A SIMPLE PROOF FOR HOCHSTER'S THEOREM

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Abstract. We give a conceptual proof for Hochster's Theorem, which asserts that each spectral space is homeomorphic to the spectrum of a ring. Given a ground field and a spectral space, our ring is constructed as filtered direct limit of prime-finite ring, which are attached in a functorial way to finite Kolmogoroff spaces. The construction simplifies an argument of Ershov along these lines. Our crucial ingredient is an assembly of finite Kolmogoroff spaces in terms of coequalizers and pushouts of one-dimensional spaces, and Schwede's observation on prime ideals in cartesian squares of rings.

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Introduction

For a ring R, the Zariski topology $\mathfrak{O}$ for the space $\operatorname{Spec}(R)$ of prime ideals is generated by the basic open sets $D(f) = \{\mathfrak{p} \mid f \not\in \mathfrak{p}\}$, where f ranges over the ring elements. This topology has two intrinsic properties: (i) The family of quasicompact open sets U_λ, $\lambda \in L$ is stable under finite intersections, and forms a basis. (ii) Each irreducible closed set Z has exactly one generic point. Note that that first condition implies that the space is quasicompact, by taking the empty intersection. Also note that spaces satisfying the second condition are called *sober spaces*. These are instances of *Kolmogoroff spaces*, where for each pair of points $a \neq b$ there is an open set that contains one but not the other point.

The topological spaces X satisfying both (i) and (ii) is called *spectral spaces*. They play an important role in the theory of locales and topoi ([8] and [13]), and the recent monograph of Dickmann, Schwartz and Tressl [4] is entirely devoted to them. Hochster's famous theorem [7] asserts that *each spectral space is homeomorphic to the spectrum of a ring*. This was established by an ingenious application of valuation theory, but one may fairly say that Hochster's proof is long and complicated. Moreover, the construction depends on choices, and does not seem to depend on the space in a functorial way.

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The finite spectral spaces X are precisely the finite Kolmogoroff spaces. Somewhat surprisingly, these have important applications in homotopy theory ([12], [16], [2], [11]). Using certain cartesian squares of rings, Lewis [10] showed that each finite Kolmogoroff space X is homeomorphic to the spectrum of a prime-finite ring R, and explicitly raises the question whether one can make R functorial in X. This indeed suggests a more conceptual and intrinsic approach to Hochster's Theorem, because the spectral spaces are precisely the filtered inverse limits of finite Kolmogoroff spaces. If one could realize such a system $(X_\lambda)_{\lambda\in L}$ with a filtered direct system $(R_\mu)_{\lambda\in L}$ of prime-finite rings, the inverse limit of spaces would be obtained from the direct limit of rings. This idea was indeed carried out by Ershov [5], but the beauty and simplicity of the reasoning is somewhat buried.

The goal of this paper is to give a simpler and clearer version of Erschov's arguments. Our approach is based on assembling finite Kolmogoroff spaces in an intrinsic way via finite Kolmogoroff spaces of dimension at most one, together with a beautiful observation of Schwede [15] on prime ideals in cartesian square of rings. Given a finite Kolmogoroff space X and a ground field k, we construct the ring $R(X)$ whose spectrum recovers the space as a certain subring

$$R(X) \subset \prod_{x\in X} k(T_U)_{U\ni x}. \tag{1}$$

Here the factors on the right are purely transcendental field extensions, where the indeterminates T_U correspond to the open neighborhoods U of the point x. The entries $(P_x)_{x\in X}$ for the ring elements in $R(X)$ belong to certain semilocal Dedekind domains with field of fractions $k(T_U)_{U\ni x}$, and their classes modulo the Jacobson radical is related to the entries P_σ, where $\sigma \in \overline{\{x\}}$, as detailed in Section 2.

I find the construction somewhat analogous to *toric varieties*, where one attaches to the combinatorial datum of a fan (N, Δ), together with the choice of a ground field k, a collection of rings $k[N\cap\sigma^\vee]$ indexed by the cones $\sigma\in\Delta$, resulting in the toric variety $\mathrm{Temb}_N(\Delta) = \bigcup_\sigma \operatorname{Spec} k[N\cap\sigma^\vee]$. Another analogy is the construction of *Chevalley groups* from root data. In our setting, the combinatorial datum is a finite ordered set $(|X|, \preccurlyeq)$, or equivalently a finite Kolmogoroff space $(|X|, \mathfrak{O})$, and the output is a prime-finite ring $R(X)$.

The kernels for the projections in (1) define a map $X \to \operatorname{Spec}(R(X))$. The construction is canonical and does not involve choices, and our main result it:

Theorem A. *(see Thm. 2.3) The maps* $X \to \operatorname{Spec}(R(X))$ *are homeomorphisms, which are natural with respect to continuous surjections of finite Kolmogoroff spaces.*

Since each spectral space X is a filtered inverse limit of finite Kolmogoroff spaces X_λ with surjective transition maps, Hochster's theorem is an immediate corollary.

We take opportunity to shed additional light on these inverse limits from a down-to-earth perspective (avoiding the patch topology and locales): Given an arbitrary topological space $X = (|X|, \mathfrak{O})$ we consider the spectral space $X^{\mathrm{sprl}} = \varprojlim X_\lambda$, where the inverse limit runs over all *finite topologies* $\mathfrak{O}_\lambda \subset \mathfrak{O}$, and the X_λ are the Kolmogoroffizations. For quasicompact X, one can project onto $X^{\mathrm{sprl}}_{\mathrm{qc}} = \varprojlim X_\mu$, where all $\mathfrak{O}_\mu$-open sets are $\mathfrak{O}$-quasicompact. This has a certain universality property:

Theorem B. *(see Thm. 3.3) For each quasicompact space X, the canonical map $X \to X^{\mathrm{sprl}}_{\mathrm{qc}}$ is universal for quasicompact continuous maps to spectral spaces.*

Note that for an arbitrary scheme S, the underlying topological space $X = |S|$ is spectral if and only if the scheme is quasicompact and quasiseparated. Also note that each morphism $f : \mathrm{Spec}(R') \to \mathrm{Spec}(R)$ of affine schemes, the underlying continuous map is quasicompact. This already suggests that in the category of spectral spaces, the quasicompact continuous maps play an important role, as already observed, for example, in [1] and [14].

The paper is organized as follows: In Section 1 we analyze how finite Kolmogoroff spaces arise via coequalizers and pushouts from finite Kolmogoroff spaces of dimension at most one. In Section 2 we attach in a functorial way to each finite Kolmogoroff space X a prime-finite ring $R(X)$ whose spectrum recovers the space. In Section 3 we discuss how spectral spaces arise as inverse limits of finite Kolmogoroff spaces, and how Hochster's Theorem follows from this.

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1. Assembly for finite Kolmogoroff spaces

Recall that a topological space X is called *Kolmogoroff* if for each pair of points $a \neq b$ there is an open set U that contains one but not the other point. This is exactly the class of spaces for which one may reconstruct the topological space from the lattice of open sets, and thus from the topos of sheaves. Note that the finite Kolmogoroff spaces $(|X|, \mathfrak{O})$ correspond to the finite ordered sets $(|X|, \preccurlyeq)$, where the order relation $a \preccurlyeq b$ corresponds to $a \in \overline{\{b\}}$. In this section we are interested in the finite Kolmogoroff spaces X.

The finite Kolmogoroff spaces X that are irreducible of dimension at most one take the form $X = \{\eta, \sigma_1, \ldots, \sigma_r\}$ with some $r \geq 0$, and the non-empty open sets are precisely the sets containing the generic point η. In turn, every finite set X with a distinguished point η can be viewed as such a Kolmogoroff space. Let us call such spaces *forks*. Note that singletons counts as forks, and that the *pointed forks* $X^* = X \smallsetminus \{\eta\}$ are discrete. Also note that each fork is homeomorphic to the spectrum of either a suitable semilocal Dedekind domain, or a field.

Let X be a finite Kolmogoroff space. Given a point $x \in X$, we define

$$\mathrm{Frk}(x) = \overline{\{x\}} = \{x, \sigma_1, \ldots, \sigma_r\}$$

and endow it with the topology that turns the set into fork with generic point x. Note that the canonical inclusion $\mathrm{Frk}(x) \to X$ is continuous, but not necessarily an embedding. Consider the diagram

$$\dot{\bigcup}_{x\in X} \mathrm{Frk}^*(x) \rightrightarrows \dot{\bigcup}_{x\in X} \mathrm{Frk}(x) \longrightarrow X, \tag{2}$$

where the upper arrow sends $\sigma \in \mathrm{Frk}^*(x)$ to the closed point $\sigma \in \mathrm{Frk}(x)$, the lower arrow sends σ to the generic point in $\mathrm{Frk}(\sigma)$, and the map on the right stems from the inclusion of forks.

Lemma 1.1. *For each finite Kolmogoroff space X, the above is a coequalizer diagram in the category of topological spaces.*

Proof. Write $f : \dot{\bigcup}_{x\in X} \operatorname{Frk}(x) \to X$ for the continuous mapping in question, and let $g : \dot{\bigcup}_{x\in X} \operatorname{Frk}(x) \to Y$ be a continuous map to another topological space Y where the two restrictions to $\dot{\bigcup}_{x\in X} \operatorname{Frk}^*(x)$ coincide, and set $g_x = g|\operatorname{Frk}(x)$. We have to show that there is a unique continuous map $h : X \to Y$ with $g = h \circ f$. Uniqueness follows from surjectivity of f. For existence, we are forced to set $h(\sigma) = g_x(\sigma)$, where $x \in X$ is a chosen point containing σ in its closure. This is indeed well-defined: Viewing σ as a generic point in $\operatorname{Frk}(\sigma)$ and as closed point in $\operatorname{Frk}(x)$ we see $g_x(\sigma) = g_\sigma(\sigma)$, which does not depend on x.

It remains to check continuity: Let $U = h^{-1}(V)$ be the preimage of an open set $V \subset Y$. Then $f^{-1}(U)$ is open, the two intersections with $\dot{\bigcup}_{x\in X} \operatorname{Frk}^*(x)$ coincide, and the task is to show that U is open. For our finite Kolmogoroff space X, this means for U is stable under generization. Fix $\sigma \in U$, and let $x \in X$ be a point with $\sigma \in \overline{\{x\}}$. Then the open set $\operatorname{Frk}(x) \cap f^{-1}(U)$ contains σ, hence also x, and thus $x \in U$. □

This expresses a given finite Kolmogoroff space X in terms of such spaces of dimension at most one. The latter immediately arise as spectra from semilocal Dedekind domains and field products, which already explains our strategy to construct prime-finite rings. To carry this out we need further information that allows induction on dimension:

Suppose the finite Kolmogoroff space X is non-empty, write $X^0 = \{\eta_1, \ldots, \eta_n\}$ for the open set of generic points, and $X' = X \smallsetminus X^0$ for the complementary closed set. The former is discrete, and the latter has $\dim(X') < \dim(X)$. Consider the commutative diagram

$$\begin{array}{ccc} \dot{\bigcup}_{\eta\in X^0} \operatorname{Frk}^*(\eta) & \longrightarrow & \dot{\bigcup}_{\eta\in X^0} \operatorname{Frk}(\eta) \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array} \tag{3}$$

where all maps stem from the canonical inclusions.

Lemma 1.2. *For each non-empty finite Kolmogoroff space X, the above diagram is cocartesian in the category of topological spaces.*

Proof. Set $W = \dot{\bigcup}_{\eta\in X^0} \operatorname{Frk}(\eta)$ and $W^* = \dot{\bigcup}_{\eta\in X^0} \operatorname{Frk}^*(\eta)$, form the amalgamated sum $P = X' \cup_{W^*} W$, and write $f : P \to X$ for the continuous map defined by the diagram (3). The upper horizontal map is an inclusion, and its complement is the set of generic points $X^0 \subset X$. We thus have a disjoint union $P = X' \cup X^0$ of sets, and conclude that the map $f : P \to X$ is bijective. It remains to verify that this continuous map is open. Let $U \subset X$ be a subset whose preimages on X' and W are open. Fix some $u \in U$ and let $x \in X$ be a generization. If $x = \eta$ is a generic point we have $x \in U$ by considering the preimage on W. If x is non-generic we likewise have $x \in U$, by looking at the intersection with X', and it follows that $U \subset X$ is open. □

2. Functorial construction of prime-finite rings

Fix a ground field k. The goal of this section is to construct for each finite Kolmogoroff space X a ring $R(X)$ together with a homeomorphisms $X \to \operatorname{Spec}(R(X))$. The crucial point of the construction is that the ring must be natural in the space, at least with respect to continuous surjections. A naive choice is the ring of all functions

$X \to K$ with values in some field extension K. This ring is obviously functorial, but fails to reflect the topology on X.

To achieve our goals choose for each open set U a formal symbol T_U. Given a point $x \in X$, we write $k[T_U]_{U \ni x}$ for the polynomial ring whose indeterminates correspond to the open neighborhoods of the point at hand. This yields a ring of finite type

$$A(X) = \prod_{x \in X} k[T_U]_{U \ni x}.$$

The construction is functorial: Given any continuous map $f : X \to Y$ of finite Kolmogoroff spaces we obtain for each $x \in X$ a ring homomorphism

$$A(Y) = \prod_{y \in Y} k[T_V]_{V \ni y} \xrightarrow{\mathrm{pr}_{f(x)}} k[T_V]_{V \ni f(x)} \longrightarrow k[T_U]_{U \ni x}, \tag{4}$$

where the map on the right sends the indeterminate T_V to the indeterminate $T_{f^{-1}(V)}$. By the universal property of products, this defines the desired ring homomorphism

$$f^* : A(Y) \longrightarrow A(X),$$

which turn $X \mapsto A(X)$ into a functor.

Next recall that a ring element is called *regular*, if the corresponding homothety is injective (the term "non-zero divisor" is also in frequent usage). For the ring $A(X)$, an element $(P_x)_{x \in X}$ is regular if and only if the entries P_x are non-zero.

Proposition 2.1. *If the continuous map $f : X \to Y$ is surjective, the ring homomorphism $f^* : A(Y) \to A(X)$ sends regular elements to regular elements.*

Proof. Let $(Q_y)_{y \in Y} \in A(Y)$ be an elements whose image $(P_x)_{x \in X} \in A(X)$ has vanishing entry P_a for some $a \in X$. From (4) we see that this entry depends only on Q_b, where $b = f(a)$. Let $V_i \subset Y$, $1 \leq i \leq r$ be the open neighborhoods of $b \in Y$. Their preimage $U_i = f^{-1}(V_i)$ are pairwise different, because $f : X \to Y$ is surjective. In turn, the elements $T_{U_i} \in k[T_U]_{U \ni x}$ are algebraically independent. Since $P_a = Q_b(T_{U_1}, \ldots, T_{U_r})$ vanishes, the same holds for $Q_b = Q_b(T_{V_1}, \ldots, T_{V_r})$. So if the image $(P_x)_{x \in X}$ is non-regular, the same holds for the argument $(Q_y)_{y \in Y}$. $\square$

Passing to localizations with respect to the multiplicative system $S(X)$ of all regular elements, we see that the formation of the field products

$$F(X) = \prod_{x \in X} k(T_U)_{U \ni x} = S(X)^{-1} A(X)$$

is functorial with respect to continuous surjections. Its spectrum gives back the *discretization* X^{dsc}, where every subset of the underlying topological space $|X|$ is open.

Given $x \in X$, we consider $\mathrm{Frk}(x) = \{x, \sigma_1, \ldots, \sigma_r\}$, whose underlying set is $\overline{\{x\}}$. For each $\sigma = \sigma_i$ we chosen an open set U_x that contains x but not σ. Following Ershov ([5], Section 2) we consider inside $k(T_U)_{U \ni x}$ the discrete valuation ring

$$R^x_\sigma = S^{-1}_{x,\sigma} \left(k(T_U, \frac{T_{U'}}{T_{U''}})[T_{U_x}] \right),$$

where U ranges over all open sets containing σ, while U', U'' run over all open sets containing x but not σ, and $S_{x,\sigma}$ is the complementary multiplicative system for the maximal ideal $\mathfrak{m}_{x,\sigma} = (T_{U_x})$. From $T_{U'} = \frac{T_{U'}}{T_{U''}} \cdot T_{U_x}$ with $U'' = U_x$ we see $\mathrm{Frac}(R^x_\sigma) = k(T_U)_{U \ni x}$. Also note that the above definition does not depend on the choice of U_x, because for any other such $\tilde{U}_x$ we have $T_{\tilde{U}_x} = \frac{T_{V'}}{T_{V''}} \cdot T_{U_x}$, with the open sets $V' = U_x$ and $V'' = \tilde{U}_x$. Of course, one may choose U_x as the smallest open neighborhood of x, but the extra flexibility will be useful in what follows. We also remark in passing that $R^x_\sigma \subset k(T_U)_{U\ni x}$ appear as localizations of some blow-up rings of $k[T_U]_{U\ni x}$. It might be interesting to pursue this line of thought.

We now can form the the intersection ring

$$R^x = R^x_{\sigma_1} \cap \ldots \cap R^x_{\sigma_r}.$$

If $x \in X$ is already a closed point, the index set is empty, and the above has to be interpreted as the field $k(T_U)_{U\ni x}$. Otherwise, the intersection ring is semilocal, with maximal ideals $\mathfrak{m}_i = \mathfrak{m}_{R^x_{\sigma_i}} \cap R^x$, having $(R^x)_{\mathfrak{m}_i} = R^x_{\sigma_i}$ and thus $\operatorname{Frac}(R^x) = k(T_U)_{U\ni x}$, as explained in [3], Chapter V, §7, No. 1. One infers with the Nakayama Lemma that every ideal $\mathfrak{a} \subset R^x$ is finitely generated, hence R^x is a semilocal Dedekind domain, and in particular *prime-finite*, meaning that the rings contains only finitely many prime ideals. The quotient by the Jacobson radical takes the form

$$\overline{R^x} = R^x / \operatorname{Jac}(R^x) = \prod_{\sigma\in\operatorname{Frk}^*(x)} k(T_U, \frac{T_{U'}}{T_{U''}}). \tag{5}$$

Note that in each factor on the right, the collection of open sets U, U', U'' depends on the index σ, although this is not reflected by our notation. Also note that the spectrum of $\prod_{x\in X} R^x$ gives back $X^{\text{frk}} = \dot{\bigcup}_{x\in X} \operatorname{Frk}(x)$, which one may call the *forkification*.

We now exploit that (5) contains $\prod_{\sigma\in\operatorname{Frk}^*(x)} k(T_U)_{U\ni\sigma}$ as a subring, and therefore also the $\prod_{\sigma\in\operatorname{Frk}^*(x)} R^\sigma$. This allows us to form the subring

$$R(X) \subset \prod_{x\in X} R^x$$

of all tuples $(P_x)_{x\in X}$ such that the entries for all the non-closed $x \in X$ satisfies the congruence condition

$$\overline{P_x} = (P_{\sigma_1}, \ldots, P_{\sigma_r}), \tag{6}$$

where the left hand side is the residue class in $\overline{R^x} = R^x / \operatorname{Jac}(R^x)$, and the tuple on the right is indexed by the points $\sigma_i \in \operatorname{Frk}^*(x)$. The subring keeps the desired functoriality properties:

Proposition 2.2. *For each continuous surjection $f : X \to Y$ of finite Kolmogoroff spaces, the induced homomorphism $f^* : F(Y) \to F(X)$ sends the subring $R(Y)$ to the subring $R(X)$.*

Proof. We first check that the homomorphism f^* sends the product ring $\prod_{y\in Y} R^y$ to the product ring $\prod_{x\in X} R^x$. Let $(P_x)_{x\in X}$ be the image of some tuple $(Q_y)_{y\in Y}$ with entries $Q_y \in R^y$. Fix a point $x \in X$. The task is to verify $P_x \in R^x$. According to (4), the entry P_x depends only on Q_y for $y = f(x)$. There is nothing to show if $x \in X$ is closed. Suppose now that x is non-closed, and fix some $\sigma = \sigma_i$ from $\operatorname{Frk}(x) = \{x, \sigma_1, \ldots, \sigma_r\}$. Continuity ensures that $\tau = f(\sigma)$ is a specialization of y. Recall that Q_y is a rational function in the T_V, $V \ni y$ and P_x is the corresponding rational function in the $T_{f^{-1}(V)}$. In case $\tau = y$ we have $x \in f^{-1}(V)$, which immediately gives $P_x \in R^x_\sigma$. Suppose now that $\tau \neq y$, choose an open set V_y that contains y but not τ, and write $Q_y = \sum_{j=0}^n \beta_j T_{V_y}$. The coefficients β_j are rational functions in T_V

and $T_{V'}/T_{V''}$, where V ranges over the open neighborhoods of τ, and V', V'' run over the open sets that contain y but not σ. The preimages

$$U_x = f^{-1}(V_y), \quad U = f^{-1}(V), \quad U' = f^{-1}(V') \quad \text{and} \quad U'' = f^{-1}(U'').$$

have the analogous properties with respect to $x, \sigma \in X$, which gives $P_x \in R^x_\sigma$. This holds for all $\sigma = \sigma_i$, and thus $P_x \in R^x$.

It remains to show that $(P_x)_{x\in X}$ satisfies the conditions (6), provided that the analogous conditions hold for $(Q_y)_{y\in Y}$. As in the previous paragraph fix $x, \sigma \in X$ and form the images $y, \tau \in Y$. The task is to verify that the residue class $\overline{P_x} \in R^x_\sigma/\operatorname{Rad}(R^x_\sigma)$ coincides with P_σ. For $\tau = y$ this is obvious, because Q_y is a rational function in the T_V, $V \ni y$ and both P_x and P_σ are the corresponding rational function in the $T_{f^{-1}(V)}$. Suppose now $\tau \neq y$, and write $Q_y = \sum_{j=0}^n \beta_j T_{V_y}$ as in the preceding paragraph, now with $\beta_0 = Q_\sigma$. Then $U_x = f^{-1}(V_y)$ is an open set that contains x but not σ, whence T_{U_x} is a uniformizer in R^x_σ, and therefore

$$\overline{P_x} = \beta_0(T_{f^{-1}(V)}, T_{f^{-1}(V')}/T_{f^{-1}(V'')}) = Q_\sigma(T_{f^{-1}(W)}) = P_\sigma,$$

with $\beta_0 = \beta_0(T_V, T_{V'}/T_{V''})$ and $Q_\sigma = Q_\sigma(T_W)$, $W \ni \sigma$. □

From the projections $\operatorname{pr}_a : F(X) \to k(T_U)_{U\ni a}$ we obtain a canonical map

$$g_X : X \longrightarrow \operatorname{Spec}(R(X)), \quad a \longmapsto \operatorname{Ker}(\operatorname{pr}_a | R(X)). \tag{7}$$

We now can formulate the main result of this paper:

Theorem 2.3. *For each finite Kolmogoroff space X, the above map is a homeomorphism. Moreover, the formation of the ring $R(X)$ is functorial and the homeomorphism g_X is natural, both with respect to continuous surjections $f : X \to Y$ of finite Kolmogoroff spaces.*

Proof. The formation of $R(X)$ is functorial in continuous surjections of finite Kolmogoroff spaces by Proposition 2.2. One easily sees that the diagram

$$\begin{array}{ccc} X^{\text{dsc}} & \xrightarrow{g_X^{\text{dsc}}} & \operatorname{Spec}(F(X)) \\ \downarrow & & \downarrow \\ X & \xrightarrow[g_X]{} & \operatorname{Spec}(R(X)) \end{array}$$

is commutative, where X^{dsc} is the discretization of X, and the upper horizontal map g_X^{dsc} is defined in the same way as g_X. The naturality of the latter is obvious. Since $X^{\text{dsc}} \to X$ is surjective, naturality of g_X follows.

It remains to verify that g_X is a homeomorphism. To this end consider the descending chain of closed sets $X = X_0 \supset X_1 \supset \ldots \supset X_n \supset X_{n+1} = \varnothing$, where $X_{i+1} = X_i \smallsetminus X_i^0$ is complement of the set of generic points $\eta \in X_i$, and $n = \dim(X)$. Consider the subrings $R_i \subset \prod_{x\in X_i} R^x$ defined by the condition (6), and the resulting maps

$$g_i : X_i \longrightarrow \operatorname{Spec}(R_i), \quad a \longmapsto \operatorname{Ker}(\operatorname{pr}_a | R_i),$$

as in (7), which has $g_X = g_0$. We now show by descending induction on $i \leq n$ that the g_i are homeomorphisms. This is obvious for $i = n$, because the space X_n is discrete and the ring R_n is a matching product of fields. Suppose now $i < n$, and that g_{i+1} is a homeomorphism. By definition, our rings sit in a cartesian square

$$\begin{array}{ccc} \prod_{\eta\in X_i^0} \overline{R^\eta} & \longleftarrow & \prod_{\eta\in X_i^0} R^\eta \\ \uparrow & & \uparrow \\ R_{i+1} & \longleftarrow & R_i. \end{array}$$

The upper horizontal map is surjective. We arrive at two cocartesian squares

$$\begin{array}{ccc} \dot\bigcup \operatorname{Frk}^*(\eta) & \longrightarrow & \dot\bigcup \operatorname{Frk}(\eta) \\ \downarrow & & \downarrow \\ X_{i+1} & \longrightarrow & X_i. \end{array} \quad \text{and} \quad \begin{array}{ccc} \operatorname{Spec}(\prod \overline{R^\eta}) & \longrightarrow & \operatorname{Spec}(\prod R^\eta) \\ \downarrow & & \downarrow \\ \operatorname{Spec}(R_{i+1}) & \longrightarrow & \operatorname{Spec}(R_i) \end{array}$$

of spaces, the right by Lemma 1.2, and the left according to [15], Theorem 3.3. Here the unions and products run over all generic points $\eta \in X_i$. The obvious identification

$$\dot\bigcup \operatorname{Frk}^*(\eta) = \operatorname{Spec}(\prod \overline{R^\eta}) \quad \text{and} \quad \dot\bigcup \operatorname{Frk}(\eta) = \operatorname{Spec}(\prod R^\eta),$$

together with our g_{i+1} and g_i, define a map of squares. By our induction hypothesis, g_{i+1} is a homeomorphism, and the universal property of cocartesian squares ensures that g_i is a homeomorphism. □

3. Spectralization functors and Hochster's Theorem

Let X be a topological space. Consider the equivalence relation $x \sim x'$ given by the condition that for every open set U we have $x \in U \Leftrightarrow x' \in U$, and write X^{kol} for the ensuing quotient space. One easy checks that this is a Kolmogoroff space, and that the projection $X \to X^{\text{kol}}$ is universal for continuous maps to Kolmogoroff spaces. The goal of this section is to construct in a related way certain spectral spaces X^{sprl} and $X^{\text{sprl}}_{\text{qc}}$, and examine their universal properties.

Write $\mathfrak{O}$ for the topology on X, and consider the collection of all coarser topologies $\mathfrak{O}_\lambda \subset \mathfrak{O}$, $\lambda \in L$ subject to $|\mathfrak{O}_\lambda| < \infty$. We regard L as an ordered set, where $\lambda \preccurlyeq \mu$ means $\mathfrak{O}_\lambda \subset \mathfrak{O}_\mu$. The ordered set L is filtered, because the topology generated by two topologies with only finitely many open sets has only finitely many open sets. The ensuing $X_\lambda = (|X|, \mathfrak{O}_\lambda)^{\text{kol}}$ form a filtered inverse system of Kolmogoroff spaces, and we define

$$X^{\text{sprl}} = \varprojlim_{\lambda\in L} X_\lambda.$$

The quotient maps $X \to X_\lambda$ are compatible, and therefore induce a continuous map $X \to X^{\text{sprl}}$. The construction is functorial: Given a continuous mapping $f : X' \to X$ we obtain $f^* : L \to L'$ by writing $f^{-1}(\mathfrak{O}_\lambda) = \mathfrak{O}_{f^*(\lambda)}$, with resulting continuous mapping $f_\lambda : X'_{f^*(\lambda)} \to X_\lambda$. The ensuing compositions

$$\prod_{\lambda'\in L'} X'_{\lambda'} \xrightarrow{\text{pr}} X'_{f^*(\lambda)} \xrightarrow{f_\lambda} X_\lambda$$

define a continuous map from the products of the $X'_{\lambda'}$ to the product of the X_λ, which induces the desired map continuous map $f^{\text{sprl}} : X'^{\text{sprl}} \to X^{\text{sprl}}$. One easily checks that the diagram

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ X'^{\mathrm{sprl}} & \xrightarrow[f^{\mathrm{sprl}}]{} & X^{\mathrm{sprl}} \end{array}$$

is commutative.

Proposition 3.1. *In the above situation, the X_λ, $\lambda \in L$ are finite Kolmogoroff spaces, the transition maps $X_\mu \to X_\lambda$, $\lambda \preccurlyeq \mu$ are surjective, the inverse limit X^{sprl} is a spectral space, and the map $X \to X^{\mathrm{sprl}}$ has dense image.*

Proof. By construction, the $Y = X_\lambda$ are Kolmogoroff, and have only finitely many open sets. The latter ensures that each point y has a smallest open neighborhood V_y, the former tells us that the map $y \mapsto V_y$ is injective, and we conclude that Y is finite. The surjectivity of the projections $X \to X_\lambda$ ensures that the transition maps are surjective. Given an irreducible closed set $Z \subset X^{\mathrm{sprl}}$, the images under projection have a unique generic point η_λ. One easily checks that the family $\eta = (\eta_\lambda)$ is compatible, defines a generic point of Z, and that there is no other generic point. To see density of the image we fix a point $(x_\lambda) \in X^{\mathrm{sprl}}$. The basic open neighborhoods take the form

$$V = \mathrm{pr}_{\mu_1}^{-1}(V_{\mu_1}) \cap \ldots \cap \mathrm{pr}_{\mu_r}^{-1}(V_{\mu_r})$$

for some indices $\mu_1, \ldots, \mu_r \in L$ and some open neighborhoods V_{μ_i} of $x_{\mu_i} \in X_\mu$. Passing to some $\mu \succcurlyeq \mu_1, \ldots, \mu_r$, we are reduced to the case $r = 1$, and set $\mu = \mu_1$. Then $V = \mathrm{pr}_\mu^{-1}(V_\mu)$ contains image points, because $X \to X_\mu$ is surjective. □

Now let X be a quasicompact space. We repeat the above construction, but allow only $\mathfrak{O}_\lambda$ whose members U are quasicompact with respect to the original topology $\mathfrak{O}$. Write $L_{\mathrm{qc}} \subset L$ for the corresponding set of indices, and define

$$X_{\mathrm{qc}}^{\mathrm{sprl}} = \varprojlim_{\lambda \in L_{\mathrm{qc}}} X_\lambda.$$

As above, this defines a spectral space. One easily checks that the canonical continuous map $X \to X_{\mathrm{qc}}^{\mathrm{sprl}}$ is quasicompact, and that the construction is functorial for continuous maps $f : X' \to X$ that are quasicompact.

Proposition 3.2. *For each spectral space X, the ordered set L_{qc} is filtered, and the canonical map $p : X \to X_{\mathrm{qc}}^{\mathrm{sprl}}$ is a homeomorphism.*

Proof. Given $\lambda, \lambda' \in L_{\mathrm{qc}}$, let $\mathfrak{O}_\mu$ be the topology formed by the intersections $U \cap U'$ with $U \in \mathfrak{O}_\lambda$ and $U' \in \mathfrak{O}_{\lambda'}$. These intersections are quasicompact, because X is spectral, hence $\mu \in L_{\mathrm{qc}}$, and the index set L_{qc} is filtered.

To proceed, we write $p_\lambda : X \to X_\lambda$ for the quotient maps and $p_{\mu\lambda} : X_\mu \to X_\lambda$ for the transition maps. We first check that $p : X \to X_{\mathrm{qc}}^{\mathrm{sprl}}$ is injective: Given points $a \neq b$, we find some quasicompact open set U with $a \in U$ and $b \notin U$, after swapping the points if necessary. The three open sets $X, U, \varnothing$ constitute a topology of the form $\mathfrak{O}_\mu$ for some $\mu \in L_{\mathrm{qc}}$, and $X_\lambda = \{\eta, \sigma\}$ can be identified with the *Sierpinski space.* Then $p_\mu(a) = \eta \neq \sigma = p_\mu(b)$, hence p is injective.

The main task is to verify that $p : X \to X^{\rm sprl}_{\rm qc}$ surjective: Let $(z_\lambda)_{\lambda\in L_{\rm qc}}$ be a point from $X^{\rm sprl}_{\rm qc}$. Then the irreducible closed sets $Z_\lambda = \overline{\{z_\lambda\}}$ satisfy

$$p_{\mu\lambda}(Z_\mu) \subset Z_\lambda \quad \text{and} \quad p_{\mu\lambda}^{-1}(X_\lambda \smallsetminus Z_\lambda) \subset X_\mu \smallsetminus Z_\mu. \tag{8}$$

Consider the quasicompact open sets $U_\lambda = p_\lambda^{-1}(X_\lambda \smallsetminus Z_\lambda)$ and the closed set $Z = X \smallsetminus \bigcup U_\lambda$. We first observe that Z is non-empty: Otherwise X is covered by the U_λ, and quasicompactness gives $X = U_{\lambda_1} \cup \ldots \cup U_{\lambda_r}$. Using that $L_{\rm qc}$ is filtered together with (8), we are reduced to the case $r = 1$. Setting $\mu = \lambda_1$, we see that X equals the preimage of $X_\mu \smallsetminus Z_\mu$, in contradiction to the surjectivity of $X \to X_\mu$.

We next check that Z is irreducible: Otherwise there are two quasicompact open sets $V, V' \subset X$, both intersecting Z but having $V \cap V' \subset U$. The open sets $X, V, V', \varnothing$ constitute a topology of the form $\mathfrak{O}_\lambda$ for some $\lambda \in L_{\rm qc}$, and the resulting $X_\lambda = \{\eta, \zeta, \zeta', \sigma\}$ has

$$p_\lambda^{-1}(\{\eta,\zeta\}) = V \quad \text{and} \quad p_\lambda^{-1}(\{\eta,\zeta'\}) = V' \quad \text{and} \quad p_\lambda^{-1}(\{\eta\}) = V \cap V'.$$

From this we infer $z_\lambda = \eta$. Using that $V \cap V'$ is quasicompact, we find some $\mu \succcurlyeq \lambda$ in $L_{\rm qc}$ such that $V \cap V' \subset U_\mu$. Choose $x \in X$ with $p_\mu(x) = z_\mu$. Then x does not belong to $U_\mu = X \smallsetminus p_\mu^{-1}(Z_\mu)$, therefore $x \notin V \cap V'$, consequently $p_\lambda(x) \neq \eta = z_\lambda = p_{\mu\lambda}(z_\mu) = p_{\mu\lambda}(p_\mu(x)) = p_\lambda(x)$, contradiction.

Since X is spectral, the irreducible closed set Z has a unique generic point z, and we indeed have $p(z) = (z_\lambda)_{\lambda\in L_{\rm qc}}$. To see this, fix an index λ. Since X_λ embeds into a finite product of Sierpinski spaces, it suffices to treat the case that $X_\lambda = \{\eta, \sigma\}$ equals the Sierpinski space, where the map p_λ is already determined by the quasicompact open set $W = p_\lambda^{-1}(\eta)$. Choose $x \in X$ with $p_\lambda(x) = z_\lambda$. If $x, z \in W$ or $x, z \notin W$ we have $p_\lambda(z) = p_\lambda(x) = z_\lambda$. The case $z \in W, x \notin W$ is impossible, because then $z_\lambda = p(x) = \sigma$ and thus $W = p_\lambda^{-1}(X \smallsetminus \{z_\lambda\}) = U_\lambda \subset U$, in contradiction to $z \in W$. It remains to rule out $z \notin W, x \in W$, when $p_\lambda(x) = \eta$. Using that W is quasicompact, we find some $\mu \succcurlyeq \lambda$ in $L_{\rm qc}$ such that $W \subset U_\mu$. Choose $x' \in X$ with $p_\mu(x') = z_\mu$. Obviously, x' is not contained in $U_\lambda = p_\lambda^{-1}(X_\lambda \smallsetminus Z_\lambda)$ and in particular not in W, and thus $p_\lambda(x') = \sigma$. On the other hand, we have $p_\lambda(x') = p_{\mu\lambda}(p_\mu(x')) = p_{\mu\lambda}(z_\mu) = z_\lambda = p_\lambda(x) = \eta$, contradiction.

Summing up, $p : X \to X^{\rm sprl}_{\rm qc}$ is a continuous bijection, and it remains to check that for each open $U \subset X$, the image is open. It suffices to treat the case that U is quasicompact, and furthermore $U \neq X, \varnothing$. Then $X, U, \varnothing$ constitute a topology of the form $\mathfrak{O}_\lambda$ for some $\lambda \in L_{\rm qc}$, the resulting $X_\lambda = \{\eta, \sigma\}$ is a copy of the Sierpinski space, and $U = p_\lambda^{-1}(\eta)$. Surjectivity of p gives $p(p_\lambda^{-1}(\eta)) = p_\lambda^{-1}(\eta)$, which is open. □

In different form, the following universality property already appears in [1], Proposition 4 and [14], Theorem 5.1:

Theorem 3.3. *For each quasicompact space X, the map $X \to X^{\rm sprl}_{\rm qc}$ is universal for quasicompact continuous maps to spectral spaces.*

Proof. Let Y be a spectral space and $f : X \to Y$ be quasicompact and continuous. We then have a commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & X^{\text{sprl}} & \dashrightarrow & X^{\text{sprl}}_{\text{qc}} \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & Y^{\text{sprl}} & \longrightarrow & Y^{\text{sprl}}_{\text{qc}} \end{array}$$

The desired factorization of f over $X^{\text{sprl}}_{\text{qc}}$ is given applying the vertical map on the right, followed by the inverse of the lower composition. Note that in the diagram, the undashed arrows exists in general, while the dashed arrows rely on quasicompactness of the space X and the map $f : X \to Y$.

It remains to verify uniqueness. In light of the identification $Y = \varprojlim Y_\lambda$ it suffices to treat the case that Y is finite. Since a finite Kolmogoroff spaces sits inside a finite product of the Sierpinski space $S = \{\eta, \sigma\}$, we may assume that Y itself equals the Sierpinski space. Now continuous maps to Y can be identified with open sets in the domain, and our task is to show that two quasicompact open sets $V, W \subset X^{\text{sprl}}_{\text{qc}}$ whose preimages on X coincide already coincide. Write

$$V = \bigcup_{i=1}^{r} \text{pr}_{\alpha_i}^{-1}(V_i) \quad \text{and} \quad W = \bigcup_{j=1}^{s} \text{pr}_{\beta_j}^{-1}(W_j) \tag{9}$$

for certain indices $\alpha_i, \beta_j \in L_{\text{qc}}$ and open sets $V_i \subset X_{\alpha_i}$ and $W_j \subset X_{\beta_j}$. Choose some $\lambda \in L_{\text{qc}}$ so that $\mathfrak{O}_\lambda$ is finer than all the $\mathfrak{O}_{\alpha_i}$ and $\mathfrak{O}_{\beta_j}$. Then the common preimage $U \subset X$ of (9) belongs to $\mathfrak{O}_\lambda$, and thus corresponds to an open set $U_\lambda \subset X_\lambda$. By construction, both open sets in (9) coincide with $\text{pr}_\lambda^{-1}(U_\lambda)$. $\square$

Now fix a ground field k. For each spectral space X, the filtered inverse system $(X_\lambda)_{\lambda \in L_{\text{qc}}}$ of finite Kolmogoroff spaces with surjective transition maps yields a filtered direct system $(R_\lambda)_{\lambda \in L_{\text{qc}}}$ of prime-finite rings $R_\lambda = R(X_\lambda)$, coming with compatible homeomorphisms $X_\lambda \to \text{Spec}(R_\lambda)$, as constructed in Section 2. We thus get continuous maps

$$X \longrightarrow \varprojlim X_\lambda \longrightarrow \varprojlim \text{Spec}(R_\lambda) \longrightarrow \text{Spec}(\varinjlim R_\lambda).$$

where the indices run over L_{qc}. The map in the middle is a homeomorphism (Theorem 2.3). The same holds for the map on the right ([6], Corollary 8.2.10) and the map on the left (Proposition 3.2). This gives the desired simple proof of Hochster's theorem ([7], compare also [4], Section 12.6 and [5]):

Theorem 3.4. *Each spectral space is homeomorphic to the spectrum of a ring.*

Note that the ring $R(X) = \varinjlim R(X_\lambda)$ is canonically constructed out of nothing but the spectral space X and the ground field k. One may wonder whether there are any ties with reconstructions of algebraic schemes from the Zariski topology [9], or what happens if one works over other ground rings rather than ground fields, in particular with the ring of integers $\mathbb{Z}$.

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