Rend. Circ. Mat. Palermo, II. Ser (2018) 67:409–419
https://doi.org/10.1007/s12215-017-0322-x

On equivariant formal deformation theory

Stefan Schröer[1] · **Yukihide Takayama**[2]

Received: 29 June 2017 / Accepted: 8 September 2017 / Published online: 25 September 2017

Abstract Using the set-up of deformation categories of Talpo and Vistoli, we re-interpret, elucidate and generalize, in the context of cartesian morphisms in abstract categories, some results of Rim concerning obstructions against extensions of group actions in infinitesimal deformations.

Keywords Formal deformation theory · Group actions · Fibered categories

Mathematics Subject Classification 14D15

Contents

Introduction

In deformation theory, one often seeks to extend automorphisms along infinitesimal extensions. This is not always possible: for example, Serre [13] showed that there are flat families of

✉ Stefan Schröer
schroeer@math.uni-duesseldorf.de

Yukihide Takayama
takayama@se.ritsumei.ac.jp

1 Mathematisches Institut, Heinrich-Heine-Universität, 40204 Düsseldorf, Germany

2 Department of Mathematical Sciences, Ritsumeikan University, 1-1-1 Nojihigashi, Kusatsu, Shiga 525-8577, Japan

smooth hypersurfaces $X \subset \mathbb{P}^4$ over $\Lambda = \mathbb{Z}_p$ whose closed fiber X_0 comes with a free action of some elementary abelian p-group G that does not extend to all infinitesimal neighborhoods X_n. Furthermore, the resulting quotient $Y_0 = X_0/G$ then does not lift to characteristic zero. The effect of automorphisms on pro-representability is discussed, for example, in the monographs of Sernesi ([12], Section 2.6) and Hartshorne ([7], Section 3.18; see also Section 3.22 for a discussion of Serre's example).

Rim [10] developed a formalism that explains the *obstructions* in terms of certain group cohomology in degree one and two. Our motivation for this note is to elucidate and perhaps simplify Rim's arguments by extending them into a purely categorical setting, merely using Grothendieck's notion of *cartesian morphisms* for functors $p : \mathscr{F} \to \mathscr{E}$ between arbitrary categories [6], much in the spirit of Talpo and Vistoli [15].

Recall that a morphism $f : \xi \to \xi'$ in $\mathscr{F}$ over a morphism $S \to S'$ in $\mathscr{E}$ is cartesian if, intuitively speaking, ξ behaves like a "base-change" of ξ' to S. Now let $\xi \in \mathscr{F}$ be an object over some $S \in \mathscr{E}$, and $G \to \mathrm{Aut}_S(\xi)$ be a homomorphism of groups. Write $\mathrm{Lif}(\xi, S')$ for the set of isomorphism classes of cartesian morphisms $\xi \to \xi'$ over $S \to S'$. This set is endowed with a G-action, by transport of structure. Fix a cartesian morphism $f : \xi \to \xi'$, and write $\mathrm{Aut}_\xi(\xi') \subset \mathrm{Aut}_{S'}(\xi')$ for the subgroup of automorphisms that induce the identity of ξ over S. Our main result is the following:

Theorem (See Theorem 1.2) *In the above setting, suppose that the group* $\mathrm{Aut}_\xi(\xi')$ *is abelian. Then the G-action on ξ extends to a G-action on ξ' if and only if the following two conditions hold:*

(i) *The isomorphism class* $[f] \in \mathrm{Lif}(\xi, S')$ *is fixed under the G-action.*
(ii) *The resulting cohomology class* $[\tilde{G}] \in H^2(G, \mathrm{Aut}_\xi(\xi'))$ *is trivial.*

Here $\tilde{G} = \mathrm{Aut}_{S'}(\xi') \times_{\mathrm{Aut}_S(\xi)} G$ *is the induced extension of G by* $\mathrm{Aut}_\xi(\xi')$*, and* $[\tilde{G}]$ *is the resulting cohomology class.*

We then apply this to the following algebro-geometric setting, using the set-up of Talpo and Vistoli [15]: let Λ be a complete local noetherian ring, with residue field $k = \Lambda/\mathfrak{m}_\Lambda$, and $\mathscr{F} \to (\mathrm{Art}_\Lambda)^{\mathrm{op}}$ be a *deformation category*, that is, a category fibered in groupoids that satisfies the *Rim–Schlessinger Condition*. The latter is a technical condition that comes from the structure theory of flat schemes over Artin rings. Note that the ring Λ may be of mixed characteristics, which was not allowed in [10].

Let $\xi \in \mathscr{F}(A)$ be an object, and $A' \to A$ be a small extension of rings, and $\xi_0 = \xi|_k$. We then use an observation of Serre from [14] and regard the set $\mathrm{Lif}(\xi, A')$, if nonempty, as a *torsor with a group of operators* G, to get a cohomology class

$$[\mathrm{Lif}(\xi, A')] \in H^1(G, I \otimes_k T_{\xi_0}(\mathscr{F})). \tag{1}$$

Here $T_{\xi_0}(\mathscr{F})$ is the *tangent space* defined by Eq. (7) in Sect. 3. The cohomology class is trivial if and only if there is an extension $\xi \to \xi'$ whose *isomorphism class* is G-fixed. The actual G-action on ξ extends to such an object $\xi' \in \mathscr{F}(A')$ if and only if the ensuing cohomology class

$$[\tilde{G}] \in H^2(G, \mathrm{Aut}_\xi(\xi')) = H^2(G, I \otimes_k \mathrm{Aut}_{\xi_0}(\xi_{k[\epsilon]})) \tag{2}$$

vanishes. Summing up, we have a primary obstruction (1), which deals with G-actions on isomorphism classes, and a secondary obstruction (2), which takes care of the actual G-action on objects.

If G is finite and the residue field $k = \Lambda/\mathfrak{m}_\Lambda$ has characteristic $p > 0$, then the above obstructions actually lie in the corresponding cohomology groups for a Sylow p-subgroup $P \subset G$. Consequently, the G-action extends if and only if the P-action extends.

1 Cartesian morphisms and extensions of group actions

In this section, we recall Grothendieck's notion of *cartesian morphisms* ([6], Exposé VI), and examine the problem of extending group actions along cartesian morphisms, using the relation between second group cohomology and extensions of groups. Our motivation was to clarify and perhaps simplify some arguments of Rim [10], by putting them to this categorical setting.

Let $p : \mathscr{F} \rightarrow \mathscr{E}$ be a functor between categories $\mathscr{F}$ and $\mathscr{E}$. For each object $S \in \mathscr{E}$, we write $\mathscr{F}(S) \subset \mathscr{F}$ for the subcategory of objects ξ with $p(\xi) = S$, and morphisms $h : \xi \rightarrow \zeta$ with $p(h) = \mathrm{id}_S$. The hom sets in this category are written as $\mathrm{Hom}_S(\xi, \zeta)$. If $\xi \in \mathscr{F}(S)$ and $\xi' \in \mathscr{F}(S')$, and $S \rightarrow S'$ is a morphism in $\mathscr{E}$, we write $\mathrm{Hom}_{S\rightarrow S'}(\xi, \xi')$ for the set of morphisms $f : \xi \rightarrow \xi'$ inducing the given $S \rightarrow S'$.

Let $f : \xi \rightarrow \xi'$ be a morphism in $\mathscr{F}$, with induced morphism $S \rightarrow S'$ in $\mathscr{E}$. One says that $f : \xi \rightarrow \xi'$ is *cartesian* if the map

$$\mathrm{Hom}_S(\zeta, \xi) \longrightarrow \mathrm{Hom}_{S\rightarrow S'}(\zeta, \xi'), \quad h \longmapsto f \circ h$$

is bijective, for each $\zeta \in \mathscr{F}(S)$. Intuitively, this means that ξ is obtained from ξ' by "base-change" along $S \rightarrow S'$.

We also say that a cartesian morphism $f : \xi \rightarrow \xi'$ is a *lifting* of ξ over $S \rightarrow S'$. Let $\mathscr{L}\!\mathit{if}(\xi, S')$ be the set of all such liftings; by abuse of notation, we suppress the morphism $S \rightarrow S'$ from notation. The group elements $\sigma \in \mathrm{Aut}_S(\xi)$ act on $\mathscr{L}\!\mathit{if}(\xi, S')$ from the left by transport of structure, written as ${}^{\sigma}f = f \circ \sigma^{-1}$. We may regard $\mathscr{L}\!\mathit{if}(\xi, S')$ also as a category, where a morphism between $f : \xi \rightarrow \xi'$ and $g : \xi \rightarrow \zeta'$ is an S'-morphism $h : \xi' \rightarrow \zeta'$ with $h \circ f = g$. Write $\mathrm{Lif}(\xi, S')$ for the set of isomorphism classes $[f]$ of lifting. Obviously, the action of $\mathrm{Aut}_S(\xi)$ descends to an action ${}^{\sigma}[f] = [f \circ \sigma^{-1}]$ from the left on $\mathrm{Lif}(\xi, S')$.

Every S'-morphism $\sigma' : \xi' \rightarrow \xi'$ yields the morphism $\sigma' \circ f$ over $S \rightarrow S'$, which in turn corresponds to a unique S-morphism $\sigma : \xi \rightarrow \xi$, which makes the diagram

$$\begin{array}{ccc} \xi & \xrightarrow{\ f\ } & \xi' \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma'} \\ \xi & \xrightarrow[\ f\ ]{} & \xi' \end{array} \tag{3}$$

commutative. The map $\sigma' \mapsto \sigma$ is compatible with compositions and respects identities, whence yields a homomorphism of groups

$$\mathrm{Aut}_{S'}(\xi') \longrightarrow \mathrm{Aut}_S(\xi), \quad \sigma' \longmapsto \sigma.$$

We call it the *restriction map*. Its kernel $\mathrm{Aut}_\xi(\xi')$ equals the group of automorphisms for the lifting $f : \xi \rightarrow \xi'$.

Now let G be a group acting on the object $\xi \in \mathscr{F}(S)$, via a homomorphism of groups $G \to \mathrm{Aut}_S(\xi)$. We seek to extend this action on $\xi \in \mathscr{F}(S)$ to an action on $\xi' \in \mathscr{F}(S')$. In other words, we want to complete the diagram

$$\begin{array}{ccc} & & G \\ & \swarrow & \downarrow \\ \mathrm{Aut}_{S'}(\xi') & \longrightarrow & \mathrm{Aut}_S(\xi) \end{array}$$

with some dashed arrow. A necessary condition is that the image of G in $\mathrm{Aut}_S(\xi)$ is contained in the image of $\mathrm{Aut}_{S'}(\xi')$. This can be reformulated as a fixed point problem:

Proposition 1.1 *The image of the homomorphism $G \to \mathrm{Aut}_S(\xi)$ is contained in the image of $\mathrm{Aut}_{S'}(\xi') \to \mathrm{Aut}_S(\xi)$ if and only if $[f] \in \mathrm{Lif}(\xi, S')$ is a fixed point for the G-action.*

Proof If the isomorphism class $[f]$ is fixed, then for each $\sigma \in G$, there exists an isomorphism $\sigma' : \xi' \to \xi'$ making the diagram (3) commutative. Since f is cartesian, the uniqueness of the arrow σ ensures that $\sigma' \mapsto \sigma$ under the restriction map $\mathrm{Aut}_{S'}(\xi') \to \mathrm{Aut}_S(\xi)$. Conversely, if the image of G lies in the image of $\mathrm{Aut}_{S'}(\xi')$, diagram (3) shows that the isomorphism class of the lifting $f : \xi \to \xi'$ is G-fixed. □

Now suppose that $[f] \in \mathrm{Lif}(\xi, S')$ is a fixed point for the G-action, such that the image of G in $\mathrm{Aut}_S(\xi)$ lies in the image of $\mathrm{Aut}_{S'}(\xi')$. Setting $\tilde{G} = \mathrm{Aut}_{S'}(\xi') \times_{\mathrm{Aut}_S(\xi)} G$, we get an induced extension of groups

$$1 \longrightarrow \mathrm{Aut}_\xi(\xi') \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1. \tag{4}$$

The splittings for this extension correspond to the extensions of the G-action to ξ'. To express this in cohomological terms, we now make the additional assumption that the kernel $\mathrm{Aut}_\xi(\xi')$ is abelian. This abelian group becomes a G-module, via ${}^\sigma h = \Phi_\sigma \circ h \circ \Phi_\sigma^{-1}$, where the $\Phi_\sigma \in \tilde{G}$ map to $\sigma \in G$. This indeed satisfies the axioms for actions, and does not depend on the choices of Φ_σ, because $\mathrm{Aut}_\xi(\xi')$ is abelian. Now the formula $c_{\sigma,\tau}\Phi_{\sigma\tau} = \Phi_\sigma\Phi_\tau$ defines a cochain $c : G^2 \to \mathrm{Aut}_\xi(\xi')$. As explained in [1], Chapter IV, Section 3, this cochain is a cocycle, and the resulting cohomology class

$$[\tilde{G}] \in H^2(G, \mathrm{Aut}_\xi(\xi'))$$

does not depend on the choice of the Φ_σ. Moreover, the extension of groups (4) splits if and only if $[\tilde{G}] = 0$. In this case, the extension is a semidirect product $\mathrm{Aut}_\xi(\xi') \rtimes G$. Indeed, the group $H^2(G, \mathrm{Aut}_\xi(\xi'))$ corresponds to isomorphism classes of group extensions of G by $\mathrm{Aut}_\xi(\xi')$ inducing the given G-module structure. Summing up, we have shown the following "abstract nonsense" result:

Theorem 1.2 *Let $p : \mathscr{F} \to \mathscr{E}$ be a functor, $f : \xi \to \xi'$ be a cartesian morphism in $\mathscr{F}$, and $S \to S'$ be the resulting morphism in $\mathscr{E}$. Let $G \to \mathrm{Aut}_S(\xi)$ be a homomorphism of groups, and assume that the group $\mathrm{Aut}_\xi(\xi')$ is abelian. Then the G-action on $\xi \in \mathscr{F}(S)$ extends to a G-action on $\xi' \in \mathscr{F}(S')$ if and only if the following two conditions holds:*

(i) *The isomorphism class $[f] \in \mathrm{Lif}(\xi, S')$ is fixed under the G-action.*
(ii) *The resulting cohomology class $[\tilde{G}] \in H^2(G, \mathrm{Aut}_\xi(\xi'))$ is trivial.*

Of particular practical importance are the *fibered categories* $p : \mathscr{F} \to \mathscr{E}$. This means that for each morphism $S \to S'$ in $\mathscr{E}$ and each object $\xi' \in \mathscr{F}(S')$, there is a cartesian morphism

$f : \xi \to \xi'$ in $\mathscr{F}$ over $S \to S'$, and the composition of cartesian morphisms in $\mathscr{F}$ is again cartesian. A *cleavage* is the choice, for each $S \to S'$ and $\xi' \in \mathscr{F}(S')$, of such a cartesian morphism $f : \xi \to \xi'$, which are called *transport morphisms*. If the transport morphisms for identities are identities, one calls the cleavage *normalized*. We also write $\xi'|_S = \xi$ for the domains. Intuitively, one should regard it as a "restriction", "pull-back" or "base-change" of ξ' along $S \to S'$. In fact, the transport morphisms induce restriction or pull-back functors

$$\mathscr{F}(S') \longrightarrow \mathscr{F}(S), \quad \xi' \longmapsto \xi'|_S.$$

In particular, for every amalgamated sum $S' \amalg_S S''$ in $\mathscr{E}$, we get a functor

$$\mathscr{F}(S' \amalg_S S'') \longrightarrow \mathscr{F}(S') \times_{\mathscr{F}(S)} \mathscr{F}(S''), \quad \xi \longmapsto (\xi|_{S'}, \xi|_{S''}, \varphi), \tag{5}$$

where $\varphi : (\xi|_{S'})|_S \longrightarrow (\xi|_{S''})|_S$ is the unique *comparison isomorphism*, compare [6], Exposé VI, Proposition 7.2, and the right hand side in (5) is the *2-fiber product of categories*, as explained in [15], Appendix C.

A *category fibered in groupoids* is a fibered category $p : \mathscr{F} \to \mathscr{E}$ so that the categories $\mathscr{F}(S)$, with $S \in \mathscr{E}$ are groupoids. These are the fibered categories that occur in moduli problems or deformation theory. They have the property that every morphism in $\mathscr{F}$ is cartesian, compare [6], Exposé VI, Remark after Definition 6.1.

2 Torsors with a group of operators

In this section we set up further notation, recall Serre's interpretation of first group cohomology in terms of torsors [14], §5.2, and relate it to fixed point problems. Let G be a group that acts from the left via automorphisms on another group T and on a set L. We write these actions as $t \mapsto {}^\sigma t$ and $\xi \mapsto {}^\sigma \xi$, where $\sigma \in G$. Suppose we have an action on the right

$$\mu : L \times T \longrightarrow L, \quad (\xi, t) \longmapsto \xi \cdot t,$$

such that the set L is a *principal homogeneous space* for the group T, that is, a right *T-torsor*. In other words, the set L is non-empty, and for each point $\xi_0 \in L$ the resulting map $T \to L$, $t \mapsto \xi_0 \cdot t$ is bijective. We assume throughout that this action is compatible with the G-action in the sense

$${}^\sigma(\xi \cdot t) = {}^\sigma\xi \cdot {}^\sigma t,$$

for all $\sigma \in G$, $\xi \in L$ and $t \in T$. One says that the T-torsor L is *endowed with a group of operators* G. They are the objects of a category, where the morphisms $(L, T) \to (L', T')$ are pairs (f, h), where $f : L \to L$ is a G-equivariant map, and $h : T \to T'$ is a G-equivariant homomorphism, which satisfy

$$f(\xi \cdot t) = f(\xi) \cdot h(t).$$

In this situation, we want to decide whether or not the G-set L has a fixed point. To this end, one may construct a cohomology class $[L] \in H^1(G, T)$ as follows: choose some $\xi \in L$. Then the equation ${}^\sigma\xi = \xi \cdot t_\sigma$ defines a map

$$G \longrightarrow T, \quad \sigma \longmapsto t_\sigma,$$

which we regard as a 1-cochain. The equation

$$\xi \cdot t_{\eta\sigma} = {}^{\eta\sigma}\xi = {}^\eta(\xi \cdot t_\sigma) = {}^\eta\xi \cdot {}^\eta t_\sigma = (\xi \cdot t_\eta) \cdot {}^\eta t_\sigma = \xi \cdot (t_\eta {}^\eta t_\sigma)$$

implies $t_{\eta\sigma} = t_\eta{}^\eta t_\sigma$, and it follows that the cochain is a cocycle. For every other point $\xi' \in L$, the equation ${}^\sigma\xi' = \xi \cdot t'_\sigma$ defines another cocycle $\sigma \mapsto t'_\sigma$. We have $\xi' \cdot s = \xi$ for some $s \in T$, and thus

$$\xi' \cdot (t'_\sigma{}^\sigma s) = {}^\sigma\xi' \cdot {}^\sigma s = {}^\sigma(\xi' \cdot s) = {}^\sigma\xi = \xi \cdot t_\sigma = (\xi' \cdot s) \cdot t_\sigma = \xi' \cdot (st_\sigma).$$

It follows that $t'_\sigma = st_\sigma{}^\sigma(s^{-1})$, whence the two cocycles are cohomologous. We thus get a well-defined cohomology class

$$[L] \in H^1(G, T).$$

In this general non-abelian setting, we regard $H^1(G, T)$ as a pointed set, where the distinguished point $\star \in H^1(G, T)$ is the cohomology class of the constant cocycle $\sigma \mapsto e$. It is also called the *trivial cohomology class*. According to [14], Proposition 33, this gives a pointed bijection between the set of isomorphism classes of T-torsors L with a group of operators G, and the set $H^1(G, T)$. We need the following consequence:

Lemma 2.1 *The cohomology class $[L] \in H^1(G, T)$ is trivial if and only if the set of fixed points L^G is nonempty.*

Proof The condition is clearly sufficient: if $\xi \in L$ is G-fixed, then the resulting cocycle is $t_\sigma = e$, so the cohomology class $[L]$ is trivial. Conversely suppose that the cocycle t_σ attached to a point $\xi \in L$ satisfies $st_\sigma{}^\sigma(s^{-1}) = e$ for some $s \in T$. Then

$${}^\sigma(\xi \cdot s^{-1}) = {}^\sigma\xi \cdot {}^\sigma(s^{-1}) = \xi \cdot t_\sigma \cdot {}^\sigma(s^{-1}) = (\xi \cdot s^{-1}) \cdot (st_\sigma{}^\sigma(s^{-1})) = \xi \cdot s^{-1},$$

whence $\xi' = \xi \cdot s^{-1}$ is the desired fixed point. □

3 Deformation categories and group actions

Let k be a field of characteristic $p \geq 0$, and let Λ be a complete local noetherian ring with residue field $k = \Lambda/\mathfrak{m}_\Lambda$. We write (Art_Λ) for the category of local Artin Λ-algebras A such that the structure homomorphism $\Lambda \to A$ is local and the induced map $k = \Lambda/\mathfrak{m}_\Lambda \to A/\mathfrak{m}_A$ on residue fields is bijective. Let $\mathscr{F} \to (\mathrm{Art}_\Lambda)^{\mathrm{op}}$ be a category fibered in groupoids satisfying the *Rim–Schlessinger condition*. Recall that the latter means that for every cartesian square

$$\begin{array}{ccc} A' \times_A A'' & \longrightarrow & A'' \\ \downarrow & & \downarrow \\ A' & \longrightarrow & A \end{array} \tag{6}$$

in the category (Art_Λ), the resulting functor

$$\mathscr{F}(A' \times_A A'') \longrightarrow \mathscr{F}(A') \times_{\mathscr{F}(A)} \mathscr{F}(A'')$$

is an equivalence of categories. Note that this functor corresponds to (5), and is actually defined with the help of a chosen cleavage, but the fact that it is an equivalence does not depend on this choice. Such a condition was first introduced by Schlessinger [11], who considered functors of Artin rings, and extended to fibered categories by Rim [9]. Following Talpo and Vistoli [15], we say that such a category fibered in groupoids $\mathscr{F} \to (\mathrm{Art}_\Lambda)^{\mathrm{op}}$ is a *deformation category*.

Note that one should regard the opposite category $(\mathrm{Art}_\Lambda)^{\mathrm{op}}$ as a full subcategory of the category (Sch/Λ) of schemes. The morphisms in this category are thus $\mathrm{Spec}(A) \to \mathrm{Spec}(A')$, and correspond to algebra homomorphisms $A' \to A$. The spectrum of a fiber product $A' \times_A A''$ of rings in (6) becomes an amalgamated sum of affine schemes $\mathrm{Spec}(A') \amalg_{\mathrm{Spec}(A)} \mathrm{Spec}(A'')$. The transport morphisms over an algebra homomorphism $B \to C$, that is $\mathrm{Spec}(C) \to \mathrm{Spec}(B)$, could also be written in tensor product notation $\zeta \otimes_B C \to \zeta$ instead of $\zeta|_C \to \zeta$. Indeed, in praxis the deformation category $\mathscr{F} \to (\mathrm{Art}_\Lambda)^{\mathrm{op}}$ often consists of flat morphisms $X \to \mathrm{Spec}(C)$ of certain schemes, and the transport morphisms are given by projections $\mathrm{pr}_1 : X \otimes_B C = X \times_{\mathrm{Spec}(B)} \mathrm{Spec}(C) \to X$.

Let $A \in (\mathrm{Art}_\Lambda)$, and $\xi \in \mathscr{F}(A)$ be an object. Suppose that G is a group endowed with a homomorphism $G \to \mathrm{Aut}_A(\xi)$. In other words, G acts on the object $\xi \in \mathscr{F}$ so that the induced action on $A \in (\mathrm{Art}_\Lambda)$ is trivial. In what follows,

$$0 \longrightarrow I \longrightarrow A' \longrightarrow A \longrightarrow 0$$

is a *small extension* with ideal $I \subset A'$. This means that $I \cdot \mathfrak{m}_{A'} = 0$, so we may regard the Λ-module I simply as a k-vector space.

We now ask whether there exists a lifting $f : \xi \to \xi'$ over $\mathrm{Spec}(A) \subset \mathrm{Spec}(A')$ to which the G-action extends. Of course, the category of all liftings may be empty, and then nothing useful can be said. But if one assumes that some lifts exist, a natural question is whether some possibly different liftings can be endowed with a G-action. To this end, we apply Theorem 1.2 to our situation. Recall that $\mathscr{Lif}(\xi, A')$ denotes the category of all liftings $f : \xi \to \xi'$ over $\mathrm{Spec}(A) \subset \mathrm{Spec}(A')$, and let $\mathrm{Lif}(\xi, A')$ be the set of isomorphism classes $[f]$, endowed with the canonical G-action

$${}^{\sigma}[f] = [f \circ \sigma^{-1}].$$

To proceed, choose a morphism $\xi_0 \to \xi$ over $\mathrm{Spec}(k) \subset \mathrm{Spec}(A)$, and consider the resulting *tangent space*

$$T_{\xi_0}\mathscr{F} = \mathrm{Lif}(\xi_0, k[\epsilon]), \tag{7}$$

where ϵ denotes an indeterminate subject to the relation $\epsilon^2 = 0$. In other words, $k[\epsilon] \in (\mathrm{Art}_\Lambda)$ is the *ring of dual numbers*, with ideal $k\epsilon$.

Given a k-vector space I, we likewise write $k[I] = k \oplus I$ for the resulting k-algebra whose ideal I satisfies $I^2 = 0$. The Rim–Schlessinger condition ensures that the functor $I \mapsto \mathrm{Lif}(\xi_0, k[I])$ of finite-dimensional k-vector spaces preserves finite products, and as a consequence $\mathrm{Lif}(\xi_0, k[I])$ and in particular the tangent space $T_{\xi_0}\mathscr{F}$ acquire the structure of an abelian group, and actually become k-vector spaces. As explained in [15], Appendix A, the natural transformation in I given by

$$I \otimes_k \mathrm{Lif}(\xi_0, k[\epsilon]) \longrightarrow \mathrm{Lif}(\xi_0, k[I]), \quad v \otimes [\xi f \!\to\! \psi] \longmapsto \big[\xi \alpha \!\to\! \psi|_{k[I]}\big] \tag{8}$$

is a natural isomorphism, where $v \in I$ runs over all vectors. The object $\psi|_{k[I]}$ arises from the transport morphism $\psi|_{k[I]} \to \psi$ over the morphism $\mathrm{Spec}(k[I]) \to \mathrm{Spec}(k[\epsilon])$ induced from the linear map $k\epsilon \to I$ with $\epsilon \mapsto v$, and $\alpha : \xi_0 \to \psi|_{k[I]}$ is the transport morphism over the inclusion $\mathrm{Spec}(k) \subset \mathrm{Spec}(k[I])$ given by $I \to 0$. Clearly, this natural isomorphism respects the action of the $\mathrm{Aut}_k(\xi_0)$, where the group elements $\sigma \in \mathrm{Aut}_k(\xi_0)$ act via transport of structure

$$v \otimes [\xi_0 \xrightarrow{f} \psi] \longmapsto v \otimes [\xi_0 \xrightarrow{f\sigma^{-1}} \psi] \quad \text{and} \quad [\xi_0 \xrightarrow{\alpha\sigma^{-1}} \psi|_{k[I]}]$$

Note that the action on $v \in I$ is trivial. In what follows, we regard the above natural isomorphism as an identification $I \otimes_k T_{\xi_0}(\mathscr{F}) = \mathrm{Lif}(\xi_0, k[I])$. Furthermore, the underlying abelian group acts on $\mathrm{Lif}(\xi, A')$ in a canonical way, via the map

$$\mathrm{Lif}(\xi, A') \times (I \otimes_k T_{\xi_0}\mathscr{F}) \longrightarrow \mathrm{Lif}(\xi, A') \tag{9}$$

recalled in (10) below. The G-action on ξ induces a G-action on ξ_0, and we also get a linear G-action on the tangent space $T_{\xi_0}\mathscr{F}$, as described above.

Proposition 3.1 *Suppose the set $L = \mathrm{Lif}(\xi, A')$ is non-empty. With respect to the action of $T = I \otimes_k T_{\xi_0}\mathscr{F}$, the set L is a T-torsor with a group of operators G.*

Proof As explained in [15], Theorem 3.15, the Rim–Schlessinger condition ensures that the set L becomes a T-torsor. Our task is merely to check that this structure is compatible with the G-actions. To this end, we have to unravel the action of T on L. Let $f : \xi \to \xi'$ be lifting of $\xi \in \mathscr{F}(A)$ over A', and $g : \xi_0 \to \tilde{\xi}$ be a lifting of $\xi_0 \in \mathscr{F}(k)$ over the ring of dual numbers $\tilde{A} = k[I]$ with ideal I. We have to describe $[f] + [g] \in \mathrm{Lif}(\xi, A')$ and understand how the group G acts on this.

To proceed, choose a cleavage for the fibered category $\mathscr{F} \to (\mathrm{Art}_\Lambda)^{\mathrm{op}}$. In other words, we fix for each object $\zeta \in \mathscr{F}(C)$ and each homomorphism $B \to C$ a transport morphism $\zeta|_C \to \zeta$ over $\mathrm{Spec}(C) \to \mathrm{Spec}(B)$ and regard the domain $\zeta|_C$ as the restriction of ζ. We do this so that $\xi_0 = \xi|_k$ holds. In what follows, we simply write $\alpha : \zeta|_C \to \zeta$ for these transport morphisms. Now the morphism f and g correspond to isomorphisms

$$\bar{f} : \xi \longrightarrow \xi'|_A \quad \text{and} \quad \bar{g} : \xi_0 \longrightarrow \tilde{\xi}|_k,$$

and we can form the composite morphism

$$\psi : \xi'|_k \xrightarrow{\bar{f}^{-1}|_k} \xi_0 \xrightarrow{\bar{g}} \tilde{\xi}|_k.$$

This gives us a triple $(\xi', \tilde{\xi}, \psi)$, which we regard as an object in the fiber product category

$$\mathscr{F}(A') \times_{\mathscr{F}(k)} \mathscr{F}(k[I]).$$

Now recall that we have isomorphisms of rings

$$A' \times_A A' \longrightarrow A' \times_k (k[I]), \quad (a_1, a_2) \longmapsto (a_1, (a_1 \bmod \mathfrak{m}_{A'}, a_2 - a_1)).$$

Here we use $k[I] = k \oplus I$, and write $a_1 \bmod \mathfrak{m}_{A'}$ for the residue class in k, and regard $a_2 - a_1$ as element of I. The Rim–Schlessinger condition yields equivalences of categories

$$\mathscr{F}(A') \times_{\mathscr{F}(A)} \mathscr{F}(A') \longleftarrow \mathscr{F}(A' \times_A A') \longrightarrow \mathscr{F}(A' \times_k k[I]) \longrightarrow \mathscr{F}(A') \times_{\mathscr{F}(k)} \mathscr{F}(k[I]),$$

where the restriction functors are defined in terms of the chosen cleavage. Choose adjoint equivalences, to get an equivalence of categories

$$\mathscr{F}(A') \times_{\mathscr{F}(k)} \mathscr{F}(k[I]) \longrightarrow \mathscr{F}(A') \times_{\mathscr{F}(A)} \mathscr{F}(A').$$

We may choose this functor so that it commutes with the projections onto the first factor $\mathscr{F}(A')$. Applying this functor to the object $(\xi', \tilde{\xi}, \psi)$ yields an object (ξ', ζ', φ), where $\xi', \zeta' \in \mathscr{F}(A')$ and $\varphi : \xi'|_A \to \zeta'|_A$ is an isomorphism. In turn, we get a lifting from the composite morphism

$$h : \xi \xrightarrow{\bar{f}} \xi'|_A \xrightarrow{\varphi} \zeta'|_A \xrightarrow{\alpha} \zeta'. \tag{10}$$

Here $\alpha : \zeta'|_A \to \zeta'$ is a transport morphism. The T-action on L is given by $[f] + [g] = [h]$, as explained in [15], Theorem 3.15.

Now we are in the position to unravel the G-action. Let $\sigma \in G$. By definition, ${}^{\sigma}[f] = [f \circ \sigma^{-1}]$ and ${}^{\sigma}[g] = [g \circ \sigma^{-1}]$. Using $f \circ \sigma^{-1}$ and $g \circ \sigma^{-1}$ rather than f and g in the preceding paragraph, we get

$$\overline{f \circ \sigma^{-1}} = \bar{f} \circ \sigma^{-1}, \quad \text{and} \quad \overline{g \circ \sigma^{-1}} = \bar{g} \circ (\sigma^{-1}|_k) = \bar{g} \circ (\sigma|_k)^{-1},$$

which implies

$$\overline{g \circ \sigma^{-1}} \circ \overline{f \circ \sigma^{-1}}^{-1}|_k = \bar{g} \circ (\sigma|_k)^{-1} \circ (\sigma|_k) \circ \bar{f}^{-1}|_k = \bar{g} \circ \bar{f}^{-1}|_k.$$

It follows that the resulting morphism $\psi : \xi'|_k \to \tilde{\xi}|_k$ is the same, whether computed with $f \circ \sigma^{-1}$ and $g \circ \sigma^{-1}$, or with f and g. In turn, the image of the object $(\xi', \tilde{\xi}, \psi)$ remains the object (ξ', ζ', φ). The resulting lifting is thus given by the composite

$$\xi \xrightarrow{\bar{f}\sigma^{-1}} \xi'|_A \xrightarrow{\varphi} \zeta'|_A \xrightarrow{\alpha} \zeta',$$

which equals $h \circ \sigma^{-1}$. This shows that ${}^{\sigma}[f] + {}^{\sigma}[g] = {}^{\sigma}[h]$. In other words, the T-torsor L is endowed with a group of operators G. □

As described in Sect. 2, this L-torsor T endowed with a group of operators G yields a cohomology class

$$[\mathrm{Lif}(\xi, A')] \in H^1(G, I \otimes_k T_{\xi_0}\mathscr{F}),$$

and Lemma 2.1 immediately gives:

Theorem 3.2 *Suppose* $\mathrm{Lif}(\xi, A')$ *is non-empty. Then the there is a G-fixed isomorphism class* $[f] \in \mathrm{Lif}(\xi, A')$ *of liftings* $f : \xi \to \xi'$ *over* $\mathrm{Spec}(A) \subset \mathrm{Spec}(A')$ *if and only if the cohomology class* $[\mathrm{Lif}(\xi, A')] \in H^1(G, I \otimes_k T_{\xi_0}\mathscr{F})$ *is trivial.*

Now suppose that there exists a lifting $f : \xi \to \xi'$ whose isomorphism class $[f] \in \mathrm{Lif}(\xi, A')$ is fixed under the G-action. As discussed in Sect. 1, we get an extension of groups

$$1 \longrightarrow \mathrm{Aut}_{\xi}(\xi') \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1, \tag{11}$$

and this extension of groups splits if and only if the G-action on ξ extends to ξ'.

Now choose a morphism $\xi_{k[\epsilon]} \to \xi_0$ over the morphism $\mathrm{Spec}(k[\epsilon]) \to \mathrm{Spec}(k)$ corresponding to the canonical inclusion $k \subset k[\epsilon]$. Since $\mathscr{F} \to (\mathrm{Art}_{\Lambda})^{\mathrm{op}}$ is a category fibered in groupoids, there is a unique morphism $\xi_0 \to \xi_{k[\epsilon]}$ over the closed embedding $\mathrm{Spec}(k) \subset \mathrm{Spec}(k[\epsilon])$ corresponding to $\epsilon \mapsto 0$ such that the composite morphism $\xi_0 \to \xi_{k[\epsilon]} \to \xi_0$ is the identity. As explained in [15], Proposition 4.5, we have a canonical identification

$$I \otimes_k \mathrm{Aut}_{\xi_0}(\xi_{k[\epsilon]}) = \mathrm{Aut}_{\xi}(\xi'), \tag{12}$$

and this group carries the structure of k-vector space. In particular, it is abelian. In fact, (12) is an incarnation of (9), for the deformation theory $\mathscr{A} \to (\mathrm{Art}_{\Lambda})^{\mathrm{op}}$ whose objects over A are the automorphisms of $\xi_0|_A$, as explained in [15], Section 4.

Since the isomorphism class of $f : \xi \to \xi'$ is G-fixed, we have a natural G-action on $\mathrm{Aut}_{\xi}(\xi')$, coming from the extension (11) or equivalently from diagram (3). The same applies for $\xi_0 \to \xi_{k[\epsilon]}$, and we thus get a G-action on $\mathrm{Aut}_{\xi_0}(\xi_{k[\epsilon]})$. Taking the trivial G-action on I, both sides in (12) acquire a G-action, and these action coincide under the identification. We thus may regard the extension class for (11) as an element in

$$[\tilde{G}] \in H^2(G, \mathrm{Aut}_{\xi}(\xi')) = H^2(G, I \otimes_k \mathrm{Aut}_{\xi_0}(\xi_{k[\epsilon]})).$$

Now Theorem 1.2 yields:

Theorem 3.3 *Suppose that* $\mathrm{Lif}(\xi, A')^G$ *is non-empty, and let* $f : \xi \to \xi'$ *be a lifting over* $\mathrm{Spec}(A) \subset \mathrm{Spec}(A')$ *whose isomorphism class is fixed under the* G*-action. Then the* G*-action on* ξ *extends to an action on* ξ' *if and only if the resulting cohomology class* $[\tilde{G}] \in H^2(G, I \otimes_k \mathrm{Aut}_{\xi_0}(\xi_{k[\epsilon]}))$ *vanishes.*

In the following applications, we assume that the group G is finite, and write $n = \mathrm{ord}(G)$ for its order.

Proposition 3.4 *Suppose* $\mathrm{Lif}(\xi, A')$ *is non-empty and that the group order* $n \geq 1$ *is invertible in the residue field* k*. Then the* G*-action on* ξ *extends to an action on* ξ' *for some lifting* $f : \xi \to \xi'$*.*

Proof The cohomology group $H^1(G, I \otimes_k T_{\xi_0}\mathscr{F})$ is a vector space over the field k, and at the same time an abelian group annihilated by $n = \mathrm{ord}(G)$. Thus it must be the zero group, and Theorem 3.2 ensures that there is a lifting $\xi \to \xi'$ over $\mathrm{Spec}(A) \subset \mathrm{Spec}(A')$ whose isomorphism class is fixed under the G-action. Arguing as above, the cohomology group $H^2(G, I \otimes_k \mathrm{Aut}_{\xi_0}(\xi_{k[\epsilon]}))$ vanishes, and Theorem 3.3 tells us that we may extend the G-action from ξ to ξ'. □

Proposition 3.5 *Suppose* $\mathrm{Lif}(\xi, A')$ *is non-empty and that the residue field* k *has characteristic* $p > 0$*. Let* $P \subset G$ *be a Sylow* p*-subgroup. Then* $\mathrm{Lif}(\xi, A')$ *has a* G*-fixed point if and only if it has a* P*-fixed point. Moreover, for each* $[\xi'] \in \mathrm{Lif}(\xi, A')^G$*, the* G*-action on* ξ *extends to* ξ' *if and only if the* P*-action extends.*

Proof According to [1], Chapter III, Proposition 10.4 the restriction map

$$H^1(G, I \otimes_k T_{\xi_0}\mathscr{F}) \longrightarrow H^1(P, I \otimes_k T_{\xi_0}\mathscr{F})$$

is injective, and the first assertion follows from Theorem 3.2. If there is a lifting $\xi \to \xi'$ whose isomorphism class is G-invariant, we again have an injective restriction map

$$H^2(G, I \otimes_k \mathrm{Aut}_{\xi_0}(\xi_{k[\epsilon]})) \longrightarrow H^2(P, I \otimes_k \mathrm{Aut}_{\xi_0}(\xi_{k[\epsilon]})),$$

and the second assertion follows from Theorem 3.3. □

Recall that a finitely generated free kP-modules V have trivial cohomology groups $H^i(P, V)$, for all $i \geq 1$. We thus get:

Corollary 3.6 *Assumptions as in the proposition. Then* $\mathrm{Lif}(\xi, A')$ *has a* G*-fixed point if* $\mathrm{Lif}(\xi, A')$ *is free as* kP*-module. Moreover, for each* $[\xi'] \in \mathrm{Lif}(\xi, A')^G$*, the* G*-action on* ξ *extends to* ξ' *if* $Aut_{\xi_0}(\xi_{k[\epsilon]})$ *is free as* kP*-module.*

In some sense, this seems to be the best possible general result: according to [1], Chapter VI, Theorem 8.5, for every finite p-group P and every field k of characteristic $p > 0$, the following holds for kP-modules V:

$$H^1(P, V) = 0 \iff H^2(P, V) = 0 \iff \text{the } kP\text{-module } V \text{ is free.}$$

If P is cyclic of order p^ν and V is finitely generated, then the action of any generator $\sigma \in P$ can be viewed as a direct sum $\sigma = J_{r_1} \oplus \cdots \oplus J_{r_m}$ of Jordan matrices $J_r \in \mathrm{GL}_r(k)$ with eigenvalue $\lambda = 1$. In this case, the kP-module V is free if and only if all summands have maximal size $r_i = p^\nu$.

Acknowledgements We like to thank the referee for useful suggestions.

References

1. Brown, K.: Cohomology of Groups. Springer, Berlin (1982)
2. Grothendieck, A., Dieudonné, J.: Éléments de géométrie algébrique I: Le langage des schémas. Springer, Berlin (1970)
3. Grothendieck, A.: Éléments de géométrie algébrique II: Étude globale élémentaire de quelques classes de morphismes. Publ. Math., Inst. Hautes Étud. Sci. 8 (1961)
4. Grothendieck, A.: Éléments de géométrie algébrique III: Étude cohomologique des faiscaux cohérent. Publ. Math., Inst. Hautes Étud. Sci. 11 (1961)
5. Grothendieck A.: Éléments de géométrie algébrique IV: Étude locale des schémas et des morphismes de schémas. Publ. Math., Inst. Hautes Étud. Sci. 32 (1967)
6. Grothendieck, A.: Revêtements étales et groupe fondamental (SGA 1). Springer, Berlin (1971)
7. Hartshorne, R.: Deformation Theory. Springer, New York (2010)
8. Matsumura, H.: Commutative Algebra, 2nd edn. Benjamin/Cummings, Reading (1980)
9. Rim, D.: Formal deformation theory. In: SGA 7 I, pp. 32–132. Springer, Berlin (1972)
10. Rim, D.: Equivariant G-structure on versal deformations. Trans. Am. Math. Soc. **257**, 217–226 (1980)
11. Schlessinger, M.: Functors of Artin rings. Trans. Am. Math. Soc. **130**, 208–222 (1968)
12. Sernesi, E.: Deformations of algebraic schemes. Grundlehren der Mathematischen Wissenschaften 334. Springer, Berlin (2006)
13. Serre, J.-P.: Exemples de variétés projectives en caractéristique p non relevables en caractéristique zéro. Proc. Natl. Acad. Sci. USA **47**, 108–109 (1961)
14. Serre, J.-P.: Cohomologie galoisienne. Fifth edition. Lecture Notes on Maththematics, vol. 5. Springer, Berlin (1994)
15. Talpo, M., Vistoli, A.: Deformation theory from the point of view of fibered categories. In: Farkas, G., Morrison, I. (eds.) Handbook of Moduli, vol. III. International Press, Somerville (2013)