K3 SURFACES OVER SMALL NUMBER FIELDS AND KUMMER CONSTRUCTIONS IN FAMILIES

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ABSTRACT. We construct K3 surfaces over number fields that have good reduction everywhere. These do not exists over the rational numbers, by results of Abrashkin and Fontaine. Our surfaces exist for three quadratic number fields, and an infinite family of S_3-number fields. To this end we develop a theory of Kummer constructions in families, based on Romagny's notion of the effective models, here applied to sign involutions. This includes quotients of non-normal surfaces by infinitesimal group schemes in characteristic two, as developed by Kondo and myself. By the results of Brieskorn and Artin, the resulting families of normal K3 surfaces admit simultaneous resolutions of singularities, at least after suitable base-changes. These resolutions are constructed in two ways: First, by blowing-up families of one-dimensional centers that acquire embedded components. Second, by computing various ℓ-adic local systems in terms of representation theory, and invoking Shepherd-Barrons results on the resolution functor, which is representable by a highly non-separated algebraic space.

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INTRODUCTION

Besides obvious examples like projective spaces $\mathbb{P}^n$, toric varieties $\mathrm{Temb}_N(\Delta)$ stemming from regular fans, or flag varieties G/P coming from Chevalley groups, it is apparently exceedingly difficult to specify schemes X over the ring $R = \mathbb{Z}$ so that the structure morphism $X \to \mathrm{Spec}(\mathbb{Z})$ is smooth and proper. The Minkowski

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discriminant bound for number fields shows that there are no finite étale morphisms $\mathrm{Spec}(\mathscr{O}_F) \to \mathrm{Spec}(\mathbb{Z})$ of degree at least two ([53], Section 42 or [55], Chapter III, Theorem 2.17). By deep results of Abrashkin and Fontaine ([1] and [29]), there are no families of abelian varieties $A \to \mathrm{Spec}(\mathbb{Z})$ of relative dimension at least one. More generally, they independently established that for any smooth proper $X \to \mathrm{Spec}(\mathbb{Z})$, the complex fiber $X_{\mathbb{C}}$ satisfies severe restrictions on the Hodge numbers ([2] and [30]). As a consequence, their are no families of K3 surfaces over the integers. Building on this, the author proved that there no such families of Enriques surface either [73].

Recall that the defining properties of *K3 surfaces* are $c_1 = 0$ and $b_2 = 22$, with quartic surfaces in $\mathbb{P}^3$ as simplest examples. Given the extraordinary role of K3 surfaces in algebraic geometry, it is natural to ask *if they exists in families over number rings with small degree and discriminant.* In other words, we seek to understand for which number fields $\mathbb{Q} \subset F$ the stack of K3 surfaces

$$\mathscr{M}_{\mathrm{K3}} \longrightarrow (\mathrm{Aff}/S)$$

starts to have non-empty fiber category $\mathscr{M}_{\mathrm{K3}}(\mathscr{O}_F)$. André ([3], Theorem 1.3.1) showed that for fixed F and $R = \mathscr{O}_F[1/a]$, up to isomorphism there are only finitely many polarized families $X \to \mathrm{Spec}(R)$ belonging to $\mathscr{M}_{\mathrm{K3}}(R)$, an instance of the so-called *Shafarevich Problem* [79]. Analogues of the Néron–Ogg–Shafarevich Criterion for good reduction of abelian varieties where developed for K3 surfaces by Matsumoto and Liedtke [50], [51], [48], [52] and Bragg and Yang [14], see also [20]. Reductions of Kummer surfaces in characteristic two, a case of particular relevance for us, where studied by Overkamp [58] and Lazda and Skorobogatov [47]. Our first main result reveals that the fiber categories for $\mathscr{M}_{\mathrm{K3}}$ are non-empty sooner than one perhaps might expect:

Theorem A. *(See Thm. 8.2 and 9.1) The fiber category $\mathscr{M}_{\mathrm{K3}}(\mathscr{O}_F)$ is non-empty if $\mathbb{Q} \subset F$ is*

(i) *the S_3-number field arising as Galois closure from a cubic number field K whose discriminant takes the form $d_K = -3f^2$ for some even factor f;*
(ii) *one of the three quadratic number field with discriminant $d_F \in \{28, 41, 65\}$.*

In turn, the smallest examples are $F = \mathbb{Q}(\sqrt[3]{2}, e^{2\pi i/3})$ and $F = \mathbb{Q}(\sqrt{7})$. The desired families of K3 surfaces will be constructed from families of normal Kummer surfaces. In the case of quadratic number fields we also get:

Theorem B. *(See Thm. 8.3) The fiber category $\mathscr{M}_{\mathrm{Enr}}(\mathscr{O}_F)$ for the stack of Enriques surfaces is non-empty for the three quadratic number field F with discriminant $d_F \in \{28, 41, 65\}$.*

In some sense, the above results are neat by-products. My main motivation is to develop a *theory for Kummer constructions in families.* The classical Kummer construction goes back to the nineteenth century: In modern terms, it attaches to each abelian surface A the quotient $A/\{\pm 1\}$ by the sign involutions. This is a normal surface with rational double points and the minimal resolution is a K3 surface. This remains only partly true in characteristic $p = 2$: Shioda [75] and Katsura [40] observed that if A is supersingular, the quotient by the sign involution yields rational surfaces rather than K3 surfaces. To remedy this confusing situation, the author replaced A by the self-product $C \times C$ of the rational cuspidal curve, a non-normal genus-one curve, and the action of $\{\pm 1\}$ by an action of the infinitesimal group scheme α_2, and showed that $(C \times C)/\alpha_2$ is indeed a K3 surface with rational double points [68]. More recently, Kondo and the author [45] considered actions of the multiplicative group scheme μ_2, and showed that $(C \times C)/\mu_2$ is a K3 surface with only sixteen rational double points of type A_1, as in the classical situation. In both cases, however, there is one additional D_4-singularity.

We now put ourself into a relative situation, over a Dedekind scheme S. Let E_F be an elliptic curve over the function field $F = \kappa(\eta)$, with resulting Weierstraß model $E \to S$. At the residue fields of characteristic two, we allow additive reduction, but discard supersingular good reduction. The central idea of this paper is to use Romagny's powerful theory of *effective models of group scheme actions* [65], and replace the sign involution by its effective model

$$G = \overline{\{\pm 1\}_\eta} \subset \mathrm{Aut}_{E/S}\,.$$

This is a family of group schemes of order two, acting faithfully on the fibers of the Weierstraß model, even at the points of bad reduction. It is then possible to characterize the condition that G acts freely at the scheme of non-smoothness $\mathrm{Sing}(E/S)$, the crucial prerequisite for infinitesimal Kummer constructions. This leads to the notion of elliptic curves E_F and E'_F that are *admissible for the Kummer construction* (Definition 5.1). The crucial conditions happens at the primes of bad reduction with residue characteristic two: The reduction type must be additive, and the action of the effective model at the singular point is free. This can be re-formulated in terms of the local Weierstraß equation, and is equivalent to the opaque condition $\mathrm{val}(b_4a_2 + b_6) = 2\,\mathrm{val}(2, a_1, a_3)$. Our third main result:

Theorem C. *(See Thm. 5.5) If the elliptic curves E_K is admissible for the Kummer construction, then the categorical quotient $V = (E \times E)/G$ is a family of normal K3 surfaces over the Dedekind scheme S.*

The existence of such categorical quotients easily follows from the Keel-Mori Theorem [41], and was further analyzed by Rydh [67]. The above also works for pairs E_F, E'_F of elliptic curves that are admissible for the Kummer construction. In the future, we hope to extend the the theory to general abelian surfaces over F.

Let $g : V \to S$ be the structure morphism for $V = (E \times E)/G$. Then the singularities in the geometric fibers are at most rational double points. The problem now is to construct a *simultaneous minimal resolution* $r : X \to V$. The main insight of Brieskorn ([15], [16], [17]) was that this indeed exists, at least after making a base-change along some finite surjective $S' \to S$. Building on this, Artin [6] showed that the functor of resolutions is actually representable by an algebraic space $\mathrm{Res}_{V/S}$, coming with a *universal simultaneous resolution*

$$f_{\mathrm{univ}} : X_{\mathrm{univ}} \to \mathrm{Res}_{V/S}$$

Usually, the base and therefore also the total space is highly non-separated, in a very un-schematic way. On the other hand, one may argue that the base is rather close to being a scheme, because the continuous map $|\mathrm{Res}_{V/S}| \to |S|$ is bijective.

In Lemma 1.1 we observe directly that the residue field extensions $\kappa(s) \subset \kappa(s')$ are purely inseparable. Here $s' \in |\operatorname{Res}_{V/S}|$ signifies the point corresponding to $s \in S$.

Shepherd-Barron [80] makes a detailed analysis of obstructions in ℓ-adic cohomology for the existence of a simultaneous minimal resolution $X \to V$, and relates them to the Weyl groups W attached to the RDP singularities via dual graphs as Dynkin diagrams. Applying his abstract results for our concrete $V = (E \times E')/G$, we are able to pin-point the geometric meaning of the obstruction, and thus explain exactly what base-changes are needed. The relevant part in $\underline{H}^2(X_\eta, \mathbb{Z}_\ell)$ is a Kronecker product arising from the Künneth Theorem, which has to match with the part in $\underline{H}^2(X_\sigma, \mathbb{Z}_\ell(1))$ stemming from the cycle class map for the exceptional divisors of the additional D_4-singularities $v_{\text{crit}} \in V$. Using the representation-theoretic fact that Kronecker products are almost never permutation representations, we get sufficient control over the situation:

Theorem D. *(See Thm. 7.5) A simultaneous minimal resolution of singularities $r : X \to V$ exists provided the following conditions hold:*

(i) *The field F contains a primitive third root of unity.*
(ii) *The purely inseparable extension $k \subset \kappa(v_{\text{crit}})$ is an equality.*
(iii) *The elliptic curves $E \otimes F^{\text{sh}}$ and $E' \otimes F^{\text{sh}}$ are isomorphic, and acquire good reduction over a quadratic extension $F^{\text{sh}} \subset L$.*

An important intermediate step that produces *simultaneous partial resolutions* are blowing-ups $Y = \operatorname{Bl}_Z(V)$ with families of one-dimensional schemes, where the closed fiber acquires embedded points, a new technique that should be useful in many other contexts. Our main result in this direction is Theorem 6.1.

Coming back to the stack of K3 surfaces $\mathscr{M}_{\text{K3}}$, the problem now is to find elliptic curves E_F over number fields F of small degree and discriminant that are admissible for the Kummer construction, and for which our results on simultaneous resolutions apply. Ogg [56] already classified the elliptic curves over $F = \mathbb{Q}$ where bad reduction occurs only at $p = 2$. Over more general number fields, such questions attracted a lot of attention, for example in the work of Setzer [78], Stroecker [81], Rohrlich [64], Ishii [37], Pinch [60], [61], [62], Bertolini and Canuto [7], Kida and Kagawa [38], [44], [42], [39], [43], Cremona and Lingham [24], Clemm and Trebat-Leder [21], and myself [72]. In this regard, however, little is known about general curves, surfaces or higher-dimensional schemes.

The work of Pinch [60] reveals that over $L = \mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(e^{2\pi i/3})$ there is precisely one conjugate pair of elliptic curves E_L that are admissible for the Kummer construction. It turns out that this indeed leads to families $X \to \operatorname{Spec}(\mathscr{O}_F)$ of K3 surfaces stemming from S_3-number fields F. The work of Comalada [23] tells us that up to isomorphism, there are exactly eight elliptic curves E_F over quadratic number fields F that have good reduction everywhere, and the occurring discriminants of the number fields are $d_F \in \{28, 41, 64\}$. Since the 2-torsion in E_F is constant, no further base-change is needed to construct $X \to \operatorname{Spec}(\mathscr{O}_F)$.

The paper is organized as follows: In Section 1 we review the foundational results of Brieskorn, Artin and Shepherd-Barron on simultaneous resolutions of surface singularities, and the role of ℓ-adic cohomology in this regard. Our central topic, the families of K3 surfaces, together with the non-existence results of Abrashkin and Fontaine are discussed in Section 2. We then focus on families of elliptic curves, and introduce in Section 3 the effective model for the sign involution, following the general construction of Romagny. In Section 4 we compute for which families the effective model acts freely at the singular points. This is a crucial prerequisite for the relative Kummer construction $V = (E \times E')/G$, which is introduced in Section 5. In Section 6 we obtain partial resolutions of singularities, via blowing-ups where the centers are Weil divisors on the total space, whose closed fibers may acquire embedded points. In Section 7 we study monodromy representation stemming from ℓ-adic cohomology, in order to construct the simultaneous resolution for the additional critical D_4-singularities in the geometric fibers of characteristic two. In the remaining sections we search for elliptic curves that are admissible for the Kummer construction: In Section 8 we examine quadratic number fields, and find three of them. It turns out that no further base-change is needed for the simultaneous resolution of singularities. In the final Section 9 we look at certain S_3-number fields. Now the families of normal K3 surfaces exists over the quadratic subfield, to which a third root of unity must be added to achieve simultaneous resolution.

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1. Simultaneous resolutions and ℓ-adic cohomology

In this section we discuss the fundamental notations and results pertaining to simultaneous resolutions of surface singularities, and the role of ℓ-adic cohomology in this regard.

Let S be a base scheme. So from now on, all schemes are endowed with a structure morphism to S, and likewise all rings R come with a $\mathrm{Spec}(R) \to S$, usually suppressed from notation. A *family of normal surfaces* is a triple (R, V, h) where R is a ring, V is an algebraic space, $h : V \to \mathrm{Spec}(R)$ is a proper flat morphism of finite presentation such that for each geometric point $s : \mathrm{Spec}(k) \to \mathrm{Spec}(R)$, the base change $V_s = V \otimes_R k$ is integral, normal, and two-dimensional. Such families form a category, where the morphisms $(R', V', h') \to (R, V, h)$ are given by commutative diagrams

$$\begin{array}{ccc} V' & \longrightarrow & V \\ \big\downarrow & & \big\downarrow \\ \mathrm{Spec}(R') & \longrightarrow & \mathrm{Spec}(R) \end{array}$$

such that the canonical map $V' \to V \otimes_R R'$ is an isomorphism. One easily checks that the forgetful functor $(R, V, h) \mapsto \mathrm{Spec}(R)$ defines a category fibered in groupoids over the category of affine schemes (Aff/S) that satisfies the stack axioms with regard to the fppf topology. The above notion of families immediately generalizes to the situation where the ring R is replaced by an arbitrary scheme or algebraic space.

Let V be a family of normal surfaces over S. We consider the category

$$\underline{\mathrm{Res}}_{V/S} = \{(R, X, f, r)\}$$

where (R, X, f) is a family of normal surfaces and $r : X \to V \otimes R$ is an R-morphism such that the following holds: For each geometric point $s : \mathrm{Spec}(k) \to \mathrm{Spec}(R)$, the fiber X_s is regular, the morphism $r_s : X_s \to V_s$ is birational, and the dualizing sheaf satisfies $(\omega_{X_s} \cdot E) \geq 0$ for each irreducible curve E in the exceptional locus $\mathrm{Exc}(X_s/V_s)$. Then every other resolution of singularities for V_s uniquely factors over X_s. Therefore, the objects are called *simultaneous minimal resolutions of singularities*. Morphisms are defined as in the previous paragraph.

Clearly, the forgetful functor

$$\underline{\mathrm{Res}}_{V/S} \longrightarrow (\mathrm{Aff}/S), \quad (R, X, f, r) \longmapsto \mathrm{Spec}(R)$$

is a category fibered in groupoids, and satisfies the stack axioms. The fiber categories $\underline{\mathrm{Res}}_{V/S}(R)$ are actually *setoids*, that is, every arrow is an isomorphism and every self-arrow is an identity. Indeed, $\underline{\mathrm{Res}}_{V/S}$ is equivalent to the comma category of affine schemes over an *algebraic space* $\mathrm{Res}_{V/S}$, according to Artin's fundamental insight ([6], Theorem 1), which builds on Brieskorn's results ([15], [16], [17]). In particular, there is a *universal simultaneous minimal resolution of singularities*

$$r_{\mathrm{univ}} : X_{\mathrm{univ}} \longrightarrow \mathrm{Res}_{V/S}\,.$$

Usually the base, and thus also the total space, is highly *non-separated in an unschematic way*, compare the illustration in [6], Figure 1.1. From another perspective the algebraic space $\mathrm{Res}_{V/S}$ is rather close to being a scheme:

Lemma 1.1. *For each $s \in S$, the fiber $(\mathrm{Res}_{V/S})_s$ is the spectrum of a finite purely inseparable field extension of the residue field $\kappa(s)$.*

Proof. Without loss of generality we may assume $S = \mathrm{Spec}(k)$, and have to verify that $\mathrm{Res}_{V/k}$ is the spectrum some finite purely inseparable extension k'. We start with some preliminary observations:

Let $k \subset L$ be some field extension, and write $r : X \to V \otimes L$ for the minimal resolution of singularities. Note that the scheme X is regular, but not necessarily geometrically regular. Suppose the latter holds. The closed subscheme $Z = \mathrm{Sing}(X/V \otimes L)$ is one-dimensional, but may have embedded components. Write E for the closed subscheme corresponding to the sheaf of ideals $\mathscr{I} \subset \mathscr{O}_Z$ comprising the torsion sections. Let $\eta_1, \ldots, \eta_r \in E$ be the generic points and E_i be the schematic closure of the monomorphism $\mathrm{Spec}(\mathscr{O}_{E,\eta_i}) \to X$. The intersection matrix $(E_i \cdot E_j)$ is negative-definite, and we find some $D = -\sum \lambda_i E_i$ such that $(D \cdot E_j) > 0$. Fix a numerically ample Weil divisor $C \subset V$. For some sufficiently large $\mu \geq 0$, we may assume that Mumford's rational pullback $\mu r^*(C)$ has integral coefficients (see [54] and [69] for details), and furthermore $\mu r^*(C) + D$ is ample on X. Let $\mathscr{L}$ be the resulting ample sheaf. Passing to some tensor power, it becomes very ample, and defines a closed embedding $X \subset \mathbb{P}^m_L$. Let $k \subset k' \subset L$ be the common field of definition for the closed subscheme X and the graph Γ_r.

We claim that $k \subset k'$ is finite and purely inseparable. For this me may enlarge L, and assume that L is algebraically closed. Each member of $\Gamma = \mathrm{Aut}(L/k)$ stabilizes

$\mathrm{Reg}(V/k)\otimes L$, and therefor also its closure X inside $\mathbb{P}^m_L$. Thus k' belongs to the fixed field L^Γ. If $t\in L$ is transcendental, the automorphism group of $k(t)$ equals $\mathrm{PGL}_2(k)$, and each element extends an element of Γ by [18], Lemma 1. Likewise, for each separable $\omega\in L$ the resulting finite Galois extension has non-trivial automorphisms groups, and its member can be extended to Γ. It follows that L^Γ and hence k' is purely inseparable. Being a field of definition, k' is finitely generated, and thus finite.

Suppose now that $k\subset L$ is purely inseparable. We claim that the extension $k\subset k'$ does not depend on L, up to unique isomorphism. To see this, fix a perfect closure k^{perf}, and regard L as a subfield of k^{perf}. The formation of $Z=\mathrm{Sing}(X/V\otimes L)$ commutes with all field extensions. It follows from [32], Proposition 4.2.7 that the same holds for the formation of the curves E. Also the formation of E_i does not change under purely inseparable extension. Thus k' is the same, whether formed with L or $L'=k^{\mathrm{perf}}$.

We finally check that the minimal resolution of singularities $r':X'\to V\otimes k'$ is indeed universal, and thus $\mathrm{Res}_{V/k}=\mathrm{Spec}(k')$: The preceding arguments already reveal that $|\mathrm{Res}_{V/k}|$ is a singleton. The remaining task boils to to checking that each simultaneous minimal resolution $X\to V\otimes R$ with an Artin local ring R is constant. This easily follows, by viewing X as a deformation of $X_0=X\otimes R/\mathfrak{m}_R$ containing the constant family $\mathrm{Reg}(V/k)\otimes R$. □

We may regard the topological space $|\mathrm{Res}_{V/S}|$, which comprises equivalence classes of morphisms $\mathrm{Spec}(K)\to\mathrm{Res}_{V/S}$, as the set $|S|$, but endowed with a possibly finer topology. It is convenient to write $s'\in|\mathrm{Res}_{V/S}|$ for the point corresponding to $s\in S$, and $\kappa(s')$ for purely inseparable extension in Proposition 1.1, and

$$r_{s'}:X_{s'}=X_{\mathrm{univ}}\times_{\mathrm{Res}_{V/S}}\mathrm{Spec}\,\kappa(s')\longrightarrow V\times_S\mathrm{Spec}\,\kappa(s')=V_{s'} \tag{1}$$

for the resulting minimal resolution of singularities. As already remarked, the proper normal surface $W=V_s$ has a minimal resolution of singularities $h:\tilde W\to W$, whose total space is regular. With imperfect $\kappa(s)$, this may fail to be geometrically regular, for example is W is regular but not geometrically regular. This deserves further study; the following two observation are already useful:

Lemma 1.2. *Notation as above. If $R^1r_{s',*}(\mathscr{O}_{X_{s'}})=0$ and $V_{s'}$ is Gorenstein, then $R^1h_*(\mathscr{O}_{\tilde W})=0$ and $W=V_s$ is Gorenstein. Furthermore, the minimal resolution (1) factors over the base-change $\tilde W\otimes\kappa(s')$, and the latter is normal.*

Proof. The canonical map $\omega_{W\otimes\kappa(s')}\to\omega_W\otimes\kappa(s')$ is bijective, so the dualizing sheaf ω_W is invertible and the scheme W is Gorenstein. Let $E_1,\ldots,E_r\subset\tilde W$ be the exceptional divisors, and write $K_{\tilde W/W}=\sum\lambda_iE_i$. Since the resolution is minimal, we have $K_{\tilde W/W}\cdot E_i\geq 0$ and consequently $\lambda_i\leq 0$ for all indices $1\leq i\leq r$, confer [69], Proposition 2.4. Seeking a contradiction, we assume that $R^1h_*(\mathscr{O}_{\tilde W})$ is non-zero. By [4], Theorem 3 the fundamental cycle $Z\subset\tilde W$ has $\chi(\mathscr{O}_Z)\leq 0$. From

$$0\leq -2\chi(\mathscr{O}_Z)=\deg(\omega_Z)=(K_{\tilde W}+Z)\cdot Z<K_{\tilde W/W}\cdot Z$$

we see that $\lambda_j<0$ for at least one $1\leq j\leq r$. In turn, the same holds for the fundamental cycle stemming from the normalization of the base-change $\tilde W\otimes$

$\kappa(s')$. This normalization is dominated by some regular modification of $X_{s'}$. By our assumptions we have $K_{X_{s'}/V_{s'}} = 0$, hence on the regular modification all coefficients are ≥ 0, contradiction.

This establishes $R^1 h_*(\mathscr{O}_{\tilde{W}}) = 0$. We infer that all coefficients λ_i vanish, and also that the base-change $\tilde{W} \otimes \kappa(s')$ is normal. If the latter is dominated by some modification rather then the minimal resolution $X_{s'}$, then $\lambda_j > 0$ for at least one $1 \leq j \leq r$, contradiction. $\square$

Sometimes the condition on the singularities are enforced by a global hypothesis:

Lemma 1.3. *Notation as above. If the dualizing sheaf $\omega_{\tilde{W}}$ is anti-nef, we have $R^1 h_*(\mathscr{O}_{\tilde{W}}) = 0$. If $\omega_{\tilde{W}}$ is numerically trivial, $W = V_s$ is furthermore Gorenstein.*

Proof. Let $Z \subset W$ be the fundamental cycle. Then $-2\chi(\mathscr{O}_Z) = (K_{\tilde{W}} \cdot Z) + Z^2 < 0$, so [4], Theorem 3 ensures that $R^1 h_*(\mathscr{O}_{\tilde{W}}) = 0$. Suppose now that $K_{\tilde{W}} = r^* K_W + K_{\tilde{W}/W}$ is numerically trivial, and let $E_1, \ldots, E_r \subset W$ be the irreducible components. Then $(K_{\tilde{W}/W} \cdot E_i) = 0$, hence $\omega_{\tilde{W}}$ is numerically trivial on Z. Using $R^1 h_*(\mathscr{O}_{\tilde{W}}) = 0$ we see that it is actually trivial on Z and all its infinitesimal neighborhoods, and infer that ω_W is invertible. $\square$

Recall that on normal surfaces, the *rational Gorenstein singularities* are precisely the *rational double points*. The terms *RDP singularities*, *ADE singularities*, *Kleinian singularities* and *canonical singularities* are also used in the literature. Let us say that $V \to S$ is an *RDP family of normal surfaces* if all geometry fibers V_s are normal surfaces with at most rational double points.

We next examine the local henselian ring $\mathscr{O}^h_{\mathrm{Res}_{V/S}, s'}$ whose residue field $\kappa(s')$ is a finite purely inseparable extension of $\kappa(s)$. the ring comes with a homomorphism

$$\mathscr{O}_{S,s} \subset \mathscr{O}^h_{S,s} \longrightarrow \mathscr{O}^h_{\mathrm{Res}_{V/S}, s'}.$$

In general, this arrow is neither injective nor surjective, and it could well be that the local ring for s' coincides with the residue field. Now Artin's reformulation ([6], Theorem 2) of Brieskorn's result is:

Theorem 1.4. *If $V \to S$ is an RDP family of normal surfaces, the induced maps $\mathrm{Spec}(\mathscr{O}^h_{\mathrm{Res}_{V/S}, s'}) \to \mathrm{Spec}(\mathscr{O}^h_{S,s})$ are surjective.*

In what follows, we freely use the theory of ℓ-adic cohomology, also in the context of algebraic spaces, with a summary of the basic facts included at the end of this section. *To simplify the exposition, we now assume that $V \to S$ is an RDP family of normal surfaces, and that S is the spectrum of a henselian discrete valuation ring R, with generic point η and closed point σ.*

Note that $|S| = \{\eta, \sigma\}$ and $|\mathrm{Res}_{V/S}| = \{\eta', \sigma'\}$, that the canonical map between these topological spaces is a homeomorphisms, and that there is a universal simultaneous minimal resolution $f_{\mathrm{univ}} : X_{\mathrm{univ}} \to \mathrm{Res}_{V/S}$. We now fix a prime $\ell > 0$ that is invertible in R, and consider the ensuing ℓ-adic sheaf $R^2 f_{\mathrm{univ},*}(\mathbb{Z}_\ell(1))$ on the algebraic space $\mathrm{Res}_{V/S}$. The cycle classes for the exceptional divisors on $X_{\sigma'} \otimes \kappa(\sigma')^{\mathrm{sep}}$ defines an ℓ-adic subsheaf inside $\underline{H}^2(X_{\sigma'}, \mathbb{Z}_\ell(1))$. The latter denotes the ℓ-adic sheaf on the spectrum of $\kappa(s')$. By Lemma 1.6 below, the inclusion extends to an ℓ-adic subsheaf

$$\Psi_{V/S} \subset R^2 f_{\mathrm{univ},*}(\mathbb{Z}_\ell(1)).$$

This is an ℓ-adic local system whose rank $r \geq 0$ is given by the number of exceptional divisors in $X_{\sigma'} \otimes \kappa(\sigma')^{\mathrm{sep}}$. If the generic fiber of V is already smooth, restriction to $\eta \in S$ gives an ℓ-adic subsheaf

$$(\Psi_{V/S})_\eta \subset \underline{H}^2(X_{\eta'}, \mathbb{Z}_\ell(1)) = \underline{H}^2(V_\eta, \mathbb{Z}_\ell(1))$$

over the field of fractions $F = \mathrm{Frac}(R)$. A powerful result of Shepherd-Barron ([80], Corollary 2.14) now can be reformulated in the following way:

Theorem 1.5. *In the above situation, suppose the following conditions:*

(i) *The generic fiber V_η is smooth.*
(ii) *The minimal resolution of singularities for V_σ is smooth.*
(iii) *All exceptional divisors for $r_\sigma : X_\sigma \to V_\sigma$ are isomorphic to $\mathbb{P}^1_\sigma$.*
(iv) *The local system $(\Psi_{V/S})_\eta$ is constant.*

Then the family $V \to S$ admits a simultaneous resolution of singularities.

Note that for the assumptions and the conclusion alike there is no need to mention the algebraic space $\mathrm{Res}_{V/S}$. To apply the result, the crucial point is to identify the local system $(\Psi_{V/S})_\eta$ in condition (iv) from the geometry of the family $V \to S$. Also note that condition (ii) simply means that the purely inseparable extension $\kappa(\sigma) \subset \kappa(\sigma')$ is an equality.

We now recall and summarize some basic facts from ℓ-adic cohomology. Let Y be a scheme or algebraic space, and $\ell > 0$ be a prime that is invertible on Y. An abelian sheaf F_ν on the étale site (Et/Y) is called *ℓ^ν-local system* of rank $r \geq 0$ if it is a twisted form of the constant sheaf $(\mathbb{Z}/\ell^\nu\mathbb{Z})_Y^{\oplus r}$. An *$\ell$-adic local system* is an inverse system $F = (F_\nu)$, where the entries are ℓ^ν-local systems, and the transition maps yield identifications $F_\nu = F_\mu \otimes \mathbb{Z}/\ell^\nu\mathbb{Z}$ whenever $\mu \geq \nu$. One defines the ℓ-adic cohomology groups as $H^i(Y, F) = \varprojlim_{\nu \geq 0} H^i(Y, F_\nu)$. Note that these are actually $\mathbb{Z}_\ell$-modules.

Suppose now that our scheme or algebraic space Y is connected and non-empty. Choose a morphism $y_0 : \mathrm{Spec}(K) \to Y$ with some separably closed field K. This yields a *monodromy representation* $\pi_1(Y, y_0) \to \mathrm{GL}(F_\nu(K))$, and actually gives an equivalence between the exact category of ℓ^ν-local systems of rank $r \geq 0$ and the exact category of continuous representations $\pi_1(Y, a) \to \mathrm{GL}_r(\mathbb{Z}/\ell^\nu\mathbb{Z})$. For ℓ-adic local systems $F = (F_\nu)$, one obtains an equivalence to the exact category of continuous representations

$$\pi_1(Y, y_0) \longrightarrow \varprojlim_{\nu} \mathrm{GL}_r(\mathbb{Z}/\ell^\nu\mathbb{Z}) = \mathrm{GL}_r(\mathbb{Z}_\ell).$$

Composing with the canonical map

$$\mathrm{Gal}(K^{\mathrm{sep}}/K)^{\mathrm{op}} = \pi_1(\mathrm{Spec}\,\kappa(y), y_0) \longrightarrow \pi_1(Y, y_0), \tag{2}$$

our representation of the fundamental group becomes a Galois $\mathbb{Z}_\ell$-module.

Lemma 1.6. *Let F be an ℓ-adic local system on Y, with resulting representation $M = \varprojlim F_\nu(y_0)$ of the fundamental group $\pi_1(Y, y_0)$, and $M' \subset M$ be Galois $\mathbb{Z}_\ell$-submodule. If the canonical map (2) is surjective, then there is a unique ℓ-adic subsystem $F' \subset F$ with $F'(y_0) = M'$.*

Proof. The surjectivity of $\mathrm{Gal}(F^{\mathrm{sep}}/F)^{\mathrm{op}} \to \pi_1(Y, y_0)$ ensures that the $\mathbb{Z}_\ell$-submodule $M' \subset M$ is stable by the action of the fundamental group. Being a subrepresentation with respect to $\pi_1(Y, y_0)$, it stems from a local subsystem. □

Note that the above lemma applies in particular if Y is a henselian local scheme, according to [33], Proposition 18.5.15, and thus also to the algebraic space $Y = \mathrm{Res}_{V/S}$ in the construction of the local subsystem $\Psi_{V/S}$ above.

2. Families of K3 surfaces

For the moment we fix an algebraically closed ground field k of characteristic $p \geq 0$. Recall that the smooth proper surfaces X with $h^0(\mathscr{O}_X) = 1$ and $c_1 = 0$ fall into four classes: K3 surfaces, Enriques surfaces, abelian surfaces, and bielliptic surfaces, which are distinguished by their second Betti number ([8] and [9]). In turn, the *K3 surfaces* are the smooth proper surfaces X with $h^0(\mathscr{O}_X) = 1$ and $c_1 = 0$ and $b_2 = 22$. An important variant: A *normal K3 surface* is a normal proper surface Y such that the minimal resolution of singularities $r : X \to Y$ is a K3 surface. Using Lemma 1.3 we then get

$$h^0(\mathscr{O}_Y) = h^2(\mathscr{O}_Y) = 1,\ h^1(\mathscr{O}_Y) = 0 \quad \text{and} \quad R^1 r_*(\mathscr{O}_X) = 0 \quad \text{and} \quad \omega_Y = \mathscr{O}_Y.$$

In particular, the exceptional divisors $E_i \subset X$, $1 \leq i \leq r$ have dual graph whose connected component correspond to the Dynkin diagram A_n or D_n or E_6, E_7, E_8.

We now replace the ground field k by a base scheme S, and examine the ensuing relative notions: A *family of K3 surfaces* over S is an algebraic space X, together with a proper flat morphisms $f : X \to S$ of finite presentation, such that for all geometric points $s : \mathrm{Spec}(k) \to S$ the fiber X_s is a K3 surfaces in the sense of the preceding paragraph. Likewise, a *family of normal K3 surfaces* is some Y and $g : Y \to S$ as above, such that the the geometric fibers Y_s are normal K3 surfaces. Note that these are automatically RDP families. Furthermore, the notions immediately extend when we replace the base scheme S by an algebraic space.

Write (Aff/S) for the category of affine schemes $U = \mathrm{Spec}(R)$ endowed with a structure morphism $U \to S$. The triples (R, X, f) where R is a ring and $f : X \to \mathrm{Spec}(R)$ is a family of K3 surfaces form a category $\mathscr{M}_{\mathrm{K3}}$. Morphisms are given by the commutative diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(R') \arrow[r] & \mathrm{Spec}(R) \end{tikzcd}$$

such that the induced map $X' \to X \otimes_R R'$ is an isomorphism. Then the forgetful functor

$$\mathscr{M}_{\mathrm{K3}} \longrightarrow (\mathrm{Aff}/S), \quad (R, X, f) \longmapsto \mathrm{Spec}(R)$$

is a category fibered in groupoids, which satisfies the stack axioms with respect to the fppf topology on (Aff/S). The fiber categories $\mathscr{M}_{\mathrm{K3}}(R)$ over the rings $R = \mathbb{Z}[1/2]$ and $R = \mathbb{Z}_{(2)}$ are non-empty: The respective equations

$$T_0^4 + T_1^4 + T_2^4 + T_3^4 = 0 \quad \text{and} \quad T_0T_1^3 + T_1T_2^3 + T_2T_3^3 + T_0^4 = 0$$

define the desired $X \subset \mathbb{P}^3_R$ as families of smooth quartic surfaces. However, these become singular over $\mathbb{Z}$. This indeed must hold for any other way to describe families of K3 surfaces over a localization $\mathbb{Z}[1/n]$, by the following deep results of Abrashkin ([1], Theorem in §7, Section 6) and Fontaine ([29], Theorem 1). The former also gives information on a few other number rings:

Theorem 2.1. *The fiber categories $\mathscr{M}_{\mathrm{K3}}(\mathscr{O}_F)$ are empty for the ring of integers $\mathscr{O}_F$ in the number fields $F = \mathbb{Q}$, and also for $F = \mathbb{Q}(\sqrt{d})$ with $d \in \{-1, -3, 5\}$.*

How to construct number fields F for which $\mathscr{M}_{\mathrm{K3}}(\mathscr{O}_F)$ becomes non-empty? What are the smallest degrees $[F : \mathbb{Q}]$ or discriminants d_F for which this happens? Our goal of this paper is to answer these questions.

The strategy is simple: First obtain families of normal K3 surfaces $V \to \mathrm{Spec}(\mathscr{O}_F)$ in a systematic way. Then exploit the geometry at hand to construct simultaneous partial resolutions of singularities $Y \to V$. For the remaining singularities, the Brieskorn–Artin Theorem then gives the desired family over the ring of integers for some extension F', and we invoke Shepherds-Barron's results to control the degree $[F' : F]$. We shall obtain the initial V via a relative Kummer construction. To get small degrees and discriminants, this relies on the infinitesimal Kummer constructions of Kondo and myself, and the theory of Néron models for elliptic curves.

3. Effective models for sign involutions

Let R be a discrete valuation ring, with maximal ideal $\mathfrak{m}$, residue field $k = R/\mathfrak{m}$, and field of fractions $F = \mathrm{Frac}(R)$. Write $p \geq 0$ for the characteristic of the residue field, and choose a uniformizer $\pi \in R$. Let

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{3}$$

be a Weierstraß equation with coefficients $a_i \in R$. One also has auxiliary values and discriminant $b_i, c_i, \Delta \in R$, as defined in [25], Section 1. The homogeneous cubic equation

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3 \tag{4}$$

defines a family of cubic curves $E \subset \mathbb{P}^2_R$ containing the *zero section*

$$\mathrm{Zero}(E/R) = V_+(X, Z) = \{(0 : 1 : 0)_R\}.$$

The dehomogenization via $x = X/Z$ and $y = Y/Z$ gives back our Weierstraß equation, which defines the affine part $E \cap D_+(Z)$.

Following [28], Section 1, we write $\mathrm{Sing}(E/R)$ for the closed subscheme defined by the first Fitting ideal of $\Omega^1_{E/R}$, and call it the *scheme of non-smoothness*. The complementary open subscheme $\mathrm{Reg}(E/R)$ carries a group law in a canonical way. Its fibers are either elliptic curves, or twisted forms of the multiplicative group $\mathbb{G}_m$, or twisted forms of the additive group $\mathbb{G}_a$, in dependence on $\Delta, c_4 \in R$. Note that we have

$$2, a_1, a_3 = 0 \quad \Longleftrightarrow \quad 2, \Delta, c_4 = 0, \tag{5}$$

which follows from the congruences $c_4 \equiv a_1^4$ modulo 2, together with $\Delta \equiv a_3^4$ modulo the ideal $(2, a_1)$.

Throughout, we assume that at least one of the elements $2, a_1, a_3 \in R$ *is non-zero*, or equivalently one of $2, c_4, \Delta \in R$ is non-zero. Another way to phrase this is that the matrix

$$N = \begin{pmatrix} 1 & -a_1 & 0 \\ 0 & -1 & 0 \\ 0 & -a_3 & 1 \end{pmatrix} \in \mathrm{GL}_3(R)$$

has order two, and the same holds for the image in $\mathrm{PGL}_3(R) = \mathrm{Aut}(\mathbb{P}^2_R)$. The resulting involution of $\mathbb{P}^2_R$ stabilizes the closed subscheme E, and induces the negation map on the commutative group scheme $\mathrm{Reg}(E/R)$. We therefore call $N : E \to E$ the *sign involution*. On the affine part, it is given by $x \to x$ and $y \to -(y+a_1x+a_3)$.

According to [31], Section 4 the functor of relative automorphisms for the family $E \to \mathrm{Spec}(R)$ is representable by a relative group scheme $\mathrm{Aut}_{E/R}$ that is separated and of finite type, and the sign involution defines a homomorphism $(\mathbb{Z}/2\mathbb{Z} \cdot N)_R \to \mathrm{Aut}_{E/R}$. The following terminology follows Romagny ([65], Definition 4.3.3):

Definition 3.1. The *effective model* for the sign involution if the schematic closure

$$G = \overline{(\mathbb{Z}/2\mathbb{Z} \cdot N)_F} \subset \mathrm{Aut}_{E/R}$$

of the generic sign involution inside the relative automorphism group scheme.

According to loc. cit., Theorem 4.3.4 the structure morphism $G \to \mathrm{Spec}(R)$ is finite and flat, of degree two, and the inclusion $G \subset \mathrm{Aut}_{E/R}$ is a closed subgroup scheme.

Recall that over any ring A, given two integers $a, b \in A$ with $ab = 2$ we get the group valued functor

$$G^{a,b}(A') = \{f \in A' \mid f^2 - af = 0\}$$

with group law $f_1 * f_2 = f_1 + f_2 - bf_1f_2$. According to the Tate–Oort Classification ([84], Theorem 2), any family of group schemes over A of order two is isomorphic to some $G^{a,b}$. Moreover, we have $G^{a,b} \simeq G^{a',b'}$ if and only (a,b) and (a',b') lie in the same orbit of the action of $A^\times$ on $A \times A$ given by $\lambda \cdot (u,v) = (\lambda u, \lambda^{-1} v)$. Note that the pairs $(a,b) = (1,2)$ and $(a,b) = (2,1)$ define the constant group scheme $(\mathbb{Z}/2\mathbb{Z})_A$ and the multiplicative group scheme $\mu_{2,A}$, respectively. If $2 = 0$ in A, we have a further pair $(a,b) = (0,0)$, which yields the unipotent group scheme α_2.

Proposition 3.2. *The canonical map* $(\mathbb{Z}/2\mathbb{Z} \cdot N)_R \to G$ *is not an isomorphism if and only if* $2, a_1, a_3 \in \mathfrak{m}$. *In this case, the residue field* k *has characteristic two, and the closed fiber* G_k *of the effective model is a copy of* $\mu_{2,k}$ *or* $\alpha_{2,k}$*, and the genus-one curve* E_k *is a twisted form of the rational cuspidal curve* $C = \mathrm{Spec}\, k[u^2, u^3] \cup \mathrm{Spec}\, k[u^{-1}]$.

Proof. Clearly, the map is not an isomorphism if and only if the sign involution on the affine part of E_k is trivial, in other words, if $-(y + a_1x + a_3) = y$. This means $-1 \equiv 1$ and $-a_1 \equiv 0$ and $-a_3 \equiv 0$ modulo $\mathfrak{m}$, and the first assertion follows.

Suppose this equivalent conditions hold. Then the residue field k has characteristic two. Since the generic fiber $(\mathbb{Z}/2\mathbb{Z})_F$ is dense in both $(\mathbb{Z}/2\mathbb{Z})_R$ and G, it follows that G_k is infinitesimal. Up to isomorphism, the only infinitesimal group schemes of order two are $\mu_{2,k}$ or $\alpha_{2,k}$. From (5) we get $\Delta, c_4 \in \mathfrak{m}_R$. So by [25], Proposition 5.1 the closed fiber E_k is a twisted form of the rational cuspidal curve. $\square$

Recall that the *scheme of N-fixed points* is the intersection of graphs $\Gamma_N \cap \Delta$ inside $\mathbb{P}^2_R \times \mathbb{P}^2_R$, regarded as a closed subscheme of $\mathbb{P}^2_R = \Delta$. Obviously, its formation commutes with base-change in R.

Proposition 3.3. *The scheme of N-fixed points $\Gamma_N \cap \Delta$ is defined inside $\mathbb{P}^2_R$ by the homogeneous ideal $\mathfrak{a} = (X, Z) \cdot (2Y + a_1X + a_3Z)$. Moreover, the generic fiber of $V_+(\mathfrak{a}) \cap E$ is finite of degree four.*

Proof. First consider the invariant open set $D_+(Z)$. With respect to $x = X/Z$ and $y = Y/Z$, the action is given by $x \mapsto x$ and $y \mapsto -y - a_1x - a_3$. The ideal for the scheme of fixed points defined $2y + a_1x + a_3 = 0$. In the same way one sees that on $D_+(X)$ the scheme of fixed points is defined by $2y' + a_1 + a_3z' = 0$. It follows that $\Gamma_N \cap \Delta = V_+(\mathfrak{a})$ holds over the open set $D_+(Z) \cup D_+(X)$.

The complementary closed set is $\operatorname{Zero}(E/R)$. Let σ be its closed point, and write $A = \widehat{R}[[X/Y, Z/Y]]$ for the formal completion of the local ring $\mathscr{O}_{\mathbb{P}^2_R,\sigma}$. Then the monomorphism $\operatorname{Spec}(A) \to \mathbb{P}^2_R$ is stable with respect to N, and the matrix acts via

$$\frac{X}{Y} \longmapsto \frac{X}{Y} \cdot \frac{-1}{1 + a_1X/Y + a_3Z/Y} \quad \text{and} \quad \frac{Z}{Y} \longmapsto \frac{Z}{Y} \cdot \frac{-1}{1 + a_1X/Y + a_3Z/Y}.$$

Now the scheme of fixed points is defined by $-1 = 1 + a_1X/Y + a_3Z/Y$. This shows $\Gamma_N \cap \Delta = V_+(\mathfrak{a})$ after base-changing with respect to $\operatorname{Spec}(A) \to \mathbb{P}^2_R$. The disjoint union $D_+(Z) \cup D_+(X) \cup \operatorname{Spec}(A)$ is an fpqc covering, and our assertion follows.

It remains to check the assertion on the generic fiber. If $2 \neq 0$ in R, then $V_+(\mathfrak{a}) \otimes F$ is the disjoint union of the point $(0 : 1 : 0)$ and the line $V_+(2Y + a_1X + a_3Z)$. Using Bezout's Theorem, we infer that the intersection with E_F is finite of degree $4 = 1 + 3$. If $\Delta \neq 0$ the intersection is the 2-torsion part of the elliptic curve E_F, which has degree $4 = 2^2$. Finally, suppose that both $2, \Delta \in R$ vanish. Using (5) we get $c_4 \neq 0$. From [25], Proposition 5.1 we see that $\operatorname{Reg}(E_F/F)$ is a twisted form of $\mathbb{G}_{m,F}$. The two-torsion part μ_2 has degree two. The formal completion at the singular point in E_F is a twisted form of $F[[u, v]]/(uv)$, and sign involution permutes the indeterminates. The fixed scheme is given by the relations $uv = 0$ and $u = v$, which also has degree two. Again our fixed scheme has degree $4 = 2 + 2$. □

Recall that *schemes of fixed points* exits in surprising generality ([27], Chapter II, §1, Theorem 3.6). In particular, we can form the scheme of fixed points

$$\operatorname{Fix}(E/R) = E^G$$

for the effective model G of the sign involution on the family of cubics E. It commutes with base-change, and is related to the scheme of N-fixed points, which may contain the closed fiber and therefor fail to be flat, as follows: Inside the principal ideal domain R one has $\gcd(2, a_1, a_3) = \pi^d$ for some unique $d \geq 0$. As in [85], page 48 we write

$$2_d = 2/\pi^d \quad \text{and} \quad a_{1,d} = a_1/\pi^d \quad \text{and} \quad a_{3,d} = a_3/\pi^d,$$

and introduce the homogeneous ideal $\mathfrak{b} = (X, Z) \cdot (2_dY + a_{1,d}X + a_{3,d}Z)$.

Proposition 3.4. *We have $\operatorname{Fix}(E/R) = E \cap V_+(\mathfrak{b})$. Moreover, this scheme is finite and flat over R, of relative degree four.*

Proof. According to Proposition 3.3, the statements holds after base-changing along $R \subset F$. We next examine the closed fiber of $Z = E \cap V_+(\mathfrak{b})$. If $2_d \notin \mathfrak{m}$ we can argue as in the proof for Proposition 3.3. Suppose now 2_d vanishes in $k = R/\mathfrak{m}$. Then $V_+(\mathfrak{b})_k$ contains the line $L = V_+(a_{1,d}X + a_{3,d}Z)_k$, and the corresponding sheaf of ideal is the torsion sheaf supported by $(0:1:0)$ with stalk k, so its degree is $l = 1$. Using Bezout's Theorem, we see that Z_k is finite, of degree at most $3 + l = 4$. Since $Z \to \operatorname{Spec}(R)$ is proper with finite fibers, it must be finite. Its coordinate ring $A = \Gamma(Z, \mathscr{O}_Z)$ is a finitely generated R-module of rank four with $\dim_k(A \otimes_R k) \leq 4$. Using the structure theory for such modules, we see that A is free.

Consider the morphism $h : G \times Z \to E$ stemming from the G-action on E. Since $(G \times Z)_F$ is schematically dense in $G \times Z$, the schematic image $h(G \times Z) \subset E$ is the closure of Z_F. This equals Z, by flatness of Z, and we infer that $Z \subset E$ is G-stable. By construction, the G-action is trivial on Z_F. We thus have $h_F = (\mathrm{pr}_2)_F$. Using schematic density of $(G \times Z)_F$ again, we see $h = \mathrm{pr}_2$, thus the G-action on Z is trivial.

Summing up, we have an inclusion $Z \subset \operatorname{Fix}(E/R)$, which is generically an equality. It remains to check that the closed fiber of the scheme of fixed points has degree four. This follows as in the proof for Proposition 3.3 if one of Δ, c_4 does not belong to $\mathfrak{m}$. Otherwise, E_k is a twisted form of the rational cuspidal curve $C = \operatorname{Spec} k[u^2, u^3] \cup \operatorname{Spec} k[u^{-1}]$, and it suffices to treat the case $E_k = C$. If $2 \notin \mathfrak{m}$ the sign involution is given by $u^{-1} \mapsto -u^{-1}$, and one immediately sees that the fixed scheme is given on the two charts by the respective relations $u^3 = 0$ and $u^{-1} = 0$, so its length is $3 + 1 = 4$. The case $2 \in \mathfrak{m}$ follows from Proposition 3.5 below. □

Consider the rational cuspidal curve $C = \operatorname{Spec} k[u^2, u^3] \cup \operatorname{Spec} k[u^{-1}]$ over a field k of characteristic two. Despite the singularity, the tangent sheaf $\Theta_{C/k}$ is invertible, having degree four ([68], Section 3). The scheme of fixed points already determines the closed fiber G_k of the effective model, at least in the case that matters to us:

Proposition 3.5. *In the above setting, the map $H \mapsto C^H$ gives a bijection between the infinitesimal subgroup schemes $H \subset \operatorname{Aut}_{C/k}$ of order two, and the effective Cartier divisors $D \subset C$ of degree four with $\mathscr{O}_C(D) \simeq \Theta_{C/k}$. Moreover, we have $H \simeq \mu_2$ if and only if D is étale at some point.*

Proof. The one-dimensional vector subspaces $L = k\delta$ inside $H^0(C, \Theta_{C/k})$ correspond to the effective Cartier divisors $D \subset C$ with $\mathscr{O}_C(D) \simeq \Theta_{C/k}$. Every such L is stable with respect to Lie bracket and p-map ([68], Section 3 and [45], Section 2). So by the Demazure–Gabriel Correspondence ([27], Chapter II, §7, No. 4), the L also correspond to the infinitesimal subgroup schemes H of order two. Moreover, the scheme of fixed points coincides with the zero scheme for $\delta \in H^0(C, \Theta_{C/k})$, in other words $D = C^H$.

As explained in [68], proof of Proposition 3.2 the derivations $u^{-2}D_u, uD_u, D_u, u^2D_u$ form a basis for $\mathfrak{g} = H^0(C, \Theta_{C/k})$. The discussion in [45], Section 2 reveal that $H \simeq \mu_2$ if and only if C^H is étale. □

The following observation on the rational cuspidal curve $C = \operatorname{Spec} k[u^2, u^3] \cup \operatorname{Spec} k[u^{-1}]$ will also be useful:

Lemma 3.6. *The scheme C has no twisted form $C' \not\simeq C$ if and only if the ground field k is perfect.*

Proof. Up to isomorphism, twisted forms correspond to classes in the non-abelian cohomology set $H^1(S, \mathrm{Aut}_{C/S})$, where $S = \mathrm{Spec}(k)$. According to [36], Theorem 8.1 the automorphism group scheme is an iterated semi-direct product $G = \mathbb{G}_a \rtimes \alpha_2 \rtimes \mathbb{G}_m$. By loc. cit., Theorem 11.7 we have $H^1(S, G) = \bigcup_\alpha K/\{u^2 - v + \alpha v^2 \mid u, v \in k\}$, where the disjoint union runs over all $\alpha \in k/k^2$. Obviously, this is a singleton if and only if k is perfect. □

Suppose k is imperfect. Let R be some Cohen ring ([11], Chapter IX, §2), and choose some $a_4 \in R$ whose class in $k = R/\mathfrak{m}_R$ is not a square. Consider the family $E \subset \mathbb{P}^2_R$ of cubic curves given by the Weierstraß equation $y^2 = x^3 + a_4 x$. Then $c_4 = -48a_4$ and $\Delta = -64a_4^3$, so E_k is of additive type. Moreover, $\mathrm{Sing}(E_k/k)$ is defined by $x^2 + a_4 = 0$, and $y = 0$ defines a reduced effective Cartier divisor containing $\mathrm{Sing}(E_k/k)$. We conclude that the twisted form E_k of the rational cuspidal curve C is regular, that is, a *quasi-elliptic curve*. For more in this direction, see the work of Szydlo [82].

4. Freeness at the singular locus

Keep the setting of the previous section, such that E is a family of cubic curves inside $\mathbb{P}^2_R = \mathrm{Proj}\, R[X, Y, Z]$, with at least one of the $2, a_1, a_3 \in R$ non-zero, and G is the effective model of the sign involution. To simplify exposition, we also assume that the generic fiber E_F is smooth, in other words, the discriminant $\Delta \in R$ is non-zero. The zero scheme, the scheme of non-smoothness, and the scheme of fixed points

$$\mathrm{Zero}(E/R) \quad \text{and} \quad \mathrm{Sing}(E/R) \quad \text{and} \quad \mathrm{Fix}(E/R),$$

define closed subschemes of E, all of which are finite over R. The goal of this section is to clarify how these three schemes intersect. Recall that we write $2_d = 2/\pi^d$ and $a_{1,d} = a_1/\pi^d$ and $a_{3,d} = a_3/\pi^d$, where $d = \mathrm{val}(2, a_1, a_3)$. One says that the closed fiber E_k has *additive type* if it is a twisted form of the rational cuspidal curve.

Proposition 4.1. *The scheme* $\mathrm{Fix}(E/R) \cap \mathrm{Sing}(E/R)$ *is disjoint from* $\mathrm{Zero}(E/R)$*, and is given as a closed subscheme in $\mathbb{A}^2_R$ by the equations*

$$(6)\quad -y^2 = x^3 + a_2x^2 + a_4x + a_6, \quad a_1y = 3x^2 + 2a_2x + a_4, \quad 2_dy + a_{1,d}x + a_{3,d} = 0.$$

If the residue field $k = R/\mathfrak{m}$ has characteristic two and the closed fiber E_k has additive type, the intersection is set-theoretically described by

$$\pi = 0, \quad x^2 = a_4, \quad y^2 = a_2a_4 + a_6, \quad 2_dy + a_{1,d}x + a_{3,d} = 0.$$

Proof. Taking the partial derivative of the homogeneous Weierstraß equation (4), one already sees that $\mathrm{Sing}(E/R)$ is disjoint from $\mathrm{Zero}(E/R)$. In turn, the intersection $\mathrm{Fix}(E/R) \cap \mathrm{Sing}(E/R)$ is contained in $\mathbb{A}^2_R = D_+(Z)$. Thus $\mathrm{Sing}(E/R)$ is defined by the Weierstraß equation $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$, together with the equations $a_1y = 3x^2 + 2a_2x + a_4$ and $2y + a_1x + a_3 = 0$ coming from the partial derivatives. The latter is a multiple of $2_dy + a_{1,d}x + a_{3,d} = 0$, which by Proposition 3.4 defines $\mathrm{Fix}(E/R) \smallsetminus \mathrm{Zero}(E/R)$. Thus we already have the second and third equation in (6). The first is obtained from the Weierstraß equation by subtracting a multiple of the third.

For the remaining statement, suppose that k has characteristic two and E_k has additive type. Since $\mathrm{Sing}(E/R)$ is set-theoretically contained in the closed fiber, we can enlarge (6) by the equation $\pi = 0$ and still describe $\mathrm{Fix}(E/R) \cap \mathrm{Sing}(E/R)$ as closed set. Moreover, we have $\Delta, c_4 \in \mathfrak{m}$ according to [25], Proposition 5.1. From (5) we get $a_1, a_3 \in \mathfrak{m}$. Using this for substitution in (6), the second equation becomes $x^2 = a_4$, and the first $y^2 = a_2a_4 + a_6$. ☐

As in [85], Section 7, it is convenient to write $\bar{a}_i, \bar{a}_{j,d} \in k$ et cetera for the residue classes for $a_i, a_{j,d} \in R$. We write $\mathrm{val}(a)$ for the valuation of the elements $a \in R$: We now can determine when the intersection in Proposition 4.1 is empty:

Proposition 4.2. *Suppose the closed fiber E_k is non-smooth. The subschemes* $\mathrm{Fix}(E/R)$ *and* $\mathrm{Sing}(E/R)$ *are disjoint if and only the following three conditions hold:*

(i) *The closed fiber E_k is of additive type.*
(ii) *The residue field $k = R/\mathfrak{m}$ has characteristic two.*
(iii) *We have* $\mathrm{val}(b_2a_4 + b_6) = 2\,\mathrm{val}(2, a_1, a_3)$*.*

Furthermore, condition (iii) is equivalent to $\mathrm{val}(a_2a_4 + a_6) = 0$*, provided* $a_1 = a_3 = 0$*.*

Proof. We start with some preliminary observations: First note that $\mathrm{Sing}(E/R) = \{z\}$ is a singleton whose residue field extension $k \subset \kappa(z)$ is purely inseparable. It follows that z belongs to the fixed scheme of any automorphism. Next note that from the very definition of b-values in [85], Section 1 we get the relation

$$b_2a_4 + b_6 = 2^2(a_2a_4 + a_6) + a_1^2a_4 + a_3^2. \tag{7}$$

So if $a_1 = a_3 = 0$ we have $\mathrm{val}(b_2a_4 + b_6) = 2\,\mathrm{val}(2, a_1, a_3) + \mathrm{val}(a_2a_4 + a_6)$ and see that (iii) simplifies as claimed. Finally, suppose that $k = R/\mathfrak{m}_R$ has characteristic two, and let $Z \subset \mathbb{A}^2_k$ be the closed subscheme defined by $x^2 = \bar{a}_4$ and $y^2 = \bar{a}_2\bar{a}_4 + \bar{a}_6$ and $2_dy + \bar{a}_{1,d}x + \bar{a}_{3,d} = 0$. Substituting the former two in the square of the latter, we immediately see

$$Z = \varnothing \quad \Longleftrightarrow \quad (2_d)^2(\bar{a}_2\bar{a}_4 + \bar{a}_6) + \bar{a}_{1,d}^2\bar{a}_4 + \bar{a}_{3,d}^2 \neq 0.$$

We now can easily check that conditions (i)–(iii) are necessary: Suppose that $\mathrm{Fix}(E/R)$ and $\mathrm{Sing}(E/R)$ are disjoint. If the closed fiber is of multiplicative type, or if the residue field $k = R/\mathfrak{m}$ has characteristic $p \neq 2$. then the effective model of the sign involution is $G = (\mathbb{Z}/2\mathbb{Z} \cdot N)_R$, and thus $z \in \mathrm{Fix}(E/R)$, contradiction. This gives (i) and (ii). From Proposition 4.1, it then follows $Z = \mathrm{Fix}(E/R) \cap \mathrm{Sing}(E/R)$. This is empty, hence the fraction

$$\frac{b_2a_4 + b_6}{\pi^{2d}} = \frac{2^2(a_2a_4 + a_6) + a_1^2a_4 + a_3^2}{\pi^{2d}}$$

belongs to $R^\times$, and consequently (iii) holds.

The conditions (i)–(iii) are also sufficient: The third gives $Z = \varnothing$, by the observations in the preceding paragraph. Condition (i) and (ii) ensure that Z coincides with the intersection $\mathrm{Fix}(E/R) \cap \mathrm{Sing}(E/R)$, according to Proposition 4.1. ☐

Let us finally determine when the closed fiber of the effective model is a multiplicative group scheme:

Proposition 4.3. *Suppose the closed fiber E_k is non-smooth, and* $\mathrm{Fix}(E/R)$ *and* $\mathrm{Sing}(E/R)$ *are disjoint. Then the following are equivalent:*

(i) *The fixed scheme* $\mathrm{Fix}(E/R)$ *is étale over R.*
(ii) *The group scheme G_k is isomorphic to $\mu_{2,k}$.*
(iii) *We have* $\mathrm{val}(2) \leq \mathrm{val}(a_i)$ *for $i = 1$ and $i = 3$.*

Proof. First observe that we already saw (i)$\Leftrightarrow$(ii) in Proposition 3.5, and that condition (iii) is equivalent to $2_d \notin \mathfrak{m}$. By Proposition 3.4, the fixed scheme $\mathrm{Fix}(E/R)_k$ has multiplicity $m \geq 2$ at the origin $(0:1:0)$ if and only if $2_d \in \mathfrak{m}$. So (iii)$\Leftrightarrow$(ii) follows from Proposition 3.5. □

5. Kummer constructions in families

Throughout this section, S denotes an irreducible Dedekind scheme with generic point $\eta \in S$, and E_F is an elliptic curve over the function field $F = \kappa(\eta)$. Note that we allow imperfect residue fields $\kappa(s)$. To simplify exposition we assume that S is *excellent*, which here means that for each closed point $s \in S$ the formal completion $\mathscr{O}_{S,s} \subset \mathscr{O}_{S,s}^\wedge$ induces a separable extension on field of fractions (compare [70], Proposition 4.1). This obviously holds if $F = \mathrm{Frac}(\mathscr{O}_{S,s})$ has characteristic zero, the case we are mainly interested in. The goal of this section is to introduce the *Kummer construction* in a relative setting over S, which leads to families of normal K3 surfaces.

Let $E \to S$ be the *Weierstraß model* for the elliptic curve E_F. This is obtained from the *Néron model* over S by passing to the minimal regular compactification, and contracting all curves disjoint from the zero section. For each closed point $s \in S$ the base change $E \otimes \mathscr{O}_{S,s}$ becomes a family of cubic curves defined by a Weierstraß equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

over the discrete valuation ring $R = \mathscr{O}_{S,s}$, with auxiliary values and discriminant $b_i, c_i, \Delta \in R$. The *invariant* $j = c_4^3/\Delta$ belongs to F and depends only on the isomorphism class of the geometric generic fiber $E_{F^{\mathrm{alg}}}$. Since the valuation $v_s = \mathrm{val}_s(\Delta) \geq 0$ is minimal among all possible Weierstraß equations for E_F with coefficients from R, this integer depends only on the isomorphism class of E_F. We call $\sum v_s \cdot s$ the *discriminant divisor*, and the members of its support the *points of bad reduction*.

By the universal property of Néron models ([10], Section 1.2), the sign involution on the generic fiber E_F extends to a homomorphism $(\mathbb{Z}/2\mathbb{Z})_S \to \mathrm{Aut}_{E/S}$. As in Section 3, the schematic closure

$$G = \overline{(\mathbb{Z}/2\mathbb{Z})_F} \subset \mathrm{Aut}_{E/S}$$

is called the *effective model of the sign involution*. This closure can be computed locally, over the discrete valuation rings $R = \mathscr{O}_{S,s}$, and we see from the discussion in Section 3 that $G \to S$ is a family of group schemes of order two. Likewise, the scheme of G-fixed points $\mathrm{Fix}(E/S)$ and the scheme of non-smoothness $\mathrm{Sing}(E/S)$ can be computed locally. The following terminology is crucial for the whole paper:

Definition 5.1. The elliptic curve E_F is called *admissible for the Kummer construction* if the following four conditions hold:

(i) The intersection $\mathrm{Sing}(E/S) \cap \mathrm{Fix}(E/S)$ is empty.
(ii) For each $s \in S$ of bad reduction, the residue field $\kappa(s)$ has characteristic two, and the reduction is additive.
(iii) For each closed point $s \in S$ of characteristic two, the fiber $\mathrm{Fix}(E/S)_s$ is geometrically disconnected.

Note that if $s \in S$ is a point of characteristic two and good reduction, the fiber $\mathrm{Fix}(E/S)_s$ coincides with $E_s[2]$, so condition (iii) means that the elliptic curve E_s is *ordinary*, or equivalently has invariant $j \neq 0$.

There is a variant for pairs: Let E'_F be another elliptic curve over the field of fractions F, and form the Weierstraß model E' and the effective model G' of the sign involution as above.

Definition 5.2. The pair of elliptic curves E_F, E'_F is called *admissible for the Kummer construction* if they have the same points of bad reduction, each elliptic curve satisfies (i) and (ii) of the previous definition, and furthermore the following conditions holds:

(iii*) For each point $s \in S$ of characteristic two, at least one of the fibers $\mathrm{Fix}(E/S)_s$ and $\mathrm{Fix}(E'/S)_s$ is geometrically disconnected.
(iv) The relative group schemes G and G' are isomorphic.

Of course, if the single elliptic curve E_F is admissible for the Kummer construction, then the pair given by $E'_F = E_F$ is admissible. Conversely, if a pair E_F and E'_F is admissible and there is at most one point $s \in S$ of characteristic two, then at least one member of the pair is admissible.

In what follows we assume that E_F, E'_F is a pair of elliptic curves that is admissible for the Kummer construction. Using condition (iv) we choose an identification $G = G'$ between the effective models for the sign involutions and consider the resulting diagonal G-action on $A = E \times_S E'$. Let us first summarize what is known about the fiber-wise quotients:

Proposition 5.3. *Let $s : \mathrm{Spec}(k) \to S$ be a geometric point of characteristic $p \geq 0$. Then the categorical quotient A_s/G_s is a normal K3 surface, and its configuration of rational double points is given by the following table:*

characteristic	effective model G_s	configuration of RDP
$p \neq 2$	$\mu_2 = \mathbb{Z}/2\mathbb{Z}$	$16A_1$
$p = 2$	μ_2	$16A_1 + D_4$
	α_2	$4D_4 + D_4$ or $2D_8 + D_4$
	$\mathbb{Z}/2\mathbb{Z}$	$4D_4$ or $2D_8$

Proof. Suppose first that E_s is smooth. Then E'_s is smooth as well. If $p \neq 2$ then $G_s = (\mathbb{Z}/2\mathbb{Z})_s$, and the quotient A_s/G_s is the classical Kummer construction: Its singular locus comprises sixteen rational double points of type A_1, corresponding to the sixteen points in $A_s[2]$. If $p = 2$ the number $n \geq 1$ of points in $A_s[2]$ satisfies

$n \mid 4$. By condition (iii) we may assume that E_s has invariant $j \neq 0$, hence is ordinary, and thus $2 \mid n$. This ensures that A_s/G_s is a normal K3 surface, by [75], Proposition 1. For $n = 4$ it has four rational double points of type D_4, whereas for $n = 2$ it has two rational double points of type D_8, see [40], Theorem C, compare also [75], Remark in §6.

Suppose next that E_s is singular. We saw in Proposition 4.2 that $p = 2$, and both E_s and E'_s are copies of the rational cuspidal curve $C = \operatorname{Spec} k[u^2, u^3] \cup \operatorname{Spec} k[u^1]$. According to Proposition 3.2 the group scheme G is infinitesimal. Suppose first that $G_s = \mu_{2,k}$. By [45], Proposition 3.2 we see that A_s/G_s is a normal K3 surface, and the configuration of rational double points is $16A_1 + D_4$. Finally suppose $G_s = \alpha_{2,k}$. Using loc. cit. together with assumption (ii), we see that A_s/G_s is a again normal K3 surfaces, and now the configuration of rational double points is either $4D_4 + D_4$ or $2D_8 + D_4$. Compare also [68]. □

Of course, one may already form categorical quotients A_s/G_s for each $s \in S$. By the above, these are normal K3 surfaces over the residue field $k = \kappa(s)$. Note, however, that for the points $z \in A_s/G_s$ in the locus of non-smoothness the residue field extension $k \subset \kappa(z)$ could be non-trivial, the exceptional curves on the resolution could have further constant field extension, or the local ring $\mathscr{O}_{A/G,z}$ might actually be regular.

Next, we form *categorical quotients* over the Dedekind scheme S, and obtain a commutative diagram

$$
\begin{tikzcd}
E \arrow[d] & A \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] \arrow[d, "q"] & E' \arrow[d] \\
E/G & A/G \arrow[l] \arrow[r] & E'/G.
\end{tikzcd}
\tag{8}
$$

Such quotients indeed exists as algebraic spaces, for example by [67], Corollary 5.4. For each $s \in S$, the universal properties of quotients gives comparison maps

$$E_s/G_s \to (E/G)_s \quad \text{and} \quad A_s/G_s \to (A/G)_s \quad \text{and} \quad E'_s/G'_s \to (E'/G')_s. \tag{9}$$

Proposition 5.4. *In the diagram (8), all algebraic spaces are projective and flat over S. Moreover, the comparison maps in (9) are isomorphisms, for all $s \in S$.*

Proof. The zero section for the Weierstraß models yield relatively ample invertible sheaves on E and E', hence $A = E \times E'$ is projective. According to [67], Proposition 4.7 the categorical quotient A/G must be projective as well. For each affine open set $V \subset S$, each collection of points $a_1, \ldots, a_r \in A_V$ admits a common affine open neighborhood. It follows that there is an affine open covering $A = \bigcup U_i$, $i \in I$ where the $U_i = \operatorname{Spec}(R_i)$ are G-stable. Moreover, the categorical quotient $V = A/G$ is obtained by gluing the spectra of the ring of invariants R_i^G, compare [67], Theorem 4.1. It follows that V is integral and flat, and satisfies Serre's condition (S_2). Consequently the fibers V_s satisfy Serre's condition (S_1), in other words, contain no embedded components. Moreover, the quotient map $q : A \to A/G = V$ is integral and surjective.

On the complement $A^0 = A \smallsetminus \operatorname{Fix}(A/S)$ of the scheme of fixed points, the G-action is free. The ensuing categorical quotient A^0/G is the actually a quotient of fppf sheaves, and commutes with arbitrary base-changes, see for example [67], Theorem 2.16. It comes with an open embedding $A^0/G \subset A/G$, whose complement is the image $q(\mathrm{Fix}(A/G))$. We see that the canonical maps $A_s/G_s \to (A/G)_s = V_s$ are finite and birational, and an isomorphism outside a finite set of closed points. Using that V_s has no embedded components, we get a short exact sequence

$$0 \longrightarrow \mathscr{O}_{V_s} \longrightarrow \mathscr{O}_{A_s/G_s} \longrightarrow \mathscr{F}_s \longrightarrow 0$$

for some coherent sheaf $\mathscr{F}$ with finite support and Euler characteristic $\chi(\mathscr{F}_s) = \chi(\mathscr{O}_{A_s/G_s}) - \chi(\mathscr{O}_{V_s})$. This difference is zero provided $s = \eta$. It actually vanishes for all $s \in S$ by flatness of $V \to S$. In turn, the comparison map $A_s/G_s \to (A/G)_s$ are isomorphisms. $\square$

Combining the above two propositions, we arrive at:

Theorem 5.5. *The categorical quotient $V = A/G$ is a family of normal K3 surfaces over the Dedekind scheme S.*

All fibers are non-smooth, and have only RDP singularities by Lemma 1.3. By the Brieskorn–Artin Theorem [6], there is a finite covering $\tilde{S} \to S$ of Dedekind schemes such that the base-change $\tilde{V} = V \times_S S'$ admits a simultaneous minimal resolution of singularities. We seek to understand when $\tilde{S} = S$ works, or when one can choose $\deg(\tilde{S}/S)$ small.

In the proof for Proposition 5.3 we already described the singularities in the geometric fibers for $V \to S$. This immediately gives:

Proposition 5.6. *As closed set, the image of the canonical map*

$$\mathrm{Fix}(A/S) \cup \mathrm{Sing}(A/S) \longrightarrow A/G = V$$

coincides with the the locus of non-smoothness $\mathrm{Sing}(V/S)$*.*

Note that $\mathrm{Fix}(A/S) = \mathrm{Fix}(E/S) \times \mathrm{Fix}(E'/S)$ is finite and flat over S, whereas the closed set

$$\mathrm{Sing}(A/S) = \mathrm{Sing}(E/S) \times \mathrm{Sing}(E') \subset A$$

comprises only finitely many points $a_1, \ldots, a_r$. Their images $s_1, \ldots, s_r \in S$ are precisely the closed points of characteristic two where the G has infinitesimal fiber. We like to call the images $v_1, \ldots, v_r \in V$ the *critical points*. The residue field extensions $\kappa(s_i) \subset \kappa(v_i)$ are purely inseparable, and the correspond singularities on $V \otimes \kappa(s_i)^{\mathrm{perf}}$ is a rational double point of type D_4 or B_3. In [68], Section 5 these are called *singularities coming from the quadruple point*, since they stem from the rational point of multiplicity four on $(E \times E') \otimes \kappa(s_i)^{\mathrm{perf}}$. For imperfect residue fields $\kappa(s_i)$, however, the local rings $\mathscr{O}_{V,v_i}$ also may be of type G_2 or A_1, or be regular.

In contrast, the scheme $\mathrm{Fix}(A/S)$ yields families of singularities inside the family of normal K3 surfaces $V = A/G$. It turns out that they are more easy to understand. Following [68], Section 5 we refer to the points $v \in V$ lying in the image of $\mathrm{Fix}(A/S)$ as the *singularities stemming from the fixed points*. This is indeed justified:

Proposition 5.7. *For every $v \in V$ belonging to the image of* $\mathrm{Fix}(A/S)$*, the local ring $\mathscr{O}_{V,v}$ is singular.*

Proof. It suffices to check that the two-dimensional complete local ring $R = \mathscr{O}^\wedge_{V_s,v}$ is singular, where $s \in S$ is the image of $v \in V$. Set $k = \kappa(s)$ and $k' = \kappa(v)$. By construction, R is the ring of invariants for some action of the group $\mathbb{Z}/2\mathbb{Z}$ on a formal power series ring $A = k'[[u_1, u_2]]$. The action is linear over k, and free outside the closed point of $\operatorname{Spec}(A)$. As in [49], Proposition 3.2, one sees that the ring of invariants is not regular. ☐

6. Blowing-up centers with embedded components

We keep the set-up as in the preceding section, and form the family of normal K3 surface $V = (E \times E')/G$ over the Dedekind scheme S, for a pair of elliptic curves E_F, E'_F that is admissible for the Kummer construction, and the resulting effective model G for the sign involution. *Here we also assume that the function field $F = \mathscr{O}_{S,\eta}$ has characteristic zero.* Let $s_1, \ldots, s_r \in S$ be the finitely many points where the effective model G has unipotent fiber, and set $F_i = \operatorname{Frac}(\mathscr{O}^\wedge_{S,s_i})$. The goal of this section is to establish the following:

Theorem 6.1. *Suppose the finite étale group schemes $E_F[2]$ and $E'_F[2]$ of degree four become constant over all F_i, for $1 \leq i \leq r$. Then there is a simultaneous partial resolution $r : Y \to V$ such that for each geometric point $s : \operatorname{Spec}(k) \to S$ the induced map $Y_s \to V_s$ coincides with the minimal resolution of singularities for the singularities stemming from the fixed points.*

In other words, $\operatorname{Sing}(Y/S)$ comprises only the critical points $y_1, \ldots, y_r \in Y$, which correspond to the finitely many closed points $s_1, \ldots, s_r \in S$ where G becomes infinitesimal. We shall see that outside these s_i, the family Y can be obtained by a blowing-up, where the center is a family of one-dimensional schemes that are not Cartier, a technique already used in [68], Section 10 and 11. Here this will involve curves with embedded components, a novel feature which might be useful in other contexts as well. The proof requires some preparation and will be completed at the end of this section.

The projection $\operatorname{pr}_1 : E \times E' \to E$ induces a fibration $V \to E/G$, which can be seen as a family of genus-one fibrations parametrized by S, and will be denoted by the same symbol. Write $\widetilde{\operatorname{Fix}}(E/S) \subset E/G$ for the schematic image of the fixed scheme $\operatorname{Fix}(E/S) \subset E$, and form the fiber product

$$\begin{array}{ccc} \tilde{Z} & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle \operatorname{pr}_1} \\ \widetilde{\operatorname{Fix}}(E/S) & \longrightarrow & E/G. \end{array}$$

We now use the reduced closed subscheme $Z = (\tilde{Z})_{\mathrm{red}}$ on the family of normal K3 surfaces $V = (E \times E')/G$ as center for some blowing-up

$$\operatorname{Bl}_Z(V) \longrightarrow V. \tag{10}$$

The projection $Z \to S$ is flat of relative dimension one, and the total space is reduced and thus satisfies Serre Condition (S_1). Note that the closed fibers Z_s, however, may acquire embedded components.

To see that this behavior is very common, just identify two closed points in the relative projective line $\mathbb{P}^1_{\mathbb{C}[[t]]}$. The resulting closed fiber has as reduction a nodal genus-one curve, and the sheaf of nilradicals is given by the residue field of the singularity. The following crucial observation takes advantage of this phenomenon:

Lemma 6.2. *Let $s : \operatorname{Spec}(k) \to S$ be a geometric point such that V_s contains no D_8-singularity. Then $\operatorname{Bl}_Z(V)_s \to V_s$ coincides with the blowing-up of the singularities stemming from the fixed points.*

Proof. It suffices to treat the case that S is the spectrum of a complete discrete valuation ring R with algebraically closed residue field $k = R/\mathfrak{m}$, and that $s = \sigma$ is the closed point. Replacing R be a finite extension, we may also assume that the finite étale scheme $E_F[2]$ of degree four over $F = \operatorname{Frac}(R)$ is constant. Thus $\operatorname{Fix}(E/S)$ has four irreducible components $\operatorname{Fix}_i(E/S)$, $1 \leq i \leq 4$. Write

$$\widetilde{\operatorname{Fix}}_i(E/S) \subset \widetilde{\operatorname{Fix}}(E/S) \quad \text{and} \quad \tilde{Z}_i \subset \tilde{Z} \quad \text{and} \quad Z_i \subset Z$$

for the ensuing irreducible components. Then $\widetilde{\operatorname{Fix}}_i(E/S)$ is a section for $V \to E/G = \mathbb{P}^1_R$, and $\tilde{Z}_i \subset V$ is the corresponding family of fibers. These fibers are irreducible, but come with multiplicity two. In fact, $\tilde{Z}_i = \mathbb{P}^1_R \oplus \mathscr{O}_{\mathbb{P}^1_R}(-2)$ are *ribbons* in the sense of Bayer and Eisenbud [12]. The $Z_i \subset V$ is the ensuing family of half-fibers, and thus $Z_i = \mathbb{P}^1_R$.

Suppose first that the group scheme G_σ is multiplicative. Then the $\operatorname{Fix}_i(E/S)$ are pairwise disjoint, and likewise $Z_i = \mathbb{P}^1_R$ are pairwise disjoint families of half-fibers with respect to $V \to E/G = \mathbb{P}^1_R$. Using [68], Proposition 10.5 we infer that $\operatorname{Bl}_Z(V) \to V$ gives fiberwise the minimal resolution of singularities, and the assertion follows.

It remains to treat the case that G_σ is unipotent. Since V_σ contains no D_8-singularity, $\widetilde{\operatorname{Fix}}(E/S)_\sigma = \{a, b\}$ comprises two k-rational points, each with multiplicity two. Without loss of generality $\widetilde{\operatorname{Fix}}_1(E/S) \cap \widetilde{\operatorname{Fix}}_2(E/S) = \{a\}$. In turn, $(Z_1 \cup Z_2)_\sigma$ contains the fiber

$$V_a = V \times_{E/G} \operatorname{Spec} \kappa(a) = \mathbb{P}^1_k \oplus \mathscr{O}_{\mathbb{P}^1_k}(-2),$$

and the inclusion $V_a \subset (Z_1 \cup Z_2)_\sigma$ is an equality outside $\operatorname{Sing}(V/S)$. From

$$\chi(\mathscr{O}_{(Z_1 \cup Z_2)_\sigma}) = \chi(\mathscr{O}_{(Z_1 \cup Z_2)_\eta}) = 2 \quad \text{and} \quad \chi(V_a) = \chi(\mathscr{O}_{\mathbb{P}^1_k}) \oplus \mathscr{O}_{\mathbb{P}^1_k}(-2)) = 0$$

we see that in the short exact sequence

$$0 \longrightarrow \mathscr{I} \longrightarrow \mathscr{O}_{(Z_1 \cup Z_2)_\sigma} \longrightarrow \mathscr{O}_{V_a} \longrightarrow 0$$

the kernel is a skyscraper sheaf with $h^0(\mathscr{I}) = 2$, which is supported by the two D_4-singularities $z_1, z_2 \in V_\sigma$ lying on the fiber $V_a = \operatorname{pr}_1^{-1}(a)$. Since $V_a \subset V$ is Cartier, and $Z_1 \cup Z_2 \subset V$ is not Cartier at each of the A_1-singularities on V_η specializing to z_1 or z_2, it follows from the Nakayama Lemma that $\mathscr{I}$ is non-zero at both z_1, z_2. In light of $h^0(\mathscr{I}) = 2$ we must have $\mathscr{I}_{z_i} \simeq \kappa(z_i)$. In turn, the blowing-up $Y \to \operatorname{Bl}_Z(V)$ becomes near $z_i \in V_\sigma$ the blowing-up of an ideal of the form $\mathfrak{m} \cdot \mathfrak{a} \subset \mathscr{O}_{V_\sigma, z_i}$ for some invertible $\mathfrak{a}$. This result is the same as the blowing-up of $\mathfrak{m} \subset \mathscr{O}_{V_\sigma, z_i}$. Summing up, $Y_\sigma \to V_\sigma$ coincides with the blowing-up of the singularities $z \in V_\sigma$, endowed with reduced scheme structure. $\square$

We continue to examine the above blowing-up $\mathrm{Bl}_Z(V)$, *now under the additional assumption that the finite étale group schemes* $E_\eta[2]$ *and* $E'_\eta[2]$ *of degree four are constant.* Then all singularities in V_η are F-valued, and we write $\Delta_1, \ldots, \Delta_{16}$ for the exceptional divisors on the resolution of singularities $\mathrm{Bl}_Z(V)_\eta$, arranged in an arbitrary order. Write

$$\mathrm{Bl}_{Z,16}(V) = V_{16} \longrightarrow V_{15} \longrightarrow \ldots \longrightarrow V_1 \longrightarrow V_0 = \mathrm{Bl}_Z(V)$$

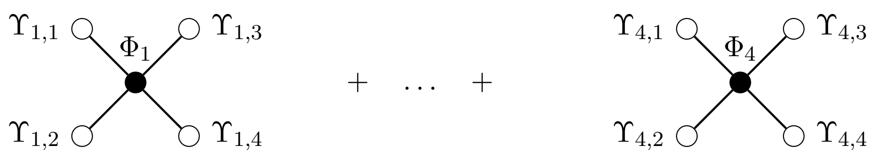
for the sequence of blowing-ups, where for $V_i = \mathrm{Bl}_{Z_i}(V_{i-1})$ the center is taken as the schematic closure of the $\Delta_i \subset (V_{i-1})_\eta = \mathrm{Bl}_Z(V)_\eta$. Of course, such a center may already be Cartier, in which case the blowing-up becomes an identity.

Proposition 6.3. *For each geometric point* $s : \mathrm{Spec}(k) \to S$ *such that* V_s *contains no* D_8*-singularity, the composite morphism* $\mathrm{Bl}_{Z,16}(V)_s \to V_s$ *is the minimal resolution of all singularities stemming from the fixed points, and an isomorphism otherwise.*

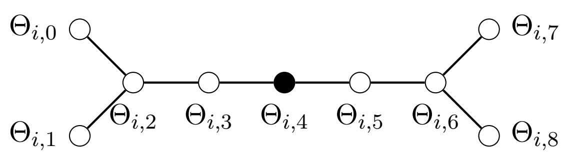
Proof. It suffices to treat the case that S is the spectrum of a complete discrete valuation ring, with closed point $\sigma = s$. In light of Proposition 6.2, there is nothing to prove if G_σ is multiplicative. So we assume that the residue field k has characteristic two, and that G_σ is unipotent.

Write $r_s : X_s \to V_s$ for the minimal resolutions of singularities. The exceptional divisors on X_η, together with the strict transform of the half-fibers over $(E/G)_\eta = \mathbb{P}^1_\eta$, take the form:

(11)



Here the black vertex signifies the strict transform of the half-fiber, and the $\Upsilon_{i,j}$ correspond to the $\Delta_1, \ldots, \Delta_{16}$. The exceptional divisors on X_σ form two configurations, with following dual graphs and $i = 1, 2$:



Recall that $\mathrm{Bl}_Z(V) \to V$ is the blowing-up of the schematic image of $\Phi_1 + \ldots + \Phi_4$. Then degenerate fibers in $\mathrm{Bl}_Z(V)_\sigma \to (E/G)_\sigma = \mathbb{P}^1_k$ are the images of

$$\Theta_{1,2} + \Theta_{1,4} + \Theta_{1,6} \quad \text{and} \quad \Theta_{2,2} + \Theta_{2,4} + \Theta_{2,6}.$$

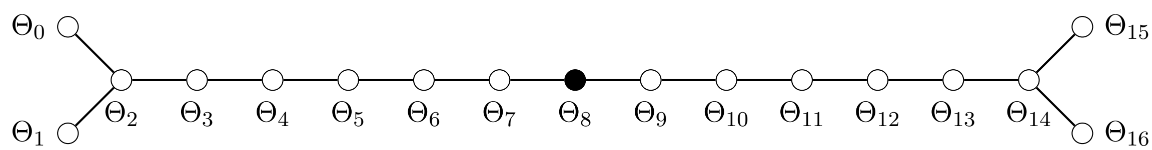
The remaining singularities on $\mathrm{Bl}_Z(V)_\sigma$ are merely A_1-singularities. We also see that each $\Upsilon_{i,j} \subset X_\eta = (\mathrm{Res}_{V/S})_\eta$ specializes to the schematic image of some of the remaining $\Theta_{r,s}$ with $r = 1, 2$ and $s = 0, 1, 3, 5, 7, 8$. In turn, the blowing-up of the schematic image of $\Upsilon_{i,j}$ resolves the singularities that lie on $\Theta_{r,s}$. The assertion follows, because the Υ_{ij} are exactly the $\Delta_1, \ldots, \Delta_{16}$. ∎

Proof of Theorem 6.1. We proceed by induction on the number $r \geq 0$ of points $\sigma_1, \ldots, \sigma_r \in S$ where the effective model G is unipotent. For $r = 0$, the group scheme G is multiplicative, the singularities $v \in V_s$ in the geometric fibers stemming from the fixed points are of type A_1, and the assertion follows from Lemma 6.2.

Suppose now $r \geq 1$, and that the assertion holds for $r-1$. Let $U \subset S$ be the complement of $\sigma = \sigma_r$. By the induction hypothesis, we find the desired simultaneous partial resolution $r_U : Y_U \to V_U$. Suppose we also have the desired simultaneous partial resolution over $\mathscr{O}_{S,\sigma}^\wedge$. According to [5], Theorem 3.2 this formal modification already exists over $\mathscr{O}_{S,\sigma}$, and thus extends to some $r_{U'} : Y_{U'} \to V_{U'}$ for some affine open neighborhood U' of $\sigma \in S$. Shrinking if necessary, we may assume that it coincides with Y_U on the overlap $U \cap U'$. The desired $r : Y \to V$ is then obtained by gluing.

This reduces us to the case that S is the spectrum of a complete discrete valuation ring R, with closed point $\sigma \in S$, residue field $k = \kappa(\sigma)$, and unipotent G_σ. If the geometric closed fiber contains no D_8-singularities, the assertion follows from Proposition 6.3. It remains to treat the case that there is a D_8-singularity. Recall that by assumption, the finite étale group schemes $E_F[2]$ and $E'_F[2]$ are constant. Without loss of generality we may assume that $\mathrm{Fix}(E/S)$ is connected and $\mathrm{Fix}(E'/S)$ is disconnected.

Write $X_s \to V_s$, $s \in S$ for the minimal resolution of singularities. The exceptional divisors on X_σ, together with the strict transform Θ_8 of the corresponding half-fiber over $(E/G)_\sigma = \mathbb{P}^1_\sigma$, form a configuration with the following dual graph:



In the above, the black vertices signify the strict transform of half-fibers. The exceptional divisors on X_η, together with the strict transform of the half-fibers over $(E/G)_\eta = \mathbb{P}^1_\eta$ are already given in (11).

We now construct iterated blowing-ups of $Y_4 \to \ldots \to Y_1 \to V$, and it is most convenient to specify the centers as schematic images of certain combinations of Φ_i and $\Upsilon_{i,j}$ with respect to the canonical morphism $X_\eta \to Y_i$. We proceed in four steps:

Step 1. Let $Y_1 = \mathrm{Bl}_Z(V)$ be the blowing-up where the center $Z \subset V$ is the schematic image of $\Phi_1 \cup \Phi_2$. This pair of curves specializes to the schematic images of $2\Theta_8$, with embedded components at the singularities. The situation is essential as in Lemma 6.2: The degenerate fiber in $Y_{1,\sigma} \to (E/G)_\sigma = \mathbb{P}^1_\sigma$ comprises the images of

$$\Theta_6 + \Theta_8 + \Theta_{10},$$

and the singularities are $(D_6 + A_1) + (A_1 + D_6)$, contained in the images of $\Theta_6 + \Theta_{10}$.

Step 2. Let $Y_2 = \mathrm{Bl}_{Z_1}(Y_1)$ be the blowing-up where the center $Z_1 \subset Y_1$ is the schematic image of $\Upsilon_{1,1} + \ldots + \Upsilon_{1,4}$. These four curves special in pairs to the schematic images of $2\Theta_6$ and $2\Theta_{10}$, with embedded components at the four singularities. Again the situation is analogous to Lemma 6.2, and the degenerate fiber in $Y_{2,\sigma} \to (E/G)_\sigma = \mathbb{P}^1_\sigma$ comprises the image of

$$\Theta_4 + \left(\sum_{i=6}^{10} \Theta_i \right) + \Theta_{12},$$

now with singularities are $(D_4 + A_1) + (A_1 + D_4)$, which are contained in the images of $\Theta_4 + \Theta_{12}$.

Step 3. Let $Y_3 = \mathrm{Bl}_{Z_2}(Y_2)$ be the blowing-up where the center $Z_2 \subset Y_2$ is the schematic image of $\Upsilon_{2,1} + \ldots + \Upsilon_{2,4}$. These four curves specialize pairwise to the images of $2(\Theta_4 + \Theta_6 + \Theta_7)$ and $2(\Theta_9 + \Theta_{10} + \Theta_{12})$, with embedded components at the four singularities. Again the situation is analogous to Lemma 6.2. The degenerate fiber in $Y_{3,\sigma} \to \mathbb{P}^1_\sigma$ comprises the images of

$$\Theta_2 + \left(\sum_{i=4}^{12} \Theta_i \right) + \Theta_{14},$$

and the singularities are $(A_1 + A_1 + A_1) + (A_1 + A_1 + A_1)$, contained in the images of $\Theta_2 + \Theta_{14}$. After changing the enumeration of the $\Psi_{2,i}$, we may assume that the image of $\Psi_{2,1}$ specializes to the image of Θ_2, and the image of $\Psi_{2,4}$ specializes to the image of Θ_{14}, now without any embedded components.

Step 4. Let $Y = \mathrm{Bl}_{Z_3}(Y_3)$ be the blowing-up where the center $Z_3 \subset Y_3$ is the schematic image of $\Psi_{2,1} + \Psi_{2,4}$. It then follows from [68], Proposition 10.5 that the composite map $Y \to V$ is indeed the desired simultaneous partial resolution of singularities. □

7. Monodromy representations

Let S be the spectrum of a discrete valuation ring R whose field of fractions $F = \mathrm{Frac}(R)$ has characteristic zero, and whose residue field $k = R/\mathfrak{m}$ has characteristic two. Write $\eta, \sigma \in S$ for the generic and closed point, respectively. Let E_η, E'_η be a pair of elliptic curves over $F = \kappa(\eta)$ that are admissible for the Kummer construction and form the the family of normal K3 surface $V = (E \times E')/G$, where G is the effective model of the sign involution. Write

$$r_\eta : X_\eta \longrightarrow V_\eta \quad \text{and} \quad r_\sigma : X_\sigma \longrightarrow V_\sigma$$

for the fiber-wise minimal resolution of singularities. Then X_η is a K3 surface; the same is true for X_σ provided that k is perfect. In this section we introduce various ℓ-adic sheaves on the spectra of F and k that encode crucial information, and establish both necessary and sufficient conditions for the existence of a simultaneous minimal resolution.

Throughout, we assume that there is a simultaneous partial resolution $Y \to V$ that fiberwise resolves all singularities stemming from the fixed points. Note that by Lemma 6.2, this automatically holds if the effective model G is multiplicative. The assumption is mainly for the sake of exposition, and could be avoided at the expanse of using ℓ-adic cohomology with compact supports.

Write $g : Y \to S$ and $f_s : X_s \to \operatorname{Spec} \kappa(s)$, $s \in S$ for the structure morphisms, which sit in commutative diagrams

$$\begin{tikzcd} X_s \arrow[rr, "r_s"] \arrow[dr, "f_s"'] & & Y_s \arrow[dl, "g_s"] \\ & S. & \end{tikzcd}$$

As customary, set $A = E \times E'$ and let $q : A \to A/G = V$ be the quotient map. We then get a diagram

$$\begin{tikzcd} \underline{H}^1(E_\eta, \mathbb{Z}_\ell) \otimes \underline{H}^1(E'_\eta, \mathbb{Z}_\ell) \arrow[r, "\cup"] & \underline{H}^2(A_\eta, \mathbb{Z}_\ell) & \\ & \underline{H}^2(V_\eta, \mathbb{Z}_\ell) \arrow[u, "q_\eta^*"'] \arrow[r, "r_\eta^*"'] & H^2(X_\eta, \mathbb{Z}_\ell) \end{tikzcd} \tag{12}$$

of ℓ-adic systems on the scheme $\operatorname{Spec}(F)$. Here $\ell > 0$ is an odd prime, and the map on the left stems from the cup product via $\alpha \otimes \alpha' \mapsto \operatorname{pr}_1^*(\alpha) \cup \operatorname{pr}_2^*(\alpha')$.

Proposition 7.1. *In the above diagram, both horizontal maps are injective, and the vertical map is bijective.*

Proof. The cup product is injective by the Künneth Theorem. The first direct images $R^1 r_{\eta,*}(\mathbb{Z}_\ell)$ vanish, because the exceptional divisors are projective lines. Now fix an algebraic closure F^{alg}. The Leray–Serre spectral sequence gives an exact sequence

$$0 \longrightarrow H^2(V \otimes F^{\mathrm{alg}}, \mathbb{Z}_\ell) \xrightarrow{q^*} H^2(X \otimes F^{\mathrm{alg}}, \mathbb{Z}_\ell) \longrightarrow H^0(V_\eta, R^2 r_{\eta,*}(\mathbb{Z}_\ell)). \tag{13}$$

This already yields the injectivity of r_η^*. We also see that the arrow on the right has finite cokernel, because the intersection form on exceptional divisors is negative definite. Consequently $H^2(V \otimes F^{\mathrm{alg}}, \mathbb{Z}_\ell)$ has rank $22 - 16 = 6$. Moreover, its cup product is non-degenerate, in light of Poincaré Duality for $X \otimes F^{\mathrm{alg}}$, with discriminant a 2-power, because the exceptional curves are (-2)-curves. Bijectivity of q_η^* is now clear, because the groups have the same rank, and the discriminant is an ℓ-adic unit. $\square$

We now introduce certain ℓ-adic subsystems

$$\Psi_\eta \subset \underline{H}^2(X_\eta, \mathbb{Z}_\ell) \quad \text{and} \quad \Psi_\sigma \subset \underline{H}^2(X_\sigma, \mathbb{Z}_\ell).$$

The former is the image of the composition stemming from diagram (12), formed with the inverse of q_η^*. The latter is defined is defined in terms of its Tate twist, by letting $\Psi_\sigma(1) \subset \underline{H}^2(X_\sigma, \mathbb{Z}_\ell)(1) = \underline{H}^2(X_\sigma, \mathbb{Z}_\ell(1))$ be the subsheaf generated by the cycle classes that are supported by the exceptional locus $\operatorname{Exc}_{X_\sigma/V_\sigma}$ arising on finite separable extensions of $\kappa(\sigma)$. Obviously $\operatorname{rank}(\Psi_\eta) = 2 \cdot 2 = 4$ and $\operatorname{rank}(\Psi_\sigma) \leq 4$.

Lemma 7.2. *Suppose there is a simultaneous resolution of singularities $r : X \to V$, for some family $f : X \to S$ of K3 surfaces. Then there is a local subsystem*

$$\Psi_S \subset f_*(\mathbb{Z}_{\ell,X})$$

whose restrictions the points $s \in S$ coincide with the Ψ_s introduced above.

Proof. One may view the Tate twist $\Psi_\eta(1) \subset \underline{H}^2(X_\eta, \mathbb{Z}_\ell(1))$ as the orthogonal complement of the sheaf of cycle classes curves that are vertical with respect to the two projections of $X_\eta = E_\eta \times E'_\eta$. Likewise, $\Psi_\sigma(1)$ is the orthogonal complement stemming from the two projections of $X_\sigma = E_\sigma \times E'_\sigma$, but now only with the cycle classes not stemming from the critical point. In both cases, the number of cycle classes is $18 = 16 + 2$, and all belong to irreducible curves mapping the image of

$$\mathrm{Fix}(E/S) \to E/G \quad \text{or} \quad \mathrm{Fix}(E'/S) \to E'/G.$$

Since the fibrations and the fixed loci exists over the base, so does the orthogonal complements Ψ_S. $\square$

We now translate the situation into representation theory and eventually into linear algebra. Fix separable closures k^{sep} and F^{sep}. The former yields the strict henselization R^{sh}. Then $F^{\mathrm{sh}} = \mathrm{Frac}(R^{\mathrm{sh}})$ is a separable extension of F, and we choose an F-embedding $F^{\mathrm{sh}} \subset F^{\mathrm{sep}}$. With respect to these field extensions we now form the Galois groups $\Gamma_\eta = \mathrm{Gal}(F^{\mathrm{sep}}/F)$ and $\Gamma_\sigma = \mathrm{Gal}(k^{\mathrm{sep}}/k)$, and consider the monodromy representations

$$\rho_\eta : \Gamma_\eta \longrightarrow \mathrm{GL}(\Psi_\eta(F^{\mathrm{sep}})) \quad \text{and} \quad \rho_\sigma : \Gamma_\sigma \longrightarrow \mathrm{GL}(\Psi_\sigma(k^{\mathrm{sep}})).$$

By definition, the former is a *Kronecker product* $\rho_\eta = \epsilon_\eta \otimes \epsilon'_\eta$, where the tensor factors are the monodromy representations

$$\epsilon_\eta : \Gamma_\eta \longrightarrow \mathrm{GL}(H^1(E \otimes F^{\mathrm{sep}}, \mathbb{Z}_\ell)) \quad \text{and} \quad \epsilon'_\eta : \Gamma_\eta \longrightarrow \mathrm{GL}(H^1(E' \otimes F^{\mathrm{sep}}, \mathbb{Z}_\ell)) \tag{14}$$

stemming from the elliptic curves E_η and E'_η. Finally, write

$$\chi_\sigma : \Gamma_\sigma \longrightarrow \mathrm{Aut}\left(\bigcup_{\nu \geq 0} \mu_{\ell^\nu}(k)\right) = \varprojlim_{\nu \geq 0} (\mathbb{Z}/\ell^\nu\mathbb{Z})^\times = \mathbb{Z}_\ell^\times$$

for the ℓ-adic *cyclotomic character*. Its kernel corresponds to the subfield $k^{\mathrm{cyc}} \subset k^{\mathrm{sep}}$ generated by the ℓ^ν-th roots of unity, $\nu \geq 0$. Note that the corresponding field $F^{\mathrm{cyc}} \subset F^{\mathrm{sep}}$ is already contained in F^{sh}.

Suppose that the minimal resolution X_σ is smooth, and let $\Theta_0, \ldots, \Theta_3 \subset X_\sigma \otimes k^{\mathrm{sep}}$ be the four exceptional divisors mapping to $v_{\mathrm{crit}} \in V$, indexed as in the Bourbaki tables ([13], page 256). The cycle classes $\mathrm{cl}(\Theta_i) \in H^2(X_\sigma \otimes k^{\mathrm{sep}}, \mathbb{Z}_\ell(1))$ form a $\mathbb{Z}_\ell$-basis in $\Psi_\sigma(1)(k^{\mathrm{sep}})$, and the monodromy action permutes the basis members. In turn, the Tate twist becomes a *permutation representation*

$$\rho_\sigma \otimes \chi_\sigma : \Gamma_\sigma \longrightarrow S_4 \subset \mathrm{GL}_4(\mathbb{Z}_\ell).$$

Proposition 7.3. *Suppose there is a simultaneous minimal resolution of singularities $r : X \to V$. Then the following holds:*

(i) *Both $F = \mathrm{Frac}(R)$ and $k = R/\mathfrak{m}$ contain a primitive third root of unity.*
(ii) *All three monodromy representations ϵ_η, ϵ'_η and ρ_σ are scalar representations, and we have $\rho_\sigma = \chi_\sigma^{-1}$.*
(iii) *The restrictions of ϵ_η and ϵ'_η to $\mathrm{Gal}(F^{\mathrm{sep}}/F^{cyc})$ are isomorphic, and their eigenvalues belong to $\{\pm 1\} \subset \mathbb{Z}_\ell^\times$.*
(iv) *The base changes $E \otimes F^{\mathrm{sh}}$ and $E' \otimes F^{\mathrm{sh}}$ acquire good reduction over a common field extension $F^{\mathrm{sh}} \subset L$ of degree at most two.*

(v) *If the elliptic curves E_η, E'_η have the same j-invariant, then their base-changes to F^{sh} are already isomorphic, at least if $j \neq 1728$ or F^{sh} contains a primitive fourth root of unity.*

Proof. First note that restricting a finite étale scheme over S to its fiber over the closed point σ gives an equivalence of Galois categories. Our Galois groups, which are the automorphism groups for the fiber functors for F^{sep} and k^{sep}, are thus related by a surjective homomorphism $h : \Gamma_\eta \to \Gamma_\sigma$, compare the discussion in [66], Section 2. In light of Lemma 7.2, the representations

$$\rho_\eta = \epsilon_\eta \otimes \epsilon'_\eta \quad \text{and} \quad \rho_\sigma \circ h$$

are isomorphic. Let $v \in V_s$ be the critical singularity. According to [68], Proposition 5.3 this is a rational double point of type D_4 if the field k contains a primitive third root of unity, and of type B_3 otherwise. In turn, for each $g \in \Gamma_\sigma$ the ensuing matrix $(\rho_\sigma \otimes \chi_\sigma)(g)$ is similar to one of the permutation matrices

$$\begin{pmatrix} E_2 & & \\ & 1 & 0 \\ & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} E_2 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}. \tag{15}$$

Both are diagonalizable in characteristic $\ell \neq 2$, and we conclude that the characteristic polynomial for $\rho_\sigma(g)$ takes the form

$$\chi_{\rho_\sigma(g)}(T) = (T - \alpha)^3 (T \pm \alpha),$$

with eigenvalue $\alpha = \chi_\sigma^{-1}(g)$. By Lemma 7.6, the factors in the Kronecker product $\epsilon_\eta(g) \otimes \epsilon'_\eta(g)$ are homotheties. Thus actually $\rho_\sigma(g) = \chi_\sigma^{-1}(g)$, which already establishes assertion (ii). Furthermore, $(\rho_\sigma \otimes \chi_\sigma)(g)$ is given by the first matrix in (15), which yields (i).

To verify (iii), suppose that $g \in \Gamma_\eta$ fixes the cyclotomic field F^{cyc}. Then the respective eigenvalues $\alpha, \alpha' \in \mathbb{Q}_\ell$ of the homotheties $\epsilon(g), \epsilon'(g)$ satisfy $\alpha\alpha' = 1$. On the other hand, the Galois actions respect the Weil pairings on $E[\ell^\nu]$ and $E'[\ell^\nu]$, so the $\epsilon(g), \epsilon'(g)$ are symplectic, and thus have determinant $d = 1$. For the eigenvalues, this means $\alpha^2 = \alpha'^2 = 1$, and therefore $\alpha, \alpha' \in \{\pm 1\}$. Using $\alpha\alpha' = 1$ we also get $\alpha = \alpha'$, which establishes (iii).

The cyclotomic field F^{cyc} is contained in F^{sh}. The common restriction of ϵ and ϵ' define a homomorphism $\mathrm{Gal}(F^{\mathrm{sep}}/F^{\mathrm{sh}}) \to \{\pm 1\}$, so the corresponding field extension $F^{\mathrm{sh}} \subset L$ has degree at most two. By construction, the finite étale group schemes $E_F[\ell^\nu]$ and $E'_F[\ell^\nu]$ become constant over L, for all $\nu \geq 0$. According to the Criterion of Néron–Ogg–Shafarevich ([77], Theorem 1), the elliptic curves E_L and E'_L have good reduction, thus (iv) holds.

It remains to verify (iv), and for this we may assume that $S = \mathrm{Spec}(R)$ is strictly henselian, such that $F^{\mathrm{sh}} = F$. If the Weierstraß models E, E' are smooth, the closed fibers have the same j-invariant, and we find an isomorphism $h : E_\sigma \to E'_\sigma$. Since R is henselian, we can lift it to an isomorphism $h : E \to E'$.

Suppose now that E_F, E'_F have bad reduction. Let $F \subset L$ be the quadratic extension over which they acquire good reduction, and set $\Gamma = \mathrm{Gal}(L/F)$. By the previous paragraph, we find an identification $E_L = E'_L$. In other word, E'_F is a twisted form of E_F. We thus have $E'_F = Q \wedge^\Gamma E_F$ via some torsor Q with respect to $\Gamma = \mathrm{Aut}_{E_F/F}$. The latter is isomorphic to μ_{2n} with some $1 \leq n \leq 3$, according to [25], Proposition 5.9. Using the norm map for the separable quadratic extension $F \subset L$, we see that the torsor is induced from P with respect to μ_2. The latter corresponds to the constant group $\Gamma = \{\pm 1\}$ given by the sign involution. Applying Lemma 7.7 below with $(X, X_0, e) = (E_F, E_F[\ell], e)$ and using (iii), we infer that $E'_F \simeq E_F$. □

Let us record the following consequence, which is not obvious from Definition 5.2:

Corollary 7.4. *Both elliptic curves E_η and E'_η have potentially good reduction.*

We now obtain a sufficient condition for resolutions of singularities:

Theorem 7.5. *A simultaneous minimal resolution of singularities $r : X \to V$ exists provided the following conditions hold:*

(i) *The field F contains a primitive third root of unity.*
(ii) *The purely inseparable extension $k \subset \kappa(v_{\mathrm{crit}})$ is an equality.*
(iii) *The elliptic curves $E \otimes F^{\mathrm{sh}}$ and $E' \otimes F^{\mathrm{sh}}$ are isomorphic, and acquire good reduction over a quadratic extension $F^{\mathrm{sh}} \subset L$.*

Proof. Let $r_\sigma : X_\sigma \to Y_\sigma$ be the minimal resolution of singularities. Assumption (ii) ensures that X_σ is smooth. We now seek to apply Shepherd-Barron's result ([80], Corollary 2.14). For this we have to ensure that all exceptional curves in X_σ are isomorphic to $\mathbb{P}^1_\sigma$, and assumption made in loc. cit., beginning of Section 2. This indeed holds by our assumption (i), as in [68], Proposition 5.3.

Consider the monodromy representations

$$\varphi_\eta : G_\eta \longrightarrow \varprojlim \mathrm{GL}(E[\ell^\nu](F^{\mathrm{sep}})) \quad \text{and} \quad \varphi'_\eta : G_\eta \longrightarrow \varprojlim \mathrm{GL}(E'[\ell^\nu](F^{\mathrm{sep}}))$$

stemming from the Tate modules. By Hensel's Lemma, the restrictions of the above to $\mathrm{Gal}(F^{\mathrm{sep}}/F^{\mathrm{sh}})$ factor over the quotient $\mathrm{Gal}(L/F^{\mathrm{sep}})$, which is cyclic of order two. Since 2 is an ℓ-adic unit, for each $g \in \mathrm{Gal}(F^{\mathrm{sep}}/F^{\mathrm{sh}})$, the images $\varphi_\eta(g)$ and $\varphi'_\eta(g)$ are diagonalizable, and have $N = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$ as common Jordan normal form, for some $\alpha, \beta \in \mu_2(\mathbb{Q}_\ell) = \{\pm 1\}$. Since F^{sep} contains the cyclotomic field F^{cyc}, the monodromy representation respects the Weil pairing, so the above representation matrix must be symplectic, and thus has determinant $\alpha\beta = 1$. Consequently $\alpha = \beta$, so the above matrix takes the form $N = \pm E_2$.

Using [63], Proposition 6.2.1 we get an identification of $\mathbb{Z}/\ell^\nu\mathbb{Z}$-local systems

$$A_F[\ell^\nu] = \underline{H}^1(A, \mathbb{Z}/\ell^\nu\mathbb{Z}(1)),$$

and we infer that the ℓ-adic representations (14) are trivial when restricted to F^{sep}. By [80], Corollary 2.14 the desired simultaneous minimal resolution $r : X \to Y$ exists. □

Building on an observation of Serre ([76], page 55), Zarhin analyzed the special role of roots of unity in ℓ-adic cohomology [86], and it would be interesting to see if there are deeper connections.

In the proof for Proposition 7.3, we have used some auxiliary facts. The first comes from linear algebra: Let V, W be two non-zero finite-dimensional vector spaces over some field K, and set $m = \dim(V)$ and $n = \dim(V)$. For $f \in \mathrm{End}(V)$

and $g \in \mathrm{End}(W)$ we form the *Kronecker product* $f \otimes g \in \mathrm{End}(V \otimes W)$. Recall that the characteristic polynomials are related by

$$\chi_f(T) = \prod_{i=1}^m (T - \lambda_i), \quad \chi_g(T) = \prod_{j=1}^m (T - \mu_j) \quad \text{and} \quad \chi_{f\otimes g}(T) = \prod_{i,j}(T - \lambda_i\mu_j),$$

for certain $\lambda_i, \mu_j \in K^{\mathrm{alg}}$. Also recall that endomorphisms that become diagonalizable after some field extension are called *semi-simple*.

Obviously, if f and g are diagonalizable or semi-simple, the same property holds for the Kronecker product. Using Jordan normal forms, one sees that if $f \otimes g$ is semisimple, the same holds for the factors. On the other hand, $f \otimes g$ can be diagonalizable without f and g being so: This already happens when both f and g have rational normal form $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, over the field $K = \mathbb{R}$. Suitable assumptions preclude such phenomena:

Lemma 7.6. *Suppose that the Kronecker product $f \otimes g$ has an eigenvalue $\alpha \in K^{\mathrm{alg}}$ with algebraic multiplicity*

$$\dim\left(\sum_{l\geq 0} \mathrm{Ker}(\alpha - f\otimes g)^l\right) > mn - \min(m,n).$$

Then both factors f and g are homotheties.

Proof. It suffices to treat the case that K is algebraically closed. Set $I = \{1, \ldots, m\}$ and $J = \{1, \ldots, n\}$. Consider the subset $S \subset I \times J$ comprising the pairs (i, j) such that $\lambda_i\mu_j \neq \alpha$, and let $E \subset U \otimes V$ be the generalized eigenspace with respect to the eigenvalue $\alpha \in K$. Then

$$|S| = mn - \dim(E) < mn - (mn - \min(m,n)) = \min(m,n).$$

In turn, none of the projections $\mathrm{pr}_1 : S \to I$ and $\mathrm{pr}_2 : S \to J$ is surjective. In particular, there is $1 \leq r \leq m$ with $\alpha = \lambda_r\mu_j$ for all j. Using $\alpha \neq 0$ we obtain $\mu_1 = \ldots = \mu_r$, hence g is a homothety. By symmetry, the same holds for f. □

We also have used a general fact from non-abelian cohomology: Let (X, e) be a pointed proper scheme over a ground field F, endowed with a finite group $\Gamma \subset \mathrm{Aut}(X, e)$, and a Γ-stable closed subscheme X_0 containing e. Then each Γ-torsor P defines a twisted form (X', X_0', e') of the pair (X, X_0, e), via

$$X' = P \wedge^\Gamma X \quad \text{and} \quad X_0' = P \wedge^\Gamma X_0,$$

compare for example the discussion in [74], Section 3.

Lemma 7.7. *In the above situation, suppose that $\mathrm{Aut}_{(X_0,e)/F}$ is constant, and that the homomorphism $\Gamma \to \mathrm{Aut}(X_0, e)$ is injective, with normal image. Let P be a Γ-torsor such that the resulting twisted form (X', X_0', e') satisfies $(X_0', e') \simeq (X_0, e)$. Then $(X', e) \simeq (X, e)$.*

Proof. Write $G = \mathrm{Aut}(X_0, e)$ and $\bar{G} = G/\Gamma$. The corresponding constant group schemes sit in a short exact sequence $1 \to \Gamma_F \to G_F \to \bar{G}_F \to 1$. This is enough to yield in non-abelian cohomology a sequence of pointed sets

$$G \longrightarrow \bar{G} \xrightarrow{\partial} H^1(F, \Gamma_F) \longrightarrow H^1(F, G_F).$$

By assumption, the class of P in $H^1(F, \Gamma_F)$ maps to the class of the trivial torsor $\bar{G}_F$ in $H^1(F, \bar{G}_F)$. As discussed in [36], Section 10 it lies in the image of the coboundary map ∂. The latter sends a point $a \in \bar{G}$ to its schematic fiber in G_F. All such fibers have a rational point, thus P is trivial, and hence $(X', e) \simeq (X, e)$. $\square$

8. K3 surfaces over quadratic number fields

Let $\mathbb{Q} \subset F$ be a number field, $\mathscr{O}_F$ be the ring of integers, and $S = \operatorname{Spec}(\mathscr{O}_F)$ the resulting Dedekind scheme. Write

$$\sigma_1, \ldots, \sigma_r \in S \quad \text{and} \quad F_i^\wedge = \operatorname{Frac}(\mathscr{O}_{F,\sigma_i}^\wedge) \quad \text{and} \quad F_i^{\mathrm{sh}} = \operatorname{Frac}(\mathscr{O}_{F,\sigma_i}^{\mathrm{sh}})$$

for the points of characteristic two and the ensuing valued fields, the $F_i^\wedge$ being complete, whereas the F_i^{sh} are strictly henselian. We now examine the existence of elliptic curves E_F that are admissible for the Kummer construction, and the ensuing question whether the resulting family of normal K3 surfaces $V = (E \times E)/G$ admits a simultaneous resolution of singularities. As usual, $E \to S$ is the Weierstraß model, and $G \subset \operatorname{Aut}_{E/S}$ denotes the effective model for the sign involutions. By Corollary 7.4, the admissible E_F have good reduction outside $\sigma_1, \ldots, \sigma_r \in S$, potentially good reduction everywhere, and the j-invariant is non-zero at each of the σ_i. Let us summarize our general results of the preceding two sections for the situation at hand:

Theorem 8.1. *Notation as above. Suppose the elliptic curve E_F is admissible for the Kummer construction. Assume furthermore that for each $1 \leq i \leq r$, one of the following two conditions hold:*

(i) *If the elliptic curve E_F has good reduction at $\sigma_i \in S$, the group scheme $E_F[2]$ becomes constant over $F_i^\wedge$.*
(ii) *If E_F has bad reduction at σ_i, it acquires good reduction over some quadratic extension of F_i^{sh}, the effective model G_{σ_i} is multiplicative, and the field F contains a primitive third root of unity.*

Then the family of normal K3 surfaces $V = (E \times E)/G$ admits simultaneous resolutions of singularities, and in particular $\mathscr{M}_{\mathrm{K3}}(\mathscr{O}_F) \neq \varnothing$.

Proof. This follows from Theorem 6.1 and Theorem 7.5. $\square$

Parsing the literature on elliptic curves over number one indeed finds examples. Here we consider quadratic number fields, and will turn to S_3-number fields in the next section.

Recall that every quadratic extension $\mathbb{Q} \subset L$ is cyclic, and takes the form $L = \mathbb{Q}(\sqrt{m})$ for some unique square-free integer $m \neq 0, 1$. Up to isomorphism, the field L is determined by its discriminant

$$d_L = \begin{cases} m & \text{if } m \equiv 1 \text{ modulo } 4; \\ 4m & \text{else.} \end{cases}$$

The occurring values, sometimes called "fundamental discriminants", are precisely the numbers

$$d_L = \epsilon \left(\frac{-1}{p_1 \ldots p_r} \right) \cdot 2^\nu p_1 \ldots p_r \tag{16}$$

for pairwise different odd primes p_i and $\nu \in \{0,2,3\}$ and $\epsilon = \pm 1$, subject to the constraint $\nu = 2 \Rightarrow \epsilon = -1$. Note that $d_L = 12$ is impossible.

Consider the following eight elliptic curves $E_F : y^2 = x^3 + a_2x^2 + a_4x$ over certain quadratic number rings $F = \mathbb{Q}(\sqrt{m})$:

d_F	m	ϵ	a_2	a_4	j
28	7	$8+3\sqrt{7}$	$-(1+2\epsilon^2)$	$16\epsilon^3$	255^3
			$-(1+2\epsilon^{-2})$	$16\epsilon^{-3}$	
41	41	$32+5\sqrt{41}$	$(3\epsilon-1)/2$	$(\epsilon^2-\epsilon)/2$	$(\epsilon-16)^3/\epsilon$
			$(-3\epsilon^{-1}-1)/2$	$(\epsilon^{-2}+\epsilon^{-1})/2$	
65	65	$8+\sqrt{65}$	$2\epsilon^2-1$	$16\epsilon^3$	257^3
			$10\epsilon^2-5$	$400\epsilon^3$	
			$8\epsilon+1$	$16\epsilon^2$	17^3
			$40\epsilon+5$	$400\epsilon^2$	

Here d_F is the discriminant and $\epsilon \in \mathscr{O}_F^\times$ is a fundamental unit. According to [23], Theorem 2, these E_F have good reduction everywhere and constant group scheme $E_F[2]$. Moreover, every such elliptic curve over a quadratic number field with these property belongs to the list. Clearly, $j(\sigma) \neq 0$ for each point $s \in S$ of characteristic two.

Let F and a pair E_F, E'_F be from the table, with resulting Weierstraß models E and E'. According to Theorem 8.1, the family of normal K3 surfaces $V = (E \times E')/G$ admits a simultaneous resolution of singularities $X \to V$. This shows:

Theorem 8.2. *We have $\mathscr{M}_{\mathrm{K3}}(\mathscr{O}_F) \neq \varnothing$ for the quadratic number fields F with discriminant $d_F \in \{28, 41, 65\}$.*

We actually get more: Write $\mathscr{M}_{\mathrm{Enr}} \to (\mathrm{Aff}/\mathbb{Z})$ for the *stack of Enriques surfaces*. The objects are triples (R, Y, f) where R is a ring, Y is an algebraic space, and $f : Y \to \mathrm{Spec}(R)$ is a proper flat morphism of finite presentation whose geometric fibers are Enriques surfaces.

Theorem 8.3. *We have $\mathscr{M}_{\mathrm{Enr}}(\mathscr{O}_F) \neq \varnothing$ for the quadratic number fields F with discriminant $d_F \in \{28, 41, 65\}$.*

Proof. Let $\zeta_1, \ldots, \zeta_3 \in E(F)$ be the three points of order two. At each point $\sigma \in S$ with residue field $k = \kappa(\sigma)$ of characteristic two, the kernel for the specialization map $E(F) \to E(k)$ contains exactly one of the ζ_i. Since there are at most two such σ, we find some $\zeta = \zeta_j$ such that $\zeta \in E(\mathscr{O}_F)$ has order two in all fibers. Likewise, we choose such $\zeta' \in E'(\mathscr{O}_F)$. On the family of abelian surfaces $A = E \times E'$, this defines an involution $(x, x') \mapsto (x + \zeta, -x' + \zeta')$. These are often called *Lieberman involutions*. The resulting action of $H = (\mathbb{Z}/2\mathbb{Z})_S$ is free, because it is free on the first factor, and commutes with the sign involution, since ζ and ζ' coincide with their negatives. In turn, we get an induced H-action on the family $Y = A/G$ of normal K3 surfaces.

Let $Y = \mathrm{Bl}_Z(V)$ be the blowing-up as in (10), which be Proposition 6.2 is fiber-wise given by blowing-up the singularities. Our H-action on V uniquely extends to Y, because the center $Z \subset V$ is H-stable. In turn, the exceptional divisors $\Delta_1, \ldots, \Delta_{16}$ on the generic fiber Y_η are H_s-stable, and so are their schematic images in Y. By Proposition 6.3, their successive blowing-ups yield a simultaneous resolution of singularities $X \to Y$, and the H-action again extends uniquely to X.

We claim that the H-action on X is free. The induced action on $\mathrm{Exc}(X/V)$ is free, because this holds for $\mathrm{Sing}(V/S)$. It remains to verify that the action on $\mathrm{Reg}(V/S)$ is free. This may be checked fiberwise, with geometric points. Suppose that (x, x') is such a point on A_s that maps to a H_s-fixed point of $\mathrm{Reg}(V/S)_s$. Then $-(x, x') = (x+\zeta, -x'+\zeta')$, and thus $\zeta' = 0$, contradiction. In turn, the free quotient X/H yields the desired family of Enriques surfaces. $\square$

We close this section with the following observation:

Proposition 8.4. *There are no elliptic curves over $\mathbb{Q}$ or $F = \mathbb{Q}(\sqrt{m})$ for $m \in \{-2, -1, 2\}$ that are admissible for the Kummer construction.*

Proof. Ogg classified the elliptic curves $E_\mathbb{Q}$ whose discriminant take the form $\Delta = \pm 2^v$, there are twenty-four isomorphism classes ([56], Table 1). In each case E has a Weierstraß equation of the form $y^2 = x^3 + a_2x^2 + a_4x$ with $a_2 \in 2\mathbb{Z}$ and $\nu \geq 1$. Thus both Δ and $a_2a_4 + a_6$ belong to $2\mathbb{Z}$, and Proposition 4.2 tells us that E_L is not admissible.

We next examine $F = \mathbb{Q}(\sqrt{-1})$. Write $i = \sqrt{-1}$ for the imaginary number. The ring of integers is given by $\mathscr{O}_F = \mathbb{Z}[i]$, with group of units $\mathscr{O}_F^\times = \{i^n \mid 0 \leq n \leq 3\}$, and uniformizer $\pi = 1+i$ at the point $\sigma \in S$ of characteristic two. Pinch classified the E_F whose discriminant take the form $i^n\pi^\nu$, now there are sixty-four isomorphism classes ([60], Table 2). In all cases $\nu \geq 10$, so $\Delta \in \pi\mathscr{O}_F$. All but one have a Weierstraß equation of the form $y^2 = x^3 + a_2x^2 + a_4x + a_6$. Going through these cases and using that both 2 and $1+i$ belong to the maximal ideal $\pi\mathscr{O}_F$, one finds $a_2a_4 + a_6 \in \pi\mathscr{O}_F$. These E_F are not admissible by Proposition 4.2. The remaining case has

$$y^2 + (1+i)xy = x^3 + ix^2 + 2x + 3i.$$

Using (7) we compute $b_2a_4 + b_6 = 24i$, thus $\mathrm{val}(b_2a_4 + b_6) = 6$. In light of $2 = i(1+i)^2$ one gets $\mathrm{val}(2, a_1, a_3) = 1$, so E_F is not admissible for the Kummer construction by Proposition 4.2.

Finally consider the real quadratic number field $L = \mathbb{Q}(\sqrt{-2})$. Now the ring of integers is $\mathscr{O}_F = \mathbb{Z}[\sqrt{-2}]$, with group of units $\mathscr{O}_F^\times = \{\pm 1\}$, and uniformizer $\pi = \sqrt{-2}$ at the point $\sigma \in S$ of characteristic two. Pinch also classified the E_F whose discriminant take the form $\pm\pi^\nu$, now there are forty cases. All but two have Weierstraß equation of the form $y^2 = x^3 + a_2x^2 + a_4x + a_6$. Going through these cases, one finds $a_2a_4 + a_6 \in \pi\mathscr{O}_F$, so these E_F are not admissible by Proposition 4.2. The remaining two cases have Weierstraß equations

$$y^2 + \pi xy = x^3 - x^2 - 2x + 3 \quad \text{and} \quad y^2 + \pi xy + \pi y = x^3 - x^2 - x.$$

Using (7) we compute the respective values $b_4a_2 + b_6 = 24$ and $b_4a_2 + b_6 = 4$. In both cases we see $\mathrm{val}(b_4a_2 + b_6) \geq 4$ and $\mathrm{val}(2, a_1, a_3) = 1$. Again by Proposition 4.2, the E_F are not admissible for the Kummer construction. $\square$

9. K3 surfaces over S_3-number fields

Recall that an S_3-*number field* is a finite Galois extension $\mathbb{Q} \subset F$ whose Galois group is isomorphic to the symmetric group S_3. The goal of this final section is to establish:

Theorem 9.1. *Let $\mathbb{Q} \subset F$ be the S_3-number field arising as Galois closure from a cubic number field K with discriminant $d_K = -3f^2$ for some even $f \geq 1$. Then $\mathscr{M}_{\mathrm{K3}}(\mathscr{O}_F) \neq \varnothing$.*

The proof appears at the end of this section, after we we have introduced certain elliptic curves and reviewed the relevant facts about S_3-number fields.

Consider the imaginary quadratic number field $L = \mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(e^{2\pi i/3})$ and set $\omega = e^{2\pi i/6} = (1+\sqrt{-3})/2$. Note that the ring of integers is $\mathscr{O}_L = \mathbb{Z}[\omega]$, the sixth root of unity $\omega \in \mathscr{O}_L^\times$ is a generator, the prime $p = 2$ is inert, and the resulting $\mathfrak{p} = 2\mathscr{O}_L$ has residue field $\mathbb{F}_4$. Consider the two elliptic curves

$$E_L: \quad y^2 = x^3 \pm (1+\omega)x^2 + \omega x. \tag{17}$$

These are the first entries in [60], Table 4, where Pinch classified all elliptic curves over L that have good reduction outside $p = 2$.

Proposition 9.2. *Up to isomorphism,* (17) *are the only elliptic curves over L that are admissible for the Kummer construction. For both of them, the effective model G of the sign involution is isomorphic to $\mu_{2,S}$. Moreover, the invariant is $j = 0$, the discriminant is $\Delta = 2^4$, the reduction has Kodaira symbol* II*, and the finite étale group scheme $E_L[2]$ becomes constant over the henselization $\mathscr{O}_L^h$ with respect to $\mathfrak{p} = 2\mathscr{O}_L$.*

Proof. By the classification of Pinch in loc. cit., there are fifty-four cases of elliptic curves over L with discriminant $\Delta = \pm\omega^{\pm 1}2^\nu$, each given by a Weierstraß equation of the form $y^2 = x^3 + a_2x^2 + a_4x + a_6$. We thus have $2_d = 1$; so by Proposition 4.2 admissibility is equivalent to $a_2a_4 + a_6 \not\equiv 0$ modulo $2\mathscr{O}_L$.

For (17) we have $a_2a_4 + a_6 = \pm(1+\omega)\omega \equiv 1$ modulo $2\mathscr{O}_L$, so our E_L are admissible for the Kummer construction. One directly computes the discriminant and j-invariant, and the Tate Algorithm [85] reveals that the Kodaira symbol is II. Since $2 \in \mathscr{O}_L$ is prime, the Tate–Oort classification ([84], Theorem 2) ensures that G is isomorphic to either $(\mathbb{Z}/2\mathbb{Z})_S$ or $\mu_{2,S}$. By Proposition 4.3 the effective model G of the sign involution is multiplicative, and we get $G = \mu_{2,S}$. The group scheme $E_L[2]$ is given by the equations $x = 0$ and $x^2 \pm (1+\omega)x + \omega = 0$. Over the residue field $\mathbb{F}_4$, the latter factors into $(x-\omega)(x-1)$, and the assertion follows from Hensel's Lemma.

For all remaining Weierstraß equations from Pinch's table, one checks $a_2a_4+a_6 \equiv 0$ modulo $2\mathscr{O}_L$, so there are no further admissible curves. $\square$

Choose a separable closure for the residue field $\mathscr{O}_L/\mathfrak{p} = \mathbb{F}_4$, form the ensuing strictly henselian ring $\mathscr{O}_L^{\mathrm{sh}}$ with respect to $\mathfrak{p} = 2\mathscr{O}_F$, and let L^{sh} be its field of fractions. According to [77], Corollary 3 for Theorem 2, there is a smallest extension $L^{\mathrm{sh}} \subset L^{\mathrm{good}}$ over which $E_{L^{\mathrm{sh}}}$ acquires good reduction, and this extension is Galois. Moreover, the Galois group is isomorphic to a subgroup of $\mathrm{Sp}_2(\mathbb{F}_3) = \mathrm{SL}_2(\mathbb{F}_3)$, a group of order twenty-four. It can also be seen as the semidirect product $Q \rtimes C_3$ of the quaternion group $Q = \{\pm E, \pm I, \pm J, \pm K\}$ by a cyclic group of order three.

Proposition 9.3. *The group* $\mathrm{Gal}(L^{\mathrm{good}}/L^{\mathrm{sh}})$ *is isomorphic to* $C_2 \times C_3$.

Proof. We have $j = 0$, hence $c_4 = 0$, and furthermore $\Delta = 2^4$. The ramification index for $p = 2$ in $\mathscr{O}_F$ is $e = 1$. We now apply [46], Theorem 2: Since $3\,\mathrm{val}(c_4) = \infty \geq 12e + \mathrm{val}(\Delta)$ and $3 \nmid \mathrm{val}(\Delta)$ and the Kodaira symbol is not IV or IV*, the Galois group has order six. It must be $C_2 \times C_3$, because $-E \in Q$ is the only element of order two in $Q \rtimes C_3$. □

We see that the cyclic extension $L^{\mathrm{sh}} \subset L^{\mathrm{good}}$ contains four intermediate fields. Of particular interest is what we call L^{bad}, the intermediate field with $[L^{\mathrm{bad}} : L^{\mathrm{sh}}] = 3$.

Proposition 9.4. *For the base-change* $E \otimes L^{\mathrm{bad}}$*, the Weierstraß equations* (17) *remain minimal, and the Kodaira symbol changes to* I_0^**. Moreover, it acquires good reduction over the quadratic extension* $L^{\mathrm{bad}} \subset L^{\mathrm{good}}$.

Proof. Write $R^{\mathrm{bad}} \subset L^{\mathrm{bad}}$ for the integral closure of $R^{\mathrm{sh}} = \mathscr{O}_L^{\mathrm{sh}}$. First note that by construction, $E \otimes L^{\mathrm{bad}}$ has bad reduction, that $R^{\mathrm{sh}} \subset R^{\mathrm{bad}}$ is totally ramified, and that the Weierstraß equation (17) has $\mathrm{val}_{R^{\mathrm{sh}}}(\Delta) = 4$ and $\mathrm{val}_{R^{\mathrm{bad}}}(\Delta) = 12$. If follows that the equations remains minimal. The extension $L^{\mathrm{bad}} \subset L^{\mathrm{good}}$ has degree two. By definition, good reduction occurs over this extension.

Recall that for additive reduction, Ogg's Formula ([57], see also the discussion in [71]) takes the form $\mathrm{val}(\Delta) = 2 + \delta + (m - 1)$, where $m \geq 1$ is the number of irreducible component in the closed fiber of the minimal regular model, and δ is the wild part for the conductor of the Galois representation on ℓ-torsion points in the generic fiber. Over R^{sh}, this becomes $4 = 2 + \delta + (1 - 1)$, hence $\delta = 6$. The extension $R^{\mathrm{sh}} \subset R^{\mathrm{bad}}$ is tamely ramified, so the δ does not change when passing to R^{bad}. Over this ring, Ogg's Formula yields $12 = 2 + 6 + (m' - 1)$, so $m' = 5$. The only Kodaira symbol coming with five irreducible components in the closed fiber is I_0^*. □

Recall that over a given field k, the Galois group of an irreducible separable polynomial $P(T)$ of degree three is isomorphic to S_3 if and only if $\mathrm{disc}(P) \in k^\times$ is not a square. By the Galois Correspondence, the resulting splitting field $k \subset F$ then contains three cubic fields K_1, K_2, K_3 and another quadratic field L. Moreover, F coincides with the Galois closure for each of the $K = K_i$. For $k = \mathbb{Q}$ the finite extension $\mathbb{Q} \subset F$ is called an *S_3-number field*.

Suppose $k \subset K$ is a separable cubic extension. Such extensions are twisted forms of the product ring $k \times k \times k$, hence the group scheme $\mathrm{Aut}_{K/k}$ is an inner form of the constant group scheme $(S_3)_k$, as described in [74], Lemma 3.1. Since $A_n \subset S_n$, $n \geq 3$ is the commutator subgroup, we get an induced twisted form $\mathrm{Aut}'_{K/k}$ of $(A_3)_{\mathbb{Q}}$. The underlying scheme takes the form $\mathrm{Aut}'_{K/k} = \{e\} \cup \mathrm{Spec}(L)$ for some étale F-algebra L of degree two.

Lemma 9.5. *The separable cubic extension* $k \subset K$ *is non-normal if and only if the étale algebra* L *is a field. In this case, the Galois closure is* $K \otimes_k L$*, and the following are equivalent:*

(i) *The cubic extension takes the form* $K = k(\sqrt[3]{\alpha})$ *for some* $\alpha \in k^\times$.

(ii) *The extension $k \subset L$ is obtained by adjoining a primitive third root of unity.*
(iii) *The group scheme $\mathrm{Aut}'_{K/k}$ is isomorphic to μ_3.*

In characteristic $p \neq 2$, we actually have $L = k(\sqrt{-3})$.

Proof. If not a field, L must be isomorphic to $k \times k$. It follows that $\mathrm{Aut}(K/k)$ contains a subgroup of order $[K : k] = 3$, hence the extension $F \subset K$ is Galois. Conversely, if the separable extension $k \subset K$ is normal, the Galois group must be cyclic of order three. But $\mathrm{Aut}'_{K/F}$ is the only subgroup scheme of order three inside $\mathrm{Aut}_{K/F}$, and it follows that $L = F \times F$. This establishes the first assertion.

Suppose now that K is non-normal, so L is a field. For degree reasons, $\tilde{K} = K \otimes_F L$ remains a field, and by the preceding paragraph the cubic extension $L \subset \tilde{K}$ is cyclic. Moreover, $F \subset \tilde{K}$ is separable of degree three. It follows that $\mathrm{Aut}(\tilde{K}/F)$ contains elements of order two and three, hence $F \subset \tilde{K}$ is a Galois extension of degree six. The Galois group is non-abelian, because the intermediate extension $F \subset K$ is non-normal, and it follows that $\tilde{K}$ is the Galois closure.

It remains to establish the equivalence of (i)–(iii). First note that each of the three conditions implies $p \neq 3$, and we thus may disallow this characteristic. Suppose first $K = F(\sqrt[3]{\alpha})$. Then the Galois closure $\tilde{K}$ is obtained by adjoining a primitive third root of unity ζ, and thus $L = K(\zeta)$. Then $\omega = -\zeta$ has minimal polynomial $Q(T) = T^2 - T + 1$, and consequently $(2\omega + 1)^2 = -3$. ☐

We now restrict to $k = \mathbb{Q}$, that is, *cubic number fields* $\mathbb{Q} \subset K$. For more details we refer to the seminal work of Hasse [35] and the monographs of Delone and Faddeev [26], Cohen [22], and Hambleton and Williams [34]. If normal, the Galois group is cyclic, and in $d_K = f^2$ one has $f = 3^\nu p_1 \ldots p_r$ with pairwise distinct primes $p_i \equiv 1$ modulo 3, and $\nu \in \{0, 2\}$, as explained in [22], Theorem 6.4.6. Up to isomorphism, there are exactly 2^r such cyclic cubic extensions.

Suppose now that $\mathbb{Q} \subset K$ is non-normal. Write K_1, K_2, K_3 for the conjugate cubic number fields in the Galois closure F, say with $K = K_1$, and L for the further quadratic number field. Then $d_K = d_L f^2$ for some number f. The latter induces the discriminant ideal $\mathfrak{d} = N_{\mathscr{O}_{\tilde{K}}/\mathscr{O}_L}(\mathrm{Ann}(\Omega^1_{\mathscr{O}_{\tilde{K}}/\mathscr{O}_L}))$ inside $\mathscr{O}_L$, but it is far from clear which numbers d_L and f are actually possible ([35], Section 1 and Satz 6, compare also [26], pp. 159–161). Moreover, we have $\mathrm{val}_p(d_F) \leq 3\,\mathrm{val}_p(d_K)$, with equality whenever $\mathscr{O}_K$ is not totally ramified at p, and some correction terms in the remaining cases ([83], Lemma 2.1).

The situation simplifies for *pure cubic fields*, which can be written as $K = \mathbb{Q}(\sqrt[3]{m})$ where $m = \pm ab^2$ with coprime $a = p_1 \ldots p_s$ and $b = p_{s+1} \ldots p_r$. Of course, swapping a and b yields an isomorphic fields.

Lemma 9.6. *A cubic number field $\mathbb{Q} \subset K$ is pure if and only if the discriminant takes the form $d_K = -3f^2$. Then K is non-normal, with quadratic field $L = \mathbb{Q}(\sqrt{-3})$. Moreover,*

$$d_L = -3 \quad \text{and} \quad f = \begin{cases} 3p_1 \ldots p_r & \text{if } a^2 \not\equiv b^2 \text{ modulo } 9; \\ p_1 \ldots p_r & \text{else} \end{cases}$$

when $K = \mathbb{Q}(\sqrt[3]{m})$ with $m = \pm ab^2$ and $a = p_1 \ldots p_s$ and $b = p_{s+1} \ldots p_r$ as above.

Proof. Suppose K is pure. Using that the Galois closure contains a primitive third root of unity and that K has a real embedding, we see that it is non-normal, with $L = \mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(e^{2\pi i/3})$ and $d_L = -3$. The statement about f follows from [22], Theorem 6.4.13.

Conversely, suppose that $d_K = -3f^2$. Writing $d_K = d_L g^2$ we get $d_L = -3m^2$ where $m = f/g$. From (16) we see $d_L = -3$ or $d_L = -12$, and already remarked that the latter is impossible. Thus $L = \mathbb{Q}(\sqrt{-3})$, since quadratic number fields are determined by their discriminants. Lemma 9.5 tells us that the cubic number field K is pure. $\square$

Proof of Theorem 9.1. We start with the quadratic number field $L = \mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(e^{2\pi i/3})$, set $\omega = e^{2\pi i/6} = (1+\sqrt{-3})/2$, and fix one of the elliptic curves

$$E_F : y^2 = x^3 \pm (\omega+1)x^2 + \omega x.$$

We saw in Proposition 9.2 that E_F is admissible for the Kummer construction, with $G = \mu_2$ as effective model for the sign involution. In turn, the categorical quotient $V = (E\times E)/G$ yields a family of normal K3 surfaces over $S = \operatorname{Spec}(\mathscr{O}_L)$. According to Lemma 6.2, there is a simultaneous partial resolution $Y \to V$ for all singularities stemming from the fixed points. Now $\operatorname{Sing}(Y/S) = \{y_{\text{crit}}\}$ is a singleton, which is a D_4-singularity in its fiber over the residue field $\mathscr{O}_L/2\mathscr{O}_L = \mathbb{F}_4$.

The given cubic number field $\mathbb{Q} \subset K$ has discriminant $d_K = -3f^2$ with some even $f \geq 1$, By Lemma 9.6 we have $K = \mathbb{Q}(\sqrt[3]{m})$ for some $m \geq 1$, and the Galois closure is given by $F = K \otimes L$. The extension $L \subset F$ is cyclic of degree three. The assumption on f ensures that $\mathscr{O}_L \subset \mathscr{O}_F$ is totally ramified over $\mathfrak{p} = 2\mathscr{O}_L$.

Fix separable closure for $\mathscr{O}_L/\mathfrak{p} = \mathbb{F}_4$, with ensuing strict henselization $\mathscr{O}_L^{\text{sh}}$ and field of fractions $L^{\text{sep}} = \operatorname{Frac}(\mathscr{O}_L^{\text{sh}})$. Also choose F^{sep} and an embedding $\mathscr{O}_L^{\text{sep}} \subset F^{\text{sep}}$. We now want to apply Theorem 7.5, and have to verify its assumptions (i)–(iii). Obviously, our field $L = \mathbb{Q}(e^{2\pi i/6})$ contains a primitive third root of unity, and the purely inseparable extension $\mathbb{F}_4 \subset \kappa(v_{\text{crit}})$ is an equality. Over $p = 2$, the extension $\mathbb{Z} \subset \mathscr{O}_L$ is étale and $\mathbb{Z} \subset \mathscr{O}_K$ is totally ramified, hence $\mathscr{O}_L \subset \mathscr{O}_F$ is totally ramified over $\mathfrak{p} = 2\mathscr{O}_L$. In turn, the induced extension $L^{\text{sh}} \otimes_L F$ coincides with $L^{\text{sh}} \subset L^{\text{good}}$. By Proposition 9.4, the elliptic curve $E \otimes F^{\text{sh}}$ acquires good reduction over some quadratic extensions. So the theorem applies, and the base-change $V \otimes \mathscr{O}_{\tilde{K}}$ admits a simultaneous minimal resolution of singularities. $\square$

According to [45], Theorem 6.1, the geometric fiber $X \otimes \mathbb{F}_4^{\text{alg}}$ admits a contraction to a normal K3 surface that is a K3-like covering of an Enriques surface. It would be interesting carry out this construction in the family $X \to \operatorname{Spec}(\mathscr{O}_F)$.

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