ON FLASQUE SHEAVES AND FLASQUE MODULES

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Abstract. We show that each sheaf of modules admits a flasque hull, such that homomorphisms into flasque sheaves factor over the flasque hull. On the other hand, we give examples of modules over non-noetherian rings that do not inject into flasque modules. This reveals the impossibility to extend the proof of Serre's vanishing result for affine schemes with flasque quasicoherent resolutions to the non-noetherian setting. However, we outline how hypercoverings can be used for a reduction to the noetherian case.

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Introduction

Let X be a ringed space. An $\mathscr{O}_X$-module $\mathscr{F}$ is called *flasque* if every local section extends to a global sections, that is, all restrictions maps $\Gamma(X,\mathscr{F}) \to \Gamma(U,\mathscr{F})$ are surjective. This notion is extraordinary useful in sheaf theory, because flasque sheaves are acyclic, such that sheaf cohomology $H^i(X,\mathscr{F})$ may be computed with flasque resolutions. In fact, each $\mathscr{O}_X$-module $\mathscr{F}$ sits in a canonical way in its Godement sheaf $\mathscr{F}^{\mathrm{gdm}}$, which is a flasque sheaf defined via stalk products $\Gamma(U,\mathscr{F}^{\mathrm{gdm}}) = \prod_{a\in U}\mathscr{F}_a$. We refer to Godement's monograph [5] for generalities on sheaf theory. Note that the term *flabby* instead of flasque is also commonly used, for example in Bredon's book [3].

The first result of this note is that each $\mathscr{O}_X$-module $\mathscr{F}$ admits a *flasque hull* $\mathscr{F} \subset \mathscr{F}^{\mathrm{fls}}$, such that each homomorphism $\mathscr{F} \to \mathscr{G}$ into some flasque $\mathscr{O}_X$-module uncanonically factors over $\mathscr{F}^{\mathrm{fls}}$. The flasque hull is defined via an intermediate construction, the *globalizing sheaf* $\mathscr{F} \subset \mathscr{F}^{\circ}$, in which all local sections of $\mathscr{F}$ become restrictions of global sections. Using transfinite induction, one obtains a direct

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system of sheaves $\mathscr{F}_\alpha$, indexed by ordinal numbers $\alpha \geq 0$. We then show that these sheaves become flasque for sufficiently large enough limit ordinals, and the smallest ordinal $\alpha \geq 0$ producing a flasque sheaf gives the flasque hull $\mathscr{F}^{\text{fls}} = \mathscr{F}_\alpha$.

Now suppose that X is a scheme and $\mathscr{F}$ is a quasicoherent sheaf. A natural question is whether or not there is some injection $\mathscr{F} \subset \mathscr{G}$ into some flasque quasicoherent sheaf $\mathscr{G}$. This indeed holds if X is noetherian. In fact, for every noetherian ring R, any injective module I is a *flasque module*, in the sense that the corresponding quasicoherent sheaf $\mathscr{I} = \tilde{I}$ is flasque ([9], Corollary 2.7, compare also [8], Exposé II). A self-contained direct argument was given by Campbell [4]. Among other things, these facts give a quick proof, in the noetherian case, that quasicoherent sheaves over affine schemes are acyclic ([10], Chapter III). However, Verdier showed that over certain ring of dual numbers of the form $R = k[[x, y]] \oplus \mathfrak{b}$, with some non-finitely generated $\mathfrak{b}$, there must be injective modules I that are not flasque ([2], Exposé II, Appendix I). The relation between injectivity and flasqueness was further analyzed by Verra [17].

Nevertheless, it remains natural to ask whether or not each module M admits a flasque module extension $M \subset N$. In light of the existence of flasque hulls for sheaves, one might indeed expect such a result. However, the second main observation of this note is that this fails, and we provide *explicit counterexamples*: The simplest situation seems to be the ring

$$R = k[T, S_0, S_1, \ldots]/(TS_0^1, TS_1^2 \ldots),$$

which admits a fraction $1/f$ that stays a "true fraction" in each larger module $R \subset N$. In particular, each injective module I containing R fails to be flasque. More generally, we attach to each module M over some ring R and each local section $s \in \Gamma(U, \mathscr{F})$ of the sheaf $\mathscr{F} = \tilde{M}$ over some quasicompact open set U a descending chain of submodules $M_0 \supset M_1 \supset \ldots$, and show that there is a module extension $M \subset N$ in which the local section s becomes a global section if and only if $M_r = 0$ for sufficiently large indices $r \geq 0$.

In light of these observations, the quick proof for acyclicity of quasicoherent sheaves over noetherian affine schemes does not carry over to the general situation, and different arguments are needed ([7], Theorem 1.3.1 or Kempf's beautiful argument [13]). The third observation of this note is that *the general case nevertheless follows easily from the noetherian case*, by using *hypercoverings*. The latter go back to ideas of Cartier and are explained in [1], Exposé V, Section 7. Here the principal fact is that, due to their combinatorial nature, any hypercovering for $X = \text{Spec}(R)$ is defined, after refinement, in terms of only finitely many ring elements $g_1, \ldots, g_m \in R$, hence already lives over some noetherian subring.

The paper is organized as follows: In the first section, we construct the globalizing sheaves $\mathscr{F}^\circ$ and the flasque hull $\mathscr{F}^{\text{fls}}$. In the second section, we examine nonexistence of flasque module extensions $M \subset N$. In the last section we discuss how hypercoverings can be used to reduce cohomological questions for arbitrary affine schemes $X = \text{Spec}(R)$ to the noetherian case.

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1. Flasque hulls of sheaves

Let X be a ringed space. Recall that an $\mathscr{O}_X$-module $\mathscr{F}$ is called *flasque* if for each open set $U \subset X$, the restriction map

$$\Gamma(X, \mathscr{F}) \longrightarrow \Gamma(U, \mathscr{F}), \quad s \longmapsto s|U \tag{1}$$

is surjective. We now use that one may view the local sections $t \in \Gamma(U, \mathscr{F})$ as homomorphisms $t : i_!(\mathscr{O}_U) \to \mathscr{F}$. Here $i_!(\mathscr{O}_U)$ is the *extension-by-zero sheaf*, defined as the sheafification of the presheaf

$$V \longmapsto \begin{cases} \Gamma(V, \mathscr{O}_U) & \text{if } V \subset U; \\ 0 & \text{else,} \end{cases}$$

and $i : U \to X$ is the inclusion map. This interpretation reveals that injective sheaves are flasque. Moreover, flasque sheaves are acyclic, by [5], Chapter II, Theorem 4.4.3. Note that flasqueness depends only on the underlying set-valued sheaf. Furthermore, it is a local property, as explained in loc. cit., Chapter II, Section 3.1. Each $\mathscr{O}_X$-module $\mathscr{F}$ is contained in the *Godement sheaf* $\mathscr{F}^{\text{gdm}}$, defined via stalk products $\Gamma(U, \mathscr{F}^{\text{gdm}}) = \prod_{a \in U} \mathscr{F}_a$, which is flasque.

One may refine the notation of flasqueness as follows: Let us write (Open/X) for the collection of all open subsets $U \subset X$, and let $\mathfrak{U} \subset (\text{Open}/X)$ be some subcollection. We say that $\mathscr{F}$ is $\mathfrak{U}$-*flasque* if the restriction maps (1) are surjective for all open sets $U \subset X$ belonging to $\mathfrak{U}$. This becomes the usual notional of flasqueness if $\mathfrak{U}$ is the collection of all open sets. The utility of the generalization relies in the following observation:

Proposition 1.1. *Suppose the space X is quasicompact, and that $\mathfrak{U} \subset (\text{Open}/X)$ is a basis of quasicompact open sets that is stable under finite intersections and finite unions. Then every $\mathfrak{U}$-flasque $\mathscr{O}_X$-module $\mathscr{F}$ is acyclic.*

Proof. Choose an exact sequence $0 \to \mathscr{F} \to \mathscr{I} \to \mathscr{G} \to 0$ with $\mathscr{I}$ injective. We first check that $H^1(X, \mathscr{F}) = 0$, or equivalently that $H^0(X, \mathscr{I}) \to H^0(X, \mathscr{G})$ is surjective. Let $t \in \Gamma(X, \mathscr{G})$ be a global section. Since X is quasicompact, there is an open covering $X = U_1 \cup \ldots \cup X_n$ by members of the basis $\mathfrak{U}$ and local section $s_i \in \Gamma(U_i, \mathscr{I})$ mapping to $t|U_i$. We now show by induction on $0 \leq m \leq n$ that after changing the $s_1, \ldots s_m$, they coincide on the overlaps $U_{ij} = U_i \cap U_j$, whence glue to a local section over $U_1 \cup \ldots \cup U_m$. This is trivial for $m = 0$. Assume now that $m \geq 1$, and that the assertion holds for $m-1$. Set $V = U_1 \cup \ldots \cup U_{m-1}$ and $V' = U_m$, let $r \in \Gamma(V, \mathscr{I})$ be the local section with $r|U_i = s_i$, and write $r' = s_m$. The difference $r|V \cap V' - r'|V \cap V'$ lies in $\Gamma(V \cap V', \mathscr{F})$. By assumption, $V \cap V' = (U_1 \cup \ldots \cup U_{m-1}) \cap U_m$ belongs to $\mathfrak{U}$, so we may extend the difference to $V' = U_m$. Changing s_m by this extension, we may assume that the difference vanishes, and can glue r and r'.

Next we check that $\mathscr{G}$ is $\mathfrak{U}$-flasque: Let $t_U \in \Gamma(U, \mathscr{G})$ be a local section over some open set U belonging to $\mathfrak{U}$. Applying the preceding paragraph with the quasicompact space U instead of X, we see that t_U comes from a local section of $\mathscr{I}$. Using that $\mathscr{I}$ is flasque we see that t_U is the restriction of a global section.

The long exact cohomology sequence for $0 \to \mathscr{F} \to \mathscr{I} \to \mathscr{G} \to 0$ gives identifications $H^{i+1}(X, \mathscr{F}) = H^i(X, \mathscr{G})$ for all $i \geq 1$, and the assertion now follows by induction on the integer $i \geq 1$. $\square$

Now let $\mathscr{F}$ be an $\mathscr{O}_X$-module, and $s_U \in \Gamma(U, \mathscr{F})$ be local sections for some open $U \subset X$ belonging to $\mathfrak{U}$. Such local sections can be viewed as a homomorphisms of $\mathscr{O}_X$-modules $s_U : i_!(\mathscr{O}_U) \to \mathscr{F}$, where $i : U \to X$ is the inclusion map. Now form the cocartesian diagram

$$\begin{array}{ccc} \bigoplus i_!(\mathscr{O}_U) & \longrightarrow & \bigoplus \mathscr{O}_X \\ \downarrow & & \downarrow \\ \mathscr{F} & \longrightarrow & \mathscr{F}^\circ, \end{array}$$

which defines an $\mathscr{O}_X$-module $\mathscr{F}^\circ$. Here the direct sums run over the set of all local sections $s_U \in \Gamma(U, \mathscr{F})$ with $U \in \mathfrak{U}$, and the vertical map on the left is given by the corresponding homomorphisms $s_U : i_!(\mathscr{O}_U) \to \mathscr{F}$. By definition, the sheaf $\mathscr{F}^\circ$ sits in a short exact sequence

$$0 \longrightarrow \bigoplus i_!(\mathscr{O}_U) \longrightarrow \mathscr{F} \oplus \bigoplus \mathscr{O}_X \longrightarrow \mathscr{F}^\circ \longrightarrow 0. \tag{2}$$

The map on the left is the diagonal map, and the map on the right is the difference map coming from the preceding diagram. The map on the left is indeed injective, because the $i_!(\mathscr{O}_U) \to \mathscr{O}_X$ are injective, and the abelian category ($\mathscr{O}_X$-Mod) of all $\mathscr{O}_X$-modules satisfies Grothendieck's axiom (AB4), which means that sums of monomorphisms remain monomorphisms ([6], Section 1.4). According to [12], Lemma 8.3.11 this ensures that the projection $\mathscr{F} \to \mathscr{F}^\circ$ is injective, and we thus may regard the projection as an inclusion $\mathscr{F} \subset \mathscr{F}^\circ$. By construction, the local sections $s_U \in \Gamma(U, \mathscr{F})$ become the restriction of global sections $s_U^\circ \in \Gamma(X, \mathscr{F}^\circ)$ corresponding to the projections $\mathscr{O}_X \to \mathscr{F}^\circ$. From the universal property of the cocartesian square, we get:

Proposition 1.2. *For each homomorphism $\varphi : \mathscr{F} \to \mathscr{G}$ to some $\mathscr{O}_X$-module $\mathscr{G}$ and each collection $t_U^\circ \in \Gamma(X, \mathscr{G})$ with $t_U^\circ | U = \varphi(s_U)$, there is a unique homomorphism $\varphi^\circ : \mathscr{F}^\circ \to \mathscr{G}$ with $\varphi^\circ | \mathscr{F} = \varphi$ and $\varphi^\circ(s_U^\circ) = t_U^\circ$.*

In particular, if $\mathscr{F}$ is $\mathfrak{U}$-flasque, any choice of global sections extending the local sections $s_U \in \Gamma(U, \mathscr{F})$ defines a retraction for the inclusion $\mathscr{F} \subset \mathscr{F}^\circ$. By abuse of notation, one may also write

$$\mathscr{F}^\circ = \mathscr{F} + \sum \mathscr{O}_X s_U^\circ,$$

and we call it the *globalizing sheaf* for all the local sections $s_U \in \Gamma(U, \mathscr{F})$ with $U \in \mathfrak{U}$. Intuitively, we have enlarged the sheaf $\mathscr{F}$ by adding formal symbols s_U° as global sections that extend all the local sections s_U. Stalk-wise, this extends by free modules:

Proposition 1.3. *For each point $a \in X$, the cokernel for the inclusion $\mathscr{F}_a \subset (\mathscr{F}^\circ)_a$ is a free $\mathscr{O}_{X,a}$-module.*

Proof. The exact sequence of sheaves (2) induces an exact sequence of stalks. If $a \in U$, the canonical inclusion $i_!(\mathscr{O}_U)_a \subset \mathscr{O}_{X,a}$ is an equality. If $a \notin U$, we have $i_!(\mathscr{O}_U)_a = 0$. Now the assertion follows from the exact sequence (2). □

Recall that Grothendieck constructed injective objects in rather general abelian categories via transfinite induction ([6], Theorem 1.10.1). Likewise, we define a direct system of $\mathscr{O}_X$-module $\mathscr{F}_\alpha$ indexed by ordinal numbers $\alpha \geq 0$. The induction starts with $\mathscr{F}_0 = \mathscr{F}$. If $\beta = \alpha + 1$ is a successor ordinal, define $\mathscr{F}_\beta = (\mathscr{F}_\alpha)^\circ$ as globalizing sheaf. The transition maps arise from the canonical inclusion of $\mathscr{F}_\alpha$ into its globalizing sheaf. If $\lambda = \{\alpha \mid \alpha < \lambda\}$ is a limit ordinal we set $\mathscr{F}_\lambda = \varinjlim_{\alpha<\lambda} \mathscr{F}_\alpha$. The transition maps arise from the canonical inclusions into the direct limit. These are injective, according to [6], Proposition 1.8.

Proposition 1.4. *The $\mathscr{O}_X$-modules $\mathscr{F}_\lambda$ are $\mathfrak{U}$-flasque for certain limit ordinals λ.*

Proof. Let $U \subset X$ be an open set belonging to $\mathfrak{U}$, and $s_U \in \Gamma(U, \mathscr{F}_\lambda)$ be a local section. The main step is to check that it comes from some local section $\Gamma(U, \mathscr{F}_\alpha)$ with $\alpha < \lambda$. Note that taking global sections over non-noetherian topological spaces does not commute with filtered direct limits, and this is where one has to make an assumption on the ordinal λ. We need the following notion from set theory: The *cofinality* $\mathrm{cf}(P)$ of any totally ordered set P is the smallest ordinal that is order isomorphic to a cofinal subset $P' \subset P$. This applies in particular if P is a cardinal number. Let τ be the cardinality of the collection (Open/X) of open sets, and suppose that our limit ordinal λ has cofinality $\mathrm{cf}(\lambda) > \tau$. The following reasoning is a special case of Quillen's small object argument, in the form of [11], Theorem 2.1.14. See [15], Chapter II, §3, Lemma 3 for the original version.

Recall that filtered direct limits in the category of $\mathscr{O}_X$-module sheaves are obtained from the filtered direct limits in the category of $\mathscr{O}_X$-module presheaves by sheafification. Thus s_U is represented by a tuple $([s_i])_{i\in I}$ of equivalence classes

$$[s_i] \in \varinjlim_{\alpha<\lambda} \Gamma(U_i, \mathscr{F}_\alpha)$$

for some open covering $U = \bigcup_{i\in I} U_i$, and the entries coincide locally on the overlaps $U_{ij} = U_i \cap U_j$. By omitting repetitions, we may assume that the map $I \to (\mathrm{Open}/X)$ given by $i \mapsto U_i$ is injective, such that $\mathrm{Card}(I) \leq \tau$. For each $i \in I$, choose some representative $s_i \in \Gamma(U_i, \mathscr{F}_{\alpha_i})$. By our assumption $\tau < \mathrm{cf}(\lambda)$, the set $\{\alpha_i \mid i \in I\}$ is not cofinal in λ viewed as the well-ordered set $\{\alpha \mid \alpha < \lambda\}$. Thus there is some ordinal $\alpha < \lambda$ with $\alpha_i \leq \alpha$ for all $i \in I$. Replacing our representatives s_i by their images in $\Gamma(U_i, \mathscr{F}_\alpha)$, we may assume that the $\alpha_i = \alpha$ for all $i \in I$.

Choose open coverings $U_{ij} = \bigcup_k U_{ijk}$ so that the differences $s_i|U_{ijk} - s_j|U_{ijk}$ become zero in the filtered direct limit $\varinjlim_{\alpha\leq\beta<\lambda} \Gamma(U_{ijk}, \mathscr{F}_\beta)$. Again using $\mathrm{cf}(\lambda) > \tau$, we find some β independent of (i,j,k) so that the differences are zero in $\Gamma(U_{ijk}, \mathscr{F}_\beta)$. Replacing α by β, we thus may assume that $s_i|U_{ijk} = s_j|U_{ijk}$. Since $\mathscr{F}_\alpha$ is a sheaf, this means that $s_i|U_{ij} = s_j|U_{ij}$ inside $\Gamma(U_{ij}, \mathscr{F}_\alpha)$, and the tuple $(s_i)_{i\in I}$ defines a local section $s_U \in \Gamma(U, \mathscr{F}_\alpha)$. Its image in $\Gamma(U, \mathscr{F}_\beta)$, for the successor ordinal $\beta = \alpha + 1$, becomes the restriction of a global section, because $\mathscr{F}_\beta = (\mathscr{F}_\alpha)^\circ$ is a globalizing sheaf. In particular, there is a global section $s \in \Gamma(X, \mathscr{F}_\lambda)$ with $s|U = s_U$. Thus $\mathscr{F}_\lambda$ is $\mathfrak{U}$-flasque. □

Since any set of ordinals is well-ordered, there is a smallest ordinal $\alpha \geq 0$ such that $\mathscr{F}_\alpha$ is $\mathfrak{U}$-flasque. We call $\mathscr{F}^{\mathrm{fls}} = \mathscr{F}_\alpha$, together with the canonical inclusion $\mathscr{F} \subset \mathscr{F}^{\mathrm{fls}}$, the *$\mathfrak{U}$-flasque hull* of $\mathscr{F}$. Note that $\mathscr{F}^{\mathrm{fls}} = \mathscr{F}_0 = \mathscr{F}$ if $\mathscr{F}$ is already $\mathfrak{U}$-flasque. In any case, we obtain a *canonical resolution*

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}^0 \longrightarrow \mathscr{F}^1 \longrightarrow \ldots$$

by $\mathfrak{U}$-flasque sheaves $\mathscr{F}^i$, which are inductively defined as $\mathfrak{U}$-flasque hulls $\mathscr{F}^i = (\mathscr{F}^{i-1}/\mathscr{F}^{i-2})^{\mathrm{fls}}$. Here we set $\mathscr{F}^{-1} = \mathscr{F}$ and $\mathscr{F}^{-2} = 0$. It is an acyclic resolution, under the assumption of Proposition 1.1, hence $H^p(X, \mathscr{F}) = H^p(\Gamma(X, \mathscr{F}_\bullet))$. The term $\mathfrak{U}$-flasque hull is justified as follows:

Proposition 1.5. *Each homomorphism $\mathscr{F} \to \mathscr{G}$ into some $\mathfrak{U}$-flasque sheaf $\mathscr{G}$ factors over the flasque hull $\mathscr{F} \subset \mathscr{F}^{\mathrm{fls}}$.*

Proof. Choose for all local sections $t_U \in \Gamma(U, \mathscr{G})$ with $U \subset X$ belonging to $\mathfrak{U}$ some global section t_U° restricting to t_U. Using transfinite induction, together with Proposition 1.2 and the universal property of direct limits, one gets the factorization $\mathscr{F}^{\mathrm{fls}} \to \mathscr{G}$. □

We already remarked that the Godement sheaf $\mathscr{F}^{\mathrm{gdm}}$ is flasque. Moreover, each local section $t_U \in \Gamma(U, \mathscr{F}^{\mathrm{gdm}}) = \prod_{a \in U} \mathscr{F}_a$, which takes the form $t_U = (t_a)_{a \in U}$, has a canonical extension to a global section $t_U^\circ = (t_a)_{a \in X}$, by setting $t_a = 0$ for $a \notin U$. As explained above, these extensions define a canonical map $\mathscr{F}^{\mathrm{fls}} \to \mathscr{F}^{\mathrm{gdm}}$.

2. Counterexamples for modules

Given a scheme X and a quasicoherent sheaf $\mathscr{F}$, it is is natural to ask whether there is an injective homomorphism $\mathscr{F} \to \mathscr{G}$ into some quasicoherent sheaf $\mathscr{G}$ that is flasque. This indeed holds if X is noetherian. In fact, for noetherian rings R every injective R-module I gives a flasque quasicoherent sheaf $\mathscr{F} = \tilde{I}$ over $X = \mathrm{Spec}(R)$, compare [10], Chapter III, Section 3.

Now fix an arbitrary ring R. Let us call an R-module M *flasque* if the corresponding quasicoherent sheaf $\mathscr{F} = \tilde{M}$ on the affine scheme $X = \mathrm{Spec}(R)$ is flasque. By Verdier's Counterexample in [2], Exposé II, Appendix I, over the non-noetherian ring of dual numbers

$$R = k[[x, y]] \oplus \mathfrak{b} \quad \text{with} \quad \mathfrak{b} = \bigoplus_{n \geq 0} k[[x, y]]/(x, y)^n$$

there must exist some injective module I that is not flasque.

We now provide a stronger counterexample by giving an explicit non-noetherian ring R and some fraction $1/f$ that stays a "true fraction" in all module extension $R \subset N$. Fix a ground field k, and consider the residue class ring

$$R = k[T, S_0, S_1, \ldots]/\mathfrak{a},$$

where $\mathfrak{a}$ is the ideal generated by the monomials $S_i T^{i+1}$ for all $i \geq 0$. We write $f, \lambda_i \in R$ for the residue classes of the indeterminates T and S_i. Set $X = \mathrm{Spec}(R)$, and consider the basic open set $U = X_f = \mathrm{Spec}(R_f)$. The fraction $1/f \in R_f$ defines a local section $s_U \in \Gamma(U, \mathscr{O}_X)$. We refer to Neeman's textbook ([14], Chapter 3) for a nice discussion of the structure sheaf $\mathscr{O}_X$ for affine schemes.

Proposition 2.1. *Notation as above. Then there is no R-module extension $R \subset N$ so that the image of the fraction $1/f \in R_f$ under the induced map $R_f \to N_f$ takes the form $b/1 \in N_f$ for some $b \in N$.*

Proof. Seeking a contradiction, we assume that such a module extension exists. The equality of fractions $1/f = b/1$ in N_f means that $f^{r+1}b = f^r$ in N for some integer

$r \geq 0$. Now form the free R-module $R \oplus RB$, where B is a formal symbol, and consider the homomorphism $R \oplus RB \to N$ extending the inclusion $R \subset N$ via $B \mapsto b$. This gives a factorization

$$R \longrightarrow (R \oplus RB)/R(f^r - f^{r+1}B) \longrightarrow N.$$

The map on the left given by $g \mapsto (g, 0)$ is injective, because the composition is injective. In the residue class module $M' = (R \oplus RB)/R(f^r - f^{r+1}B)$, we have $0 = \lambda_r(f^r - f^{r+1}B) = \lambda_r f^r$. It follows that $\lambda_r f^r \in R$ lies in the kernel of the inclusion $R \subset N$, whence $\lambda_r f^r = 0$. On the other hand, we have a canonical projection

$$R \longrightarrow k[T, S_r]/(S_r T^{r+1})$$

given by $S_i \mapsto 0$ for $i \neq r$. By unique factorization in polynomial rings, the monomial $S_r T^r$ is not divisible by $S_r T^{r+1}$. Thus the residue class of $S_r T^r$, which is the image of $\lambda_r f^r = 0$, is non-zero, contradiction. $\square$

Note that the radical of our ideal is $\sqrt{\mathfrak{a}} = (T) \cdot (S_0, S_1, \ldots)$, such that the reduction $X_{\text{red}} = \mathbb{A}^\infty \cup \mathbb{A}^1$ is the union of an infinite-dimensional affine space and an affine line, glued at the origin. The arguments in the above proof still work if one uses the larger ideal $\mathfrak{a}' = \mathfrak{a} + (S_0, S_1, \ldots)^2$, such that $X_{\text{red}} = \mathbb{A}^1$.

Now let R be an arbitrary ring, $X = \operatorname{Spec}(R)$ the corresponding affine scheme, M be some R-module, $\mathscr{F} = \tilde{M}$ resulting quasicoherent sheaf, $U \subset X$ be a quasicompact open subset, and $s_U \in \Gamma(U, \mathscr{F})$ be some local section. Choose some open covering $U = U_1 \cup \ldots \cup U_n$ by basic open sets $U_i = X_{f_i} = \operatorname{Spec}(R_{f_i})$, for certain elements $f_i \in R$. Write $s_U | U_i = a_i / f_i^{d_i}$ with $a_i \in M$ and $d_i \geq 0$. Since $X_{f_i} = X_{f_i^{d_i}}$, we may assume $d_1 = \ldots = d_n = 1$. Furthermore, we have $a_i f_j (f_i f_j)^d = a_j f_i (f_i f_j)^d$ for some common $d \geq 0$. Writing the fractions a_i / f_i as $a_i f_i^d / f_i^{d+1}$, we finally reduce to the case $d = 0$.

We now want to describe the obstruction for the existence of some module extension $M \subset N$ so that $s_U \in \Gamma(U, \mathscr{F})$ becomes the restriction of a global section for $\mathscr{G} = \tilde{N}$. Suppose for a moment that such an extension exists. Choose some element $b \in N$ with $b/1 = a_i / f_i$ in the localizations N_{f_i}. This means that $f_i^{r+1} b = f^r a_i$ inside N for all integers $r \gg 0$. This said, we pass to the universal situation: Let B be some formal symbol, and consider the homomorphism

$$\psi_r : M \longrightarrow (M \oplus RB) / \sum_{i=1}^n R(f_i^r a_i - f_i^{r+1} B), \quad g \longmapsto (g, 0). \tag{3}$$

The kernels $M_r = \operatorname{Ker}(\psi_r)$ form a decreasing chain of submodules in M, because the $\sum_{i=1}^n R(f_i^r a_i - f_i^{r+1} B)$ form a decreasing chain of submodules in $M \oplus RB$. Since RB is a free summand, the M_r are generated by the linear combinations

$$\sum_{i=1}^n \lambda_i f_i^r a_i \in M \quad \text{satisfying} \quad \sum_{i=1}^n \lambda_i f_i^{r+1} = 0 \quad \text{in } R.$$

with coefficients $\lambda_i \in R$. This yields the following characterization:

Theorem 2.2. *Notation as above. We have $M_r = 0$ for some integer $r \geq 0$ if and only if there exists an injective homomorphism $\varphi : \mathscr{F} \to \mathscr{G}$ of quasicoherent* *sheaves on $X = \operatorname{Spec}(R)$ so that the image $\varphi_U(s_U) \in \Gamma(U, \mathscr{G})$ of the local section $s_U \in \Gamma(U, \mathscr{F})$ is the restriction of some global section.*

Proof. The condition is sufficient: If the submodule M_r vanishes, we simply take

$$N = (M \oplus RB) / \sum_{i=1}^{n} R(f_i^r a_i - f_i^{r+1} B).$$

The canonical map $M \to N$ given by $g \mapsto (g, 0)$ is then injective, and the residue class $b \in N$ of the formal symbol B satisfies $b/1 = a_i/f_i$ inside N_{f_i}. In turn, $b \in N$ defines a global section of $\mathscr{G} = \tilde{N}$ with $b|U_i = a_i/f_i = s_i$. The sheaf axiom ensures that $b|U = s_U$, were $U = U_1 \cup \ldots \cup U_n$.

The condition is also necessary: If such an extension $\mathscr{F} \subset \mathscr{G}$ exists, we set $N = \Gamma(X, \mathscr{G})$ and get an element $b \in N$ with $b/1 = a_i/f_i$ inside N_{f_i}. Then there is an integer $r \geq 0$ with $f_i^r a_i = f_i^{r+1} b$ inside N. Thus the inclusion $M \subset N$ factors over the map (3), consequently $M_r = 0$. □

From this, one may pass to the *generic situation*: Fix some $n \geq 0$, let $T_1, \ldots, T_n$ be indeterminates, and choose further indeterminates S_{ir} for $1 \leq i \leq n$ and $r \geq 0$. Consider the ring

$$R = \mathbb{Z}[T_0, \ldots, T_n, S_{ir}]/\mathfrak{a}$$

obtained by adjoining all the S_{ir} for $1 \leq i \leq n$ and $r \geq 0$, and where the ideal $\mathfrak{a}$ is generated by the relations $\sum_{i=1}^{n} S_{ir} T_i^{r+1}$ with $r \geq 0$. Write f_i, λ_{ir} for the residue classes of the indeterminates T_i, S_{ir}. Furthermore, let a_i be formal symbols, and consider the R-module

$$M = (\bigoplus_{i=1}^{n} R a_i)/U,$$

where U is the submodule generated by the elements $f_j a_i - f_i a_j$, with $i \neq j$. Let $\mathscr{F} = \tilde{M}$ be the resulting quasicoherent sheaf over $X = \operatorname{Spec}(R)$. By definition, the fractions $a_i/f_i \in M_{f_i}$ glue together and thus define a local section $s_U \in \Gamma(U, \mathscr{F})$ over $U = X_{f_1} \cup \ldots \cup X_{f_n}$. For each $r \geq 0$, the submodule $M_r \subset M$ contains $\sum_{i=1}^{n} \lambda_{ir} f_i^r a_i$. One easily sees that these elements are non-trivial, by setting $f_j = \lambda_{jr} = 0$ for $j \neq 1$. The upshot is that the R-module M does not embed into any flasque module.

Let us close this section with the following remarks: Recall that the inclusion $\operatorname{QCoh}(X) \subset (\mathscr{O}_X\text{-Mod})$ of the category of quasicoherent sheaves into the category of all $\mathscr{O}_X$-modules admits a right adjoint

$$\mathscr{G} \longmapsto Q(\mathscr{G}) = \widetilde{\Gamma(X, \mathscr{G})},$$

the *coherator*, as discussed in [2], Exposé II, Lemma 3.2 and [16], Appendix B. Given a quasicoherent sheaf $\mathscr{F}$, the inclusion $\mathscr{F} \subset \mathscr{F}^{\text{fls}}$ into the flasque hull thus corresponds to an inclusion $\mathscr{F} \subset Q(\mathscr{F}^{\text{fls}})$. Here, however, one looses control over flasqueness, because $Q(i_!(\mathscr{O}_U)) = 0$ for each open $U \subsetneq X$, at least if X is irreducible.

3. Cohomology over non-noetherian affine schemes

Let R be a ring, M be some R-module, and $\mathscr{F} = \tilde{M}$ the resulting quasicoherent sheaf on the affine scheme $X = \operatorname{Spec}(R)$. According to [7], Theorem 1.3.1 the $\mathscr{O}_X$-module $\mathscr{F}$ is acyclic. A very nice alternative proof was given by Kempf [13].

In light of the previous section, it is in general impossible to embed M into some flasque module, hence the direct arguments of [10], Chapter III, Section 3 with flasque quasicoherent resolutions over noetherian rings do not carry over to the non-noetherian situation.

However, it is easy to reduce the general case to the noetherian situation by using *hypercoverings*, and I want to outline this reduction here. As explained in [1], Exposé V, Section 7, there is a bijection

$$\varinjlim H^p(\operatorname{Hom}(K_\bullet, \mathscr{F})) \longrightarrow H^p(X, \mathscr{F})$$

of universal δ-functors. Here the direct limit runs over the filtered category of all hypercoverings $K_\bullet$ for X, the hom set is taken in the category of presheaves on X, and regarded as cochain complex. Hypercoverings are certain *semi-simplicial coverings* $K_\bullet : \Delta \to (\mathrm{Open}/X)'$, defined on the category Δ of finite sets $[n] = \{0, \ldots, n\}$ and monotonous injections $[m] \to [n]$, and taking values in the category of disjoint unions of open sets. They generalize the usual semi-simplical coverings from Čech cohomology, which are given as an $(n+1)$-fold fiber products $K_n = U \times_X \times \ldots \times_X U$, where $U = \bigcup U_i$ is the disjoint union attached to an open covering $X = U_1 \cup \ldots \cup U_r$. The $n+1$ face maps $[n-1] \to [n]$ induce the $n+1$ projections $U^{n+1} \to U^n$.

Hypercoverings $K_\bullet$ are constructed inductively as follows. One formally starts with $K_{-1} = X$. If for some degree $n \geq -1$ the terms $K_{-1}, K_0, \ldots, K_n$ are already given, one extends this *truncated semi-simplical* covering $K_{\leq n}$ to a full semi-simplical complex $L_\bullet$ by taking fiber products in the universal way, such that $\operatorname{Hom}(T_{\leq n}, K_{\leq n}) = \operatorname{Hom}(T_\bullet, L_\bullet)$ for all other semi-simplicial coverings $T_\bullet$. Then one is allowed to choose an open covering $L_{n+1} = \bigcup U_i$ and defines $K_{n+1} = \bigcup U_i$ as the corresponding disjoint union. Furthermore, one demands that $K_\bullet = L_\bullet$ for some $n \geq 0$. We refer to [1], Exposé V, Section 7 for details.

Note that in our situation $X = \operatorname{Spec}(R)$, by passing to some refinement we may assume that in each step of the construction, the open coverings for the L_{n+1} are of the form $U_1 \cup \ldots \cup U_s$ for some basic open subsets $U_i = X_{f_i}$ and some $s \geq 0$. Regarding K_{n+1} as presheaf on X, we form the hom set $\operatorname{Hom}(K_{n+1}, \mathscr{F})$ in the category of presheaves. With the preceding notation, the Yoneda Lemma gives

$$\operatorname{Hom}(K_{n+1}, \mathscr{F}) = \prod_i^s \Gamma(U_i, \mathscr{F}) = M_{f_1} \oplus \ldots \oplus M_{f_s}.$$

This construction yields the *cochain complex* $\operatorname{Hom}(K_\bullet, \mathscr{F})$, where the coboundary maps $d : \operatorname{Hom}(K_n, \mathscr{F}) \to \operatorname{Hom}(K_{n+1}, \mathscr{F})$ are the usual alternating sums of maps induced from the face maps $[n] \to [n+1]$.

Now let $[\alpha] \in H^p(X, \mathscr{F})$ be a cohomology class, for $p > 0$. Represent it by some cocycle $\alpha \in \operatorname{Hom}(K_p, \mathscr{F})$, with respect to some hypercovering $K_\bullet$ of X. In light of the description of $K_\bullet$ above, there are only *finitely many* ring elements $g_1, \ldots, g_m \in R$ needed to specify the basic open sets and their inclusion relations occurring in the successive formation of the hypercovering $\ldots \Rrightarrow K_1 \rightrightarrows K_0 \longrightarrow X$. Form the subring $R' \subset R$ generated by these $g_1, \ldots, g_m$, and consider the hypercovering $K'_\bullet$ of $X' = \operatorname{Spec}(R')$ constructed with the $g_1, \ldots, g_m \in R'$ in the same way as the hypercovering $K_\bullet$ of $X = \operatorname{Spec}(R)$. To proceed, view M as a module over R'. The resulting quasicoherent sheaf is the direct image $\mathscr{F}' = f_*(\mathscr{F})$ for the canonical morphism $f : X \to X'$. We see that our cocycle α lies in the image of the canonical map

$$\mathrm{Hom}(K'_p, \mathscr{F}') \longrightarrow \mathrm{Hom}(K_p, \mathscr{F}).$$

These maps are actually bijective, since the localizations M_{g_j} do not depend on whether one regards g_j as an element of R' or R. Thus we have an identification of cochain complexes $\mathrm{Hom}(K'_\bullet, \mathscr{F}') = \mathrm{Hom}(K_\bullet, \mathscr{F})$, and may regard α as a cocycle in $\mathrm{Hom}(K'_p, \mathscr{F}')$. Consequently, the cohomology class $[\alpha]$ lies in the image of $H^p(X', \mathscr{F}') \to H^p(X, \mathscr{F})$, which is an edge map for the Leray–Serre spectral sequence for $f : X \to X'$. Since R is noetherian and $\mathscr{F}'$ is quasicoherent, we have $H^p(X', \mathscr{F}') = 0$, by the arguments with flasque quasicoherent resolutions in the noetherian case. The upshot is that $[\alpha] = 0$ and consequently $H^p(X, \mathscr{F}) = 0$ also in the non-noetherian case.

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