in F^\times$. Starting from this information, we shall show that the regular del Pezzo surface V is a Cartier divisor inside the three-dimensional weighted projective space $\mathbb{P}(1, 1, 1, 2)$, and that its defining equation must produce a singularity $v \in V$.

To achieve this goal, we first establish that $H^1(V, \mathscr{O}_V) = 0$. The conductor square (5) gives an exact sequence

$$H^0(\mathscr{O}_X) \oplus H^0(\mathscr{O}_C) \to H^0(\mathscr{O}_R) \to H^1(\mathscr{O}_Y) \to H^1(\mathscr{O}_X) \oplus H^1(\mathscr{O}_C).$$

The map on the left is surjective, because the ramification curve $R = 2H$ is the split ribbon $\mathbb{P}^1 \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$, hence $H^0(R, \mathscr{O}_R) = F$. The contracted Hirzebruch surface $X = \mathbb{P}(1, 1, 2)$ has $H^1(X, \mathscr{O}_X) = 0$, and it suffices to check $H^1(C, \mathscr{O}_C) = 0$. According to Proposition 5.6, the curve C is geometrically integral. The two curves $C, D_K \subset Y$ have the same support, and thus $C \subset (D_K)_{\mathrm{red}}$. By Proposition 9.1, we have $(D_K)_{\mathrm{red}} = \mathbb{P}^1$ and thus $C = \mathbb{P}^1$.

Next, consider the invertible sheaf $\mathscr{L} = \mathscr{O}_V(D)$: the sheaf $\mathscr{L}$ is ample because its pullback to $X = \mathbb{P}_K(1, 1, 2)$ is ample. The restriction has $\deg(\mathscr{L}_D) = 2$, whereas the dualizing sheaf has $\deg(\omega_D) = -2\chi(\mathscr{O}_D) = 0$. In turn, $h^1(\mathscr{L}_D) = 0$ and $h^0(\mathscr{L}_D) = 2$. Choose a vector space basis $s, s' \in H^0(D, \mathscr{L}_D)$. Since there is no rational point, the sections vanish at some non-rational points $a, a' \in D$ whose residue fields must be isomorphic to $\tilde{F} = F^{1/2}$. The local rings $\mathscr{O}_{D,a}$ and $\mathscr{O}_{D,a'}$ then must be regular. Consequently we have $a \neq a'$ because $Fs \neq Fs'$ are different linear systems. It follows that $\mathscr{L}_D$ is globally generated.

The short exact sequence $0 \to \mathscr{O}_V \to \mathscr{L} \to \mathscr{L}_D \to 0$ yields an exact sequence

$$0 \longrightarrow H^0(V, \mathscr{O}_V) \longrightarrow H^0(V, \mathscr{L}) \longrightarrow H^0(D, \mathscr{L}_D) \longrightarrow H^1(V, \mathscr{O}_V).$$

The term on the right vanishes, and we conclude that $\mathscr{L}$ is globally generated, with $h^0(\mathscr{L}) = 3$. We get a finite morphism $f : V \to \mathbb{P}^2$. It must be surjective, of degree $\deg(f) = \deg(\mathscr{L}) = 2$. This morphism is actually flat, because $\mathbb{P}^2$ is regular and V is Cohen–Macaulay. Setting $\mathscr{A} = f_*(\mathscr{O}_V)$, we get a short exact sequence

$$(20) \qquad 0 \longrightarrow \mathscr{O}_{\mathbb{P}^2} \longrightarrow \mathscr{A} \longrightarrow \mathscr{O}_{\mathbb{P}^2}(d) \longrightarrow 0$$

for some integer d. We claim that $d = -2$. To see this, choose a global section $s \in H^0(V, \mathscr{L})$ with zero-locus $D \subset V$. Restricting the above sequence to the line $\mathbb{P}^1 = V_+(s)$ gives a short exact sequence $0 \to \mathscr{O}_{\mathbb{P}^1} \to \nu_*(\mathscr{O}_D) \to \mathscr{O}_{\mathbb{P}^1}(d) \to 0$, thus $d + 1 = \chi(\mathscr{O}_{\mathbb{P}^1}(d)) = \chi(\mathscr{O}_D) - \chi(\mathscr{O}_{\mathbb{P}^1}) = -1$, and therefore $d = -2$. The short exact sequence (20) splits as an extension of coherent sheaves because

$$\mathrm{Ext}^1(\mathscr{O}_{\mathbb{P}^2}(d), \mathscr{O}_{\mathbb{P}^1}) = H^1(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(-d)) = 0.$$

Consequently $\mathscr{A} = \mathscr{O}_{\mathbb{P}^2} \oplus \mathscr{O}_{\mathbb{P}^2}(-2)$, and the algebra structure is determined by some linear map

$$(\Psi, \Phi) : \mathscr{O}_{\mathbb{P}^2}(-2) \otimes \mathscr{O}_{\mathbb{P}^2}(-2) \longrightarrow \mathscr{A} = \mathscr{O}_{\mathbb{P}^2} \oplus \mathscr{O}_{\mathbb{P}^2}(-2),$$

where we regard $\Psi, \Phi \in F[x, y, z]$ as homogeneous polynomials of degree four and two, respectively. It follows that V is isomorphic to the Cartier divisor in the weighted projective space $\mathbb{P}(1, 1, 1, 2)$ defined by the equation $P(x, y, z, t) = 0$, for the homogeneous polynomial

$$P(x, y, z, t) = t^2 - \Phi(x, y, z)t - \Psi(x, y, z)$$

of degree four, where $\deg(t) = 2$. Without restriction we may assume that the curve $D \subset V$ is given by the equation $z = 0$. The locus of non-smoothness $\mathrm{Sing}(V/F) \subset \mathbb{P}(1, 1, 1, 2)$ is given by the Jacobian ideal $\mathfrak{a} = (P, P_t, P_x, P_y, P_z)$ generated by P and its partial derivatives, because the unique singularity $(0 : 0 : 0 : 1)$ of the weighted projective space lies outside V, and the reflexive sheaf $\mathscr{O}_{\mathbb{P}(1,1,1,2)}(-4)$ is invertible. According to Proposition 7.1, the divisorial part is given by $2D \subset \mathrm{Sing}(V/F)$, such that $\mathfrak{a} \subset (z^2)$. Using $P_t = \Phi$, we see that $z^2 | \Phi$, and thus the homogeneous quadratic polynomial is of the form $\Phi = \alpha z^2$ for some scalar $\alpha \in F$. According to Theorem 9.5, we have

$$\Psi(x, y, z) = \beta(x^2 - \beta y^2)^2 + zR(x, y, z)$$

for some homogeneous polynomial $R(x, y, z)$ of degree three. Thus $P_z = zR_z + R$. Since the Jacobian ideal $\mathfrak{a}$ is contained in (z^2) and in particular in (z), we infer that $z | R$. Thus the polynomial P defining our regular del Pezzo surface $V \subset \mathbb{P}(1, 1, 1, 2)$ takes the explicit form

$$P(x, y, z, t) = t^2 - \alpha z^2 t - \beta(x^2 - \beta y^2)^2 + z^2 Q(x, y, z)$$

for some homogeneous quadratic polynomial $Q(x, y, z)$. This finally leads to the desired contradiction: Set $\mathbb{P} = \mathbb{P}(1, 1, 1, 2)$, and consider the non-rational closed point $v = (\beta^{1/2} : 1 : 0 : 0)$. Then $v \in V \cap D_+(y)$, and the local ring is given by $\mathscr{O}_{V,v} = \mathscr{O}_{\mathbb{P},v}/(Py^{-4})$. But we have

$$\frac{t}{y^2}, \frac{z}{y}, \left(\frac{x}{y}\right)^2 - \beta \in \mathfrak{m}_{\mathbb{P},v},$$

which implies $Py^{-4} \in \mathfrak{m}_{\mathbb{P},v}^2$. Consequently, the local ring $\mathscr{O}_{V,v}$ is not regular, a contradiction. $\qquad\square$

Similar arguments apply to the twin case $h^0(\mathscr{O}_D) = h^1(\mathscr{O}_D) = 1$ in characteristic three. Then the curve D is given by a homogeneous equation $z^3 - \varepsilon(x^3 - \varepsilon y^3) = 0$ for some non-cube $\varepsilon \in F^\times$. However, we shall take another approach in Section 13.

11. Non-existence with irregularity in characteristic two

Let F be a ground field of characteristic $p = 2$. Throughout, we use the notation from Section 5: Let V be a regular del Pezzo surface that is geometrically integral but geometrically non-normal. Let $N \subset \mathrm{Sing}(V/F)$ be the divisorial part of the locus of non-smoothness, and let $D = N_{\mathrm{red}}$ be the underlying reduced scheme.

In this section we will rule out the other subcase from Proposition 7.1 occurring in Proposition 7.2. Throughout, we assume that $h^0(\mathscr{O}_D) = h^1(\mathscr{O}_D) = 2$, and that over the algebraic closure $\Omega = \bar{F}$, the normalization of V_Ω is the contracted Hirzebruch surface $\mathbb{P}(1,1,2)$. We then have $\nu^{-1}(D_\Omega) = R = 2H$, such that $D^2 = 2$. It appears to be the most challenging case, and the main result of this section is the following.

Theorem 11.1. *Regular del Pezzo surfaces V as above do not exist.*

Throughout, we assume that such a surface V exists. To reach the desired contradiction we may assume that the ground field F is separably closed. The main idea is to study two-dimensional linear systems $H^0(V, \mathscr{L})$ coming from invertible sheaves $\mathscr{L} = \mathscr{N}(D)$ with $\mathscr{N}$ numerically trivial. It turns out that these define genus-one fibration on blow-ups, and the Canonical Bundle Formula will yield the desired contradiction. We start by computing some relevant cohomology groups.

Proposition 11.2. *The conductor curve $C \subset Y$ has $h^0(\mathscr{O}_C) = h^1(\mathscr{O}_C) = 1$.*

Proof. The conductor curve C is geometrically integral by Proposition 5.6, thus $h^0(\mathscr{O}_C) = 1$. It also follows that we have an inclusion $C \subset D_K$ as subschemes inside $Y = V_K$, and get a short exact sequence

$$0 \longrightarrow \mathscr{I} \longrightarrow \mathscr{O}_{D_K} \longrightarrow \mathscr{O}_C \longrightarrow 0$$

for some sheaf of ideals $\mathscr{I} \subset \mathscr{O}_{D_K}$ with $h^0(\mathscr{I}) = 1$. The long exact cohomology sequence yields surjections

$$(21) \qquad H^0(D_K, \mathscr{O}_{D_K}) \longrightarrow H^0(C, \mathscr{O}_C) \quad \text{and} \quad H^1(D_K, \mathscr{O}_{D_K}) \longrightarrow H^1(C, \mathscr{O}_C).$$

The former is not injective. Let $f \in H^0(D_K, \mathscr{O}_{D_K})$ be an element in the kernel. By assumption, $H^1(\mathscr{O}_D)$ is a one-dimensional vector space over the field $H^0(\mathscr{O}_D)$, so $H^1(D_K, \mathscr{O}_{D_K})$ is a free module of rank one over the ring $H^0(D_K, \mathscr{O}_{D_K})$. For each cohomology class α, the element $f\alpha$ becomes zero in $H^1(C, \mathscr{O}_C)$. In turn, the surjection $H^1(D_K, \mathscr{O}_{D_K}) \to H^1(C, \mathscr{O}_C)$ is also not injective. This already gives $h^1(\mathscr{O}_C) \leq 1$.

Seeking a contradiction, we assume that $h^1(\mathscr{O}_C) = 0$. Since $D_K^2 = 2$, the invertible sheaf $\mathscr{L} = \mathscr{O}_Y(D_K)$ has $\deg(\mathscr{L}_C) = 1$, and it follows that $\mathscr{L}_C$ is very ample with $h^0(\mathscr{L}_C) = 2$, thus defining an isomorphism $C \to \mathbb{P}^1$. Consequently $h^1(\mathscr{I}) = 2$. Let $\eta \in D_K$ be the generic point. Since $D_K^2 = 2$, the length of the local Artin ring $\mathscr{O}_{D_K,\eta}$ is two, and it follows that $\mathscr{I}$ annihilates itself. Hence $\mathscr{I}$ becomes a torsion-free $\mathscr{O}_C$-module, thus $\mathscr{I} = \mathscr{O}_{\mathbb{P}^1}(n)$ for some integer n. Using $h^0(\mathscr{I}) = 1$ we see $n = 0$, and this gives $h^1(\mathscr{I}) = 0$, a contradiction. The only remaining possibility is $h^1(\mathscr{O}_C) = 1$. $\qquad\square$

Using Proposition 4.6, we infer the cohomological invariants for $Y = V \otimes_F K$ and thus also for V.

Proposition 11.3. *The regular del Pezzo surface V has cohomological invariants $h^0(\mathscr{O}_V) = h^1(\mathscr{O}_V) = 1$ and $h^2(\mathscr{O}_V) = 0$.*

From this we deduce the structure of the Picard scheme.

Proposition 11.4. *The connected component $\mathrm{Pic}^0_{V/F}$ is a twisted form of $\mathbb{G}_{a,F}$, and it contains infinitely many rational points. In turn, there are infinitely many isomorphism classes of numerically trivial invertible sheaves $\mathscr{N}$ on V.*

Proof. In light of $h^2(\mathscr{O}_V) = 0$, the Picard scheme $P = \mathrm{Pic}^0_{V/F}$ is smooth ([47, Lecture 27]). It is a one-dimensional scheme, because its tangent space $H^1(V, \mathscr{O}_V)$ is a one-dimensional vector space. The base-change to K lies in the kernel for the pull-back map $\mathrm{Pic}_{Y/K} \to \mathrm{Pic}_{X/K} = \mathbb{Z}$. Using [8, Exposé XII, Theorem 1.1] we infer that P is quasiaffine. By the classification of one-dimensional commutative affine group schemes in [68], it must be a twisted form of $\mathbb{G}_a$ or $\mathbb{G}_m$.

Now regard P as a smooth connected algebraic curve, write $P \subset \bar{P}$ for the regular compactification, and choose some very ample invertible sheaf $\mathscr{L}$ on $\bar{P}$. Since the projective curve $\bar{P}$ is geometrically reduced, we may apply Bertini Theorems (for example [31, Proposition 4.3] or [37, Chapter I, Theorem 6.3]) and deduce that there are infinitely many reduced Cartier divisors $A \subset \bar{P}$ disjoint from the points at infinity. Since our ground field F is separably closed, these Cartier divisors are sums of rational points. They correspond to invertible sheaves on V because $\mathrm{Br}(F) = 0$. In other words, there are infinitely many isomorphism classes of numerically trivial sheaves on V.

Seeking a contradiction, we assume that P is a twisted form of $\mathbb{G}_m$. Choose some prime number $\ell \neq p$. The kernel $P[\ell]$ is a twisted form of the finite étale group scheme μ_ℓ. Since F is separably closed, we conclude that $P[\ell] = \mu_\ell \simeq \mathbb{Z}/\ell\mathbb{Z}$. In turn, we find an invertible sheaf $\mathscr{L}$ of order ℓ. Choose a trivialization $\mathscr{L}^{\otimes -\ell} \to \mathscr{O}_V$, such that the locally free sheaf $\mathscr{A} = \mathscr{O}_V \oplus \mathscr{L}^{\otimes -1} \oplus \ldots \oplus \mathscr{L}^{\otimes 1-\ell}$ acquires the structure of a finite étale $\mathscr{O}_V$-algebra. This shows that the algebraic fundamental group $\pi_1(V)$ is non-trivial. However, the morphism $\nu : X \to V$ is a universal homeomorphism, so the induced map $\pi_1(X) \to \pi_1(V)$ is bijective by [32, Exposé IX, Theorem 4.10]. However, $X = \mathbb{P}(1,1,2)$ is simply-connected, a contradiction. $\square$

Note that the structure of twisted forms of the additive group scheme $\mathbb{G}_a$ were determined in [54]. We now consider invertible sheaves $\mathscr{L} = \mathscr{N}(D)$ on V, where $\mathscr{N}$ is numerically trivial.

Proposition 11.5. *The linear system $H^0(V, \mathscr{L})$ is two-dimensional.*

Proof. We compute this with the induced invertible sheaf $\mathscr{L}_Y$ on the non-normal del Pezzo surface $Y = V \otimes_F K$, by using the exact sequence

$$(22) \qquad 0 \longrightarrow H^0(Y, \mathscr{L}_Y) \longrightarrow H^0(X, \mathscr{L}_X) \oplus H^0(C, \mathscr{L}_C) \longrightarrow H^0(R, \mathscr{L}_R)$$

explained in (7). The contracted Hirzebruch surface $X = \mathbb{P}(1,1,2)$ has $H^1(X, \mathscr{O}_X) = 0$, and it follows that $\mathscr{L}_X = \mathscr{O}_X(R) = \mathscr{O}_X(2H)$, regardless of the numerically trivial sheaf $\mathscr{N}$. Pulling back along the resolution of singularities $S \to X$ and working on the Hirzebruch surface $S \to \mathbb{P}^1$, one easily sees $h^0(\mathscr{L}_X) = 4$.

Since $\nu^{-1}(D_K) = R = 2H$, we have $(D_K \cdot C) = 1$, so $\deg(\mathscr{L}_C) = 1$. Serre duality gives $h^1(\mathscr{L}_C) = 0$ and thus $h^0(\mathscr{L}_C) = 1$ by Riemann–Roch. In turn, the base locus

for the invertible sheaf $\mathscr{L}_C$ comprises a unique rational base point $b \in C$, contained in the regular locus of the curve C.

Finally, the ramification divisor is the split ribbon $R = \mathbb{P}^1 \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$. In turn, we get $\mathscr{L}_R = \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}$, and therefore $h^0(\mathscr{L}_R) = 3$. The short exact sequence $0 \to \mathscr{O}_X(-R) \to \mathscr{O}_X \to \mathscr{O}_R \to 0$ yields an exact sequence

$$H^0(X, \mathscr{O}_X) \longrightarrow H^0(R, \mathscr{O}_R) \longrightarrow H^1(X, \mathscr{O}_X(-R)).$$

We have $h^2(\mathscr{O}_X(-R)) = h^0(\mathscr{O}_X(-R)) = 0$ by Serre duality. Riemann–Roch gives

$$\chi(\mathscr{O}_X(-R)) = \frac{R^2 + (R \cdot K_X)}{2} + \chi(\mathscr{O}_X) = \frac{R^2 - 2R^2}{2} + 1 = 0,$$

and thus $h^1(\mathscr{O}_X(-R)) = 0$. In turn, the restriction map $H^0(X, \mathscr{L}_X) \to H^0(R, \mathscr{L}_R)$ is surjective. Using the exact sequence (22), we infer $h^0(\mathscr{L}) = 2$. $\qquad \square$

Next, we examine the two-dimensional linear system on $Y = V_K$ induced by $\mathscr{L}_Y$.

Proposition 11.6. *There is a non-zero global section $s' \in H^0(Y, \mathscr{L}_Y)$ so that the resulting effective Cartier divisor $D' \subset Y$ has $\mathrm{Supp}(D') = \mathrm{Supp}(D_K)$. Such a section is unique up to invertible scalar. The cohomological invariants for the scheme D' are given by*

$$h^0(\mathscr{O}_{D'}) = h^1(\mathscr{O}_{D'}) = \begin{cases} 2 & \text{if } \mathscr{L} \simeq \omega_V^\vee; \\ 1 & \text{else.} \end{cases}$$

Proof. According to Proposition 5.12, the global sections of $\mathscr{L}_Y$ vanishing along C correspond to the global sections of $\mathscr{L}_X(-R)$. In our situation, the normalization X is the contracted Hirzebruch surface $\mathbb{P}(1, 1, 2)$ and $\mathscr{L}_X(-R) = \mathscr{O}_X$. Existence and uniqueness follow.

Since the conductor curve C is reduced, we have an inclusion $C \subset D'$ inside the scheme Y. In turn, there is a short exact sequence

$$(23) \qquad\qquad 0 \longrightarrow \mathscr{I} \longrightarrow \mathscr{O}_{D'} \longrightarrow \mathscr{O}_C \longrightarrow 0$$

for some sheaf of ideals $\mathscr{I} \subset \mathscr{O}_{D'}$. By definition of the conductor square, $\nu^{-1}(C) = R$ holds as subschemes on X. By assumption on our regular del Pezzo surface V we also have $\nu^{-1}(D') = R$. In turn, the morphism $\nu : R \to Y$ factors over D', and the image of the homomorphism $\mathscr{O}_{D'} \to \mathscr{O}_R$ is the subsheaf $\mathscr{O}_C \subset \mathscr{O}_R$. Furthermore, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{O}_Y(-D') & \longrightarrow & \mathscr{O}_Y & \longrightarrow & \mathscr{O}_{D'} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{O}_X(-R) & \longrightarrow & \mathscr{O}_X & \longrightarrow & \mathscr{O}_R & \longrightarrow & 0. \end{array}$$

The cokernels for the two vertical maps on the right are the same, and the Snake Lemma gives an identification $\mathscr{I} = \mathscr{O}_X(-R)/\mathscr{O}_Y(-D')$. In turn, we have a second short exact sequence

$$(24) \qquad\qquad 0 \longrightarrow \mathscr{O}_Y(-D') \longrightarrow \mathscr{O}_X(-R) \longrightarrow \mathscr{I} \longrightarrow 0.$$

Riemann–Roch gives

$$\chi(\mathscr{O}_X(-R)) = \frac{(R^2) + (R \cdot K_X)}{2} + \chi(\mathscr{O}_X) = \frac{(R^2) - 2(R^2)}{2} + 1 = 0$$

and

$$\chi(\mathscr{O}_Y(-D')) = \frac{(R^2) + (R \cdot \nu^{-1}(K_Y))}{2} + \chi(\mathscr{O}_Y) = \frac{(R^2) - (R^2)}{2} + \chi(\mathscr{O}_Y) = 0.$$

Consequently $\chi(\mathscr{I}) = 0$ and $\chi(\mathscr{O}_{D'}) = \chi(\mathscr{O}_C) + \chi(\mathscr{I}) = 0$. Thus it suffices to verify the statement on $h^0(\mathscr{O}_{D'})$.

Obviously $h^0(\mathscr{O}_X(-R)) = 0$. Using $\omega_X(R) = \mathscr{O}_X(-R)$ also $h^2(\mathscr{O}_X(-R)) = 0$, and thus $h^1(\mathscr{O}_X(-R)) = 0$. Likewise, we have $h^0(\mathscr{O}_Y(-D')) = 0$. If $\mathscr{L} = \omega_Y^\vee$, then Serre duality gives $h^2(\mathscr{O}_Y(-D')) = 1$ and thus $h^1(\mathscr{O}_Y(-D')) = 1$. On the other hand, if $\mathscr{L} \neq \omega_Y^\vee$, the invertible sheaf $\omega_Y(D')$ is numerically trivial but non-trivial, and thus $h^i(\mathscr{O}_Y(-D')) = 0$ for all $i \geq 0$. From (24) we get an exact sequence

$$0 \longrightarrow H^0(\mathscr{I}) \longrightarrow H^1(Y, \mathscr{O}_Y(-D')) \longrightarrow H^1(X, \mathscr{O}_X(-R)).$$

We just saw that the term on the right vanishes, and thus $h^0(\mathscr{I}) = 1$ if $\mathscr{L} = \omega_Y^\vee$, and $h^0(\mathscr{I}) = 0$ if $\mathscr{L} \neq \omega_Y^\vee$. To finish the argument, we use the exact sequence

$$0 \longrightarrow H^0(\mathscr{I}) \longrightarrow H^0(D', \mathscr{O}_{D'}) \longrightarrow H^0(C, \mathscr{O}_C)$$

stemming from (23). We have $h^0(\mathscr{O}_C) = 1$, because C is geometrically reduced. It follows that the map on the right is surjective, and the assertion follows. $\square$

Note that we do not assert that the above Cartier divisor D' on $Y = V_K$ arises from an effective Cartier divisor on V. In fact, this is impossible if $\mathscr{N} \neq \mathscr{O}_V$, because on the normal scheme V effective Cartier divisors are determined by their support and multiplicities.

By assumption on our regular del Pezzo surface V, the divisor $D = N_{\mathrm{red}}$, where $N \subset \mathrm{Sing}(V/F)$ is the divisorial part of the locus of non-smoothness, has $h^0(\mathscr{O}_D) = h^1(\mathscr{O}_D) = 2$. Let us record the following consequence.

Corollary 11.7. *The dualizing sheaf for the del Pezzo surface is $\omega_V = \mathscr{O}_V(-D)$.*

Proposition 11.8. *Suppose that $\mathscr{N} \neq \mathscr{O}_V$. For each non-zero $s' \in H^0(V, \mathscr{L})$ the resulting effective Cartier divisor $D' \subset V$ has cohomological invariants $h^0(\mathscr{O}_{D'}) = h^1(\mathscr{O}_{D'}) = 1$.*

Proof. Riemann–Roch gives

$$\chi(\mathscr{O}_{D'}) = \chi(\mathscr{O}_V) - \chi(\mathscr{O}_V(-D')) = \chi(\mathscr{O}_V) - \chi(\mathscr{O}_V(-D)) = \chi(\mathscr{O}_D) = 0,$$

so it suffices to check $h^0(\mathscr{O}_{D'}) = 1$. The curve $D' \subset V$ is integral, because its class generates $\mathrm{NS}(V)$. In turn, the preimage $\nu^{-1}(D_K') \subset X$ is irreducible. Since this is linearly equivalent to $R = 2H$, it must be isomorphic to the split ribbon $\mathbb{P}^1 \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$. Using that $\nu^{-1}(D_K') \to D_K'$ is schematically dominant, we infer that $h^0(\mathscr{O}_{D_K'}) = 1$. $\square$

For each vector space basis $s_1, s_2 \in H^0(V, \mathscr{L})$ the resulting two effective Cartier divisor $D_1, D_2 \subset V$ define the base-locus $\mathrm{Bs}(\mathscr{L}) = D_1 \cap D_2$ as a closed subscheme.

Proposition 11.9. *The base-locus $\mathrm{Bs}(\mathscr{L})$ consists of a unique closed point $b \in V$ contained in $D \subset V$, and its residue field has $[\kappa(b) : F] = 2$.*

Proof. First note that formation of base-loci commute with extension of ground fields. The Néron–Severi group $\mathrm{NS}(Y)$ is cyclic, with $\mathscr{L}$ as generator. It follows that every non-zero global section of $\mathscr{L}$ defines an integral Cartier divisor. From this we infer that the base-locus $\mathrm{Bs}(\mathscr{L}) = D_1 \cap D_2$ is finite. It must have degree two because $D^2 = 2$.

Using the restriction map $H^0(\mathscr{L}_Y) \to H^0(C, \mathscr{L}_C)$ and the fact that $\mathscr{L}_C$ has degree one on the integral curve C of genus one, we see that $\mathrm{Bs}(\mathscr{L})$ consists of a unique closed point $b \in V$ contained in D. Since our del Pezzo surface V is regular and $D \subset N$ is the divisorial part of the locus of non-smoothness, we see that the point $b \in V$ is not rational. Since the base-locus has length two as closed subscheme, we infer that the field extension $F \subset \kappa(b)$ has degree two. $\qquad\square$

The two-dimensional linear system $H^0(V, \mathscr{L})$ defines a rational map

$$X \dashrightarrow \mathbb{P}^1 = \mathbb{P}H^0(V, \mathscr{L})$$

defined on the complement of the base-point $b \in V$. Let $\tilde{V} \to V$ be the blowing-up with center $b \in V$. The exceptional divisor $E \subset \tilde{V}$ is a projective line over the residue field $\kappa(b)$, and we get a fibration $f : \tilde{V} \to \mathbb{P}^1$. Note that all this depends on the numerically trivial sheaf $\mathscr{N}$ in $\mathscr{L} = \mathscr{N}(D) = \mathscr{N} \otimes \omega_V^\vee$, but we do not indicate this in notation.

Proposition 11.10. *If $\mathscr{N} \neq \mathscr{O}_V$, then $f : \tilde{V} \to \mathbb{P}^1$ is a genus-one fibration with the following property: for every point $a \in \mathbb{P}^1$, the schematic fiber $\tilde{V}_a = f^{-1}(a)$ is integral.*

Proof. We first establish that $f : \tilde{V} \to \mathbb{P}^1$, which is proper, surjective, and flat is indeed a genus-one fibration. According to Proposition 11.8, for each rational point $a \in \mathbb{P}^1$ the fiber $\tilde{V}_a = f^{-1}(a)$ has $h^0(\mathscr{O}_{\tilde{V}_a}) = h^1(\mathscr{O}_{\tilde{V}_a}) = 1$. For the generic fiber this implies $\chi(\mathscr{O}_{\tilde{V}_\eta}) = 0$. Semicontinuity gives $h^0(\mathscr{O}_{\tilde{V}_\eta}) = 1$, and thus also $h^1(\mathscr{O}_{\tilde{V}_\eta}) = 1$. Zariski's Main Theorem ensures that the canonical injection $\mathscr{O}_{\mathbb{P}^1} \subset f_*(\mathscr{O}_{\tilde{V}})$ is an equality. Summing up, the morphism $f : \tilde{V} \to \mathbb{P}^1$ is a genus-one fibration.

Now fix a closed point $a \in \mathbb{P}^1$. It remains to verify that the scheme $\tilde{V}_a = f^{-1}(a)$ is integral. This is clear if $a \in \mathbb{P}^1$ is a rational point, because the image of the fiber in V generates $\mathrm{NS}(V)$. For arbitrary $a \in \mathbb{P}^1$, the fiber $\tilde{V}_a$ is at least irreducible, because the group $\mathrm{NS}(\tilde{V})$ has rank $\rho = 2$. Write $\tilde{V}_a = m\Theta$ for some integral curve $\Theta \subset \tilde{V}$ and some multiplicity $m \geq 1$. The exceptional divisor $E \subset \tilde{V}$ for the blowing-up $\tilde{V} \to V$ is a projective line over the quadratic field extension $F \subset \kappa(b)$ and the projection $E \to \mathbb{P}^1$ has degree two; we must have $m \leq 2$.

Seeking a contradiction, we now assume $m = 2$. In particular, the closed point $a \in \mathbb{P}^1$ is not rational. The intersection $E \cap \tilde{V}_a$ is a finite scheme, which has degree two over the residue field $\kappa(a)$. In turn, $E \cap \Theta$ is isomorphic to $\mathrm{Spec}\,\kappa(a)$. We infer that the field extension $\kappa(a) \subset H^0(\mathscr{O}_\Theta)$ is bijective, and that the curve Θ is regular at the intersection point $E \cap \Theta$. It follows that the composite morphism $\Theta \subset \tilde{V} \to V$ is a closed embedding. Its image $D'' \subset V$ passes through the center $b \in V$, which is also the base-point for the invertible sheaf $\mathscr{L} = \mathscr{N}(D)$, and has $[\kappa(b) : F] = 2$. Choose a non-zero global section of $H^0(V, \mathscr{L})$. Recall that $s_1, s_2 \in H^0(V, \mathscr{L})$ is a vector space basis, with corresponding effective Cartier divisors $D_1, D_2 \subset V$. Since $(D_1 \cdot D_2) = 2$, the intersection $D_1 \cap D_2$ is transversal, and in particular the curves D_1, D_2 are regular at the intersection point $b \in D_1 \cap D_2$. Without restriction, we may assume that the intersection $D_1 \cap D''$ is transversal as well, and thus $(D_1 \cdot D'') = 2$. It follows that $\mathscr{L} = \mathscr{N}(D)$ is numerically equivalent to $\mathscr{O}_V(D'')$. Furthermore, the pullback to the conductor curve $C \subset Y = V_K$ becomes isomorphic. Since both pullback maps $\mathrm{Pic}(V) \to \mathrm{Pic}(Y)$ and restriction map $\mathrm{Pic}(V) \to \mathrm{Pic}(C)$ are injective, we conclude that $\mathscr{O}_V(D'') \simeq \mathscr{L}$. It follows that Θ is a fiber over some rational point in $\mathbb{P}^1$, a contradiction. $\qquad\square$

Proof for Theorem 11.1. According to Corollary 11.7, we have $\omega_V = \mathscr{O}_V(-D)$. Choose some numerically trivial invertible sheaf $\mathscr{N} \not\simeq \mathscr{O}_V$, and consider the genus-one fibration $f : \tilde{V} \to \mathbb{P}^1$ constructed above from the invertible sheaf $\mathscr{L} = \mathscr{N}(D) = \mathscr{O}_V(D')$. According to Proposition 11.10, all fibers over closed points of $\mathbb{P}^1$ are integral. So we have $K_{\tilde{V}} = \sum m_i F_i$, where m_i are integers and F_i are fibers. Since the base of the fibration is $\mathbb{P}^1$, we deduce that $K_{\tilde{V}} = mF$, for the fiber F over the rational point $\infty \in \mathbb{P}^1$ and some integer $m < 0$. With $E^2 = -2$ and $(F \cdot E) = 2$, we deduce that $m = -1$. Using that $\omega_V = f_*(\omega_{\tilde{V}})$, we infer that $\omega_V = \mathscr{O}_V(-D')$. On the other hand, we have $\omega_V = \mathscr{O}_V(-D)$ by Corollary 11.7. Thus the two divisors $D, D' \subset V$ are linearly equivalent, hence $\mathscr{N} \simeq \mathscr{O}_V$, a contradiction. $\square$

12. PECULIAR CURVES OF HIGHER GENUS

In this section we shall study the geometry of certain peculiar algebraic curves of higher genus that show up in our study of del Pezzo surfaces in characteristic three that are regular and geometrically non-normal. The geometry of such curves, however, merits a study in all characteristics. Our methods break down in characteristic two, so we exclude this from consideration.

Let F be an imperfect ground field of characteristic $p \geq 3$. For the sake of exposition, we assume that F is separably closed and has $\mathrm{pdeg}(F) = 1$. In turn, every finite field extension is of the form F^{1/p^n}, which has degree p^n. We write $\tilde{F} = F^{1/p}$ for the unique field extension of degree p. Throughout, D denotes a proper integral curve with $h^0(\mathscr{O}_D) = 1$ satisfying the following conditions:

(i) The genus of the curve is $h^1(\mathscr{O}_D) = p + 1$.
(ii) The local rings $\mathscr{O}_{D,a}$, $a \in D$ are Gorenstein and geometrically unibranch.
(iii) There is no rational point on D.
(iv) The normalization $\tilde{D}$ has genus $h^1(\mathscr{O}_{\tilde{D}}) = 0$.

In turn, we get the numerical invariants

$$\chi(\mathscr{O}_D) = -p, \quad \deg(\omega_D) = 2p, \quad h^0(\omega_D) = p + 1, \quad \text{and} \quad h^1(\omega_D) = 1.$$

Note that for each closed point $x \in D$, the residue field $\kappa(x)$ is a purely inseparable finite extension of F, so its degree is a p-power. As usual, we form the conductor square

(25)
$$\begin{array}{ccc} A & \longrightarrow & \tilde{D} \\ \downarrow & & \downarrow{\scriptstyle\nu} \\ B & \longrightarrow & D \end{array}$$

for the normalization morphism $\nu : \tilde{D} \to D$.

Proposition 12.1. *The normalization $\tilde{D}$ is isomorphic to the projective line $\mathbb{P}^1_{\tilde{F}}$, and the conductor loci satisfy $h^0(\mathscr{O}_B) = 2p$ and $h^0(\mathscr{O}_A) = 4p$. More precisely, we have*

$$\mathscr{O}_A \simeq \tilde{F}[\epsilon] \times \tilde{F}[\epsilon] \ \text{ and } \ \mathscr{O}_B \simeq \tilde{F} \times \tilde{F} \qquad \text{or} \qquad \mathscr{O}_A \simeq \tilde{F}[\eta] \ \text{ and } \ \mathscr{O}_B \simeq \tilde{F}[\epsilon].$$

Here ϵ and η denote indeterminates subject to the relation $\epsilon^2 = 0$ and $\eta^4 = 0$.

Proof. Seeking a contradiction, suppose that $h^0(\mathscr{O}_{\tilde{D}}) = 1$. According to [60, Theorem 2.3] the scheme $\tilde{D}$ is geometrically reduced, and thus the same holds for D. According to the well-known Bertini Theorems (for example [31, Proposition 4.3]

or [37, Chapter I, Theorem 6.3]), we find an effective Cartier divisor $C \subset D$ that is also geometrically reduced. Since F is separably closed, such a subscheme consists of rational points, a contradiction. So we have $h^0(\mathscr{O}_{\tilde{D}}) = p^\nu$ for some exponent $\nu \geq 1$. The invertible sheaf $\omega_D|\tilde{D}$ has degree $2p$, which ensures $\nu = 1$, because p is odd. Since $\text{pdeg}(F) = 1$, the two field extensions $\tilde{F}$ and $H^0(\tilde{D}, \mathscr{O}_{\tilde{D}})$ coincide.

Now regard $\tilde{F}$ as the ground field for the curve $\tilde{D}$, which is geometrically reduced. Applying Bertini Theorems as above, we find a rational point $x \in \tilde{D}$, and set $\mathscr{L} = \mathscr{O}_{\tilde{D}}(x)$. The long exact sequence coming from $0 \to \mathscr{O}_{\tilde{D}} \to \mathscr{L} \to \kappa(x) \to 0$ shows that $\mathscr{L}$ is globally generated with $\dim_{\tilde{F}} H^0(\tilde{D}, \mathscr{L}) = 2$, and it follows that the ensuing $\tilde{D} \to \mathbb{P}^1_{\tilde{F}}$ is an isomorphism.

From the conductor square (25), we get an exact sequence

$$0 \to H^0(\mathscr{O}_D) \to H^0(\mathscr{O}_{\tilde{D}}) \oplus H^0(\mathscr{O}_B) \to H^0(\mathscr{O}_A) \to H^1(\mathscr{O}_D) \to 0.$$

Write $h^0(\mathscr{O}_A) = pa$ for some integer $a \geq 1$. Since the curve D is Gorenstein, we must have $h^0(\mathscr{O}_B) = pa/2$. In turn, we get $1 - (p + pa/2) + pa - (p + 1) = 0$. Thus $a = 4$, and the values for $h^0(\mathscr{O}_A)$ and $h^0(\mathscr{O}_B)$ follow.

Now decompose $A = \sum n_i a_i$ into prime divisors on the regular curve $\tilde{D}$, and write $[\kappa(a_i) : \tilde{F}] = p^{\nu_i}$. Then $4 = \sum n_i p^{\nu_i}$. Each summand is even, because the local rings on D are Gorenstein and geometrically unibranch, and p is odd. One easily sees that the only solutions are $4 = 2 + 2$ or $4 = 4$, and the structure for $\mathscr{O}_A$ follows. The Artin ring $\mathscr{O}_B$ of length $h^0(\mathscr{O}_B) = 2p$ has an induced decomposition, with corresponding lengths $2p = p + p$ or $2p = 2p$, since the local rings on B are geometrically unibranch. Since D contains no rational point, all residue fields of $\mathscr{O}_B$ are copies of $\tilde{F}$. The structure of $\mathscr{O}_B$ follows.   □

Choose an identification $\tilde{D} = \mathbb{P}^1_{\tilde{F}}$. We now construct some partial denormalization $\hat{D}$, with morphisms $\mathbb{P}^1_{\tilde{F}} = \tilde{D} \to \hat{D} \to D$. First suppose that $\mathscr{O}_A = \tilde{F}[\epsilon] \times \tilde{F}[\epsilon]$. Then define $\hat{D}$ as the denormalization of $\tilde{D} = \mathbb{P}^1_{\tilde{F}}$ with respect to $\mathscr{O}_B/(0 \times \tilde{F})$ inside the ring $\mathscr{O}_A/(0 \times \tilde{F}[\epsilon])$. Then $\hat{D}$ is either a peculiar genus-one curve studied in Section 9, or the rational cuspidal curve $\text{Spec} \, \tilde{F}[T^2, T^3] \cup \text{Spec} \, \tilde{F}[1/T]$.

Now suppose that $\mathscr{O}_A = \tilde{F}[\eta]$, with $\eta^4 = 0$, and write $\tilde{F} = F(\beta^{1/p})$. Since $h^0(\mathscr{O}_D) = 1$, the subalgebra $\mathscr{O}_B = \tilde{F}[\epsilon] \subset \tilde{F}[\eta] = \mathscr{O}_A$ is given by

$$\beta^{1/p} \mapsto \beta^{1/p} + \lambda_1 \eta + \lambda_2 \eta^2 + \lambda_3 \eta^3 \quad \text{and} \quad \epsilon = \mu_2 \eta^2 + \mu_3 \eta^3,$$

where neither $(\lambda_1, \lambda_2, \lambda_3)$ nor (μ_2, μ_3) are zero tuples. Consider $\mathscr{O}_A/\mathfrak{m}_A^2 = \tilde{F}[\eta]/(\eta^2)$ and the resulting subring $\mathscr{O}_B/(\mathscr{O}_B \cap \mathfrak{m}_A^2)$. Both residue class rings have residue fields isomorphic to $\tilde{F}$, with $h^0(\mathscr{O}_A/\mathfrak{m}_A^2) = 2p$ and $h^0(\mathscr{O}_B/(\mathscr{O}_B \cap \mathfrak{m}_A^2)) = p$. More precisely, $\mathscr{O}_B/(\mathscr{O}_B \cap \mathfrak{m}_A^2)$ coincides with the image of $\tilde{F} = H^0(\tilde{D}, \mathscr{O}_{\tilde{D}})$ inside $\mathscr{O}_A/\mathfrak{m}_A^2$ if and only if $\lambda_1 = 0$. Now define $\hat{D}$ as the denormalization of $\tilde{D} = \mathbb{P}^1_{\tilde{F}}$ with respect to $\mathscr{O}_B/(\mathscr{O}_B \cap \mathfrak{m}_A^2)$ inside $\mathscr{O}_A/\mathfrak{m}_A^2$. Again $\hat{D}$ is either the rational cuspidal curve over $\tilde{F}$ or a peculiar genus-one curve studied in Section 9. The latter holds if and only if $\lambda_1 \neq 0$.

The following statement, valid in all the above cases, is immediate if $\hat{D}$ is a rational cuspidal curve over $\tilde{F}$, and follows directly from Proposition 9.1 if $\hat{D}$ is a peculiar genus-one curve.

Proposition 12.2. *The reduction $C = (D \otimes_F \tilde{F})_{\mathrm{red}}$ of the base-change has*

$$\dim_{\tilde{F}} H^0(C, \mathscr{O}_C) = 1 \quad and \quad \dim_{\tilde{F}} H^1(C, \mathscr{O}_C) \leq 1.$$

In combination with Maddock's bound (9), this will play a crucial role in our analysis of del Pezzo surfaces V in characteristic three that are regular but geometrically non-normal. To proceed, we study the linear system attached to the dualizing sheaf.

Proposition 12.3. *The invertible sheaf ω_D is globally generated.*

Proof. Choose a non-zero global section $s \in H^0(D, \omega_D)$. The zero-locus is an effective Cartier divisor $C \subset D$ of $\mathrm{length}(\mathscr{O}_C) = 2p$. As a set, it consists of either two points $c_1, c_2 \in D$ or of a single point $c \in D$, because F is separably closed and $C(F) = \varnothing$. In all cases, the respective residue fields $\tilde{F}_i$ are isomorphic to $\tilde{F}$, and $\mathscr{O}_C = \tilde{F} \times \tilde{F}$ or $\mathscr{O}_C = \tilde{F}[\epsilon]$, where ϵ denotes an indeterminate subject to the relation $\epsilon^2 = 0$. The case that $\mathscr{O}_C$ is a field does not occur, by our assumption $p \neq 2$.

First consider the case that C is reduced, such that $C = \{c_1, c_2\}$. The short exact sequence $0 \to \mathscr{O}_D \to \omega_D \to \mathscr{O}_C \to 0$ induces an exact sequence

$$0 \longrightarrow H^0(D, \mathscr{O}_D) \longrightarrow H^0(D, \omega_D) \longrightarrow H^0(C, \mathscr{O}_C) = \tilde{F} \times \tilde{F}.$$

The linear map on the right has rank p, so we may assume without restriction that $c_2 \in D$ is not a base point. Seeking a contradiction, we assume that $c_1 \in D$ is a base point. Set $\mathscr{L} = \mathscr{O}_D(c_2) = \omega_D(-c_1)$. Then the inclusion $H^0(D, \mathscr{L}) \subset H^0(D, \omega_D)$ is an equality, in particular $h^0(\mathscr{L}) = p + 1$, and furthermore $\mathscr{L}$ is globally generated. The short exact sequence $0 \to \mathscr{L}(-c_1) \to \mathscr{L} \to \kappa(c_1) \to 0$ yields an exact sequence

$$0 \longrightarrow H^0(D, \mathscr{L}(-c_1)) \longrightarrow H^0(D, \mathscr{L}) \longrightarrow \tilde{F}.$$

The map on the right is not injective, and we conclude that the numerically trivial sheaf $\mathscr{L}(-c_1) = \mathscr{O}_D(c_2 - c_1)$ admits a non-zero section. In turn, $\mathscr{O}_D(c_1) \simeq \mathscr{O}_D(c_2)$ is globally generated, a contradiction.

Now suppose that C is non-reduced, such that $\mathscr{O}_C = \tilde{F}[\epsilon]$. Seeking a contradiction, we assume that $c \in D$ is a base point. Then the image of the restriction map $H^0(D, \omega_D) \to H^0(C, \mathscr{O}_C)$ factors over the maximal ideal $\tilde{F}\epsilon$ of the local Artin ring $\mathscr{O}_C$, and we get a short exact sequence

$$0 \longrightarrow H^0(D, \mathscr{O}_D) \longrightarrow H^0(D, \omega_D) \longrightarrow \tilde{F}\epsilon \longrightarrow 0.$$

Choose two sections s', s'' of ω_D vanishing at $c \in D$, whose images in $\tilde{F}\epsilon$ are linearly independent over F. Then the resulting closed subschemes $D', D'' \subset D$ do not coincide, but we have $D' \cap C = D'' \cap C$. It follows that the local ring $\mathscr{O}_{D,c}$ is singular, with embedding dimension two.

We now distinguish two cases, according to the singularities on D. First suppose that the ramification divisor $A \subset \tilde{D}$ is of the form $\mathscr{O}_A = \tilde{F}[\epsilon] \times \tilde{F}[\epsilon]$, such that $\mathscr{O}_B$ is isomorphic to $\tilde{F} \times \tilde{F}$. We may assume that the first projection corresponds to the inclusion $c \in B$. We now regard the global sections $s \in H^0(D, \omega_D)$ as pairs $(s|\tilde{D}, s|B)$ whose entries coincide in $\mathscr{O}_A$. Since $c \in B$ is a base point, this means that $s|B \in 0 \times \tilde{F}$ and $s|\tilde{D}$ is a global section of $\mathscr{O}_{\tilde{D}} \subset \omega_D|\tilde{D} = \mathscr{O}_{\tilde{D}}(2)$. In other words, we have an identification of $H^0(D, \omega_D)$ with the intersection of two copies of $\tilde{F}$ as subalgebras inside $0 \times \tilde{F}[\epsilon]$. Such an intersection is a subalgebra inside $\tilde{F}$, which has dimension $d = p$ or $d = 1$, contradicting $h^0(\omega_D) = p + 1$.

Now suppose that $\mathscr{O}_B = \tilde{F}[\epsilon]$ inside $\mathscr{O}_A = \tilde{F}[\eta]$ with $\eta^4 = 0$. The inclusion is given by $\epsilon = \alpha\eta^2 + \beta\eta^3$ for some scalars $\alpha, \beta \in F$, not both zero. The residue class ring $\mathscr{O}_A/\epsilon\mathscr{O}_A$ is thus $\tilde{F}[\eta]/(\eta^n)$ with $n = 2$ or $n = 3$. In both cases, we may regard the global section $s \in H^0(D, \omega_D)$ as pairs $(s|\tilde{D}, s|B)$ where the first entry is a global section of $\mathscr{O}_{\tilde{D}} \subset \omega_D|\tilde{D}$ and the second entry lies is $\tilde{F}\epsilon$. As in the preceding paragraph, this contradicts $h^0(\omega_D) = p + 1$. $\square$

Proposition 12.4. *There is a non-zero section of ω_D whose zero-scheme is reduced.*

Proof. Suppose this would not hold. Then for each non-zero global section $s \in H^0(D, \omega_D)$, the zero-scheme $Z = Z(s) \subset D$ is local, with $\mathscr{O}_Z = \tilde{F}[\epsilon]$. Choose two global sections s, s' that generate ω_D, and consider the resulting finite flat morphism $f : D \to \mathbb{P}^1$ of degree $\deg(f) = 2p$. The morphism is not purely inseparable, by our standing assumption $p \neq 2$. Hence the fiber $f^{-1}(y)$ is geometrically disconnected for the generic point $y = \eta$. By [29, Proposition 9.7.8] this holds for almost all points $y \in \mathbb{P}^1$. Since F is separably closed, whence infinite, there must be a rational point $y \in \mathbb{P}^1$ with $f^{-1}(y)$ geometrically disconnected. On the other hand, this fiber is the spectrum of $\mathscr{O}_Z = \tilde{F}[\epsilon]$, which is geometrically connected, a contradiction. $\square$

13. Non-existence in characteristic three

Let F be a ground field of characteristic $p = 3$, and let V be a regular del Pezzo surface that is geometrically integral but geometrically non-normal. In Section 8, we already narrowed down the possibilities. Here our main result is the following.

Theorem 13.1. *If a del Pezzo surface V as above exists, the ground field F necessarily has $\mathrm{pdeg}(F) \geq 2$.*

Seeking a contradiction, we assume throughout that such a del Pezzo surface V exists over a ground field F with $\mathrm{pdeg}(F) \leq 1$. Without restriction, we may assume that the structure morphism $V \to \mathrm{Spec}(F)$ is adapted, and that F is separably closed. As usual, we use the notation from Section 5, such that $Y = V \otimes_F K$ is a non-normal del Pezzo surface whose normalization X is geometrically normal, with conductor square

$$
\begin{array}{ccc}
R & \longrightarrow & X \\
\downarrow & & \downarrow \\
C & \longrightarrow & Y.
\end{array}
$$

Recall that the normalization is either the projective plane $X = \mathbb{P}^2$ or the weighted projective plane $X = \mathbb{P}(1, 1, 3)$. We write $N \subset \mathrm{Sing}(V/F)$ for the divisorial part of the locus of non-smoothness and $D = N_{\mathrm{red}}$ for its reduction.

Proposition 13.2. *The conductor curve C is isomorphic to $\mathbb{P}^1$. Furthermore, we have $h^1(\mathscr{L}) = 0$ for each $\mathscr{L} \in \mathrm{Pic}(V)$, and $h^0(\mathscr{O}_A) = 1$ for each effective Cartier divisor $A \subset V$, and the restriction map $\mathrm{Pic}(V) \to \mathrm{Pic}(X)$ is injective.*

Proof. According to Propositions 5.5 and 5.8, the ramification divisor R is the split ribbon $\mathbb{P}^1 \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$. Furthermore, C is geometrically integral by Proposition 5.6, hence $C = (D_K)_{\mathrm{red}}$. To conclude that $C = \mathbb{P}^1$, it suffices to check that $h^1(\mathscr{O}_C) = 0$.

By Proposition 4.6, we have $h^1(\mathscr{O}_V) = h^1(\mathscr{O}_C)$. In light of Maddock's bound (9), it thus suffices to check $h^1(\mathscr{O}_C) \leq 1$. Since the canonical map

$$H^1(D_K, \mathscr{O}_{D_K}) \longrightarrow H^1((D_K)_{\mathrm{red}}, \mathscr{O}_{(D_K)_{\mathrm{red}}}) = H^1(C, \mathscr{O}_C)$$

is surjective, it is enough to verify that one of the two groups on the left is at most one-dimensional. Indeed, we have $h^1(\mathscr{O}_D) = 1$ if the normalization is a weighted projective plane $X = \mathbb{P}(1,1,3)$, according to the results in Section 8.

Now suppose that $X = \mathbb{P}^2$, and consider the integral curve $D \subset V$. We first check that this is a peculiar curve of higher genus studied in Section 12. According to the result in Section 8, we have $h^0(\mathscr{O}_D) = 1$ and $h^1(\mathscr{O}_D) = 4 = p + 1$. This curve is an effective Cartier divisor in a regular surface, so all local rings $\mathscr{O}_{D,a}$ are Gorenstein. They are also geometrically unibranch, since $R \to D$ is a universal homeomorphism. The curve D contains no rational points, because it lies in the locus of non-smoothness $\mathrm{Sing}(V/F)$, by Corollary 2.6. It remains to check that the normalization $\tilde{D}$ has $h^1(\mathscr{O}_{\tilde{D}}) = 0$. The conductor square

$$\begin{array}{ccc} A & \longrightarrow & \tilde{D} \\ \downarrow & & \downarrow \\ B & \longrightarrow & D \end{array}$$

for the normalization yields an exact sequence

$$0 \to H^0(\mathscr{O}_D) \to H^0(\mathscr{O}_{\tilde{D}}) \oplus H^0(\mathscr{O}_B) \to H^0(\mathscr{O}_A) \to H^1(\mathscr{O}_D) \to H^1(\mathscr{O}_{\tilde{D}}) \to 0.$$

The field extension $\tilde{F} = H^0(\tilde{D}, \mathscr{O}_{\tilde{D}})$ is purely inseparable, of degree 3^ν for some exponent $\nu \geq 0$. We have $h^1(\mathscr{O}_{\tilde{D}}) = 3^\nu g$ and $h^0(\mathscr{O}_A) = 3^\nu a$ for some integers $g \geq 0$ and $a \geq 1$. Moreover, $h^0(\mathscr{O}_B) = 3^\nu a/2$, according to Proposition A.2. The above exact sequence gives $1 - (3^\nu + 3^\nu a/2) + 3^\nu a - 4 + 3^\nu g = 0$, in other words

$$(26) \qquad\qquad\qquad 3^\nu(a/2 + g - 1) = 3.$$

In particular $0 \leq \nu \leq 1$. If $\nu = 0$, the normal curve $\tilde{D}$ is geometrically normal, according to [60, Theorem 2.3]. Here we use that $\mathrm{pdeg}(F) \leq 1$. It follows that D is also geometrically normal, because the normalization map $\tilde{D} \to D$ is birational. This contradicts that the Cartier divisor $D \subset V$ is contained in the locus of non-smoothness $\mathrm{Sing}(V/F)$, by Corollary 2.7. We thus have $\nu = 1$, and equation (26) only has the following two solutions:

$$g = 0, a = 4 \quad \text{and} \quad g = 1, a = 2.$$

In the former case, D is a peculiar curve of higher genus, and Proposition 12.2 tells us that $h^1(\mathscr{O}_C) \leq 1$. In the latter case, $\tilde{F} = H^0(\tilde{D}, \mathscr{O}_{\tilde{D}})$ is a field extension of the form $\tilde{F} = F(\beta^{1/3})$ for some non-cube $\beta \in F^\times$. Note that this field extension is unique up to isomorphism, because now $\mathrm{pdeg}(F) = 1$. Moreover, the cohomology group $H^1(\tilde{D}, \mathscr{O}_{\tilde{D}})$ is a one-dimensional vector space over $\tilde{F}$, and we fix a cohomology class $\alpha \neq 0$. Now consider the base-change $\tilde{F} \otimes_F K = K[t]/(t^3)$. The resulting nilpotent element $t \in H^0(\tilde{D}_K, \mathscr{O}_{\tilde{D}_K})$ vanishes on the reduction $C = (\tilde{D}_K)_{\mathrm{red}}$, and in turn the image of α in the cohomology group $H^1(C, \mathscr{O}_C)$ is annihilated by t. It follows that the image of the surjective restriction map

$$H^1(D_K, \mathscr{O}_{D_K}) \longrightarrow H^1((D_K)_{\mathrm{red}}, \mathscr{O}_{(D_K)_{\mathrm{red}}}) = H^1(C, \mathscr{O}_C)$$

is at most one-dimensional, and we get $h^1(\mathscr{O}_C) \leq 1$ again.

Now let $\mathscr{L}$ be an invertible sheaf on the del Pezzo surface V. The short exact sequence $0 \to \mathscr{L}_Y \to \mathscr{L}_X \oplus \mathscr{L}_C \to \mathscr{L}_R \to 0$ from the conductor square yields an exact sequence

$$H^0(\mathscr{L}_X) \oplus H^0(\mathscr{L}_C) \to H^0(\mathscr{L}_R) \to H^1(\mathscr{L}_Y) \to H^1(\mathscr{L}_X) \oplus H^1(\mathscr{L}_C) \to H^1(\mathscr{L}_R).$$

Now we use some general facts: For each weighted projective space $\mathbb{P} = \mathbb{P}(q_0, \ldots, q_n)$ with $\gcd(q_0, \ldots, q_n) = 1$, all tautological sheaves $\mathscr{O}_{\mathbb{P}}(n)$ have trivial cohomology in intermediate degrees $0 < i < n$, according to [21, Theorem in Section 1.4]. Moreover, it is well known that $\mathscr{O}_{\mathbb{P}}(1)$ generates the group of reflexive rank one sheaves $\mathrm{APic}(\mathbb{P})$; compare [17, Section 4.1]. It follows that $H^1(X, \mathscr{L}_X) = 0$, and also that the restriction map $H^0(X, \mathscr{L}_X) \to H^0(R, \mathscr{L}_R)$ is surjective.

Recall that $R = \mathbb{P}^1 \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$ is the split ribbon, hence $R_{\mathrm{red}} \to C$ is an isomorphism. As a consequence, the pull-back map $H^1(\mathscr{L}_C) \to H^1(\mathscr{L}_R)$ is injective. In light of the above long exact sequence, the group $H^1(Y, \mathscr{L}_Y)$ vanishes, thus $h^1(\mathscr{L}) = 0$. Applying this to $\mathscr{L} = \mathscr{O}_X$ we infer that the Picard scheme $\mathrm{Pic}^0_{V/F} = 0$, such that $\mathrm{Pic}(V) = \mathrm{NS}(V)$. The injectivity of $\mathrm{Pic}(V) \to \mathrm{Pic}(X)$ follows from Proposition 4.6. $\qquad\square$

For $n \geq 0$, Serre duality gives $h^2(\omega_V^{\otimes -n}) = 0$, because $\omega_V^{\otimes n+1}$ is antiample, so Riemann–Roch yields

$$(27) \qquad h^0(\omega_V^{\otimes -n}) = \frac{(-nK_V)^2 - (-nK_V) \cdot K_V}{2} + \chi(\mathscr{O}_V) = \frac{n(n+1)}{2} K_V^2 + 1.$$

To proceed, we consider the invertible sheaves

$$\mathscr{L} = \begin{cases} \omega_V^{\otimes -2} & \text{if } X = \mathbb{P}^2; \\ \omega_V^{\otimes -1} & \text{if } X = \mathbb{P}(1,1,3); \end{cases}$$

which both have $h^0(\mathscr{L}) = 4$.

Proposition 13.3. *The ample invertible sheaf $\mathscr{L}$ is globally generated.*

Proof. Suppose first that $X = \mathbb{P}^2$. Consider first the invertible sheaf $\omega_X^{\otimes -1}$. Since it generates the Picard group, the effective Cartier divisor $A \subset V$ attached to any non-zero $s \in H^0(X, \omega_X^{\otimes -1})$ is integral. Moreover, the selfintersection number is $K_V^2 = 1$, and $h^0(\omega_V^{\otimes -1}) = 2$. Thus there is another non-zero global section s' such that the corresponding effective Cartier divisor A' has $A \cap A' = \mathrm{Spec}\,\kappa(v)$ for some rational point $a \in V$. Moreover, the local ring $\mathscr{O}_{A,a}$ is regular. This already shows that $\mathrm{Bs}(\mathscr{L}) \subset \{a\}$. The restriction map $H^0(V, \mathscr{L}) \to H^0(A, \mathscr{L}_A)$ is surjective because $H^1(X, \mathscr{L}(-A)) = 0$.

It thus suffices to check that $a \in A$ is not a base point for the restriction $\mathscr{L}_A$. This would hold if $h^1(\mathscr{L}_A(-a)) = h^0(\omega_A \otimes \mathscr{L}^{\otimes -1}(a))$ vanishes. The adjunction formula shows that the degree of $\omega_A \otimes \mathscr{L}_A^{\otimes -1}(a)$ is

$$(K_V + A) \cdot A - (\mathscr{L} \cdot A) + 1 = 0 - 2 + 1 = -1.$$

In turn, the invertible sheaf $\omega_A \otimes \mathscr{L}^{\otimes -1}(a)$ on the integral scheme A has no non-zero global sections. This shows that $\mathscr{L}$ is globally generated. In the case $X = \mathbb{P}(1,1,3)$, one argues as above with $\omega_V^{\otimes -1} = \mathscr{L}$. $\qquad\square$

Consider the finite morphism $V \to \mathbb{P}^3$ resulting from the linear system $H^0(V, \mathscr{L})$. Its image is an integral surface $V' \subset \mathbb{P}^2$, and we get a finite surjection $f: V \to V'$. There are only a few possibilities.

Proposition 13.4. *The finite morphism $f : V \to V'$ is birational. The image $V' \subset \mathbb{P}^3$ is a quartic surface if $X = \mathbb{P}^2$, and a cubic surface if $X = \mathbb{P}(1,1,3)$.*

Proof. This follows from $(\mathscr{L} \cdot \mathscr{L}) = \deg(f) \cdot \deg(V')$ and $\deg(V') > 1$. In case $X = \mathbb{P}(1,1,3)$, the selfintersection is $(\mathscr{L} \cdot \mathscr{L}) = 3$, and the assertion is immediate.

Now suppose that $X = \mathbb{P}^2$. Seeking a contradiction, we assume that $f : V \to V'$ is a double covering over some quadric surface V' in $\mathbb{P}^3 = \operatorname{Proj} F[T_0, \ldots, T_3]$. As $p \neq 2$, we may assume that V' is defined by the homogeneous equation $T_0^2 + \ldots + T_n^2 = 0$ for some $0 \leq n \leq 3$. We have $n \neq 0$, because V' is reduced. Moreover, $n \neq 1$ because V' is irreducible. In case $n = 3$, we would have $V' \simeq \mathbb{P}^1 \times \mathbb{P}^1$, in contradiction to $\rho(V) = 1$. Thus $n = 2$, and $V' = \mathbb{P}(1,1,2)$ is isomorphic to the contracted Hirzebruch surface with numerical invariant $e = 2$.

To simplify notation, set $\mathbb{P} = \mathbb{P}(1,1,2)$ and write $\mathscr{O}_{\mathbb{P}}(1)$ for the tautological sheaf on the weighted homogeneous spectrum $\mathbb{P} = \operatorname{Proj} k[T_0, T_1, T_2]$. This sheaf generates the group $\operatorname{APic}(\mathbb{P})$ of isomorphism classes of reflexive rank one sheaves. Note that $\mathscr{O}_{\mathbb{P}}(2)$ corresponds to the restriction of the invertible sheaf $\mathscr{O}_{\mathbb{P}^3}(1)$ to the quadric surface $\mathbb{P} = V'$, and that $\mathscr{O}_{\mathbb{P}}(1)$ comes from the fiber on the resolution of singularities, which is a Hirzebruch surface. The latter shows that the rational selfintersection number of $\mathscr{O}_{\mathbb{P}}(1)$ is $1/2$, hence the induced map $f^* : \operatorname{APic}(\mathbb{P}) \to \operatorname{Pic}(V)$ under the double covering is bijective. In particular, $\omega_V^{\otimes -1}$ is the reflexive hull of the rank one sheaf $f^* \mathscr{O}_{\mathbb{P}}(1)$.

Since $p \neq 2$, we may view V as the relative spectrum of $\mathscr{A} = \mathscr{O}_{\mathbb{P}} \oplus \mathscr{O}_{\mathbb{P}}(-d)$ for some integer d. The multiplication law for $\mathscr{A}$ is given by some homomorphism $\varphi : \mathscr{O}_{\mathbb{P}}(-2d) \to \mathscr{O}_{\mathbb{P}}$, which can also be viewed as a global section $\varphi \in \Gamma(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(2d))$. We have $d > 0$ because $h^0(\mathscr{O}_V) = 1$. Moreover, the integer d is odd because V is regular whereas $\mathbb{P}$ is singular, such that the finite morphism $f : V \to \mathbb{P}$ is not flat. We have $\omega_{\mathbb{P}} = \mathscr{O}_{\mathbb{P}}(-4)$, and $H^2(V, \mathscr{O}_V) = H^2(\mathbb{P}, \mathscr{A})$ is Serre dual to $\operatorname{Hom}(\mathscr{A}, \omega_{\mathbb{P}}) = H^0(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(-4) \oplus \mathscr{O}_{\mathbb{P}}(d-4))$, hence $d \leq 3$. Furthermore, for each integer t the Projection Formula yields $f_*(\omega_V^{\otimes -t}) = \mathscr{O}_{\mathbb{P}}(t) \oplus \mathscr{O}_{\mathbb{P}}(t-d)$, thus

$$2 = h^0(\omega_V^{\otimes -1}) = h^0(\mathscr{O}_{\mathbb{P}}(1)) + h^0(\mathscr{O}_{\mathbb{P}}(1-d)).$$

Using $h^0(\mathscr{O}_{\mathbb{P}}(1)) = 2$ we conclude $1 - d < 0$. Thus $d = 3$ is the only remaining possibility.

Summing up, $\mathscr{A} = \mathscr{O}_{\mathbb{P}} \oplus \mathscr{O}_{\mathbb{P}}(-3)$, and the multiplication law comes from some non-zero section $\varphi \in \Gamma(\mathbb{P}, \mathscr{O}_{\mathbb{P}}(6))$. The latter defines an effective Cartier divisor $B \subset \mathbb{P}$. In light of the multiplication law on $\mathscr{A}$, the schematic preimage takes the form $f^{-1}(B) = 2A$ for some effective Cartier divisor $A \subset V$. Let $U \subset \mathbb{P}$ be the complement of $B \cup \operatorname{Sing}(\mathbb{P})$. The scheme U is smooth and the double covering $f : V \to \mathbb{P}$ is étale over U, which implies $D \subset A$. We actually have $D = A$, because $\mathscr{O}_V(D) = \omega_V^{\otimes -3} = \mathscr{O}_V(A)$. In turn, the induced morphism $f : D \to B$ is birational. Since D is singular, the integral curve B is singular as well.

We claim that B does not pass through the singular point $y \in \mathbb{P}$, where the local ring $\mathscr{O}_{\mathbb{P}, y}$ is a rational double point of type A_1, whose local fundamental group $\pi_1^{\operatorname{loc}}(\mathscr{O}_{\mathbb{P}, y})$ is cyclic of order two. Let $x \in f^{-1}(y)$ be the preimage, and suppose that $y \in B$. Applying [36, Proposition 2.3] for the inclusion of local rings $\mathscr{O}_{\mathbb{P}, y} \subset \mathscr{O}_{V, x}$, we conclude that $\pi_1^{\operatorname{loc}}(\mathscr{O}_{\mathbb{P}, y})$ is trivial, a contradiction.

Fix a closed point $b \in B$ where the local ring $\mathscr{O}_{B,b}$ is singular. By the preceding paragraph we have $\varphi_b \in \mathfrak{m}_b^2 \mathscr{O}_{\mathbb{P},b}(2d)$. Now choose an identification $\mathscr{O}_{\mathbb{P},b}(2d) = \mathscr{O}_{\mathbb{P},b}$ and a regular system of parameters $x, y \in \mathscr{O}_{\mathbb{P},b}^\wedge$, and write $\varphi_b = g(x,y)$ as a formal power series with neither constant nor linear terms. For the point $a \in f^{-1}(b)$, we get $\mathscr{O}_{V,a}^\wedge = \kappa(b)[[x,y,z]]/(z^2 - g)$. Since $z^2 - g \in (x,y,z)^2$, the local ring $\mathscr{O}_{V,a}$ is singular, a contradiction. $\qquad\square$

Proof for Theorem 13.1. Consider first the case that $X = \mathbb{P}^2$. Then $\mathscr{L} = \omega_V^{\otimes -2}$ defines a finite birational morphism $f : V \to V'$ onto some quartic surface $V' \subset \mathbb{P}^3$, with $\mathscr{L} = f^*(\mathscr{O}_{V'}(1))$. The Adjunction Formula gives $\omega_{V'} = \mathscr{O}_{V'}$, and consequently $\omega_V = \mathscr{O}_V(-A)$ where $A \subset V$ is the ramification divisor. We obtain a contradiction by showing that $f : V \to V'$ does not ramify along A. Since $\omega_V^{\otimes -1}$ generates the Picard group and V is regular, the curve A must be integral. The long exact sequence for $0 \to \omega_V \to \mathscr{O}_V \to \mathscr{O}_A \to 0$, together with Proposition 13.2 yields $h^0(\mathscr{O}_A) = 1$, and the Adjunction Formula gives $\omega_A = \mathscr{O}_A$, whence $h^1(\mathscr{O}_A) = 1$. The restriction $\mathscr{L}|A$ has degree $(\mathscr{L} \cdot A) = 2$, and Riemann–Roch gives $h^0(\mathscr{L}|A) = 2$. So the globally generated sheaf $\mathscr{L}|A$ defines a morphism $A \to \mathbb{P}^1$ of degree two. Consequently, the morphism $f : V \to V'$ does not ramify along A, a contradiction.

Now consider the case $X = \mathbb{P}(1,1,3)$. Then $\mathscr{L} = \omega_V^{\otimes -1}$ defines a finite birational morphism $f : V \to V'$ onto some cubic surface $V' \subset \mathbb{P}^3$. The Adjunction Formula gives $\omega_{V'} = \mathscr{O}_{V'}(-1)$, and we have $\omega_V = f^*(\omega_{V'}) \otimes \mathscr{O}_V(-R)$, where $R \subset V$ is the ramification divisor. Here we have $(\omega_V \cdot \omega_V) = 3$ and $(\omega_{V'} \cdot \omega_{V'}) = 3$. Together with $\mathrm{Pic}(V) = \mathbb{Z}$, one easily infers that the ramification divisor is empty, such that $V = V'$ itself is a cubic surface. For each anticanonical divisor $A \subset V$, we have $h^0(\mathscr{O}_A) = 1$ and $\omega_A = \mathscr{O}_A$, whence also $h^1(\mathscr{O}_A) = 1$. Riemann–Roch gives $h^0(\mathscr{L}|A) = 3$, showing that A becomes a quadric curve inside some plane $\mathbb{P}^2 \subset \mathbb{P}^3$ under the embedding $V \subset \mathbb{P}^3$.

To proceed, we consider the special case that $A = D$ is the reduced locus of non-smoothness, which is geometrically non-reduced. The cubic curve $D \subset \mathbb{P}^2$ is given by some irreducible homogeneous polynomial $\bar{P} \in F[T_0, T_1, T_2]$ of degree three. Since $\mathrm{Sing}(D/F) = D$, all partial derivatives $\bar{P}_i = \partial \bar{P}/\partial T_i$ are multiples of $\bar{P}$. For degree reasons we must have $\bar{P}_i = 0$, and thus our polynomial takes the form $\bar{P} = \sum_{i=0}^2 \lambda_i T_i^3$ for some scalars $\lambda_i \in F$. Thus $D \subset \mathbb{P}^2$ is a p-Fermat hypersurface, which were studied in [60, §3].

In turn, the cubic surface $V \subset \mathbb{P}^3$ is given by a homogeneous polynomial of the form

$$P(T_0, T_1, T_2, T_3) = \sum_{i=0}^3 \lambda_i T_i^3 + T_3^2 L + T_3 Q$$

for some linear term $L = L(T_0, T_1, T_2)$ and some quadratic term $Q = Q(T_0, T_1, T_2)$. Here the curve $D \subset V$ is given by $T_3 = 0$. Since $D \subset \mathrm{Sing}(V/F)$, all partial derivatives $\partial P/\partial T_i$ are multiples of T_3. Using $\partial P/\partial T_3 = 2T_3 L + Q$, we infer $T_3|Q$ and thus may assume $Q = 0$.

The curve D is integral with $h^0(\mathscr{O}_D) = 1$, and our ground field has $\mathrm{pdeg}(F) \le 1$. According to [60, Theorem 2.3] there must be some closed point $a \in D$ where the local ring $\mathscr{O}_{D,a}$ is singular. This point has homogeneous coordinates $(\alpha_0 : \alpha_1 : \alpha_2 : 0)$ with $\alpha_i \in F^{\mathrm{alg}}$. Without restriction, we may assume that $\alpha_0 = 1$. Then the homogenization $\bar{P}T_0^{-3}$ lies in the square of the maximal ideal $\mathscr{O}_{\mathbb{P}^2,a}$. In light of the

form $P = \sum \lambda_i T_i^3 + T_3^2 L$, it follows that PT_0^{-3} lies in the square of the maximal ideal of $\mathscr{O}_{\mathbb{P}^3,a}$, thus the local ring $\mathscr{O}_{V,a}$ is singular, a contradiction. $\square$

14. DEL PEZZO SURFACES AND FIBRATIONS

We now can state and prove the main result of our paper.

Theorem 14.1. *Let F be a ground field of characteristic $p > 0$ with p-degree $\mathrm{pdeg}(F) \leq 1$. Then every regular del Pezzo surface V over F with Picard number $\rho(V) = 1$ is geometrically normal.*

Proof. Seeking a contradiction, assume there exists a del Pezzo surface V with Picard number $\rho(V) = 1$ that is geometrically non-normal. Since $\mathrm{pdeg}(F) \leq 1$, we can apply [60, Theorem 2.3] and conclude that V is geometrically integral. According to Theorem 5.9, we must have $p \leq 3$. Choose some finite separable field extension $F \subset F'$ and some finite purely inseparable extension $F \subset K'$ so that $V' = V \otimes_F F'$ and $F' \subset F' \otimes_F K'$ are adapted, as explained in Section 4.

The geometry of V' was described in the tables of Sections 6, 7, and 8. In all but one case, both surfaces V and V' have Picard number $\rho = 1$. The case with $\rho(V') > 1$ is treated in Theorem 6.4, and then $\rho(V') = 2$. In this case, we also have $\rho(V) = 2$, as remarked at the end of Section 6, a contradiction.

Thus we may assume from the start that F is separably closed, and that we have a finite purely inseparable field extension $F \subset K$ such that V is adapted. The possibilities for V with arbitrary p-degree were first narrowed down in Propositions 6.2, 7.2, and 8.2. Finally, the non-existence of the remaining possibilities for $\mathrm{pdeg}(F) = 1$ follow from Theorems 6.4, 10.1, 11.1, and 13.1. $\square$

The previous result is sharp: In the rest of this section we will describe two new examples V and W of regular del Pezzo surfaces defined over any imperfect field F in characteristic $p = 2$ with the following properties:

(i) W geometrically non-regular, with Picard number $\rho(W) = 1$;
(ii) V geometrically non-normal, with Picard number $\rho(V) = 2$.

Recall that $z^p - xy = 0$ is the equation, in normal form, for the rational double point of type A_{p-1}. In what follows we shall use twisted forms of the corresponding local ring that are regular. It is very easy to determine when this happens.

Lemma 14.2. *Let R be a local ring that is essentially of finite-type over F. Suppose there is a finite purely inseparable extension $F \subset K$ and an isomorphism*

$$\hat{R} \otimes_F K \simeq K[[x, y, z]]/(z^p - xy).$$

Then the degree $d = [\kappa : F]$ of the residue field $\kappa = R/\mathfrak{m}_R$ is either $d = 1$ or $d = p$. The latter holds if and only if the local R is regular.

Proof. Let $J \subset R$ be the Jacobian ideal. Then $M = R/J$ has finite length. Computing with $K[[x, y, z]]/(z^p - xy)$, one sees that its vector space dimension is $\dim_F(M) = p$. Since there is a filtration on M whose subquotients are copies of the unique simple R-module $\kappa = R/\mathfrak{m}_R$, we conclude that either $d = 1$ or $d = p$. In the latter case, we must have $M \simeq \kappa$, so the Jacobian ideal coincides with the maximal ideal. Computing again with $K[[x, y, z]]/(z^p - xy)$, we see that M has finite projective dimension. By the homological characterization of regularity, the local ring R must be regular. $\square$

If a local ring R is regular and satisfies the assumption of the lemma, we call it a *regular twisted form of the rational double point of type A_{p-1}*. See [59] for more on this. If the ground field F is imperfect, such rings indeed do exist: Choose a scalar $\beta \in F$ that is not a pth power, and consider the local ring R coming from the residue class ring $K[x, y, z]/(z^p - xy - \beta)$ with respect to the maximal ideal $\mathfrak{m} = (x, y)$. If β becomes a pth power in the field extension $F \subset K$, then the substitution $z = z' + \beta^{1/p}$ yields the desired isomorphism to the rational double point of type A_{p-1}.

Suppose that R is a regular twisted form of a rational double point of type A_{p-1}, with residue field $\kappa = R/\mathfrak{m}_R$. Let $r : X \to \mathrm{Spec}(R)$ be the blowing-up of the reduced closed point, and write $D \subset X$ for the resulting exceptional divisor.

Proposition 14.3. *As a scheme, the exceptional divisor D is isomorphic to $\mathbb{P}^1_\kappa$, and its selfintersection number is $D^2 = -p$. The surface X is regular but not geometrically normal. For $p = 2$, the locus of non-smoothness $N = \mathrm{Sing}(X/F)$ is given by $N = 2D$. Otherwise, it contains two embedded associated points, and its divisorial part is $N_{\mathrm{div}} = D$.*

Proof. Regarding the blowing-up as a closed subscheme $X \subset \mathbb{P}^1 \otimes_F R$ as in [8, Exposé VII, Proposition 1.8], we see that the exceptional divisor is given by $D = \mathbb{P}^1_\kappa$, and it follows that the scheme X is regular. To see that it is not geometrically normal, we may compute with $A = K[[x, y, z]]/(z^p - xy)$. According to the proof of Lemma 14.2, the center of the blowing-up is defined by the Jacobian ideal, which is generated by $x, y \in A$. The x-chart of the blowing-up is given by the variables $x, y/x, z$ modulo the relation $z^p - x^2(y/x) = 0$, and the exceptional divisor has equation $x = 0$. The module $\Omega^1_{A/K}$ is generated by the differentials $dx, d(y/x), dz$ modulo the relation $2x(y/x)dx + x^2 d(y/x) = 0$. Consequently, the locus of non-smoothness is defined by the ideal $(2x(y/x), x^2) \subset A$, and the assertion on $N = \mathrm{Sing}(X/F)$ follows. $\square$

Now fix a scalar $\beta \in F$, and consider the hypersurfaces $W = W_\beta$ inside $\mathbb{P}^3$ of degree $\deg(W) = p$ over the ground field F given by the homogeneous equation

$$(28) \qquad\qquad T_2^p + T_0 T_1 T_3^{p-2} - \beta T_3^p = 0.$$

These are projective Gorenstein surfaces with $h^0(\mathscr{O}_W) = 1$ and $h^1(\mathscr{O}_W) = 0$. Furthermore, the dualizing sheaf is $\omega_W = \mathscr{O}_W(p - 4)$. Note that W is a del Pezzo surface if and only if $p \leq 3$. In any case, $K_W^2 = (p - 4)^2 p$. In characteristic two, this becomes $K_W^2 = 8$.

Clearly, the substitution $T_2 = T_2' + \beta^{1/p} T_3'$ shows that the subschemes $W_\beta, W_0 \subset \mathbb{P}^3$ become projectively equivalent over the field extension $K = F(\beta^{1/p})$. In particular, each $W = W_\beta$ is a twisted form of W_0.

Proposition 14.4. *In characteristic $p = 2$, the restriction map $\mathrm{Pic}(\mathbb{P}^2) \to \mathrm{Pic}(W)$ is bijective. For $p \geq 3$, this holds true up to torsion in $\mathrm{Pic}(W)$.*

Proof. Since $H^1(W, \mathscr{O}_W) = 0$, the Picard scheme is zero-dimensional, thus the canonical map $\mathrm{Pic}(W) \to \mathrm{NS}(W)$ is bijective, and this group is finitely generated.

Consider the hyperplane section $H = W \cap V_+(T_3) \subset W$. Clearly, the affine scheme $U = W \smallsetminus H$ is isomorphic to the spectrum of $k[x, y, z]/(z^p - xy)$. This can be regarded as an affine toric variety, and thus its Picard group vanishes ([38, Theorem 9 on page 28]; compare also [17, Proposition 4.2.2]). Consequently, every Cartier

divisor $D \in \mathrm{Div}(W)$ is linearly equivalent to some Cartier divisor supported by the effective Cartier divisor $H \subset W$.

Using equation (28) and treating the case $p = 2$ and $p \geq 3$ separately, one easily sees that the scheme H is irreducible. Let $\eta \in H$ be the generic point, and set $n = \mathrm{length}(\mathscr{O}_{H,\eta})$. Then $(D \cdot H) = n \deg(\mathscr{L}|H_{\mathrm{red}})$, where $\mathscr{L} = \mathscr{O}_W(D)$. Since W is projective, the intersection form on $\mathrm{NS}(W)$ modulo its torsion subgroup is non-degenerate ([8, Exposé XIII, Corollary 7.4]), and we infer that every Cartier divisor $D \in \mathrm{Div}(W)$ is numerically equivalent to some rational multiple of H. Since $H^2 = p$, the element $H \in \mathrm{Pic}(W)$ is primitive, and thus $H \in \mathrm{NS}(W)$ is a generator up to torsion. In case $p = 2$, the hyperplane H is isomorphic to the quadric curve in $\mathbb{P}^2$ given by $T_2^2 + T_0 T_1 = 0$, which is integral, and it follows that $H \in \mathrm{NS}(W)$ indeed is a generator. $\square$

Consider the closed point $x = (0 : 0 : \beta^{1/p} : 1)$. If the scalar $\beta \in F$ is a pth power, then $x \in W$ is a rational point, and the local ring $R = \mathscr{O}_{W,x}$ is a rational double point of type A_{p-1}. Now suppose that $\beta \in F^\times$ is not a pth power. Then $x \in W$ is non-rational, with residue field $K = F(\beta^{1/p}) = \kappa(x)$. By Lemma 14.2, the local ring $R = \mathscr{O}_{W,x}$ is a regular twisted form of the rational double point of type A_{p-1}.

The two hyperplanes $H_0 = V_+(T_0)$ and $H_1 = V_+(T_1)$ yield two linearly equivalent Cartier divisors $C_i = W \cap H_i$ whose intersection $C_1 \cap C_2$ consists of this point $x \in W$, viewed as a reduced subscheme. Let $r : V \to W$ be the blowing-up with reduced center $x \in W$. The exceptional divisor $D = r^{-1}(x)$ is a copy of $\mathbb{P}^1_K$, and the strict transforms of the Cartier divisors C_1, C_2 yield a fibration $\varphi : V \to \mathbb{P}^1$, for which $D \subset X$ is horizontal, of relative degree $\deg(D/\mathbb{P}^1) = p$.

Proposition 14.5. *Suppose $p = 2$. Then V is a regular del Pezzo surface of degree $K_V^2 = 6$. Its Picard number is $\rho(V) = 2$, and the Picard group is freely generated by the fiber $F = \varphi^{-1}(\infty)$ and the exceptional divisor $D = r^{-1}(x)$. The Gram matrix is $\left(\begin{smallmatrix} 0 & 2 \\ 2 & -2 \end{smallmatrix}\right)$, and we have $K_V = -(D + 2F)$. Furthermore, $\mathrm{Sing}(V/F) = D$, and the locus of non-smoothness is $N = 2D$.*

Proof. Write $K_{V/W} = nD$ for some integer $n \in \mathbb{Z}$. The Adjunction Formula gives

$$-2(n+1) = (n+1)D^2 = (K_{V/W} + D) \cdot D = \deg(K_D) = -2\chi(\mathscr{O}_{\mathbb{P}^1_K}) = -4,$$

thus $n = 1$. Consequently $K_V^2 = K_W^2 + K_{V/W}^2 = 6$. Using $r^{-1}(H_0) = F + D$ and $K_W = -2H_0$, we infer $K_V = -(D + 2F)$. By Proposition 14.4, the Picard group of W is generated by H_0, so $\mathrm{Pic}(V)$ is generated by the strict transform F and the exceptional divisor D. We compute $(K_V \cdot D) = D^2 = -2$ and
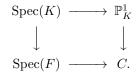
$$(K_V \cdot F) = K_V \cdot (r^{-1}(H_0) - D) = (K_W \cdot H_0) - D^2 = -4 + 2 = -2.$$

The cone of curve is generated by the two extremal rays coming from $D, F \subset V$. By the Nakai Criterion ([34, Chapter I, Theorem 5.1]), the anticanonical divisor $-K_V$ is ample. $\square$

Let us examine the fibration $\varphi : V \to \mathbb{P}^1$ for $p = 2$ in more detail. The fiber over a rational point $(\lambda_0 : \lambda_1) \in \mathbb{P}^1$, say with $\lambda_0 = 1$, can be regarded as the zero-scheme inside $\mathbb{P}^2$ for the equation

$$T_2^2 + \lambda_1 T_1^2 - \beta T_3^2 = 0.$$

According to [60, Theorem 3.3] this is regular provided that the field extension $F \subset F(\lambda_1^{1/2}, \beta^{1/2})$ has degree four. In our situation, this means $\lambda_1^{1/2} \notin K$. On the other hand, if the condition does not hold, the curve $C = \varphi^{-1}(\lambda_0 : \lambda_1)$ is at least integral, and its normalization is given by the conductor square

$$
\begin{array}{ccc}
\mathrm{Spec}(K) & \longrightarrow & \mathbb{P}^1_K \\
\downarrow & & \downarrow \\
\mathrm{Spec}(F) & \longrightarrow & C.
\end{array}
$$

Roughly speaking, a K-rational point on $\mathbb{P}^1_K$ is replaced by an F-rational point on C. In any case, the fiber is a twisted form of the ribbon $\mathbb{P}^1 \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$. In particular, all fibers are proper curves C with $h^0(\mathscr{O}_C) = 1$ and $h^1(\mathscr{O}_C) = 0$.

Let us call a proper morphism between integral scheme $q : Z \to B$ a *genus-zero fibration* if $q_*(\mathscr{O}_Z) = \mathscr{O}_B$ and the generic fiber $C = Z_\eta$ is a curve with $h^0(\mathscr{O}_C) = 1$ and $h^1(\mathscr{O}_C) = 0$. If the generic fiber is smooth, we say that the fibration is a *ruling*. Otherwise, we call it a *quasiruling*, in analogy to the quasielliptic fibrations.

The regular del Pezzo surfaces W and V constructed above actually occur as generic fiber of some *del Pezzo fibrations*. Fix some algebraically closed ground field k of characteristic $p = 2$. Consider the divisor $Y \subset \mathbb{P}^1 \times \mathbb{P}^3$ of bidegree $\deg(Y) = (1, 2)$ given by the bihomogeneous equation

$$
(29) \qquad\qquad S_0(T_2^2 + T_0 T_1) - S_1 T_3^2 = 0,
$$

where the S_0, S_1 and $T_0, \ldots, T_3$ are the homogeneous coordinates for $\mathbb{P}^1$ and $\mathbb{P}^3$, respectively. Write $\mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(d_1, d_2) = \mathrm{pr}_1^*(\mathscr{O}_{\mathbb{P}^1}(d_1)) \otimes \mathrm{pr}_2^*(\mathscr{O}_{\mathbb{P}^1}(d_2))$ to simplify notation. The Adjunction Formula gives

$$
\omega_Y = \mathscr{O}_Y(-1, -2) = \mathscr{O}_Y(-Y),
$$

so our Y is a Fano threefold of degree $-K_Y^3 = (A + 2B)^4 = A \cdot (2B)^3 = 8$, where $A, B \subset \mathbb{P}^1 \times \mathbb{P}^3$ are divisors corresponding to $\mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(1, 0)$ and $\mathscr{O}_{\mathbb{P}^1 \times \mathbb{P}^3}(0, 1)$.

Proposition 14.6. *The Fano threefold Y is normal. Its singular scheme $\mathrm{Sing}(Y)$ is given by $V_+(S_0, T_3, T_2^2 + T_0 T_1)$, which is isomorphic to $\mathbb{P}^1$. Moreover, for each closed point $a \in \mathrm{Sing}(Y)$, the corresponding complete local ring $\mathscr{O}_{Y,a}^\wedge$ is isomorphic to $k[[x, y, z, w]]/(z^2 - xy)$. Furthermore, the blowing-up $Y' \to Y$ with reduced center $\mathrm{Sing}(Y)$ is a crepant resolution of singularities.*

Proof. The assertion on $\mathrm{Sing}(Y)$ follows from computing the partial derivatives in (29). We see that Y is regular in codimension one. Being a hypersurface, it is also Cohen–Macaulay, and Serre's Criterion ensures that Y is normal.

Each singular point $a \in Y$ must have homogeneous coordinates $(0 : 1)$ in the homogeneous coordinates with respect to the first factor $\mathbb{P}^1$. Without restriction, we may assume that its homogeneous coordinates for the second factor $\mathbb{P}^3$ are of the form $(1 : \lambda^2 : \lambda : 0)$. Setting

$$
z = T_3/T_0, \quad x = S_0/S_1, \quad y = (T_2/T_0)^2 - (T_1/T_0), \quad \text{and} \quad w = T_2/T_0
$$

yields the assertion on $\mathscr{O}_{Y,a}^\wedge$. This is basically the equation for a rational double point of type A_1, up to the additional variable w, so the blowing-up of the reduced singular locus gives a crepant resolution of singularities. $\qquad\square$

The singularities $a \in Y$ are very simple special cases of the so-called *compound du Val singularities* introduced in [50]. In particular, these are rational and canonical singularities.

The first projection induces a morphism $f : Y \to \mathbb{P}^1$. All fibers $f^{-1}(\lambda_0 : \lambda_1)$ are hypersurfaces in $\mathbb{P}^3$, and it follows that the map $\mathscr{O}_{\mathbb{P}^1} \to f_*(\mathscr{O}_Y)$ is bijective, and $R^1 f_*(\mathscr{O}_Y)$ vanishes. Moreover, the dualizing sheaf $\omega_Y = \mathscr{O}_Y(-1, -2)$ is f-antiample. Set $F = k(\mathbb{P}^1)$.

Proposition 14.7. *The generic fiber Y_F is the regular geometrically normal but non-smooth del Pezzo surface $W = W_\beta$ of degree $K_W^2 = 8$ and Picard number $\rho(W) = 1$ constructed above, with scalar $\beta = S_1/S_0 \in F$. The closed fiber $f^{-1}(\infty)$ over the point $\infty = (0 : 1)$ is isomorphic to the split ribbon $\mathbb{P}^2 \oplus \mathscr{O}_{\mathbb{P}^2}(-1)$. All other closed fibers are isomorphic to the normal del Pezzo surface $W = W_0$.*

Proof. This follows immediately from the defining equation (29), and we merely need to clarify the structure of the closed fiber $f^{-1}(\infty) \subset Y$. By definition, this is the hypersurface $H \subset \mathbb{P}^3$ given by $T_3^2 = 0$, thus $H_{\mathrm{red}} = \mathbb{P}^2$. The structure sheaf sits in a short exact sequence $0 \to \mathscr{L} \to \mathscr{O}_H \to \mathscr{O}_{\mathbb{P}^2} \to 0$, with $\mathscr{L} = \mathscr{O}_{\mathbb{P}^2}(-1)$. In other words, H is a ribbon on $\mathbb{P}^2$ with invertible sheaf $\mathscr{L}$. According to [5, Theorem 1.2], these are classified by elements in $\mathrm{Ext}^1(\Omega^1_{\mathbb{P}^2/k}, \mathscr{L}) = H^1(\mathbb{P}^2, \Theta_{\mathbb{P}^2/k}(-1))$. But this cohomology group vanishes, which easily follows from the long exact cohomology sequence for the Euler sequence $0 \to \mathscr{O}_{\mathbb{P}^2}(-1) \to \mathscr{O}_{\mathbb{P}^2}^{\oplus 3} \to \Theta_{\mathbb{P}^2/k}(-1) \to 0$. Thus $H = f^{-1}(\infty)$ is isomorphic to the split ribbon $\mathbb{P}^2 \oplus \mathscr{O}_{\mathbb{P}^2}(-1)$. $\qquad\square$

Note that W_0 is a contracted Hirzebruch surface with invariant $e = 2$. Now consider the closure

$$Z = Y \cap V_+(T_0, T_1) = V_+(T_0, T_1, S_0 T_2^2 - S_1 T_3^2) \subset Y$$

of $\mathrm{Sing}(Y_F/F)$ inside the total space Y. Clearly, $Z \subset Y$ is a smooth complete intersection of two effective Cartier divisors. We may regard this as a divisor in $\mathbb{P}^1 \times \mathbb{P}^1$, and the second projection gives an isomorphism $\mathrm{pr}_2 : Z \to \mathbb{P}^1$, whereas the first projection $\mathrm{pr}_1 : Z \to \mathbb{P}^1$ can be seen as the relative Frobenius morphism. Note that $Z \subset \mathrm{Reg}(Y)$, which is also a consequence of Proposition 14.6.

Now let $X \to Y$ be the blowing-up with reduced center $Z \subset Y$. This gives a commutative diagram

$$
\begin{array}{ccc}
X' & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y' & \longrightarrow & Y,
\end{array}
$$

where the horizontal morphisms are crepant resolutions described in Proposition 14.6. Now consider the induced fibrations $f : Y' \to \mathbb{P}^1$ and $g : X' \to \mathbb{P}^1$. Using Proposition 14.5, we immediately get the following.

Theorem 14.8. *The morphism $g : X' \to \mathbb{P}^1$ is a fibration from a smooth threefold, whose generic fiber is a regular and geometrically non-normal del Pezzo surface with Picard number two. The morphism $f : Y' \to \mathbb{P}^1$ is a fibration from a smooth threefold whose generic fiber is a regular, geometrically normal, and geometrically non-regular del Pezzo surface with Picard number one.*

15. Mori fiber spaces

We now examine our results on del Pezzo surfaces in the context of the minimal model program. We refer to the monographs of Kollár and Mori [40] and Matsuki [44] for general expositions, and the book of Kollár [41] on the singularities occurring in the minimal model program. In this section, k denotes an algebraically ground field of characteristic $p \geq 0$.

Let us call a morphism $f : Z \to B$ between proper normal integral schemes a *fibration* if $f_*(\mathscr{O}_Z) = \mathscr{O}_B$. A fibration is called a *Mori fibration* if the following four properties hold:

(i) The generic fiber $V = Z_\eta$ has dimension $\dim(V) \geq 1$.
(ii) The local rings $\mathscr{O}_{Z,a}$, $a \in Z$ are $\mathbb{Q}$-factorial klt singularities.
(iii) The Picard numbers satisfy $\rho(Z) = \rho(B) + 1$.
(iv) The $\mathbb{Q}$-Cartier divisor K_Z is relatively antiample for $f : Z \to B$.

In this situation, one also says that the total space Z is a *Mori fiber space*. Mori fibrations $f : Z \to B$ arise from the contractions of *extremal rays of fiber-type* in the cone of curves $\overline{\mathrm{NE}}(Z)$, and play a central role in the minimal model program.

A local noetherian ring R, such as $R = \mathscr{O}_{Z,a}$, is called a *klt singularity* if it is normal, $\mathbb{Q}$-Gorenstein, and for every proper birational modification $r : X \to \mathrm{Spec}(R)$ with X normal, the *discrepancies* $\mu_i \in \mathbb{Q}$ defined by

$$K_X = r^*(K_R) + \sum \mu_i E_i$$

are bounded by $\mu_i > -1$. Here $E_i \subset X$ are the exceptional divisors, and the equality holds in $\mathrm{Pic}(X) \otimes \mathbb{Q}$. If the discrepancies satisfy the stronger condition $\mu_i \geq 0$, the local ring R is a *canonical singularity*. For $\mu_i > 0$, the local ring is a *terminal singularity*. For further details on the singularities appearing in the context of the minimal model program, see [41, Section 2].

For every fibration $f : Z \to B$, the generic fiber $V = Z_\eta$ is a proper normal scheme over the function field $F = \mathscr{O}_{B,\eta} = \kappa(\eta) = k(B)$, with $h^0(\mathscr{O}_V) = 1$. It has dimension $n = \dim(Z) - \dim(B)$. In Mori fiber spaces, it is $\mathbb{Q}$-factorial, with Picard number $\rho(V) = 1$. The latter holds because the scheme Z is $\mathbb{Q}$-factorial with $\rho(Z) - \rho(B) = 1$. Moreover, the $\mathbb{Q}$-Cartier divisor K_V is antiample. If the non-Gorenstein locus of the total space Z does not dominate the base B, then the V is an n-dimensional Fano variety. For $n = 2$, the generic fiber V is a del Pezzo surface. Let us record the following general fact.

Lemma 15.1. *Suppose that Z is a proper normal integral scheme, and let $Z \to B$ be a fibration of relative dimension $\dim(Z) - \dim(B) = 2$. Then the generic fiber $V = X_\eta$ is normal and geometrically irreducible over the function field $F = k(B)$. If Z has only canonical or terminal singularities, then V is Gorenstein or regular, respectively. The scheme V is geometrically reduced, provided that $\dim(B) = 1$.*

Proof. According to the arithmetical proof in [1], each two-dimensional canonical singularity is Gorenstein, and each two-dimensional terminal singularity is regular. The last statement follows from [60, Theorem 2.3]. □

Combining this with Theorem 14.1, we get the following immediate consequence, which answers the questions originating in Kollár's study of extremal rays on three-folds ([39, Remark 1.2]).

Theorem 15.2. *Let Z be a threefold with only terminal singularities, and let $f : Z \to B$ be a Mori fibration of relative dimension $n = 2$. Then the generic fiber $V = X_\eta$ is a del Pezzo surface over the function field $F = k(B)$ that is geometrically normal.*

Appendix A. Finite modifications and Gorenstein conditions

In this appendix, we collect and slightly generalize some well-known facts pertaining to finite modifications, gluings of schemes, and Gorenstein conditions that go back to Serre [63], and were further developed by Reid [51].

Suppose that Y is a noetherian scheme, and let $\nu : X \to Y$ be a *finite modification*. This means that the morphism is finite and schematically dominant, and the injective homomorphism $\mathscr{O}_Y \to \nu_*(\mathscr{O}_X)$ is generically bijective. The *conductor ideal* $\mathfrak{C} \subset \mathscr{O}_Y$ is the annihilator of the coherent sheaf $\nu_*(\mathscr{O}_X)/\mathscr{O}_Y$. This is a quasi-coherent ideal sheaf, and the corresponding closed subscheme $C \subset Y$ is called the *conductor scheme*. The subsheaf $\mathfrak{C} \subset \nu_*(\mathscr{O}_X)$ is an ideal sheaf, in fact the largest ideal sheaf in $\mathscr{O}_Y$ that is also an ideal sheaf in $\nu_*(\mathscr{O}_X)$. We call the resulting closed subscheme $R \subset X$ the *ramification locus*. The diagram

$$
\begin{array}{ccc}
R & \longrightarrow & X \\
\downarrow & & \downarrow{\scriptstyle\nu} \\
C & \longrightarrow & Y
\end{array}
\tag{30}
$$

is both cartesian and cocartesian. Note that the finite morphisms $X \to Y$ and the induced morphism $R \to C$ are schematically dominant. We refer to the restriction $R \to C$ as the *gluing map*, and call the above diagram the *conductor square*. The sequence

$$
0 \longrightarrow \mathscr{O}_Y \longrightarrow \nu_*(\mathscr{O}_X) \oplus \mathscr{O}_C \longrightarrow \nu_*(\mathscr{O}_R) \longrightarrow 0
\tag{31}
$$

of coherent sheaves is exact, where the arrow on the left is the diagonal map, and the arrow on the right is the difference map $(s,t) \mapsto s_R - t_R$. Using the Snake Lemma, one infers that the sequence

$$
0 \longrightarrow \mathscr{O}_Y \longrightarrow \nu_*(\mathscr{O}_X) \longrightarrow \nu_*(\mathscr{O}_R)/\mathscr{O}_C \longrightarrow 0
\tag{32}
$$

is exact as well. As explained in [57, Proposition 4.1] the exact sequence (31) of coherent sheaves induces an exact sequence of multiplicative abelian sheaves

$$
1 \longrightarrow \mathscr{O}_Y^\times \longrightarrow \nu_*(\mathscr{O}_X)^\times \oplus \mathscr{O}_C^\times \longrightarrow \nu_*(\mathscr{O}_R)^\times \longrightarrow 1,
\tag{33}
$$

where the map on the right is $(s,t) \mapsto s_R/t_R$.

Conversely, suppose we start with a noetherian scheme X, a closed subscheme $R \subset X$ that contains no generic point $\eta \in X$, and a schematically dominant finite morphism $\nu : R \to C$. Then there exists a morphism $\nu : X \to Y$ onto a noetherian *algebraic space* Y, making the diagram (30) cocartesian ([4, Theorem 6.1]). The sequence (31) is exact, and the diagram is also cartesian. After replacing $C \subset Y$ and $R \subset X$, we may assume that these subschemes are the conductor scheme and the ramification locus. One also says that Y is obtained by X by *gluing* with respect to the gluing map $R \to C$. For a general discussion of this process, we refer to Ferrand's paper [24].

In this paper, we are mainly interested in the case where $X \to Y$, or equivalently the gluing map $R \to C$, are universal homeomorphisms (Proposition 4.7). In this

case, the algebraic space Y is automatically a scheme, as follows from [48, Theorem 6.2.2]. To simplify the exposition, we have decided to restrict our discussion to schemes. In what follows, we often write $\mathscr{O}_X$ and $\mathscr{O}_R$ instead of the more precise $\nu_*(\mathscr{O}_X)$ and $\nu_*(\mathscr{O}_R)$, which should not cause any confusion.

If the the ramification locus R has no embedded components, it is customary to write $\mathscr{M}_R$ for the quasicoherent sheaf of meromorphic functions on R. In other words, $\mathscr{M}_R = i_*(\mathscr{O}_{R^{(0)}})$ where $i : R^{(0)} = \amalg_{\zeta \in R} \operatorname{Spec}(\mathscr{O}_{R,\zeta}) \to R$ is the inclusion of the scheme of generic points. The gluing map induces an inclusion $\mathscr{O}_C \subset \mathscr{M}_C \cap \mathscr{O}_R$, where the intersection takes place in $\mathscr{M}_R$. The following is a generalization of Reid's observation [51, Proposition 2.2].

Proposition A.1. *Suppose that both X and Y satisfy Serre's Condition (S_2). Then the following three conditions hold:*

(i) *The ramification locus R has no embedded components.*
(ii) *For each generic point $\zeta \in R$, the local ring $\mathscr{O}_{X,\zeta}$ has dimension one.*
(iii) *For each point $y \in C$ with $\dim(\mathscr{O}_{Y,y}) \geq 2$, the inclusion $\mathscr{O}_{C,y} \subset (\mathscr{M}_C \cap \mathscr{O}_R)_y$ is an equality.*

Proof. For the first assertion, suppose there is an associated point $x \in \operatorname{Ass}(\mathscr{O}_R)$ that is embedded. Let $y \in Y$ be its image. Clearly, $\dim(\mathscr{O}_{X,x}) = \dim(\mathscr{O}_{Y,y}) \geq 2$. Replacing Y by the spectrum of the ring $\mathscr{O}_{Y,y}$, we may assume that $y \in Y$ is a closed point. Applying local cohomology to the short exact sequence (31) yields an exact sequence of groups

$$H_y^0(\mathscr{O}_Y) \longrightarrow H_y^0(\mathscr{O}_X) \oplus H_y^0(\mathscr{O}_C) \longrightarrow H_y^0(\mathscr{O}_R) \longrightarrow H_y^1(\mathscr{O}_Y).$$

The outer terms and $H_y^0(\mathscr{O}_X)$ vanish, because both X and Y satisfy (S_2). It follows that $H_y^0(\mathscr{O}_C) \to H_y^0(\mathscr{O}_R)$ is bijective. Now choose some non-zero $\bar{s} \in \mathscr{O}_{R,y}$ with support $\operatorname{Supp}(\bar{s}) = \{x\}$. Then $\bar{s} \in H_y^0(\mathscr{O}_R) = H_y^0(\mathscr{O}_C)$, in particular $\bar{s} \in \mathscr{O}_{C,y}$. Let $s \in \mathscr{O}_{X,y}$ be a representative. For every $f \in \mathscr{O}_{X,y}$, the short exact sequence (32) implies $sf \in \mathscr{O}_{Y,y}$, thus $s \in \mathfrak{C}$ and finally $\bar{s} = 0$, a contradiction.

For the second assertion, suppose that $\zeta \in R$ is a generic point, with image $y \in C$. Again we may assume that $y \in Y$ is closed. Since $\nu : X \to Y$ is birational, we have $\dim(\mathscr{O}_{X,x}) \geq 1$. In the above exact sequence, $H_y^0(\mathscr{O}_X) = 0$ and the inclusion $H_y^0(\mathscr{O}_C) = \mathscr{O}_{C,y} \subset \mathscr{O}_{R,y} = H_y^0(\mathscr{O}_R)$ is not an equality. Therefore $H_y^1(\mathscr{O}_Y) \neq 0$, which implies that $\dim(\mathscr{O}_{Y,y}) = 1$.

Finally, we check Condition (iii). Again we may assume that Y is local, with closed point $y \in Y$. Suppose that $s \in \mathscr{O}_{R,y}$ lies in $\mathscr{O}_C$ at all generic points $\zeta \in R$. Choose an open dense subset $V \subset Y$ so that $s \in H^0(V, \mathscr{O}_C)$, and write $A = Y \smallsetminus V$ for the complementary closed subset. Furthermore, we have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(C, \mathscr{O}_C) & \longrightarrow & H^0(V, \mathscr{O}_C) & \longrightarrow & H_A^1(\mathscr{O}_C) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(R, \mathscr{O}_R) & \longrightarrow & H^0(V, \mathscr{O}_R) & \longrightarrow & H_A^1(\mathscr{O}_R) & \longrightarrow & 0.
\end{array}
$$

The rows are exact, because the schemes C and R are affine. Since $H_A^1(\mathscr{O}_Y) = H_A^1(\mathscr{O}_X) = 0$, the long exact sequence coming from (31) ensures that the vertical map on the right is injective. Since C is affine, the vertical map on the left has $H^0(C, \mathscr{F})$ as cokernel, with $\mathscr{F} = \mathscr{O}_R/\mathscr{O}_C$. The cokernel of the vertical map in the middle at least is contained in $H^0(V, \mathscr{F})$. In turn, the Snake Lemma implies that

the restriction map $H^0(C, \mathscr{F}) \to H^0(V, \mathscr{F})$ is injective. Thus the class of s in $\mathscr{F}$ vanishes, in other words $s \in \mathscr{O}_{C,y}$. $\square$

Next, we discuss Gorenstein conditions. Let B be a complete local noetherian ring. Then the contravariant functor

$$M \longmapsto \mathrm{Hom}_B(H^d_{\mathfrak{m}}(M), E)$$

on the category of B-modules is representable. Here $\mathfrak{m} = \mathfrak{m}_B$ is the maximal ideal, $d = \dim(B)$ is the Krull dimension, and E is the injective hull of the residue field $\kappa(B)$. According to Aoyama [2], any ω_B representing this functor is called a *canonical module*. Note that ω_B is unique up to isomorphism, and that we do not demand that B is Cohen–Macaulay. The complete local ring B is called *quasi-Gorenstein* if the canonical module ω_B is invertible. If B is additionally Cohen–Macaulay, one says that B is *Gorenstein*.

For local noetherian rings B that are not necessarily complete, a module ω_B is called canonical if $\omega_{\hat{B}} \simeq \omega_B \otimes_B \hat{B}$. We then say that B *admits a canonical module*. Note that this is not always the case. However, this condition obviously holds if $\hat{B}$ is quasi-Gorenstein. Extending the notions from the complete local case to the local case, we say that B is Gorenstein or quasi-Gorenstein if the respective property holds for the completion $\hat{B}$.

We say that our noetherian scheme Y is Gorenstein or quasi-Gorenstein at a point $y \in Y$, if the respective property holds for the local ring $B = \mathscr{O}_{Y,y}$, or equivalently the complete local ring $\hat{B} = \mathscr{O}^\wedge_{Y,y}$. We say that Y is *Gorenstein or quasi-Gorenstein* if the respective property holds for each point $y \in Y$. If this holds for all points with $\dim(\mathscr{O}_{Y,y}) \leq n$, we say that Y is *Gorenstein or quasi-Gorenstein in codimension n*, or that Y satisfies condition (G_n) or (qG_n), respectively.

If the point $y \in Y$ of codimension one is contained in the conductor scheme, $\mathscr{O}_{C,y} \subset \mathscr{O}_{R,y}$ is a finite extension of Artin rings, and we denote by

$$\mathrm{length}(\mathscr{O}_{C,y}) \leq \mathrm{length}(\mathscr{O}_{R,y})$$

their lengths as modules over $\mathscr{O}_{C,y}$. The following two results, under various additional assumptions, are due to Samuel ([56, Theorem 5]), Gorenstein ([25, Theorem 6]), Rosenlicht ([53, Theorem 14]), Roquette [52], Serre ([63, Chapter IV, §3, Section 11]), Kunz [42], and Reid ([51, Theorem 3.2]). In our general form, we merely demand suitable Gorenstein assumptions and make no reference to ground rings.

Proposition A.2. *Suppose that Y is Gorenstein in codimension one. Then the length formula*

$$\mathrm{length}(\mathscr{O}_{R,y}) = 2 \, \mathrm{length}(\mathscr{O}_{C,y})$$

holds for each point $y \in Y$ of codimension one contained in C.

Proof. It suffices to treat the case that $Y = \mathrm{Spec}(B)$ is the spectrum of a complete one-dimensional local noetherian ring. Set $l = \mathrm{length}(B/\mathfrak{C})$ and choose a sequence of ideals

$$(34) \qquad \mathfrak{C} = \mathfrak{b}_0 \subset \mathfrak{b}_1 \subset \ldots \subset \mathfrak{b}_l = B$$

with simple subquotients. Note that this corresponds to a Jordan–Hölder sequence for $B/\mathfrak{C}$, which is a module of finite length. Clearly, each $\mathfrak{b}_i$ is a maximal Cohen–Macaulay B-module. According to [23, Theorem 21.21], the contravariant functor

$M \mapsto \operatorname{Hom}_B(M, \omega_B)$ is an exact antiequivalence from the category of maximal Cohen–Macaulay modules to itself. In fact, the biduality maps

$$M \longrightarrow \operatorname{Hom}_B(\operatorname{Hom}_B(M, \omega_B), \omega_B), \quad m \longmapsto (f \mapsto f(m)),$$

are bijective. Since B is Gorenstein, the module $\omega_B = B$ is canonical, and we conclude that (34) induces another sequence

(35) $$\operatorname{Hom}_B(\mathfrak{b}_l, B) \subset \operatorname{Hom}_B(\mathfrak{b}_{l-1}, B) \subset \ldots \subset \operatorname{Hom}_B(\mathfrak{b}_0, B)$$

of the same length, with simply subquotients. The term on the left is $\operatorname{Hom}_B(B, B)=B$. Thus we may splice the two sequences (34) and (35) and obtain a sequence of length $2l$ with simple subquotients, starting with $\mathfrak{C} = \mathfrak{b}_0$ and ending with $\operatorname{Hom}_B(\mathfrak{C}, B) = \operatorname{Hom}_B(\mathfrak{b}_0, B)$. It remains to identify the latter with A, in such a way that the resulting inclusion $\mathfrak{C} \subset A$ coincides with the canonical inclusion. To achieve this, consider the commutative diagram

$$
\begin{array}{ccc}
B & \xrightarrow{\text{ can }} & A \\
\downarrow & & \downarrow \\
\operatorname{Hom}_B(B, B) & \xrightarrow[\text{res}]{} & \operatorname{Hom}_B(\mathfrak{C}, B),
\end{array}
$$

where the upper map is the canonical inclusion, the lower map is given by restriction, and the vertical maps are given by $a \mapsto (x \mapsto ax)$. We need to verify that the map $A \to \operatorname{Hom}_B(\mathfrak{C}, B)$ is bijective. Applying the antiequivalence of categories again and using biduality, we have to check that

$$\mathfrak{C} = \operatorname{Hom}_B(\operatorname{Hom}_B(\mathfrak{C}, B), B) \longrightarrow \operatorname{Hom}_B(A, B), \quad b \longmapsto (y \mapsto by)$$

is bijective. Clearly, the map $f \mapsto f(1)$ is a left inverse. Thus $\mathfrak{C} \to \operatorname{Hom}_B(A, B)$ is injective and admits a complement $M \subset \operatorname{Hom}_B(A, B)$. Since B is Cohen–Macaulay, thus torsion-free, the Hom module and thus also M are torsion-free. But the canonical inclusions $\mathfrak{C} \subset B \subset A$ become equalities after inverting any regular $s \in \mathfrak{m}_B$. It follows that $M_s = 0$, and thus $M = 0$. Summing up, we have shown that $\operatorname{length}(A/\mathfrak{C}) = 2l$. $\qquad\square$

The converse statement takes the following form.

Proposition A.3. *Let $y \in Y$ be a point of codimension one. Suppose that the following three conditions hold:*

(i) *The semilocal ring $\mathscr{O}_{X,y}$ is Gorenstein.*
(ii) *The conductor ideal $\mathfrak{C}_y \subset \mathscr{O}_{X,y}$ is invertible.*
(iii) *The modules $\mathscr{O}_{C,y}$ and $(\mathscr{O}_R/\mathscr{O}_C)_y$ are free of the same rank over some local Artin subring $W \subset \mathscr{O}_{C,y}$ with W Gorenstein.*

Then the local ring $\mathscr{O}_{Y,y}$ is Gorenstein.

Proof. To simplify notation, set $A = \mathscr{O}_{X,y}$ and $B = \mathscr{O}_{Y,y}$. It suffices to treat the case that the rings in question are complete, whence A is a product of complete local rings. By assumption, the conductor ideal has the form $fA = \mathfrak{C}_y$ for some regular element $f \in A$, and we write $\bar{A}, \bar{B}$ for the resulting residue class rings. Now the exact sequence (32) takes the form

(36) $$0 \longrightarrow B \longrightarrow A \longrightarrow \bar{A}/\bar{B} \longrightarrow 0.$$

Since the semilocal ring A and the subring $W \subset \bar{B}$ are Gorenstein, the modules $\omega_A = A$ and $\omega_W = W$ are dualizing. In turn, both modules

$$\mathrm{Ext}_A^1(\bar{A}, A) = \mathrm{Ext}_A^1(\bar{A}, \omega_A) \quad \text{and} \quad \mathrm{Hom}_W(\bar{A}, W) = \mathrm{Hom}_W(\bar{A}, \omega_W)$$

are dualizing for the semilocal ring $\bar{A}$, and it follows that these $\bar{A}$-modules are isomorphic. Likewise, $\omega_{\bar{B}} = \mathrm{Hom}_W(\bar{B}, W)$ is dualizing for $\bar{B}$. Using the short exact sequence $0 \to fA \to A \to \bar{A} \to 0$, we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}(A, \omega_A) \longrightarrow \mathrm{Hom}(fA, \omega_A) \overset{\partial}{\longrightarrow} \mathrm{Ext}^1(\bar{A}, \omega_A) \longrightarrow 0.$$

According to [51, the theorem in 2.6], the kernel of the composite map

$$(37) \qquad \mathrm{Hom}(fA, \omega_A) \overset{\partial}{\longrightarrow} \mathrm{Ext}^1(\bar{A}, \omega_A) \simeq \mathrm{Hom}_W(\bar{A}, \omega_W) \overset{\mathrm{res}}{\longrightarrow} \mathrm{Hom}_W(\bar{B}, W)$$

is a dualizing module ω_B for B. The arrow on the left is the connecting map, and the map on the right the restriction map, which is often referred to as the *trace map*. Up to isomorphism, the kernel does not depend on the chosen isomorphism in the middle, because the connecting map is isomorphic to the residue class map $A \to \bar{A}$, and the induced homomorphism of multiplicative groups $A^\times \to \bar{A}^\times$ for the semilocal rings is surjective. The idea now is to choose a particular isomorphism in the middle of (37) that is adapted to our problem.

By assumption, the finitely generated W-modules $\bar{B}$ and $\bar{A}/\bar{B}$ are free. Hence the inclusion $\bar{B} \subset \bar{A}$ admits a complement, and we may write $\bar{A} = \bar{B} \oplus \bar{A}/\bar{B}$ as W-modules. Choosing a W-basis in these summands, we obtain a unimodular symplectic form $\Phi : \bar{A} \times \bar{A} \to W$ so that $\bar{A}/\bar{B} \subset \bar{A}$ becomes a Lagrangian, with orthogonal complement $\bar{B} \subset \bar{A}$. This crucial step hinges on the assumption that $\mathrm{rank}_W(\bar{A}) = \mathrm{rank}_W(\bar{A}/\bar{B})$. In turn, the diagram

$$(38) \qquad \begin{array}{ccc} \bar{A} & \overset{\Phi}{\longrightarrow} & \mathrm{Hom}_W(\bar{A}, W) \\ \mathrm{pr} \downarrow & & \downarrow \mathrm{res} \\ \bar{A}/\bar{B} & \underset{\Phi}{\longrightarrow} & \mathrm{Hom}_W(\bar{B}, W) \end{array}$$

becomes commutative, where the horizontal maps are given by $a \mapsto (a' \mapsto \Phi(a, a'))$, the map on the left is the canonical projection, and the map on the right is given by restriction.

We now can make the sequence (37) explicit: Using the homomorphism $f \mapsto 1$ as an A-basis $e \in \mathrm{Hom}(fA, \omega_A)$ and its image under the connecting map as an $\bar{A}$-basis $\bar{e} \in \mathrm{Ext}^1(\bar{A}, \omega_A)$, where $\omega_A = A$, we may regard ω_B as the kernel for the composite map

$$Ae \longrightarrow \bar{A}\bar{e} = \bar{A} \overset{\Phi}{\longrightarrow} \mathrm{Hom}_W(\bar{A}, W) \overset{\mathrm{res}}{\longrightarrow} \mathrm{Hom}_W(\bar{B}, W).$$

By diagram (38), this composite mapping is isomorphic to the canonical projection $A \to \bar{A}/\bar{B}$. The exact sequence (36) now gives the desired isomorphism $\omega_B \simeq B$. $\square$

Note that if the local Artin ring $\mathscr{O}_{C,y}$ contains a field, it is natural to choose for $W \subset \mathscr{O}_{C,y}$ a coefficient field, and Condition (iii) becomes equivalent to the length condition $\mathrm{length}(\mathscr{O}_{R,y}) = 2\,\mathrm{length}(\mathscr{O}_{C,y})$.

Proposition A.4. *Suppose that X and Y satisfies Condition (S_2), and that the three assumptions of Proposition A.3 hold for each point $y \in Y$ of codimension one. Then Y is quasi-Gorenstein if for each closed point $x \in X$, the classes of ω_A and*

R in the local Class group $\mathrm{Cl}(A)$ for the local ring $A = \mathscr{O}_{X,x}$ are inverse to each other.

Proof. According to [2, Corollary 2.4] it suffices to treat the case that $Y = \mathrm{Spec}(B)$ is a complete local scheme, with closed point $y \in Y$. The canonical module ω_B and B satisfies Serre's Condition (S_2) (see [2, 1.10]). Using the arguments for [35, Theorem 1.12], it suffices to check that the coherent sheaves $\mathscr{F} = \widetilde{\omega_B}$ and $\mathscr{O}_Y$ are isomorphic over some open subset $V \subset Y$ that contains all points of codimension one. Furthermore, by the assumption on the local class group we may choose as canonical module $\omega_A = \mathfrak{C}$, such that we get an identification $A = \mathrm{Hom}(\mathfrak{C}, \omega_A)$. As in the proof of Proposition A.3, we consider the composite map

$$(39) \qquad A = \mathrm{Hom}_A(\mathfrak{C}, \omega_A) \longrightarrow \mathrm{Ext}^1_A(\bar{A}, \omega_A) \simeq \mathrm{Hom}_{\bar{B}}(\bar{A}, \omega_{\bar{B}}) \longrightarrow \omega_{\bar{B}}.$$

The map on the left is the connecting map from $0 \to \mathfrak{C} \to A \to \bar{A} \to 0$, whereas the map on the right is the trace map. Note that the term on the right is supported by the conductor $C \subset Y$. According to [51, 2.6, the theorem], the kernel I of the above composition is a canonical module ω_B.

By assumption, both schemes X and Y satisfy Serre's Condition (S_2). According to Proposition A.1, the ramification locus $R \subset X$ is purely one-codimensional, and contains no embedded components. Let $\zeta_1, \ldots, \zeta_r \in C$ be the generic points. Then $\dim(\mathscr{O}_{X,\zeta_i}) = 1$, and we saw in the proof for Proposition A.3 that the $\bar{B}$-module $\mathrm{Ext}^1(\bar{A}, \omega_A)$ and $\bar{A}/\bar{B}$ are isomorphic at these points $\zeta_i \in Y$. It follows that there is an open neighborhood $V \subset Y$ containing all points of codimension one so that the quasicoherent sheaf attached to $\omega_{\bar{B}}$ and the sheaf $\mathscr{O}_R/\mathscr{O}_C$ have isomorphic restriction to V. Let $\mathscr{I} = \widetilde{I}$ be the quasicoherent sheaf on Y corresponding to the B-module $I = \omega_B$. By our construction, $\mathscr{I}|V \simeq \mathscr{O}_V$. Thus we see that $\widetilde{\omega_B}|V \simeq \mathscr{O}_V$. $\qquad \square$

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