ENRIQUES SURFACES WITH NORMAL K3-LIKE COVERINGS

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Abstract. We analyze the structure of simply-connected Enriques surfaces in characteristic two whose K3-like coverings are normal, building on the work of Ekedahl, Hyland and Shepherd-Barron. We develop general methods to construct such surfaces and the resulting twistor lines in the moduli stack of Enriques surfaces, including the case that the K3-like covering is a normal rational surface rather then a normal K3 surface. Among other things, we show that elliptic double points indeed do occur. In this case, there is only one singularity. The main idea is to apply flops to Frobenius pullbacks of rational elliptic surfaces, to get the desired K3-like covering. Our results hinge on Lang's classification of rational elliptic surfaces, the determination of their Mordell–Weil lattices by Shioda and Oguiso, and the behavior of unstable fibers under Frobenius pullback via Ogg's Formula. Along the way, we develop a general theory of Zariski singularities in arbitrary dimension, which is tightly interwoven with the theory of height-one group schemes actions and restricted Lie algebras. Furthermore, we determine under what conditions tangent sheaves are locally free, and introduce a theory of canonical coverings for arbitrary proper algebraic schemes.

Contents

Introduction

A central result in the classification of algebraic surfaces is that there are only four types of algebraic surfaces S with first Chern class $c_1 = 0$. These are the abelian surfaces,

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the bielliptic surfaces, the K3 surfaces and the Enriques surfaces. They are distinguished by their second Betti numbers, which take the respective values

$$b_2 = 6, \quad b_2 = 2, \quad b_2 = 22 \quad \text{and} \quad b_2 = 10.$$

This holds regardless of the characteristic. Over the complex numbers, the bielliptic surfaces are quotients of abelian surfaces by a finite free group action, and the Enriques surfaces $Y = X/G$ are quotients of K3 surfaces X by free involutions.

This no longer holds true literally when the characteristic $p \geq 0$ of the ground field k divides the order of the group G. A fundamental insight of Bombieri and Mumford [7] was that much carries over if one replaces finite groups by *finite group schemes*. This comes with the price that singularities appear on the covering X, although the quotient $Y = X/G$ is smooth. In some sense, the non-smoothness of X and G cancel each other. This phenomenon has numerous applications. For example, I have used this effect to give the correct Kummer construction in characteristic $p = 2$, by replacing an abelian surface by the self-product of a rational cuspidal curve [49].

For Enriques surfaces Y, the numerically trivial part $P = \operatorname{Pic}^{\tau}_{Y/k}$ of the Picard scheme is a group scheme of order two. By the Tate–Oort classification of finite group schemes of prime order [54], there are three possibilities in characteristic $p = 2$. In the case where the group scheme P is unipotent, one says that Y is a *simply-connected Enriques surface*. These surface come along with a torsor $\epsilon : X \to Y$ with respect to the Cartier dual $G = \underline{\operatorname{Hom}}(P, \mathbb{G}_m)$, which here indeed is a local group scheme. The torsor takes over the role of the universal covering, and is therefore called the *K3-like covering*. This is a very adept terminology, because the K3-like covering has the same cohomological properties of a K3 surface, yet is never a smooth surface.

The goal of this paper is to analyze the structure of K3-like coverings X, and the ensuing simply connected Enriques surfaces Y. These where already investigated, among others, by Bombieri and Mumford [6], Blass [4], Lang [28, 29], Cossec and Dolgachev [12], Ekedahl and Shepherd-Barron [16] in the non-normal case, Ekedahl, Hyland and Shepherd Barron [17] in the normal case, Katsura and Kondo [26], and Liedtke [33]. There are, however, many open foundational questions. In this paper, we shall concentrate on *normal K3-like coverings*, although much of our theory works in general. An open question was whether non-rational singularities could appear on a K3-like covering X. One main result of this paper is that they do:

Theorem. (Compare Theorem 13.3) *There are normal K3-like coverings X containing an elliptic double point, which is obtained from the contraction of a rational cuspidal (-1)-curve.*

These were subsequently further studied by Matsumoto [36]. We actually give a systematic way to produce normal K3-like coverings, not based on techniques using explicit equations and rational vector fields, but on a procedure combining Frobenius base-changes with flops, a method borrowed from the birational geometry of higher-dimensional varieties. If a non-rational singularity appears, the situation is rather special:

Theorem. (Compare Theorem 14.1) *If a normal K3-like covering X contains a non-rational singularity, then there are no other singularities on X.*

Each elliptic fibration $\varphi : Y \to \mathbb{P}^1$ on the Enriques surface comes with a jacobian fibration, which is a rational elliptic surface $J \to \mathbb{P}^1$. According to the result of Liu, Lorenzini and Raynaud [34], the Kodaira types of all fibers Y_b and J_b coincide. The latter were classified by Persson [44] and Miranda [38] over the complex numbers, which was extended by Lang in [30, 31] to characteristic two. There are 110 families with only reduced fibers. We can show that, with a few uncovered cases, all these arise from simply-connected Enriques surfaces:

Theorem. (Compare Theorem 15.4) *For each rational elliptic surface $J \to \mathbb{P}^1$ with only reduced fibers, with the possible exception of six cases, there is a simply-connected Enriques surface Y whose K3-like covering X is birational to the Frobenius pullback $X' = J^{(2/\mathbb{P}^1)}$, and having an elliptic fibration $\varphi : Y \to \mathbb{P}^1$ whose jacobian is $J \to \mathbb{P}^1$. Moreover, the induced fibration $f : X \to \mathbb{P}^1$ induces an injection $H^0(X, \Theta_{X/k}) \subset H^0(\mathbb{P}^1, \Theta_{\mathbb{P}^1/k})$.*

This relies on Shioda's theory of Mordell–Weil lattices [51] and their classification for rational elliptic surfaces by Oguiso and Shioda [42]. The passage

$$X' \longleftarrow S \longrightarrow X$$

from the Frobenius pullback $X' = J^{(2/\mathbb{P}^1)}$ to a K3-like covering X is the fundamental idea of the paper. It should be seen as a *flop*, a notion coming from the minimal model program for the classification of higher-dimensional algebraic varieties. Roughly speaking, it describes the passage from one minimal model to another, without changing the canonical class.

Ekedahl, Hyland and Shepherd Barron showed in their groundbreaking paper [17] that for simply-connected Enriques surfaces Y whose K3-like covering X has only rational double points, the tangent sheaf $\Theta_{X/k} = \underline{\mathrm{Hom}}(\Omega^1_{X/k}, \mathscr{O}_X)$ is locally free, in fact isomorphic to $\mathscr{O}_X \oplus \mathscr{O}_X$. We generalize this to arbitrary normal K3-like coverings. Indeed, the main role of the flop $X' \leftarrow S \to X$ is to make the tangent sheaf trivial.

It turns out that in the restricted Lie algebra $\mathfrak{g} = H^0(X, \Theta_{X/k})$, each vector is p-closed, that is, each line is invariant under the p-map $x \mapsto x^{[p]}$. Using the correspondence between finite-dimensional restricted Lie algebras $\mathfrak{l}$ and finite height-one groups schemes $G = \mathrm{Spec}(U^{[p]}(\mathfrak{l})^\vee)$, we get a rational map

$$\mathbb{P}(\mathfrak{g}) \dashrightarrow \mathscr{M}_{\mathrm{Enr}}, \qquad \mathfrak{l} \longmapsto X/G$$

from the *twistor curve* $\mathbb{P}(\mathfrak{g})$ into the moduli stack of Enriques surfaces $\mathscr{M}_{\mathrm{Enr}}$. It was introduced in [17] to study this moduli stack. We like to call it the *twistor construction*, because it resembles the 2-sphere of complex hyperkähler manifolds obtained by rotating the complex structure while keeping the underlying topological space. Whether this is more than an analogy remains to be seen.

The twistor map is not everywhere defined. This is explained by our general theory of *Zariski singularities*, which is crucial to understand the non-smoothness of the K3-like coverings $\epsilon : X \to Y$ and their generalizations. We therefore develop a theory of Zariski singularities in full generality for arbitrary dimensions $n \geq 2$ and characteristics $p > 0$. These are hypersurface singularities $A = R/(g)$, where $R = k[[x_1, \ldots, x_n, z]]$ where the power series can be chosen of the form $g = z^p - f(x_1, \ldots, x_n)$. Using methods from commutative algebra involving free resolutions and projective dimensions, we get the following criterion for the freeness of the tangent module:

Theorem. (See Theorem 1.2) *A hypersurface singularity $A = R/(g)$ has free tangent module $\Theta_{A/k}$ of rank n if and only if A is geometrically reduced, and the module of first-order deformations $T^1_{A/k}$ has projective dimension* $\mathrm{pd} \leq 2$.

This automatically holds for the 2-dimensional geometrically reduced Zariski singularities. Note that it is usually difficult to recognize a Zariski singularity, if it is not given in the usual form $g = z^p - f(x_1, \ldots, x_n)$. We therefore develop a homological characterization of Zariski singularities, which is analogous to Serre's characterization of regularity in local rings. Roughly speaking: A hypersurface singularity $A = R/(g)$ is a Zariski singularity if and only if there is an α_2-action so that the structure sheaf $\mathscr{O}_Z$ of the closed orbit $Z \subset \operatorname{Spec}(A)$ has finite projective dimension.

Furthermore, we develop easy tools, involving Tjurina numbers, local class groups, local fundamental groups, and length formulas to recognize Zariski singularities. The following are the Zariski singularities among the rational double points:

$$A_1, D_{2n}^0, E_7^0, E_8^0\ (p=2), \quad A_2, E_6^0, E_8^0\ (p=3), \quad A_4, E_8^0\ (p=5), \quad A_{p-1}\ (p \geq 7).$$

Finally, we develop a theory of *canonical coverings* for arbitrary proper algebraic schemes Y with $h^0(\mathscr{O}_Y) = 1$, using the Raynaud Correspondence $H^1(Y_{\text{fppf}}, G_Y) = \operatorname{Hom}(\hat{G}, \operatorname{Pic}^\tau_{Y/k})$. There is indeed a canonical choice for G, namely the Cartier dual of the Frobenius kernel $\operatorname{Pic}_{Y/k}[F]$. To understand the singularities on the resulting canonical coverings $\epsilon : X \to Y$, we restrict the covering to closed subscheme $A \subset Y$ that have certain singularities, or on which the G-torsor trivializes. For example, for ADE-configurations A on smooth surfaces Y we introduce the *fundamental singular locus*, which consist of the singular locus of the fundamental cycle, and show: If G is non-reduced, then the G-torsor $X \to Y$ is singular over the fundamental singular locus of each ADE-configuration. Under the assumption that X is normal, this gives some crucial restrictions on the geometry of curves, in particular if Y is an Enriques surface.

The paper is organized as follows: In Section 1, we develop the theory of Zariski singularities in arbitrary dimensions. Section 2 contains a homological characterization of such singularities. This is applied to rational double points in Section 3. In Section 4, we give a general theory of canonical coverings, based on the Raynaud correspondence. In Section 5, we review the notion of simply-connected Enriques surfaces and discuss their basic properties. In Section 6, we introduced six conditions that normal K3-like coverings satisfy, and that indeed allow the construction of K3-like coverings. The resulting twistor curves are discussed in Section 7. In Section 8, we relate K3-like coverings X to elliptic fibrations on Enriques surfaces Y and rational surfaces J. To put this to work, we use Ogg's Formula to understand the behavior of Kodaira types under Frobenius base-change in Section 9. In Section 10, we give a geometric interpretation of the Tate Algorithm, which computes minimal Weierstraß equations, in order to describe certain elliptic singularities. All this is put into action in Section 11, which deals with the Frobenius pullback of rational elliptic surfaces. In Section 12, we discuss Lang's Classification of rational elliptic surfaces in characteristic $p = 2$, which will be crucial for our goals. Section 13 contains the construction of normal K3-like coverings with an elliptic singularity. The absence of other singularities is established in Section 14. In Section 15 we study normal K3-like coverings with rational singularities.

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1. Zariski singularities

In this section we systematically develop a general theory of Zariski singularities, a class of local rings introduced by Ekedahl, Hyland and Shepherd–Barron in dimension two [17].

Let k be a ground field, and A be a complete local noetherian ring such that the canonical map $k \to A/\mathfrak{m}_A$ is bijective. We then get an identification $k = A/\mathfrak{m}_A$ of the ground field with the residue field. Let $\dim(A) \leq \operatorname{edim}(A)$ be the dimension and embedding dimension. Equality holds if and only if the local noetherian ring R is regular. We are interested in the following situation:

Proposition 1.1. *The two conditions below are equivalent:*

(i) *The ring A is equidimensional, contains no embedded associated primes, and we have* $\operatorname{edim}(A) \leq \dim(A) + 1$.
(ii) *The ring A is isomorphic to $R/(g)$, with $R = k[[x_1, \ldots, x_n, z]]$ and some power series $g \in R$ that is neither a unit nor zero.*

If this holds, then $\dim(A) = n$*, and the ring A is Cohen–Macaulay and Gorenstein.*

Proof. If (ii) holds, then Krull's Principal Ideal Theorem ensures that every irreducible component of $\operatorname{Spec}(A)$ is n-dimensional. Moreover, since R is Cohen–Macaulay and Gorenstein, the same holds for $A = R/(g)$. In particular, A contains no embedded associated prime.

To see (i)⇒(ii), set $n = \dim(A)$. Using the completeness and the assumption on the embedding dimension, we write $A = R/\mathfrak{a}$ for some ideal $\mathfrak{a}$ in the formal power series ring $R = k[[x_1, \ldots, x_n, z]]$, which is neither the zero nor the unit ideal. According to [20], Proposition 21.7.2 the subscheme $\operatorname{Spec}(A) \subset \operatorname{Spec}(R)$ is a Weil divisor. Then $\mathfrak{a}$ is generated by a non-zero non-unit $g \in R$, because the ring R is factorial. □

It is convenient to say that the local scheme $\operatorname{Spec}(A)$, and also the local ring A as well, is a *hypersurface singularity* if the equivalent conditions of the Proposition hold. Note that this includes the case where the local ring A is regular, when we may choose $g = z$. An isomorphism $A \simeq R/(g)$ as above is called a *presentation*. Note also that in light of condition (i), the ring $R' = R \otimes_k k'$ is a hypersurface singularity if and only if R is a hypersurface singularity, where $k \subset k'$ is a finite field extension. Let us write

$$J = (g_{x_1}, \ldots, g_{x_n}, g_z) \subset A$$

for the *jacobian ideal* in $A = R/(g)$ generated by residue classes of the partial derivatives $g_{x_i} = \partial g/\partial x_i$, $1 \leq i \leq n$ and $g_z = \partial g/\partial z$. The primes $\mathfrak{p} \subset A$ not containing the jacobian ideal are exactly those for which the local noetherian rings $A_\mathfrak{p}$ are geometrically regular over k. By abuse of language, we call the closed subset $\operatorname{Spec}(A/J) \subset \operatorname{Spec}(A)$ the *locus of non-smoothness*. By the preceding observation, this subset does not depend on the choice of presentation $A = R/(g)$. Note that

$$A/J = R/(g, g_{x_1}, \ldots, g_{x_n}, g_z),$$

where the ideal on the right is the *Tjurina ideal* in R. We observe that the ring A is reduced or integral if and only if the power series $g \in R$ is squarefree or irreducible, respectively. This holds because $R = k[[x_1, \ldots, x_n, z]]$ is factorial. The ring A is regular if and only if one of the partial derivatives of $g(x_1, \ldots, x_n, z)$ is a unit, and then it is even geometrically regular. We say that the R is a *geometrically isolated hypersurface singularity* if $\dim(A/J) = 0$, such that $\operatorname{Spec}(A)$ is geometrically regular outside the closed point.

Now suppose that the ground field k has positive characteristic $p > 0$. Choose some algebraically closed extension field Ω, and let $k' = k^{1/p}$ be the intermediate extension comprising all elements $\omega \in \Omega$ with $\omega^p \in k$. Then $k \subset k'$ is a purely inseparable extension of height one, and the vector space dimension

$$\operatorname{pdeg}(k) = \dim_k(\Omega^1_{k'/k}) \in \mathbb{N} \cup \{\infty\}$$

is called the *p-degree* of k. This is also known as the *degree of imperfection*, and can also be seen as the cardinality of a p-basis for $k \subset k^{1/p}$ or equivalently $k^p \subset k$. If k is the field of fractions for a polynomial ring or formal power series ring in d indeterminates over a perfect field k_0, then $\operatorname{pdeg}(k) = n$. In what follows, we suppose that *our ground field k has finite p-degree*, which holds in particular for perfect fields. This assumption will save us some troubles when it comes to Kähler differentials of formal power series, which becomes apparent as follows:

Let $kR^p \subset R$ be the join subring generated by the subrings k and R^p. Clearly, we have an equality $R^p = k^p[[x_1^p, \ldots, x_n^p, z^p]]$ inside R. Since the field extension $k^p \subset k$ is finite, the canonical maps

$$k \otimes_{k^p} k^p[[x_1^p, \ldots, x_n^p, z^p]] \longrightarrow kR^p \subset k[[x_1^p, \ldots, x_n^p, z^p]]$$

are bijective. It follows that the module of Kähler differentials

$$\Omega^1_{R/k} = \Omega^1_{R/kR^p}$$

is finitely generated, in fact free, with basis $dx_1, \ldots, dx_n, dz$. For ground fields of characteristic zero, or of infinite p-degree, the module of Kähler differentials is usually not finitely generated, not even $\mathfrak{m}_R$-adically separated.

In our situation, the standard exact sequence $gR/g^2R \to \Omega^1_{R/k} \otimes_R A \to \Omega^1_{A/k} \to 0$ becomes

$$A \longrightarrow A^{n+1} \longrightarrow \Omega^1_{A/k} \longrightarrow 0,$$

where the mappin on the right sends the standard basis vectors in A^{n+1} to the differentials $dx_1, \ldots, dx_n, dz$ and the map on the left is the $(n+1) \times 1$-matrix of partial derivatives given by the transpose of $(g_{x_1}, \ldots, g_{x_n}, g_z)$. The latter is non-zero if and only if the ring A is geometrically reduced. Since A contains no embedded associated primes, the map is then injective. Dualizing the short exact sequence we get an exact sequence

$$0 \longrightarrow \Theta_{A/k} \longrightarrow A^{\oplus n+1} \longrightarrow A \longrightarrow \operatorname{Ext}^1(\Omega^1_{A/k}, A) \longrightarrow 0. \tag{1}$$

The term on the right is also called $T^1_{A/k} = \operatorname{Ext}^1(\Omega^1_{A/k}, A)$, the *module of first order deformations*. By the exact sequence, this module coincides with the residue class ring A/J, whose spectrum is the locus of non-smoothness, and the tangent module $\Theta_{A/k} = \operatorname{Hom}(\Omega^1_{A/k}, A)$ on the left is a syzygy for A/J. This reveals that the scheme structure on the locus of non-smoothness does not depend on the presentation $A = R/(g)$. We also immediately get:

Theorem 1.2. *Suppose that A is a hypersurface singularity. Then the tangent module $\Theta_{A/k}$ is free of rank $n = \dim(A)$ if and only if A is geometrically reduced and the A-module $T^1_{A/k} = A/J$ has projective dimension ≤ 2.*

The above property is rather peculiar: Write $A = R/(g)$ with some power series $g(x_1, \ldots, x_n, z)$. Consider the *Frobenius power* $J^{[p]} = (g^p_{x_1}, \ldots, g^p_{x_n}, g^p_z) \subset A$ of the jacobian ideal $J = (g_{x_1}, \ldots, g_{x_n}, g_z) \subset A$. The following condition was already used in [50] in dimension two with the computer algebra system Magma [35]:

Proposition 1.3. *Suppose that A is geometrically isolated hypersurface singularity of dimension $n = \dim(A)$. Then the tangent module $\Theta_{A/k}$ is free of rank n if and only if $n \leq 2$ and the length formula* $\operatorname{length}(A/J) = p^n \cdot \operatorname{length}(A/J^{[p]})$ *holds.*

Proof. For geometrically isolated hypersurface singularities, the jacobian ideal $J \subset A$ is $\mathfrak{m}_A$-primary. According to a result of Miller ([37], Corollary 5.2.3), an $\mathfrak{m}_A$-primary ideal $\mathfrak{a} \subset A$ has finite projective dimension if and only if the length formula $\operatorname{length}(A/\mathfrak{a}) = p^n \cdot \operatorname{length}(A/\mathfrak{a}^{[p]})$ holds.

It follows that our condition is sufficient: If $n \leq 2$ and the length formula holds, A/J has finite projective dimension, which a priori is $\operatorname{pd}(A/J) \leq n \leq 2$, hence $\Theta_{A/k}$ is free of rank n by Theorem 1.2. The condition is also necessary: If the tangent module $\Theta_{X/k}$ is free, then the module A/J has finite projective dimension ≤ 2, and the length formula holds by Miller's result. Moreover, the Auslander–Buchsbaum Formula $\operatorname{pd}(A/J) + \operatorname{depth}(A/J) = \operatorname{depth}(A)$ immediately gives $n = \operatorname{depth}(A) = \operatorname{pd}(A/J) \leq 2$. □

We say that our complete local ring A is a *Zariski singularity* if it is a hypersurface singularity admitting a presentation $A = R/(g)$ with a power series of the form

$$g(x_1, \ldots, x_n, z) = z^p - f(x_1, \ldots, x_n),$$

where $f \in k[[x_1, \ldots, x_n]]$ is not a unit. This term was coined by Ekedahl, Hyland and Shepherd-Barron [17] in the case of isolated surface singularities, apparently referring to the theory of *Zariski surfaces*, compare Blass and J. Lang [5].

Clearly, the Zariski singularity A is reduced if and only if f is not a p-power. Using [9], Chapter VIII, §7, No. 4, Proposition 7, we immediately get:

Proposition 1.4. *The Hilbert–Samuel multiplicity $e(A) \geq 1$ of a Zariski singularity satisfies the inequality $e(A) \leq p$.*

Now set $A_0 = k[[x_1, \ldots, x_n]]$ and $J_0 = (f_{x_1}, \ldots, f_{x_n}) \subset A_0$. Then $J = J_0 A$ is the jacobian ideal for A. This has the following consequence:

Proposition 1.5. *Suppose that A is a geometrically reduced Zariski singularity. Then the tangent module $\Theta_{A/k}$ is free of rank $n = \dim(A)$ if and only if the ring A_0/J_0 has depth $\geq n-2$.*

Proof. Clearly, A is a finite flat algebra over A_0 of degree p, so the projective dimension of $A/J = A/J_0A = A_0/J_0 \otimes_{A_0} A$ as R-module coincides with the projective dimension of A_0/J_0 as an A_0-module. By the Auslander–Buchsbaum Formula, we have

$$\operatorname{pd}(A_0/J_0) + \operatorname{depth}(A_0/J_0) = \operatorname{depth}(A_0) = n,$$

and the result follows from Theorem 1.2. □

In dimension $n = 2$, the condition becomes vacuous, and we get:

Corollary 1.6. *The tangent module $\Theta_{A/k}$ of any two-dimensional geometrically reduced Zariski singularity is free of rank two.*

If A is a geometrically isolated hypersurface singularity, that is, the A-module $T^1_{A/k}$ has finite length, its length $\tau = \operatorname{length} T^1_{A/k}$ is called the the *Tjurina number*. This is also the colength of the jacobian ideal $J \subset A$, or equivalently the colength of the Tjurina ideal in R. For Zariski singularities, we get:

Proposition 1.7. *If A is a geometrically isolated Zariski singularity given by a formal power series $g = z^p - f(x_1, \ldots, x_n)$, then the Tjurina number satisfies*

$$\tau = p \cdot \operatorname{length} k[[x_1, \ldots, x_n]]/(f_{x_1}, \ldots, f_{x_n}).$$

In particular, the Tjurina number $\tau \geq 0$ is a multiple of the characteristic $p > 0$.

Finally, we state some useful facts on Zariski-singularities pertaining to fundamental groups and class groups:

Proposition 1.8. *Let A be a normal Zariski singularity, and $A \subset B$ be a finite ring extension, with* $\mathrm{Spec}(B)$ *normal and connected. Assume that* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is étale in codimension one. Then there is a finite separable field extension $k \subset k'$ with $B = A \otimes_k k'$.*

Proof. Write $A = R/(g)$ as usual, and set $A_0 = k[[x_1, \ldots, x_n]]$. The resulting morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(A_0)$ is a universal homeomorphism that is finite and flat of degree p. Let $U_0 \subset \mathrm{Spec}(A_0)$ be the open subset corresponding to the regular locus $U \subset \mathrm{Spec}(A)$. According to [21], Exposé IX, Theorem 4.10 every finite étale covering $U' \to U$ is the pullback of some finite étale covering $U_0' \to U_0$. The latter extends to $\mathrm{Spec}(A_0)$, by the Zariski–Nagata Purity Theorem [22], Exposé X, Theorem 3.4. Since the complete local ring A_0 is henselian, the extension must have the form $A_0' = A_0 \otimes_k k'$ for some finite separable field extension $k \subset k'$. Consider the pullback $A' = A \otimes_{A_0} A_0' = A \otimes_k k'$. Then $\mathrm{Spec}(A')$ is regular in codimension one and Cohen–Macaulay, thus normal. By construction, the two morphisms $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ and $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ coincide over $U \subset \mathrm{Spec}(A)$, whose complement has codimension $d \geq 2$. Now [23], Theorem 1.12 ensures that there is an isomorphism $B \simeq A'$ of A-algebras. □

Corollary 1.9. *Suppose that A is a normal two-dimensional Zariski singularity, with algebraically closed ground field k and resolution of singularities $X \to \mathrm{Spec}(A)$. Then each irreducible component E_i of the exceptional divisor $E \subset X$ is a rational curve.*

Proof. Fix some prime $l \neq p$, and suppose that some E_i is non-rational. Consequently one finds an invertible sheaf $\mathscr{L}_E \not\simeq \mathscr{O}_E$ on E with $\mathscr{L}_E^{\otimes l} \simeq \mathscr{O}_E$. The short exact sequences $0 \to \mathscr{O}_E(-nE) \to \mathscr{O}_{(n+1)E}^\times \to \mathscr{O}_{nE}^\times \to 1$ induces long exact sequences

$$H^1(E, \mathscr{O}_E(-nE)) \longrightarrow \mathrm{Pic}((n+1)E) \longrightarrow \mathrm{Pic}(nE) \longrightarrow 0$$

where the term on the left is a p-group. It follows that there is an invertible sheaf $\mathscr{L}$ on X, restricting to $\mathscr{L}_E$, with $\mathscr{L}^{\otimes l} \simeq \mathscr{O}_X$. Then the ensuing $\mathscr{O}_X$-algebra $\mathscr{A} = \mathscr{O}_X \oplus \mathscr{L}^{\otimes -1} \oplus \ldots \oplus \mathscr{L}^{\otimes 1-l}$ of rank $l > 1$ defines a ring extension $A \subset B$ of degree l as in the Proposition. However, since k is separably closed, we must have $B = A$, contradiction. □

Proposition 1.10. *Let A be a normal Zariski singularity. Then the class group* $\mathrm{Cl}(A)$ *is an abelian p-group, and every element is annihilated by p^{d+n-1}, where $d = \mathrm{pdeg}(k)$ and $n = \dim(A)$.*

Proof. Write $A = k[[x_1, \ldots, x_n, z]]/(g)$ for some power series $g = z^p - f(x_1, \ldots, x_n)$, and consider the subring $A_0 = [[x_1, \ldots, x_n]]$ and the finite ring extension $A_0 \subset B$ with $B = k^{1/p}[[x_1^{1/p}, \ldots, x_n^{1/p}]]$. The inclusion $A_0 \subset B$ is finite and flat, of degree p^{n+d}. Clearly, the power series $f \in A_0$ admits a p-th root $h \in B$, which gives an integral homomorphism $A \to B$ of A_0-algebras via $z \mapsto h$. This map must be injective, because A is integral and $\dim(A) = \dim(B_0)$. The composite inclusion $A_0 \subset A \subset B$ has degree p^{d+n}, and the first inclusion has degree p, whence $A \subset B$ has degree $d = p^{d+n-1}$. Let $U \subset \mathrm{Spec}(A)$ be the regular locus, and $V \subset \mathrm{Spec}(B)$ be its preimage. Then $\mathrm{Cl}(A) = \mathrm{Pic}(U)$ and $\mathrm{Cl}(B) = \mathrm{Pic}(V) = \mathrm{Pic}(B) = 0$. The composition $\mathrm{Pic}(U) \to \mathrm{Pic}(V)$ with the norm map $\mathrm{Pic}(V) \to \mathrm{Pic}(U)$ is multiplication by $\deg(V/U) = p^{d+n-1}$, and the result follows. □

Corollary 1.11. *Suppose that A is a normal two-dimensional Zariski singularity, with resolution of singularities $X \to \mathrm{Spec}(A)$. Let $E \subset S$ be the exceptional divisor, $E = E_1 \cup \ldots \cup E_r$ be the irreducible components, and $N = (E_i \cdot E_j)$ the intersection matrix. If the ground field k is algebraically closed, then the Smith group $\mathbb{Z}^r/N\mathbb{Z}^r$ is annihilated by p. In particular, the absolute value of* $\det(N)$ *is a p-power.*

Proof. The Smith group $\mathbb{Z}^r/N\mathbb{Z}^r$ is finite, because the intersection matrix is negative-definite. Any effective Weil divisor on $\mathrm{Spec}(A)$ can be seen as an effective Weil divisor

$D \subset X$ that contains no exceptional curve E_i, and we get intersection numbers $(D \cdot E_i)$. This yields a well-defined homomorphism $\mathrm{Cl}(A) \to \mathbb{Z}^r/N\mathbb{Z}^r$. In light of the Proposition, we merely have to check that the map is surjective. Each element of the Smith group is represented by a difference of vectors with non-negative entries. Suppose we have a vector of non-negative integers $(n_1, \ldots, n_r) \in \mathbb{Z}^r$. As the ground field k is algebraically closed, there is an effective Cartier divisor $D_0 \subset E$ such that $\mathscr{L}_0 = \mathscr{O}_E(D)$ has intersection numbers $(\mathscr{L}_0 \cdot E_i) = n_i$. Since A is complete and in particular henselian, we may extend it to some invertible sheaf $\mathscr{L}$ on X by [20], Corollary 21.9.12. This yields an element in $\mathrm{Cl}(A)$ having the same class as $(n_1, \ldots, n_r)$ in the Smith group $\mathbb{Z}^n/N\mathbb{Z}^r$. ☐

2. Homological characterization

Since a complete local noetherian ring A may arise in very many different ways, it is a priori not clear how to decide whether or not it is a Zariski singularity, even if it comes with a presentation $A = R/(g)$. In this section, we establish a homological characterization using p-closed derivations and actions of height one group schemes that is both intrinsic and practical. We start by recalling the equivalence between finite groups schemes of height one and finite-dimensional restricted Lie algebras. For details, we refer to the monograph of Demazure and Gabriel [13], Chapter II, §7. A very brief summary is contained in [49], Section 1.

Fix a ground field k of characteristic $p > 0$. Let $\mathfrak{g}$ be a *restricted Lie algebra*, with Lie bracket $[x, y]$ and p-map $x^{[p]}$. The latter is an additional structure that turns a Lie algebra into a restricted one. It satisfies the semilinearity property $(\lambda x)^{[p]} = \lambda^p x^{[p]}$, and the two Jacobson Formulas

$$\mathrm{ad}(x^{[p]}) = \mathrm{ad}(x)^p \quad \text{and} \quad (x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{r=1}^{p-1} s_r(x, y),$$

where $\mathrm{ad}(x)(y) = [x, y]$ is the adjoint representation, and $s_r(x, y)$ is some universal expression in Lie brackets of x and y, for example explained in [13], Chapter II, §7, Definition 3.3. For more details, see the book of Strade and Farnsteiner [52].

Let $U(\mathfrak{g})$ be the universal enveloping algebra, $U^{[p]}(\mathfrak{g})$ be the quotient by the ideal generated by the elements $x^p - x^{[p]} \in U(\mathfrak{g})$ with $x \in \mathfrak{g}$, and $U^{[p]}(\mathfrak{g})^\vee$ the dual vector space. The diagonal map $\mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$ induces the comultiplication $\Delta : U^{[p]}(\mathfrak{g}) \to U^{[p]}(\mathfrak{g}) \otimes U^{[p]}(\mathfrak{g})$ and dually a multiplication in $U^{[p]}(\mathfrak{g})^\vee$. The latter becomes a commutative k-algebra, and we may form

$$G = \mathrm{Spec}(U^{[p]}(\mathfrak{g})^\vee).$$

As explained in loc. cit. Proposition 3.9, this acquires the structure of a group scheme of height one, with $\mathrm{Lie}(G) = \mathfrak{g}$. Its R-valued points $G(R)$ is the group of elements $g \in U^{[p]}(\mathfrak{g}) \otimes_k R$ satisfying $\Delta(g) = g \otimes g$. Such elements g are also called *group-like*. Furthermore, the correspondences

$$G \longmapsto \mathrm{Lie}(G) \quad \text{and} \quad \mathfrak{g} \longmapsto \mathrm{Spec}(U^{[p]}(\mathfrak{g})^\vee)$$

are adjoint equivalences between the categories of finite groups schemes of height one and the category of finite-dimensional restricted Lie algebras, by loc. cit. Proposition 4.1.

If X is a scheme, the right G-actions $X \times G \to X$ correspond to the homomorphisms $\mathfrak{g} = \mathrm{Lie}(G) \to \mathrm{Der}_k(\mathscr{O}_X, \mathscr{O}_X)$ of restricted Lie algebras. Here

$$\mathrm{Der}_k(\mathscr{O}_X, \mathscr{O}_X) = \mathrm{Hom}_{\mathscr{O}_X}(\Omega^1_{X/k}, \mathscr{O}_X) = H^0(X, \Theta_{A/k})$$

is the restricted Lie algebra of derivations $D : \mathscr{O}_X \to \mathscr{O}_X$, where the p-map is the p-fold composition $D^p = D \circ \ldots \circ D$ in the associative algebra of all differential operators

$\mathscr{O}_X \to \mathscr{O}_X$. In the case that $X = \mathrm{Spec}(A)$ is affine, this boils down to specify a derivation $D : A \to A$.

If the restricted Lie algebra $\mathfrak{g}$ is one-dimensional, with basis vector $x \in \mathfrak{g}$, then the Lie bracket must be trivial, and the p-map is determined via $x^{[p]} = \lambda x$ by some unique scalar $\lambda \in k$. Let us write $\mathfrak{g}_\lambda$ and G_λ for the resulting restricted Lie algebra and height-one group scheme. The right actions of G_λ on the scheme X thus correspond to derivations $D : A \to A$ satisfying $D^p = \lambda D$. Such derivations are called *p-closed*.

To make this more explicit, write $U^{[p]}(\mathfrak{g}_\lambda) = k[x]/(x^p - \lambda x)$, and choose as a vector space basis the divided powers $u_i = x^i/i!$ with $0 \leq i \leq p-1$. This basis has the advantage that the comultiplication is given by $\Delta(u_i) = \sum_{r+s=i} u_r \otimes u_s$. Denote the dual basis by $t_i \in U^{[p]}(\mathfrak{g}_\lambda)^\vee$. Inside the ring $U^{[p]}(\mathfrak{g}_\lambda)^\vee$, we get $t_r \cdot t_s = t_{t+s}$, and in particular $t_1^p = 0$. With $t = t_1$, one obtains $U^{[p]}(\mathfrak{g}_\lambda)^\vee = k[t]/(t^p)$. Consequently, the action $X \times G_\lambda \to X$ takes the explicit form

$$A \longrightarrow A \otimes_k U^{[p]}(\mathfrak{g}_\lambda)^\vee, \quad f \longmapsto \sum_{i=0}^{p-1} \frac{D^i(f)}{i!} \otimes t^i. \tag{2}$$

Now suppose we have a right G_λ-action on an affine scheme $X = \mathrm{Spec}(A)$, and let $a \in \mathrm{Spec}(A)$ be a k-rational point, corresponding to a maximal ideal $\mathfrak{m} \subset A$. Let $\mathfrak{a} \subset A$ be the intersection of the kernels for the k-linear maps

$$\mathfrak{m} \xrightarrow{D^i} A \longrightarrow A/\mathfrak{m}, \quad 1 \leq i \leq p-1. \tag{3}$$

One easily checks that this vector subspace $\mathfrak{a} \subset A$ is an ideal. We call it the *orbit ideal* for the G_λ-action, or the corresponding p-closed derivation $D : A \to A$.

Proposition 2.1. *The closed subscheme $X' = \mathrm{Spec}(A/\mathfrak{a})$ is a G_λ-invariant Artin scheme consisting only of the point $a \in X$, and its length is either $l = 1$ or $l = p$. In the latter case, X' is a trivial G_λ-torsor over $\mathrm{Spec}(k)$, and their is an open neighborhood $U \subset X$ of the point $a \in X$ on which the G_λ-action is free.*

Proof. For each $f \in \mathfrak{m}$, we obviously have $f^p \in \mathfrak{a}$, whence $X' = \{a\}$ holds as a set. From (2) we infer that $X' \subset X$ is invariant under the G_λ-action. To compute its length, we may assume that $\mathfrak{a} = 0$, by replacing X with X'. Then the composite map

$$A \longrightarrow k \otimes U^{[p]}(\mathfrak{g}_\lambda)^\vee = k[t]/(t^p)$$

obtained from (2) is injective, and thus $\mathrm{length}(A) \leq p$. Furthermore, we may regard A as a subalgebra of $B = k[t]/(t^p)$. If $\mathfrak{m}_A = 0$, then $\mathrm{length}(A) = 1$. Now suppose that $\mathfrak{m}_A \neq 0$. Fix some non-zero $f \in \mathfrak{m}_A$. Since $A \subset B$, there is some $0 \leq i \leq p-1$ so that $D^i(f)$ does not vanish in $A/\mathfrak{m}_A = k$. Hence $D^i(f) \in A$ is a unit. This already ensures that $\mathrm{Spec}(A) \to \mathrm{Spec}(A^D)$ is a torsor, for example by [46], Theorem 4.1. Here A^D is the kernel of the derivation $D : A \to A$, which is a k-subalgebra $A^D \subset A$. It follows that $\mathrm{length}(A) \geq p$. In light of the inclusion $A \subset B$, we must have $\mathrm{length}(A) = p$.

Now suppose that $\mathrm{length}(A) = p$. Then $X \times G \to X$ is a G-torsor containing a rational point, so the torsor is trivial. To show that there is an open neighborhood on which the action is free, we revert to the original situation $X = \mathrm{Spec}(A)$, with $X' = \mathrm{Spec}(A/\mathfrak{a})$. Let $A^D = \mathrm{Ker}(D)$ be the kernel of the additive map $D : A \to A$, which is a subring of A. Since $\mathfrak{a} \neq \mathfrak{m}$, there is some $f \in \mathfrak{m}$ so that $D^i(f)$ is non-zero in $A/\mathfrak{m}$, whence a unit in the local ring $A_\mathfrak{p}$. Replacing A by some suitable localization A_g, we may assume that $D^i(f) \in A$ is a unit. This already ensures that $\mathrm{Spec}(A) \to \mathrm{Spec}(A^D)$ is a torsor, again by [46], Theorem 4.1. $\square$

The kernel of the derivation $D : A \to A$ is a subring, and we call this subring $A^D \subset A$ the *ring of invariants*. Then $X/G_\lambda = \operatorname{Spec}(A^D)$ is a categorical quotient in the category (Aff/k) of all affine schemes, and in the category of all schemes (Sch/k) as well. If the action is free, that is, the canonical map $X \times G \to X \times_{X/G_\lambda} X$ given by $(a,g) \mapsto (a, ag)$ is an isomorphism, and the canonical morphism $X \to X/G_\lambda$ is a G_λ-torsor.

We say that a rational point $a \in X = \operatorname{Spec}(A)$ is a *fixed point* if the orbit ideal $\mathfrak{a} \subset A$ is maximal. Otherwise, the G_λ-action is free in an open neighborhood. Being an A-module, the orbit ideal has a projective dimension $\operatorname{pd}(\mathfrak{a}) \geq 0$, which may be finite or infinite. From this we get the desired *homological characterization* of Zariski singularities:

Theorem 2.2. *Suppose that A is a complete local noetherian ring with $A/\mathfrak{m}_A = k$.*

(i) *If A is a Zariski singularity, there is a derivation $D : A \to A$ satisfying $D^p = 0$ such that the orbit ideal $\mathfrak{a} \subset A$ is not the maximal ideal, and has finite projective dimension.*

(ii) *If there is a p-closed derivation $D : A \to A$ such that the orbit ideal is not the maximal ideal and has finite projective dimension, then $A \otimes_k k'$ is a Zariski singularity for some finite separable extension $k \subset k'$.*

Proof. (i) If $A = R/(g)$ is a Zariski singularity, say given by the power series $g = z^p - f(x_1, \ldots, x_n)$, we may take the standard derivation $D = D_z$, which satisfies $D^p = 0$. Clearly, $x_1, \ldots, x_n, z^p$ are contained in the orbit ideal $\mathfrak{a} \subset A$, but not z, because $D(z) = 1$ is nonzero in $A/\mathfrak{m}_A$. Thus the orbit ideal is not maximal, and by Proposition 2.1 we must have $\mathfrak{a} = (x_1, \ldots, x_n, z^p)$. This ideal is induced from the maximal ideal for the finite flat extension $k[[x_1, \ldots, x_n]] \subset A$, whence has finite projective dimension.

(ii) Write $D^p = \lambda D$ for some scalar $\lambda \in k$. Choose some $\mu = \lambda^{1/(p-1)}$ in some algebraic closure. This gives a finite separable extension $k' = k(\mu)$. Replacing A by $A \otimes_k k'$ we may assume $k = k'$. If $\lambda \neq 0$, we replace D by μD. The upshot is that either $D^p = D$ or $D^p = 0$. Then the corresponding height one group scheme is $G = \mu_p$ or $G = \alpha_p$, respectively. According to Proposition 2.1, the G-action on X is free. Since the ring of invariants A^D contains the ring of p-th powers A^p, the projection $X \to X/G$ is a universal homeomorphism, whence the ring of invariants A^D is local. According to the Eakin–Nagata Theorem, or equivalently by faithful flatness, it is also noetherian. It then also must be complete. It follows from [49], Proposition 2.2 that the invariant ring is also regular.

Choose some regular system of parameters $x_1, \ldots, x_n \in A^D$, and write the invariant ring as $A^D = k[[x_1, \ldots, x_n]]$. Now set $S = \operatorname{Spec}(A^D)$, and suppose that $G = \mu_p$. From the short exact sequence $0 \to G \to \mathbb{G}_m \xrightarrow{F} \mathbb{G}_m \to 0$, we get an exact sequence

$$H^0(S, \mathbb{G}_m) \longrightarrow H^1(S, G) \longrightarrow H^1(S, \mathbb{G}_m).$$

The term on the right vanishes, because $\operatorname{Pic}(S) = 0$. Whence the total space $X = \operatorname{Spec}(A)$ of the torsor takes the form $A = A^D[z]/(z^p - f)$ for some invertible formal power series $f(x_1, \ldots, x_n)$. Its constant term $\alpha \in k$ must be a p-power, because $A^D \subset A$ has trivial residue field extension. Replacing z by $z - \beta$ with $\beta^p = \alpha$ reveals that A is a Zariski singularity. In the case $G = \alpha_p$, one uses the short exact sequence $0 \to G \to \mathbb{G}_a \xrightarrow{F} \mathbb{G}_a \to 0$ and argues similarly. $\square$

According to Proposition 1.6, the tangent module of a 2-dimensional geometrically reduced Zariski singularity is free. It is possible to give an explicit basis:

Proposition 2.3. *Let $A = k[[x, y, z]]/(g)$ be a two-dimensional geometrically isolated Zariski singularity, given by an formal power series $g = z^p - f(x, y)$. Then the derivations*

$$D_z \quad \text{and} \quad f_y D_x - f_x D_y \tag{4}$$

form an A-basis for the tangent module $\Theta_{A/k} = \mathrm{Der}_k(A, A) = \mathrm{Hom}(\Omega^1_{A/k}, A)$.

Proof. In our situation, the exact sequence (1) becomes

$$0 \longrightarrow \Theta_{A/k} \longrightarrow A^{\oplus 3} \xrightarrow{(f_x, f_y, 0)} A.$$

Clearly, the transpose of the vectors $(0, 0, 1)$ and $(f_y, -f_x, 0)$ lie in the kernel of the map on the right, so the derivations in (4) indeed can be regarded as elements of the tangent module $\Theta_{A/k}$. Moreover, $D_z \in \Theta_{A/k}$ generates a free direct summand, and the kernel of the linear map $(f_x, f_y) : A^2 \to A$ gives a complement.

By assumption, A is not geometrically regular, such that $f_x, f_y \in \mathfrak{m}_A$. Regard A as a finite flat algebra over $A_0 = k[[x, y]]$. Since the jacobian ideal (f_x, f_y) is $\mathfrak{m}_{A_0}$-primary, the elements $f_x, f_y \in A_0$ form a regular sequence. Thus the Koszul complex

$$0 \longrightarrow A_0 \xrightarrow{\binom{f_y}{-f_x}} A_0^2 \xrightarrow{(f_x, f_y)} A_0$$

is exact. Tensoring with the flat algebra A shows that the kernel of $(f_x, f_y) : A^2 \to A$ is free of rank one, with generator $f_y D_x - f_x D_y$. □

Proposition 2.4. *Let $A = k[[x, y, z]]/(g)$ be a two-dimensional geometrically isolated Zariski singularity, given by an formal power series $g = z^p - f(x, y)$. Let*

$$D = uD_z + v(f_y D_x - f_x D_y) \in \Theta_{A/k}$$

be a p-closed derivation, for some coefficients $u, v \in A$. Then the corresponding G-action on $\mathrm{Spec}(A)$ is free if and only if $u \in A^\times$. If this is the case, the orbit ideal is given by $\mathfrak{a} = (x, y)$, and the ring of invariants A^D is regular.

Proof. The orbit ideal $\mathfrak{a} \subset A$ obviously contains z^p. Since the local ring A is geometrically singular, we have $f_x, f_y \in \mathfrak{m}_A$. Thus also $x, y \in \mathfrak{a}$. If the coefficient $u \in A$ is not invertible, that is, $u \in \mathfrak{m}_A$, we also have $z \in \mathfrak{a}$, and the G-action on $\mathrm{Spec}(A)$ is not free. Now suppose that $u \in A^\times$. Then $D(z) = u$ does not vanish in the residue field, and thus $z \notin \mathfrak{a}$. Consequently, the G-action on $\mathrm{Spec}(A)$ is free, and the inclusion $(x, y, z^p) \subset \mathfrak{a}$ is an equality. In light of the relation $z^p = f(x, y)$ in A, we get $\mathfrak{a} = (x, y)$, and this has finite projective dimension. It follows from [49], Proposition 2.2 that the invariant ring A^D is regular. □

This is quite remarkable, because the ideal $\mathfrak{a} = (x, y)$ in the ring A neither depends on the choice of the regular system of parameters $x, y, z \in R$, nor the choice of the formal power series $g = z^p - f(x, y)$, nor the choice of the p-closed derivation $D \in \Theta_{A/k}$ giving a free group scheme action. We simply call it the *orbit ideal* of the two-dimensional geometrically reduced Zariski singularity. It defines a canonical zero-dimensional subscheme of embedding dimension one on the singular scheme $X = \mathrm{Spec}(A)$, called the *orbit subscheme*. In turn, it yields a unique tangent vector

$$\mathrm{Spec}\, k[\epsilon] \subset \mathrm{Spec}\, k[[x, y, z]]/(x, y, z^p) \subset \mathrm{Spec}(A) \tag{5}$$

that indicates the direction of any p-closed vector field corresponding to a free action. We call it the *orbit direction*. Here and throughout, ϵ denotes an indeterminate satisfying $\epsilon^2 = 0$.

Theorem 2.5. *Let A be a two-dimensional geometrically reduced Zariski singularity that is not regular, and $\mathfrak{g} \subset \Theta_{A/k}$ be a restricted Lie subalgebra with $\dim_k(\mathfrak{g}) = 2$ for which every element is p-closed, and $A\mathfrak{g} = \Theta_{A/k}$. Then there is a line $\mathfrak{l} \subset \mathfrak{g}$ so that for each element $D \in \mathfrak{g}$, the corresponding group scheme action on $\mathrm{Spec}(A)$ is free if and only if $D \notin \mathfrak{l}$. If these conditions holds, the invariant ring A^D is regular.*

Proof. Suppose A is given by the formal power series $g = z^p - f(x, y)$. Choose a k-basis $D_1, D_2 \in \mathfrak{g}$. Then $D_1, D_2 \in \Theta_{A/k}$ form an A-basis as well. According to Proposition 2.4, the $D_z, f_y D_x - f_x D_y \in \Theta_{A/k}$ yield another A-basis. Write

$$D_z = r_{11} D_1 + r_{12} D_2 \quad \text{and} \quad f_y D_x - f_x D_y = r_{21} D_1 + r_{22} D_2$$

for some base-change matrix $(r_{ij}) \in \mathrm{GL}_2(A)$. Let $\bar{r}_{ij} \in A/\mathfrak{m}_A = k$ be the residue classes of the matrix entries, and let $\mathfrak{l} \subset \mathfrak{g}$ be the line generated by the vector $\bar{r}_{21} D_1 + \bar{r}_{22} D_2$. Each derivations $D = \lambda D_1 + \mu D_2 \in \mathfrak{g}$ with coefficients $\lambda, \mu \in k$ can be written as $D = u D_z + v(f_y D_x - f_x D_y)$ with coefficients $u, v \in A$. Obviously, $D \notin \mathfrak{l}$ means that $u \in A^\times$. The assertion thus follows from Proposition 2.4. □

In the above situation, we call $\mathfrak{l} \subset \mathfrak{g}$ the *canonical line*. This also can be regarded in the following way: The composite map $\mathfrak{g} \subset \Theta_{A/k} \to \Theta_{A/k} \otimes k$ is bijective. Under the canonical map

$$\Theta_{A/k} \otimes k = \mathrm{Hom}_A(\Omega^1_{A/k}, A) \otimes_A k \longrightarrow \mathrm{Hom}_k(\Omega^1_{A/k} \otimes k, k) = \mathrm{Hom}_A(\Omega^1_{A/k}, k),$$

the elements $D \in \mathfrak{g}$ are turned into derivations $D : A \to k$, which correspond to homomorphisms $A \to k[\epsilon]$. Note, however, that the vector spaces above have dimension two and three, and that the canonical line $\mathfrak{l} \subset \mathfrak{g}$ is the kernel of the above linear map. Hence the image is one-dimensional, and indeed induces the orbit direction $\mathrm{Spec}\, k[\epsilon] \subset \mathrm{Spec}(A)$.

In Theorem 2.5, the 2-dimensional restricted Lie algebra $\mathfrak{g}$ has the property that every vector is p-closed. Up to isomorphism, there are exactly two such restricted Lie algebras of dimension two:

Lemma 2.6. *Suppose k is algebraically closed. Then the 2-dimensional restricted Lie algebras $\mathfrak{g}$ for which every vector is p-closed are up to isomorphism given by $\mathfrak{g} = kx \oplus ky$ with*

$$[x, y] = x^{[p]} = y^{[p]} = 0 \qquad \text{or} \qquad [x, y] = y,\ x^{[p]} = x,\ y^{[p]} = 0.$$

Proof. This follows from the classification of 2-dimensional restricted Lie algebras [56], Proposition A3. □

3. Rational double points

Let k be an algebraically closed ground field of characteristic $p > 0$, and A be a complete local ring that is normal and of dimension two. We say that A is a *rational double point* if it is a rational singularity of Hilbert–Samuel multiplicity $e(A) = 2$. Equivalently, it is a rational Gorenstein singularity. Recall that in characteristic $p = 2$, the *rational double points* are classified into

$$A_n, \quad D^r_{2n}, D^r_{2n+1}, \quad E^0_6, E^1_6, \quad E^0_7, \ldots, E^3_7, \quad \text{and} \quad E^0_8, \ldots, E^4_8,$$

with $0 \leq r \leq n - 1$, according to Artin's analysis [2]. We now can determined which of them are Zariski:

Theorem 3.1. *In characteristic $p = 2$, the rational surface singularities that are Zariski singularities are precisely the rational double points of type A_1, D^0_{2m}, E^0_7 and E^0_8.*

Proof. It follows from the defining equations in [2] that the given rational double points are Zariski. The converse is not at all obvious. Let $A = k[[x, y, z]]/(g)$ be a rational Zariski singularity, given by the formal power series $g = z^2 - f(x, y)$. Being a hypersurface singularity, it is Gorenstein, hence a rational double point. According to Corollary 1.11, the intersection matrix $N = (E_i \cdot E_j)$ for the exceptional divisor on a resolution of singularities has $\det(N) = \pm 2^\nu$ for some $\nu \geq 0$, whence only type A_n with $n = 2^\nu - 1$, D_n, E_7 and E_8

are possible. Furthermore, this Smith group $\Phi = \mathbb{Z}^r/N\mathbb{Z}^r$ is annihilated by $p = 2$, which rules out A_n with $n \geq 2$, and D_n with $n = 2m+1$.

Clearly, the module of Kähler differentials $\Omega^1_{A/k}$ is generated by dx, dy, dz, and $Adz \subset \Omega_{A/k}$ is an invertible direct summand. According to [50], proof of Theorem 6.3 such an invertible direct summand does not exist for D_n^r with $r > 0$. Likewise, in the proof for loc. cit. Theorem 6.4 the same is shown for the singularities E_7^r and E_8^r with $r > 0$. $\square$

Using Proposition 1.11, one shows:

Proposition 3.2. *In characteristic $p = 3$, the rational surface singularities that are Zariski singularities are the rational double points of type A_2, E_6^0, E_8^0. In characteristic $p = 5$, these are A_4 and E_8^0. For $p \geq 7$, there is only A_{p-1}.*

A rational double point that is also a Zariski singularity is called a *Zariski rational double point*. Going through their normal form [2], one obtains the following observation:

Proposition 3.3. *The Tjurina numbers for Zariski rational double points of type A_n, D_n or E_n in characteristic $p > 0$ are $\tau = pn/(p-1)$. In particular, $\tau = 2n$ in characteristic two.*

4. Canonical coverings of algebraic schemes

The goal of this section is to establish some general facts about "canonical coverings" of proper algebraic schemes. We will apply this later to the K3-like covering of simply-connected Enriques surface, but the underlying principles have a far more general relevance.

Fix a ground field k, and let Y be a proper k-scheme with $k = H^0(Y, \mathscr{O}_Y)$, and $\mathrm{Pic}_{Y/k}$ be its Picard scheme. Let G be a finite commutative group scheme, and $\hat{G} = \underline{\mathrm{Hom}}(G, \mathbb{G}_m)$ be the Cartier dual. This is also a finite commutative group scheme, of the same order $h^0(\mathscr{O}_G) = h^0(\mathscr{O}_{\hat{G}})$, and the biduality map

$$G \longrightarrow \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(G, \mathbb{G}_m), \mathbb{G}_m), \quad g \longmapsto (f \mapsto f(g))$$

is an isomorphism of group schemes.

Set $S = \mathrm{Spec}(k)$, let $f : Y \to S$ be the structure morphism, and write $G_Y = f^*(G)$ for the induced relative group scheme over Y. According to [45], Proposition 6.2.1 applied to the Cartier dual $\hat{G}$, we have a a canonical bijection

$$R^1 f_*(G_Y) \longrightarrow \underline{\mathrm{Hom}}(\hat{G}, \mathrm{Pic}_{Y/k}) = \underline{\mathrm{Hom}}(\hat{G}, \mathrm{Pic}^\tau_{Y/k})$$

of abelian sheaves on the site (Sch/k) of all k-schemes, endowed with the fppf-topology. Taking global sections yields an identification

$$H^0(S, R^1 f_*(G_Y)) = \mathrm{Hom}(\hat{G}, \mathrm{Pic}^\tau_{Y/k}).$$

The term on the left sits in the exact sequence coming from the Leray–Serre spectral sequence

$$0 \longrightarrow H^1(S, f_*(G_Y)) \longrightarrow H^1(Y, G_Y) \longrightarrow H^0(S, R^1 f_*(G)) \longrightarrow H^2(S, f_*(G_Y)).$$

If the ground field k is algebraically closed, every non-empty k-scheme of finite type has a rational point, by Hilbert's Nullstellensatz, and it follows that $H^r(S, F) = 0$ for every degree $r \geq 1$ and every abelian fppf-sheaf F. Thus the outer terms in the preceding sequence vanish. This yields:

Proposition 4.1. *If k is algebraically closed, then the canonical map*

$$H^1(Y, G_Y) \longrightarrow \operatorname{Hom}(\hat{G}, \operatorname{Pic}^\tau_{Y/k})$$

is bijective. In other words, the isomorphism classes of G-torsors $\epsilon : X \to Y$ correspond to homomorphisms of group scheme $\hat{G} \to \operatorname{Pic}_{Y/k}$.

Often there are *canonical choices* for G with respect to Y and the resulting numerical trivial part $P = \operatorname{Pic}^\tau_{Y/k}$. For example, if Y is Gorenstein with numerically trivial dualizing sheaf ω_Y, we may take for $\hat{G} \subset P$ the discrete subgroup generated by the class of ω_Y. One may call the resulting $\epsilon : X \to Y$ the *ω-canonical covering.* If the Picard scheme is zero-dimensional, then one may choose G with $\hat{G} = P$ or $\hat{G} = P^0$, and the resulting torsors are called the *τ-canonical covering* and *0-canonical covering*, respectively. For every $n \neq 0$ that is prime to the characteristic exponent $p \geq 1$, we furthermore get the *n-canonical coverings* with $\hat{G} = P[n]$, the kernel of the multiplication by n-map. In characteristic $p > 0$, we may also use the kernel of the relative Frobenius map $F : P \to P^{(p)}$, and get the *F-canonical covering.* Later, we will apply this to simply-connected Enriques surfaces Y. Then we have $\operatorname{Pic}^0_{Y/k} = \operatorname{Pic}^\tau_{Y/k} = \operatorname{Pic}_{Y/k}[F]$, and the ensuing torsor $\epsilon : X \to Y$ is called the *K3-like covering.*

In what follows, we will assume that the ground field k is algebraically closed, such that we have $H^1(Y, G_Y) = \operatorname{Hom}(\hat{G}, \operatorname{Pic}^\tau_{Y/k})$. This identification is natural in Y and G, and the following obvious consequence will play a key role throughout:

Lemma 4.2. *Let $\epsilon : X \to Y$ be a G-torsor, corresponding to a homomorphism $\hat{G} \to \operatorname{Pic}_{Y/k}$. Let $Z \to Y$ be a proper morphism. If $h^0(\mathscr{O}_Z) = 1$, and the composite homomorphism $\hat{G} \to \operatorname{Pic}_{X/k} \to \operatorname{Pic}_{Z/k}$ is zero, then the induced torsor $X \times_Y Z \to Z$ is trivial.*

Proof. The condition $h^0(\mathscr{O}_Z) = 1$ ensures that $H^1(Z, G) = \operatorname{Hom}(\hat{G}, \operatorname{Pic}_{Z/k})$, and the naturality of this identification implies that the induced torsor becomes trivial. □

As a consequence, we get a statement on the singularities of the total space of torsors:

Proposition 4.3. *Let $\epsilon : X \to Y$ be a G-torsor with G non-reduced, $Z \subset Y$ be a Weil divisor with $h^0(\mathscr{O}_Z) = 1$, and assume that the composite map $\hat{G} \to \operatorname{Pic}^\tau_{Y/k} \to \operatorname{Pic}_{Z/k}$ is zero. Let $x \in X$ be a closed point mapping to a point $z \in Z$ where the local ring $\mathscr{O}_{Z,z}$ is singular. Then $\mathscr{O}_{X,x}$ is singular as well.*

Proof. Set $n = \dim(\mathscr{O}_{X,x})$, such that the local ring $\mathscr{O}_{Z,z}$ on the Weil divisor has dimension $n-1$. Since this local ring is not regular, its embedding dimension is $\operatorname{edim}(\mathscr{O}_{Z,z}) \geq n$. The induced torsor is $T \simeq Z \times G$. The local ring at the origin $0 \in G$ has embedding dimension $\operatorname{edim}(\mathscr{O}_{G,0}) \geq 1$, because the group scheme G is non-reduced. Thus we have

$$\operatorname{edim}(\mathscr{O}_{X,x}) \geq \operatorname{edim}(\mathscr{O}_{T,x}) = \operatorname{edim}(\mathscr{O}_{Z,z}) + \operatorname{edim}(\mathscr{O}_{G,0}) \geq n+1,$$

hence the local ring $\mathscr{O}_{X,x}$ is not regular. □

Sometimes, one may verify the assumption by looking at cohomology groups:

Corollary 4.4. *Assumptions as in the previous proposition. Suppose further $\hat{G}$ is local, that Y is normal of dimension $n \geq 1$, and that the canonical mapping $H^{n-1}(Y, \omega_Y) \to H^{n-1}(Y, \omega_Y(Z))$ is injective. Let $x \in X$ be a closed point mapping to a point $z \in Z$ where $\mathscr{O}_{Z,z}$ is singular. Then $\mathscr{O}_{X,x}$ is singular as well.*

Proof. The short exact sequence $0 \to \mathscr{O}_Y(-Z) \to \mathscr{O}_Y \to \mathscr{O}_Z \to 0$ gives a long exact sequence

$$H^1(\mathscr{O}_X(-Z)) \longrightarrow H^1(X, \mathscr{O}_X) \longrightarrow H^1(Z, \mathscr{O}_Z).$$

The map on the left hand side is surjective, because it is Serre dual to the injection $H^{n-1}(Y, \omega_Y) \to H^{n-1}(Y, \omega_Y(Z))$. Whence the map on the right is zero. The latter is the tangent map for the restriction $\mathrm{Pic}_{Y/k} \to \mathrm{Pic}_{Z/k}$. Suppose for a moment that $\hat{G}$ has height one. Then this group scheme is determined by its restricted Lie algebra $\mathrm{Lie}(\hat{G})$, the composite map $\hat{G} \to \mathrm{Pic}_{Y/k} \to \mathrm{Pic}_{Z/k}$ is trivial, and the assertion follows from the proposition. In the general case, the Frobenius kernel $\hat{H} = \hat{G}[F]$ has height one and corresponds to a surjection $G \to H$. The result thus holds for the associated H-torsor $X \times_G H$, and therefore also on X. $\square$

The following connection to singularity theory follows immediately from Theorem 2.2, our homological characterization of Zariski singularities, or a direct local computation:

Proposition 4.5. *Let $\epsilon : X \to Y$ be a G-torsors. Suppose that Y is smooth, and that $\hat{G}$ is unipotent of order p, in other words either $\hat{G} = \mathbb{Z}/p\mathbb{Z}$ or $\hat{G} = \alpha_p$. Then for each closed point $x \in X$, the complete local ring $\mathscr{O}^\wedge_{X,x}$ is a Zariski singularity.*

Now suppose that Y is a smooth surface. A curve $E \subset Y$ is called *negative-definite* if the intersection matrix $N = (E_i \cdot E_j)$ attached to the integral components $E_1, \ldots, E_r \subset E$ is negative-definite. This are precisely the *exceptional divisors* for some contraction $Y \to Y'$, where Y' is a normal 2-dimensional algebraic space. To each such negative-definite curve we have the *fundamental cycle* $Z = \sum n_i E_i$ with certain coefficients $n_i \geq 1$. It can be defined as the smallest such cycle for which all intersection numbers are $(Z \cdot E_i) \leq 0$, and we have Artin's Algorithm [1] to compute it from the intersection matrix N. We call the singular locus $\mathrm{Sing}(Z)$ of the fundamental cycle the *fundamental-singular locus* of $E \subset Y$. Note that

$$\mathrm{Sing}(Z) = \bigcup_{i \neq j} (E_i \cap E_j) \quad \cup \quad \bigcup_{n_i > 1} E_i.$$

This coincides with the singular locus of the reduced curve E if the fundamental cycle is reduced, but usually is much larger.

We define the *fundamental-genus* of the negative-definite curve $E \subset Y$ as the natural number $h^1(\mathscr{O}_Z) \geq 0$. A negative-definite curve $E \subset Y$ is called of *rational type* if it has fundamental-genus $h^1(\mathscr{O}_Z) = 0$. In other words, it contracts to a rational singularity. In the special case of rational double points, the intersection matrix N then corresponds to one of the Dynkin diagrams A_n, D_n, E_6, E_7 or E_8, and we call the curve an *ADE-configuration.*

Proposition 4.6. *Let Y be a smooth surface, and $\epsilon : X \to Y$ be a G-torsor for some non-reduced G. Then the local ring $\mathscr{O}_{X,x}$ is singular for each point $x \in X$ mapping to the fundamental-singular locus* $\mathrm{Sing}(Z) \subset X$ *of a negative-definite curve of rational type $A \subset Y$.*

Proof. Since G is non-reduced, there is a subgroup scheme $H \subset G$ so that $G' = G/H$ is either μ_p or α_p. Set $X' = X/H$. By construction, $X \to X'$ is a H-torsor, and $X' \to Y$ is a G'-torsor, in fact coming from the composite $\hat{G}' \subset \hat{G} \to \mathrm{Pic}_{Y/k}$. According to [19], Proposition 6.7.4 it thus suffices to treat the case $G = G'$. Then the Cartier dual $\hat{G}$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ or α_p, respectively. The Picard scheme of the fundamental cycle is of the form $\mathrm{Pic}_{Z/k} = \mathbb{Z}^{\oplus r}$, where $r \geq 1$ is the number of irreducible components of Z. In turn, the homomorphism $\hat{G} \to \mathrm{Pic}_{Z/k}$ must be trivial, and the assertion follows from Proposition 4.3. $\square$

We say that a curve A is *semi-normal* at some closed point $a \in A$ if the irreducible components in $\mathrm{Spec}(\mathscr{O}^\wedge_{A,a})$ are normal and meet like coordinate axes in $\mathbb{A}^n$. For example, this holds if $\mathscr{O}_{A,a}^\wedge$ is isomorphic to $k[[x,y]]/(xy)$, that is, if the curve has *normal crossings* at $a \in A$.

Proposition 4.7. *Let Y be a normal surface, and $\epsilon : X \to Y$ be a G-torsor for some non-reduced G. Let $A \subset Y$ be a curve whose normalization is a disjoint union of copies of $\mathbb{P}^1$. Then the local ring $\mathscr{O}_{X,x}$ is singular for each point $x \in X$ that maps to a point in $a \in A$ where A is singular and semi-normal.*

Proof. Again it suffices to treat the case that G is either μ_p or α_p. Clearly, we may assume that each irreducible component of A contains the point $a \in A$. Let $A' \to A$ be the morphism that is the normalization outside $a \in A$, and an isomorphism over an open neighborhood of $a \in A$. We have $h^0(\mathscr{O}_{A'}) = 1$, because the curve A' is connected and reduced. Moreover, its Picard scheme then sits in a short exact sequence

$$0 \longrightarrow \mathbb{G}_m^{\oplus r} \longrightarrow \mathrm{Pic}_{A'/k} \longrightarrow \mathbb{Z}^{\oplus s} \longrightarrow 0$$

for some integers $r, s \geq 0$. Since $\hat{G}$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ or α_p, there is only the zero homomorphism $\hat{G} \to \mathrm{Pic}_{A'/k}$, whence the torsor $X \times_Y A' \to A'$ is trivial. It follows that the torsor $X \times_Y A \to A$ is trivial on some open neighborhood of $a \in A$. By Proposition 4.3, the local ring $\mathscr{O}_{X,x}$ is singular. □

Proposition 4.8. *Suppose Y is a normal Gorenstein surface, and $\epsilon : X \to Y$ is a G-torsor for some non-reduced G. Let $A \subset Y$ be a curve isomorphic to $\mathbb{P}^1$. If the invertible sheaf ω_X is nef, then the preimage $\epsilon^{-1}(A)$ passes through some point $x \in X$ where the local ring $\mathscr{O}_{X,x}$ is not factorial, in particular not regular.*

Proof. It suffices to treat the case that G is either μ_p or α_p. Since $\mathrm{Pic}_{A/k}$ is torsion free and reduced, the torsor is trivial over A, whence the preimage $\epsilon^{-1}(A)$ is isomorphic to $A \times G$. In turn, $C = \epsilon^{-1}(A)_{\mathrm{red}}$ is isomorphic to $\mathbb{P}^1$. Seeking a contradiction, we assume that this preimage lies in the locus where X is locally factorial. The conormal sheaf of C in $\epsilon^{-1}(A) = A \times G$ obviously is $\mathscr{I}/\mathscr{I}^2 = \mathscr{O}_A$, and this is also the conormal sheaf of $C \subset X$. It follows that $C^2 = \deg(\mathscr{O}_C(C) = -\deg(\mathscr{I}/\mathscr{I}^2) = 0$. The Adjunction Formula gives $-2 = (K_X \cdot C) + C^2 = (K_X \cdot C)$, contradicting $(K_X \cdot C) \geq 0$. □

We finally turn to some global invariants. Let us formulate the following fact:

Proposition 4.9. *Suppose that Y is normal and proper, and that $P = \mathrm{Pic}^\tau_{Y/k}$ is a finite unipotent group scheme of order p. Let $\epsilon : X \to Y$ be the G-torsor with $\hat{G} = P$. Then*

$$h^0(\mathscr{O}_X) = 1, \quad \chi(\mathscr{O}_X) = p\chi(\mathscr{O}_Y) \quad \text{and} \quad \omega_X = \epsilon^*(\omega_Y).$$

Proof. The assumption on $P = \mathrm{Pic}^\tau_{Y/k}$ means that the group scheme G of order p is local. In turn, $\epsilon : X \to Y$ is a universal homeomorphism. Let $\eta \in Y$ be the generic point. Then $\mathscr{O}_{X,\eta}$ is an Artin local ring, and has degree p as an algebra over the function field $k(Y) = \mathscr{O}_{Y,\eta}$. Moreover, the field extension $k(Y) \subset k(X)$ is purely inseparable, hence its degree $d \geq 1$ is either $d = 1$ or $d = p$.

Seeking a contradiction, we assume $h^0(\mathscr{O}_X) > 1$. Since the ground field k is algebraically closed, the algebra $H^0(X, \mathscr{O}_X)$ and whence the scheme X is non-reduced. Since X contains no embedded associated points, the Artin local ring $\mathscr{O}_{X,\eta}$ is non-reduced. We infer that $d = 1$, such that $X_{\mathrm{red}} \to Y$ is birational. By Zariski's Main Theorem, the finite birational $X_{\mathrm{red}} \to Y$ is an isomorphism, hence the G-torsor $\epsilon : X \to Y$ admits a section, contradiction.

The statement on the Euler characteristic is a special case of [40], Theorem 2 on page 121. It remains to establish the assertion on the dualizing sheaves. The relative dualizing sheaf for the finite dominant morphism $\epsilon : X \to Y$ is given by $\omega_{X/Y} = \underline{\mathrm{Hom}}_{\mathscr{O}_Y}(\mathscr{O}_X, \mathscr{O}_Y)$.

In light of $\omega_X = \epsilon^*(\omega_Y) \otimes \omega_{X/Y}$, it suffices to show that $\omega_{X/Y} \simeq \mathscr{O}_X$. Since $\epsilon : X \to Y$ is a G-torsor, the finite flat $\mathscr{O}_Y$-algebra $\mathscr{O}_X$ of degree p locally admits a p-basis consisting of a single element. It follows that the sheaf of Kähler differentials $\Omega^1_{X/Y}$ is invertible. According to [27], Satz 9 we have $\omega_{X/Y} = (\Omega^1_{X/Y})^{\otimes(1-p)}$, so it suffices to check that $\Omega^1_{X/Y} \simeq \mathscr{O}_X$.

Recall from [20], Section 16.3 that the sheaf of Kähler differentials can be defined by the exact sequence

$$0 \longrightarrow \Omega^1_{X/Y} \longrightarrow \mathscr{O}_{X^{(1)}} \longrightarrow \mathscr{O}_X \longrightarrow 0,$$

where $X = X \times_X X$ is the diagonal inside $X \times_Y X$, and $X^{(1)}$ is its first infinitesimal neighborhood. Since the morphism $\mathrm{can} : X \times G \longrightarrow X \times_Y X$ given by $(x, g) \mapsto (x, xg)$ is an isomorphism, we get a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & X \times G \\ {\scriptstyle\Delta}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{can}} \\ X \times_X X & \longrightarrow & X \times_Y X, \end{array}$$

where the upper map is given by $x \mapsto (x, e)$. Using that the underlying scheme of G is isomorphic to the spectrum of $k[t]/(t^p)$, we infer that the conormal sheaf of $X \subset X \times_Y X$ is isomorphism to $\mathscr{O}_X$, in other words $\Omega^1_{X/Y} \simeq \mathscr{O}_X$. $\square$

Corollary 4.10. *Assumptions as in the proposition. Suppose furthermore that Y is Gorenstein, of even dimension $n \geq 2$, with ω_Y numerically trivial, and such that*

$$\sum_{i=3}^{n} (-1)^i h^i(\mathscr{O}_Y) \geq 0 \quad \text{and} \quad \sum_{j=2}^{n-1} (-1)^j h^j(\mathscr{O}_X) \leq 0.$$

Then the above inequalities are equalities, and we have $p = 2$, $\omega_X = \mathscr{O}_X$, $h^1(\mathscr{O}_X) = 0$ and $h^0(\mathscr{O}_X) = h^n(\mathscr{O}_X) = 1$.

Proof. The invertible sheaf $\omega_X = \epsilon^*(\omega_Y)$ is numerically trivial, so Serre Duality gives $h^n(\mathscr{O}_X) \leq 1$. If the Picard scheme is non-smooth, we have $\mathrm{Pic}^\tau_{Y/k} = \alpha_p$, thus $h^1(\mathscr{O}_Y) = 1$ and the obstruction group $H^2(\mathscr{O}_Y)$ must be non-zero, according to [39], Corollary on page 198. If the Picard scheme is smooth, we have $\mathrm{Pic}^\tau_{Y/k} = \mathbb{Z}/p\mathbb{Z}$, consequently $h^1(\mathscr{O}_Y) = 0$. In both case, the inequality $1 - h^1(\mathscr{O}_Y) + h^2(\mathscr{O}_Y) \geq 1$ holds. This gives a chain of inequalities

$$p \leq p(1 - h^1(\mathscr{O}_Y) + h^2(\mathscr{O}_Y)) \leq p\chi(\mathscr{O}_Y) = \chi(\mathscr{O}_X) \leq 1 - h^1(\mathscr{O}_X) + h^n(\mathscr{O}_X) \leq 2 - h^1(\mathscr{O}_X).$$

The only possibility is $p = 2$, $h^1(\mathscr{O}_X) = 0$ and $h^n(\mathscr{O}_X) = 1$. The latter shows that ω_X has a non-zero global section $s : \mathscr{O}_X \to \omega_X$. This map is necessarily bijective, because ω_X is numerically trivial. $\square$

5. Simply-connected Enriques surfaces and K3-like coverings

Fix an algebraically closed ground field k of characteristic $p = 2$, and let Y be an *Enriques surface*. By definition, this is a smooth proper connected surface with

$$c_1(Y) = 0 \quad \text{and} \quad b_2(Y) = 10.$$

This implies that Y is minimal, and that the dualizing sheaf ω_Y is numerically trivial. They form one of the four classes of surfaces with $c_1 = 0$, the other being the abelian, bielliptic and K3-surfaces, which respective Betti number $b_2 = 6$, $b_2 = 2$ and $b_2 = 22$, according to the classification of surfaces.

For Enriques surfaces, the numerically trivial part $P = \mathrm{Pic}^\tau_{Y/k}$ is a group scheme of order two, and its group of rational points $P(k)$ is generated by ω_Y. The Enriques surfaces come in three types: *Ordinary, classical and supersingular*, which means that P is isomorphic to the respective group schemes μ_2, $\mathbb{Z}/2\mathbb{Z}$ and α_2. Note that there are several other designations in the literature.

Let $G = \underline{\mathrm{Hom}}(P, \mathbb{G}_m)$ be the Cartier dual, such that $\hat{G} = P$, and denote by $\epsilon : X \to Y$ the resulting G-torsor, as discussed in Section 4. If $P = \mu_2$ is diagonalizable, this is an étale covering, and X is a K3 surface endowed with an free involution, which can also be viewed as the universal covering of Y. If $P = \mathbb{Z}/2\mathbb{Z}, \alpha_2$ is unipotent, then the Cartier dual $G = \mu_2, \alpha_2$ is local. In this case we say that Y is a *simply-connected Enriques surface*, and the G-torsor $\epsilon : X \to Y$, which is a universal homeomorphism, is called the *K3-like covering*. Let us say that a connected reduced surface X is a *K3-like covering* if it is isomorphic to the K3-like covering of some Enriques surface Y. We record:

Proposition 5.1. *Suppose that a simply-connected reduced surface X with numerically trivial ω_X admits a free action of a local group scheme G of order two whose quotient $Y = X/G$ is smooth. Then Y is an Enriques surface, the quotient map $\epsilon : X \to Y$ is the K3-like covering, and $\mathrm{Pic}^\tau_{Y/k} \simeq \hat{G}$.*

Proof. First note that since X is reduced, the G-torsor $\epsilon : X \to Y$ is non-trivial, hence the corresponding homomorphism $\hat{G} \to \mathrm{Pic}^\tau_{Y/k}$ is non-zero. This already ensures that Y is not a K3 surface. The scheme X is Cohen–Macaulay and Gorenstein, because this holds for Y, and $\epsilon : X \to Y$ is a G-torsor. In particular, ω_X is invertible. The proof for Proposition 4.9 shows that $\epsilon^*(\omega_Y) = \omega_X$, so ω_Y is numerically trivial. By the classification of surface, Y is either Enriques, abelian or bielliptic. Since G is local, the map $\epsilon : X \to Y$ is a universal homeomorphism, consequently Y is simply-connected. Therefore, Y is neither abelian nor bielliptic, and the only remaining possibility is that Y is an Enriques surface. Thus $P = \mathrm{Pic}^\tau_{Y/k}$ has order two. It follows that the non-zero homomorphism $\hat{G} \to P$ is an isomorphism, and $\epsilon : X \to Y$ must be the K3-like covering. □

From now on, we assume that Y is a simply-connected Enriques surface. We then have the following well-known facts on the K3-like covering X:

Proposition 5.2. *The scheme X is integral, the complete local rings $\mathscr{O}^\wedge_{X,x}$ at the closed points $x \in X$ are reduced Zariski singularities, and we have*

$$\omega_X = \mathscr{O}_X, \quad h^0(\mathscr{O}_X) = h^2(\mathscr{O}_X) = 1 \quad \text{and} \quad h^1(\mathscr{O}_X) = 0.$$

Moreover, the scheme X is not smooth.

Proof. The first statement follows from Proposition 4.5, the second from Corollary 4.10. Suppose X would be smooth. By the classification of surfaces, X is either an abelian surface or a K3 surface, thus the second Betti number is either $b_2 = 22$ or $b_2 = 6$. However, by the topological invariance of Betti numbers, we have $b_2(X) = b_2(Y) = 10$, contradiction. □

An integral curve $A \subset Y$ with $A^2 = -2$ is called a (-2)-*curve*. By the Adjunction Formula, these are the curves on the Enriques surface isomorphic to $\mathbb{P}^1$. An integral curve $C \subset Y$ with $C^2 = 0$ and $\mathrm{Pic}^\tau_{C/k} = \mathbb{G}_a$ is called a *rational cuspidal curve*. These are the curves on Y that are isomorphic to

$$\mathrm{Spec}(k[t^2, t^3]) \cup \mathrm{Spec}(k[t^{-1}]).$$

An integral curve $F \subset Y$ with $F^2 = 0$ and $\mathrm{Pic}^\tau_{F/k} = \mathbb{G}_m$ is called a *rational nodal curve.* These are the curves on Y that can be seen as

$$\mathrm{Spec}(k[[s,t]]/(st)) \cup \mathrm{Spec}(k[s^{-1}, t^{-1}]/(s^{-1}t^{-1} - 1)).$$

Proposition 5.3. *Each (-2)-curve $A \subset Y$, each rational cuspidal curve $C \subset Y$ and each rational nodal curve $F \subset Y$ passes through the image of* $\mathrm{Sing}(X)$*. Each negative-definite curve $E \subset Y$ is an ADE-configuration, and its fundamental-singular locus is contained in the image of* $\mathrm{Sing}(X)$*.*

Proof. To see that $E \subset Y$ is an ADE-configuration, let Z be its fundamental cycle, and consider the exact sequence

$$H^1(Y, \mathscr{O}_Y) \longrightarrow H^1(Z, \mathscr{O}_Z) \longrightarrow H^2(Y, \mathscr{O}_Y(-Z)) \longrightarrow H^2(Y, \mathscr{O}_Y).$$

The map on the right is Serre dual to $H^0(Y, \omega_Y) \to H^0(Y, \omega_Y(Z))$. Any non-zero global section of $\omega_Y(Z)$ would define a curve $Z' \subset Y$ numerically equivalent to $Z \subset Y$. Since the latter is negative definite, the only possibility is $Z' = Z$, whence $\omega_Y \simeq \mathscr{O}_Y$. In this case, we conclude that the inclusion $H^0(Y, \omega_Y) \subset H^0(Y, \omega_Y(Z))$ is bijective, again because $Z \subset Y$ is negative-definite. The upshot is that in the above exact sequence, the map on the right is injective, whence the map on the left is surjective.

If Y is ordinary then $h^1(\mathscr{O}_Y) = 0$, hence $h^1(\mathscr{O}_Z) = 0$ and $E \subset Y$ is an ADE-configuration. If Y is supersingular, we have $h^1(\mathscr{O}_Y) = 1$ and $\mathrm{Pic}^\tau_{Y/k} = \alpha_2$. Seeking a contradiction, we assume that the surjection $H^1(Y, \mathscr{O}_Y) \to H^1(Z, \mathscr{O}_Z)$ is non-zero, hence bijective. Using that curves have smooth Picard groups, we infer that $\mathrm{Pic}^\tau_{E/k} = \mathbb{G}_a$. Each irreducible component $E_i \subset E$ must be a (-2)-curve, whence $E_i \simeq \mathbb{P}^1$ is smooth. It follows that there are two irreducible components $E_i \neq E_j$ with intersection number $n = (E_i \cdot E_j) \geq 2$. Then the intersection matrix $\begin{pmatrix} -2 & n \\ n & -2 \end{pmatrix}$ is not negative-definite, contradicting that $E \subset Y$ is negative-definite. Again we see that $h^1(\mathscr{O}_Z) = 0$, and the curve $E \subset Y$ is an ADE-configuration.

The assertions on the curves A, F, E and the image of $\mathrm{Sing}(X)$ follow from Propositions 4.8, 4.7. and 4.6, respectively. It remains to treat the rational cuspidal curve C, which has $\mathrm{Pic}^\tau_{C/k} = \mathbb{G}_a$. If the restriction map $\mathrm{Pic}^\tau_{Y/k} \to \mathrm{Pic}^\tau_{C/k}$ is trivial, we may apply Proposition 4.3. Now suppose that the restriction map is non-zero. Then $\tilde{C} = \epsilon^{-1}(C)$ is a non-trivial G-torsor over C. It becomes trivial after pulling-back along the normalization $\nu : \mathbb{P}^1 \to C$, according to Proposition 4.2. We thus get a cartesian diagram

$$\begin{tikzcd} \mathbb{P}^1 \oplus \mathscr{O}_{\mathbb{P}^1} \arrow[r] \arrow[d] & \tilde{C} \arrow[d] \\ \mathbb{P}^1 \arrow[r] & C. \end{tikzcd}$$

where the horizontal arrows are birational. Consequently, the Weil divisor $\tilde{C} \subset X$ is of the form $\tilde{C} = 2B$, where $B = \tilde{C}_{\mathrm{red}}$ is an irreducible Weil divisor, and the morphism $B \to C$ is birational. Since C is the rational cuspidal curve, the morphism $B \to C$ is either the normalization or the identity map. To see that the letter does not happen, note that the G-torsor $\epsilon : X \to Y$ becomes trivial after pulling-back to itself. Since $h^0(\mathscr{O}_X) = 1$, this means that the restriction map $\mathrm{Pic}^\tau_{Y/k} \to \mathrm{Pic}^\tau_{X/k}$ is trivial. It follows that the composite map $\mathrm{Pic}^\tau_{Y/k} \to \mathrm{Pic}^\tau_B$ is trivial, contradiction. Thus we must have $B = \mathbb{P}^1$. Consider the short exact sequence

$$0 \longrightarrow \mathscr{N} \longrightarrow \mathscr{O}_{\tilde{C}} \longrightarrow \mathscr{O}_B \longrightarrow 0.$$

The sheaf $\mathscr{N}$ is torsion free, and therefore invertible when regarded as module over $\mathscr{O}_B$. By Riemann–Roch, its degree is

$$\deg(\mathscr{N}) = \chi(\mathscr{N}) - \chi(\mathscr{O}_B) = \chi(\mathscr{O}_{\tilde{C}}) - 2\chi(\mathscr{O}_B) = 2\chi(\mathscr{O}_C) - 2\chi(\mathscr{O}_B) = -2.$$

Seeking a contradiction, we now assume that the surface X is locally factorial along B. Then $B \subset X$ is a Cartier divisor isomorphic to $\mathbb{P}^1$ with selfintersection number $B^2 = -\deg(\mathscr{N}) = 2$. Riemann–Roch yields $-2 = \deg(K_B) = (K_X + B)\cdot B = 2$, contradiction. $\square$

As the referee noticed, an alternative argument may use a genus-one fibration $\varphi : Y \to \mathbb{P}^1$ and Katsura's result ([25], Proposition 3.2) that for some non-zero rational 1-form ω on the projective line, in suitable coordinates actually given by $\omega = dt/t$ or $\omega = dt$, the pull-back $\varphi^*(\omega_Y)$ to the Enriques surface extends to a global section of $\Omega^1_{Y/k}$. The zero-locus of this global 1-form on Y describes the the image of the singular locus of the K3-like covering X, and from this the assertion can be inferred. In some sense, the above proof is a coordinate-free version for this.

We now furthermore assume that the K3-like covering X has only isolated singularities, that is, the surface X is normal. Let $r : S \to X$ be a resolution of singularities. The *geometric genus* of a singularity $x \in X$ is the length $p_g(\mathscr{O}_{X,x}) = \operatorname{length} R^1r_*(\mathscr{O}_S)_x$. A singularity $x \in X$ is called *rational* if its geometric genus is $p_g = 0$, and we call it *elliptic* if $p_g = 1$. Note that this term has various meanings in the literature. For example, Wagreich [55] uses it for singularities with arithmetic genus $p_a = 1$, which is the largest integer of the form $1 - \chi(\mathscr{O}_Z)$, where $Z \subset S$ ranges over the exceptional divisors. According to loc. cit., this includes all singularities with fundamental genus $p_f = 1$, which is defined as $p_f(\mathscr{O}_{X,x}) = h^1(\mathscr{O}_Z)$, where $Z \subset S$ is the fundamental divisor [1]. A related class of singularities are the minimally elliptic singularities studied by Laufer [32]. A singularity with Hilbert–Samuel multiplicity $e(\mathscr{O}_{X,x}) = 2$ is called a *double point*.

Proposition 5.4. *All rational singularities on the K3-like covering X are rational double points, and the only possible types are A_1, E_7^0, E_8^0 and D_{2n}^0. There is at most one non-rational singularity $x \in X$. If present, it is an elliptic double point.*

Proof. The singularities on the K3-like covering are Zariski singularities, whence have multiplicity $e = 2$, by Proposition 1.4. Furthermore, they are Gorenstein, thus all rational singularities must be rational double points. The only rational double points that are Zariski singularities are of type $A_1, D_{2n}^0, E_7^0, E_8^0$, according to Proposition 3.1. Next, consider the Leray–Serre spectral sequence for the minimal resolution of singularities $r : S \to X$. It gives an exact sequence

$$0 \longrightarrow H^1(X, \mathscr{O}_X) \longrightarrow H^1(S, \mathscr{O}_S) \longrightarrow H^0(X, R^1r_*\mathscr{O}_S) \longrightarrow H^2(X, \mathscr{O}_X). \tag{6}$$

The term on the left vanishes, and the term on the right is one-dimensional. Moreover, we have $K_S = K_{S/X}$, which is a Cartier divisor supported on the exceptional curve with coefficients ≤ 0.

Now suppose that there is a non-rational singularity $x \in X$. Then $K_{S/X} < 0$, and Serre Duality gives $H^2(S, \mathscr{O}_S) = 0$. Thus the Picard scheme is smooth, and the connected component $\operatorname{Pic}^0_{S/k}$ is an abelian variety. On the other hand, each integral curve $E \subset S$ contained in the exceptional locus is a curve of genus zero, so the restriction map $\operatorname{Pic}^0_{S/k} \to \operatorname{Pic}^0_{E/k}$ is trivial. In turn, the Albanese morphism $S \to A$ contracts each such $E \subset S$, and thus factors over Y. Using that $\operatorname{Pic}_{Y/k}$ is 0-dimensional, we conclude that $\dim(A) = 0$, and thus $H^1(S, \mathscr{O}_S) = 0$. It now follows from the above exact sequence that $R^1r_*(\mathscr{O}_S)$

has length ≤ 1. In turn, there is precisely one non-rational singularity, and its geometric genus is one. $\square$

We will later see that if there is an elliptic singularity $x \in X$, there are no further singularities. In any case, the structure of the smooth surface S depends on the nature of the singularities on X:

Proposition 5.5. *The smooth surface S is a K3 surface with $\rho(S) = b_2(S) = 22$ if the K3-like covering X contains only rational singularities and the resolution $r : S \to X$ is minimal, and is a rational surface with $\rho(S) = b_2(S) \geq 11$ otherwise.*

Proof. If all singularities are rational, the exact sequence (6) shows that $h^1(\mathscr{O}_S) = 0$. Furthermore, we have $\omega_{S/X} = \mathscr{O}_S$, whence $\omega_S = \mathscr{O}_S$. By the classification of algebraic surfaces, S is a K3 surface. It then has Betti number $b_2 = 22$, whereas our Enriques surface Y and its K3-like covering have $b_2(X) = b_2(Y) = 10$. It follows that the exceptional divisor for $r : S \to X$ consist of $12 = b_2(S) - b_2(X)$ irreducible components. This ensures $\rho(S) = b_2(S)$.

If there is a non-rational singularity $x \in X$, then $-K_S = -K_{S/X}$ is an effective Cartier divisor, and we still have $H^1(S, \mathscr{O}_S) = 0$. By the classification of surfaces, S is rational. In any case, the Picard number satisfies $\rho(S) > \rho(X) \geq \rho(Y) = 10$, because there is at least one singularity, and thus $b_2(S) \geq 11$. $\square$

Every Enriques surface Y admits at least one genus-one fibration $\varphi : Y \to \mathbb{P}^1$. It has one or two multiple fibers $Y_b = 2C$, and their multiplicity is $m = 2$. Two multiple fibers occur precisely for the classical Enriques surfaces, and we then have $\omega_Y = \mathscr{O}_Y(C_1 - C_2)$, where $C_1, C_2 \subset Y$ are the two half-fibers. Two fibrations $\varphi, \varphi' : Y \to \mathbb{P}^1$ are called *orthogonal* if for respective half-fibers $C, C' \subset Y$ have intersection number $(C \cdot C') = 1$.

For our simply-connected Enriques surface Y whose K3-like covering X is normal, there are some strong restrictions on the nature of these fibrations. The following were already observed by Cossec and Dolgachev (see [12], Proposition 5.7.3 and its Corollary):

Theorem 5.6. *For every genus-one fibration $\varphi : Y \to \mathbb{P}^1$, the following holds:*

(i) *The fibration $\varphi : Y \to \mathbb{P}^1$ is elliptic, and not quasielliptic.*
(ii) *Each singular fiber $Y_b = \varphi^{-1}(b)$ is of Kodaira type* I_n *for some $1 \leq n \leq 9$, or of Kodaira type* II, III *or* IV.
(iii) *There are at least two other elliptic fibrations $\varphi', \varphi'' : Y \to \mathbb{P}^1$ so that $\varphi, \varphi', \varphi''$ are mutually orthogonal.*

Proof. Suppose there is some quasielliptic fibration $\varphi : Y \to \mathbb{P}^1$. Then almost all fibers Y_a, $a \in \mathbb{P}^1$ are rational cuspidal curves. According to Proposition 5.3, the normal surface X must contain infinitely many singularities, contradiction.

The Kodaira types in (ii) correspond to the curves of canonical type that are reduced. If there would be another Kodaira type, the corresponding fiber contains an ADE-configuration of type D_4. Its fundamental cycle is non-reduced, and again X contains a curve of singularities, contradiction.

The last property holds for all Enriques surfaces with a few exceptions, according to [12], Theorem 3.5.1. The exceptions are called *extra special* in loc. cit., and each of them contains a genus-one fibration containing a fiber of type II^*, I_4^* or III^*, which are non-reduced. This contradicts Property (ii). $\square$

An integral curve $E \subset Y$ is called *non-movable* if $h^0(\mathscr{O}_Y(E)) = 1$.

Proposition 5.7. *Let $E \subset Y$ be a non-movable elliptic curve. Then the schematic preimage $\epsilon^{-1}(E) \subset X$ is an elliptic curve, and the K3-like covering X is smooth along this preimage.*

Proof. The curve $2E$ is movable, and defines an elliptic fibration $\varphi : Y \to \mathbb{P}^1$ for which $E \subset Y_b$ is a half-fiber. Suppose first that Y is classical. Then there is another half-fiber C, and $\omega_Y = \mathscr{O}_Y(E - C)$ has order two in $\operatorname{Pic}(Y)$. According to [12], Theorem 5.7.2 multiple fibers are not wild, so the restriction $\omega_Y|E = \mathscr{O}_E(E)$ has order two in $\operatorname{Pic}(E)$. It follows that the restriction map $\operatorname{Pic}^\tau_{Y/k} \to \operatorname{Pic}^\tau_{E/k}$ is injective. Now suppose that Y is supersingular, such that $\omega_Y = \mathscr{O}_Y$. The short exact sequence $0 \to \mathscr{O}_Y(-E) \to \mathscr{O}_Y \to \mathscr{O}_E \to 0$ yields a long exact sequence

$$H^1(Y, \mathscr{O}_Y) \longrightarrow H^1(E, \mathscr{O}_E) \longrightarrow H^2(Y, \mathscr{O}_Y(-E)) \longrightarrow H^2(Y, \mathscr{O}_Y).$$

The map on the right is Serre dual to $H^0(Y, \mathscr{O}_Y) \to H^0(Y, \mathscr{O}_Y(E))$. The latter is bijective, because the curve $E \subset Y$ is non-movable. In turn, the map on the left $H^1(Y, \mathscr{O}_Y) \to H^1(E, \mathscr{O}_E)$ is surjective, and again we conclude that the restriction map $\operatorname{Pic}^\tau_{Y/k} \to \operatorname{Pic}^\tau_{E/k}$ is injective.

In both cases, we see that the G-torsor $E' = \epsilon^{-1}(E) \to E$ is non-trivial. According to Proposition 4.9, the curve E' is integral, with $h^0(\mathscr{O}_{E'}) = h^1(\mathscr{O}_{E'}) = 1$. This curve is normal. If not, the normalization $E'' \to E'$ must have $h^1(\mathscr{O}_{E''}) < 1$, giving an integral surjection $\mathbb{P}^1 \to E$, contradiction. Thus E' is a smooth Cartier divisor in the surface X, and we conclude that X is smooth along E'. $\square$

Let us write $x_1, \ldots, x_m \in X$ for the singularities on the K3-like coverings. The complete local rings $\mathscr{O}_{X,x_i}$ are Zariski singularity, thus given by a formal power series of the form $g_i = z^2 - f_i(x, y)$. Recall that the *Tjurina number* satisfies

$$\tau_i = 2 \cdot \operatorname{length} k[[x, y]]/(f_i, \partial f_i/\partial x, \partial f_i/\partial y),$$

by Proposition 1.7. Note that this is always an even number.

Proposition 5.8. *We have $\Theta_{X/k} = \mathscr{O}_X^{\oplus 2}$, and the Tjurina numbers for the singularities satisfy $\sum_{i=1}^m \tau_i = 24$. In particular, if S is a K3 surface, then the exceptional divisor for the resolution of singularities $r : S \to X$ has 12 irreducible components.*

Proof. If S is K3, this is due to Ekedahl, Hyland and Shepherd-Barron [17], and their arguments generalize as follows: Since $\epsilon : X \to Y$ is a torsor for the Cartier dual $G = \underline{\operatorname{Hom}}(\operatorname{Pic}^0_{Y/k}, \mathbb{G}_m)$, we have a short exact sequence

$$0 \longrightarrow \mathscr{O}_X \longrightarrow \Theta_{X/k} \longrightarrow \mathscr{L} \longrightarrow 0$$

for some invertible sheaf $\mathscr{L}$. The map on the left corresponds to the p-closed vector field defining the G-action on X. Using that $\det(\Theta_{X/k}) = \omega_X = \mathscr{O}_X$, we conclude that $\mathscr{L} \simeq \mathscr{O}_X$. The preceding extension splits, because we have $\operatorname{Ext}^1(\mathscr{O}_X, \mathscr{O}_X) = H^1(X, \mathscr{O}_X) = 0$. Summing up, $\Theta_{X/k} = \mathscr{O}_X^{\oplus 2}$.

Now suppose that S is a K3 surface. It then has Betti number $b_2 = 22$, whereas our Enriques surface Y and its K3-like covering have $b_2(X) = b_2(Y) = 10$. It follows that the exceptional divisor for $r : S \to X$ consist of $12 = b_2(S) - b_2(X)$ irreducible components. Since the resulting singularities are rational double points, we must have $\sum \tau_i = 24$.

Finally suppose that S is rational. Here we have Betti number $b_2(S) = 10 - K_S^2$ and Chern number $c_2(S) = 2 + b_2(S) = 12 - K_S^2$. The Chern numbers are related, according to [17], Proposition 3.12 and Corollary 3.13, in the following way:

$$c_2(S) = c_2(X) + \gamma \quad \text{and} \quad c_2(\Theta_{X/k}) = c_2(X) - \tau.$$

Here $c_2(X) = c_2(L^\bullet_{X/k})$ is defined as the second Chern class of the cotangent complex $L^\bullet_{X/k}$, which in our situation is a complex of length one comprising locally free sheaves of finite rank. It can be defined with the help of an embedding $X \subset \mathbb{P}^n$ into some projective

n-space. Moreover, $\tau = \sum \tau_i$, and the other correction term γ is given by the Local Noether Formula

$$-1 = -\operatorname{length} R^1 g_*(\mathscr{O}_S) = \frac{K_S^2 + \gamma}{12},$$

from [17], Proposition 3.12. In other words $\gamma = -12 - K_S^2$. Combining these equations, one gets

$$0 = c_2(\Theta_{X/k}) = c_2(S) - \gamma - \tau = (12 - K_S^2) + (12 + K_S^2) - \tau = 24 - \tau.$$

Again, we have $\sum \tau_i = 24$. □

6. Normal surfaces with trivial tangent sheaf

Let k be an algebraically closed ground field of characteristic $p = 2$. In order to understand and construct simply-connected Enriques surface Y and their K3-like coverings X, we now impose eight axiomatic conditions on certain elliptic fibrations.

Let $h : S \to \mathbb{P}^1$ be a smooth elliptic surface, whose total space S is either a rational surface or a K3 surface. An curve $C \subset S$ is called *vertical* if each irreducible component is contained in some fiber $h^{-1}(a)$, $a \in \mathbb{P}^1$. It is *horizontal* if each irreducible component dominates $\mathbb{P}^1$. The same locution is used for fibrations with normal total space. Let $S \to S' \to \mathbb{P}^1$ be the relative minimal model, obtained by successively contracting vertical (-1)-curves. Note that if S is a K3 surface, there are no (-1)-curves at all, such that $S = S'$.

Let $E \subset S$ be a vertical negative-definite curve, and $r : S \to X$ be its contraction. Then X is a proper normal surface, with an induced elliptic fibration $f : X \to \mathbb{P}^1$, satisfying $h = f \circ r$. We denote by $x_1, \ldots, x_r \in X$ the images of the connected components $E_1, \ldots, E_r \subset E$, and assume that each local ring $\mathscr{O}_{X,x_i}$ is singular. Then $r : S \to X$ is a resolution of singularities, but we do not assume that it is the minimal resolution. Our conditions are:

- **(E0)** *The fibers $f^{-1}(a) \subset X$ are reduced for all closed points $a \in \mathbb{P}^1$.*
- **(E1)** *The singular local rings $\mathscr{O}_{X,x_i}$, $1 \leq i \leq r$ are complete intersections with free tangent modules.*
- **(E2)** *The Tjurina numbers τ_i for the local rings $\mathscr{O}_{X,x_i}$ add up to $\sum \tau_i = 24$.*
- **(E3)** *If S is a rational surface, the skyscraper sheaf $R^1 r_*(\mathscr{O}_S)$ has length one; it is supported in the multiple fiber if $h : S \to \mathbb{P}^1$ has a multiple fiber.*
- **(E4)** *If S is a rational surface, then $\omega_S = \mathscr{N}(D)$, where D is a divisor supported on the exceptional divisor $E \subset S$, and the invertible sheaf $\mathscr{N}$ is non-fixed.*
- **(E5)** *There is a horizontal Cartier divisor $F \subset X$ that is an elliptic curve.*
- **(E6)** *For each singularity $x_i \in X$, the local rings $\mathscr{O}_{X,x_i}$ are Zariski singularities.*
- **(E7)** *There is a horizontal Cartier divisor $C \subset X$ that is a rational cuspidal curve, and the singular point on C is a regular point on X.*

Here the invertible sheaf $\mathscr{N} \in \operatorname{Pic}(S)$ is called *non-fixed* if $h^0(\mathscr{N}) \geq 1$, and furthermore $h^0(\mathscr{N}) = 1$ implies $\mathscr{N} = \mathscr{O}_S$. Note that (E1) is a consequence of (E6), according to Proposition 1.6. However, it is useful to have (E1) as a separate condition, because it will be handy when the Enriques surface acquires rational double points. Our conditions indeed help to analyze and construct simply-connected Enriques surface, because we have:

Proposition 6.1. *Suppose Y is a simply-connected Enriques surface whose K3-like covering X is normal, $r : S \to X$ is a resolution of singularities, $E \subset S$ is the exceptional divisor, and $h : S \to \mathbb{P}^1$ is the fibration induced from some elliptic fibration $\varphi : Y \to \mathbb{P}^1$. Then conditions* (E1)–(E6) *do hold, whereas condition* (E7) *does not hold.*

Proof. Condition (E4) is true, because $\omega_X = \mathscr{O}_X$, and thus $\omega_S = \mathscr{O}_S(K_{S/X})$, where the relative canonical divisor $K_{S/X}$ is supported by the exceptional divisor. Property (E6) and thus also (E1) hold according to Theorem 2.2. Proposition 5.8 gives (E2), and Theorem 5.6 ensures (E5).

Condition (E3) pertains to the case that S is rational. According to Proposition 5.5 together with Proposition 5.4, there is exactly one elliptic singularity $x_1 \in X$, whence $R^1r_*(\mathscr{O}_S)$ has length one. Now suppose that the relatively minimal rational elliptic surface $S' \to \mathbb{P}^1$ has a multiple fiber S'_a. Write C' for the underlying indecomposable curve of canonical type. Then $K_{S'} = -C'$, according to [12], Proposition 5.6.1. Seeking a contradiction, we assume that the elliptic singularity $x_1 \in X$ does not lie on X_a. Without restriction, we may assume that $r : S \to X$ is the minimal resolution of singularities. Then $K_S = -E$, where $E \subset S$ is a negative-definite curve mapping to $x_1 \in X$. In turn, $K_{S'} = -E'$, where $E' \subset S'$ is the image of E. Thus the curves $E' \neq C'$ are linearly equivalent. But since S'_a is a multiple fiber, the indecomposable curve of canonical type C' is non-movable, contradiction.

Finally, we have to verify that condition (E7) does not hold. Suppose to the contrary that there is a Cartier divisor $C \subset X$ that is a rational cuspidal curve such that the local ring $\mathscr{O}_{X,c}$ is regular, where $c \in C$ is the singular point. The Adjunction Formula $\deg(\omega_C) = (K_X + C) \cdot C$ implies $C^2 = 0$. Consequently, the image $D = \epsilon(C)$ on the Enriques surface is an integral rational curve with $D^2 = 0$. It follows that D is a rational cuspidal curve. By Proposition 5.3, there is a singular point $x \in X$ mapping to D. But for each closed point $x \in C$, the local ring $\mathscr{O}_{X,x}$ is regular, because either $x = c$ or $\mathscr{O}_{C,x}$ is regular, contradiction. □

The goal of this section is to establish a converse for Proposition 6.1. Let us write $\mathscr{O}_X(n)$ for the preimage of the invertible sheaves $\mathscr{O}_{\mathbb{P}^1}(n)$ under the elliptic fibration $f : X \to \mathbb{P}^1$, and likewise we write $\mathscr{O}_S(n)$ and $\mathscr{O}_{S'}(n)$. The following is the key step in producing Enriques surfaces with normal K3-like coverings:

Theorem 6.2. *Suppose that $r : S \to X$ satisfies the conditions* (E1)–(E4). *Then the following holds:*

(i) *The dualizing sheaf is given by $\omega_X = \mathscr{O}_X$.*
(ii) *The tangent sheaf $\Theta_{X/k}$ is locally free of rank two, and there is an extension $0 \to \mathscr{F}^\vee \to \Theta_{X/k} \to \mathscr{F} \to 0$ for some coherent subsheaf $\mathscr{F}$ inside the invertible sheaf $f^*(\Theta_{\mathbb{P}^1/k}) = \mathscr{O}_X(2)$.*
(iii) *If also condition* (E0) *holds, the inclusion $\mathscr{F} \subset \mathscr{O}_X(2)$ is an equality, and we have $\Theta_{X/k} = \mathscr{O}_X(-n) \oplus \mathscr{O}_X(n)$ for some $0 \leq n \leq 2$.*
(iv) *We have $n = 0$ if furthermore* (E5) *holds, wheras $n \neq 0$ if we have* (E7) *instead.*

Proof. Our first step is to show that $\omega_X = \mathscr{O}_X$. The case that S is K3 is obvious: Every negative-definite curve on S produces a rational double point on X, thus our X is Gorenstein with $\omega_X = \mathscr{O}_X$. The more interesting case is that S is rational: Then $H^1(S, \mathscr{O}_S) = H^2(S, \mathscr{O}_S) = 0$, and the Leray–Serre spectral sequence for the resolution of singularities $r: S \to X$ gives an exact sequence

$$0 \to H^1(X, \mathscr{O}_X) \to H^1(S, \mathscr{O}_S) \to H^0(X, R^1 r_*(\mathscr{O}_S)) \to H^2(X, \mathscr{O}_X) \to H^2(S, \mathscr{O}_S).$$

It follows that $h^1(\mathscr{O}_X) = 0$ and $h^2(\mathscr{O}_X) = 1$, the latter by condition (E3). Serre duality gives $h^0(\omega_X) = 1$. Suppose that ω_X is non-trivial. Then $\omega_X = \mathscr{O}_X(C)$ for some Cartier divisor $C \subset X$ that is not linearly equivalent to any other effective Cartier divisor. According to condition (E4), we have $\omega_S = \mathscr{N}(D)$, where $D \subset S$ is supported by the exceptional divisor $E \subset S$ and $\mathscr{N}$ is non-fixed. If $\mathscr{N} = \mathscr{O}_S$, then ω_X is trivial outside the singularities, whence everywhere trivial, contradiction. Thus there are two curves $A \neq B$ on S both giving $\mathscr{N}$. Write $A = A_1 + A_2$ and $B = B_1 + B_2$ where A_2, B_2 are the parts supported by the exceptional divisor $E \subset S$. Using $h^0(\omega_X) = 1$, we infer that both $A_1, B_1 \subset S$ map to $C \subset X$, hence $A_1 = B_1$. It follows that $A_2 \neq B_2$ are linearly equivalent and supported by the exceptional divisor $E \subset S$. This contradicts the fact that the intersection form $(E_i \cdot E_j)$ is negative-definite. Summing up, we have $\omega_X = \mathscr{O}_X$.

The tangent sheaf $\Theta_{X/k}$ is locally free of rank two, according to condition (E1). Note that the dual of $\Theta_{X/k}$ coincides with the bidual of $\Omega^1_{X/k}$. Consequently we have $\det(\Theta_{X/k}) = \omega_X^\vee = \mathscr{O}_X$. The next step is to verify that the second Chern class $c_2(\Theta_{X/k})$ vanishes. The case of K3 surfaces was already treated in [17], Section 3. We proceed in a similar way: In both cases on has

$$c_2(\Theta_{X/k}) = c_2(X) - \tau \quad \text{and} \quad c_2(X) = c_2(S) - \nu,$$

where $\tau = \sum \tau_i = 24$ is the sum of Tjurina numbers, and ν is a correction term

$$\nu = c_1^2(X) - c_1^2(S) - 12 \cdot \operatorname{length} R^1 r_*(\mathscr{O}_S).$$

according to the Local Noether Formula [17], Proposition 3.12. In our situation $c_1^2(X) = c_1^2(\Theta_X) = 0$. If S is K3, then $c_1^2(S) = 0$, $c_2(S) = 24$ and $\nu = 0$, which gives $c_2(\Theta_{X/k}) = 0$. If S is rational, we have $c_2(S) = 2 + b_2 = 12 - c_1^2(S)$ and $\nu = c_1^2(S) - 12$, also giving

$$c_2(\Theta_{X/k}) = c_2(S) - \tau - \nu = (12 - c_1^2(S)) - 24 - (-c_1^2(S) - 12) = 0.$$

Our next step is to regard the tangent sheaf as an extension. Consider the short exact sequence $f^*(\Omega^1_{\mathbb{P}^1/k}) \to \Omega^1_{X/k} \to \Omega^1_{X/\mathbb{P}^1} \to 0$. The map on the left is injective, because the function field extension $k(\mathbb{P}^1) \subset k(X)$ is separable and X has no embedded components. Thus we have a short exact sequence

$$0 \longrightarrow f^*(\Omega^1_{\mathbb{P}^1/k}) \longrightarrow \Omega^1_{X/k} \longrightarrow \Omega^1_{X/\mathbb{P}^1} \longrightarrow 0.$$

The sheaf $\Omega^1_{X/\mathbb{P}^1}$ is invertible at each point $x \in X$ at which the fiber $X_{f(x)}$ is regular. Dualizing, we get an exact sequence $0 \to (\Omega^1_{X/\mathbb{P}^1})^\vee \to \Theta_{Y/k} \to \mathscr{F} \to 0$, for some coherent subsheaf $\mathscr{F}$ of $f^*(\Theta_{\mathbb{P}^1/k}) = \mathscr{O}_X(2)$. Both outer terms are invertible in codimenson one, and the term on the left is reflexive. From $\det(\Theta_{Y/k}) = \mathscr{O}_X$ we infer that $(\Omega^1_{X/\mathbb{P}^1})^\vee = \mathscr{F}^\vee$, and get the desired extension

$$0 \longrightarrow \mathscr{F}^\vee \longrightarrow \Theta_{X/k} \longrightarrow \mathscr{F} \longrightarrow 0. \tag{7}$$

This establishes assertions (i) and (ii).

Now assume that condition (E0) holds, which means that all closed fibers $f^{-1}(a)$ are reduced. Then $\Omega^1_{X/\mathbb{P}^1}$ is invertible in codimension one, hence the inclusion $\mathscr{F} \subset \mathscr{O}_X(2)$ is an isomorphism outside finitely many closed points. It follows that the dual map $\mathscr{O}_X(-2) \to \mathscr{F}^\vee$ is an isomorphism and that $\mathscr{F} = \mathscr{I}\mathscr{O}_X(2)$ for some sheaf of ideals $\mathscr{I} \subset \mathscr{O}_X$ that defines a finite subscheme $Z \subset X$. Note that the structure sheaf $\mathscr{O}_Z$ has finite projective dimension, by the exact sequence (7). Rewrite this sequence as $0 \to \mathscr{L}^\vee \to \Theta_{Y/k} \to \mathscr{I}\mathscr{L} \to 0$, with $\mathscr{L} = \mathscr{O}_X(2)$. Then

$$0 = c_2(\Theta^\vee_{X/k}) = -c_1^2(\mathscr{L}) + \operatorname{length}(\mathscr{O}_X/\mathscr{I}).$$

Since $\mathscr{L}$ comes from $\mathbb{P}^1$, we have $c_1^2(\mathscr{L}) = 0$ and whence $\mathscr{I} = \mathscr{O}_X$. Summing up, $\mathscr{F} = \mathscr{L}$, and the tangent sheaf is an extension of $\mathscr{L} = \mathscr{O}_X(2)$ by $\mathscr{L}^\vee = \mathscr{O}_X(-2)$. The extension class lies in

$$\operatorname{Ext}^1(\mathscr{O}_X(2), \mathscr{O}_X(-2)) = H^1(X, \mathscr{O}_X(-4)).$$

The latter can be computed with the Leray–Serre spectral sequence, together with the Projection Formula: We get an exact sequence

$$0 \longrightarrow H^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(-4)) \longrightarrow H^1(X, \mathscr{O}_X(-4)) \longrightarrow H^0(\mathbb{P}^1, R^1 f_*(\mathscr{O}_X)(-4)).$$

The term on the right vanishes: The higher direct image sheaf $R^1 f_*(\mathscr{O}_X)$ commutes with base-change. Since $S \to \mathbb{P}^1$ has no wild fibers, according to [12], Proposition 5.6.1, it follows that the map $t \mapsto h^1(\mathscr{O}_{X_t})$ takes the constant values one, and thus $R^1 f_*(\mathscr{O}_X)$ is invertible. The Leray–Serre spectral sequence yields

$$\chi(R^1 f_*(\mathscr{O}_X)) = \chi(\mathscr{O}_{\mathbb{P}^1}) - \chi(\mathscr{O}_X) = 1 - 2 = -1,$$

thus $R^1 f_*(\mathscr{O}_X) = \mathscr{O}_{\mathbb{P}^1}(-2)$, and consequently $R^1 f_*(\mathscr{O}_X)(-4)$ has no non-zero global section. We infer that the extension of $\mathscr{O}_X(2)$ by $\mathscr{O}_X(-2)$ giving $\Theta^\vee_{X/k}$ comes from an extension of $\mathscr{O}_{\mathbb{P}^1}(-2)$ by $\mathscr{O}_{\mathbb{P}^1}(2)$, in particular $\Theta_{X/k}$ is the preimage of some locally free sheaf $\mathscr{E}$ of rank two on $\mathbb{P}^1$. But all such sheafs on $\mathbb{P}^1$ are sums of line bundles, according to Grothendieck (see for example [43]). The splitting type of $\mathscr{E}$ is of the form $(-n, n)$ for some integer $n \geq 0$, because $c_1(\mathscr{E}) = 0$. Since $\mathscr{E}$ surjects onto $\mathscr{O}_{\mathbb{P}^1}(2)$, only the three possibilities $n = 0, 1, 2$ exists. This gives assertion (iii).

Now suppose that furthermore condition (E5) holds: Consider the horizontal Cartier divisor $F \subset X$ that is an elliptic curve, and let $d > 0$ be the degree of the map $F \to \mathbb{P}^1$. We have an exact sequence

$$0 \longrightarrow \mathscr{O}_F(-F) \longrightarrow \Omega^1_{X/k}|_F \longrightarrow \Omega^1_{F/k} \longrightarrow 0.$$

The scheme X is regular along the regular Cartier divisor F, so the above sheaves are locally free on F. The sheaves $\Omega^1_{X/k}$ and $\Theta^\vee_{X/k}$ have the same restriction to F, which is thus of form $\mathscr{M} \oplus \mathscr{M}^\vee$, where $\deg(\mathscr{M}) = nd$. Seeking a contradiction, we assume $n \neq 0$, such that $\deg(\mathscr{M}) > 0$. This implies that there is only the zero map $\mathscr{M} \to \Omega^1_{F/k} = \mathscr{O}_F$. It follows that $\mathscr{M}^\vee \to \Omega^1_{F/k}$ is surjective, thus bijective, contradicting $\deg(\mathscr{M}^\vee) < 0$.

Finally, suppose that condition (E7) holds: Consider the horizontal Cartier divisor $C \subset X$ that is a rational cuspidal curve, and let $d > 0$ be the degree of the map $C \to \mathbb{P}^1$. Now we have an exact sequence

$$0 \longrightarrow \mathscr{O}_C(-C) \longrightarrow \Omega^1_{X/k}|_C \longrightarrow \Omega^1_{C/k} \longrightarrow 0.$$

To compute the term on the right, write the rational cuspidal curve with two affine charts as

$$C = \operatorname{Spec} k[t^2, t^3] \cup \operatorname{Spec} k[t^{-1}].$$

On the first chart, the Kähler differentials are generated by dt^2, dt^3 modulo the relation $t^4 dt^2$. On the second chart, dt^{-1} is a generator. On the overlap, we have $dt^3 = t^2 dt = t^4 \cdot dt^{-1}$. It follows that the coherent sheaf $\mathscr{N} = \Omega^1_{C/k}/\text{Torsion}$ is invertible with $\deg(\mathscr{N}) = -4$. As in the preceding paragraph, $\Omega^1_{X/k}|_C = \mathscr{M} \oplus \mathscr{M}^\vee$ with $\deg(\mathscr{M}) = nd$. Hence the map $\mathscr{M} \to \mathscr{N}$ vanishes, and $\mathscr{M}^\vee \to \mathscr{N}$ is surjective, thus bijective. It follows that $nd = 4$, therefore $n \neq 0$. $\square$

Proposition 6.3. *Suppose that $r : S \to X$ satisfies conditions* (E0)–(E5)*. Then the canonical map of restricted Lie algebras $H^0(X, \Theta_{X/k}) \to H^0(\mathbb{P}^1, \Theta_{\mathbb{P}^1/k})$ is injective, and every non-zero vector in $H^0(X, \Theta_{X/k})$ is p-closed.*

Proof. According to Theorem 6.2, the canonical map for the sheaves of Kähler differentials $f^*(\Omega^1_{\mathbb{P}^1/k}) \to \Omega^1_{X/k}$ induces a short exact sequence

$$0 \longrightarrow \mathscr{O}_X(-2) \longrightarrow \Theta_{X/k} \longrightarrow f^*(\Theta_{\mathbb{P}^1/k}) \longrightarrow 0.$$

In turn, we get an exact sequence $0 \to \mathscr{O}_{\mathbb{P}^1}(-2) \to f_*(\Theta_{X/k}) \to \Theta_{\mathbb{P}^1/k}$, and thus an inclusion $H^0(X, \Theta_{X/k}) \subset H^0(\mathbb{P}^1, \Theta_{\mathbb{P}^1/k})$.

Since $\Theta_{\mathbb{P}^1/k} = \mathscr{O}_{\mathbb{P}^1}(2)$, the restricted Lie algebra $\mathfrak{h} = H^0(\mathbb{P}^1, \Theta_{\mathbb{P}^1/k})$ is 3-dimensional. It can be seen as a semidirect product $\mathfrak{h} = \mathfrak{a} \rtimes \mathfrak{b}$, where $\mathfrak{a} = k^{\oplus 2}$ with trivial Lie bracket and p-map, and $\mathfrak{b} = kb$ with $b^{[p]} = b$. The semidirect product structure comes from the homomorphism $\mathfrak{b} \to \mathfrak{gl}(\mathfrak{a})$ given by $b \mapsto \mathrm{id}_{\mathfrak{a}}$, compare the discussion in [49], Section 3. Explicitely, Lie bracket and p-map are given by

$$[a + \lambda b, a' + \lambda' b] = \lambda a' - \lambda' a \quad \text{and} \quad (a + \lambda b)^{[2]} = \lambda(a + \lambda b).$$

It is easy to see that every non-zero vector in $\mathfrak{a} \rtimes \mathfrak{b}$ is p-closed, whence the same holds for the restricted Lie algebra $H^0(X, \Theta_X)$. □

Now suppose that $S \to X$ satisfies the condition (E0)–(E6). Then the restricted Lie algebra $\mathfrak{g} = H^0(X, \Theta_{X/k})$ is two-dimensional, and every non-zero vector is p-closed. Moreover, the singularities $x_1, \ldots, x_r \in X$ are Zariski. Let $A_i = \mathscr{O}^\wedge_{X,x_i}$ be the corresponding complete local ring. Then $A_i\mathfrak{g} = \Theta_{A_i/k}$. In turn, Proposition 2.5 yields canonical lines $\mathfrak{l}_i \subset \mathfrak{g}$, for $1 \leq i \leq r$.

Theorem 6.4. *Suppose $S \to X$ satisfies conditions* (E0)–(E6)*. Then X is a K3-like covering. More precisely, for each vector field $D \in \mathfrak{g} = H^0(X, \Theta_{X/k})$ not contained in the union of the canonical lines $\mathfrak{l}_i \subset \mathfrak{g}$, the ensuing quotient $Y = X/G$ is a simply-connected Enriques surface. The Picard group $\mathrm{Pic}^\tau_{Y/k}$ is the Cartier dual of G, and the projection $\epsilon : X \to Y$ is the K3-like covering.*

Proof. Since the vector field D avoids the canonical lines, the G-action is free and the quotient $Y = X/G$ is smooth, according to Proposition 2.5. Furthermore, $\omega_X = \mathscr{O}_X$ by Theorem 6.2. In the resolution of singularities $r : S \to X$, the smooth surface S is either K3 or rational, and in both cases the algebraic fundamental group $\pi_1(S)$ vanishes. It follows that $\pi_1(X)$ vanishes as well. Now Proposition 5.1 ensures that Y is a simply-connected Enriques surface, the quotient morphism $\epsilon : X \to Y$ is the K3-like covering, and $\mathrm{Pic}^\tau_{Y/k}$ is the Cartier dual of G. □

7. The twistor curves in the moduli stack

Twistor curves where introduced and studied by Ekedahl, Hyland and Shepherd-Barron [17]. We now investigate further their remarkable discovery. Let X be a normal K3-like covering, and suppose that there is an elliptic fibration $f : X \to \mathbb{P}^1$ whose fibers are reduced. Combining Proposition 6.1 and Theorem 6.4 one obtains:

Proposition 7.1. *The tangent sheaf $\Theta_{X/k}$ is isomorphic to $\mathscr{O}_X^{\oplus 2}$, every vector in the 2-dimensional restricted Lie algebra $\mathfrak{g} = H^0(X, \Theta_{X/k})$ is p-closed, and for every point $x \in X$, we have $\mathscr{O}_{X,x}\mathfrak{g} = \Theta_{X/k,x}$. Moreover, the morphism $f : X \to \mathbb{P}^1$ induces an inclusion of restricted Lie algebras $\mathfrak{g} \subset H^0(\mathbb{P}^1, \Theta_{\mathbb{P}^1/k})$.*

Proof. Choose an Enriques surface Y so that $\epsilon : X \to Y$ is the K3-like covering. Let $\varphi : Y \to \mathbb{P}^1$ be the elliptic fibration induced from $f : X \to \mathbb{P}^1$. According to Proposition 6.1, conditions (E1)–(E6) hold. The claim on $\Theta_{X/k}$ now follows from Theorem 6.2. Moreover, every vector space basis $D_1, D_2 \in \mathfrak{g}$ yields an isomorphism $(D_1, D_2) : \mathscr{O}_X^{\oplus 2} \to \Theta_{X/k}$, whence an $\mathscr{O}_{X,x}$-basis $D_1, D_2 \in \Theta_{X/k,x}$ at each point $x \in X$. The assertions on the restricted Lie algebra $\mathfrak{g}$ come from Proposition 6.3. $\square$

Let $x_1, \ldots, x_r \in X$ be the singularities. The corresponding local rings $A_i = \mathscr{O}_{X,x_i}$ are Zariski singularities, and we have $A_i\mathfrak{g} = \Theta_{A_i/k}$. According to Proposition 2.5, this gives canonical lines $\mathfrak{l}_i \subset \mathfrak{g}$, $1 \leq i \leq r$ in the two-dimensional restricted Lie algebra $\mathfrak{g} = H^0(X, \Theta_{X/k})$.

We call the projective line $\mathbb{P}(\mathfrak{g}) \simeq \mathbb{P}^1$ the *twistor curve* attached to the K3-like covering X, in analogy to twistor spaces coming from the two-spheres of complex structures on complex hyperkähler manifolds. The closed points $t_i \in \mathbb{P}(\mathfrak{g})$ corresponding to the canonical lines $\mathfrak{l}_i \subset \mathfrak{g}_i$ are called the *boundary points*. We write

$$\mathbb{P}(\mathfrak{g})^\circ = \mathbb{P}(\mathfrak{g}) \smallsetminus \{t_1, \ldots, t_r\}$$

for the complementary open subset of *interior points*. We saw in the last section that each interior point $t \in \mathbb{P}(\mathfrak{g})$ of the twistor line gives a simply connected Enriques surface $Y_t = X/G_t$, where G_t is the height-one group scheme corresponding to the non-zero vector field $D \in H^0(X, \Theta_{X/k})$ giving the interior point $t \in \mathbb{P}(\mathfrak{g})$. It is easy to see that this construction extends to a flat family

$$\mathfrak{Y} \longrightarrow \mathbb{P}(\mathfrak{g})^\circ$$

of Enriques surfaces $\mathfrak{Y}_t = X/G_t$. This family can be seen as a morphism

$$\mathbb{P}(\mathfrak{g})^\circ \longrightarrow \mathscr{M}_{\mathrm{Enr}}$$

into the moduli stack $\mathscr{M}_{\mathrm{Enr}}$ of Enriques surface. We now seek to understand this map more closely, in particular how to verify that this map is non-constant.

The elliptic fibration $f : X \to \mathbb{P}^1$ on the K3-like covering induces a family of elliptic fibrations $\mathfrak{Y} \to \mathbb{P}^1 \times \mathbb{P}(\mathfrak{g})^\circ$ over the interior of the twistor line. Moreover, the nonzero vector field

$$D \in \mathfrak{g} = H^0(X, \Theta_{X/k}) \subset H^0(\mathbb{P}^1, \Theta_{\mathbb{P}^1/k})$$

induce non-zero vector fields on $\mathbb{P}^1$. In turn, we get a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\epsilon"] \arrow[d, "f"'] & X/G \arrow[d, "\varphi"] \\ \mathbb{P}^1 \arrow[r] & \mathbb{P}^1/G. \end{tikzcd} \tag{8}$$

The vertical map on the right is the induced elliptic fibration $\varphi : Y \to \mathbb{P}^1$ on the simply-connected Enriques surface $Y = X/G$. Since $\Theta_{\mathbb{P}^1/k} = \mathscr{O}_{\mathbb{P}^1}(2)$, this induced vector field vanishes along a divisor $A \subset \mathbb{P}^1$ of degree two. Geometrically speaking, the tangent vectors comprising the vector field $D \in H^0(X, \Theta_{X/k})$ are vertical at all points $x \in X$ lying over a point $a \in A$. The divisor $A \subset \mathbb{P}^1$ of degree $\deg(A) = 2$ consists either of one or two closed points.

Proposition 7.2. *If $A \subset \mathbb{P}^1$ consist of two points, then the simply-connected Enriques surface $Y = X/G$ is classical. If this divisor consists of only one point, Y is supersingular.*

Proof. Write $\tilde{\mathfrak{g}} = H^0(\mathbb{P}^1, \Theta_{\mathbb{P}^1/k})$ for the three-dimensional restricted Lie algebra that contains $\mathfrak{g} = H^0(X, \Theta_{X/k})$. With $\mathbb{P}^1 = \operatorname{Spec} k[t] \cup \operatorname{Spec} k[t^{-1}]$, we may regard

$$D_0 = \partial/\partial t, \quad D_1 = t\partial/\partial t \quad \text{and} \quad D_2 = t^2\partial/\partial t$$

as a vector space basis for $\tilde{\mathfrak{g}}$. Given a linear combination $D = \sum \lambda_i D_i$, one computes $D(t) = \sum \lambda_i t^i$. A direct computation shows that $D^2 = 0$ holds if and only if $\lambda_1 = 0$. The condition $D^2 = 0$ means that $G = \alpha_p$, in other words that the Enriques surface Y is supersingular. On the other hand, the condition $\lambda_1 = 0$ means that D, regarded as a section of $\Theta_{\mathbb{P}^1/k} = \mathscr{O}_{\mathbb{P}^1}(2)$, vanishes at a single point. $\square$

Proposition 7.3. *The fibers $Y_b \subset Y$ over the images $b \in \mathbb{P}^1 = \mathbb{P}^1/G$ of the points $a \in A \subset \mathbb{P}^1$ are precisely the multiple fibers for $\varphi : Y \to \mathbb{P}^1$.*

Proof. Write $B \subset \mathbb{P}^1/G$ for the image of $A \subset \mathbb{P}^1$, and consider the complementary open subsets $U = \mathbb{P}^1 \smallsetminus A$ and $V = \mathbb{P}^1/G \smallsetminus B$. The G-action on U is free, and the projection $U \to V$ is a G-torsor. The diagram (8), which is G-equivariant, yields a morphism $X_U \to U \times_V Y_V$ of G-torsors over Y_V. This must be an isomorphism, by the general fact that categories of torsors are groupoids.

Now let Y_b be some multiple fiber, for some closed point $b \in \mathbb{P}^1/G$. Then for each closed point $y \in Y_b$, the local ring has $\operatorname{edim}(\mathscr{O}_{Y_b,y}) = 2$. Consequently, the fiber product $\mathbb{P}_1 \times_{\mathbb{P}^1/G} Y_b$ has embedding dimension ≥ 3 at each closed point. In light of the preceding paragraph, it follows that $b \notin V$, because our K3-like covering X is normal. Summing up, we have shown that multiple fibers for $\varphi : X \to \mathbb{P}^1$ may occur at most over points $b \in B$.

If $A \subset \mathbb{P}^1$ consists of only one point, the Y is supersingular by Proposition 7.2, and there is precisely one multiple fiber. If A consists of two points, then Y is classical, and there are precisely two multiple fibers. In both case, we conclude that for all points $b \in B$, the fiber Y_b must be multiple. $\square$

It follows that if the restricted Lie algebra $\mathfrak{g} \subset H^0(\mathbb{P}^1, \Theta_{\mathbb{P}^1/k})$ contains global vectors fields D with $D^2 = 0$ and $D^2 \neq 0$, and the former defines an interior point in the twistor curve, then the flat family $\mathfrak{Y} \to \mathbb{P}(\mathfrak{g})^\circ$ of simply-connected Enriques surfaces contains both supersingular and classical members, whence the morphism $\mathbb{P}(\mathfrak{g})^\circ \to \mathscr{M}_{\text{Enr}}$ is non-constant. Moreover, one may use the position of the points $b \in \mathbb{P}^1/G$ where Y_b is multiple, relative to other points $b' \in \mathbb{P}^1/G$ where $Y_{b'}$ is singular, to deduce further results on the non-constancy of the twistor construction $\mathbb{P}(\mathfrak{g})^\circ \to \mathscr{M}_{\text{Enr}}$.

Note that some simply-connected Enriques surface Y admit non-zero global vector fields. This is automatically the case if Y is supersingular, according to [12], Proposition 1.4.2. For Y classical it holds if and only if it is a so-called *exceptional Enriques* surface, according to [16], Theorem B. The problem to describe global 1-forms rather then vector fields was solved in [25], Section 3.

8. K3-like coverings and elliptic fibrations

Let k be an algebraically closed ground field of characteristic $p = 2$, and Y be a simply-connected Enriques surface, with K3-like covering $\epsilon : X \to Y$. The latter is a torsor for the Cartier dual G of $\operatorname{Pic}^\tau_{X/k}$. Assume that X has only isolated singularities. We now want to study the geometry and the singularities of X in more detail, by choosing an elliptic fibration $\varphi : Y \to \mathbb{P}^1$.

Proposition 8.1. *The Stein factorization for the composite map $\varphi \circ \epsilon : X \to \mathbb{P}^1$ is the Frobenius map $F : \mathbb{P}^1 \to \mathbb{P}^1$.*

Proof. This could be deduced from the diagram (8). Let us give a direct, independent argument: Write C for the Stein factorization, such that we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\epsilon} & Y \\ {\scriptstyle\psi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi} \\ C & \xrightarrow[s]{} & \mathbb{P}^1. \end{array}$$

The surjection $s : C \to \mathbb{P}^1$ is radical, because this holds for $\epsilon : X \to Y$. Whence $C = \mathbb{P}^1$, and the morphism is a power of the relative Frobenius map $F : \mathbb{P}^1 \to \mathbb{P}^1$. Since $\deg(\epsilon) = 2$, we either have $s = F$ or $s = \mathrm{id}$. To rule out the latter is suffices to check that $\mathscr{A} = \varphi_*(\epsilon_*(\mathscr{O}_X))$ has rank ≥ 2 as $\mathscr{O}_{\mathbb{P}^1}$-module. Consider first the case that Y is classical, such that $G = \mu_2$. Then $\epsilon_*(\mathscr{O}_X) = \mathscr{O}_Y \oplus \mathscr{L}$ for some invertible sheaf $\mathscr{L} \in \mathrm{Pic}(Y)$ of order two, whence $\mathscr{L} = \omega_Y = \mathscr{O}_Y(C_1 - C_2)$, where $C_1, C_2 \subset Y$ are the two half-fibers. It immediately follows that $\mathrm{rank}(\mathscr{A}) \geq 2$.

Finally, suppose that Y is supersingular, such that $G = \alpha_2$. Then we have a short exact sequence $0 \to \mathscr{O}_Y \to \epsilon_*(\mathscr{O}_X) \to \mathscr{O}_Y \to 0$, according to [15], Proposition 1.7. In turn, we get a long exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^1} \longrightarrow \mathscr{A} \longrightarrow \varphi_*(\mathscr{O}_Y) \longrightarrow R^1\varphi_*(\mathscr{O}_Y).$$

Write $R^1\varphi_*(\mathscr{O}_Y) = \mathscr{L} \oplus \mathscr{T}$ for some invertible sheaf $\mathscr{L}$ and some torsion sheaf $\mathscr{T}$. The Canonical Bundle Formula ([7], Theorem 2) ensures that $\mathscr{L} = \mathscr{O}_{\mathbb{P}^1}(-2)$. It follows that the coboundary map $\mathscr{O}_{\mathbb{P}^1} = \varphi_*(\mathscr{O}_Y) \to R^1\varphi_*(\mathscr{O}_Y)$ has nontrivial kernel, and again $\mathrm{rank}(\mathscr{A}) \geq 2$. □

Write $f : X \to \mathbb{P}^1$ for the Stein factorization, which sits in the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\epsilon} & Y \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle\varphi} \\ \mathbb{P}^1 & \xrightarrow[F]{} & \mathbb{P}^1. \end{array}$$

We thus get a canonical morphism $X \to Y^{(2/\mathbb{P}^1)}$ from the K3-like covering to the *Frobenius pullback* $Y^{(2/\mathbb{P}^1)} = Y \times_{\mathbb{P}^1} \mathbb{P}^1$.

Proposition 8.2. *The morphism $X \to Y^{(2/\mathbb{P}^1)}$ is the normalization. Its ramification locus consists of the preimages $\epsilon^{-1}(Y_b) \subset X$ of the multiple fibers Y_b, $b \in \mathbb{P}^1$.*

Proof. Both maps $X \to Y$ and $Y^{(2/\mathbb{P}^1)} \to Y$ have degree two, whence $X \to Y^{(2/\mathbb{P}^1)}$ is birational. It thus must be the normalization, because we assume throughout that X is normal. For each closed point $y \in Y_b$ lying on a multiple fiber, the embedding dimension is $\mathrm{edim}(\mathscr{O}_{Y_b,y}) = 2$, so the local rings for each closed point on the preimage of Y_b in the Frobenius pullback $Y^{(2/\mathbb{P}^1)}$ has embedding dimension three. It follows that the preimage $\epsilon^{-1}(Y_b) \subset X$ belongs to the ramification locus, which is the locus on X defined by the conductor ideal. On the other hand, $Y^{(2/\mathbb{P}^1)}$ is smooth at each point mapping to a point $y \in Y$ where the morphism $\varphi : Y \to \mathbb{P}^1$ is smooth. Since all fibers of $\varphi : Y \to \mathbb{P}^1$ are reduced by Theorem 5.6, it follows that the ramification locus for $X \to Y^{(2/\mathbb{P}^1)}$ consists precisely of the preimages of the multiple fibers. □

Next, consider the *jacobian fibration* $J \to \mathbb{P}^1$ attached to the elliptic fibration $\varphi : Y \to \mathbb{P}^1$, which is a relatively minimal elliptic fibration endowed with a section $O \subset J$. According to [12], Theorem 5.7.2 the smooth surface J is rational. Moreover, all such rational elliptic surfaces $J \to \mathbb{P}^1$ were classified by Lang [31]. In some sense, we completely understand such jacobian fibrations. Our strategy throughout is to relate the rational elliptic surface $J \to \mathbb{P}^1$ to the fibration $f : X \to \mathbb{P}^1$ on the K3-like covering.

Proposition 8.3. *For each closed point $b \in \mathbb{P}^1$, the fibers Y_b and J_b have the same Kodaira type. In particular, the rational elliptic surface $J \to \mathbb{P}^1$ has only reduced fibers, and the possible Kodaira types for the singular fibers are I_n with $1 \leq n \leq 9$, II, III or IV.*

Proof. If $b \in \mathbb{P}^1$ is a closed point whose fiber Y_b is non-multiple, then $J \to \mathbb{P}^1$ admits a section over the formal completion $\mathrm{Spec}(\mathscr{O}^\wedge_{\mathbb{P}^1,b})$, which yields an isomorphisms

$$J \otimes_{\mathbb{P}^1} \mathrm{Spec}(\mathscr{O}^\wedge_{\mathbb{P}^1,b}) \simeq Y \otimes_{\mathbb{P}^1} \mathrm{Spec}(\mathscr{O}^\wedge_{\mathbb{P}^1,b}).$$

For the multiple fibers Y_b, one can at least say that J_b and Y_b have the same Kodaira type, according to a general result of Liu, Lorenzini and Raynaud [34], Theorem 6.6. The second assertion follows from Theorem 5.6. □

To proceed, consider for the rational elliptic surface $J \to \mathbb{P}^1$ the Frobenius pullback

$$X' = J^{(2/\mathbb{P}^1)} = J \times_{\mathbb{P}^1} \mathbb{P}^1.$$

According to Proposition 9.1 below, this is a normal surface, having only isolated Zariski singularities, and the dualizing sheaf is $\omega_X = \mathscr{O}_X$. Clearly, the elliptic fibration $X' \to \mathbb{P}^1$ admits a section. Write $\eta \in \mathbb{P}^1$ for the generic point.

Proposition 8.4. *We have $X'_\eta \simeq X_\eta$ if and only if the fibration $f : X \to \mathbb{P}^1$ admits a section.*

Proof. The condition is obviously necessary. Conversely, suppose that $f : X \to \mathbb{P}^1$ admits a section. By definition, the generic fiber of $J \to \mathbb{P}^1$ is the jacobian for the generic fiber of $Y \to \mathbb{P}^1$. In turn, $X'_\eta = (J^{(2/\mathbb{P}^1)})_\eta$ is the jacobian for $(Y^{(2/\mathbb{P}^1)})_\eta$. The latter coincides with X_η, by Proposition 8.2. Thus $X'_\eta \simeq X_\eta$. □

We need to understand this condition better, in order to exploit the connection between the K3-like covering X of the Enriques surface Y and the Frobenius pullback X' of the rational elliptic surface J. Let us call a curve $A \subset Y$ on the Enriques surface a *two-section* if A is an irreducible curve that is horizontal with respect to the elliptic fibration $\varphi : Y \to \mathbb{P}^1$, of relative degree two. We say that $A \subset Y$ is a *radical two-section* if the surjection $A \to \mathbb{P}^1$ is radical, hence a universal homeomorphism.

Proposition 8.5. *Let $A \subset Y$ be a radical two-section for $\varphi : Y \to \mathbb{P}^1$. Then $B = \epsilon^{-1}(A)_{\mathrm{red}}$ is a section for $f : X \to \mathbb{P}^1$. Conversely, the image $A = \epsilon(B)$ for any section $B \subset X$ is a radical two-section. Moreover, such A and B exists if and only if $X'_\eta \simeq X_\eta$.*

Proof. The first assertion is obvious, because X is the normalization of $Y^{(2/\mathbb{P}^1)}$ and the two morphisms $F : \mathbb{P}^1 \to \mathbb{P}^1$ and $\varphi : A \to \mathbb{P}^1$ coincide generically. Conversely, suppose that $B \subset X$ is a section. Its image $A = \epsilon(B)$ is an integral Cartier divisor, and the composition $B \to A \to \mathbb{P}^1$ is radical of degree two. We thus have either $\deg(B/A) = 1$ and $\deg(A/\mathbb{P}^1) = 2$, or $\deg(B/A) = 2$ and $\deg(A/\mathbb{P}^1) = 1$. The latter case is impossible, because the elliptic fibration $\varphi : Y \to \mathbb{P}^1$ has no sections. Thus $A \subset Y$ must be a two-section. The last statement follows from Proposition 8.4. □

A curve $A \subset Y$ is called *rational* if it is reduced, and its normalization is isomorphic to the projective line $\mathbb{P}^1$.

Proposition 8.6. *A two-section $A \subset Y$ is radical if and only if the curve A is rational.*

Proof. The condition is clearly necessary. Conversely, suppose that the two-section $A \subset Y$ is rational, and let $\mathbb{P}^1 \to A$ be the normalization map. The induced G-torsor $X_{\mathbb{P}^1} = \mathbb{P}^1 \times G$ is trivial. It follows that the Weil divisor $\epsilon^{-1}(A) \subset X$ is non-reduced, thus its reduction B yields a section for $f : X \to \mathbb{P}^1$. According to Proposition 8.5, the image $A = \epsilon(B)$ is a radical two-section. □

Proposition 8.7. *Suppose there is another elliptic fibration $\varphi' : Y \to \mathbb{P}^1$ that is orthogonal to $\varphi : Y \to \mathbb{P}^1$, and has a singular half-fiber $C \subset \varphi'^{-1}(b')$. Then some irreducible component $A \subset C$ is a radical two-section for $\varphi : Y \to \mathbb{P}^1$.*

Proof. By the classification of singular fibers, each irreducible component $A \subset C$ is a rational curve. Let $F = \varphi^{-1}(b)$ be a fiber, such that $(C \cdot F) = 2$. Since F is movable, there is some irreducible component $A \subset C$ with $1 \leq (A \cdot F) \leq 2$. We must have $(A \cdot F) = 2$, because $\varphi : Y \to \mathbb{P}^1$ admits no section. The assertion now follows form Proposition 8.6. □

For the applications we have in mind, it is permissible to replace the chosen elliptic fibration $\varphi : Y \to \mathbb{P}^1$ by another, more suitable one:

Proposition 8.8. *Suppose that the simply-connected Enriques surface Y contains a (-2)-curve $A \subset Y$, or a rational cuspidal curve C that is not movable. Then there is an elliptic fibration $\varphi : Y \to \mathbb{P}^1$ admitting a radical two-section.*

Proof. Suppose there is a (-2)-curve $A \subset Y$. Then the desired elliptic fibration exists. In characteristic $p \neq 2$, Cossec showed that the desired genus-one fibration exists ([11], Theorem 4). This carries over to characteristic $p = 2$, according to Lang ([29], Theorem A.3. The fibration is indeed elliptic by Theorem 5.6.

Now suppose that there is a non-movable rational cuspidal curve $C \subset Y$. Then it is a half-fiber of some genus-one fibration $\psi : Y \to \mathbb{P}^1$. Again by Theorem 5.6, there is an orthogonal fibration $\varphi : Y \to \mathbb{P}^1$. In both cases, the two-sections are radical, by Proposition 8.6. □

We do not know if the conditions of the preceding proposition holds for *all* simply-connected Enriques surfaces. Note that there are Enriques surfaces Y without (-2)-curves, according to [28], Theorem 4.3. In any case, the exceptions must be restricted in the following sense:

Proposition 8.9. *Suppose the simply-connected Enriques surface Y has no elliptic fibration admitting a radical two-section. Then for every elliptic fibration $\varphi : Y \to \mathbb{P}^1$, the following holds: The singular fibers have Kodaira type I_1 or II, and only the following configurations are possible:*

$$1^{12}, \quad \mathrm{II} + 1^8, \quad \mathrm{II} + 1^6, \quad \mathrm{II} + 1^5, \quad \mathrm{II} + \mathrm{II} + \mathrm{II}, \quad \mathrm{II} + \mathrm{II}, \quad \textit{and} \quad \mathrm{II}.$$

Moreover, the half-fibers are smooth.

Proof. In light of Proposition 8.8 and Proposition 8.7, the Enriques surface Y contains neither (-2)-curves nor non-movable rational cuspidal curves. In turn, only I_0, I_1 or II are possible Kodaira types, and the half-fibers are smooth. According to Proposition 8.3, the Kodaira types of the fibers Y_b and J_b coincide, for all closed points $b \in \mathbb{P}^1$. Looking at Lang's classification [31] of rational elliptic surfaces, we obtain the list of possible configurations. □

Let us now assume that $\varphi : Y \to \mathbb{P}^1$ admits a radical two-section. Then we have an isomorphism $X_\eta = X'_\eta$, an in particular the normal surfaces X and X', which both have trivial canonical class, are birational. Using terminology from the *minimal model program* in dimension three and higher, one may regard X as a *flop* of X', because their canonical class neither became more negative nor more positive.

Choose a common resolution of singularities $X' \stackrel{r'}{\leftarrow} S \stackrel{r}{\rightarrow} X$. The induced projection $h : S \to X' \to \mathbb{P}^1$ is an elliptic fibration on the smooth surface S, and we write $S \to J'$ for the relative minimal model. Locally, J' is obtained by applying the Tate Algorithm to the Frobenius base-change of the Weierstraß equation for J. Summing up, we have the following commutative diagram:

$$\begin{tikzcd}
 & S \arrow[ddl] \arrow[dd, "r'"'] \arrow[dr, "r"] & & & \\
 & & X \arrow[rr, "\epsilon"] \arrow[dd, "f"'] & & Y \arrow[dd, "\varphi"] \\
J' & X' \arrow[rr] \arrow[dr] & & J \arrow[dr] & \\
 & & \mathbb{P}^1 \arrow[rr, "F"'] & & \mathbb{P}^1
\end{tikzcd}$$

If X and X' have only rational singularities, then their minimal resolution of singularities coincides with the relatively minimal model of $X_\eta \simeq X'_\eta$, whence both are given by S. It follows that S is a K3-surface, and the morphism $S \to J'$ is an isomorphism. In this case, $X' \leftarrow S \to X$ are both crepant resolutions. On the other hand, if X has an elliptic singularity, such that S and hence J' is a rational surface, we have $K_{J'} = -F$ for some fiber $F \subset J'$. Now one should regard J' as a *flip* of both X' and X, because the canonical class became more negative. We observe:

Lemma 8.10. *For all closed points $a \in \mathbb{P}^1$, with Frobenius image $b = F(a) \in \mathbb{P}^1$, the four curves Y_b, X_a, X'_a and J_b have the same dual graph.*

Proof. The curves X_a and Y_b have the same dual graph, because the K3-like covering $\epsilon : X \to Y$ is a universal homeomorphism. The same argument applies to the curves X'_a and J_b. Finally, the curves Y_b and J_b have the same dual graph according [34], Theorem 6.6. □

The singularities on the K3-like covering X of the Enriques surface Y are closely related to the singularities on the Frobenius pull-back X' of the rational elliptic surface J, at least outside the multiple fibers:

Proposition 8.11. *For each closed point $a \in \mathbb{P}^1$ whose Frobenius image $b \in \mathbb{P}^1$ has a non-multiple fiber Y_b, there is an isomorphism of two-dimensional schemes*

$$X \times_{\mathbb{P}^1} \operatorname{Spec}(\mathscr{O}^\wedge_{\mathbb{P}^1,a}) \simeq X' \times_{\mathbb{P}^1} \operatorname{Spec}(\mathscr{O}^\wedge_{\mathbb{P}^1,a}).$$

Proof. Write $R = \mathscr{O}^\wedge_{\mathbb{P}^1,b}$ for the complete local ring of the point $b \in \mathbb{P}^1$. Since Y_b is non-multiple, the induced projection $Y \times_{\mathbb{P}^1} \operatorname{Spec}(R) \to \operatorname{Spec}(R)$ admits a section, which in turn induces an identification $Y \times_{\mathbb{P}^1} \operatorname{Spec}(R) = J \times_{\mathbb{P}^1} \operatorname{Spec}(R)$. In light of Proposition 8.2, taking the fiber product with the Frobenius morphism $F : \mathbb{P}^1 \to \mathbb{P}^1$ yields the assertion. □

9. Ogg's Formula and Frobenius base-change

In this section let k be an algebraically closed ground field of arbitrary characteristic $p > 0$, and $J \to B$ be a smooth elliptic surface that is jacobian and relatively minimal. For simplicity, we assume that B is a smooth proper connected curve, such that J is a smooth proper connected surface, although the analysis also applies to local situations as well.

Let us write $B^{(p)}$ for the scheme B, endowed with the new structure morphism $B^{(p)} \to \mathrm{Spec}(k) \xrightarrow{F} \mathrm{Spec}(k)$, obtained by transport of structure with the absolute Frobenius of k. Then the absolute Frobenius on B becomes the *relative Frobenius morphism* $B^{(p)} \to B$. The cartesian square

$$\begin{array}{ccc} J^{(p/B)} & \longrightarrow & J \\ \downarrow & & \downarrow \\ B^{(p)} & \longrightarrow & B \end{array}$$

defines an integral proper connected surface $J^{(p/B)}$ endowed with an elliptic fibration. We call $J^{(p/B)}$ the *Frobenius pullback*. Let us first record:

Lemma 9.1. *The singularities on the Frobenius pullback $J^{(p/B)}$ are Zariski singularities.*

Proof. Fix a closed point $x \in J^{(p/B)}$, and write $a \in B^{(p/B)}$, $y \in J$ and $b \in B$ for its images. Choose a uniformizer $\pi \in \mathscr{O}_{B,b}$, and write $z = \pi$ for the resulting uniformizer $z \in \mathscr{O}_{B^{(p)},a}$. Then the morphism of complete local k-algebras

$$k[[z]] \longrightarrow \mathscr{O}^\wedge_{B^{(p)},b}, \quad \sum \lambda_i z^i \longmapsto \sum \lambda_i^p z^i$$

is bijective. Using this identification, we may regard the extension $\mathscr{O}^\wedge_{B,b} \subset \mathscr{O}^\wedge_{B^{(p)},a}$ as $k[[\pi]] \subset k[[\pi, z]]/(z^p - \pi)$. In turn, we get

$$\mathscr{O}^\wedge_{J^{(p/B)},x} = \mathscr{O}^\wedge_{J,y}[z]/[z^p - \pi],$$

where the uniformizer $\pi \in \mathscr{O}^\wedge_{B,b}$ becomes a non-unit $\pi \in \mathscr{O}^\wedge_{J,y}$. Since the local ring $\mathscr{O}_{J,y}$ is regular, the local ring $\mathscr{O}^\wedge_{J^{(p/B)},x}$ is a Zariski singularity. □

Let $S \to J^{(p/B)}$ be a resolution of singularities, which inherits an elliptic fibration $S \to B$, and write $S \to J'$ for the contraction to the relative minimal model. Then we have a new smooth surface J' endowed with an elliptic fibration $J' \to B^{(p)}$, which is jacobian and relatively minimal. We call J' the *smooth Frobenius pullback*. Note that in general, the Kodaira dimension of the surfaces J and J' are different.

Fix a section $O \subset J$ for the jacobian elliptic fibration $J \to B$, and let $W \to \mathbb{P}^1$ be the ensuing Weierstraß fibration. The normal surface W is obtained by contracting all vertical irreducible curves disjoint to the section. Locally at each point $b \in B$, it is given by some minimal Weierstraß equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \qquad a_i \in \mathscr{O}_{B,b}.$$

We denote by $v_b \geq 0$ the *valuation of the discriminant* $\Delta_b \in \mathscr{O}_{B,b}$ for any such minimal Weierstraß equation, by $m_b \geq 1$ the *number of irreducible components* in the fiber J_b, and by $\delta_b \geq 0$ the *wild part of the conductor* for the Galois representation on the geometric generic l-torsion points. Here $l \neq p$ is any prime different from the characteristic. If the fiber J_b is of semistable, that is of type I_m, then $\delta_b = 0$ and $v_b = m$. If the fiber is unstable, Ogg's Formula gives

$$v_b = 2 + \delta_b + (m_b - 1). \tag{9}$$

For a proof, see the original paper [41] and also [48], which containes a complete proof and generalizations.

The closed point $b \in B$ may also be seen as a closed point on the scheme $B^{(p)}$. To avoid confusion, we denote it by $a \in B^{(p)}$. This gives numerical invariants v_a, δ_a, m_a for the smooth Frobenius pullback $J' \to B$. Now observe that we may obtain J' from the base-changed Weierstraß fibration $W^{(p/B)}$ as follows: Run the Tate Algorithm [53] until the Weierstraß equation becomes minimal and then do the minimal resolution of singularities. This gives another invariant: The *length of the Tate Algorithm* $\lambda_a \geq 0$, that is, the number of repetitions that are necessary to finish the Tate Algorithm. The case $\lambda_a = 0$ means that the Weierstraß equation is already minimal.

If the fiber is semistable, we have $\lambda_a = 0$ and $m_a = pm_b$. Indeed, a local computation shows that on $W^{(p/B)}$ we get rational double points of type A_{p-1} lying over the singular points of the semistable fiber J_b. The situation is more interesting at the unstable fibers:

Lemma 9.2. *Suppose the fiber J_b is unstable. Then J'_a is unstable as well, and its numerical invariants are given by the formulas*

$$v_a = pv_b - 12\lambda_a, \quad \delta_a = \delta_b \quad \textit{and} \quad m_a = pm_b + (p-1)(\delta_b + 1) - 12\lambda_a.$$

In characteristic two, this means $v_a = 2v_b - 12\lambda_a$ and $m_a = 2m_b + \delta_b + 1 - 12\lambda_a$.

Proof. Among all function field extensions that achieve semistable reduction, there is a smallest one, and this smallest one is a Galois extension. This follows from the Néron–Ogg–Shafarevich Criterion (see for example [8], Section 7.4, Theorem 5). In particular, unstable fibers stay unstable under purely inseparable field extensions. Each round of the Tate Algorithm reduces the valuation of the discriminant by 12. The wild part of the conductor depends on a Galois representation, and is thus not affected by purely inseparable extensions. The equation for the number $m_a \geq 1$ of irreducible components is now a consequence of Ogg's Formula (9). □

By construction, the Weierstraß model W is normal, and the fibers $W \to B$ contain at most one non-smooth point, which is a rational double point. In turn, the base-change $W^{(p/B)}$ is normal, and contains fiber-wise at most one singularity.

Proposition 9.3. *We have $\lambda_a > 0$ if and only if the normal surface $W^{(p/B)}$ contains a non-rational singularity lying over the point $a \in B^{(p)}$.*

Proof. There is nothing to prove if $W^{(p/B)}$ is smooth along the fiber. Now suppose that a singularity $x \in W^{(p/B)}$ is present, such that the fiber $C = W_a^{(p/B)}$ is a rational cuspidal curve. Let $S \to W^{(p/B)}$ be the minimal resolution of this singularity. It the singularity is rational, then S is the smooth model at $a \in B^{(b)}$, thus the Weierstraß equation was minimal, whence $\lambda_a = 0$. Conversely, suppose that the singularity is non-rational. Then $W^{(p/B)}$ cannot be a Weierstraß model, so the Weierstraß equation was not minimal, that is, $\lambda_a > 0$. □

Proposition 9.4. *If both J and J' are rational surfaces, then we have $\sum \lambda_a = p - 1$, where the sum runs over all closed points $a \in B^{(p)} = \mathbb{P}^1$.*

Proof. Let us write $v = \sum v_b$, $v' = \sum v_a$ and $\lambda = \sum \lambda_a$ for the sums of local invariants. Since J and J' are rational, $v = v' = 12$ must hold, according to [31], Lemma 0.1. On the other hand, we have $v' = pv - 12\tau$, which forces $\tau = p - 1$. □

In particular, in characteristic $p = 2$ there is precisely one fiber that does not stay minimal, and it becomes minimal after one round of the Tate Algorithm. We shall analyze this particular situation in the next section.

10. Geometric interpretation of the Tate Algorithm

The *Tate Algorithm* turns arbitrary Weierstraß equations over discrete valuation rings into minimal Weierstraß equations [53]. This algorithm is of paramount importance for the arithmetic theory of elliptic curves, as well as the structure of elliptic surface. The goal of this section is to describe the geometry behind the algorithm. As pointed out by one referee, similar results are discussed in Conrad's expository notes ([10], Section 4). For the sake of the reader, we provide independent arguments that emphasize the role of Weierstraß equations.

Let R be a discrete valuation ring with residue field $k = R/\mathfrak{m}_R$ and field of fractions $F = \operatorname{Frac}(R)$, and choose a uniformizer $\pi \in R$. Suppose we have a Weierstraß equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

with coefficients $a_i \in R$ and discriminant $\Delta \neq 0$. Setting $y = X/Z$, $x = X/Z$ and multiplying with Z^3 gives its homogenization

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3,$$

which defines a relative cubic $J \subset \mathbb{P}^2_R = \operatorname{Proj} R[X, Y, Z]$ endowed with a section $O \subset J$ given by the equations $X = Z = 0$, such that $\mathscr{O}_J(1) = \mathscr{O}_J(3O)$.

Theorem 10.1. *Let $S \to J$ be the blowing-up with center given by $\pi = X = Z = 0$, and $S \to J'$ be the blowing-down of the strict transform of the closed fiber J_0. Then $J' \to \operatorname{Spec}(R)$ is a relative cubic given by the Weierstraß equation*

$$y^2 + a'_1xy + a'_3y = x^3 + a'_2x^2 + a'_4x + a'_6 \tag{10}$$

whose coefficients are $a'_i = a_i\pi^i$.

Proof. Write the closed fiber of the blowing-up as $S_0 = C \cup E$, where C is the strict transform of J_0, and E is the (-1)-curve. In light of the exact sequence

$$1 \longrightarrow \mathscr{O}^\times_{S_0} \longrightarrow \mathscr{O}^\times_C \oplus \mathscr{O}^\times_E \longrightarrow \kappa(s)^\times \longrightarrow 1,$$

where $s \in C \cap E$ is the rational intersection point, we easily see that the restriction map $\operatorname{Pic}(S_0) \to \operatorname{Pic}(C) \oplus \operatorname{Pic}(E)$ is injective. Consider the invertible sheaf $\mathscr{L} = \mathscr{O}_S(3O + 2E)$, which has $(\mathscr{L} \cdot C) = 2$ and $(\mathscr{L} \cdot E) = 1$. Using Riemann–Roch and the Theorem on Formal Functions, one easily infers that such $\mathscr{L}$ is globally generated, with $R^1h_*(\mathscr{L}) = 0$ and $h_*(\mathscr{L})$ free of rank three. Here $h : S \to \operatorname{Spec}(R)$ is the structure morphism. We thus get a morphisms $S \to \mathbb{P}^2_R$. Its image X is an effective Cartier divisor. Base-changing to the generic fiber, we see that $X \subset \mathbb{P}^2_R$ is a relative cubic.

Me may compute the cubic equation for $X \subset \mathbb{P}^2_R$ by regarding X as a closure of some affine cubic. Consider first the affine open subset $D_+(Y) \subset \mathbb{P}^2_R$. Here the relative cubic J is given by the dehomogenized cubic equation

$$\frac{Z}{Y} + a_1\frac{X}{Y}\frac{Z}{Y} + a_3\left(\frac{Z}{Y}\right)^2 = \left(\frac{X}{Y}\right)^3 + a_2\left(\frac{X}{Y}\right)^2\frac{Z}{Y} + a_4\frac{X}{Y}\left(\frac{Z}{Y}\right)^2 + a_6\left(\frac{Z}{Y}\right)^3.$$

The blowing-up $S \to J$ is covered by two affine charts, and we look at the π-chart given by the indeterminates $X/\pi Y, Z/\pi Y$ over R, subject to the relation

$$\frac{Z}{\pi Y} + a_1\pi\frac{X}{\pi Y}\frac{Z}{\pi Y} + a_3\pi\left(\frac{Z}{\pi Y}\right)^2 = \pi^2\left(\frac{X}{\pi Y}\right)^3 + a_2\pi^2\left(\frac{X}{\pi Y}\right)^2\frac{Z}{\pi Y} + a_4\pi^2\frac{X}{\pi Y}\left(\frac{Z}{\pi Y}\right)^2 + a_6\pi^2\left(\frac{Z}{\pi Y}\right)^3.$$

Via the substitutions $X/\pi Y = U/V$ and $Z/\pi Y = W/V$, we may view this as a closed subscheme inside $D_+(V) \subset \mathbb{P}^2_R = \operatorname{Proj} R[U, V, W]$. Its closure in $\mathbb{P}^2_R$ is given by the homogeneous cubic equation

$$V^2W + a_1\pi UVW + a_2\pi VW^2 = \pi^2U^3 + \pi^2a_2U^2W + \pi^2a_4UW^2 + \pi^2a_6W^3. \tag{11}$$

The closed fiber is thus given $\pi = V^2W = 0$, which is the union of the line $W = 0$ and the double line $V^2 = 0$ in $\mathbb{P}^2_k$. Indeed, since $S \smallsetminus C$ coincides with the π-chart of the blowing-up, $S \smallsetminus C \to X$ is an open embedding, and the above closure equals X.

Now consider the automorphism

$$f = \begin{pmatrix} \pi^{-2} & & \\ & \pi^{-2} & \\ & & 1 \end{pmatrix} \in \mathrm{PGL}_3(F) = \mathrm{Aut}(\mathbb{P}^2_F).$$

Over the function field, it gives a new cubic $f(J_F) \subset \mathbb{P}^2_F$. Applying f to the cubic equation (11) and multiplying by π^4, we get the homogeneous equation

$$V^2W + a'_1UVW + a'_3VW = U^3 + a'_2U^2W + a'_4UW^2 + a'_6W^3,$$

where $a'_i = \pi^i a_i$. Dehomogenizing the above homogeneous equation with $x = U/W$ and $y = V/W$ gives the desired Weierstraß equation (10). Thus the relative cubic $J' \subset \mathbb{P}^2_R$ defined by the above homogeneous equation is the schematic closure of $f(J_F)$. Note that the closed fiber J'_k is a rational cuspidal curve, whence J' is normal.

Clearly, the rational map $f : \mathbb{P}^2_R \dashrightarrow \mathbb{P}^2_R$ is defined on the generic fiber $\mathbb{P}^2_K$ and the standard affine open subsets $D_+(U)$ and $D_+(V)$, whence it is defined everywhere except at the point $z = (0 : 0 : 1) \in \mathbb{P}^2_k$. In turn, the rational map $f : J \dashrightarrow J'$ is defined outside $z \in J$. The latter is nothing but the intersection of the line $\pi = W = 0$ and the double line $\pi = V^2 = 0$ on the closed fiber. Every section $(a : b : 0)$ for $J \to \mathrm{Spec}(R)$ is mapped to itself under the rational map $f : \mathbb{P}^2_R \dashrightarrow \mathbb{P}^2_R$, whence $f : J \dashrightarrow J'$ is a bijective morphism on the complement of the double line $\pi = V^2 = 0$. Now choose a modification $S' \to J$ with center $z \in J$ so that the rational map $J \dashrightarrow J'$ comes from a morphism $S' \to J'$. Let $E \subset S'_0$ be an irreducible component different from the strict transform of the line $\pi = W = 0$. By Zariski's Main Theorem, it cannot map to a regular point on J'_0, because these are already images of points form the line $\pi = W = 0$. In turn, it must map to the singular point of the rational cuspidal curve J'_0. Thus the morphism $S' \to J'$ factors over J, and furthermore the induced morphism $J \to J'$ contracts the double line $\pi = V^2 = 0$. It follows that $S \to J'$ is the contraction of $C \subset S$. □

This observation gives some information on elliptic singularities, which turns out very useful. Let us call a Weierstraß equation

$$y^2 + a'_1xy + a'_3y = x^3 + a'_2x^2 + a'_4x + a'_6 \tag{12}$$

almost minimal if its discriminant $\Delta \in R$ is nonzero, and the Weierstraß equation becomes minimal after one round in the Tate Algorithm.

Proposition 10.2. *Every almost minimal Weierstraß equation defines an elliptic singularity, for which the exceptional divisor on the minimal resolution of singularities coincides with the closed fiber on the regular minimal model $J \to \mathrm{Spec}(R)$. The intersection numbers on the exceptional divisor and the fiber divisor coincide, except that for one component that has multiplicity one in the fiber, the intersection number 0 or -2 become -1 respective -3 on the exceptional divisor.*

Proof. Running through the Tate Algorithm, we may assume that $\pi^i | a'_i$. Now Theorem 10.1 describes how the singularity on J' arises from J via the birational correspondence $J \leftarrow S \to J'$. The statement on the exceptional divisor and the intersection numbers follow. It remains to check that the singularity is indeed elliptic. Write the closed fiber as $S_0 = E + C + D$, where E is the (-1)-curve, C is the irreducible component that intersects E, and D be the union of the remaining irreducible components. The contraction $r' : S \to J'$ factors over the contraction $S' \to J'$ of D, which introduces only rational double points. In light of the Leray–Serre spectral sequence, it suffices to show that for the induced map $g : S' \to J'$ the higher direct image $R^1 g_*(\mathscr{O}_{S'})$ has length one.

Changing notation, we write the closed fiber $S'_0 = E + C$, where $E = \mathbb{P}^1$ and C is a rational cuspidal curve. The intersection point $E \cap C$ is smooth in S', whence both $E, C \subset S'$ are Cartier divisors, with intersection matrix $N = \left(\begin{smallmatrix} -1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$, so $\deg \mathscr{O}_C(-C) = 1$. By the Theorem on Formal Functions, we have to show that $h^1(\mathscr{O}_{nC}) = 1$ for all $n \geq 1$. This is obvious for $n = 1$. We proceed by induction, using the long exact sequence attached to the short exact sequence

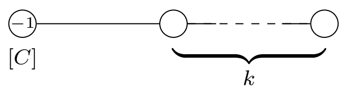
$$0 \longrightarrow \mathscr{O}_C(-nC) \longrightarrow \mathscr{O}_{(n+1)C} \longrightarrow \mathscr{O}_{nC} \longrightarrow 0,$$

together with the fact that $\mathscr{L} = \mathscr{O}_C(-nC)$ has degree $n \geq 1$, whence $H^1(C, \mathscr{L}) = 0$ by Riemann–Roch. $\square$

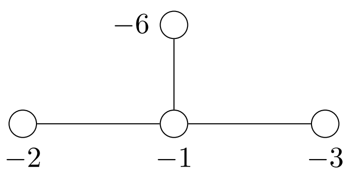
Note that blowing-ups of elliptic singularities are usually non-normal. Furthermore, the sequence of normalized blowing-ups of reduced singular points usually produces a non-minimal resolution. This makes it often so difficult to compute a resolution of singularities explicitly. The above result avoids all these complications and makes it unnecessary to cope with explicite equations. For our goals, the cases of Kodaira type II is particularly important:

Corollary 10.3. *Every almost minimal Weierstraß equation of type* II *defines an elliptic singularity, for which the exceptional divisor E on the minimal resolution of singularities is a rational cuspidal curve with intersection matrix $N = (-1)$.*

This is one of the simplest cases of Wagreich's classification of elliptic double points [55], Theorem 3.5. It belongs to the family of elliptic double points, whose exceptional divisors are symbolically described by Wagreich in the following way:



Here the chain of length $k \geq 0$ to the right consists of (-2)-curves. Our singularity is the special case $k = 0$. We simply say that it is the *elliptic singularity obtained by contracting a cuspidal* (-1)*-curve*. One may also describe it in terms of the minimal good resolution, where the exceptional divisor has simple normal crossings. This is obtained from the minimal resolution S by three further blowing-ups, such that E has four irreducible components, which are smooth rational curves. The dual graph takes the following form:



11. Frobenius pullback of rational elliptic surfaces

Let k be an algebraically closed ground field of arbitrary characteristic $p > 0$. Suppose that $f : J \to \mathbb{P}^1$ is a rational elliptic surface that is relatively minimal and jacobian. In this section, we examine the Frobenius pullback $J^{(p/\mathbb{P}^1)}$, which is defined by the cartesian diagram

$$\begin{array}{ccc} J^{(p/\mathbb{P}^1)} & \longrightarrow & J \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \mathbb{P}^1 & \xrightarrow[F]{} & \mathbb{P}^1, \end{array}$$

where $F : \mathbb{P}^1 \to \mathbb{P}^1$ is the relative Frobenius. Here we collect some elementary general facts, although eventually we are mainly interested in characteristic two. First note that the singularities on $J^{(p/B)}$ are Zariski, according to Proposition 9.1. In particular, the Frobenius pullback is locally of complete intersection, hence Gorenstein, and its tangent sheaf is locally free. Let us write $\mathscr{O}_J(n)$ and $\mathscr{O}_{J^{(p/\mathbb{P}^1)}}(n)$ for the invertible sheaves $f^*\mathscr{O}_{\mathbb{P}^1}(n)$ and $g^*\mathscr{O}_{\mathbb{P}^1}(n)$, respectively.

Proposition 11.1. *The dualizing sheaf is $\omega_{J^{(p/\mathbb{P}^1)}} = \mathscr{O}_{J^{(p/\mathbb{P}^1)}}(p-2)$.*

Proof. By assumption, $f : J \to \mathbb{P}^1$ is a rational elliptic surface that is relatively minimal and jacobian. According to [12], Proposition 5.6.1 we have $\omega_J = \mathscr{O}_J(-1)$. Using $\omega_{\mathbb{P}^1} = \mathscr{O}_{\mathbb{P}^1}(-2)$ one gets $\omega_{J/\mathbb{P}^1} = \mathscr{O}_J(1)$. Relative dualizing sheaves commute with flat base change. It follows that the relative dualizing sheaf for $J^{(p/\mathbb{P}^1)} \to \mathbb{P}^1$ is the preimage of $F^*\mathscr{O}_{\mathbb{P}^1}(1) = \mathscr{O}_{\mathbb{P}^1}(p)$, and the result follows. $\square$

Proposition 11.2. *The singular locus on the surface $J^{(p/\mathbb{P}^1)}$ corresponds to the locus on the surface J where the fibration $J \to \mathbb{P}^1$ is not smooth. In particular, $J^{(p/\mathbb{P}^1)}$ is normal if and only if $J \to \mathbb{P}^1$ has only reduced fibers.*

Proof. Clearly, $J^{(p/\mathbb{P}^1)}$ is smooth over each $y \in J$ at which the fiber $F = J_{f(y)}$ is smooth. Now suppose that the local ring has $\operatorname{edim}(\mathscr{O}_{F,y}) \geq 2$. The preimage of the fiber in $J^{(p/\mathbb{P}^1)}$ is a scheme isomorphic to $F' = F \otimes_k k[T]/(T^p)$, which has $\operatorname{edim}(\mathscr{O}_{F',y}) \geq 3$. In turn, the Frobenius pullback acquires a singularity there. $\square$

Proposition 11.3. *We have $H^1(J^{(p/\mathbb{P}^1)}, \mathscr{O}_{J^{(p/\mathbb{P}^1)}}) = 0$.*

Proof. The Leray–Serre spectral sequence for $g : J^{(p/\mathbb{P}^1)} \to \mathbb{P}^1$ gives an exact sequence

$$H^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}) \longrightarrow H^1(J^{(p/\mathbb{P}^1)}, \mathscr{O}_{J^{(p/\mathbb{P}^1)}}) \longrightarrow H^0(J^{(p/\mathbb{P}^1)}, R^1g_*\mathscr{O}_{J^{(p/\mathbb{P}^1)}}), \tag{13}$$

and the term on the left vanishes. Forming higher direct images commutes with flat base-change. So to understand the term on the right, we merely have to compute the coherent sheaf $R^1f_*(\mathscr{O}_J)$. Choose a section $O \subset J$. Using $\omega_J = \mathscr{O}_J(-1)$ and Adjunction Formula, one gets $(O \cdot O) = -1$. The ensuing short exact sequence $0 \to \mathscr{O}_J \to \mathscr{O}_J(O) \to \mathscr{O}_{\mathbb{P}^1}(-1) \to 0$ induces an exact sequence

$$\mathscr{O}_{\mathbb{P}^1}(-1) \longrightarrow R^1f_*(\mathscr{O}_J) \longrightarrow R^1f_*(\mathscr{O}_J(O)).$$

The terms on the left is invertible, the term in the middle has rank one, and the term on the right vanishes, consequently $R^1f_*(\mathscr{O}_J) = \mathscr{O}_{\mathbb{P}^1}(-1)$. Thus $R^1g_*\mathscr{O}_{J^{(p/\mathbb{P}^1)}} = \mathscr{O}_{\mathbb{P}^1}(-p)$, whose group of global sections vanishes. The assertion thus follows from the exact sequence (13). $\square$

Since J is a smooth rational surface, a minimal model is either some Hirzebruch surface or the projective plane. In the latter case, J factors over the blowing-up of the projective plane, by $\rho = 10$. It follows that in all cases there is some *ruling* $r : J \to \mathbb{P}^1$, that is, fibrations whose generic fiber is a projective line.

Proposition 11.4. *For each closed fiber $D = r^{-1}(t)$ of some ruling, the induced map $f : D \to \mathbb{P}^1$ has degree two.*

Proof. Regarding this degree as an intersection number, we easily reduce to the case that the fiber D is smooth. Then $D = \mathbb{P}^1$. Now the Adjunction Formula gives $-2 = (D \cdot \omega_S) = -(D \cdot F)$, and the assertion follows. $\square$

Now suppose that the elliptic fibration $f : J \to \mathbb{P}^1$ has only reduced fibers, such that $J^{(p/\mathbb{P}^1)}$ is normal. Let $r : J \to \mathbb{P}^1$ be some ruling. Consider the Stein factorization $J^{(p/\mathbb{P}^1)} \to C$ given by the commutative diagram

$$\begin{tikzcd} J^{(p/\mathbb{P}^1)} \arrow[r] \arrow[d] & J \arrow[d, "r"] \\ C \arrow[r] & \mathbb{P}^1. \end{tikzcd}$$

Proposition 11.5. *The morphism $C \to \mathbb{P}^1$ is an isomorphism, and the generic fiber of $J^{(p/\mathbb{P}^1)} \to C$ is a regular curve R with $h^1(\mathscr{O}_R) = p - 1$. If $p \geq 3$, the geometric generic fiber is a rational curve with two singularities. In characteristic two, $J^{(p/\mathbb{P}^1)} \to C$ is a quasielliptic fibration.*

Proof. Let $D = r^{-1}(t)$ be a smooth fiber, thus $D \simeq \mathbb{P}^1$. The induced morphism $f : D \to \mathbb{P}^1$ is affine of degree two, so D is the relative spectrum of some coherent $\mathscr{O}_{\mathbb{P}^1}$-algebra whose underlying module is $\mathscr{A} = \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$. Likewise, the relative Frobenius $F : \mathbb{P}^1 \to \mathbb{P}^1$ is the relative spectrum of some $\mathscr{B} = \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-1)^{\oplus(p-1)}$. In turn, the preimage of D in $J^{(2/\mathbb{P}^1)}$ is the spectrum of $\mathscr{C} = \mathscr{A} \otimes \mathscr{B}$, and we see $h^0(\mathscr{C}) = 1$ and $h^1(\mathscr{C}) = p - 1$. For $p = 2$, the double covering $\mathbb{P}^1 = D \to \mathbb{P}^1$ is ramified at two points, so the preimage of D has precisely two singularities.

In light of $h^0(\mathscr{C}) = 1$, the Theorem on Formal Functions ensures that the composite mapping $J^{(p/\mathbb{P}^1)} \to \mathbb{P}^1$ equals its own Stein factorization, thus $C = \mathbb{P}^1$. Its generic fiber is normal, because this holds for $J^{(p/\mathbb{P}^1)}$, and its arithmetic genus equals $h^1(\mathscr{C}) = p - 1$. In characteristic $p = 2$, this means that the fibration is quasielliptic. In odd characteristics, the non-smooth locus of the generic fiber for $J^{(p/\mathbb{P}^1)} \to C$ consist of two points, because this holds for almost all closed fibers. $\square$

12. Lang's classification and Frobenius pullback

For rational elliptic surfaces over the complex numbers, the possible configurations of Kodaira types were classified by Persson [44] and Miranda [38]. This classification was extended by Lang [31] to characteristic $p = 2$, including for each case examples of global Weierstraß equations. These equations show that all numerically possible cases indeed exist. This builds on the normal forms for unstable fibers obtained in [30], Section 2A. Lang's results will be the key to understand and construct simply-connected Enriques surfaces and their K3-like coverings.

He introduced very useful short-hand notation for configurations of Kodaira types: For example, $\mathrm{II} + 21^6$ stands for the configuration comprising one fiber of type II, one fiber of type I_2, and six fibers of type I_1. It turns out that there are 147 possible configurations of Kodaira types. Here we are only interested in cases where all fibers are reduced, for which there are 110 configurations. There are 35 cases where all fibers of J are semistable, 58 cases with semistable fibers and precisely one unstable fiber, and 17 cases with only unstable fibers. The latter is equivalent to the condition that the global j-invariant is $j(J/\mathbb{P}^1) = 0$.

Lang used alpha-numerical symbols for normal forms of *local* Weierstraß equations for unstable fibers. Let us call them the *Lang types*. One may tabulate the Lang types for reduced fibers as follows:

Lang type	1A	1B	1C	2A	2B	3	9A	9B	9C	10A	10B	10C	11
Kodaira type	II	II	II	III	III	IV	II	II	II	III	III	III	IV
m	1	1	1	2	2	3	1	1	1	2	2	2	3
v	4	6	7	4	6	4	4	8	12	4	8	12	4
δ	2	4	5	1	3	0	2	6	10	1	5	9	0

Table 1: The reduced unstable Lang types

As in Section 9, the integer $m \geq 1$ is the number of irreducible components in the fiber, $v \geq 1$ is the valuation of a minimal Weierstraß equation, and $\delta \geq 0$ is the wild part of the conductor, which satisfies $\delta = v - m - 1$ according to Ogg's Formula.

In what follows, we work over an algebraically closed ground field k of characteristic $p = 2$. Fix a rational elliptic surface $J \to \mathbb{P}^1$ that is jacobian, with only reduced fibers. Hence the Frobenius pullback $X' = J^{(2/\mathbb{P}^1)}$ is normal. Choose a section $O \subset J$, and let $W \to \mathbb{P}^1$ be the resulting Weierstraß fibration.

In what follows, fibers of Lang type 9C play an exceptional role. They have Kodaira type II and numerical invariants $v = 12$, $m = 1$ and $\delta = 10$. If present, there are no other singular fibers, hence the contraction $J \to W$ to the Weierstraß model is an isomorphism, and the whole of J is given by the global Weierstraß equation

$$y^2 + t^3\gamma_0 y = x^3 + t\gamma_1 x^2 + t\gamma_3 x + t\gamma_5, \tag{14}$$

according to [30], Section 2A. Here $\gamma_i \in k[t]$ are polynomials of degree $\leq i$, and furthermore $t \nmid \gamma_0, \gamma_5$. We regard the base of the fibration as the projective line $\mathbb{P}^1 = \operatorname{Spec} k[t] \cup \operatorname{Spec} k[t^{-1}]$. The singular fiber is located over $t = 0$.

Theorem 12.1. *The Frobenius pullback $X' = J^{(2/\mathbb{P}^1)}$ contains a non-rational singularity $x_1' \in X'$ if and only if $J \to \mathbb{P}^1$ contains a singular fiber of Lang type 9C, and the coefficient $\gamma_3 \in k[t]$ in the above Weierstraß equation is divisible by $t \in k[t]$. In this case, the singularity is an elliptic double point, the exceptional divisor on the minimal resolution of singularities is a rational cuspidal curve with intersection matrix $N = (-1)$, and there are no other singularities on X'.*

Proof. We start by collecting some general observations. Clearly a singularity $x \in J^{(2/\mathbb{P}^1)}$ is rational if the local Weierstraß equations for the corresponding fiber in the Weierstraß model $W \to \mathbb{P}^1$ of $J \to \mathbb{P}^1$ remains minimal after Frobenius pullback. This automatically holds for the semistable fibers. Let $J_a \subset J$ be an unstable fiber. The invariants for the minimal model of $J^{(2/\mathbb{P}^1)}$ satisfy $2v - 12\lambda = 2 + \delta + (m' - 1)$, where $\lambda \geq 0$ is the number of repetitions in the Tate Algorithm, as explained in Proposition 9.2. Thus $12\lambda \leq 2v - 1 - \delta$. Using Table 1, we see that the right hand side is < 12, and thus $\lambda = 0$ and the singularities on X' are rational, unless the fiber is of Lang type 9C or 10C, in which case $\lambda = 1$ may occur.

According to [31], Lemma 0.1, we have $\sum_{a \in \mathbb{P}^1} v_a = 12$. Fibers of Lang type 9C and 10C have $v_a = 12$, whence $J \to \mathbb{P}^1$ has exactly one singular fiber. Their Kodaira types are II or III, whence the projection $J \to \mathbb{P}^1$ is smooth outside a single point, so $\operatorname{Sing}(X')$ contains exactly one point. Let $x_1' \in X'$ be the unique singularity. To verify our assertion, it suffices to treat these two special cases.

Suppose first that $J \to \mathbb{P}^1$ contains a fiber of Lang type 10C. By [31], the Weierstraß equation is of the form

$$y^2 + t^3\gamma_0 y = x^3 + t\gamma_1 x^2 + t\gamma_3 x + t^2\gamma_4, \tag{15}$$

where $\gamma_i \in k[t]$ are polynomials of degree $\leq i$, subject to $t \nmid \gamma_0, \gamma_3$. The singular fiber is again located over $t = 0$. Going through the Tate Algorithm ([53], Section 7) and using $t \nmid \gamma_3$, we see that the Frobenius pullback of the Weierstraß equation remains minimal, and the singular fiber on X' has Kodaira type I_n^* for some $n \geq 0$. We have $v_a = 2v_b = 24$ and $\delta_a = \delta_b = 9$, so Ogg's Formula $v_a = 2 + \delta_a + (m_a - 1)$ gives $m_a = 14$, thus the fiber is of type I_9^*, and the singularity is a rational double point of type D_{12}.

Now suppose that $J \to \mathbb{P}^1$ contains a fiber of Lang type 9C as in (14). Again going through the Tate Algorithm, we see that the the singular fiber on X' has Kodaira type I_n^* for some $n \geq 0$ if and only if $t \nmid \gamma_3$. In this situation we argue as in the preceding paragraph to deduce that the fiber is of Kodaira type I_8^*, and the singularity is again a rational double point of type D_{12}. For $t \mid \gamma_3$, the Weierstraß equation is not minimal. This establishes the equivalence of the two conditions.

Now suppose that there is a non-rational singularity $x_1' \in X'$, hence $J \to \mathbb{P}^1$ contains a fiber of Lang type 9C. The local Weierstraß equation for X' must have $v' = 2 \cdot 12 - 12$, whence is almost minimal, and the singularity is elliptic, by Proposition 10.2. The minimal model of X' has $v' = 2 \cdot v - 12 = 12$. Ogg's Formula $12 = 2 + (m' - 1) + 10$ yields $m' = 1$, so the Kodaira type is II. The statement on the exceptional divisor now follows from Corollary 10.3. □

We now turn to the global properties: Let $r' : S \to X'$ be the minimal resolution of singularities, and write $h : S \to \mathbb{P}^1$ for the induced elliptic fibration.

Lemma 12.2. *The smooth surface S is either a K3 surface or a rational surface. The latter holds if and only if X' contains a non-rational singularity.*

Proof. According to Proposition 11.1 and Proposition 11.3, we have $\omega_{X'} = \mathscr{O}_{X'}$ and $h^1(\mathscr{O}_{X'}) = 0$. Now suppose that X' has only rational singularities. For the minimal resolution of singularities, this means $\omega_S = \mathscr{O}_S$ and $h^1(\mathscr{O}_S) = 0$. Furthermore, the rational elliptic surface J has Betti number $b_2(J) = 10$, thus $b_2(S) \geq 11$. From the classification of surfaces we infer that S is a K3 surface.

Now suppose that X' contains a non-rational singularity $x_1' \in X'$. Then $K_{S/X'} = -D'$ for some negative-definite curve $D' \subset S$ contracted by $r : S \to X'$. Consequently the plurigenera $P_m(S) = h^0(\omega_S^{\otimes n})$ vanish for all $n \geq 1$. In particular, we have $h^2(\mathscr{O}_S) = h^0(\omega_S) = 0$, so the Picard scheme $\mathrm{Pic}_{S/k}$ is smooth. Now consider the Albanese map $S \to \mathrm{Alb}_{S/k}$. This map factors over X', because the exceptional curve for the resolution of singularities $r : S \to X'$ are rational curves by Proposition 1.9. Since $X' \to J$ is a universal homeomorphism, and J is covered by rational curves, the Albanese map must be zero, and we infer $h^1(\mathscr{O}_S) = 0$. The Castelnuovo Criterion now ensures that the surface S is rational. □

Thus we are precisely in the situation studied in Section 6, and determine which of the seven condition (E1)–(E7) from Section 6 do hold:

Proposition 12.3. *The elliptic fibrations $h' : S \to \mathbb{P}^1$ and $r' : S \to X'$ satisfy conditions* (E1)–(E7)*, except condition* (E5)*, and the tangent sheaf is given by $\Theta_{X'} = \mathscr{O}_{X'}(2) \oplus \mathscr{O}_{X'}(-2)$.*

Proof. According to Proposition 11.5, each ruling $r : J \to \mathbb{P}^1$ induce a quasielliptic fibration $X' \to \mathbb{P}^1$, hence (E7) holds. The singularities on the Frobenius pullback X' are Zariski singularities by Lemma 9.1, so condition (E6) and in particular (E1) hold.

By assumption, $J \to \mathbb{P}^1$ admits a section. Thus the same holds for $h' : S \to X'$, hence the latter admits no multiple fiber. If S is rational, then by Theorem 12.1 the surface contains a single singularity, which is an elliptic singularity, so $R^1 r'_*(\mathscr{O}_S)$ has length one.

In other words, condition (E3) holds. Furthermore, we have $\omega_S = \mathscr{O}_S(-C')$ for some negative-definite curve supported on the exceptional divisor for $r' : S \to X'$, so condition (E4) holds.

It remains to check condition (E2) on the total Tjurina number. Let v_b, m_b, δ_b be the numerical invariants for $J \to \mathbb{P}^1$, and v_a, m_a, δ_a be the ensuing invariants for the minimal model $J' \to \mathbb{P}^1$ of the Frobenius pullback X'. Recall that $\delta_a = \delta_b$. Since J is a rational surface, we have $\sum v_b = 12$, according to [31], Lemma 0.1. Suppose first that X' has only rational singularities. Then $v_a = 2v_b$, and the total number of exceptional divisors for the resolution of singularities $J' \to X'$ is $\sum(m_a - m_b)$. In light of Proposition 3.3, we have to verify $\sum(m_a - m_b) = 12$. If the fiber J_b is semistable, the same holds for J_a, and Ogg's Formula gives $m_a - m_b = v_a - v_b = v_b$. If the fiber J_b is unstable, also J_a is unstable, and likewise get

$$m_a - m_b = (v_a - 1 - \delta_a) - (v_b - 1 - \delta_b) = 2v_b - v_b = v_b.$$

In any case, we obtain $\sum(m_a - m_b) = \sum v_b = 12$.

Now suppose that X' contains an elliptic singularity. Then $J \to \mathbb{P}^1$ is given by the Weierstraß equation (14), and its Frobenius pullback is described by

$$y^2 + t^6\gamma_0^2 y = x^3 + t^2\gamma_1^2 x^2 + t^2\gamma_3^2 x + t^2\gamma_5^2,$$

where $t \nmid \gamma_0$. The Tjurina ideal is generated by the given Weiererstraß polynomial $y^2 + t^6\gamma_0^2 y - \ldots$, together with the partial derivatives $t^6\gamma_0^2$ and $x^2 + t^2\gamma_3^2$. Using that γ_0 is a unit in $k[[t]]$, and regarding the generators as integral equations in t, x and y of degree $d = 6, 2, 2$, we infer that

$$k[[t, x, y]]/(t^6, x^2 + t^2\gamma_3^2, y^2 + t^6\gamma_0^2 y - \ldots)$$

has length $\tau = 6 \cdot 2 \cdot 2 = 24$.

Finally, we compute the tangent sheaf. Choose a ruling $J \to \mathbb{P}^1$, and let $X' \to \mathbb{P}^1$ be the resulting quasielliptic fibration. For each smooth fiber $D \subset J$, the preimage $C \subset X'$ has degree two with respect to the elliptic fibration $X' \to \mathbb{P}^1$, according to Proposition 11.4. Now Theorem 6.2 gives $\Theta_{X'/k} = \mathscr{O}_{X'}(-2) \oplus \mathscr{O}_{X'}(2)$. $\square$

13. K3-like coverings with elliptic double point

In this section, we shall construct simply-connected Enriques surfaces Y whose K3-like covering X is normal and contains an elliptic double point. The idea is simple: We start with a rational elliptic surface $J \to \mathbb{P}^1$ containing a singular fiber of Lang type 9C, chosen so that the Frobenius pullback $X' = J^{(2/\mathbb{P}^1)}$ stays rational. It comes with an induced elliptic fibration $X' \to \mathbb{P}^1$. Its relatively minimal smooth model J' can be seen as a flip $X' \leftarrow S \to J'$, and the desired K3-like covering arises as a flop $X' \leftarrow S \to X$. The crucial step is the construction of the flop X, starting from the flip J', so that the flop removes all quasielliptic fibrations, introduce another elliptic fibration, and yet does not change the nature of the Zariski singularities.

For this, we need as key ingredients Shioda's theory of Mordell–Weil lattices [51] and their classification for rational elliptic surface, which is due to Oguiso and Shioda [42]. We start by recalling this. Let $J \to B$ be an elliptic surface over a smooth proper curve B, with smooth total space and endowed with zero section $0 \subset J$. We also assume that it is relatively minimal. The Néron–Severi group $\mathrm{NS}(J) = \mathrm{Pic}(J)/\mathrm{Pic}^0(J)$ is endowed with the intersection pairing $(P \cdot Q)$. The group of sections for $J \to B$ is called the *Mordell–Weil group* $\mathrm{MW}(J/B) = \mathrm{Pic}^0(J_\eta)$. The *trivial sublattice* $T = T(J/B)$ inside the Néron–Severi group $\mathrm{NS}(J)$ is the subgroup generated by the zero-section $O \subset J$, together with all irreducible components $\Theta \subset J_b$, $b \in B$ that are disjoint from the zero-section. Thus we may identify $\mathrm{MW}(J/B)$, up to its torsion subgroup, as a subgroup of the orthogonal complement $T^\perp \subset \mathrm{NS}(J)\otimes\mathbb{Q}$. Consequently, it acquires a non-degenerate bilinear $\mathbb{Q}$-valued form

$$\langle P, Q\rangle = -(\bar{P}\cdot\bar{Q})$$

called the *height pairing*, which was extensively studied by Shioda [51]. The sign is a customary convention, introduced to have a positive-definite height pairing, and $P\mapsto\bar{P}$ denotes the orthogonal projection onto $T^\perp$. One also calls $\mathrm{MW}(J/B)$ the *Mordell–Weil lattice*. The subgroup comprising all sections $P\subset J$ that pass through the same fiber components as the zero section $O\subset J$ is called the *narrow Mordell–Weil lattice*.

The *explicit formula* for the height pairing is

$$\langle P, Q\rangle = \chi(\mathscr{O}_J) + (P\cdot O) + (Q\cdot O) - (P\cdot Q) - \sum \mathrm{contr}_b(P,Q),$$

according to [51], Theorem 8.6. The summands in the middle are the usual intersection numbers, and the *local contributions* $\mathrm{contr}_b(P,Q)\in\mathbb{Q}_{\geq 0}$ are certain rational numbers tabulated in [51], page 229. They reflect how the sections P,Q pass through the irreducible components of the fiber J_b, in relation to the zero-section $O\subset J$. The corresponding quadratic form can be written as

$$\langle P, P\rangle = 2\chi(\mathscr{O}_J) + 2(P\cdot O) - \sum \mathrm{contr}_b(P),$$

where $\mathrm{contr}_b(P) = \mathrm{contr}_b(P,P)$.

For each point $b\in B$, we denote by Φ_b the group of components in the Néron model of $J\otimes\mathscr{O}_{B,b}$. Its order is the number of irreducible components in S_b with multiplicity $m=1$ in the fiber. Clearly, the narrow Mordell–Weil lattice is the subgroup of $\mathrm{MW}(J/B)$ of elements P with $P\equiv O$ in Φ_b for all $b\in B$. Note that if $P\equiv O$ in Φ_b, then the local contributions $\mathrm{contr}_b(P,Q)$ in the explicit formula for the height pairing vanish.

Suppose now that $J\to\mathbb{P}^1$ be a jacobian rational elliptic surface. Then we have $\chi(\mathscr{O}_J) = 1$. Oguiso and Shioda [42] tabulated the possible isomorphism classes of Mordell–Weil lattices $\mathrm{MW}(J/\mathbb{P}^1)$ in terms of the trivial sublattices $T = T(J/\mathbb{P}^1)$, a list containing 74 cases. The arguments are purely lattice-theoretical, and apply to arbitrary characteristics.

Let us give an example: If the configuration of Kodaira types is $\mathrm{III} + 32^21$, the trivial lattice becomes $A_1^{\oplus 3}\oplus A_2$, which appears as No. 23 in the Oguiso–Shioda list; we then read off that the Mordell–Weil lattice is given by

$$\mathrm{MW}(J/\mathbb{P}^1) = A_1^\vee \oplus \frac{1}{6}\begin{pmatrix} 2 & 1\\ 1 & 2\end{pmatrix}.$$

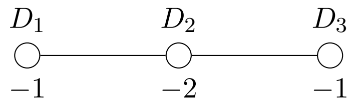
Here the second summand is a Gram matrix, and the first summand is the dual $A_1^\vee = (1/2)$ for the ADE-lattice $A_1 = (2)$. The following observation will be useful:

Proposition 13.1. *Suppose that $J\to\mathbb{P}^1$ has only reduced fibers. Then there is a section $P\subset J$ that is disjoint from the zero-section $O\subset J$ and has $P\not\equiv O$ in the group of components Φ_b for some point $b\in\mathbb{P}^1$. Furthermore, every torsion section $P\neq O$ is disjoint from the zero-section.*

Proof. According to [42], Theorem 2.5, the Mordell–Weil group is generated by sections that are disjoint from the zero-section. So we first check that the Mordell–Weil group is non-zero. According to the Oguiso–Shioda list, only No. 62 has $\mathrm{MW}(J/\mathbb{P}^1) = 0$. In this case, the vertical lattice is $T = E_8$. It follows that $J\to\mathbb{P}^1$ has a fiber of Kodaira type II^*, which is nonreduced, contradiction. Going through the Oguiso–Shioda list, one observes that in all other cases the narrow Mordell–Weil lattice is a proper sublattice, so the first assertion follows. The last statement is [42], Proposition 5.4. □

In what follows, k denotes an algebraically closed ground field of characteristic $p=2$. Choose a Weierstraß equation of Lang type 9C, as given in (14), with $t\mid\gamma_3$. The resulting rational elliptic surface $J \to \mathbb{P}^1$ then contains only one singular fiber, which is located over the origin $b \in \mathbb{P}^1$ and of Lang type 9C, and the Frobenius pullback $X' = J^{(2/\mathbb{P}^1)}$ is a normal rational surface containing a unique singularity, which is an elliptic double point obtained by contracting a cuspidal (-1)-curve, and is located over the point $a \in \mathbb{P}^1$ corresponding to b. Note that the exceptional divisor on the minimal resolution of singularities is a rational cuspidal curve with self-intersection -1.

Let $r' : S \to X'$ be the non-minimal resolution obtained by blowing-up further a closed regular point on the rational cuspidal curve. Write $h : S \to \mathbb{P}^1$ for the induced elliptic fibration. The fiber $S_a = h^{-1}(a)$ has three irreducible components D_1, D_2, D_3, where D_2 is the rational cuspidal curve and D_1, D_3 are smooth rational curves. The dual graph, with the selfintersection numbers, is



with D_2 rational cuspidal.

Proposition 13.2. *The smooth rational surface S has Picard number $\rho(S) = 12$.*

Proof. The elliptic fibration $h : S \to \mathbb{P}^1$ is relatively minimal, except at the fiber $S_a = D_1 + D_2 + D_3$. The curves D_1, D_3 are disjoint (-1)-curves. Contracting them yields the relatively minimal model. The latter has Picard number ten, by [12], Proposition 5.6.1, so our surface S has Picard number twelve. □

We choose the indices for the irreducible components $S_a = D_1 + D_2 + D_3$ so that $r' : S \to X'$ contracts $E' = D_2 + D_3$, and that $S \to J'$ contracts the (-1)-curves $D_1 + D_3$. Since J' is a rational elliptic surface, we have $K_{J'} = -J'_a$ whence $K_S = -D_2$. Now let $r : S \to X$ be the contraction of $E = D_1 + D_2$. This yields another normal surface X with $\omega_X = \mathscr{O}_X$, containing a unique elliptic singularity obtained by contracting a cuspidal (-1)-curve.

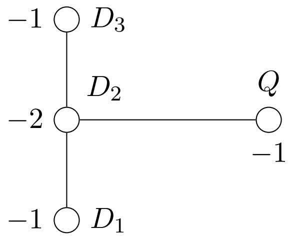
Theorem 13.3. *The normal surface X, with its elliptic singularity obtained by contracting a cuspidal (-1)-curve, is a K3-like covering.*

Proof. In light of Theorem 6.4, we have to verify that the elliptic fibration $h : S \to \mathbb{P}^1$ and the contraction $r : S \to X$ of the exceptional divisor $E = D_1 + D_2$ satisfies conditions (E1)–(E6) of Section 6. Recall that $x_1 \in X$ and $x'_1 \in X'$ are the unique singular points. It follows from Lemma 13.4 below that the complete local rings $\mathscr{O}^\wedge_{X,x_1}$ and $\mathscr{O}^\wedge_{X',x'_1}$ are isomorphic. Thus all conditions but (E5) follow from Proposition 12.3.

It remains to show that condition (E5) holds, that is, we must construct on X another elliptic fibration. For this we first consider the flip J', which is a smooth rational elliptic surface, whose sole singular fiber is of type II. Choose a zero-section $O \subset J'$, and consider the resulting Mordell–Weil lattice $\mathrm{MW}(J'/\mathbb{P}^1)$. Since the trivial lattice is $T = 0$, the Mordell–Weil lattice is E_8, according to [42]. For each section $P \subset J'$ corresponding to a root $\alpha \in E_8$, the height pairing gives $2 = \langle P, P\rangle = 2 + 2(P \cdot 0)$, whence P is disjoint from the zero-section. Now let $\alpha, \beta \in E_8$ be two adjacent simple roots, and let $P, P' \subset J'$ be the sections corresponding to the roots $\alpha, \alpha + \beta \in E_8$, such that $\alpha \cdot (\alpha + \beta) = 2 - 1 = 1$. In terms of the height pairing, this means

$$1 = \langle P, P'\rangle = 1 + (P \cdot O) + (P' \cdot O) - (P \cdot P') = 1 - (P \cdot P'),$$

thus $O, P, P' \subset J'$ are pairwise disjoint. From this we infer that at least one of the resulting three strict transforms on S is a section for $h : S \to \mathbb{P}^1$ passing through the irreducible component $D_2 \subset S_a = h^{-1}(a)$. After changing the zero-section $O \subset J$, we may assume that the strict transform $Q \subset S$ of the zero-section satisfies $(Q \cdot D_2) = 1$. The dual graph takes the form



with D_2 rational cuspidal.

Consider the negative-semidefinite curve $C = D_1 + D_2 + Q \subset S$ of arithmetic genus $h^1(\mathscr{O}_C) = 1$. Then some $\mathscr{O}_S(nC)$, with $n \geq 1$ induces another genus-one fibration $g : S \to \mathbb{P}^1$.

We claim that the curve C itself is not movable. Seeking a contradiction, we assume it were. Then the curve $C \subset S$ is *d-semistable*, which means that the normal sheaf is $\mathscr{O}_C(-C) = \mathscr{O}_C$, such that $\mathscr{O}_{D_2}(-D_2 - D_1 - Q) = \mathscr{O}_{D_2}$. In light of the fibration $h : S \to \mathbb{P}^1$ defined by the movable curve $D_1 + D_2 + D_3$, we have $\mathscr{O}_{D_2}(-D_2) = \mathscr{O}_{D_2}(D_1 + D_3)$. It follows that $\mathscr{O}_{D_2}(D_3 - Q) = \mathscr{O}_{D_2}$. By Riemann–Roch, this does not hold for the two points $D_3 \cap D_2 \neq Q \cap D_2$ on the rational cuspidal curve D_2. We conclude that C is not movable.

However, the curve $2C$ is movable. To see this, set $\mathscr{L} = \mathscr{O}_S(C)$. Since $\mathrm{Pic}^0(C) = k$ is 2-torsion and the sheaf $\mathscr{L}_C$ is numerically trivial, we must have $\mathscr{L}_C^{\otimes 2} = \mathscr{O}_C$. Consider the short exact sequence $0 \to \mathscr{L} \to \mathscr{L}^{\otimes 2} \to \mathscr{O}_C \to 0$ and the ensuing long exact sequence

$$H^0(S, \mathscr{L}^{\otimes 2}) \longrightarrow H^0(C, \mathscr{O}_C) \longrightarrow H^1(S, \mathscr{L}).$$

It thus suffices to check that the term on the right vanishes. By Serre Duality, $h^2(\mathscr{L}) = h^0(\omega_S \otimes \mathscr{L}^\vee) = 0$, because $K_S = -D_2$. Riemann–Roch gives

$$1 - h^1(\mathscr{L}) = \chi(\mathscr{L}) = C \cdot (C + D_2)/2 + \chi(\mathscr{O}_S) = 1,$$

thus $h^1(\mathscr{L}) = 0$. Summing up, $2C \subset S$ is movable, and thus defines a genus-one fibration $g : S \to \mathbb{P}^1$. The intersection number

$$C \cdot (D_1 + D_2 + D_3) = C \cdot D_3 = D_2 \cdot D_3 = 1$$

shows that $2C$ and thus also all other fibers of $g : S \to \mathbb{P}^1$ have degree two with respect to the original elliptic fiber $h : S \to \mathbb{P}^1$.

According to Proposition 13.2, the rational surface S has Picard number $\rho(S) = 12$. By construction, the irreducible components $D_1, Q \subset C$ are disjoint (-1)-curves on S. Contracting them thus gives the relatively minimal model for the genus-one fibration $g : S \to \mathbb{P}^1$. We conclude that all g-fibers but $C \subset S$ are minimal, contain at most two irreducible components, and are thus of Kodaira type II, III or I_n with $n \leq 2$.

It remains to check that our genus-one fibration is elliptic. Seeking a contradiction, we assume that $g : S \to \mathbb{P}^1$ is quasielliptic. Now we use the fact that the fibers of $h : S \to \mathbb{P}^1$, except the sole singular fiber $h^{-1}(a) = D_1 + D_2 + D_3$, contain no (-2)-curves. It follows that every component of every fiber of $g : S \to \mathbb{P}^1$ is horizontal with respect to $h : S \to \mathbb{P}^1$, with the exception of D_1 and D_2. Moreover, every fiber other than $2C$ has Kodaira type II or III, and its irreducible components necessarily pass through $D_3 \subset h^{-1}(a)$, because D_1 and D_2 are contained in the fiber $2C$.

Since S is rational, all other fibers beside $2C$ are non-multiple ([12], Proposition 5.6.1), whence there type coincides with the type of the corresponding fiber on the associated jacobian quasielliptic fibration. According to Ito's analysis of jacobian quasielliptic fibrations in characteristic two ([24], Proposition 5.1), there is at least one reducible fiber, which in our situation must be of type III. In fact, the Mordell–Weil lattice for a quasielliptic fibration is 2-torsion, whence the trivial lattice has rank ten, so there are eight fibers of Kodaira type III. Each irreducible component $R, R' \subset S$ of such a fiber is a section for $h : S \to \mathbb{P}^1$, both pass through $D_3 \subset S$, and they meet tangentially somewhere. Consequently, their images define two sections $P, P' \subset J'$ with intersection matrix $\left(\begin{smallmatrix} -1 & 3 \\ 3 & -1 \end{smallmatrix}\right)$. Obviously, P, P' are disjoint from the zero-section $O \subset J'$, which is the image of $Q \subset S$. To derive a contradiction, we compute the Gram matrix for the height paring on J'. As noted above, the sections $P, P' \neq O$ are disjoint from the zero-section. Thus $\langle P, P\rangle = 2 + 2(P \cdot O) = 2$, likewise $\langle P', P'\rangle = 2$, and finally

$$\langle P, P'\rangle = 1 + (P \cdot O) + (P' \cdot O) - (P \cdot P') = -2.$$

So the Gram matrix with respect to the height pairing is $\left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$, which has zero determinant. On the other hand, the Mordell–Weil lattice is non-degenerate, contradiction. Thus $g : S \to \mathbb{P}^1$ and the induced fibration on X are elliptic rather than quasielliptic fibrations. This means that the condition (E5) indeed holds. □

In the preceding proof, we have used the following observation:

Lemma 13.4. *Let R be a discrete valuation ring, and $J \to \mathrm{Spec}(R)$ a relatively minimal genus-one fibration, endowed with two disjoint sections $P_1, P_2 \subset J$. Let $S \to J$ be the blowing-up whose center consists of the two closed points $z_i \in P_i$, and let $E_i \subset S$ be their preimage. Write $S \to X_i$ for the blowing-down so that E_i surjects onto the closed fiber of X_i, and let $x_i \in X_i$ be the resulting singular point. Then the two local rings $\mathscr{O}_{X_1,x_1}$ and $\mathscr{O}_{X_2,x_2}$ are isomorphic.*

Proof. Let $f : J_\eta \to J_\eta$ be the isomorphism that sends the the generic fiber of P_1 to the generic fiber of P_2. By relative minimality, it extends to an isomorphism $f : J \to J$. It has $f(z_1) = z_2$, and by the universal property of blowing-ups induced an isomorphism $f : S \to S$ with $f(E_1) = E_2$. In turn, it maps the exceptional locus for $S \to X_1$ to the exceptional locus for $S \to X_2$. By the universal property of contractions [18], Lemma 8.11.1, it yields the isomorphism between the two local rings in question. □

As the referee pointed out, the resulting simply-connected Enriques surfaces $Y = X/G$ must be supersingular, rather then classical. Indeed, the elliptic fibration $J \to \mathbb{P}^1$ has constant j-invariant zero, according to Lang's classification [31], and all fibers but J_b are smooth. In turn, this also holds for the induced fibrations on J', S and X. If Y would be classical, the induced elliptic fibration $\varphi : Y \to \mathbb{P}^1$ would have two multiple fibers that are tame, and one of which has the supersingular elliptic curve $F \subset Y$ as reduction. This implies $\mathscr{O}_F(F) \simeq \mathscr{O}_F$, and the fiber must be wild, contradiction.

14. Uniqueness of the elliptic double point

Let Y be a simply-connected Enriques surface over an algebraically closed ground field k of characteristic $p = 2$, with normal K3-like covering. The goal of this section is to establish the following uniqueness result:

Theorem 14.1. *If there is an elliptic singularity $x_1 \in X$, then there are no other singularities on X.*

Proof. Seeking a contradiction, we assume that there is another singularity $x_2 \in X$. Let $y_1, y_2 \in Y$ be the images of the two singularities $x_1, x_2 \in X$.

First, we show that there are no elliptic fibrations $\varphi : Y \to \mathbb{P}^1$ admitting radical two-sections $A \subset Y$. Seeking a contradiction, we assume that such a fibration exists. Let $J \to \mathbb{P}^1$ be its jacobian fibration, which is a rational elliptic surface, and write $X' =$

$J^{(2/\mathbb{P}^1)}$ for the Frobenius pullback. By Proposition 8.5, the normal surfaces X' and X are birational. It follows that the surface X' is rational, whence contains an elliptic singularity. According to Theorem 12.1, $J \to \mathbb{P}^1$ must contain a fiber of Lang type 9C, and this is the only singular fiber J_b. In turn, the Frobenius base-change $X' = J^{(2/\mathbb{P}^1)}$ has but one singularity, lying on the fiber X'_a, where $a \in \mathbb{P}^1$ corresponds to $b \in \mathbb{P}^1$ under the relative Frobenius morphism.

To proceed, recall that if $2F \subset Y$ is a multiple fiber with F smooth, the K3-like covering X is smooth along $\epsilon^{-1}(F)$, according to Proposition 5.7. If the fiber Y_b is not a multiple fiber, then all half-fibers must be smooth, and by Proposition 8.11 there is but one singularity on X, namely the point corresponds to the singular point in Y_b, contradiction. Thus Y_b is a multiple fiber. According to Proposition 8.3, the two fibers Y_b, J_b have the same Kodaira symbol II. Write $Y_b = 2F$, with a rational curve F. By the above discussion, the two singularities $x_1, x_2 \in X$ map to F. This analysis applies not only to $\varphi : Y \to \mathbb{P}^1$, but also to all other elliptic fibrations admitting a radical two-section.

According to Theorem 5.6, there is another elliptic fibration $\varphi' : Y \to \mathbb{P}^1$ with $(F \cdot F') = 1$. Then the rational curve F is a radical two-section for the new fibration φ', according to Proposition 8.6. In turn, we have $x_1, x_2 \in F'$. Now $x_1, x_2 \in F \cap F'$ contradict the intersection number $(F \cdot F') = 1$. Consequently, there is no elliptic fibration on Y admitting a radical two-section. According to Proposition 8.8, there are no (-2)-curves on Y.

Now choose again an elliptic fibration $\varphi : Y \to \mathbb{P}^1$. Then all fibers are irreducible, according to Proposition 8.8. By the result of Liu, Lorenzini and Raynaud [34], Theorem 6.6 the same holds for the associated jacobian fibration $J \to \mathbb{P}^1$. Suppose for all elliptic fibrations on Y the jacobian would have only one singular fiber J_b. Arguing as above, this fiber $Y_b = 2F$ is multiple, and we find two orthogonal elliptic fibrations with $(F \cdot F') = 1$ and thus $y_1, y_2 \in F \cap F'$, contradiction. Consequently, we may choose an elliptic fibration $\varphi : Y \to \mathbb{P}^1$ so that its jacobian $J \to \mathbb{P}^1$ contains at least two singular fibers.

Since all fibers are irreducible, the trivial lattice $T \subset \mathrm{NS}(J)$ is zero, and the Mordell–Weil group $\mathrm{MW}(J/\mathbb{P}^1)$ is isomorphic to E_8. It follows that the Picard group of the generic fiber Y_η is a free abelian group of rank nine. Since $\epsilon : X \to Y$ is a universal homeomorphism, the same holds for the generic fiber X_η. Now let $S \to X$ be the resolution of singularities, and $S \to S'$ be the relative minimal model over $\mathbb{P}^1$. Then $S' \to \mathbb{P}^1$ is a rational elliptic surface, which is not necessarily jacobian. In turn, $\mathrm{Pic}(S')$ has rank ten. However, we just saw $\mathrm{Pic}(S'_\eta/\eta)$ has rank nine, whence all fibers of $S' \to \mathbb{P}^1$ are irreducible. Again by the result of Liu, Lorenzini and Raynaud [34], Theorem 6.6 the same holds for the smooth model $J' \to \mathbb{P}^1$ of the Frobenius pullback $X' = J^{(2/\mathbb{P}^1)}$. If a singular fiber J'_a is semistable, then also the corresponding fiber J_b is semistable, and $m_a = 2m_b \geq 2$, contradiction. Thus all singular fibers J'_a are of Kodaira type II, with numerical invariant $m_a = 1$. According to Theorem 12.1, the presence of two singular fibers on J ensures that all singularities on the Frobenius pullback $X' = J^{(2/\mathbb{P}^1)}$ are rational. By Lemma 9.2, each singular fiber J'_a has numerical invariant $m_a = 2m_b + \delta + 1 \geq 3$, contradiction. □

15. K3-like coverings with rational double points

Let k be an algebraically closed ground field of characteristic $p = 2$. In this section, we construct K3-like coverings X that are normal with only rational singularities. The main result is that for each but possibly six of the 110 configurations of Kodaira types from Lang's classification [31] of rational elliptic surfaces with reduced fibers actually appears as a configuration on some K3-like covering X and the ensuing Enriques surfaces $Y = X/G$.

Indeed, we start with a rational elliptic surface $J \to \mathbb{P}^1$ that is jacobian with only reduced fibers and assume that the Frobenius pullback $X' = J^{(2/\mathbb{P}^1)}$ has only rational singularities. In other words, the minimal resolution of singularities $r' : S \to X'$ is a K3 surface, endowed with an elliptic fibration $S \to \mathbb{P}^1$. The case leading to an elliptic singularity was already treated in Section 13.

We write $E' \subset S$ for the exceptional divisor of the minimal resolution of singularities $r' : S \to X'$. Let $E \subset S$ be another vertical negative-definite curve, and $r : S \to X$ be its contraction. We now seek to choose this E so that the resulting flop $X' \leftarrow S \to X$ yields a K3-like covering X. Roughly speaking, the new contraction must introduce another elliptic fibration, destroy all quasielliptic fibrations, yet keep the Kodaira type of the singular fibers. To achieve these conflicting tasks, we introduce for each closed point $a \in \mathbb{P}^1$ the following conditions relating the vertical negative-definite curves E_a and E'_a on the K3-surface S:

(M1) *If the fibers X'_a and S_a are semistable, then E_a either coincides with E'_a, or it is the strict transform of X'_a.*

(M2) *If the fiber X'_a has type* II *and S_a has type I_n^* for some $n \geq 0$, then either E_a coincides with E'_a, or it is the union of all irreducible components $\Theta \subset S_a$ except the terminal component nearest to the strict transform of X'_a.*

(M3) *If the fiber X'_a has type* III *and S_a has type I_n^* for some $n \geq 0$, then either E_a coincides with E'_a, or it is the union of all irreducible components $\Theta \subset S_a$ except the two terminal components contained in E'_a.*

(M4) *In all other cases, E_a coincides with E'_a.*

By abuse of notation, we here say that the fiber X_a on the normal surface X has a certain Kodaira type if this is the Kodaira type of J_a on the smooth surface J. Moreover, a *terminal irreducible component* on S_a means a component corresponding to a vertex in the dual graph that is adjacent to only one other vertex. Note that in the boundary case $n = 0$ of condition (M2), the curve E_a is the union of the irreducible components $\Theta \subset S_a$ except an arbitrary terminal component.

Definition 15.1. In the preceding situation, we say that $X' \leftarrow S \to X$ is a *mutation* if the conditions (M1)–(M4) on the vertical negative-definite curves $E, E' \subset S$ holds for all closed points $a \in \mathbb{P}^1$. A mutation is a *good mutation* if in addition there is a horizontal Cartier divisor $F \subset X$ that is an elliptic curve.

Good mutations lead to the desired flops:

Proposition 15.2. *Suppose $X' \leftarrow S \to X$ is a good mutation of the Frobenius pullback $X' = J^{(2/\mathbb{P}^1)}$. Then the normal surface X is a K3-like covering with only rational singularities. The induced elliptic fibration $\varphi : Y \to \mathbb{P}^1$ on the simply-connected Enriques surface $Y = X/G$ has the same configuration of Kodaira types as the original rational elliptic surface $J \to \mathbb{P}^1$. Furthermore, the induced fibration $f : X \to \mathbb{P}^1$ induces an inclusion $H^0(X, \Theta_{X/k}) \subset H^0(\mathbb{P}^1, \Theta_{\mathbb{P}^1/k})$.*

Proof. In our situation, S is a K3-surface. In order to apply Theorem 6.4, it suffices to check that conditions (E0), (E2), (E5) and (E6) from Section 6 hold. Condition (E5) holds by definition on good mutations. According to Proposition 12.3, the Frobenius pullback $X' = J^{(2/\mathbb{P}^1)}$ satisfies conditions (E2) and (E6).

The fibers on J and hence on X' are reduced by assumption, and this obviously also transfers to the fibers on X, so condition (E0) holds. It remains to check that a mutation $X' \leftarrow S \to X$ produces on X only Zariski singularities, and that the total Tjurina does not change. This is obvious if we are in the case (M1), when S_a is semistable. It remains to verify the cases (M2) and (M3). Then S_a is a fiber of Kodaira type I_n^* for some $n \geq 0$. The following argument shows that the ensuing singularities on X' and X are formally isomorphic. Set $R = \mathscr{O}_{\mathbb{P}^1,a}$ and $F = \mathrm{Frac}(R)$. For each terminal component $\Theta, \Theta' \subset S_a$, we may choose formal sections $A, A' \subset S \otimes \hat{R}$ passing through Θ and Θ', respectively. Since the generic fiber $U = S \otimes \hat{F}$ is an elliptic curve, there is a translation automorphism $\tau : U \to U'$ sending the $\hat{F}$-rational point $A \cap U$ to $A' \cap U$. By the functoriality of the relative minimal model, this extends to an automorphism of $S \otimes \hat{R}$ sending Θ to Θ'. In turn, the singularity arising from contractions of all components but Θ is formally isomorphic to the singularity given by the contraction of all components but Θ'. This settles the case (M2), and the case (M3) follows analogously. $\square$

It is very easy to specify mutations. The challenge, however, is to introduce another elliptic fibration as well. Rather than giving an elliptic curve, we shall construct its degenerate fibers. This boils down to finding sections for $J' \to \mathbb{P}^1$ with suitable intersection behavior. Using the following, we shall infer that the resulting fibration is elliptic rather then quasielliptic:

Proposition 15.3. *Let $X' \leftarrow S \to X$ be a mutation with respect to the vertical negative-definite $E \subset S$. Suppose that there is curve of canonical type $F \subset S$ with the following properties:*

(i) *The type of the curve F is either I_n or I_{2n+1}^* or IV or IV^*.*
(ii) *It has degree > 0 with respect to the elliptic fibration $S \to \mathbb{P}^1$.*
(iii) *Each irreducible component of E is either contained in or disjoint from F.*

Then $X' \leftarrow S \to X$ is a good mutation.

Proof. The curve of canonical type $F \subset S$ induces a genus-one fibration $g : S \to \mathbb{P}^1$ on the K3 surface S. Condition (i) ensures that the new fibration is elliptic, in light of [47], the Proposition on page 150. Condition (ii) implies that this fibration is different from our original elliptic fibration $S \to \mathbb{P}^1$. By the last condition, the new elliptic fibration on S induces a new elliptic fibration on X. One of its closed fibers is an elliptic curve $E \subset X$ that is horizontal with respect to the old elliptic fibration $X \to \mathbb{P}^1$. Thus the mutation $X' \leftarrow S \to X$ is good. $\square$

We now see that in most cases for $J \to \mathbb{P}^1$, one finds a good mutation, which yields a normal K3-like covering. The following are the configurations of Kodaira symbols for J where the approach does not work at the moment:

$$\mathrm{III},\ \mathrm{II}+5,\ \mathrm{III}+6,\ \mathrm{IV}+\mathrm{IV}+\mathrm{IV},\ \mathrm{IV}+\mathrm{II},\ \mathrm{IV}+\mathrm{III}. \tag{16}$$

The main result of this section is:

Theorem 15.4. *Suppose that the rational elliptic surface $J \to \mathbb{P}^1$ has a configuration of Kodaira symbols different from the ones in (16). Then there exists a good mutation $X' \leftarrow S \to X$, and the resulting X is a K3-like covering.*

Proof. The second statement follows from Theorem 15.2. The strategy for the first statement is to use Lang's classification for $J \to \mathbb{P}^1$ together with the Oguiso–Shioda list for the Mordell–Weil lattice $\mathrm{MW}(J/\mathbb{P}^1)$, in order to guess some vertical negative-definite curve $E \subset S$ that yields a mutation, as well as to produce suitable sections $P, P', Q, \ldots \subset J$ that allow to construct a curve of canonical type $F \subset S$ for which Proposition 15.3 applies. Throughout, $O \subset J$ denotes the zero-section. Recall from Proposition 13.1 that in any case there is a section disjoint from the zero-section O. To simplify notation, we now write $a, b \in \mathbb{P}^1$ for closed points, and S_a, S_b, X'_a, X'_b and J_a, J_b for the fibers of the fibrations $S \to \mathbb{P}^1$, $X' \to \mathbb{P}^1$ and $J \to \mathbb{P}^1$. We have to examine five cases:

Case (i): *Suppose there are at least two semistable fibers J_a and J_b.* Then define $E_a \subset S_a$ and $E_b \subset S_b$ as the strict transform of the fibers X'_a and X'_b, respectively. Let $E \subset S$ be the curve obtained from E' by replacing $E'_a + E'_b$ by $E_a + E_b$. This yields a mutation $X' \leftarrow S \to X$. Now choose a section $P \subset J$ disjoint from the zero-section $O \subset J$. Together with suitable chains of irreducible components inside J_a and J_b, they form a cycle of projective lines. Their preimage $F \subset S$ becomes a cycle of (-2)-curves, that is, a curve of canonical type I_n for some $n \geq 4$. By construction, each irreducible component of E is either disjoint from or contained in F. Thus Proposition 15.3 applies.

Case (ii): *Suppose there is exactly one semistable fiber J_b.* According to [31], Section 1 and 2, there are four possibilities for the configuration of singular fibers:

$$\mathrm{II} + 8, \quad \mathrm{III} + 8, \quad \mathrm{II} + 5, \quad \mathrm{III} + 6.$$

The respective Lang types of the unstable fiber J_a are $1A, 2A, 1C, 2B$, and the latter two configurations are excluded from consideration. Write the semistable fiber $J_b = \sum \Theta_i$ with cyclic indices in the natural way, that is, $(\Theta_0 \cdot O) = 1$ and $(\Theta_i \cdot \Theta_{i+1}) = 1$.

Suppose that the configuration is $\mathrm{II} + 8$. The fiber J_a has numerical invariants $v = 4$, $m = 1$, $\delta = 2$. On S the invariants become $v' = 8$ and $m' = 5$, whence S_a has Kodaira type I_0^*, and the corresponding singularity on X' is of type D_4. Fix some terminal irreducible component in S_a different from the strict transform of X'_a, and define E_a as the union of the remaining components $\Theta \subset S_a$. Let $E \subset S$ be the curve obtained from E' by replacing E'_a by E_a. This yields a mutation $X' \leftarrow S \to X$. The E_a, the strict transforms of the zero-section O and Θ_0, and the two components in the semistable fiber S_b adjacent to the strict tranform of Θ_0 form curve $F \subset S$ of canonical type I_3^*, and each irreducible component of E is either disjoint from or contained in F. The case that the configuration is $\mathrm{III} + 8$ is analogous and left to the reader.

From now on, we assume that all singular fibers J_a are unstable.

Case (iii): *Suppose there is a fiber J_a of Kodaira type* II*, whose Lang type is 9A or 9C.* This means $v = 4r$, $m = 1$ and $\delta = 4r - 2$, with respective values $r = 1, 3$. In turn, the invariants on S become $v' = 8r$ and $m' = 4r + 1$. The only possibility is that the fiber S_a has type I^*_{4r-4}, and the singularity on $X' = J^{(2/\mathbb{P}^1)}$ is of type D_{4r}. Fix the terminal component of S_a nearest to the strict transform of X'_a, and let $E_a \subset S_a$ be the union of the remaining irreducible components. Let $E \subset S$ be the curve obtained from $E' \subset S$ by replacing E'_a by E_a. This yields a mutation $X' \leftarrow S \to X$. Let $P \subset J$ be a section disjoint from the zero-section $O \subset J$. Together with the $4r$ components of E_a, they induce a canonical curve $F \subset S$ of type I^*_{4r-3}. By construction, each irreducible component from E is either disjoint from or contained in F.

Case (iv): *Suppose there is a fiber J_a of Kodaira type* III *with Lang type 10A.* The fiber J_a then has invariants $v = 4$, $m = 2$ and $\delta = 1$. In turn, the surface S gets a fiber with $v' = 8$ and $m' = 6$, which must be of type I_1^*, and the singularity on X' is of type D_4. Let E_a be the union of the two terminal components in X'_a disjoint from E'_a. Define $E \subset S$ as the curve obtained from $E' \subset S$ by replacing E'_a by E_a. Its contraction yields a mutation $X' \leftarrow S \to X'$.

According to Lang's classification, we have the following seven possibilities for the configuration of Kodaira types on J:

$$\mathrm{III}+\mathrm{II}+\mathrm{II},\ \mathrm{III}+\mathrm{III}+\mathrm{II},\ \mathrm{III}+\mathrm{III}+\mathrm{III},\ \mathrm{III}+\mathrm{II}+\mathrm{IV},\ \mathrm{III}+\mathrm{IV}+\mathrm{IV},\ \mathrm{III}+\mathrm{II},\ \mathrm{III}+\mathrm{III}.$$

The corresponding trivial lattices are of the form $A_1^{\oplus n}$ with $n = 1, 2, 3$ and $A_1 \oplus A_2^{\oplus m}$ with $m = 1, 2$. In the Oguiso–Shioda classification, they are No. 2,4,7,6,20. One sees that in all cases, the narrow Mordell–Weil lattice contains a vector α with $\alpha^2 = 2$. In turn, there is a section $P \subset J$ intersecting the same fiber components as the zero-section $O \subset J$, with

$$2 = \langle P, P\rangle = 2 + 2(P \cdot O).$$

It follows that O, P are disjoint. Consequently O, P and the four components of E_a induce a curve $F \subset S$ of canonical type I_1^*. By construction, each irreducible component of E is either contained in or disjoint from F.

Case (v): *Suppose there is a fiber J_a of Kodaira type* IV*.* This fiber has Lang type 11, with invariants $v = 4$, $m = 3$, $\delta = 0$. Whence the fiber S_a has invariants $v' = 8$, $m' = 7$. Their possible Kodaira types are IV^* or I_2^*. The respective singularities on X' are D_4 or A_4. The latter is not Zariski by Theorem 3.1, hence the singularity is D_4 and the Kodaira symbol is IV^*. Those containing a fiber of Lang type $9A, 9C, 10A$ were treated above. According to Lang's Classification, the only remaining possibilities are $\mathrm{IV} + \mathrm{II}$ and $\mathrm{IV} + \mathrm{III}$, which we have excluded from consideration.

This completes the proof: Case (i) applies if all fibers are semistable, according to [31], Section 1. It also covers all but four cases with $j(J/\mathbb{P}^1) \neq 0$, by [31], Section 2. The extra cases are treated in (ii). The remaining cases have global j-invariant $j(J/\mathbb{P}^1) = 0$, which were analysed in [31], Section 3. Then all fibers are unstable, and there is a fiber J_a of Lang type 9A, 9C, 10A or 11; these are treated in (iii)–(v). The exceptions that contain a fiber of Lang type $10C$ or 11 are excluded from consideration. $\square$

Another way to construct Enriques surface Y from a rational elliptic surface $J \to \mathbb{P}^1$ is via Ogg–Shafarevich theory, as exposed in [12], Chapter V, Section 4.

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