THE DELIGNE–ILLUSIE THEOREM AND EXCEPTIONAL ENRIQUES SURFACES

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ABSTRACT. Building on the results of Deligne and Illusie on liftings to truncated Witt vectors, we give a criterion for non-liftability that involves only the dimension of certain cohomology groups of vector bundles arising from the Frobenius pushforward of the de Rham complex. Using vector bundle methods, we apply this to show that exceptional Enriques surfaces, a class introduced by Ekedahl and Shepherd-Barron, do not lift to truncated Witt vectors, yet the base of the miniversal formal deformation over the Witt vectors is regular. Using the classification of Bombieri and Mumford, we also show that bielliptic surfaces arising from a quotient by a unipotent group scheme of order p do not lift to the ring of Witt vectors. These results hinge on some observations in homological algebra that relates splittings in derived categories to Yoneda extensions and certain diagram completions.

CONTENTS

INTRODUCTION

Let k be a perfect field of characteristic $p > 0$, and Y be a smooth proper k-scheme. Often it is a challenging question whether or not the scheme Y lifts to the ring of Witt vectors W, or even its truncation $W_2 = W/p^2W$. According to a famous result of Deligne and Illusie [15], the existence of such W_2-liftings implies that the Hodge–de Rham spectral sequence $E_1^{rs} = H^s(Y, \Omega_Y^r) \implies H^{r+s}(Y, \Omega_Y^\bullet)$

degenerates at the E_1-page, provided $\dim(Y) \leq p$. This result is used to show that schemes with "exotic" Hodge cohomology frequently do not admit such lifts. Consequently, the base $\mathrm{Spf}(A)$ of the miniversal formal deformation is not formally smooth over the ring W. Note that it could still be given by a regular local ring, for example $A = W[[U, V]]/(UV - p)$. Also note that failure of lifting to W_2 occurs in surprisingly simple situations, even for smooth models of inseparable covers of the projective plane ([36], Theorem 3.4).

Actually, Deligne and Illusie identified the *gerbe of liftings* of the scheme Y' to the ring W_2 with the *gerbe of splittings* for the one-term complex $F_* \mathcal{O}_Y \xrightarrow{d} Z\Omega^1_Y$. Here $Y' = Y \otimes_k k$ is the base-change with respect to the Frobenius map $\lambda \mapsto \lambda^p$, and $Z\Omega^1_Y$ is the sheaf of 1-cocycles in the push-forward $F_*(\Omega^\bullet_Y)$ of the de Rham complex with respect to the relative Frobenius $F : Y \to Y'$. To my best knowledge, this amazing result was *never used directly* to show that certain schemes do not lift to the ring W_2. The main goal of this paper is to show that such arguments are indeed feasible. For this, we establish *general numerical criteria* that ensure that the gerbes in question have no global objects, and apply this to certain *Enriques surfaces* and *bielliptic surfaces*.

This hinges on some general results in homological algebra, which ensure among other things that the above gerbes admit a global object if and only if the Yoneda class of the four-term exact sequence

$$0 \longrightarrow \mathcal{O}_{Y'} \longrightarrow F_*(\Omega_Y) \xrightarrow{d} Z\Omega^1_Y \longrightarrow \Omega^1_{Y'} \longrightarrow 0$$

in $\mathrm{Ext}^2(\Omega^1_{Y'}, \mathcal{O}_{Y'})$ vanishes. This sequence is obtained by splicing two short exact sequences

$$0 \to \mathcal{O}_{Y'} \to F_*(\Omega^1_Y) \to B\Omega^1_Y \to 0 \quad \text{and} \quad 0 \to B\Omega^1_Y \to Z(\Omega^1_Y) \to \Omega^1_{Y'} \to 0.$$

If the former splits, one says that Y is *Frobenius-split*. This notation was introduced by Mehta and Ramanathan [40], and has numerous striking consequences for the cohomology of sheaves. We refer to the monograph of Brion and Kumar [11] for a highly readable account. Let us say that Y is *Cartier-split* if the second short exact sequence splits. As Srinivas [52] and Yobuko [55] observed, this means that the scheme Y and also the morphism $F : Y \to Y'$ admits a lifting to the ring W_2.

Here we are interested in a much weaker and more flexible version: We say that the scheme Y is *pre-Cartier-split* if the connecting map $\mathrm{Hom}(B\Omega^1_Y, \mathcal{O}_{Y'}) \to \mathrm{Ext}^1(\Omega^1_{Y'}, \mathcal{O}_{Y'})$ coming from the second short exact sequence is the zero map. Our first main result is the following general numerical criterion:

Theorem. *(See 1.4.) Suppose Y is pre-Cartier-split but not Frobenius-split, and satisfies $h^1(\Theta_Y) \geq h^1(\underline{\mathrm{Hom}}(Z\Omega^1_Y, \mathcal{O}_{Y'}))$. Then the scheme Y does not lift to the ring of truncated Witt vectors W_2.*

Under the condition $c_1 = 0$ and $\dim(Y) = 2$ this simplifies further. One gets the following version, in which the differentials of the de Rham complex are eliminated:

Theorem. *(See 2.5.) Suppose Y is a surface, that the dualizing sheaf ω_Y has order $p \geq 2$ in the Picard group, and that $h^1(\Theta_Y) \geq h^1(\underline{\mathrm{Hom}}(F_*\Omega^1_Y, \mathcal{O}_{Y'}))$. Then the scheme Y does not lift to the ring W_2.*

This is amenable to vector bundle techniques, and the main idea is to put the cotangent sheaf into a short exact sequence $0 \to \mathscr{L} \otimes \omega_Y \to \Omega^1_Y \to \mathscr{I}\mathscr{L}^\vee \to 0$, and write $\underline{\mathrm{Hom}}(F_*\Omega^1_Y, \mathscr{O}_{Y'})$ on Y' as the Frobenius image of some sheaf on Y, in order to compute cohomology. In characteristic $p = 2$, we formulate in Theorem 6.1 four elementary conditions concerning the invertible sheaves ω_Y and $\mathscr{L}$ that ensure that the surface Y does not lift to the ring W_2.

We then apply our results to *exceptional Enriques surface* Y. These surfaces were introduced and studied by Ekedahl and Shepherd-Barron [19]. Despite their highly unusual geometry, which was already considered by Cossec and Dolgachev [12], they admit rather concrete descriptions in terms of equations found by Salomonsson [46]. Actually, one should treat them together with the *supersingular Enriques surfaces*, because both have $h^0(\Theta_Y) = h^2(\Theta_Y) = 1$ and $h^1(\Theta_Y) = 12$, which makes their deformation theory seemingly complicated. Ekedahl, Hyland and Shepherd-Barron [20] showed that the base of the miniversal formal deformation of a supersingular Enriques surface is the formal spectrum of $A = W[[T_1, \ldots, T_{12}]]/(2 - FG)$ where F, G lie in the maximal ideal. Before Theorem 4.5, they ask to clarify the miniversal deformation in the exceptional case. Extending their results, we obtain:

Theorem. *(See 7.2.) The base of the miniversal deformation of an Enriques surface Y in characteristic two is given by a complete local noetherian ring A that is regular, W-flat and of Krull dimension eleven or twelve. Moreover, the following are equivalent:*

   (i) *The Enriques surface Y is exceptional or supersingular.*
   (ii) *The scheme Y does not lift to the ring W_2.*
   (iii) *The absolute ramification index is $e(A) \geq 2$.*
   (iv) *The dimension is $\dim(A) = 12$.*

This is in striking contrast to general results of Liedtke [37], who showed that *normal* Enriques surfaces having a Cossec–Verra polarization are unobstructed. The non-liftability of our smooth Y thus must be caused by the necessity of base-change needed in Artin's simultaneous resolution [2] of the singularities in some normal models of Y, all of which are rational double points. This was further elucidated by Shepherd-Barron [51]. One should compare the above result with the situation in characteristic zero: Then the T^1-lifting Theorem ensures that the base of the miniversal deformation for smooth schemes with $c_1 = 0$ is given by a regular ring (confer [5], [54], [53], [34], [45], [48], [18]).

Finally, we apply our results to *bielliptic surfaces* Y, which were classified by Bombieri and Mumford [8], [7]. Their deformation theory was studied by Partsch [44], when both genus-one fibrations are elliptic. Here we examine the case that the surface is of the form $Y = (E \times C)/G$, where C is the rational cuspidal curve, G is a finite group scheme, and the characteristic is $p = 2$. We shall see that if $G = \alpha_2$, these surfaces do not lift to the ring of Witt vectors W. We also describe the tangent and cotangent sheaves and their cohomology. The assertion depends on general results about proper group schemes and Picard schemes, in particular:

Theorem. *(see 4.3.) Set $G = \mathrm{Pic}^0_{Y/k}$. Suppose the local group scheme $L = G/G_{\mathrm{red}}$ contains some α_{p^n}, $n \geq 1$ as a direct summand, and that the first Betti number satisfies $b_1 = 2(h^1(\mathscr{O}_Y) - h^2(\mathscr{O}_Y))$. Then the scheme Y does not lift to the ring W.*

All the above result hinge on certain general results from homological algebra, in which we give a new interpretation of the gerbe of splittings for a two-term complex $f : M \to N$ in some abelian category $\mathcal{A}$. Here we introduce the notion of *diagram completions*, which consists of an object E, together with two morphisms $h : M \to E$ and $g : E \to N$ making

(1)
$$
\begin{array}{ccc}
M & \xrightarrow{\mathrm{pr}} & B \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle i} \\
E & \xrightarrow[g]{} & N
\end{array}
$$

both cartesian and cocartesian. If the objects of the abelian category are abelian sheaves on some space or site, and the cohomology sheaves H^i for the one-term complex $M \xrightarrow{f} N$ are locally free of finite rank, we get the following result, which seems to be of independent interest:

Theorem. *(See 9.5.) The gerbe of diagram completions and the gerbe of splittings for $f : M \to N$ have the same class in the group*

$$
H^2(X, \underline{\mathrm{Hom}}_{\mathscr{O}_X}(H^1, H^0)) = \mathrm{Ext}^2(H^1, H^0).
$$

Moreover, either of them admits a global object if and only if the Yoneda class of the exact sequence $0 \to H^0 \to M \xrightarrow{f} N \to H^1 \to 0$ vanishes.

This paper is organized as follows: In Section 1 we introduce the notion of pre-Cartier split schemes, and give our general numerical criterion against liftings to the ring W_2. In Section 2 this is examined under the additional conditions $c_1 = 0$ and $\dim(Y) = 2$. Section 3 contains a discussion for the regularity properties of the base of the miniversal deformation in mixed characteristics. In Section 4 we consider obstructions to liftings arising from the theory of group schemes and Picard groups. Section 5 contains some computations with vector bundles and Chern classes on surfaces. These are used in Section 6, to prove our main result on non-liftability of surfaces in characteristic two. Section 7 and 8 contain the applications to Enriques and bielliptic surfaces. The final Section 9 deals with necessary homological algebra in a general abstract setting.

1. Numerical criteria against first-order liftings

Let k be a perfect field of characteristic $p > 0$, and Y be a smooth proper k-scheme of dimension $n = \dim(Y)$. Furthermore assume that $h^0(\mathscr{O}_Y) = 1$. To simplify notation, we write $\Omega^1_Y = \Omega^1_{Y/k}$ for the cotangent sheaf, $\Theta_Y = \underline{\mathrm{Hom}}(\Omega^1_Y, \mathscr{O}_Y)$ for the tangent sheaf, $\Omega^i_Y = \Lambda^i(\Omega^1_Y)$ for the sheaves of differential forms, and $Y' = Y \otimes_k k$ for the base-change with respect to the Frobenius map $\lambda \to \lambda^p$. The commutative

diagram

(2)

$$
\begin{array}{c}
\xymatrix{
Y \ar@/^2pc/[rr]^{F_Y} \ar[r]^{F} \ar[dr] & Y' \ar[r] \ar[dl] & Y \ar[d] \\
& \mathrm{Spec}(k) \ar[r]_{F_k} & \mathrm{Spec}(k),
}
\end{array}
$$

where F_Y and F_k are absolute Frobenius morphisms, defines the *relative Frobenius morphism* $F : Y \to Y'$, which is a finite flat universal homeomorphism of degree $\deg(Y/Y') = p^n$. Note that for each skyscraper sheaf $\mathscr{T}$ on Y the Frobenius push-forward $F_*(\mathscr{T})$ is a skyscraper sheaf on Y' with $h^0(F_*\mathscr{T}) = h^0(\mathscr{T})$, and that for each locally free $\mathscr{O}_Y$-module $\mathscr{E}$ of rank $r \geq 0$ the Frobenius pushforward $\mathscr{E}' = F_*(\mathscr{E})$ is a locally free $\mathscr{O}_{Y'}$-module of rank $r' = rp^n$

The latter applies in particular to the $\mathscr{E} = \Omega_Y^i$. Moreover, the k-linear differentials in the de Rham complex $\Omega_Y^\bullet$ become $\mathscr{O}_{Y'}$-linear maps in the resulting cochain complex $F_*(\Omega_Y^\bullet)$. We write $B\Omega_Y^i$ and $Z\Omega_Y^i$ for coboundaries and cocycles, viewed as coherent sheaves on Y'. These are actually locally free, and the *inverse Cartier operator* $d(f \otimes 1) \mapsto f^{p-1}df$ gives an identification $\Omega_{Y'}^i = Z\Omega_Y^i / B\Omega_Y^i$. In particular, we have a commutative diagram

(3)

$$
\xymatrix{
& 0 \ar[dr] & & 0 \\
& & B\Omega_Y^1 \ar[ur] \ar[dr] & \\
0 \ar[r] & \mathscr{O}_{Y'} \ar[r] & F_*\mathscr{O}_Y \ar[ur] \ar[rr]_{d} & & Z\Omega_Y^1 \ar[r] & \Omega_{Y'}^1 \ar[r] & 0,
}
$$

where the four-term horizontal sequence is exact, and arises via splicing of the two short exact sequences with kinks.

We now consider the second cohomology group

$$
H^2(Y', \Theta_{Y'}) = \mathrm{Ext}^2(\Omega_{Y'}^1, \mathscr{O}_{Y'}),
$$

whose elements can be regarded as equivalence classes of *gerbes banded by* $\Theta_{Y'}$, or in short $\Theta_{Y'}$-gerbes ([23], Chapter IV, Section 3.4). It contains the class of the *gerbe of splittings* $\mathscr{S}c(F_*\mathscr{O}_Y \xrightarrow{d} Z\Omega_Y^1)$ for the two-term complex $F_*\mathscr{O}_Y \xrightarrow{d} Z\Omega_Y^1$, as defined in [15], Section 3. It also contains the class of the *gerbe of liftings* $\mathscr{R}el(Y', W_2)$ for the k-scheme Y' to the ring W_2. The objects of this gerbe are the proper flat morphisms $\mathfrak{Y}' \to \mathrm{Spec}(W_2)$, together with an identification $\mathfrak{Y}' \otimes_{W_2} k = Y'$. Here $W_2 = W/p^2 W$ is the truncation of length two for the ring of Witt vectors W. We refer to [6], Chapter IV, §1 for a comprehensive treatment of Witt vectors. Since the Frobenius map $k \to k$ is bijective, it induces an automorphism of the local ring W, and the liftings of Y' to W_2 correspond to the liftings of Y. According to [15], Proposition 3.3 there is an equality

$$
\mathrm{cl}\,\mathscr{S}c(F_*\mathscr{O}_Y \xrightarrow{d} Z\Omega_Y^1) = \mathrm{cl}\,\mathscr{R}el(Y', W_2)
$$

of classes in the second cohomology group $H^2(Y', \Theta_{Y'})$. From the abstract situation treated in Theorem 9.5 below we get another, completely different interpretation:

Proposition 1.1. *There is a lifting of the scheme Y to the ring W_2 if and only if the Yoneda class of the four-term exact sequence in* (3) *vanishes in the extension group $\mathrm{Ext}^2(\Omega^1_{Y'}, \mathscr{O}_{Y'})$. Up to isomorphism, to give such a lifting amounts to factor the differential $d : F_*\mathscr{O}_Y \to Z\Omega^1_Y$ over some locally free $\mathscr{O}'_Y$-module $\mathscr{E}$ of rank $p^n + n$ such that the diagram*

$$
\begin{array}{ccc}
F_*\mathscr{O}_Y & \longrightarrow & B\Omega^1_Y \\
\downarrow & & \downarrow \\
\mathscr{E} & \longrightarrow & Z\Omega^1_Y
\end{array}
$$

becomes both cartesian and cocartesian.

In somewhat different form, this was already observed by De Clercq, Florence and Lucchini Arteche ([13], Corollary 5.18).

The scheme Y is called *Frobenius-split* if the inclusion $\mathscr{O}_{Y'} \subset F_*(\mathscr{O}_Y)$ admits a retraction [40]. In other words, the extension class

$$\mathrm{cl}(F_*\mathscr{O}_Y) \in \mathrm{Ext}^1(B\Omega^1_Y, \mathscr{O}_{Y'}) = H^1(Y', \underline{\mathrm{Hom}}(B\Omega^1_Y, \mathscr{O}_{Y'}))$$

vanishes. Let us say that Y is *Cartier-split* if the surjection $Z\Omega^1_Y \to \Omega^1_{Y'}$ admits a section, that is, the extension class $\mathrm{cl}(Z\Omega^1_Y) \in \mathrm{Ext}^1(\Omega^1_{Y'}, B\Omega^1_Y)$ is zero. Using that the *Yoneda product* $\mathrm{cl}(F_*\mathscr{O}_Y) * \mathrm{cl}(Z\Omega^1_Y)$ is the *Yoneda class* of the four-term exact sequence in (3), we get with Proposition 1.1:

Proposition 1.2. *If Y is Frobenius-split or Cartier-split, then the scheme Y lifts to the ring W_2.*

We now introduce another condition that is much weaker than Cartier-split: The short exact sequence to the right in (3) yields an exact sequence

$$(4) \qquad \mathrm{Hom}(Z\Omega^1_Y, \mathscr{O}_{Y'}) \longrightarrow \mathrm{Hom}(B\Omega^1_Y, \mathscr{O}_{Y'}) \xrightarrow{\partial} \mathrm{Ext}^1(\Omega^1_{Y'}, \mathscr{O}_{Y'}).$$

The map on the right is the *connecting map*, which may or may not vanish.

Definition 1.3. We say that Y is *pre-Cartier-split* if the connecting map ∂ in the exact sequence (4) is the zero map.

Saying that the scheme Y is Cartier-split means that the short exact sequence $0 \to B\Omega^1_Y \to Z\Omega^1_Y \to \Omega^1_{Y'} \to 0$ splits; then the connecting map is a priori zero and Y is also pre-Cartier-split. The latter also holds if the group $\mathrm{Hom}(B\Omega^1_Y, \mathscr{O}_{Y'})$ vanishes. A key observation for this paper is the following criterion:

Theorem 1.4. *Suppose Y is pre-Cartier-split but not Frobenius-split, and satisfies*

$$(5) \qquad h^1(\Theta_Y) \geq h^1(\underline{\mathrm{Hom}}(Z\Omega^1_Y, \mathscr{O}_{Y'})).$$

Then this inequality is an equality, and the scheme Y does not lift to the ring of truncated Witt vectors W_2.

Proof. Since Y is not Frobenius-split, the extension class $\mathrm{cl}(F_*\mathscr{O}_Y) \in \mathrm{Ext}^1(B\Omega^1_Y, \mathscr{O}_{Y'})$ does not vanish. The short exact sequence to the right in (3) yields a long exact sequence

$$\mathrm{Ext}^1(\Omega^1_{Y'}, \mathscr{O}_{Y'}) \longrightarrow \mathrm{Ext}^1(Z\Omega^1_Y, \mathscr{O}_{Y'}) \longrightarrow \mathrm{Ext}^1(B\Omega^1_Y, \mathscr{O}_{Y'}) \longrightarrow \mathrm{Ext}^2(\Omega^1_{Y'}, \mathscr{O}_{Y'}).$$

The connecting map in (4) vanishes by assumption, so the map on the left is injective. It is actually bijective, because by assumption the vector space dimension $h^1(\Theta_Y)$ of its domain is at least as large as the dimension of its range. In turn, the inequality (5) is an equality. Moreover, the map in the middle of the above exact sequence is zero, so the extension class $\mathrm{cl}(F_*\mathscr{O}_Y) \neq 0$ is not in the image. Hence its image in $\mathrm{Ext}^2(\Omega^1_{Y'}, \mathscr{O}_{Y'})$ is non-zero. This image equals the Yoneda class of the four-term sequence in (3), by definition of the connecting map for Yoneda extension groups. According to Proposition 1.1, the scheme Y does not lift to the ring W_2. □

This result reveals that under suitable assumptions, a mere bound on certain cohomology groups implies non-existence of liftings. Note that the scheme Y is Frobenius split or Cartier split if and only if the respective property holds for the base-change to the algebraic closure k^{alg}.

We are particularly interested in the situation where $\omega_Y^{\otimes p} \simeq \mathscr{O}_Y$ but $\omega_Y \not\simeq \mathscr{O}_Y$. Then the scheme Y comes with a *canonical covering* $\epsilon : X \to Y$, which is a torsor under the local group scheme $\mu_p = \underline{\mathrm{Hom}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{G}_m)$. This explains why we choose the symbol Y for our smooth proper scheme. Note that the canonical covering is a finite flat universal homeomorphism of degree $\deg(X/Y) = p$, and that the scheme X usually contains singularities, and easily may become non-normal. For more details, see for example [50], Section 4.

2. THE CONDITION $c_1 = 0$

We keep the assumptions of the previous section, such that Y is a smooth proper n-dimensional scheme over some perfect field k of characteristic $p > 0$, with Frobenius pullback $Y' = Y \otimes_k k$. The dualizing sheaf $\omega_Y = \Omega_Y^n$ and the relative dualizing sheaf $\omega_{Y/Y'}$ will be of paramount importance. The latter is defined as a coherent sheaf on Y by the formula $F_*(\omega_{Y/Y'}) = \underline{\mathrm{Hom}}(F_*\mathscr{O}_Y, \mathscr{O}_{Y'})$. Let us start with the following facts:

Lemma 2.1. *The fibers of the relative Frobenius $F : Y \to Y'$ are Gorenstein, and the relative dualizing sheaf is given by $\omega_{Y/Y'} = \omega_Y^{\otimes 1-p}$. If ω_Y is p-torsion in the Picard group, we have $F_*(\omega_Y) = \underline{\mathrm{Hom}}(F_*\mathscr{O}_Y, \mathscr{O}_{Y'})$.*

Proof. Disregarding the structure morphisms to $\mathrm{Spec}(k)$, we first note that the projection $Y' = Y \otimes_k k \to Y$ is an isomorphism of schemes. It thus suffices to verify the first statement for the absolute Frobenius $F_Y : Y \to Y$. Fix a point $a \in Y$, and choose a regular system of parameters $f_1, \ldots, f_r \in \mathscr{O}_{Y,a}$. The schematic fiber $Y_a = F_Y^{-1}(a)$ is the spectrum of the ring $R = \mathscr{O}_{Y,a}/(f_1^p, \ldots, f_r^p)$. Passing to formal completions, we see that $R = \kappa(a)[[T_1, \ldots, T_r]]/(T_1^p, \ldots, T_r^p)$. The socle of this local Artin ring is generated by $\prod_{i=1}^r T_i^{p-1}$, whence the local ring R is Gorenstein.

In turn, the relative dualizing sheaf is invertible, and satisfies the formula $\omega_Y = \omega_{Y/Y'} \otimes F^*(\omega_{Y'})$. We have $\omega_{Y'} = \omega_Y \otimes_k k$, for the base-change with the Frobenius map $\lambda \mapsto \lambda^p$. Since the absolute Frobenius induces multiplication-by-p on the Picard group, the diagram (2) yields $F^*(\omega_{Y'}) = \omega_Y^{\otimes p}$, and the assertion on the relative dualizing sheaf follows. Finally, if ω_Y is p-torsion, we get $\omega_{Y/Y'} = \omega_Y$, and the last statement comes from the definition of relative dualizing sheaves. □

Write $c_1 = c_1(Y) = c_1(\Omega_Y^1) = c_1(\omega_Y)$ for the *first Chern class*, say as an element in the group $\mathrm{Num}(Y)$ of invertible sheaves modulo numerical equivalence. In other words, the condition $c_1 = 0$ means that $(\omega_Y \cdot C) = 0$ for all curves $C \subset Y$.

Lemma 2.2. *Suppose $c_1 = 0$ holds. Then the vector space $\mathrm{Hom}(B\Omega_Y^1, \mathscr{O}_{Y'})$ is at most one-dimensional. It is one-dimensional if and only if the scheme Y is not Frobenius-split and the dualizing sheaf ω_Y is $(p-1)$-torsion in the Picard group.*

Proof. The short exact sequence to the left in the diagram (3) splits if and only if the dual exact sequence

$$0 \longrightarrow \underline{\mathrm{Hom}}(B\Omega_Y^1, \mathscr{O}_{Y'}) \longrightarrow F_*(\mathscr{L}) \longrightarrow \mathscr{O}_{Y'} \longrightarrow 0$$

splits, for the numerically trivial sheaf $\mathscr{L} = \omega_{Y/Y'} = \omega_Y^{\otimes 1-p}$. In turn, we get an inclusion $\mathrm{Hom}(B\Omega_Y^1, \mathscr{O}_{Y'}) \subset H^0(Y, \mathscr{L})$. Since Y is integral, we have $h^0(\mathscr{L}) \leq 1$; equality holds if and only if $\mathscr{L} \simeq \mathscr{O}_Y$. This shows that $\mathrm{Hom}(B\Omega_Y^1, \mathscr{O}_{Y'})$ is at most one-dimensional.

Suppose that it is one-dimensional. Then $h^0(\mathscr{L}) \neq 0$, hence $h^0(\mathscr{L}) = 1$, and ω_Y is $(p-1)$-torsion. Moreover, the extension does not split, because otherwise the contribution of $h^0(\mathscr{O}_{Y'}) = 1$ yields the contradiction $h^0(\mathscr{L}) = 2$. Conversely, suppose that $\omega_Y^{\otimes 1-p} = \mathscr{O}_Y$ and that the extension does not split. In the resulting exact sequence

$$0 \longrightarrow \mathrm{Hom}(B\Omega_Y^1, \mathscr{O}_{Y'}) \longrightarrow H^0(Y, \mathscr{O}_Y) \longrightarrow H^0(Y', \mathscr{O}_{Y'}),$$

the unit section $1 \in H^0(Y', \mathscr{O}_{Y'})$ is not in the image of the map on the right. So this map vanishes, and it follows that the term on the left is one-dimensional. $\square$

Proposition 2.3. *Suppose that the order of ω_Y in the Picard group is p^ν with some exponent $\nu \geq 1$. Then Y is pre-Cartier-split but not Frobenius-split.*

Proof. Suppose Y is Frobenius split, and choose a retraction for $\mathscr{O}_{Y'} \subset F_*(\mathscr{O}_Y)$. The latter defines a non-zero global section s of $\omega_{Y/Y'} = \omega_Y^{\otimes 1-p}$. This sheaf is numerically trivial, so the map $s : \mathscr{O}_Y \to \omega_{Y/Y'}$ is bijective. In turn, the order p^ν divides $p-1$, contradiction. Thus Y is not Frobenius split. By assumption we have $c_1 = 0$, and the dualizing sheaf ω_Y is not $(p-1)$-torsion in the Picard group. Consequently, Lemma 2.2 gives $\mathrm{Hom}(B\Omega_Y^1, \mathscr{O}_{Y'}) = 0$. A priori, the connecting map $\mathrm{Hom}(B\Omega_Y^1, \mathscr{O}_{Y'}) \to \mathrm{Ext}^1(\Omega_{Y'}^1, \mathscr{O}_{Y'})$ is zero, hence the scheme Y is pre-Cartier split. $\square$

Recall that $n = \dim(Y)$. We now look at the right end of the cochain complex $F_*(\Omega_Y^\bullet)$. Using $\Omega_{Y'}^n = \omega_{Y'}$ and $Z\Omega_Y^n = F_*(\omega_Y)$ we get a commutative diagram

(6)
$$
\begin{array}{ccccc}
& & 0 & & 0 \\
& \nearrow & & \nearrow & \\
& & B\Omega_Y^n & & \\
& \nearrow & & \searrow & \\
0 \to Z\Omega_Y^{n-1} \to F_*\Omega_Y^{n-1} & \xrightarrow{\quad d \quad} & F_*\omega_Y & \to & \omega_{Y'} \to 0,
\end{array}
$$

where the horizontal four-term sequence is exact, and obtained by splicing the two short exact sequences with kinks.

Proposition 2.4. *Suppose that the order of the dualizing sheaf ω_Y in the Picard group coincides with the characteristic $p \geq 2$. Then the canonical map*

$$H^{n-1}(Y', \underline{\mathrm{Hom}}(F_*\Omega_Y^{n-1}, \mathscr{O}_{Y'})) \longrightarrow H^{n-1}(Y', \underline{\mathrm{Hom}}(Z\Omega_Y^{n-1}, \mathscr{O}_{Y'}))$$

is surjective.

Proof. According to Lemma 2.1, we have $F_*(\omega_Y) = \underline{\mathrm{Hom}}(F_*\mathscr{O}_Y, \mathscr{O}_{Y'})$, so its dual sheaf gets identified with $F_*(\mathscr{O}_Y)$, via biduality. Dualizing the short exact sequence to the right in (6), we thus get

$$0 \longrightarrow \omega_{Y'}^{\otimes -1} \longrightarrow F_*\mathscr{O}_Y \longrightarrow \underline{\mathrm{Hom}}(B\Omega_Y^n, \mathscr{O}_{Y'}) \longrightarrow 0.$$

The resulting long exact sequence shows that the induced homomorphism

$$H^n(Y, \mathscr{O}_Y) \longrightarrow H^n(Y', \underline{\mathrm{Hom}}(B\Omega_Y^n, \mathscr{O}_{Y'}))$$

is surjective. The term on the left is zero: Serre duality yields $h^n(\mathscr{O}_Y) = h^0(\omega_Y)$, and the latter vanish because ω_Y is numerically trivial yet $\omega_Y \not\simeq \mathscr{O}_Y$.

The short exact sequence to the left in (6) gives an exact sequence

$$\mathrm{Ext}^{n-1}(F_*\Omega_Y^{n-1}, \mathscr{O}_{Y'}) \longrightarrow \mathrm{Ext}^{n-1}(Z\Omega_Y^{n-1}, \mathscr{O}_{Y'}) \longrightarrow \mathrm{Ext}^n(B\Omega_Y^n, \mathscr{O}_{Y'}).$$

The term on the right vanishes, as we just saw, hence the mapping on the left is surjective. $\square$

For dimension $n = 2$, the above yields information for cohomology in degree one. We get the following numerical criterion for surfaces:

Theorem 2.5. *Suppose $\dim(Y) = 2$, that the order of the dualizing sheaf ω_Y in the Picard group coincides with the characteristic $p \geq 2$, and that*

$$h^1(\Theta_Y) \geq h^1(\underline{\mathrm{Hom}}(F_*\Omega_Y^1, \mathscr{O}_{Y'})).$$

Then this inequality is an equality, and the scheme Y does not lift to the ring W_2.

Proof. The scheme Y is pre-Cartier split but not Frobenius-split, according to Proposition 2.3. With Proposition 2.4 we get the estimates

$$h^1(\Theta_Y) \geq h^1(\underline{\mathrm{Hom}}(F_*\Omega_Y^1, \mathscr{O}_{Y'})) \geq h^1(\underline{\mathrm{Hom}}(Z\Omega_Y^1, \mathscr{O}_{Y'})).$$

Thus Theorem 1.4 applies: The above inequalities must be equalities, and the scheme Y does not lift to the ring W_2. $\square$

The big advantage of the preceding result is that the differentials in the cochain complex $F_*\Omega_Y^\bullet$ do not enter anymore. We merely have to compute the first cohomology of the locally free sheaf $\underline{\mathrm{Hom}}(F_*\Omega_Y^1, \mathscr{O}_{Y'})$. The next result tells us that under certain assumptions, this dual of Frobenius pushforward remains a Frobenius pushforward, which makes the necessary computations of cohomology feasible.

We say that a quasicoherent sheaf $\mathscr{E}$ on Y admits an *F-descend* if $\mathscr{E} \simeq F^*(\mathscr{E}')$ for some quasicoherent sheaf $\mathscr{E}'$ on Y'. By fpqc-descend ([26], Exposé VIII), this means that on the fiber product $Y \times_{Y'} Y$ there is a descend datum $\varphi : \mathrm{pr}_1^*(\mathscr{E}) \to \mathrm{pr}_2^*(\mathscr{E})$. According to [33], Theorem 5.1 such a descend datum can be interpreted as an integrable connection $\nabla : \mathscr{E} \to \mathscr{E} \otimes \Omega_Y^1$ with p-curvature zero.

Proposition 2.6. *Suppose $\mathscr{E}$ is a locally free sheaf on Y that admits an F-descent. Then $\underline{\mathrm{Hom}}(F_*(\mathscr{E}^\vee), \mathscr{O}_{Y'}) = F_*(\mathscr{E} \otimes \omega_{Y/Y'})$. If moreover the dualizing sheaf ω_X is p-torsion in the Picard group, we have $\underline{\mathrm{Hom}}(F_*(\mathscr{E} \otimes \omega_Y), \mathscr{O}_{Y'}) = F_*(\mathscr{E}^\vee)$.*

Proof. Write $\mathscr{E} = F^*(\mathscr{E}')$. Then also $\mathscr{E}^\vee = F^*(\mathscr{E}'^\vee)$, and the Projection Formula gives $F_*(\mathscr{E}^\vee) = F_*(\mathscr{O}_Y) \otimes \mathscr{E}'^\vee$. With the relations between tensor products and hom modules ([10], Chapter II, §3, No.s 1–2), together with the definition of relative dualizing sheaves one obtains

$$\underline{\operatorname{Hom}}(F_*(\mathscr{E}^\vee), \mathscr{O}_{Y'}) = \underline{\operatorname{Hom}}(F_*\mathscr{O}_Y, \mathscr{O}_{Y'}) \otimes \mathscr{E}' = F_*(\omega_{Y/Y'}) \otimes \mathscr{E}'.$$

Applying the Projection Formula again, we obtain the first assertion.

Now assume that ω_Y is p-torsion in the Picard group. Then Lemma 2.1 and biduality gives

$$\underline{\operatorname{Hom}}(F_*(\omega_Y), \mathscr{O}_{Y'}) = \underline{\operatorname{Hom}}(\underline{\operatorname{Hom}}(F_*\mathscr{O}_Y, \mathscr{O}_{Y'}), \mathscr{O}_{Y'}) = F_*(\mathscr{O}_Y),$$

and we can proceed as in the preceding paragraph. $\qquad\square$

Note that an invertible sheaf $\mathscr{L}$ admits F-descend if and only if it is p-divisible in the Picard group: If $\mathscr{L} = \mathscr{N}^{\otimes p}$, we form the base-change $\mathscr{L}' = \mathscr{N} \otimes_k k$ under the Frobenius map $\lambda \mapsto \lambda^p$, and obtain $\mathscr{L} = F_Y^*(\mathscr{N}) = F^*(\mathscr{L}')$. Conversely, if $\mathscr{L} = F^*(\mathscr{L}')$, we let $\mathscr{N} = \mathscr{L}' \otimes_k k$ under the inverse $\lambda \to \lambda^{1/p}$ of the Frobenius map, and get $\mathscr{L} = F_Y^*(\mathscr{N}) = \mathscr{N}^{\otimes p}$.

3. THE BASE OF THE MINIVERSAL DEFORMATION

We now examine liftability via the miniversal formal deformation, which is also called the semi-universal formal deformation, or prorepresentable hull in the terminology of Schlessinger [47]. Let Y be proper and smooth with $h^0(\mathscr{O}_Y) = 1$ over a perfect field k of characteristic $p > 0$. Set

$$s = h^1(\Theta_Y) \quad \text{and} \quad r = h^2(\Theta_Y),$$

and let $W = W(k)$ be the ring of Witt vectors. Let $\mathfrak{Y} \to \operatorname{Spf}(A)$ be the *miniversal formal deformation*, where A is a complete local noetherian W-algebra with residue field $A/\mathfrak{m}_A = k$. Then for every lifting $\mathfrak{Y}_B$ of Y over some local Artin W-algebra B with residue field k, there is a homomorphism $A \to B$ with $\mathfrak{Y}_B = \mathfrak{Y} \otimes_A B$, and the induced map of cotangent spaces $\mathfrak{m}_A/(pA + \mathfrak{m}_A^2) \to \mathfrak{m}_B/(pB + \mathfrak{m}_B^2)$ is unique.

Lemma 3.1. *The base of the miniversal deformation is given by a ring of the form*

$$A = W[[T_1, \ldots, T_s]]/(f_1, \ldots, f_r),$$

where the f_i are formal power series with coefficients from W, and their images in $k[[T_1, \ldots, T_s]]$ have no linear terms.

Proof. The argument is parallel to the proof for Proposition 1.5 in [14]. Let me sketch Deligne's arguments for the convenience of the reader: Since Y is smooth, the isomorphism classes of deformations over the ring of dual numbers $k[\epsilon]$ correspond to vectors in $H^1(Y, \Theta_Y)$, which has dimension s. In turn, A is a quotient of the formal power series ring $R = W[[T_1, \ldots, T_s]]$ by some ideal I contained in the maximal $\mathfrak{m} = \mathfrak{m}_R$. We have to verify that the ideal can be generated by r elements. Consider the formal subschemes $S' \subset S''$ inside $S = \operatorname{Spf}(R)$ defined by the ideals $I \supset \mathfrak{m}I$. Then the obstruction to extend $\mathfrak{Y} \to S'$ to S'' lies in the vector space $H^2(Y, \Theta_Y \otimes I/\mathfrak{m}I) = H^2(Y, \Theta_Y) \otimes I/\mathfrak{m}I$. The tensor factor to the left has dimension r, so we may view the obstruction as an r-tuple with entries from $I/\mathfrak{m}I$. Choose representatives $f_1, \ldots, f_r \in I$. By construction, $\mathfrak{Y} \to S'$ extends to the

formal spectrum of $R/(\mathfrak{m}I + \sum Rf_i)$. So by the versal property of $\mathfrak{Y}$, the projection $R/(\mathfrak{m}I + \sum Rf_i) \to R/I$ admits a retraction, and it follows that $\mathfrak{m}I + \sum Rf_i = I$. By the Nakayama Lemma, the elements $f_1, \ldots, f_r \in I$ generate the ideal. These generators indeed have no linear terms modulo p, again because the isomorphism classes of deformations over the ring of dual numbers $k[\epsilon]$ corresponds to the r-dimensional vector space $H^1(Y, \Theta_Y)$. $\qquad\square$

Write $\mathfrak{m} = (p, T_1, \ldots, T_s)$ for the maximal ideal of the formal power series ring $W[[T_1, \ldots, T_s]]$. Note that $p, T_1, \ldots, T_s$ yield a basis of the cotangent space $\mathfrak{m}/\mathfrak{m}^2$, hence these elements form a regular system of parameters. Write the beginning of the formal power series as

$$(7) \qquad f_i(T_1, \ldots, T_s) \equiv \beta_{i,s+1}p + \sum_{j=1}^{s} \beta_{ij}T_j \quad \text{modulo } \mathfrak{m}^2,$$

and let $\bar{\beta}_{ij} \in k$ be the residue classes of the coefficients $\beta_{ij} \in W$. This defines two matrices

$$B_{\text{pure}} = (\bar{\beta}_{ij})_{\substack{1 \le i \le r \\ 1 \le j \le s}} \quad \text{and} \quad B_{\text{mixed}} = (\bar{\beta}_{ij})_{\substack{1 \le i \le r \\ 1 \le j \le s+1}}$$

with entries from the field k. Checking liftability to W_2 now translates into a rank computation:

Lemma 3.2. *The scheme Y lifts to W_2 if and only if* $\operatorname{rank}(B_{\text{mixed}}) = \operatorname{rank}(B_{\text{pure}})$.

Proof. By the defining properties of versal deformations, all liftings come from W-algebra homomorphism $A \to W_2$. The latter are given by $\mu_j \in pW_2$ satisfying the system of equations $f_i(\mu_1, \ldots, \mu_s) = 0$ in the ring W_2. Write these truncated Witt vectors as $\mu_j = (0, \lambda_j)$ with scalars $\lambda_j \in k$. For any lift $\tilde{\lambda}_j \in W_2$ of $\lambda_j \in W_2/pW_2$ we have $p\tilde{\lambda}_j = \mu_j$. By abuse of notation, we may also write $p\lambda_j$ for this element. The equations $f_i(p\lambda_1, \ldots, p\lambda_s) = 0$ in the ring W_2 translate into the system of linear equations

$$B_{\text{pure}} \cdot \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_s \end{pmatrix} = - \begin{pmatrix} \bar{\beta}_{1,s+1} \\ \vdots \\ \bar{\beta}_{r,s+1} \end{pmatrix},$$

over the field k, in light of (7). Since the matrix B_{mixed} is obtained from B_{pure} by adjoining the transpose of $(\bar{\beta}_{1,s+1}, \ldots, \bar{\beta}_{r,s+1})$ as additional column, solvability of the above system of linear equations means that the two matrices have the same rank. $\qquad\square$

We now consider the case where the number of relations is $r = 1$. Then we may drop the indices for the formal power series, and we write $f = f_1$ and $\bar{\beta}_j = \bar{\beta}_{ij}$.

Proposition 3.3. *Let $h^1(\Theta_Y) = s$ be arbitrary, but suppose that $h^2(\Theta_Y) = 1$. Then the scheme Y does not lift to the ring W_2 if and only if $\bar{\beta}_1 = \ldots = \bar{\beta}_s = 0$ and $\bar{\beta}_{s+1} \ne 0$. In this situation, the complete local ring $A = W[[T_1, \ldots, T_s]]/(f)$ is regular of dimension $s = h^1(\Theta_Y)$.*

Proof. Our matrix B_{mixd} becomes the vector $(\bar{\beta}_1, \ldots, \bar{\beta}_s, \bar{\beta}_{s+1})$, and the first assertion follows from Lemma 3.2. In the cotangent space $\mathfrak{m}/\mathfrak{m}^2$ for the regular local ring $W[[T_1, \ldots, T_s]]$, the classes of $p, T_1, \ldots, T_s$ form a basis, and the class of the relation

$f \in \mathfrak{m}$ coincides with the first basis vector. In turn, the residue class ring A remains regular, with $\dim(A) = (s+1) - 1 = s$. $\square$

We say that Y *formally lifts to characteristic zero* if the canonical map $W \to A$ is injective. We then regard this map as an inclusion $W \subset A$. Since $W \cap \mathrm{Nil}(A) = 0$, there is a minimal prime ideal $\mathfrak{p} \subset A$ so that $W \subset A/\mathfrak{p}$ remains injective. In turn, $C = A/\mathfrak{p}$ is a complete local ring with residue field $k = C/\mathfrak{m}_C$ that is integral, and whose field of fractions $\mathrm{Frac}(C)$ has characteristic zero. We thus obtain compatible infinitesimal deformations $\mathfrak{Y}_i \to \mathrm{Spec}(C_i)$ of the scheme Y over the residue class rings $C_i = C/p^{i+1}C$.

Corollary 3.4. *Assumptions as in the proposition. Suppose furthermore that the scheme Y does not lift to the ring W_2. Then Y formally lifts to characteristic zero if and only if the complete local ring*

$$A/pA = k[[T_1, \ldots, T_s]]/(f)$$

is singular. In this situation, we have $\dim(A/pA) = s - 1$.

Proof. The local ring $B = W[[T_1, \ldots, T_s]]$ is factorial. According to the proposition, the formal power series $f \in B$ is a prime element, and we may assume that it is of the form $f = p + g$ with some $g \in \mathfrak{m}_B^2$. Regard $p \in B$ as another prime element. If $(f) = (p)$ the ring $A/pA = A = B/pB$ is regular and the scheme Y does not lift to characteristic zero. On the other hand, if $(f) \neq (p)$ then the prime element p remains a regular element in $A = B/fB$. In turn, the map $W \to A$ is injective, so the scheme Y formally lifts to characteristic zero. By Krull's Principal Ideal Theorem, the ring $A/pA = B/(f, p)$ is of dimension $(s+1) - 2 = s - 1$. Using the description $A/pA = B/(g, p)$ we see that it has embedding dimension s, which means that A/pA is singular. $\square$

Recall that the proper homomorphic images $V/\mathfrak{m}_V^i$, $i \geq 1$ of discrete valuation rings V are exactly the local artinian principal ideal rings ([39], Theorem 3.3). Zariski and Samuel call them *special PIR's* ([57], page 245). One could also characterize them as the local noetherian rings of dimension zero and embedding dimension at most one. We propose to call them *jet rings*.

The situation of Corollary 3.4 is somewhat paradoxical, and perhaps warrants a brief discussion. The deformations of the scheme Y are *unobstructed in the following sense*: For each jet ring quotient $A/\mathfrak{a}$ there is smaller ideal $\mathfrak{a}' \subsetneqq \mathfrak{a}$ such that $A/\mathfrak{a}'$ stays a jet ring quotient. This is because one finds a regular system of parameters $f_1, \ldots, f_s \in A$ with $\mathfrak{a} = (f_1^l, f_2, \ldots, f_s)$, and then sets $\mathfrak{a}' = (f_1^{l+1}, f_2, \ldots, f_s)$. One should view the $\mathrm{Spec}(A/\mathfrak{a}) \subset \mathrm{Spec}(A/\mathfrak{a}')$ as jets of formal curves inside the base $\mathrm{Spf}(A)$ of the miniversal deformation. Note that in order to deform over rings in which $p \neq 0$, one first has to travel over some infinitesimal neighborhoods in which $p = 0$ holds. On the other hand, one may regard the deformations of Y as *obstructed*: For certain discrete valuation rings V, some jet ring $V/\mathfrak{m}_V^i$ is the homomorphic image of A, but $V/\mathfrak{m}_V^{i+1}$ is not. In fact, one may choose $V = W$ with $i = 1$, or $V = k[[T]]$.

Suppose R is any local noetherian ring. With respect to our prime $p > 0$, one may define the *absolute ramification index*

$$e(R) = \sup\{i \in \mathbb{N} \mid p \cdot 1_A \in \mathfrak{m}_R^i\} \in \mathbb{N} \cup \{\infty\}.$$

By Krull's Intersection Theorem, $e(R) = \infty$ means that $p \in R$ is the zero element, hence R is an $\mathbb{F}_p$-algebra. If $0 < e(R) < \infty$, the residue field $R/\mathfrak{m}_R$ has characteristic $p > 0$, hence all other primes $l \neq p$ become invertible, and we get an extension $\mathbb{Z}_{(p)} \subset R$ of local rings. For integral domains R, this means flatness. For discrete valuations rings R, our invariant $e(R)$ is then the usual ramification index. If complete, the ring R becomes an algebra over the ring $W(k)$ of Witt vectors. Finally, the condition $e(R) = 0$ means that $p \in R$ is invertible as well, which makes R into a $\mathbb{Q}$-algebra. We see that the absolute ramification index $e(A) \geq 0$ for the base of the miniversal formal deformation $\mathfrak{Y} \to \mathrm{Spf}(A)$ yields an numerical invariant of the scheme $Y = \mathfrak{Y}_0$ that reflects liftability.

4. Proper flat group schemes

Sometimes, first-order liftings are already precluded by Picard schemes. The goal of this section is to collect some results in this direction, which mainly rely on the theory of relative group schemes whose structure morphism is proper. We start with the following general set-up: Let R be a discrete valuation ring, with residue field $k = R/\mathfrak{m}_R$ and field of fractions $F = \mathrm{Frac}(R)$. Let $\mathfrak{G}$ be a relative commutative group scheme whose structure morphism $\mathfrak{G} \to \mathrm{Spec}(R)$ is proper and flat, and that the closed fiber $\mathfrak{G}_k = \mathfrak{G} \otimes_R k$ is connected. The Stein factorization gives an affine scheme $\mathfrak{H} = \mathrm{Spec}\,\Gamma(\mathfrak{G}, \mathscr{O}_{\mathfrak{G}})$, which is finite and flat over R. Using that global sections commute with flat base-change, one infers that $\mathfrak{H}$ inherits the structure of relative group scheme, that the canonical map $h : \mathfrak{G} \to \mathfrak{H}$ is a homomorphism, and that the closed fiber $\mathfrak{H}_k$ is local. Note that we do not assume that the structure morphism $f : \mathfrak{H} \to \mathrm{Spec}(R)$ is cohomologically flat, such that the equality $\mathscr{O}_{\mathfrak{H}} = h_*(\mathscr{O}_{\mathfrak{G}})$ may not be preserved by base-change.

Write $\mathfrak{G}_F^0 = (\mathfrak{G}_F)^0$ for the connected component of the origin for the generic fiber, and assume throughout that the reduced parts $\mathfrak{G}_{k,\mathrm{red}} = (\mathfrak{G}_k)_{\mathrm{red}}$ and $\mathfrak{G}_{F,\mathrm{red}}^0 = (\mathfrak{G}_F^0)_{\mathrm{red}}$ are geometrically reduced. This automatically holds if the residue field k is perfect and the function field F has characteristic zero. The assumption ensures that these reduced parts are subgroup schemes, which are connected, smooth and proper, hence abelian varieties. Write $\mathfrak{A}_F = \mathfrak{G}_{F,\mathrm{red}}^0$, and let $\mathfrak{A} \subset \mathfrak{G}$ be the Zariski closure, which is an integral closed subscheme that is proper and flat over R. Using that the formation of closures commutes with flat base-change, and we infer that $\mathfrak{A} \subset \mathfrak{G}$ is a relative subgroup scheme. Since $\Gamma(\mathfrak{A}_F, \mathscr{O}_{\mathfrak{A}}) = F$, the image of the subgroup scheme $\mathfrak{A}$ in the group scheme $\mathfrak{H}$ vanishes. We shall see below that the resulting sequence $0 \to \mathfrak{A} \to \mathfrak{G} \to \mathfrak{H} \to 0$ of proper flat relative group schemes is "exact". One has to exercise some care to make this precise, because the category of commutative group schemes over R is not abelian. To do so, view $\mathfrak{G}$ as a scheme over $\mathfrak{H}$, with respect to the canonical morphism $h : \mathfrak{G} \to \mathfrak{H}$. As such, it comes with an action of the induced relative group scheme $\mathfrak{A}_{\mathfrak{H}} = \mathfrak{A} \otimes_R \Gamma(\mathscr{O}_{\mathfrak{G}})$.

Proposition 4.1. *Assumptions as above. Then the following holds:*

(i) *The relative group scheme $\mathfrak{A}$ is an abelian scheme, and we have $\mathfrak{A}_k = \mathfrak{G}_{k,\mathrm{red}}$ as subgroup schemes of $\mathfrak{G}_k$.*

(ii) *The projection $h : \mathfrak{G} \to \mathfrak{H}$ is a torsor for the action of the induced abelian scheme $\mathfrak{A}_{\mathfrak{H}}$.*

(iii) *The resulting sequence of group schemes $0 \to \mathfrak{G}_{k,\mathrm{red}} \to \mathfrak{G}_k \to \mathfrak{H}_k \to 0$ is exact.*

Proof. First note that by fpqc descent, we may replace the ground ring R by any extension of discrete valuation rings. By passing to the strict localization, it suffices to treat the case that the residue field k is separably closed and that R is henselian.

We first check with the Néron–Ogg–Shafarevich Criterion that the abelian variety $\mathfrak{A}_F$ has good reduction. Consider the quotient $\Psi_F = \mathfrak{G}_F/\mathfrak{A}_F$, which is an extension of the group scheme of components by some local group scheme, and fix a prime $l > 0$ that is relatively prime to the characteristic exponent of the residue field $k = R/\mathfrak{m}_R$, and that does not divide the order of Ψ_F. Write $O_R \subset \mathfrak{G}$ for the zero section. The cartesian diagram

$$
\begin{array}{ccc}
\mathfrak{G}[l^n] & \longrightarrow & O_R \\
\downarrow & & \downarrow {\scriptstyle \mathrm{can}} \\
\mathfrak{G} & \xrightarrow{\ l^n\ } & \mathfrak{G}
\end{array}
$$

defines a relative subgroup scheme $\mathfrak{G}[l^n]$. Its formation commutes with base-change in W, and the structure morphism $\mathfrak{G}[l^n] \to \mathrm{Spec}(R)$ is proper. Since multiplication by l^n is finite on abelian varieties and finite group schemes, we see that the structure morphism is quasifinite, hence finite. Moreover, both fibers have length l^{2ng}, where $g \geq 0$ is the common dimension of the two abelian varieties $\mathfrak{G}_{k,\mathrm{red}}$ and $\mathfrak{A}_F$. With [28], Chapter III, Theorem 9.9 we conclude that the structure morphism $\mathfrak{G}[l^n]$ is locally free of degree $d = l^{2ng}$, that the generic fiber $\mathfrak{G}[l^n]_F$ is contained in $\mathfrak{A}_F$, and that the closed fiber $\mathfrak{G}[l^n]_k$ is contained in $\mathfrak{G}_{k,\mathrm{red}}$. Since the residue field k is separably closed and the ring is henselian, we see that the relative group schemes $\mathfrak{G}[l^n]$ are constant. In particular, each F^{sep}-valued point on $\mathfrak{A}_F[l^n]$ comes from a F-valued point. This ensures that the Néron model $\mathfrak{A}'$ of $\mathfrak{A}_F$ over R is an abelian scheme ([9], Chapter 7.4, Theorem 5).

We thus have mutually inverse birational map $\mathfrak{A} \dashrightarrow \mathfrak{A}'$ and $\mathfrak{A}' \dashrightarrow \mathfrak{A}$, whose domains of definitions contain the generic fiber $\mathfrak{A}_F = \mathfrak{A}'_F$. Choose some integral scheme $\mathfrak{X}$ and some proper birational morphisms $\mathfrak{A} \leftarrow \mathfrak{X} \to \mathfrak{A}'$ over R that become identities over F. The union $\bigcup_{n \geq 0} \mathfrak{A}_F[l^n]$ is Zariski dense in the generic fiber, and the union of the closures become Zariski dense in the closed fibers of $\mathfrak{A}$ and $\mathfrak{A}'$. It follows that the strict transforms of $\mathfrak{A}_k$ and $\mathfrak{A}'_k$ in $\mathfrak{X}$ coincide. Consequently, $\mathfrak{A}_k$ is generically reduced. In turn, this closed fiber is an abelian variety, and $\mathfrak{A} \to \mathrm{Spec}(R)$ is an abelian scheme. This establishes (i).

To proceed, we employ the theory of stacks. Consider the relative subgroup scheme $\mathfrak{A} \subset \mathfrak{G}$ and the resulting stack $[\mathfrak{G}/\mathfrak{A}]$. The latter is a fibered category over the category of affine schemes (Aff/R), with fibers over $U = \mathrm{Spec}(A)$ given by pairs $(\mathscr{T}, \varphi)$, where $\mathscr{T}$ is an $\mathfrak{A}|U$-torsor and $\varphi : \mathscr{T} \to \mathfrak{G}|U$ is an equivariant morphism. The pairs with $\mathscr{T} = \mathfrak{A}|U$ and φ the canonical inclusion $\mathfrak{A}|U \subset \mathfrak{G}|U$ define a 1-morphism $\mathfrak{G} \to [\mathfrak{G}/\mathfrak{A}]$. Note that the 2-fiber product $\mathfrak{G} \times_{[\mathfrak{G}/\mathfrak{A}]} \mathfrak{G}$ has fiber categories given by (g_1, g_2, ψ), where $g_i \in \mathfrak{G}(U)$, and the isomorphism ψ can be viewed as some $a \in \mathfrak{A}(U)$ with $g_2 = a + g_1$. In turn, the canonical morphism

$$
(8) \qquad \mathfrak{A} \times \mathfrak{G} \longrightarrow \mathfrak{G} \times_{[\mathfrak{G}/\mathfrak{A}]} \mathfrak{G}, \quad (a, g) \longmapsto (ag, g)
$$

is a 1-isomorphism. For the abelian scheme $\mathfrak{A}$, the structure morphism $\mathfrak{A} \to \mathrm{Spec}(R)$ is smooth, separated and of finite type, hence $[\mathfrak{G}/\mathfrak{A}]$ is an *Artin stack*, for example by [35], Example 4.6.1.

The 1-morphisms $[\mathfrak{G}/\mathfrak{A}] \to \mathrm{Spec}(R)$ is separated. To see this, we apply the Valuative Criterion (loc. cit. Proposition 7.8): Suppose $(\mathscr{T}_1, \varphi_1)$ and $(\mathscr{T}_2, \varphi_2)$ are two objects of $[\mathfrak{G}/\mathfrak{A}]$ over the spectrum U of a complete valuation ring A with algebraically closed residue field, and $\alpha_F : (\mathscr{T}_1, \varphi_1)_F \to (\mathscr{T}_2, \varphi_2)_F$ is an isomorphism over $F = \mathrm{Frac}(A)$. We have to check that there is at most one extension to an isomorphism over A. The assumptions on A ensure that the torsors admit sections, so it suffices to treat the case that both $\mathscr{T}_1, \mathscr{T}_2$ coincide with $\mathfrak{A}|U$. Then α_F can be identified with an element of $s_F \in \mathfrak{A}(F)$, and the separatedness of $\mathfrak{A} \to \mathrm{Spec}(R)$ ensures that there is at most one extension.

The algebraic stack $[\mathfrak{G}/\mathfrak{A}]$ is actually an algebraic space. To see this, we apply loc. cit. Corollary 8.1.1 and have to verify the following: Suppose $(\mathscr{T}, \varphi)$ is an object of the stack over some $U = \mathrm{Spec}(A)$, and α is an automorphism of $(\mathscr{T}, \varphi)$, then actually $\alpha = \mathrm{id}$. This problem is local, so it suffices to treat the case $\mathscr{T} = \mathfrak{A}|U$. Then α is the translation with respect to some section $s \in \mathfrak{A}(U)$. Since the $\mathfrak{A}$-action on $\mathfrak{G}$ is free, we must have $s = e$, thus $\alpha = \mathrm{id}$.

Since (8) is an isomorphism, the projection $\mathfrak{G} \to \mathfrak{X}$ is a $\mathfrak{G}_{\mathfrak{X}}$-torsor. In turn, the structure morphism $\mathfrak{X} \to \mathrm{Spec}(R)$ is of finite type. Forming the stack $[\mathfrak{A}/\mathfrak{G}]$ commutes with base-change in R, consequently the fibers of $\mathfrak{X}$ are finite. It follows that $\mathfrak{X}$ is a scheme (loc. cit., Theorem A.2). Since $\mathfrak{G} \to \mathrm{Spec}(R)$ is proper, $\mathfrak{G} \to \mathfrak{X}$ is surjective and $\mathfrak{X} \to \mathrm{Spec}(R)$ is separated and of finite type, the latter must be proper ([24], Corollary 5.4.3), hence finite. Since the projection $h : \mathfrak{G} \to \mathfrak{X}$ is a torsor for the abelian variety, we see that $\mathscr{O}_{\mathfrak{X}} = h_*(\mathscr{O}_{\mathfrak{G}})$. In turn, $\mathfrak{G}$ and $\mathfrak{X}$ have the same Stein factorization, and it follows $\mathfrak{H} = \mathfrak{X}$. This establishes (ii). Since $\mathfrak{A}_k = \mathfrak{G}_{k,\mathrm{red}}$, and forming the stack $[\mathfrak{G}/\mathfrak{A}]$ commutes with base-change in $\mathrm{Spec}(R)$, we also have (iii). $\qquad\square$

We now consider the following more special situation: Suppose that k is a perfect field, and G is a commutative group scheme over k that is proper and connected. Then the reduced part G_{red} is a subgroup scheme that is an abelian variety, and the quotient $L = G/G_{\mathrm{red}}$ is a local group scheme. Let L^{mult} be its multiplicative part, such that $U = L/L^{\mathrm{mult}}$ is unipotent. Since k is perfect, the resulting extension splits uniquely, and we have $L = L^{\mathrm{mult}} \times U$, see [17], Chapter IV, §3, Theorem 1.1. Note that L^{mult} corresponds to finite Galois modules, whereas U is given by a Dieudonné module of finite length. Write $W = W(k)$ for the ring of Witt vectors, W_2 for its truncation of length two, and $\alpha_{p^n} = \mathbb{G}_a[F^n]$ for the iterated Frobenius kernel.

Proposition 4.2. *Suppose the unipotent group scheme $U = L/L^{\mathrm{mult}}$ contains α_{p^n} as a direct summand, for some exponent $n \geq 1$. Then L does not lift to the ring W_2, and G does not lift to the ring W.*

Proof. Seeking a contradiction, we assume that there is a relative group scheme $\mathfrak{G}$ whose structure morphism $\mathfrak{G} \to \mathrm{Spec}(W)$ is proper and flat, with closed fiber $\mathfrak{G}_k = G$. According to Proposition 4.1, there is relative group scheme $\mathfrak{H}$ whose structure morphism is finite, with closed fiber $\mathfrak{H}_k = L$. In particular, L lifts to the ring W_2, which reduces the second assertion to the first.

Now suppose that we have a relative group scheme $\mathfrak{L} \to \mathrm{Spec}(W_2)$ whose structure morphism is finite, with closed fiber $\mathfrak{L}_k = L$, and consider the ensuing Hopf algebra $H = \Gamma(\mathfrak{L}, \mathscr{O}_{\mathfrak{L}})$. As a k-algebra, the fiber ring $\bar{H} = H/pH$ takes the form $\bar{H} = k[T_1, \ldots, T_r]/(T_1^{p^{n_1}}, \ldots, T_r^{p^{n_r}})$ for some integer $r \geq 0$ and some exponents $n_i \geq 0$, according to [17], Chapter III, §3, Corollary 6.3.

By assumption, we have a decomposition $L = \alpha_{p^n} \oplus L'$. The first factor is the spectrum of the local Artin ring $k[t]/(t^{p^n})$, with comultiplication $t \mapsto t \otimes 1 + 1 \otimes t$. The projection $L \to \alpha_{p^n}$ corresponds to an inclusion of Hopf algebras $k[t]/(t^{p^n}) \subset \bar{H}$, and we may assume $t = T_1$ and $n = n_1$.

Clearly, the k-algebra $\bar{H}$ is a complete intersection, and the $\bar{H}$-module $\Omega^1_{\bar{H}/k}$ is freely generated by the differentials $dT_1, \ldots, dT_r$. In turn, $\mathrm{Ext}^1(\Omega^1_{\bar{H}}, \bar{H}) = 0$. It follows that all lifts of the scheme $\mathrm{Spec}(\bar{H})$ to the ring W_2 are isomorphic, so we may write the W_2-algebra as $H = W_2[T_1, \ldots, T_r]/(T_1^{p^{n_1}}, \ldots, T_r^{p^{n_r}})$. Using multi-index notation, we observe that the monomials $T^a = \prod_{i=1}^r T_i^{a_i}$ form a basis for the underlying W_2-module, with $0 \leq a_i < p^{n_i}$. The comultiplication takes the form

$$\Delta(t) = t \otimes 1 + 1 \otimes t + p \sum \lambda_{ab} T^a \otimes T^b,$$

for certain scalars $\lambda_{ab} \in W_2$. As in [43], first example in the introduction, we now use that the map $\Delta : H \to H \otimes H$ is a homomorphism of rings: On the one hand, thanks to the relation $t^{p^n} = 0$ we get $\Delta(t^{p^n}) = \Delta(0) = 0$. On the other hand, the relation $p^2 = 0$ gives

$$\Delta(t)^{p^n} = (t \otimes 1 + 1 \otimes t + p \sum \lambda_{ab} T^a \otimes T^b)^{p^n} = (t \otimes 1 + 1 \otimes t)^{p^n}.$$

With $t^{p^n} = 0$ and the Binomial Theorem, this becomes $\sum_{i=1}^{p^n-1} \binom{p^n}{i} t^i \otimes t^{p^n-i}$. It is well-known that on binomial coefficients of the form $\binom{p^n}{i}$, the p-adic valuation $\nu_p : \mathbb{Z} \to \mathbb{N} \cup \{\infty\}$ takes the value $n - \nu_p(i)$. In particular, the binomial coefficient for $i = p^{n-1}$ does not vanish in the ring W_2. Summing up, we have the contradiction $0 = \Delta(t^{p^n}) = \Delta(t)^{p^n} \neq 0$. $\square$

In light of this, relative Picard scheme may preclude liftings to W or its truncation W_2, under suitable representability assumptions. Suppose that Y is a smooth proper scheme over k, satisfying $h^0(\mathscr{O}_Y) = 1$. Let

$$b_i = \mathrm{rank}_{\mathbb{Z}_\ell}(\varprojlim_\nu H^i(Y \otimes k^{\mathrm{sep}}, \mu_{\ell^\nu}^{\otimes i}))$$

be its ℓ-adic Betti numbers.

Theorem 4.3. *Set $G = \mathrm{Pic}^0_{Y/k}$. Suppose the local group scheme $L = G/G_{\mathrm{red}}$ contains some α_{p^n}, $n \geq 1$ as a direct summand, and that $b_1 = 2(h^1(\mathscr{O}_Y) - h^2(\mathscr{O}_Y))$ holds. Then the scheme Y does not lift to the ring W. If moreover $G_{\mathrm{red}} = 0$, the scheme does not even lift to W_2.*

Proof. This relies on some foundational results on relative Picard schemes, which we recall first. Suppose that R is an arbitrary local noetherian W-algebra, and that $\mathfrak{Y} \to \mathrm{Spec}(R)$ is a proper flat morphism with closed fiber $\mathfrak{Y} \otimes_W k = Y$. By Artin's result (see [9], Section 8.3, Theorem 1), the relative Picard functor is representable by some relative group space $P = \mathrm{Pic}_{\mathfrak{Y}/R}$, which means a group object in the category of algebraic spaces over R. Moreover, the structure morphism $P \to \mathrm{Spec}(R)$ is

separated (loc. cit. Section 8.4, Theorem 3), and the condition on the Betti number ensures that it is also flat ([20], Proposition 4.2). The inclusion of $P^\tau = \mathrm{Pic}^\tau_{\mathfrak{Y}/R}$ is representable by an open and closed embedding ([4], Exposé XIII, Theorem 4.7 together with [25], Corollary 2.3). Moreover, the structure morphism $P^\tau \to \mathrm{Spec}(R)$ is proper ([9], Section 8.4, Theorem 4 combined with Theorem 3).

Now suppose that the local ring R is henselian, of dimension $\dim(R) \leq 1$. According to [1], Theorem 4.B, the algebraic space P^τ is actually a scheme. Moreover, the connected components of the scheme P^τ correspond to the connected components of the closed fiber $P^\tau \otimes_R k$. Write $\mathfrak{G} \subset P^\tau$ for the connected component with $\mathfrak{G}_k = P^0 \otimes_R k$. This is a subgroup scheme. By construction, the structure morphism $\mathfrak{G} \to \mathrm{Spec}(R)$ is proper and flat, and the closed fiber $G = \mathfrak{G}_k$ is connected.

Seeking a contradiction, we now suppose that $R = W$ is the ring of Witt vectors. Proposition 4.1 applies, and we find some finite flat group scheme $\mathfrak{H}$ over W with closed fiber $\mathfrak{H}_k = G/G_{\mathrm{red}}$. In particular, the local group scheme $L = G/G_{\mathrm{red}}$ contains α_{p^n} and lifts to the ring W_2, in contradiction to Proposition 4.2. Finally suppose that $G_{\mathrm{red}} = 0$, and that $R = W_2$ is the ring of truncated Witt vectors. Then the group scheme L lifts to the ring W_2, and Proposition 4.2 gives again a contradiction. $\square$

5. Vector bundle computations

We now make some computations with vector bundles on surfaces that will be useful in the following sections. Suppose our smooth proper scheme Y has dimension $n = 2$, and let $\mathscr{E}$ be a locally free sheaf. The Hirzebruch–Riemann–Roch Theorem $\chi(\mathscr{F}) = \mathrm{ch}(\mathscr{F})\,\mathrm{td}(\Omega^1_Y)$ applied to $\mathscr{F} = \mathscr{E}$ and $\mathscr{F} = \mathscr{O}_Y$ yields the formula

$$(9) \qquad \chi(\mathscr{E}) = \frac{(D \cdot D) - (D \cdot K_Y)}{2} - c_2(\mathscr{E}) + \mathrm{rank}(\mathscr{E})\chi(\mathscr{O}_Y),$$

where for simplicity we set $\det(\mathscr{E}) = \mathscr{O}_Y(D)$ and $\omega_Y = \mathscr{O}_Y(K_Y)$. Moreover, $c_2(\mathscr{E}) \in \mathbb{Z}$ is the *second Chern number*. Note that this integer is uniquely defined by the above equation.

Now suppose that $\mathscr{E}$ has rank two. Let $\mathscr{E} \to \mathscr{F}$ be a surjection onto some coherent sheaf that is invertible in codimension one. Then dual sheaf $\mathscr{L} = \underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{O}_Y)$ is reflexive of rank one ([29], Corollary 1.8), whence invertible. The canonical map

$$\mathscr{E} = \mathscr{E}^{\vee\vee} \longrightarrow \mathscr{F}^{\vee\vee} = \mathscr{L}^\vee$$

is surjective in codimension one, thus yields an exact sequence $\mathscr{E} \to \mathscr{L}^\vee \to \mathscr{O}_Z \to 0$, where $Z \subset Y$ is a finite subscheme. Let $\mathscr{I} \subset \mathscr{O}_Y$ be the corresponding coherent ideal sheaf.

Proposition 5.1. *The kernel for the resulting surjection $\mathscr{E} \to \mathscr{I}\mathscr{L}^\vee$ is isomorphic to the invertible sheaf $\mathscr{L} \otimes \det(\mathscr{E})$. Moreover, for each point $a \in Z$ the local ring $\mathscr{O}_{Z,a}$ is of the form $\kappa(a)[[x,y]]/\mathfrak{a}$ for some parameter ideal $\mathfrak{a} = (f, g)$.*

Proof. Let $\mathscr{N} \subset \mathscr{E}$ be the kernel in question. The exact sequence

$$(10) \qquad\qquad 0 \longrightarrow \mathscr{N} \longrightarrow \mathscr{E} \longrightarrow \mathscr{L}^\vee \longrightarrow \mathscr{O}_Z \longrightarrow 0$$

shows that at each point $a \in Y$, the stalk $\mathscr{N}_a$ is a syzygy for the module $\mathscr{O}_{Z,z}$ over the regular local ring $R = \mathscr{O}_{Y,a}$. In turn, $\mathscr{N}$ is locally free. Taking ranks at the generic point $\eta \in Y$ we see that $\mathscr{N}$ is invertible.

On the open set $U = Y \smallsetminus Z$, we have $\mathscr{O}_Z | U = 0$, thus $\det(\mathscr{E})_U = \mathscr{N}_U \otimes \mathscr{L}_U^\vee$. This subset $U \subset Y$ contains all points of codimension one, and with [29], Theorem 1.12 we deduce that the equality already holds over Y. Thus $\mathscr{N} = \det(\mathscr{E}) \otimes \mathscr{L}$.

Now fix a point $a \in Z$. Set $L = \kappa(a)$, write $\mathscr{O}_{Y,a}^\wedge = L[[x,y]]$ and choose trivializations of $\mathscr{E}$ and $\mathscr{L}^\vee$ on some open neighborhood. The exact sequence (10) shows that $\mathscr{O}_{Z,a}$ is of the form $L[[x,y]]/\mathfrak{a}$ for some ideal $\mathfrak{a} = (f,g)$. The generators lie in the maximal ideal and form a parameter system, because $\dim(\mathscr{O}_{Z,a}) = 0$. $\square$

We thus have a short exact sequence of coherent sheaves

(11) $$0 \longrightarrow \mathscr{L} \otimes \det(\mathscr{E}) \longrightarrow \mathscr{E} \longrightarrow \mathscr{I}\mathscr{L}^\vee \longrightarrow 0,$$

This sequence gives another expression for the second Chern number:

Proposition 5.2. *In the above situation, we have the formula*

$$c_2(\mathscr{E}) + c_1^2(\mathscr{L}) + c_1(\mathscr{L})c_1(\mathscr{E}) = h^0(\mathscr{O}_Z).$$

Proof. First of all, $\chi(\mathscr{E}) = \chi(\mathscr{L} \otimes \det(\mathscr{E})) + \chi(\mathscr{L}^\vee) - h^0(\mathscr{O}_Z)$ holds by additivity of Euler characteristics. Applying Riemann–Roch this becomes

$$L^2 + (L \cdot D) + \frac{(D \cdot D) - (D \cdot K_Y)}{2} + 2\chi(\mathscr{O}_Y) - h^0(\mathscr{O}_Z),$$

where we write $\mathscr{L} = \mathscr{O}_Y(L)$ and $\det(\mathscr{E}) = \mathscr{O}_Y(D)$. Together with the Hirzebruch–Riemann–Roch formula (9), this gives the assertion. $\square$

Under suitable assumptions on the invertible sheaf $\det(\mathscr{E})$, $\mathscr{L}$, ω_Y one obtains formulas for the cohomological invariants of $\mathscr{E}$:

Proposition 5.3. *If in the above situation the dual sheaves for $\mathscr{L}$ and $\mathscr{L} \otimes \det(\mathscr{E}) \otimes \omega_Y^\vee$ have no non-zero global sections, the cohomological invariants are given by*

$$h^i(\mathscr{E}) = \begin{cases} h^0(\mathscr{L} \otimes \det(\mathscr{E})) & \text{for } i = 0; \\ h^1(\mathscr{L} \otimes \det(\mathscr{E})) + h^1(\mathscr{L}^\vee) + h^0(\mathscr{O}_Z) & \text{for } i = 1; \\ h^0(\mathscr{L} \otimes \omega_Y) & \text{for } i = 2. \end{cases}$$

If furthermore $\det(\mathscr{E}) = \omega_Y$ we obtain the values $h^0(\mathscr{E}) = h^2(\mathscr{E}) = h^0(\mathscr{L} \otimes \omega_Y)$ and $h^1(\mathscr{E}) = 2h^1(\mathscr{L} \otimes \omega_Y) + h^0(\mathscr{O}_Z)$.

Proof. The groups $H^0(Y, \mathscr{I}\mathscr{L}^\vee) \subset H^0(Y, \mathscr{L}^\vee)$ vanish, and the long exact sequence for the short exact sequence (11) gives an identification $H^0(Y, \mathscr{L} \otimes \det(\mathscr{E})) = H^0(Y, \mathscr{E})$, which establishes the case $i = 0$.

Likewise get $H^2(Y, \mathscr{E}) = H^2(Y, \mathscr{I}\mathscr{L}^\vee)$, because the group $H^2(Y, \mathscr{L} \otimes \det(\mathscr{E}))$ is Serre dual to $H^0(Y, \mathscr{L}^\vee \otimes \det(\mathscr{E})^\vee \otimes \omega_Y) = 0$. The finite subscheme $Z \subset Y$ yields an exact sequence $0 \to \mathscr{I}\mathscr{L}^\vee \to \mathscr{L}^\vee \to \mathscr{O}_Z \to 0$. In turn, we get a long exact sequence

$$H^1(Y, \mathscr{O}_Z) \longrightarrow H^2(Y, \mathscr{I}\mathscr{L}^\vee) \longrightarrow H^2(Y, \mathscr{L}^\vee) \longrightarrow H^2(Y, \mathscr{O}_Z).$$

The outer terms vanish for dimension reason, which establishes the formula for $i = 2$. For the remaining case, consider the long exact sequence

$$H^0(Y, \mathscr{L}^\vee) \longrightarrow H^0(Y, \mathscr{O}_Z) \longrightarrow H^1(Y, \mathscr{I}\mathscr{L}^\vee) \longrightarrow H^1(Y, \mathscr{L}^\vee) \longrightarrow H^1(Y, \mathscr{O}_Z).$$

The outer terms vanish, and we get $h^1(\mathscr{I}\mathscr{L}^\vee) = h^0(\mathscr{O}_Z) + h^1(\mathscr{L}^\vee)$. By Serre Duality and our assumption on $\mathscr{L} \otimes \det(\mathscr{E}) \otimes \omega_Y^\vee$, the group $H^2(Y, \mathscr{L} \otimes \det(\mathscr{E}))$ vanishes. From (11) again we get a short exact sequence

$$0 \longrightarrow H^1(\mathscr{L} \otimes \det(\mathscr{E})) \longrightarrow H^1(Y, \mathscr{E}) \longrightarrow H^1(Y, \mathscr{I}\mathscr{L}^\vee) \longrightarrow 0,$$

and the case $i = 1$ follows. The formulas for the situation $\det(\mathscr{E}) = \omega_Y$ are immediate. $\qquad\square$

A curve $H \subset Y$ is called *ample* or *semiample* if the invertible sheaf $\mathscr{O}_Y(H)$ has the respective property.

Corollary 5.4. *Suppose that the dualizing sheaf ω_Y is two-torsion in the Picard group, that $\det(\mathscr{E}) = \omega_Y$, and that $(\mathscr{L} \cdot H) > 0$ for some semiample curve $H \subset Y$. Then we have $h^0(\mathscr{E}^\vee) = h^2(\mathscr{E}^\vee) = h^0(\mathscr{L})$ and $h^1(\mathscr{E}^\vee) = 2h^1(\mathscr{L}) + h^0(\mathscr{O}_Z)$.*

Proof. We have $\det(\mathscr{E}^\vee) = \det(\mathscr{E})^\vee = \omega_Y^\vee \simeq \omega_Y$. The wedge product gives a perfect pairing $\mathscr{E} \otimes \mathscr{E} \to \Lambda^2\mathscr{E} = \omega_Y$, hence we get identifications $\mathscr{E} = \underline{\mathrm{Hom}}(\mathscr{E}, \omega_Y)$ and $\mathscr{E}^\vee = \mathscr{E} \otimes \omega_Y$. Tensoring the exact sequence (11) with $\omega_Y \simeq \omega_Y^\vee$ gives

$$0 \longrightarrow \mathscr{N} \otimes \omega_Y \longrightarrow \mathscr{E}^\vee \longrightarrow \mathscr{I}\mathscr{N}^\vee \longrightarrow 0$$

for the invertible sheaf $\mathscr{N} = \mathscr{L} \otimes \omega_Y^\vee$. We have $(\mathscr{N} \cdot H) = (\mathscr{L} \cdot H) > 0$, so the duals for the invertible sheaves $\mathscr{N} = \mathscr{N} \otimes \det(\mathscr{E}^\vee) \otimes \omega_Y$ have no global sections. Proposition 5.3 applied with $\mathscr{E}^\vee$ and the above short exact sequence yields the formulas. $\qquad\square$

Corollary 5.5. *Assumptions as in the previous corollary. Then the following are equivalent:*

   (i) $h^i(\mathscr{E}) = h^i(\mathscr{E}^\vee)$ *for some degree $0 \le i \le 2$.*
   (ii) $h^i(\mathscr{L}) = h^i(\mathscr{L} \otimes \omega_Y)$ *for some degree $0 \le i \le 1$.*

If one of these equivalent conditions is true, the equalities hold for all $0 \le i \le 2$.

Proof. The previous corollary gives $h^0(\mathscr{E}^\vee) = h^2(\mathscr{E}^\vee) = h^0(\mathscr{L})$ and $h^1(\mathscr{E}^\vee) = 2h^1(\mathscr{L}) + h^0(\mathscr{O}_Z)$. Proposition 5.3 yields $h^0(\mathscr{E}) = h^2(\mathscr{E}) = h^0(\mathscr{L} \otimes \omega_Y)$ and $h^1(\mathscr{E}) = 2h^1(\mathscr{L} \otimes \omega_Y) + h^0(\mathscr{O}_Z)$. Furthermore, both invertible sheaf $\mathscr{L}$ and $\mathscr{L} \otimes \omega_Y$ have no cohomology in degree two, and the same Euler characteristics. Suppose we have $h^i(\mathscr{L}) = h^i(\mathscr{L} \otimes \omega_Y)$ for some $0 \le i \le 1$. Then equality holds for all $i \ge 0$, and so does $h^i(\mathscr{E}) = h^i(\mathscr{E}^\vee)$. Conversely, suppose that we have $h^j(\mathscr{E}) = h^j(\mathscr{E}^\vee)$ for some degree $0 \le j \le 2$. Then $h^i(\mathscr{L}) = h^i(\mathscr{L} \otimes \omega)$ for some $0 \le i \le 1$, and the assertion follows. $\qquad\square$

6. Algebraic surfaces

Let Y be a smooth proper surface over an algebraically closed field k of characteristic $p = 2$. We now investigate in what circumstances Theorem 2.5 applies, such that the surface Y does not lift to the ring W_2. Choose some coherent quotient $\Omega_Y^1 \to \mathscr{F}$ that is invertible in codimension one. Consider the resulting invertible sheaf $\mathscr{L} = \underline{\mathrm{Hom}}(\mathscr{F}, \mathscr{O}_Y)$ and the ensuing exact sequence

$$(12) \qquad\qquad 0 \longrightarrow \mathscr{L} \otimes \omega_Y \longrightarrow \Omega_Y^1 \longrightarrow \mathscr{I}\mathscr{L}^\vee \longrightarrow 0,$$

where $\mathscr{I} \subset \mathscr{O}_Y$ is a coherent ideal sheaf corresponding to some finite subscheme $Z \subset Y$.

Theorem 6.1. *Suppose the following three assumptions hold:*

(i) *The dualizing sheaf ω_Y has order $p = 2$ in the Picard group.*
(ii) *There is an semiample curve $H \subset Y$ with $(\mathscr{L} \cdot H) > 0$.*
(iii) *The invertible sheaf $\mathscr{L} \otimes \omega_Y$ is p-divisible in the Picard group.*

Then we have

$$h^1(\Omega_Y^1) = h^1(\underline{\mathrm{Hom}}(F_*\Omega_Y^1, \mathscr{O}_{Y'})) \quad and \quad h^0(\mathscr{L}) \leq h^0(\mathscr{L} \otimes \omega_Y).$$

If the latter inequality is an equality, the surface Y does not lift to the ring W_2.

Proof. First, we establish the equality $h^1(\Omega_Y^1) = h^1(\underline{\mathrm{Hom}}(F_*\Omega_Y^1, \mathscr{O}_{Y'}))$, which is the main part of the argument. From Proposition 5.3 we know that

(13) $$h^1(\Omega_Y^1) = h^1(\mathscr{L} \otimes \omega_Y) + h^1(\mathscr{L}^\vee) + h^0(\mathscr{O}_Z).$$

Applying the Frobenius pushforward to (12) yields an exact sequence of coherent sheaves

$$0 \longrightarrow F_*(\mathscr{L} \otimes \omega_Y) \longrightarrow F_*(\Omega_Y^1) \longrightarrow F_*(\mathscr{I}\mathscr{L}^\vee) \longrightarrow 0,$$

where the two terms on the left are locally free. In turn, we get an exact sequence of coherent sheaves

(14) $$0 \to \underline{\mathrm{Hom}}(F_*(\mathscr{I}\mathscr{L}^\vee), \mathscr{O}_{Y'}) \to \underline{\mathrm{Hom}}(F_*(\Omega_Y^1), \mathscr{O}_{Y'}) \to \underline{\mathrm{Hom}}(F_*(\mathscr{L} \otimes \omega_Y), \mathscr{O}_{Y'})$$

Being duals on a regular two-dimensional scheme, all terms are locally free. The cokernel for the map on the right is the skyscraper sheaf $\mathscr{T} = \underline{\mathrm{Ext}}^1(F_*(\mathscr{I}\mathscr{L}^\vee), \mathscr{O}_{Y'})$. Moreover, the restriction map

$$\underline{\mathrm{Hom}}(F_*(\mathscr{L}^\vee), \mathscr{O}_{Y'}) \longrightarrow \underline{\mathrm{Hom}}(F_*(\mathscr{I}\mathscr{L}^\vee), \mathscr{O}_{Y'})$$

between locally free sheaves is bijective (for example [29], Theorem 1.12). Applying Proposition 2.6 with $\mathscr{E} = \mathscr{L}^\vee \otimes \omega_Y$ we get

$$\underline{\mathrm{Hom}}(F_*(\mathscr{L} \otimes \omega_Y), \mathscr{O}_{Y'}) = F_*(\mathscr{L}^\vee) \quad and \quad \underline{\mathrm{Hom}}(F_*(\mathscr{L}^\vee), \mathscr{O}_{Y'}) = F_*(\mathscr{L} \otimes \omega_Y).$$

The exact sequence (14) gives a commutative diagram
(15)

$$0 \to F_*(\mathscr{L} \otimes \omega_Y) \to \underline{\mathrm{Hom}}(F_*(\Omega_Y^1), \mathscr{O}_{Y'}) \xrightarrow{d} F_*(\mathscr{L}^\vee) \to \mathscr{T} \to 0,$$

where the horizontal four-term sequence is exact, obtained from splicing the two short exact sequences with kinks, for some coherent sheaf $\mathscr{S}$. This gives an inclusion $H^0(Y', \mathscr{S}) \subset H^0(Y, \mathscr{L}^\vee)$. Moreover, we have $H^2(Y', F_*(\mathscr{L} \otimes \omega_Y)) = H^2(Y, \mathscr{L} \otimes \omega_Y)$, which is dual to $H^0(Y, \mathscr{L}^\vee)$. All these groups vanish, by assumption (ii). So the long exact sequence for the short exact sequence to the left yields

$$h^1(\underline{\mathrm{Hom}}(F_*(\Omega_Y^1), \mathscr{O}_{Y'})) = h^1(\mathscr{L} \otimes \omega_Y) + h^1(\mathscr{S}).$$

On the other hand, the short exact sequence to the right gives

$$H^0(Y, \mathscr{L}^\vee) \to H^0(Y', \mathscr{T}) \to H^1(Y', \mathscr{S}) \to H^1(Y, \mathscr{L}^\vee) \to H^1(Y', \mathscr{T}).$$

The outer terms vanish, by assumption (ii) and for dimension reasons, such that $h^1(\mathscr{S}) = h^0(\mathscr{T}) + h^1(\mathscr{L}^\vee)$.

In light of (13), it remains to verify $h^0(\mathscr{T}) = h^0(\mathscr{O}_Z)$. Recall that we started with an exact sequence $0 \to \mathscr{L} \otimes \omega_Y \to \Omega^1_Y \to \mathscr{L}^\vee \to \mathscr{O}_Z \to 0$, which is a resolution of the skyscraper sheaf $\mathscr{O}_Z$ by locally free sheaves. In turn, we get a resolution $0 \to F_*(\mathscr{L} \otimes \omega_Y) \to F_*(\Omega^1_Y) \to F_*(\mathscr{L}^\vee) \to F_*(\mathscr{O}_Z) \to 0$ of the skyscraper sheaf $F_*(\mathscr{O}_Z)$ by locally free sheaves. Dimension shifting gives

$$\mathscr{T} = \underline{\mathrm{Ext}}^1(F_*(\mathscr{I}\mathscr{L}^\vee), \mathscr{O}_{Y'}) = \underline{\mathrm{Ext}}^2(F_*(\mathscr{O}_Z), \mathscr{O}_{Y'}).$$

Since $F : Y \to Y'$ is finite we have $h^0(F_*(\mathscr{O}_Z)) = h^0(\mathscr{O}_Z)$. Fix a closed point $b \in Y'$. It remains to check that the stalks

(16) $$M = F_*(\mathscr{O}_Z)_b \quad \text{and} \quad \underline{\mathrm{Ext}}^2(F_*(\mathscr{O}_Z), \mathscr{O}_{Y'})_b = \mathrm{Ext}^2_R(M, R)$$

have the same length over the complete local ring $R = \mathscr{O}^\wedge_{Y',b}$. But this is a general fact: Let $R/\mathfrak{m}_R \subset E$ be an injective hull, and $\mathcal{C}$ be the category of R-modules of finite length. By Matlis Duality ([38], Theorem 18.6), the functor $N \mapsto \mathrm{Hom}_R(N, E)$ induces an anti-equivalence of $\mathcal{C}$, in particular N and $\mathrm{Hom}_R(N, E)$ have the same length. Local Duality gives $\mathrm{Hom}_R(\mathrm{Ext}^2_R(N, R), E) = H^0_{\mathfrak{m}}(N) = N$, with the 2-dimensional local Gorenstein ring R and the finite R-module N (see for example [27], Theorem 6.3). Summing up, the modules in (16) have the same length, and therefore $h^1(\Omega^1_Y) = h^1(\underline{\mathrm{Hom}}(F_*\Omega^1_Y, \mathscr{O}_{Y'}))$.

Next, we establish the inequality $h^0(\mathscr{L}) \leq h^0(\mathscr{L} \otimes \omega_Y)$. By (13) and Corollary 5.4 we have

(17) $$h^1(\Omega^1_Y) = 2h^1(\mathscr{L}^\vee) + h^0(\mathscr{O}_Z) \quad \text{and} \quad h^1(\Theta_Y) = 2h^1(\mathscr{L}) + h^0(\mathscr{O}_Z).$$

Assumption (i) gives $\chi(\mathscr{L}) = \chi(\mathscr{L} \otimes \omega_Y)$, whereas assumption (ii) ensures that $h^2(\mathscr{L}) = h^2(\mathscr{L} \otimes \omega_Y) = 0$. Seeking a contradiction, we now assume $h^0(\mathscr{L}) > h^0(\mathscr{L} \otimes \omega_Y)$. Then we also have $h^1(\mathscr{L}) > h^1(\mathscr{L} \otimes \omega_Y) = h^1(\mathscr{L}^\vee)$, and with the equations in (17) we obtain $h^1(\Theta^1_Y) > h^1(\Omega^1_Y) = h^1(\underline{\mathrm{Hom}}(F_*\Omega^1_Y, \mathscr{O}_{Y'}))$. But this contradicts Theorem 2.5. Note that for this step we need the assumption that the dualizing sheaf has order two.

Finally, suppose we have $h^0(\mathscr{L}) = h^0(\mathscr{L} \otimes \omega_Y)$. From Corollary 5.5 we get $h^1(\Theta^1_Y) = h^1(\Omega^1_Y) = h^1(\underline{\mathrm{Hom}}(F_*\Omega^1_Y, \mathscr{O}_{Y'}))$, and Theorem 2.5 tells us that the scheme Y does not lift to the ring W_2. $\qquad\square$

Now suppose we have a *quasielliptic fibration* $f : Y \to B$. This means that B is a smooth proper curve, and the generic fiber Y_η is a twisted form of the *rational cuspidal curve* $\mathrm{Spec}\, k[t^2, t^3] \cup \mathrm{Spec}\, k[t^{-1}]$. The fibration gives a short exact sequence

$$0 \longrightarrow f^*(\Omega^1_Y) \longrightarrow \Omega^1_Y \longrightarrow \Omega^1_{Y/B} \longrightarrow 0.$$

The map on the left is indeed injective, because the function field extension $k(B) \subset k(Y)$ is separable. It follows that the coherent sheaf $\Omega^1_{Y/B}$ has rank one. Write $\Omega^1_{Y/B} \to \mathscr{F}$ for the quotient modulo the torsion subsheaf. Then $\mathscr{F}$ is invertible in codimension one, and we obtain a short exact sequence

$$0 \longrightarrow \mathscr{L} \otimes \omega_Y \longrightarrow \Omega^1_Y \longrightarrow \mathscr{I}\mathscr{L}^\vee \longrightarrow 0$$

attached to the quasielliptic fibration. In order to apply Theorem 6.1, one has to check that the invertible sheaves $\mathscr{L}$ and ω_Y have certain properties. Write $F = k(B)$ for the function field of the curve, and $\mathrm{Sing}(Y_F/F)$ be the scheme of non-smoothness,

as defined in [21], Section 2. This is the closed subscheme of the generic fiber defined by the first Fitting ideal of $\Omega^1_{Y_F/F}$.

Proposition 6.2. *As Cartier divisors on the generic fiber, we have* $\mathrm{Sing}(Y_F/F) = 2\xi$ *for some closed point* ξ, *and the field extension* $F \subset \kappa(\xi)$ *is purely inseparable of degree* $p = 2$.

Proof. We first make an explicit computation with the cuspidal rational curve over F. The coordinate ring of the first chart is isomorphic to $F[x,y]/(y^2 - x^3)$, by setting $x = t^2$ and $y = t^3$. The module of Kähler differentials is generated by dx and dy modulo $x^2 dx$. Hence $\mathrm{Sing}(C/F)$ is defined by an additional relation $x^2 = 0$. It becomes the spectrum of $F[x,y]/(y^2, x^2)$, which is radical of length four. In turn, $\mathrm{Sing}(Y_F/F)$ is radical of length four. It contains no rational point by [21], Corollary 2.6. Since the field F has p-degree $\mathrm{pdeg}(F) = 1$, the scheme of non-smoothness has residue field $\kappa(\xi) = F^{1/p}$, which has degree two. Our assertion follows. $\square$

The closure $C = \overline{\{\xi\}}$ inside the quasielliptic surface Y is called the *curve of cusps*.

Proposition 6.3. *Suppose that all closed fibers* $f^{-1}(b)$ *are simple, with Kodaira symbol* II. *Then we have* $\mathscr{L} = \mathscr{O}_Y(2C) \otimes f^*(\mathscr{N})$ *for some invertible sheaf* $\mathscr{N}$ *on* B.

Proof. By assumption, all geometric fibers in question are rational cuspidal curves $\mathrm{Spec}\, k[t^2, t^3] \cup \mathrm{Spec}\, k[t^{-1}]$. The sheaf of Kähler differentials modulo torsion is invertible, and generated on the first chart by dt^3, and on the second chart by dt^{-1}. On the overlap we have $dt^{-1} = t^{-2}dt = t^{-4}dt^3$, which gives the cocycle $t^{-4} \in k[t^{\pm 1}]^\times$. Its inverse is given by t^4, and the resulting divisor coincides with the locus of non-smoothness.

Consider the invertible sheaf $\mathscr{M} = \mathscr{L}(-2C)$. The restrictions to fibers $f^{-1}(b)$, $b \in B$, are trivial, by the above computation. The direct image $f_*(\mathscr{O}_Y)$ commutes with arbitrary base-change. By the Theorem of Formal Functions, the direct image $\mathscr{N} = f_*(\mathscr{M})$ is invertible. According to the Projection Formula, the adjunction map $f^*(\mathscr{N}) \to \mathscr{M}$ is bijective. $\square$

We record the following immediate consequence:

Corollary 6.4. *Assumptions as in the proposition. If furthermore the selfintersection numbers* $(\mathscr{L} \cdot \mathscr{L})$ *and* C^2 *vanish, then* $\mathscr{L}$ *is p-divisible in the Picard group.*

Proof. According to the proposition, the invertible sheaf $\mathscr{L}$ comes from a divisor of the form $2C + F$, where $F = \sum m_i f^{-1}(b_i)$ is a linear combination of fibers. From $(\mathscr{L} \cdot \mathscr{L}) = 4C^2 + 4C \cdot F$ we infer $C \cdot F = 0$. In turn, the divisor $\sum m_i b_i$ on the curve B has degree zero. But the group of rational points on the abelian variety Pic^0_C is n-divisible for any integer $n \geq 1$. It follows that $\mathscr{L} = \mathscr{O}_Y(2C + F)$ is two-divisible in $\mathrm{Pic}(Y)$. $\square$

7. ENRIQUES SURFACES

Let k be an algebraically closed ground field. Recall that a smooth surface Y with $h^0(\mathscr{O}_Y) = 1$ is called an *Enriques surface* if $c_1 = 0$ and $b_2 = 10$. We refer to the monograph of Cossec and Dolgachev [12] for a comprehensive account. The group scheme $P = \mathrm{Pic}^\tau_{Y/k}$ of numerically trivial invertible sheaves is finite of order two, and

its group of rational points is generated by the canonical class K_Y. The canonical covering $\epsilon : X \to Y$ is a torsor under the Cartier dual $G = \underline{\mathrm{Hom}}(P, \mathbb{G}_m)$, and its total space is integral, with cohomological invariants $h^1(\mathscr{O}_Y) = 0$ and $h^2(\mathscr{O}_Y) = 1$, with $\omega_Y = \mathscr{O}_Y$. In characteristic $p \geq 3$, the canonical covering is a smooth K3 surface, and the base of the miniversal formal deformation $\mathfrak{Y} \to \mathrm{Spec}(A)$ is given by the ring $A = W[[T_1, \ldots, T_{10}]]$.

From now on, we suppose the characteristic is $p = 2$. Then there are three possibilities for the group scheme $P = \mathrm{Pic}^\tau_{Y/k}$, namely μ_2 or $\mathbb{Z}/2\mathbb{Z}$ or α_2. The respective Enriques surfaces Y are aptly called *ordinary*, *classical* and *supersingular*. Ordinary Enriques surfaces behave as in odd characteristics. For classical and supersingular Enriques surfaces, the group scheme P is unipotent, its Cartier dual G is local, the canonical covering X is singular, and both schemes have trivial fundamental group. We then say that Y is a *simply-connected Enriques surface*, and X is called the *K3-like covering*.

Let Y be a simply-connected Enriques surface, and $X' \to X$ be the normalization of the K3-like covering. Ekedahl and Shepherd-Barron [19] showed that the ramification divisor for the normalization is the preimage of a curve $C \subset Y$ called the *conductrix*. They call Y an *exceptional Enriques surface* if the *biconductrix* $2C \subset Y$ has $h^1(\mathscr{O}_{2C}) \neq 0$, and give a beautiful classification of these surfaces in terms of the multiplicities $m_i \geq 1$ and intersection matrix $(C_i \cdot C_j)$ for the conductrix $C = \sum m_i C_i$, and also by properties of the Hodge ring $\bigoplus H^i(Y, \Omega^j_Y)$. Exceptional Enriques surfaces are a priori simply-connected, and both classical and supersingular cases do occur. The following fact shows that exceptional and supersingular Enriques surfaces share an important property:

Proposition 7.1. *The cohomological invariants for the tangent sheaf of an Enriques surface Y are given by the following table:*

	$h^0(\Theta_Y)$	$h^1(\Theta_Y)$	$h^2(\Theta_Y)$
exceptional/supersingular	1	12	1
otherwise	0	10	0

Proof. First note that $\chi(\Theta_Y) = -c_2 + 2\chi(\mathscr{O}_Y) = 10$. Furthermore, we have $\Theta_Y = \Omega^1_Y \otimes \omega_Y$, and Serre Duality gives $h^i(\Theta_Y) = h^{2-i}(\Theta_Y)$. So it suffices to verify the values in degree $i = 0$. Ordinary Enriques surfaces have $h^0(\Theta_Y) = 0$, whereas supersingular have $h^0(\Theta_Y) = 1$, according to [12], Proposition 1.4.2. Now suppose that Y is classical. By [19] the Enriques surface Y admits non-zero global vector fields if and only if Y is exceptional, and then $h^0(\Theta_Y) = 1$. $\square$

We now come to the main result of this paper.

Theorem 7.2. *Let Y be an Enriques surface, and $\mathfrak{Y} \to \mathrm{Spf}(A)$ be its miniversal formal deformation. Then the complete local noetherian ring A is regular with $11 \leq \dim(A) \leq 12$, and flat as W-algebra. Moreover, the following are equivalent:*

(i) *The Enriques surface Y is exceptional or supersingular.*
(ii) *The scheme Y does not lift to the ring W_2.*
(iii) *The absolute ramification index is $e(A) \geq 2$.*
(iv) *The dimension is $\dim(A) = 12$.*

Proof. Suppose first that Y is neither exceptional nor supersingular. Then we have $h^1(\Theta_Y) = 10$ and $h^2(\Theta_Y) = 0$, hence $A = W[[T_1, \ldots, T_{10}]]$, and the assertion is immediate.

Now suppose that Y is exceptional or supersingular, such that $h^1(\Theta_Y) = 12$ and $h^2(\Theta_Y) = 1$. According to Proposition 3.3, we merely have to check that the scheme Y does not lift to the ring W_2. For supersingular Enriques surfaces, this follows from [20], Proposition 4.6, compare also Theorem 4.3, and also [51], Theorem 7.1 for a description of the base of the versal deformation in terms of invariant rings.

It remains to treat the case that Y is classical and exceptional. To show that Y does not lift to the ring W_2 we now check that the assumptions of Theorem 6.1 are satisfied. Since Y is classical, the dualizing sheaf ω_Y has order $p = 2$ in the Picard group. Let $C \subset Y$ be the conductrix, and consider the invertible sheaf $\mathscr{L} = \omega_Y(2C)$, such that $\mathscr{L} \otimes \omega_Y = \mathscr{O}_Y(2C)$. Obviously, $\mathscr{L} \otimes \omega_Y$ is p-divisible in the Picard group, and $(\mathscr{L} \cdot H) > 0$ for every ample curve $H \subset Y$. According to the proof of Proposition 0.5 in [19], there is a short exact sequence

$$0 \longrightarrow \mathscr{L} \otimes \omega_Y \longrightarrow \Omega_Y^1 \longrightarrow \mathscr{I}\mathscr{L}^\vee \longrightarrow 0,$$

where $\mathscr{I}$ is the ideal sheaf of some finite subscheme $Z \subset X$. We have $h^0(\Omega_Y^1) = 1$, and the above short exact sequence immediately gives $h^0(\mathscr{L} \otimes \omega_Y) = 1$. Tensoring with ω_Y we obtain another the exact sequence $0 \to \mathscr{L} \to \Theta_Y \to \mathscr{I}\mathscr{L}^\vee \otimes \omega_Y \to 0$, which gives an exact sequence

$$0 \longrightarrow H^0(Y, \mathscr{L}) \longrightarrow H^0(Y, \Theta_Y) \longrightarrow H^0(Y, \mathscr{I}\mathscr{L}^\vee \otimes \omega_Y).$$

Clearly, the term on the right vanishes. Moreover, we have $h^0(\Theta_Y) = 1$ by Proposition 7.1 and conclude $h^0(\mathscr{L}) = 1$. In particular the equality $h^0(\mathscr{L}) = h^0(\mathscr{L} \otimes \omega_Y)$ holds. We thus may apply Theorem 6.1 and get that the scheme Y does not lift the ring W_2. $\qquad\square$

Note that the Hodge–de Rham spectral sequence $E_1^{rs} = H^s(Y, \Omega_Y^r) \Rightarrow H^{r+s}(Y, \Omega_Y^\bullet)$ degenerates on the E_1-page if and only if the Enriques surface Y is not supersingular, according to [32], Proposition 7.3.8. In particular, [15], Corollary 2.4 does not apply for such Enriques surfaces. Moreover, I do not see any ample invertible sheaf violating the Kodaira–Akizuki–Nakano Vanishing $H^r(Y, \Omega_Y^s \otimes \mathscr{L}) = 0$ for $r + s > 2$, so loc. cit. Corollary 2.8 also does not help to establish non-existence of liftings to W_2.

8. Bielliptic surfaces

Let k be an algebraically closed field, and Y be a smooth proper k-scheme with $\dim(Y) = 2$ and $h^0(\mathscr{O}_Y) = 1$. Let us say that Y is a *bielliptic surface* if $c_1 = 0$ and $b_2 = 2$. By the Enriques classification according to Bombieri and Mumford ([8] and [7]), we then have

$$b_1 = b_3 = 2 \quad \text{and} \quad c_2 = \chi(\mathscr{O}_Y) = 0.$$

Moreover, the number $h^1(\mathscr{O}_Y) = h^2(\mathscr{O}_Y) + 1$ is either one or two. Note that Bombieri and Mumford used the terms *hyperelliptic* and *quasi-hyperelliptic surfaces*.

Throughout, Y denotes a bielliptic surfaces. Then $Y = (E \times C)/G$, where the first factor E is elliptic curve, the second factor C is either another elliptic curve or the rational cuspidal curve $\operatorname{Spec} k[t^2, t^3] \cup \operatorname{Spec} k[t^{-1}]$, and the finite group scheme

G acts diagonally via inclusions $G \subset E$ and $G \subset \mathrm{Aut}_{C/k}$. The action is free on E but non-free on C, and the possible orders $\mathrm{ord}(G) = h^0(\mathscr{O}_G)$ are the numbers $d = 2, 3, 4, 6$. The two projections for the product $X = E \times C$ induce two fibrations

$$B = E/G \xleftarrow{f} Y \xrightarrow{g} C/G = \mathbb{P}^1$$

on the quotient $Y = X/G$, where B is another elliptic curve. Both projections are genus-one fibrations, and $f : Y \to B$ is quasielliptic if and only if C is the rational cuspidal curve. In this case, we are in characteristic $p = 2$ or $p = 3$, and the group scheme G is non-reduced. Moreover, all closed fibers $f^{-1}(b)$ are simple with Kodaira symbol II.

Consider the one-dimensional representation $\chi : G \to \mathrm{GL}(H^0(C, \omega_C)) = \mathbb{G}_m$, and the invertible sheaf $\mathscr{N} = \underline{\mathrm{Hom}}(R^1 f_*(\mathscr{O}_Y), \mathscr{O}_B)$ on the elliptic curve $B = E/G$.

Proposition 8.1. *The dualizing sheaf is $\omega_Y = f^*(\mathscr{N})$. Moreover, the common order of ω_Y and $\mathscr{N}$ in the Picard groups coincides with the order for the subgroup scheme $\chi(G) \subset \mathbb{G}_m$.*

Proof. The Canonical Bundle Formula ([8], Theorem 2) gives $\omega_Y = f^*(\mathscr{N})$. By the Projection Formula, the sheaves ω_Y and $\mathscr{N}$ have the same order. If C is an elliptic curve, the assertion on the order of ω_Y is given in loc. cit., page 37. If C is the rational cuspidal curve the proof for Proposition 8 in [7] gives the assertion. $\square$

Let us now examine the quasielliptic situation in the most important case $p = \mathrm{ord}(G) = 2$ in more detail:

Proposition 8.2. *Suppose that $p = 2$, and that the bielliptic surface $Y = (E \times C)/G$ is formed with the rational cuspidal curve C and the group scheme $G = \mu_2$. Then $h^1(\mathscr{O}_Y) = 1$ and $h^2(\mathscr{O}_Y) = 0$, and the cohomological invariants for the tangent and cotangent sheaves are*

$$h^i(\Theta_Y) = \begin{cases} 1 & \textit{for } i = 0, 2; \\ 2 & \textit{for } i = 1 \end{cases} \quad \textit{and} \quad h^i(\Omega_Y^1) = \begin{cases} 3 & \textit{for } i = 0, 2; \\ 6 & \textit{for } i = 1. \end{cases}$$

Proof. Since $G = \mu_2$ is simple and acts non-trivially on C, the representation $\chi : G \to \mathrm{GL}(H^0(C, \mathscr{O}_C))$ is a monomorphism, so the dualizing sheaf has order two. It follows that $h^2(\mathscr{O}_Y) = 0$ and hence $h^1(\mathscr{O}_Y) = 1$. Recall that $f : Y \to B$ denotes the quasielliptic fibration. Since $\omega_Y = f^*(\mathscr{N})$, we see that $\mathscr{N}$ has order two in $\mathrm{Pic}(B)$. The inclusion of group schemes $G \subset E$ shows that the elliptic curve E is ordinary, and the same holds for the isogeneous curve $B = E/G$. Note that up to isomorphism, $\mathscr{N}$ is the only invertible sheaf of order two.

The faithful action of the height-one group scheme $G = \mu_2$ on the rational cuspidal curve $C = \mathrm{Spec}\, k[t^2, t^3] \cup \mathrm{Spec}\, k[t^{-1}]$ corresponds to a non-zero vector field $\delta \in H^0(C, \Theta_C)$ satisfying $\delta^{[2]} = \delta$. Write $D_{t^{-1}}$ for the derivative with respect to the variable t^{-1}. As explained in [49], Section 3 the Lie algebra $\mathfrak{g} = H^0(C, \Theta_C)$ is four-dimensional, and we can write $\delta = P(t^{-1})D_{t^{-1}}$ for some polynomial of the form

$$P(t^{-1}) = \lambda_4 t^{-4} + \lambda_2 t^{-2} + \lambda_0 + \lambda_1 t^{-1}$$

The condition $\delta^{[2]} = \delta$ means $\lambda_1 \neq 0$, and the condition that the singularity of C is not a fixed point means $\lambda_4 \neq 0$. The polynomial is separable, because its derivative is $P'(t^{-1}) = \lambda_1$. As explained in [49], Section 1 (compare also [31], Section 3) its

four roots define the fixed scheme C^G. Let $c_1, \ldots, c_4 \in \mathbb{P}^1 = C/G$ be the images of the fixed points, and write $0 \in \mathbb{P}^1$ for the image of the singularity $0 \in C$. It follows that the $g^{-1}(c_i)$ are precisely the multiple fibers. These fibers are tame, with multiplicity $m = 2$. Write $B_i = g^{-1}(c_i)_{\mathrm{red}}$ for the corresponding half-fibers, and also set $E_0 = g^{-1}(0)$. We have chosen this notation because the canonical morphisms $Y \to B = E/G$ and $E \times C \to Y$ induce identifications $B_i = B$ and $E \times \{0\} = E_0$. From the Canonical Bundle Formula we get $\omega_Y = \mathscr{O}_Y(-2E_0 + B_1 + \ldots + B_4)$.

Since all fibers for the quasielliptic fibration $f : Y \to B$ are simple with Kodaira symbol II, the coherent sheaf $\Omega^1_{Y/B}$ modulo torsion is invertible. Setting $\mathscr{L} = \underline{\mathrm{Hom}}(\Omega^1_{Y/B}, \mathscr{O}_Y)$ we obtain a short exact sequence

$$(18) \qquad 0 \longrightarrow \mathscr{L} \otimes \omega_Y \longrightarrow \Omega^1_Y \longrightarrow \mathscr{L}^\vee \longrightarrow 0.$$

Note that $E_0 = g^{-1}(0)$ is the curve of cusps. According to Proposition 6.3, we have $\mathscr{L} = \mathscr{O}_Y(2E_0) \otimes f^*(\mathscr{N}')$ for some invertible sheaf $\mathscr{N}'$ on the elliptic curve $B = E/G$. We have $E_0^2 = 0$, and with $c_1 = c_2 = 0$ and Proposition 5.2 we also get $(\mathscr{L} \cdot \mathscr{L}) = 0$. It follows that $\mathscr{N}'$ is numerically trivial. We claim that it is has order two, such that $\mathscr{N}' = \mathscr{N}$ and $\mathscr{L} \otimes \omega_Y = \mathscr{O}(2E_0)$. Consider the half-fiber $B_1 = g^{-1}(c_1)_{\mathrm{red}}$ and the resulting exact sequence

$$0 \longrightarrow \mathscr{O}_{B_1}(-B_1) \longrightarrow \Omega^1_Y|B_1 \longrightarrow \Omega^1_{B_1} \longrightarrow 0.$$

With respect to the identification $B_1 = B$, the outer terms are $\mathscr{N}$ and $\mathscr{O}_B$. Since $\mathrm{Ext}^1(\mathscr{O}_B, \mathscr{N}) = 0$, we obtain $\Omega^1_Y|B_1 = \mathscr{N} \oplus \mathscr{O}_B$. Restricting the short exact sequence (18) to the curve B_1 gives a surjection $\mathscr{N} \oplus \mathscr{O}_B = \Omega^1_Y|B_1 \to \mathscr{N}'^\vee$. Since both $\mathscr{N}$ and $\mathscr{N}'$ have degree zero, it follows that either $\mathscr{N}' = \mathscr{N}$ or $\mathscr{N}' = \mathscr{O}_B$.

Seeking a contradiction, we suppose $\mathscr{N}' = \mathscr{O}_B$. Then $\mathscr{L} = \mathscr{O}_Y(2E_0)$, and the projection formula for the elliptic fibration $g : Y \to \mathbb{P}^1$ gives $h^0(\mathscr{L}) = h^0(\mathscr{O}_{\mathbb{P}^1}(2)) = 3$. Furthermore, we have $\mathscr{L} \otimes \omega_Y = \mathscr{O}_Y(B_1 + \ldots + B_4)$. Each global section vanishes only along curves that are vertical with respect to $g : Y \to \mathbb{P}^1$. Using $\mathscr{O}_{B_i}(B_i) \neq \mathscr{O}_{B_i}$, we infer $h^0(\mathscr{L} \otimes \omega_Y) = 1$. Now $h^0(\mathscr{L}) = 3 > 1 = h^0(\mathscr{L} \otimes \omega_Y)$ contradicts Theorem 6.1.

In turn, we have $\mathscr{N}' = \mathscr{N}$, hence $\mathscr{L} \otimes \omega_Y = \mathscr{O}_Y(2E_0)$ and $\mathscr{L} = \mathscr{O}_Y(B_1 + \ldots + B_4)$. As above, this gives $h^0(\mathscr{L}) = 1$ and $h^0(\mathscr{L} \otimes \omega_Y) = 3$. Now

$$h^0(\Omega^1_Y) = h^2(\Omega^1_Y) = h^0(\mathscr{L} \otimes \omega_Y) = 3 \quad \text{and} \quad h^0(\Theta^1_Y) = h^2(\Theta^1_Y) = h^0(\mathscr{L}) = 1$$

follow from Proposition 5.3 and Corollary 5.5. The Hirzebruch–Riemann–Roch Formula gives $\chi(\Omega^1_Y) = \chi(\Theta_Y) = 0$, and the values in degree $i = 1$ follow as well. $\qquad \square$

Note that we cannot deduce non-liftability from Theorem 6.1, because the inequality $h^0(\mathscr{L}) \leq h^0(\mathscr{L} \otimes \omega_Y)$ is not an equality. The situation changes if the group scheme G is unipotent. To simplify notation, we now write $\mathscr{O}_Y(n) = g^*(\mathscr{O}_{\mathbb{P}^1}(n))$ for the pullback of invertible sheaves along the elliptic fibration $g : Y \to \mathbb{P}^1 = C/G$.

Proposition 8.3. *Suppose that $p = 2$, and that the bielliptic surface $Y = (E \times C)/G$ is formed with the rational cuspidal curve C and the group scheme $G = \alpha_2$. Then the cotangent sheaf sits is a non-split short exact sequence*

$$(19) \qquad 0 \longrightarrow \mathscr{O}_Y(2) \longrightarrow \Omega^1_Y \longrightarrow \mathscr{O}_Y(-2) \longrightarrow 0,$$

in particular we have $\omega_Y = \mathcal{O}_Y$ *and* $\Theta_Y = \Omega_Y^1$. *The cohomological invariants are given by the formulas* $h^1(\mathcal{O}_Y) = 2$, $h^2(\mathcal{O}_Y) = 1$ *and*

$$h^i(\Omega_Y^1) = h^i(\Theta_Y) = \begin{cases} 3 & \text{for } i = 0, 2; \\ 6 & \text{for } i = 1. \end{cases}$$

Proof. Now the representation $\chi : G \to \mathrm{GL}(H^0(C, \omega_C))$ is trivial, whence $\omega_Y = \mathcal{O}_Y$ by Proposition 8.1, such that $h^2(\mathcal{O}_Y) = 1$ and $h^1(\mathcal{O}_Y) = 2$. Moreover, $\Theta_Y = \Omega_Y^1 \otimes \omega_Y$ is isomorphic to Ω_Y^1. Again set $\mathscr{L} = \underline{\mathrm{Hom}}(\Omega_{Y/B}^1, \mathcal{O}_Y)$, and consider the short exact sequence

$$(20) \qquad\qquad 0 \longrightarrow \mathscr{L} \longrightarrow \Omega_Y^1 \to \mathscr{L}^\vee \longrightarrow 0$$

stemming from the quasielliptic fibration $f : Y \to B$. As in the preceding proof, we have $\mathscr{L} = \mathcal{O}_Y(2E_0) \otimes f^*(\mathscr{N}')$ for some numerically trivial sheaf $\mathscr{N}'$ on the elliptic curve $B = E/G$. We claim that in the present situation $\mathscr{N}' \simeq \mathcal{O}_B$. Choose a fixed point or the G-action on C, let $b_1 \in \mathbb{P}^1$ be its image, and $B_1 = g^{-1}(b_1)_{\mathrm{red}}$ be the resulting copy of B. The Adjunction Formula shows that the conormal sheaf $\mathcal{O}_{B_1}(B_1)$ is trivial. In turn, both outer terms in the short exact sequence $0 \to \mathcal{O}_{B_1}(-B_1) \to \Omega_Y^1|B_1 \to \Omega_{B_1}^1 \to 0$ are isomorphic to $\mathcal{O}_{B_1}$. Restricting (20) to B_1 gives an inclusion $\mathscr{N}' \subset \Omega_Y^1|B_1$. We infer $\mathrm{Hom}(\mathscr{N}', \mathcal{O}_B) \neq 0$, and hence $\mathscr{N}' \simeq \mathcal{O}_B$. This gives the short exact sequence (19). The long exact cohomology sequence and the Projection Formula yields $h^0(\Omega_Y^1) = h^0(\mathcal{O}_Y(2)) = 3$. By Serre duality $h^2(\Omega_Y^1) = h^0(\Theta_Y \otimes \omega_Y) = h^0(\Omega_Y^1) = 3$. The Hirzebruch–Riemann–Roch Formula ensures $\chi(\Omega_Y^1) = 0$, and thus $h^1(\Omega_Y^1) = 6$.

It remains to check that the extension (19) does not split. For this we first compute $R^1 g_*(\mathcal{O}_Y)$, which can be written as $R^1 g_*(\mathcal{O}_Y) = \mathcal{O}_{\mathbb{P}^1}(d) \oplus \mathscr{F}$ for some integer d and some finite sheaf $\mathscr{F}$. As explained in the previous proof, the action of $G = \alpha_2$ on the rational cuspidal curve $C = \mathrm{Spec}\, k[t^2, t^3] \cup \mathrm{Spec}\, k[t^{-1}]$ is given by some derivation $\delta = P(t^{-1}) D_{t^{-1}}$ with $P(t^{-1}) = \lambda_4 t^{-4} + \lambda_2 t^{-2} + \lambda_0$ with $\lambda_4 \neq 0$. We see that the fixed scheme either consists of two points $c_1, c_2 \in C$ with multiplicity $m = 2$, or a single point $c_1 \in C$ of multiplicity $m = 4$. Write $C_i = g^{-1}(c_i)_{\mathrm{red}}$ for the ensuing reduced fibers. The Canonical Bundle Formula ([8], Theorem 2) gives $\omega_Y = g^*(\mathcal{O}_{\mathbb{P}^1}(d - 2)) \otimes \mathcal{O}_Y(\sum a_i C_i)$ with certain coefficients $0 \leq a_i \leq m - 1$. Using $\omega_Y = \mathcal{O}_Y$ we conclude that $\mathcal{O}_Y(\sum a_i C_i)$ is globally generated. If there is a single multiple fiber, we must have $a_1 = 0$. If there are two multiple fibers, the coefficients vanish as well: otherwise $a_1 = a_2 = 1$ and thus $d = 1$, thus $\mathcal{O}_Y(C_1 - C_2) = \mathcal{O}_Y$, contradicting $h^0(\mathcal{O}_Y(C_i)) = 1$. Summing up, in both cases we have $a_i = 0$ and $d = 2$. Applying the Canonical Bundle Formula again, we see that the torsion part $\mathscr{F} \subset R^1 g_*(\mathcal{O}_Y)$ has length $h^0(\mathscr{F}) = 2$.

From this information we may compute $h^1(\mathcal{O}_Y(n))$ for any integer n: The Projection Formula yields $R^1 g_*(\mathcal{O}_Y(n)) = \mathscr{F} \oplus \mathcal{O}_{\mathbb{P}^1}(n - 2)$. The Leray–Serre spectral sequence for the elliptic fibration $g : Y \to \mathbb{P}^1$ induces an exact sequence

$$0 \longrightarrow H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n - 2)) \longrightarrow H^1(Y, \mathcal{O}_Y(n)) \longrightarrow H^0(\mathbb{P}^1, \mathscr{F} \oplus \mathcal{O}_{\mathbb{P}^1}(n - 2)) \longrightarrow 0.$$

This gives $h^1(\mathcal{O}_Y(2)) = 0 + 3$, whereas $h^1(\mathcal{O}_Y(-2)) = 3 + 2$ Seeking a contradiction, we now suppose that the extension (19) splits, such that $\Omega_Y^1 = \mathcal{O}_Y(2) \oplus \mathcal{O}_Y(-2)$. This gives $6 = h^1(\Omega_Y^1) = h^1(\mathcal{O}_Y(2)) + h^1(\mathcal{O}_Y(-2)) = 8$, contradiction. $\qquad\square$

The extension class for (19) lies in $\mathrm{Ext}^1(\mathscr{O}_Y(-2), \mathscr{O}_Y(2)) = H^1(Y, \mathscr{O}_Y(4))$, which has dimension $h^1(\mathscr{O}_Y(4)) = 5$. It would be interesting to describe this extension class explicitly. Note also that the values $h^i(\Theta_Y)$ for $Y = (E \times C)/G$ where C is elliptic and the G-action on it has a fixed point where computed by Partsch ([44], Proposition 6.1). It would be interesting to understand the situation in families.

We now apply our general results on proper group schemes:

Theorem 8.4. *Suppose that $p = 2$, and that the bielliptic surface $Y = (E \times C)/G$ is formed with the rational cuspidal curve C and the group scheme $G = \alpha_2$. Then Y does not lift to the ring of Witt vectors W.*

Proof. According to [8], discussion on page 25, the group scheme $P = \mathrm{Pic}^0_{Y/k}$ has dimension one and embedding dimension two. The fibration $Y \to B = E/G$ gives an inclusion $B \subset P$, and the G-torsor $X \to Y$ yields an inclusion $\alpha_2 \subset P$. The presence of multiple fibers shows that $\alpha_2 \cap B = 0$, and we conclude that the resulting inclusion $B \times \alpha_2 \subset P$ is an equality. In turn, $P/P_{\mathrm{red}} = \alpha_2$. Moreover, we have $b_1 = 2$, $h^1(\mathscr{O}_Y) = 2$ and $h^1(\mathscr{O}_Y) = 1$. Thus Theorem 4.3 applies, and we see that the scheme Y does not lift to the ring W. $\square$

9. Some homological algebra

In this final section we discuss the relevant homological algebra used throughout the paper. Our goal is to give a concise description how splittings in the derived category, Yoneda extensions and certain diagrams are related. The material should be of independent interest.

Let $f : M \to N$ be a homomorphism between two objects M, N in some abelian category $\mathcal{A}$. We may regard it as a *two-term complex*, with f as differential. Let $H^0 = \mathrm{Ker}(f)$ and $H^1 = \mathrm{Coker}(f)$ be its cohomology, and write $B = \mathrm{Im}(f)$ for the coboundaries. Now let E be another object, and $M \xrightarrow{h} E \xrightarrow{g} N$ be some homomorphisms. We say that (E, g, h) is a *diagram completion* if the diagram

(21)
$$
\begin{array}{ccc}
M & \xrightarrow{\mathrm{pr}} & B \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle i} \\
E & \xrightarrow[g]{} & N
\end{array}
$$

is *cartesian and cocartesian.* Here $\mathrm{pr} : M \to B$ and $i : B \to N$ are the canonical projections and injections, respectively. The condition means that $f = g \circ h$, and that the sequence $0 \to M \xrightarrow{(h,\mathrm{pr})} E \oplus B \xrightarrow{(g,-i)} N \to 0$ is exact. It follows that $h : M \to E$ is a monomorphism, $g : E \to N$ is an epimorphism, and we have identifications

(22) $\mathrm{Ker}(g) = \mathrm{Ker}(\mathrm{pr}) = H^0$ and $\mathrm{Coker}(h) = \mathrm{Coker}(i) = H^1$,

according to [30], Lemma 8.3.11. The composition of the inclusion $H^0 \subset M$ with $h : M \to E$ yields an inclusion $H^0 \subset E$. In turn, we get a diagram

(23)
$$
\begin{array}{ccccc}
M & \xrightarrow{\;h\;} & E & \xleftarrow{\;\text{can}\;} & H^0 \\
f\downarrow & & (g,0)\downarrow & & \downarrow 0 \\
N & \xrightarrow[(\mathrm{id}_N,0)]{} & N \oplus H^1 & \xleftarrow[(0,\mathrm{id})]{} & H^1,
\end{array}
$$

and one easily checks that it is commutative. We now regard the vertical maps as two-term complexes, and the horizontal maps as morphisms between complexes. Using the identifications (22), we infer that these are *quasi-isomorphisms*. We thus may regard (23) as an *isomorphism*

$$H^0 \oplus H^1[-1] \longrightarrow (M \xrightarrow{f} N)$$

in the derived category $D^b(\mathcal{A})$. Note that this constitutes a *splitting* of the complex $M \xrightarrow{f} N$ in the sense of Deligne and Illusie ([15], Section 3).

Recall that by Yoneda's construction [56], the groups $\mathrm{Ext}^n(A, B)$ can be defined via equivalence classes of exact sequences $0 \to C_{n+1} \to \ldots \to C_0 \to 0$ with $C_0 = A$ and $C_{n+1} = B$. This works without the existence of injective or projective resolutions, and yields a ∂-functors in B. For details we refer to [41], Chapter VII. Write $\mathrm{cl}(C_\bullet) \in \mathrm{Ext}^n(A, B)$ for the resulting *Yoneda class*. In particular, the horizontal exact sequence in the commutative diagram

(24)
$$
\begin{array}{ccc}
 & 0 \qquad\qquad 0 & \\
 & \searrow \qquad \nearrow & \\
 & B & \\
 \mathrm{pr}\nearrow & & \searrow i \\
0 \to H^0 \to M \xrightarrow{\quad f \quad} & & N \to H^1 \to 0
\end{array}
$$

yields a Yoneda class, which we denote by $\mathrm{cl}(f) \in \mathrm{Ext}^2(H^1, H^0)$. It coincides with the *Yoneda product* $\mathrm{cl}(M) * \mathrm{cl}(N)$ of the *extension classes* for the two short exact sequences with kinks.

Lemma 9.1. *The homomorphism $f : M \to N$ admits a diagram completion (E, g, h) if and only if the Yoneda class $\mathrm{cl}(f) \in \mathrm{Ext}^2(H^1, H^0)$ vanishes.*

Proof. In somewhat different formulation, this already appears in [3], Theorem 5.1. Let me give an independent argument. The short exact sequence to the right in (24) yields an extension class $\mathrm{cl}(N) \in \mathrm{Ext}^1(H^1, B)$, whereas the short exact sequence to the left gives a long exact sequence

(25)
$$\mathrm{Ext}^1(H^1, M) \longrightarrow \mathrm{Ext}^1(H^1, B) \xrightarrow{\;\partial\;} \mathrm{Ext}^2(H^1, H^0).$$

By definition of this sequence ([41], Chapter VII, Section 5), the image of the extension class $\mathrm{cl}(N)$ under the connecting map is $\mathrm{cl}(f) \in \mathrm{Ext}^2(H^1, H^0)$.

Suppose $\mathrm{cl}(f) = 0$. Then the extension $0 \to B \to N \to H^1 \to 0$ arises from an extension $0 \to M \xrightarrow{h} E \to H^1 \to 0$, and this means that there is a cocartesian

diagram

(26)
$$
\begin{array}{ccc}
M & \xrightarrow{\ h\ } & E \\
{\scriptstyle \mathrm{pr}}\downarrow & & \downarrow{\scriptstyle g} \\
B & \xrightarrow{\ i\ } & N.
\end{array}
$$

Using that $h : M \to E$ is a monomorphism, together with [30], Lemma 8.3.11 we infer that the above cocartesian diagram is also cartesian. In turn, (E, g, h) is a diagram completion.

Conversely, suppose there is a diagram completion (E, g, h), giving a cartesian and cocartesian diagram (26). Now recall that h is a monomorphism and $\mathrm{Coker}(h) = \mathrm{Coker}(i) = H^1$. This means that the extension class $\mathrm{cl}(N)$ lies in the image of the map on the left in (25), and thus $\mathrm{cl}(f) \in \mathrm{Ext}^2(H^1, H^0)$ vanishes. □

The diagram completions form a category $\mathrm{Cp}(M \xrightarrow{f} N)$. In this category, the morphisms $(E, g, h) \to (E', g', h')$ are those homomorphisms $\varphi : E \to E'$ making the diagram

(27)
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & & E \oplus B & & N & \longrightarrow & 0 \\
& & & \overset{(h,\mathrm{pr})}{\nearrow}\ \underset{(h',\mathrm{pr}')}{\searrow} & \downarrow{\scriptstyle \varphi \oplus \mathrm{id}_B} & \overset{(g,-i)}{\searrow}\ \underset{(g',-i')}{\nearrow} & & & \\
& & & & E' \oplus B & & & &
\end{array}
$$

commutative. In other words, $\varphi \oplus \mathrm{id}_B$ is a morphism of extensions. The latter is an isomorphism, by the Five Lemma, hence the same holds for the direct summand φ, so the category $\mathrm{Cp}(M \xrightarrow{f} N)$ is a *groupoid*.

Each homomorphism $\xi : H^1 \to H^0$ yields an endomorphism

(28)
$$
E \xrightarrow{g} N \xrightarrow{\mathrm{can}} H^1 \xrightarrow{\xi} H^0 \xrightarrow{\mathrm{can}} M \xrightarrow{h} E,
$$

which we denote by $\xi_E : E \to E$. Consider the endomorphism $\xi_E \circ \xi_E$. By definition, it is a composition of the form $\ldots \to H^0 \xrightarrow{\mathrm{can}} M \xrightarrow{h} E \xrightarrow{g} N \to \ldots$. This vanishes, because the composition of $f = g \circ h$ with the inclusion of $H^0 = \mathrm{Ker}(f) \subset M$ is zero. It follows that $\mathrm{id}_E + \xi_E$ is an automorphism of E, with inverse $\mathrm{id}_E - \xi_E$. From the definition and (27) one easily infers that it is actually an automorphism of (E, g, h). In turn, we obtain an homomorphism of groups

(29)
$$
\mathrm{Hom}_{\mathcal{A}}(H^1, H^0) \longrightarrow \mathrm{Aut}(E, g, h) \subset \mathrm{Aut}_{\mathcal{A}}(E), \quad \xi \longmapsto \mathrm{id}_E + \xi_E
$$

This map is injective, because in (28), the two arrows to the left are epimorphisms, whereas the two arrows to the right are monomorphisms.

Proposition 9.2. *The inclusion* $\mathrm{Hom}_{\mathcal{A}}(H^1, H^0) \subset \mathrm{Aut}(E, g, h)$ *is an equality provided that the canonical inclusion* $H^0 \subset M$ *admits a retraction.*

Proof. Choose a retraction $r : M \to H^0$. Using the functoriality of our maps, we may assume $M = H^0 \oplus B$. The composition $h \circ r : B \to E$, together with the universal property of cocartesian diagrams shows that the surjection $E \to N$ admits a splitting. Thus we may assume $E = H^0 \oplus N$, where the morphism $h : M \to E$ is given by the matrix $\left(\begin{smallmatrix} \mathrm{id} & 0 \\ 0 & \mathrm{pr} \end{smallmatrix}\right)$, and $g : E \to N$ is given by $(0, \mathrm{id})$. Using (27), one sees

that each automorphism of (E, g, h) is of the form $\mathrm{id}_E + \xi_E$ for some homomorphism $\xi : H^1 \to H^0$. $\qquad\square$

For the dual situation, we make the following observation:

Proposition 9.3. *The category* $\mathrm{Cp}(M \xrightarrow{f} N)$ *has precisely one isomorphism class provided that the canonical surjection* $N \to H^1$ *admits a section.*

Proof. Fix a section $s : H^1 \to N$, and write $N = B \oplus H^1$. Set $E_0 = M \oplus H^1$. Let $g_0 : M \to E_0$ be the canonical inclusion and $h_0 : E = M \oplus H^1 \to B \oplus H^1 = N$ by the matrix $\left(\begin{smallmatrix} \mathrm{pr} & 0 \\ 0 & \mathrm{id} \end{smallmatrix}\right)$. One easily checks that (E_0, g_0, h_0) is a diagram completion. Let (E, g, h) be another diagram completion. Composing $g : E \to N$ with the retraction $N \to B$ and using the universal property of cartesian squares, we get $E \simeq M \oplus H^1$, and infer that (E, g, h) is isomorphic to (E_0, g_0, h_0). $\qquad\square$

Now we bring in topology. Suppose that $\mathcal{C}$ is a ringed site. For the sake of exposition, we assume that there is a final object $X \in \mathcal{C}$, and write the structure sheaf as $\mathscr{O}_X$. We regard the objects as "open sets", and write them as $U \to X$. From now on we assume that our abelian category is $\mathcal{A} = (\mathscr{O}_X\text{-Mod})$, such that our $f : M \to N$ is a homomorphism of $\mathscr{O}_X$-modules. Note also that there are enough injective objects. *Furthermore assume that* $H^1 = \mathrm{Coker}(f)$ *is locally free of finite rank, and that* $\mathrm{pr} : M \to B$ *locally admits sections,* as in [15], Section 3.2.

The first condition ensures that the contravariant functor $U \mapsto \mathrm{Hom}_{\mathscr{O}_U}(H^1|U, F|U)$ satisfies the sheaf axiom, where F is an abelian sheaf, and $U \to X$ runs over the objects of $\mathcal{C}$. We denote the resulting sheaf $\underline{\mathrm{Hom}}_{\mathscr{O}_X}(H^1, F)$. Note also that we have an identification $\mathrm{Ext}^n(H^1, F) = H^n(X, \underline{\mathrm{Hom}}_{\mathscr{O}_X}(H^1, F))$, $n \geq 0$ of universal ∂-functors in $F \in \mathcal{A}$.

Let $\mathscr{C}p(M \xrightarrow{f} N)$ be the *category fibered in groupoids* over $\mathcal{C}$, whose objects over $U \to X$ are the diagram completions (E, g, h) for the restrictions $M|U \xrightarrow{f|U} N|U$. Morphisms $(E, g, h) \to (E', g', h')$ over a given $U \to U'$ are isomorphisms $(E, g, h) \to (E'|U, g'|U, h'|U)$. One easily checks that this category is fibered and satisfies the stack axioms. Roughly speaking, this means that all Hom presheaves are sheaves, and that all descend data are effective. See [42], Chapter 2 and 3 for the relevant definitions.

For each object (E, g, h) over $U \to X$, we obtain from (29) a homomorphism of group-valued sheaves

$$\Psi_{E,g,h} : \underline{\mathrm{Hom}}_{\mathscr{O}_X}(H^1, H^0)|U \longrightarrow \mathrm{Aut}_{(E,g,h)/U}, \quad \xi \mapsto \mathrm{id}_E + \xi_E$$

We observe:

Proposition 9.4. *The above are isomorphisms, and the stack* $\mathscr{C}p(M \xrightarrow{f} N)$ *is a gerbe banded by the abelian sheaf* $\underline{\mathrm{Hom}}_{\mathscr{O}_X}(H^1, H^0)$.

Proof. We have to check that all $\Psi_{E,g,h}$ are isomorphisms, and that all objects in $\mathscr{C}p(M \xrightarrow{f} N)$ are locally isomorphic. Both are local problems, and by our overall assumptions it suffices to treat the case that $N \to H^1$ and $M \to B$ admit splittings. The assertion on the homomorphisms and the objects follow from Proposition 9.2 and 9.3, respectively. $\qquad\square$

Recall that Deligne and Illusie ([15], Section 3) defined the *gerbe of splittings*, which we denote by $\mathscr{Sc}(M \xrightarrow{f} N)$. The objects over $U \to X$ are the splittings s for the canonical projection $N|U \to H^1|U$, and the morphisms $s \to s'$ between two splittings are defined as the homomorphisms $\xi : H^1|U \to M|U$ with $s' = s + \mathrm{pr} \circ \xi$. Via the tautological map

$$\Phi_{E,g,h} : \underline{\mathrm{Hom}}_{\mathscr{O}_X}(H^1, H^0)|U \longrightarrow \mathrm{Aut}_{s/U}, \quad \xi \longmapsto \xi$$

this also becomes a gerbe banded by $\underline{\mathrm{Hom}}_{\mathscr{O}_X}(H^1, H^0)$. In turn, we have two gerbes banded by the same coefficient sheaf, giving two cohomology classes. Our main result here is:

Theorem 9.5. *The gerbe* $\mathscr{Cp}(M \xrightarrow{f} N)$ *of diagram completions and the gerbe* $\mathscr{Sc}(M \xrightarrow{f} N)$ *of splittings have the same class in the cohomology group*

$$H^2(X, \underline{\mathrm{Hom}}_{\mathscr{O}_X}(H^1, H^0)) = \mathrm{Ext}^2(H^1, H^0).$$

Moreover, either of them admits a global object if and only if the Yoneda class $\mathrm{cl}(f) \in \mathrm{Ext}^2(H^1, H^0)$ *of the exact sequence* $0 \to H^0 \to M \xrightarrow{f} N \to H^1 \to 0$ *vanishes.*

Proof. First we construct a functor from the latter category to the former. Suppose we have a global splitting $s : H^1 \to N$, and write $N = B \oplus s(H^1)$, where $s(H^1) = \mathrm{Im}(s)$. Set $E = M \oplus s(H^1)$. Let the homomorphisms g, h be defined by the diagram

$$
(30) \qquad
\begin{array}{ccc}
M & \xrightarrow{\ \mathrm{pr}\ } & B \\
{\scriptstyle (\mathrm{id},0)}\big\downarrow & & \big\downarrow {\scriptstyle (\mathrm{id},0)} \\
M \oplus s(H^1) & \xrightarrow[{(\mathrm{pr},\mathrm{id})}]{} & B \oplus s(H^1).
\end{array}
$$

Clearly, this constitutes a diagram completion. The same reasoning applies locally over $U \to X$. According to [23], Chapter IV, Corollary 2.2.7, the functor $s \mapsto (E, g, h)$ is an equivalence of categories. In turn, if one of them admits a global object, so does the other. By Proposition 9.1, the category of diagram completions contains a global object if and only if the Yoneda class vanishes.

It remains to check that the gerbe classes coincide, and do not differ by a sign, say. For this we have to check that they are banded by the coefficient sheaf $\underline{\mathrm{Hom}}_{\mathscr{O}_X}(H^1, H^0)$ in the same way. Our construction (30) is functorial in s. In particular, each homomorphism $\xi : H^1 \to H^0$, viewed as an automorphism of s, yields the automorphism of (E, g, h) with $E = M \oplus s(H^1)$ as above given by the matrix $\left(\begin{smallmatrix} \mathrm{id} & \xi \\ 0 & \mathrm{id} \end{smallmatrix}\right) = \mathrm{id}_E + \xi_E$. In light of (29), the actions of the abelian sheaf $\underline{\mathrm{Hom}}_{\mathscr{O}_X}(H^1, H^0)$ via Ψ and Φ on the objects s and (E, g, h) coincide. $\square$

Let us close with the following remark: Since the quasi-isomorphisms in the homotopy category of cochain complexes admit a calculus of left and right fractions in the sense of Gabriel and Zisman [22], any isomorphism in the derived category represented by quasi-isomorphisms as in the diagram (23) may also be represented by quasi-isomorphisms $H^0 \oplus H^1[-1] \leftarrow C^\bullet \to (M \xrightarrow{f} N)$, with arrows pointing in reverse directions. This dichotomy seems to lie at the heart of the matter for the preceding results.

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