KUMMER SURFACES ASSOCIATED WITH GROUP SCHEMES

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Abstract. We introduce Kummer surfaces $X = \mathrm{Km}(C \times C)$ with the group scheme $G = \mu_2$ acting on the self-product of the rational cuspidal curve in characteristic two. The resulting quotients are normal surfaces having a configuration of sixteen rational double points of type A_1, together with a rational double point of type D_4. We show that our Kummer surfaces are precisely the supersingular K3 surfaces with Artin invariant $\sigma \leq 3$, and characterize them by the existence of a certain configuration of thirty curves. After contracting suitable curves, they also appear as normal K3-like coverings for simply-connected Enriques surfaces.

Contents

Introduction

For each abelian surface A in characteristic $p \neq 2$ the quotient $Z = A/\{\pm 1\}$ by the sign involution is a normal surface with sixteen rational double points of type A_1, and the minimal resolution of singularities $X = \mathrm{Km}(A)$ is a K3 surface called *Kummer surface*. Over the complex numbers a K3 surface is a Kummer surface if and only if it contains sixteen disjoint (-2)-curves (Nikulin [15]). In this paper we call a non-singular rational curve on a K3 surface a (-2)-*curve* for simplicity. If $A = J$ is the jacobian variety of a curve of genus two, the Kummer surface X contains thirty-two distinguished (-2)-curves forming the so-called (16_6)-configuration (e.g. Griffiths and Harris [7], Chapter 6, page 787, Figure 21), and the existence of these thirty-two (-2)-curves characterizes the Kummer surface associated with a curve of genus 2 (Nikulin [14]).

Shioda [24] showed that a Kummer surface $X = \mathrm{Km}(A)$ in odd characteristics is supersingular if and only if the abelian surface A is supersingular. For $p = 2$, however, Shioda [23] and Katsura [9] observed that the singularities on $Z = A/\{\pm 1\}$

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are more complicated, and that X is a K3 surface if and only if A is not supersingular and then $\rho(X) \leq 20$. Indeed, for any supersingular abelian surface the quotient acquires an elliptic singularity, and X becomes a rational surface. Recall that an abelian variety is *supersingular* if it is isogenous to a product of supersingular elliptic curves.

The second author [20] obtained Kummer surfaces in characteristic $p = 2$ by replacing the supersingular abelian surface A by the self-product $C \times C$ of the rational cuspidal curve C, which is a *non-normal* surface, and the constant group $G = \{\pm 1\}$ by the additive group scheme $G = \alpha_2$, which is *non-reduced*. It turns out that for suitable G-actions, the quotients $Z = (C \times C)/G$ are normal, and the minimal resolutions of singularities X are supersingular K3 surfaces with Artin invariant $\sigma \leq 2$. The configuration of singularities is either $5D_4$ or $D_4 + 2D_8$.

Recall that for supersingular K3-surfaces, the isomorphism class of the Picard lattice is determined by a single integer $1 \leq \sigma \leq 10$ called the *Artin invariant*. Ogus [17] proved for odd primes that $\sigma \leq 2$ means that X is Kummer. For the Kummer surfaces $X = \mathrm{Km}(C \times C)$ with group scheme $G = \alpha_2$ one also $\sigma \leq 2$. Up to isomorphism, there is a unique supersingular K3 surface with $\sigma = 1$. Dolgachev and the first author [6] characterized it, among other things, by the existence of forty-two (-2)-curves. Shimada and Zhang [22] showed that every supersingular K3 surface with $\sigma \leq 2$ is isomorphic to a Kummer surface with $G = \alpha_2$, by characterizing these Kummer surfaces in terms of the configurations of twenty-six (-2)-curves with dual graph given in Figure 2.

The main goal of this paper is to extend the construction $X = \mathrm{Km}(C \times C)$ to the multiplicative group scheme $G = \mu_2$. It turns out that $Z = (C \times C)/G$ has only rational double points, and that their configuration is $16A_1 + D_4$, which is very close to the classical situation over the complex numbers. The Artin invariant now becomes $\sigma \leq 3$. Indeed, the construction of the Kummer surface with group scheme $G = \mu_2$ has two moduli coming from the possible embeddings $G \to \mathrm{Aut}_{C\times C}$. Our principal result is a characterization of such Kummer surfaces.

Theorem. *(See Thm. 4.1 and 5.2) Let X be a K3 surface in characteristic $p = 2$. Then the following are equivalent:*

(i) *There is an isomorphism $X \simeq \mathrm{Km}(C \times C)$ for a Kummer surface with group scheme $G = \mu_2$.*
(ii) *There is a configuration of thirty (-2)-curves on X with simple normal crossings and dual graph given in §3, Figure 1.*
(iii) *The K3 surface X is supersingular with Artin invariant $\sigma \leq 3$.*

Keum [10] showed that every Kummer surface $X = \mathrm{Km}(A)$ over the complex numbers is the K3-covering of some Enriques surface Y. This was extended to odd characteristics by Jang [8]. The first author [12], §3.3 established this also for our Kummer surface with Artin invariant $\sigma = 1$. Here we extend this to all Kummer surfaces $X = \mathrm{Km}(C \times C)$ with group scheme $G = \mu_2$:

Theorem. *(See Thm. 6.1) Let X be a supersingular K3 surface with Artin invariant $\sigma \leq 3$ in characteristic $p = 2$. Then there is a contraction $X \to X'$ of twelve (-2)-curves such that that the normal K3 surface X' is the K3-like covering of some simply-connected Enriques surface Y.*

The paper is organized as follows: In Section 1 we recall some facts on group schemes G of height ≤ 1 and restricted Lie algebras $\mathfrak{g}$ that will be used throughout. Section 2 contains an analysis of G-actions on the self-product $C \times C$ of the rational cuspidal curve. This is used in Section 3 to construct our Kummer surface $X = \mathrm{Km}(C \times C)$ with group scheme G, where we also determine the dual graph for the distinguished curves and compute the Artin invariant. Our characterization with such configuration of curves occupies Section 4. In Section 5 we give the characterization with Artin invariants.

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1. Some restricted Lie algebras

We start by discussing some restricted Lie algebras from a purely algebraic point of view. Let k be a ground field of characteristic $p > 0$. Recall that a *restricted Lie algebra* is a Lie algebra $\mathfrak{g}$, endowed with an additive self-map $x \mapsto x^{[p]}$ called *p-map*. The latter is related to scalar multiplication, Lie brackets and vector addition according to the following three axioms:

$$(1) \qquad (\lambda x)^{[p]} = \lambda^p x, \quad [x^{[p]}, y] = (\mathrm{ad}_x)^p(y), \quad (x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{r=1}^{p-1} s_r(x,y)$$

for all vectors $x, y \in \mathfrak{g}$ and scalars $\lambda \in k$. Here $\mathrm{ad}_x(y) = [x,y]$ is the adjoint representation, and $s_r(x,y)$ are certain universal expressions involving nested Lie brackets. For example, we have $s_1(x,y) = [x,y]$ in characteristic two, and $s_1(x,y) = [x,[x,y]]$ in characteristic three. For details, we refer to Demazure and Gabriel [5], Chapter II, §7, No. 3.

A vector $x \in \mathfrak{g}$ is called *p-closed* if it is nonzero, and $x^{[p]} \in \lambda x$ for some scalar $\lambda \in k$. In other words, the line $kx \subset \mathfrak{g}$ is a restricted Lie subalgebra. For each unit $\epsilon \in k^\times$, we get $(\epsilon x)^{[p]} = \epsilon^{p-1}\lambda(\epsilon x)$, and we see that the class of λ in $k/k^{\times p-1}$ depends only on the line, and not on the vector. One may regard this class as an "eigenvalue" for the p-map, and $x \in \mathfrak{g}$ as an "eigenvector".

For each group scheme G, the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ is endowed with a p-map in a canonical way. A group scheme G on which the relative Frobenius map $F : G \to G^{(p)}$ is trivial is called *of height* ≤ 1. In fact, the functor $G \mapsto \mathrm{Lie}(G)$ is an equivalence between the category of finite group schemes of height ≤ 1 and the category of finite-dimensional restricted Lie algebras ([5], Chapter II, §7, No. 4). Such group schemes admit p-basis, and their order $|G| = h^0(\mathscr{O}_G) = p^d$ is given by $d = \dim_k(\mathfrak{g})$. In particular, the lines generated by p-closed vectors $x \in \mathfrak{g}$ correspond to subgroup schemes $H \subset G$ of order p. These are twisted form of $H = \mu_p$ or $H = \alpha_p$. In characteristic $p = 2$ we actually have $H = \mu_2$ or $H = \alpha_2$.

We now examine a special type of restricted Lie algebras: Let $\mathfrak{a}$ be a finite-dimensional restricted Lie algebra, with trivial Lie bracket $[a, a'] = 0$ and trivial

p-map $a^{[p]} = 0$. Let $\mathfrak{b} = ke$ be the one-dimensional restricted Lie algebra, with basis vector $e \in \mathfrak{b}$ and p-map $(\lambda e)^{[p]} = \lambda^p e$. On the vector space sum $\mathfrak{a} \oplus \mathfrak{b}$, we define Lie bracket and p-map via the formulas

$$[a + \lambda e, a' + \lambda' e] = \lambda a' - \lambda' a \quad \text{and} \quad (a + \lambda e)^{[p]} = \lambda^{p-1}(a + \lambda e). \tag{2}$$

The former satisfies the axioms for Lie brackets, and the resulting Lie algebra is written as $\mathfrak{g} = \mathfrak{a} \rtimes \mathfrak{b}$. This is indeed the semi-direct product with respect to the homomorphism of Lie algebras $\mathfrak{b} \to \mathfrak{gl}(\mathfrak{a})$, $e \mapsto \mathrm{id}_\mathfrak{a}$.

Proposition 1.1. *The above p-map endows $\mathfrak{g} = \mathfrak{a} \rtimes \mathfrak{b}$ with the structure of a restricted Lie algebra, such that the inclusions of $\mathfrak{a}$ and $\mathfrak{b}$ are homomorphisms of restricted Lie algebras. Moreover, this p-map is unique, and each vector in $\mathfrak{g}$ is p-closed.*

Proof. Uniqueness follows from the axioms (1). Since the homomorphism of Lie algebras $\rho : \mathfrak{b} \to \mathfrak{gl}(\mathfrak{a})$ satisfies $\rho(x^{[p]}) = \rho(x)^p$, the existence of a p-map follows from Strade and Farnsteiner [27], Theorem 2.5. To check that it is given by our formula, it suffices to treat the case $\mathfrak{a} = ka$, which was verified in loc. cit. example 4 on page 72. From (2) we see that each vector in the semidirect product is p-closed. □

Let G be the finite group scheme of height ≤ 1 with $\mathrm{Lie}(G) = \mathfrak{g}$. Then the closed subschemes $H \subset G$ of order p correspond to lines in the vector space $\mathfrak{g}$, or points on the projectivization $\mathbb{P}(\mathfrak{g})$.

Now consider the direct sum $\mathfrak{g} \oplus \mathfrak{g}$ of restricted Lie algebras. Here the Lie bracket and p-map are given by

$$[(x, x'), (y, y')] = ([x, y], [x', y']) \quad \text{and} \quad (x, x')^{[p]} = (x^{[p]}, x'^{[p]}).$$

A straight-forward computation shows:

Proposition 1.2. *A vector $(x, x') \in \mathfrak{g} \oplus \mathfrak{g}$ whose entries are non-zero is p-closed if and only if $x = a + \lambda e$ and $x' = a' + \lambda e$ for some scalars $\lambda, \lambda' \in k$ satisfying $\lambda^{p-1} = \lambda'^{p-1}$. In this case, we have $(x, x')^{[p]} = \lambda^{p-1}(x, x')$.*

We see that the set of p-closed vectors is the union of the restricted Lie subalgebras $\mathfrak{g} \oplus 0$ and $0 \oplus \mathfrak{g}$ and $(\mathfrak{a} \oplus \mathfrak{a}) \rtimes \mathfrak{b}$, where the latter is formed with the graphs $\mathfrak{b} = \Gamma_\zeta \subset \mathfrak{b} \oplus \mathfrak{b}$ from the mulitpication by $(p-1)$-th roots of unity $\zeta \in k^\times$.

2. Diagonal actions and rational points

Let k be a ground field of characteristic $p = 2$, and consider the rational cuspidal curve

$$C = \operatorname{Spec} k[u^2, u^3] \cup \operatorname{Spec} k[u^{-1}].$$

As explained by the second author in [20], Section 3 the sheaf $\Omega^1_{C/k}$ of Kähler differentials is invertible modulo torsion, and the dual sheaf $\Theta_{C/k}$ is invertible of degree four. By Riemann–Roch, $H^0(C, \Theta_{C/k})$ is four-dimensional. As a restricted Lie algebra this is a semidirect product $\mathfrak{a} \rtimes \mathfrak{b}$ studied in the previous section, where the first factor $\mathfrak{a}$ is generated by the vector fields $u^{-2}D_u, D_u, u^2 D_u$ and the second factor $\mathfrak{b}$ is generated by uD_u. Here the derivation $D_u : \mathscr{O}_C \to \mathscr{O}_C$ is determined by $D_u(u) = 1$. Note that the basis vectors can be rewritten as

$$u^{-2}D_u = u^{-4}D_{u^{-1}}, \quad D_u = u^{-2}D_{u^{-1}}, \quad u^2 D_u = D_{u^{-1}} \quad \text{and} \quad uD_u = u^{-1}D_{u^{-1}}.$$

By Proposition 1.1, each non-zero vector field $\delta \in H^0(C, \Theta_{C/k})$ is p-closed, and thus defines a faithful action of the height-one group scheme G with $\mathrm{Lie}(G) = k\delta$. Note that we have $G \simeq \alpha_2$ if $\delta \in \mathfrak{a}$ and $G \simeq \mu_2$ else.

We now consider the self-product $C \times C$, which is a non-normal integral surface. As discussed in Section 1, the restricted Lie algebra

$$H^0(C \times C, \Theta_{C \times C/k}) = (\mathfrak{a} \rtimes \mathfrak{b}) \oplus (\mathfrak{a} \rtimes \mathfrak{b}) \tag{3}$$

contains the restricted Lie subalgebra $(\mathfrak{a} \oplus \mathfrak{a}) \rtimes \mathfrak{b}$, whose elements have the form

$$\delta = (\lambda_4 u^{-4} + \lambda_2 u^{-2} + \lambda_0) D_{u^{-1}} + (\mu_4 v^{-4} + \mu_2 v^{-2} + \mu_0) D_{v^{-1}} + \tau(uD_u + vD_v) \tag{4}$$

for some scalars λ_i, μ_i and a common $\tau = \lambda_1 = \mu_1$. Here we use the two indeterminates u, v to describe the first and second factor in $C \times C$, respectively.

Now let G be a local group scheme order p acting on $C \times C$. In light of the decomposition (3), this is the diagonal action stemming from G-actions on the factors.

We now additionally assume that the induced actions on both factors are faithful. According to Proposition 1.2, such actions arise from the non-zero vector field δ as in (4). Note that $G = \mu_2$ if and only if $\tau \neq 0$, and $G = \alpha_2$ otherwise.

Consider the quotient $Z = (C \times C)/G$, which is an integral surface. The second projection $\mathrm{pr}_2 : C \times C \to C$ induces a morphism

$$Z = (C \times C)/G \longrightarrow C/G = \mathbb{P}^1.$$

Here the projective line is given by $\mathbb{P}^1 = \operatorname{Spec} k[v^2] \cup \operatorname{Spec} k[v^{-2}]$. Write $K = k(v^{-2})$ for its function field. The generic fiber $Z_K = Z \otimes_{\mathscr{O}_{\mathbb{P}^1}} K$ is a twisted form of the rational cuspidal curve $C_K = C \otimes_k K$. We compute its K-rational points in dependence of the vector field δ:

Proposition 2.1. *The set of K-rational points in the regular locus $\mathrm{Reg}(Z_K)$ corresponds to the solution $(\alpha, \beta) \in K^2$ of the system of equations*

$$\lambda_4 \alpha^4 + \mu_4 \alpha = 0, \quad \lambda_2 \alpha^2 + \mu_2 \alpha = 0, \quad \lambda_4 \beta^4 + \lambda_2 \beta^2 + \lambda_0 + \tau\beta = \mu_0 \alpha.$$

Proof. Write $L = k(v^{-1})$ for the function field of the rational cuspidal curve $C = \operatorname{Spec} k[v^2, v^3] \cup \operatorname{Spec} k[v^{-1}]$. Each K-rational point on the regular locus of $Z_K \subset Z$ has as preimage on $C_L \subset C \times C$ a G-stable L-valued point. As explained by the second author in [20], proof for Proposition 7.2, these correspond to G-equivariant K-morphisms $\mathrm{Spec}(L) \to \mathrm{Reg}(C_K) = \operatorname{Spec} K[u^{-1}]$. To make the actions explicit we consider the polynomials

$$P(u^{-1}) = \lambda_4 u^{-4} + \lambda_2 u^{-2} + \lambda_0 + \tau u^{-1},$$
$$Q(v^{-1}) = \mu_4 v^{-4} + \mu_2 v^{-2} + \mu_0 + \tau v^{-1}$$

that appear in the vector field (4). Then G acts on C_K via the vector field $P(u^{-1})D_{u^{-1}}$ and on $\mathrm{Spec}(L)$ by the derivation $Q(v^{-1})D_{v^{-1}}$. The morphism $\mathrm{Spec}(L) \to \mathrm{Reg}(C_K)$ is given by a homomorphism of K-algebras

$$K[u^{-1}] \longrightarrow L, \quad u^{-1} \longmapsto \alpha v^{-1} + \beta.$$

for some $\alpha, \beta \in K$. The G-equivariance means that the substitution $u^{-1} = \alpha v^{-1} + \beta$ turns the derivation $P(u^{-1})D_{u^{-1}}$ into the derivation $Q(v^{-1})D_{v^{-1}}$. The latter condition boils down to the equation $P(\alpha v^{-1} + \beta) = \alpha Q(v^{-1})$. Comparing coefficients in this polynomial equation for the indeterminate v^{-1} gives the desired system of equations in α, β. □

The solutions for the first two equations $\lambda_4\alpha^4 + \mu_4\alpha = 0$ and $\lambda_2\alpha^2 + \mu_2\alpha = 0$ form a vector space over $\mathbb{F}_2$. One easily sees that the number of solutions is of the form 2^m for some integer $0 \leq m \leq 2$, provided that $\lambda_4 \neq 0$. This leads to a formula for the number of rational points on the generic fiber:

Corollary 2.2. *Suppose that $\lambda_4, \mu_4 \neq 0$, that the generic fiber Z_K is normal, and that k is algebraically closed. Let $0 \leq m \leq 2$ be as above. Then the number of rational points in the generic fiber for the morphism $Z \to \mathbb{P}^1$ is given by*

$$|Z_K(K)| = \begin{cases} 2^{2+m} & \text{if } G = \mu_2; \\ 2^{1+m} & \text{if } G = \alpha_2 \text{ and } \lambda_2 \neq 0; \\ 2^m & \text{if } G = \alpha_2 \text{ and } \lambda_2 = 0. \end{cases}$$

For $G = \mu_2$, each $0 \leq m \leq 2$ occurs for a suitable choice of $\lambda_2, \mu_2 \in k$. For $G = \alpha_2$ the occuring values are $m \in \{0, 1\}$ for $\lambda_2 \neq 0$, and $m \in \{0, 2\}$ for $\lambda_2 = 0$.

Proof. First note that all the solutions $\alpha, \beta \in K$ for the equations in Proposition 2.1 already lie in k, because this field is relatively algebraically closed in K. Moreover, the set $Z_K(K)$ is contained in $\mathrm{Reg}(Z_K)$, because the curve Z_K is normal.

Suppose first that $G = \mu_2$, in other words $\tau \neq 0$. Then the third equation $\lambda_4\beta^4 + \lambda_2\beta^2 + \lambda_0 + \tau\beta = \mu_0\alpha$ is separable, thus for each solution α of the first two equation one gets four solutions β of the third equation. This gives the desired formula $|Z_K(K)| = 2^m \cdot 4$. For λ_2 generic the non-zero solution $\alpha = \mu_2/\lambda_2$ of the second equation is not a solution of the first equation, thus $m = 0$. The other extreme $\lambda_2 = \mu_2 = 0$ yields $m = 2$, and for suitable choices of λ_2, μ_2 we also get $m = 1$.

Now suppose $G = \alpha_2$, such that $\tau = 0$. If $\lambda_2 \neq 2$, the second equation in Proposition 2.1 has two solutions $\alpha' = 0$ and $\alpha'' \neq 0$, and the third equation also has two solutions. Clearly, $\alpha' = 0$ solves the first equation, and by choosing μ_4 in a suitable way, α'' a may or may not solve it. This gives $m \in \{0, 1\}$. Finally, suppose $\lambda_2 = 0$. Then the third equation has a single root, wheras the first equation has four roots. All of them solve the second equation provided $\mu_2 = 0$, but only $\alpha = 0$ is a root if $\mu_2 \neq 0$. This gives $m \in \{0, 2\}$. □

3. Kummer surfaces associated with group schemes

We keep the assumptions of the previous section, such that the group scheme G acts on the self-product $C \times C$ of the rational cuspidal curve in characteristic $p = 2$. The action is given by some global vector field

$$\delta = (\lambda_4 u^{-4} + \lambda_2 u^{-2} + \lambda_0)D_{u^{-1}} + (\mu_4 v^{-4} + \mu_2 v^{-2} + \mu_0)D_{v^{-1}} + \tau(uD_u + vD_v) \tag{5}$$

with seven coefficients $\lambda_i, \mu_i, \tau \in k$. We assume that the action is faithful on each factor. For the sake of exposition, we also assume that k is algebraically closed. Write $Z = (C \times C)/G$ for the resulting quotient, which is an integral surface. As in [20], Proposition 4.3 one verifies:

Proposition 3.1. *The following conditions are equivalent:*

(i) *The integral scheme $Z=(C\times C)/G$ is normal.*
(ii) *Both coefficients $\lambda_4,\mu_4\in k$ are non-zero.*
(iii) *The G-actions on the two factors C are free at the singular point.*

From now on we assume that indeed $\lambda_4,\mu_4\neq 0$, and proceed to study the resulting normal surface $Z=(C\times C)/G$. The *fixed scheme* $(C\times C)^G$ for the group scheme action thus lies in the regular locus, and is thus the zero-scheme for the equations $\lambda_4u^{-4}+\lambda_2u^{-2}+\lambda_0+\tau u^{-1}=0$ and $\mu_4v^{-4}+\mu_2v^{-2}+\mu_0+\tau v^{-1}=0$. This is a finite subscheme of length $l=16$. The point on $C\times C$ that lies over the singular point in both factors is given by $u^2=u^3=v^2=v^3=0$ and called the *quadruple point*.

Proposition 3.2. *The G-action on $C\times C$ and the normal surface $Z=(C\times C)/G$ has the following properties:*

(i) *The singular locus $\operatorname{Sing}(Z)$ consists of the images of the fixed points, together with the image of the quadruple point. The latter is always a rational double point of type D_4.*
(ii) *For $G=\mu_2$, the fixed scheme $(C\times C)^G$ is reduced, and its image on Z consists of $l=16$ points, which are rational double points of type A_1.*
(iii) *For $G=\alpha_2$ the fixed scheme is non-reduced, and its image on Z either consists of four rational double points of type D_4, or two rational double points of type D_8, or one elliptic singularity. The latter holds if and only if $\lambda_2=\mu_2=0$.*
(iv) *The minimal resolution of singularities $X\to(C\times C)/G$ is a K3 surface if and only if all singularities on $(C\times C)/G$ are rational, that is, $\lambda_2,\mu_2\in k$ do not vanish simultaneously.*

Proof. The case that G is additive is already treated by the second author in [20], under the assumption $\lambda_4=\mu_4=1$ and $\lambda_0=\mu_0$. The general case works in virtually the same. Let us make the singularities at the fixed points explicit:

The fixed points are given by the vanishing of the polynomials $P=\lambda_4u^{-4}+\lambda_2u^{-2}+\lambda_0+\tau u^{-1}$ and $Q=\mu_4v^{-4}+\mu_2v^{-2}+\mu_0+\tau v^{-1}$. Moreover, the kernel $R\subset k[u^{-1},v^{-1}]$ of the derivation δ is generated as k-algebra by the elements $a=u^{-2}$ and $b=v^{-2}$ and

$$c=\tau u^{-1}v^{-1}+(\mu_4v^{-4}+\mu_2v^{-2}+\mu_0)u^{-1}+(\lambda_4u^{-4}+\lambda_2u^{-2}+\lambda_0)v^{-1}.$$

These three generators are subject to the single relation

$$c^2+\tau^2ab+(\mu_4^2b^4+\mu_2^2b^2+\mu_0^2)a+(\lambda_4^2a^4+\lambda_2^2a^2+\lambda_0^2)b=0.$$

If $G=\mu_2$ the polynomials P and Q are separable, one easily checks that this equation indeed defines sixteen rational double points of type A_1. The remaining assertions are as in [20], Section 5 and 6. $\square$

From now on, we assume that the coefficients $\lambda_2,\mu_2\in k$ do not vanish simultaneously, such that the minimal resolution $X\to(C\times C)/G$ is a supersingular K3 surface. Note that the group scheme G, the action on $C\times C$ and the resulting K3 surface X all depend on the vector field (5). By abuse of notation we write

$$X=\operatorname{Km}(C\times C)$$

and call it the *Kummer surface associated with the group scheme G*. Note that we either have $G=\mu_2$ or $G=\alpha_2$.

The vector field in (5) depends on the seven parameters $\lambda_i, \mu_i, \tau \in k$. However, the isomorphism class of the Kummer surface $X = \mathrm{Km}(C \times C)$ has only two moduli, because the image of the embedding $G \to \mathrm{Aut}_{C\times C}$ depends only on the line $k\delta \subset H^0(C \times C, \Theta_{C\times C})$ rather than the vector δ, and the canonical action of $\mathbb{G}_a \rtimes \mathbb{G}_m$ on the affine line $\mathbb{A}^1$, which extends to an action on C, yields re-parameterization of the indeterminates u^{-1} and v^{-1}.

Now suppose $X = \mathrm{Km}(C \times C)$ is a Kummer surface with group scheme $G = \mu_2$. Owing to its construction as minimal resolution of singularities $X \to (C \times C)/G$, this K3 surface contains thirty *distinguished curves*

$$E_{ij}, \quad E_r, \quad C_s, \quad C'_s \qquad (1 \leq i, j \leq 4, \quad 0 \leq r \leq 3, \quad 0 \leq s \leq 4) \tag{6}$$

defined as follows: The two projections $\mathrm{pr}_1, \mathrm{pr}_2 : C \times C \to C$ induce two fibrations $f, f' : X \to \mathbb{P}^1$. The $E_{ij} \subset X$ are the exceptional curves lying over the images $(a_i, a'_j) \in \mathbb{P}^1\times\mathbb{P}^1$ of the sixteen fixed points in $C\times C$. The $E_r \subset X$ are the exceptional curves over the image $(a_0, a'_0) = (0, 0)$ of the quadruple point. Finally, the $C_s, C'_s \subset X$ are the strict transforms of the fibers $f^{-1}(a_s)$ and $f'^{-1}(a'_s)$, respectively.

For any genus-one fibration $f : S \to \mathbb{P}^1$ on a K3 surface S with a section $O \subset S$, the *trivial lattice* $T(S/\mathbb{P}^1)$ is the sublattice inside $\mathrm{Pic}(S)$ generated by the irreducible components of the closed fibers, together with the chosen section.

Proposition 3.3. *For the group scheme $G = \mu_2$, the thirty distinguished curves on the Kummer surface $X = \mathrm{Km}(C \times C)$ listed in (6) form a configuration of (-2)-curves with simple normal crossings whose dual graph is depicted in Figure 1. Moreover, the Kummer surface X has Picard number $\rho = 22$.*

Proof. This can be checked with a local computation with rings of invariants. However, we can also argue that subconfigurations like $C_1 + E_{11} + \ldots + E_{14}$ appear as set-theoretical fibers for genus-one fibrations $f : X \to \mathbb{P}^1$. Hence the components must be (-2)-curves with simple normal crossings, and the shape of the dual graph follows.

The above genus-one fibration $f : X \to \mathbb{P}^1$ is actually induced by the first projection $\mathrm{pr}_1 : C \times C \to C$, hence it is quasielliptic. It has five reducible fibers with Kodaira symbol I_0^*, and $C'_1 \subset X$ is a section. In turn, the trivial lattice $T(X/\mathbb{P}^1) \subset \mathrm{Pic}(X)$, which is generated by the vertical curves disjoint from C'_1, together with a fiber and C'_1, has rank $22 = 5 \cdot 4 + 2$. It follows that our K3 surface X has Picard number $\rho = 22$. □

Note that removing the curves $C_0, C'_0, E_0, \ldots, E_3$ yields the dual graph that occurs in the classical Kummer surface $X = \mathrm{Km}(E_1 \times E_2)$ in characteristic 0 attached to the product of ellliptic curves, where the quotients $Z = (E_1 \times E_2)/\{\pm 1\}$ acquires sixteen rational double points of type A_1. For the sake of completeness, we depict the distinguished curves on the Kummer surface $X = \mathrm{Km}(C \times C)$ associated with the group scheme $G = \alpha_2$ and generic action in Figure 2.

A finitely generated free abelian group endowed with a non-degenerate $\mathbb{Z}$-valued bilinear form is called a *lattice*. A lattice L is *p-elementary* if the discriminant group L^*/L is an elementary abelian p-group. Here $L^* = \mathrm{Hom}(L, \mathbb{Z})$. Let L be a 2-elementary even lattice of rank $r \geq 2$ with signature $\mathrm{sign}(L) = (1, r - 1)$, and assume that the discriminant bilinear form $b_L : L^*/L \times L^*/L \to \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ is alternating,

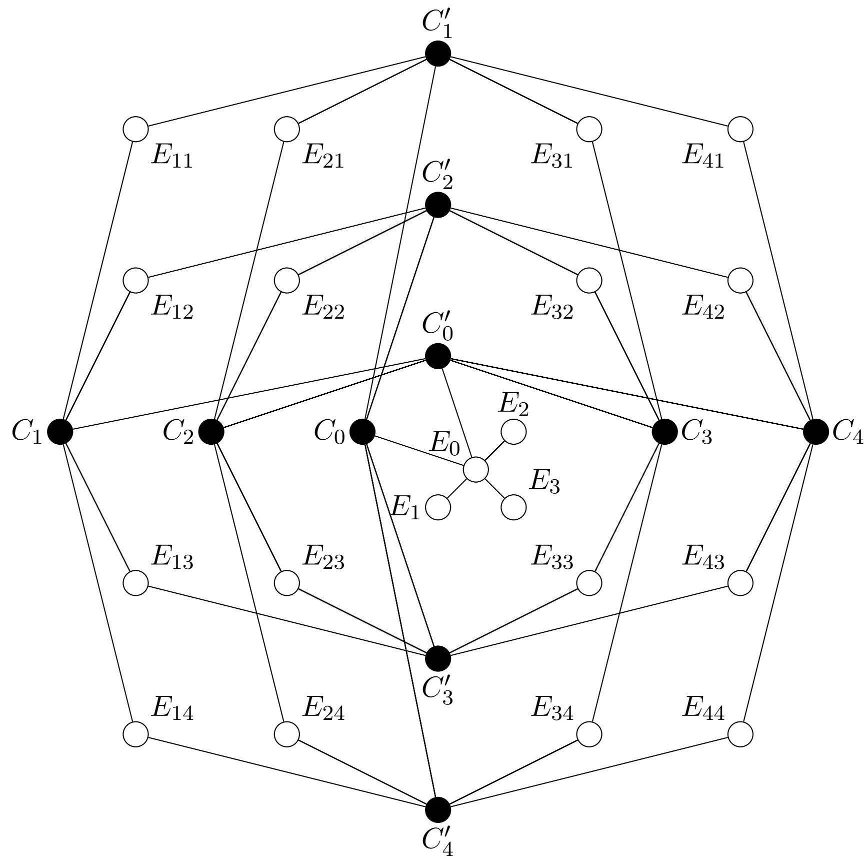

FIGURE 1. Dual graph for the thirty distinguished (-2)-curves on Kummer surface $\mathrm{Km}(C\times C)$ associated with the group scheme $G=\mu_2$

in other words, the discriminant quadratic form $q_L : L^*/L \to \frac{1}{2}\mathbb{Z}/2\mathbb{Z}$ factors over $\mathbb{Z}/2\mathbb{Z}$. Regarding the latter as a quadratic form with values in $\mathbb{Z}/2\mathbb{Z} = \frac{1}{2}\mathbb{Z}/\mathbb{Z}$, we see that b_L becomes its associated bilinear form. Thus q_L is non-degenerate and the $\mathbb{F}_2$-vector space L^*/L is symplectic and thus even-dimensional, so $\mathrm{disc}(L) = (-1)^{r-1}2^{2\sigma}$ for some integer $\sigma \geq 0$.

According to Nikulin ([16], Theorem 3.6.2), the even indefinite 2-elementary lattice L is determined up to isometry by the numerical invariants $r \geq 2$ and $\sigma \geq 0$. Moreover, such a lattice with invariants $r \geq 2$ and $\sigma \geq 0$ exists if and only if $2\sigma \leq r$ and furthermore $r \equiv 2$ modulo 4, where in the boundary cases $\sigma = 0$ and $2\sigma = r$ the congruence must hold modulo 8. The discriminant quadratic form $q_L : L^*/L \to \mathbb{Z}/2\mathbb{Z}$ is given by

$$q_L = \begin{cases} q(t_1, \ldots, t_{2\sigma}) & \text{if } r \equiv 2 \bmod 8; \\ q(t_1, \ldots, t_{2\sigma}) + t_1^2 + t_2^2 & \text{if } r \equiv 6 \bmod 8 \end{cases} \tag{7}$$

with the polynomial $q(t_1, \ldots, t_{2\sigma}) = t_1t_2 + \ldots + t_{2\sigma-1}t_{2\sigma}$. Recall that up to isometry, there are two non-degenerate quadratic forms on $V = \mathbb{F}_2^{\oplus 2\sigma}$ with $\sigma \geq 1$, which are distinguished by the *Arf invariant*, or equivalently by the *Witt index*.

An even overlattice $L \subset L'$ corresponds to a subgroup $H \subset L^*/L$ on which the discriminant quadratic form q_L vanishes, via the assignment $H = L'/L$. Such H

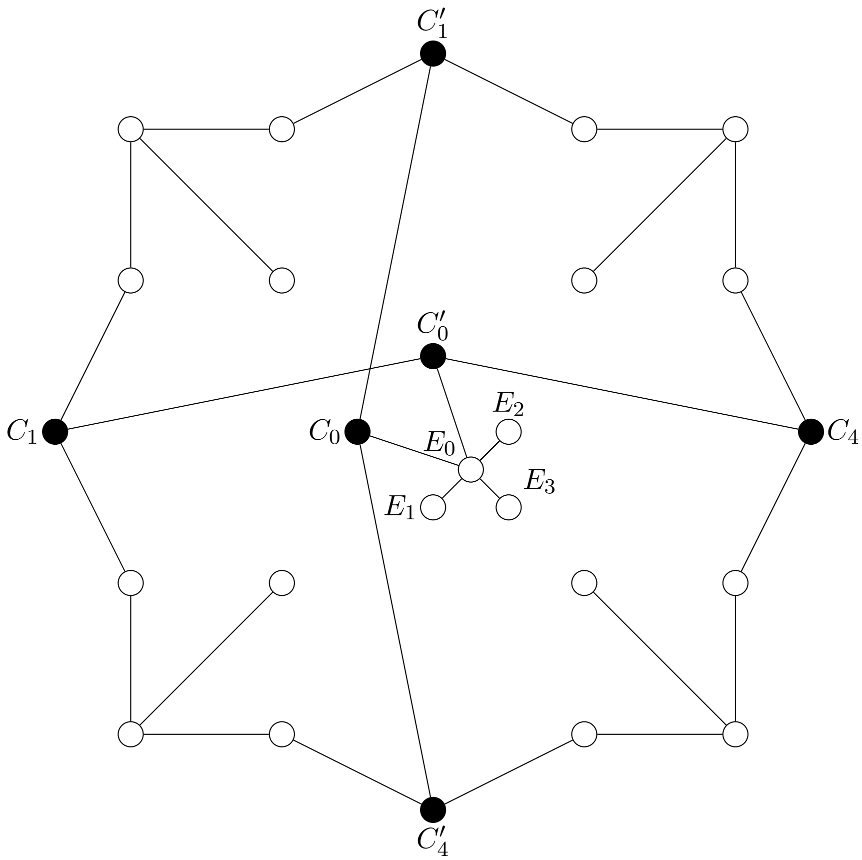

FIGURE 2. Dual graph for the twenty-six distinguished (-2)-curves on $\mathrm{Km}(C \times C)$ associated with the group scheme $G = \alpha_2$ and generic action

is called *totally singular* (Bourbaki's terminology [4], §9, No. 2, Definition 2). The discriminant group for the overlattice L' is given by the subquotient $H^\perp/H$, so the bilinear form $b_{L'}$ stays alternating. In turn, the invariants for the overlattice L' are given by the formulas $r' = r$ and $\sigma' = \sigma - \dim_{\mathbb{F}_2}(H)$. It is not difficult to count the number of overlattices:

Lemma 3.4. *The number $n \geq 0$ of overlattices $L \subset L'$ with index $[L' : L] = 2$ is given by the formula $n = 2^{2\sigma-1} + (-1)^\epsilon 2^{\sigma-1} - 1$ for the exponent $\epsilon = (r-2)/4$.*

Proof. The number of totally isotropic subgroups $H \subset L^*/L$ of dimension 1 corresponds to the number of non-trivial zeros for the quadratic equation $q_L = 0$. As above, set $q = t_1t_2 + \ldots + t_{2\sigma-1}t_{2\sigma}$. The number of zeros for $q = 0$ is given by $2^{2\sigma-1} + 2^{\sigma-1}$, whereas $q + t_1^2 + t_2^2 = 0$ has $2^{2\sigma-1} - 2^{\sigma-1}$ zeros (see for example [26], Theorem 4.16). The assertion follows from (7). □

For supersingular K3 surfaces X in arbitrary characteristics $p > 0$, the Picard lattice $L = \mathrm{Pic}(X)$ has rank $\rho = 22$, the intersection form is even, and the discriminant group has order $|L^*/L| = p^n$ for some $n \geq 0$. Artin [2] showed that the lattice L is p-elementary, with even exponent $n = 2\sigma$, at least for odd characteristics. The required results on flat cohomology where established a little later by Milne [13].

Rudakov and Shafarevich extended this to $p = 2$ and established that the discriminant bilinear form b_L is alternating (Rudakov and Shafarevich [18], Theorem 3). In turn, the Picard lattice $L = \mathrm{Pic}(X)$ is determined up to isometry by the numbers $r = 22$ and $1 \leq \sigma \leq 10$. The latter is called the *Artin invariant*.

Proposition 3.5. *For the group scheme $G = \mu_2$, the Artin invariant of the Kummer surface $X = \mathrm{Km}(C \times C)$ is an integer $1 \leq \sigma \leq 3$, and each such number occurs.*

Proof. Consider the fibration $X \to \mathbb{P}^1$ induced from $\mathrm{pr}_2 : C \times C \to C$, and write $K = k(\mathbb{P}^1)$ for the function field of the projective line. We saw in Corollary 2.2 that the number of K-rational points in the generic fiber X_K takes the form $|X_K(K)| = 2^{2+m}$ with $0 \leq m \leq 2$. Moreover, each such integer can be realized with suitable vector fields $\delta \in H^0(C \times C, \Theta_{C\times C})$.

From Proposition 3.3 we deduce that $X \to \mathbb{P}^1$ has exactly five singular fibers, all of which are of type I_0^*. In turn, the trivial lattice $L = T(X/\mathbb{P}^1)$ inside $P = \mathrm{Pic}(X)$ has discriminant $\mathrm{disc}(L) = -2^{2\cdot 5}$, whereas the full Picard lattice has $\mathrm{disc}(P) = -2^{2\sigma}$. Finally, the index for the sublattice is given by $[P : L] = 2^{2+m}$. This gives $10 = 2\sigma + 2(2 + m)$, and thus $\sigma = 3 - m$. The assertion now follows. ☐

Let $L \subset \mathrm{Pic}(X)$ be the sublattice generated by the distinguished curves in Figure 1. This lattice has invariants $r = 22$ and $\sigma = 3$, and occurs for all Kummer surfaces $X = \mathrm{Km}(C \times C)$ associated with the group scheme $G = \mu_2$. By Lemma 3.4, there are $n_2 = 27$ overlattices $L \subset L'$ with invariant $\sigma' = 2$. Moreover, for fixed L' there are five further overlattices $L' \subset L''$ with invariant $\sigma'' = 1$. In total, there are $n_1 = \frac{5n_2}{3} = 45$ such overlattices $L \subset L''$. We saw above that at least one L' and one L'' appears as Picard groups for Kummer surfaces $X = \mathrm{Km}(C \times C)$. We do not know which of them actually occur in this way.

4. Characterization with configurations of curves

Throughout this section we work over an algebraically closed ground field k of characteristic $p \geq 0$. Over the complex numbers, the classical Kummer surfaces $X = \mathrm{Km}(A)$ with group $G = \{\pm 1\}$ attached to an abelian surface A can be characterized by the existence of sixteen disjoint (-2)-curves (Nikulin [15]). Those coming from jacobians of genus-two curves are characterized by a (16_6)-configuration of thirty-two (-2)-curves (Nikulin [14]). Kummer surfaces of product type, which arise from a product $A = E_1 \times E_2$ of elliptic curves, are also characterized by a *double Kummer pencil*, which comprises twenty-four (-2)-curves, as in Figure 1 but without $E_0 + \ldots + E_3$ (see Remark 4.2 and Shioda and Inose [25], Section 2). Our main result is a characterization for the Kummer surfaces $\mathrm{Km}(C \times C)$ associated with the group scheme $G = \mu_2$:

Theorem 4.1. *Let X be a K3 surface containing a configuration of thirty (-2)-curves with normal crossings and dual graph as in Figure 1. Then the characteristic must be $p = 2$, and X is isomorphic to a Kummer surface $\mathrm{Km}(C \times C)$ associated with the group scheme $G = \mu_2$.*

Proof. We first construct a jacobian quasielliptic fibration $f : X \to \mathbb{P}^1$. Consider the following divisors:

$$C_0 + 2E_0 + E_1 + E_2 + E_3 \quad \text{and} \quad 2C_i + E_{i,1} + \ldots + E_{i,4}, \quad 1 \leq i \leq 4. \tag{8}$$

These are pairwise disjoint, and each forms a singular fiber of type I_0^*. Let $f : X \to \mathbb{P}^1$ be the resulting genus-one fibration. It is jacobian because the curve C_1' provides a section. The trivial lattice $T(X/\mathbb{P}^1)$ has rank $r \geq 22$. It follows that the K3 surface X has Picard number $\rho = 22$ and the Mordell–Weil group $\mathrm{MW}(X/\mathbb{P}^1)$ is finite. Consequently the above five divisors are the reducible fibers, and we have $p > 0$. The Picard lattice $\mathrm{Pic}(X)$ has discriminant $-p^{2\sigma}$. The sublattice $L \subset \mathrm{NS}(X)$ generated by the irreducible curves appearing in (8), together with a section is isomorphic to $U \oplus D_4^{\oplus 5}$ which has even discriminant -2^{10}, and we conclude $p = 2$. Furthermore, the fibration $f : X \to \mathbb{P}^1$ is quasielliptic (Rudakov and Shafarevich [18], Proposition on page 150).

By symmetry, the curves

$$(9) \qquad C_0' + 2E_0 + E_1 + E_2 + E_3 \quad \text{and} \quad 2C_j' + E_{1,j} + \ldots + E_{4,j}, \quad 1 \leq j \leq 4$$

give another such fibration $f' : X \to \mathbb{P}^1$. Using the dual graph, we compute the intersection number $f^{-1}(\infty) \cdot f'^{-1}(\infty)$ between the fibers as

$$(2C_1 + \sum_i E_{1,i}) \cdot (2C_1' + \sum_j E_{j,1}) = (2C_1 + E_{11}) \cdot (2C_1' + E_{11}) = 2.$$

In turn, the resulting morphism $(f, f') : X \to \mathbb{P}^1 \times \mathbb{P}^1$ is an alteration of degree two, which means a proper surjection between integral scheme whose generic fiber has length two. Let $X \to Z \to \mathbb{P}^1 \times \mathbb{P}^1$ be its Stein factorization. The morphism contracts precisely the irreducible curves $C \subset X$ that are vertical for both fibrations. These curves correspond to the white vertices in the Figure 1, thus form an ADE-configuration of the type $16A_1 + D_4$. In turn, Z is a normal K3 surface with Picard number $\rho = 2$, coming with a finite flat morphism $Z \to \mathbb{P}^1 \times \mathbb{P}^1$.

Let $z_{ij} \in Z$ be the images of the exceptional curves $E_{ij} \subset X$, and let $z \in Z$ be the image of $E_0 \cup \ldots \cup E_3$. The complement of these seventeen singular points is the regular locus $U = \mathrm{Reg}(Z)$. We now construct an invertible sheaf on U whose class in $\mathrm{Pic}(U)$ has order two, such that the resulting μ_2-torsor will lead to the desired normal Kummer surface. Let $g : Z \to \mathbb{P}^1$ be the morphism induced by the fibration $f : X \to \mathbb{P}^1$, and consider the images $F_i \subset Z$ of the curves $C_i \subset X$. Then $F_0, 2F_1, \ldots, 2F_4$ are schematic fibers for g, and it follows that the Weil divisor $A = 2F_0 - (F_1 + \ldots + F_4)$ has order two modulo principal divisors. Fix an identification $\mathscr{O}_Z(-2A) = \mathscr{O}_Z$. For the reflexive rank-one sheaf $\mathscr{F} = \mathscr{O}_Z(A)$, we obtain a canonical map $\mathscr{F}^\vee \otimes \mathscr{F}^\vee \to \mathscr{O}_Z$, which endows the coherent sheaf $\mathscr{A} = \mathscr{O}_Z \oplus \mathscr{F}^\vee$ with the structure of a $\mathscr{O}_Z$-algebra graded by the group $\mathbb{Z}/2\mathbb{Z}$. Let $\epsilon : \tilde{Z} \to Z$ be the resulting finite Z-scheme, which is irreducible and Cohen–Macaulay. Note that this construction depends on the identification $\mathscr{O}_Z(-2A) = \mathscr{O}_Z$, which is unique up to factors from $\Gamma(Z, \mathscr{O}_X)^\times = k^\times$; it follows that the Z-scheme $\tilde{Z}$ is unique up to unique isomorphism. The $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathscr{A}$ corresponds to an action of the group scheme $G = \mu_2$ on $\tilde{Z}$, with quotient $Z = \tilde{Z}/G$. Over the regular locus $U = \mathrm{Reg}(Z)$, the action is free, and the quotient map becomes a G-torsor.

Let $g' : Z \to \mathbb{P}^1$ be the morphism induced by the fibration $f' : X \to \mathbb{P}^1$. The situation is actually symmetric in g and g': Using the dual graph given in Figure 1, one sees that the divisors

$$2C_0 - (C_1 + \ldots + C_4) \quad \text{and} \quad 2C_0' - (C_1' + \ldots + C_4')$$

have the same intersection numbers with all curves occurring in the dual graph. Since these generate the Picard group up to finite index and $\mathrm{Pic}^\tau(X)=0$, we conclude that the above divisors differ by a principal divisor. Let $F_i'\subset Z$ be the images of the curves $C_i'\subset X$, and set $A'=2F_0'-(F_1'+\ldots+F_4')$. Then $2F_0-(F_1+\ldots+F_4)$ and $2F_0'-(F_1'+\ldots+F_4')$ also differ by a principal divisor. In turn, our double covering $\epsilon:\tilde{Z}\to Z$ defined with $\mathscr{F}=\mathscr{O}_Z(A)$ is equivariantly isomorphic to the double covering defined with $\mathscr{F}'=\mathscr{O}_Z(A')$.

The main task now is to identify $\tilde{Z}$ with the self-product $C\times C$ of the rational cuspidal curve. We start by computing the Euler characteristic for the structure sheaf, which boils down to compute $\chi(\mathscr{F})$ on the normal K3 surface Z. First note that the half-fibers F_i, $1\leq i\leq 4$ are copies of the projective line: To see this, write $h:X\to Z$ for the contraction. For each A_1-singularity $z_{ij}\in Z$, the schematic fiber is given by $h^{-1}(z_{ij})=C_{ij}$, according to [1], Theorem 4, and this ensures that the induced morphism $h:C_i\to F_i$ is an isomorphism. Consider the disjoint union $F=F_1\cup\ldots\cup F_4$. The short exact sequence $0\to\mathscr{O}_Z(-F)\to\mathscr{O}_Z\to\mathscr{O}_F\to 0$ immediately gives $\chi(\mathscr{O}_Z(-F))=\chi(\mathscr{O}_Z)-4\chi(\mathscr{O}_{\mathbb{P}^1})=-2$. In contrast to the half-fibers, the fiber F_0 is a a copy of the rational cuspidal curve: The D_4-singularity $z\in Z$ has schematic fiber $h^{-1}(z)=2E_0+E_1+E_2+E_3$, again by [1], Theorem 4. In turn, $h^{-1}(z)\cap C_0=C_0\cap 2E_0$ is a local Artin scheme of length two, which is mapped to the closed point $z\in F_0$ under the induced morphism $C_0\to F_0$. It follows that F_0 is the rational cuspidal curve. Since F_0 is a fiber, we furthermore have $2F_0=F_0\otimes k[\epsilon]$, where ϵ is an indeterminate subject to $\epsilon^2=0$. The short exact sequence $0\to\mathscr{O}_Z(-F)\to\mathscr{O}_Z(2F_0-F)\to\mathscr{O}_{2F_0}\to 0$ yields $\chi(\mathscr{F})=\chi(\mathscr{O}_Z(-F))+\chi(\mathscr{O}_{2F_0})=-2+2\cdot 0=-2$. In turn, we get $\chi(\mathscr{O}_{\tilde{Z}})=\chi(\mathscr{O}_Z)+\chi(\mathscr{F})=2-2=0$.

Next note that $\tilde{Z}$ is reduced: If not, the structure morphism $\epsilon:\tilde{Z}\to Z$ admits generically a section. By the Valuative Criterion for proper morphism, such a generic section extends over an open subset $V\subset Z$ containing all points of codimension one. Thus $\epsilon^{-1}(V)\to V$ is a trivial G-torsor, and it follows that the invertible sheaf $\mathscr{F}|U$ is trivial. In turn, the Weil divisor $A=2F_0-(F_1+\ldots+F_4)$ is principal over V, hence on Z. However, this Weil divisor is not principal at each rational double point of type A_1, contradiction. Summing up, our surface $\tilde{Z}$ is integral.

Furthermore, we observe that the dualizing sheaf is isomorphic to the structure sheaf. Indeed, we have $\omega_Z=\mathscr{O}_Z$, whence $\omega_{\tilde{Z}}$ is trivial over the open set $\epsilon^{-1}(U)$. Since ω_Z is Cohen–Macaulay, we must have $\omega_{\tilde{Z}}=\mathscr{O}_{\tilde{Z}}$. We now get the cohomological invariants: $h^0(\mathscr{O}_{\tilde{Z}})=1$ because $\tilde{Z}$ is integral, $h^2(\mathscr{O}_{\tilde{Z}})=1$ by Serre duality, and finally $h^1(\mathscr{O}_{\tilde{Z}})=2$ according to the computation of the Euler characteristic.

Consider the composition $\tilde{g}:\tilde{Z}\to\mathbb{P}^1$ of the double covering $\tilde{Z}\to Z$ with the fibration $g:Z\to\mathbb{P}^1$, and let $D=\operatorname{Spec}\tilde{g}_*(\mathscr{O}_{\tilde{Z}})$ be the Stein factorization. Then D is an integral curve, which turns out to be non-normal. The morphism $D\to\mathbb{P}^1$ is a finite universal homeomorphism, and we claim that it has degree two. Indeed, the sheaf $\mathscr{F}$ and the structure sheaf $\mathscr{O}_Z$ become isomorphic over the generic geometric fiber $S=f^{-1}(\bar{\eta})$, and it follows that the G-torsor $\epsilon:\tilde{Z}\to Z$ becomes trivial when pulled back to S. It follows that the locally free sheaf $\tilde{g}_*(\mathscr{O}_{\tilde{Z}})$ has rank two.

We observed at the beginning that $f':X\to\mathbb{P}^1$ is quasielliptic. So besides the five reducible fibers corresponding to (9), all closed fibers are copies of the rational cuspidal curve. The corresponding fibers $Z_a=g'^{-1}(a)$, $a\in\mathbb{P}^1$ on the normal K3 surface Z are contained in $\operatorname{Reg}(Z)$, and the induced torsor $\tilde{Z}_a = \epsilon^{-1}(Z_a) \to Z_a$ is trivial. In turn, the reduction $(\tilde{Z}_a)_{\text{red}}$ is another copy of the rational cuspidal curve. Taking degrees in the commutative diagram

$$\begin{array}{ccc} (\tilde{Z}_a)_{\text{red}} & \xrightarrow{\epsilon} & Z_a \\ {\scriptstyle \tilde{g}}\downarrow & & \downarrow{\scriptstyle g} \\ D & \xrightarrow[\text{can}]{} & \mathbb{P}^1, \end{array}$$

we see that the finite dominant morphism $\tilde{g} : (\tilde{Z}_a)_{\text{red}} \to D$ is birational. In particular $h^1(\mathscr{O}_D) \geq 1$ holds.

By symmetry, the above reasoning also applies for the composition $\tilde{g}' : \tilde{Z} \to \mathbb{P}^1$ of the double covering $\epsilon : \tilde{Z} \to Z$ with the other fibration $f' : Z \to \mathbb{P}^1$, and the ensuing Stein factorization $D' = \operatorname{Spec} \tilde{g}'_*(\mathscr{O}_{\tilde{Z}})$. Consider the resulting morphisms $\tilde{Z} \to D$ and $\tilde{Z} \to D'$ and the ensuing diagonal morphism $\varphi : \tilde{Z} \to D \times D'$ between integral schemes, which is proper and dominant. We claim that it is birational: In the commutative diagram

$$\begin{array}{ccc} \tilde{Z} & \xrightarrow{\epsilon} & Z \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle (g,g')} \\ D \times D' & \longrightarrow & \mathbb{P}^1 \times \mathbb{P}^1, \end{array}$$

the upper map has degree two, the right map has degree two, and the lower map has degree four. If follows that $\deg(\varphi) = 1$.

Next, we claim that the integral curves D, D' have $h^1(\mathscr{O}_D) = h^1(\mathscr{O}_{D'}) = 1$. Seeking a contradiction, we assume that this does not hold. Without restriction, we have $h^1(\mathscr{O}_D) \geq 2$ and $h^1(\mathscr{O}_{D'}) \geq 1$. The canonical injection $H^1(\mathscr{O}_D) \subset H^1(\tilde{Z}, \mathscr{O}_{\tilde{Z}})$ must be an equality, by dimension reasons. To proceed, consider a fiber $\tilde{Z}_a = \tilde{g}^{-1}(a)$ such that the induced projection $g' : \tilde{Z}_a \to D'$ is birational. Then the composite map $H^1(D', \mathscr{O}_{D'}) \to H^1(\tilde{Z}, \mathscr{O}_{\tilde{Z}}) \to H^1(\tilde{Z}_a, \mathscr{O}_{\tilde{Z}_a})$ is surjective. Hence there is a cohomology class $\alpha \in H^1(\tilde{Z}, \mathscr{O}_{\tilde{Z}})$ whose restriction to the fiber $\tilde{Z}_a = g^{-1}(a)$ is non-zero. On the other hand, any cohomology class lies in the image of $g^* : H^1(D, \mathscr{O}_D) \to H^1(\tilde{Z}, \mathscr{O}_{\tilde{Z}})$, whence vanishes on Z_a, contradiction. Summing up, we have $h^1(\mathscr{O}_D) = h^1(\mathscr{O}_{D'}) = 1$.

Since the morphism $D \to \mathbb{P}^1$ and $D' \to \mathbb{P}^1$ are purely inseparable, we infer that both D, D' are copies of the rational cuspidal curve. Summing up, we have a birational morphism $\varphi : \tilde{Z} \to C \times C$ between proper integral schemes, which are Gorenstein with trivial dualizing sheaves. Consider the resulting conductor square

$$\begin{array}{ccc} R & \longrightarrow & \tilde{Z} \\ \downarrow & & \downarrow \\ B & \longrightarrow & C \times C, \end{array}$$

where $B \subset C \times C$ is defined by the annihilator ideal for $\varphi_*(\mathscr{O}_{\tilde{Z}})/\mathscr{O}_{C \times C}$, and $R = \varphi^{-1}(C)$ is its schematic preimage. If non-empty, the schemes B, R are equidimensional of dimension one, and without embedded components, because both $\tilde{Z}$

and $C\times C$ are Cohen–Macaulay. The adjunction formula gives $\omega_{\tilde{Z}} = \varphi^*(\omega_{C\times C})(-R)$. It follows that R and hence also B is empty, thus $\varphi : \tilde{Z} \to C\times C$ is an isomorphism.

Summing up, our normal K3 surface Z is isomorphic to the quotient for a faithful action of $G = \mu_2$ on the self-product $\tilde{Z} = C \times C$. In other words, the K3 surface X is a Kummer K3 surface $\mathrm{Km}(C \times C)$ formed with the group scheme $G = \mu_2$. □

Note that the crucial identification $\tilde{Z} = C \times C$ could also be established with explicit computations of coordinate rings, as the referee pointed out. The above coordinate-free argument might be useful in more general situations, when $C \times C$ is replaced by non-normal surfaces without product sturctures.

Remark 4.2. Assume that a K3 surface X is defined over an algebraically closed field k of characteristic $p \neq 2$ and contains a double Kummer pencil. Recall that this is a configuration of twenty-four (-2)-curves as in Figure 1 but without $E_0+\ldots+E_3$. The divisors $E_{11} + E_{12} + E_{13} + E_{14} + 2C_1$ and $E_{11} + E_{21} + E_{31} + E_{41} + 2C'_1$ define elliptic fibrations with four singular fibers of type I_0^*. It follows that the divisor $\sum_{1\leq i,j\leq 4} E_{ij}$ is divided by 2 in $\mathrm{Pic}(X)$ which defines a double covering $Y \to X$ branched along sixteen curves E_{ij}. The preimages of C_i, C'_i are elliptic curves and those of E_{ij} are exceptional curves. By contracting these sixteen exceptional curves, we obtain an abelian surface A. The surface A contains two elliptic curves E, E' (e.g. the preimages of C_1 and C'_1) meeting transversally at one point. This implies that $A \simeq E \times E'$ and hence $X \simeq \mathrm{Km}(E \times E')$.

5. Characterization with Artin invariants

Let k be an algebraically closed ground field of characteristic $p \geq 0$, and let X be a K3 surface. Then $\mathrm{Pic}(X)$ is a free abelian group of rank $\rho \leq 20$ or $\rho = 22$ endowed with a non-degenerate intersection pairing.

Lemma 5.1. *Suppose X and X' are K3-surfaces such that there is an isometry between Picard groups. Let $C_1, \ldots, C_r \subset X$ be (-2)-curves. Then there are (-2)-curves $C'_1, \ldots, C'_r \subset X'$ so that the intersection matrices $N = (C_i \cdot C_j)$ and $N' = (C'_i \cdot C'_j)$ coincide.*

Proof. By assumption there is an isomorphism $f : \mathrm{NS}(X) \to \mathrm{NS}(X')$ respecting the intersection pairing. According to Rudakov and Shafarevich [19], Section 3, Proposition one may choose f that the induced map between real vector space yields a bijection $\mathrm{Nef}(X) \to \mathrm{Nef}(X)$ between the nef cones (see also Shimada and Zhang [22], Proposition 3.1). In turn, it induces a bijection $\overline{\mathrm{NE}}(X) \to \overline{\mathrm{NE}}(X')$ between the cones of curves, which are dual to the nef cones. But on any smooth projective surface S, the extremal rays $\mathbb{R}_{\geq 0}c \subset \overline{\mathrm{NE}}(S)$ with $c^2 < 0$ correspond to the integral curves $C \subset S$ with $C^2 < 0$, according to Kollár [11], Lemma 4.12. On K3 surfaces, these are exactly the (-2)-curves. In turn, our chosen map $f : \mathrm{NS}(X) \to \mathrm{NS}(X')$ induces a bijection between the classes of (-2)-curves on X and X', and respects their intersection numbers. □

Now suppose that X is a K3-surface with $\rho = 22$. Then we are in characteristic $p > 0$, the K3-surface is supersingular, and the Picard group $\mathrm{Pic}(X)$ is determined up to isometry by the Artin invariant $1 \leq \sigma \leq 10$.

Theorem 5.2. *Let X be a supersingular K3 surface in characteristic 2 with Artin invariant $\sigma \leq 3$. Then X is isomorphic to a Kummer surface $\mathrm{Km}(C \times C)$ associated with the group scheme $G = \mu_2$.*

Proof. According to Proposition 3.5, there is a Kummer surface X' associated with the group scheme $G = \mu_2$ whose Artin invariant coincides with the given σ. Note that X' depends on the chosen embedding $h' : G \to \mathrm{Aut}_{C\times C}$. According to Proposition 3.3 it contains thirty (-2)-curves with simple normal crossings and dual graph as in Figure 1. By Lemma 5.1 such a configuration also exists on X. According to Theorem 4.1 the K3 surface X must be a Kummer surface associated with the group scheme G, for another embedding $h : G \to \mathrm{Aut}_{C\times C}$. □

6. Enriques surfaces and K3-like coverings

Keum [10] showed that every Kummer surface $X = \mathrm{Km}(A)$ over the field $k = \mathbb{C}$ of complex numbers admits a free action of the group $H = \mathbb{Z}/2\mathbb{Z}$. Hence the quotient $Y = X/H$ is an Enriques surface, and the Kummer surface arises as its *K3-covering*. The result was extend to characteristic $p \neq 2$ by Jang [8]. If the abelian surface is a product $A = E \times E'$, the *Lieberman involutions*

$$E \times E' \longrightarrow E \times E', \quad (a, a') \longmapsto (a + \zeta, -a' + \zeta')$$

induce such free H-actions on $X = \mathrm{Km}(E \times E')$, where ζ, ζ' denote non-zero 2-division points.

We now establish an analogue for Kummer surfaces associated with group schemes. The case of Artin invariant $\sigma = 1$ was already treated by the first author [12].

Theorem 6.1. *Let X be a supersingular K3 surface with Artin invariant $\sigma \leq 3$ over an algebraically closed ground field k of characteristic $p = 2$. Then there is a contraction $X \to X'$ of twelve (-2)-curves such that the normal K3 surface X' is the K3-like covering of some simply-connected Enriques surface Y.*

Proof. According to Theorem 5.2, our K3 surface can be written as a Kummer surface $X = \mathrm{Km}(C \times C)$ associated with the group scheme $G = \mu_2$. By Theorem 4.1 it contains a configuration of thirty (-2)-curves E_{ij}, C_r, C'_r, E_s with dual graph in Figure 1. The curves

$$E_{11} + E_{14} + E_{41} + E_{42} + E_{22} + E_{23} + E_{33} + E_{34} + \sum_{i=1}^{4}(C_i + C'_i)$$

form a singular fiber of type I_{16}. Let $f : X \to \mathbb{P}^1$ be the resulting elliptic fibration. The connected curve $E_0 + \ldots + E_3$ is vertical with respect to this fibration, hence belongs to some fiber, say $f^{-1}(\infty)$. Let $s \geq 1$ be the number of irreducible components in this fiber. Then the trivial lattice $T(X/\mathbb{P}^1)$ has rank $r \geq 15 + (s-1) + 2 = s + 16$. Since this lattice has rank at most $\rho = 22$, we infer that $s \leq 6$. It follows that $f^{-1}(\infty)$ has Kodaira symbol I_n^* with $n \leq 1$, and that it is the only non-reduced fiber.

Let $X \to X'$ be the contraction of the eight disjoint curves $C_1, \ldots, C_4, C'_1, \ldots, C'_4$ together with $E_0 + \ldots + E_3$. Then X' is a normal K3 surface with eight rational double points of type A_1, together with a rational double point of type D_4. Moreover, $f : X \to \mathbb{P}^1$ descends to a fibration $f' : X' \to \mathbb{P}^1$. We will apply a result of the second author ([21], Theorem 6.4) to see that X' is a K3-like covering; for this we have to verify certain conditions (E0), (E2), (E5) and (E6):

First note that the fibers of $f' : X' \to \mathbb{P}^1$ are reduced, because $f^{-1}(\infty)$ is the only non-reduced fiber for $f : X \to \mathbb{P}^1$ and all components with multiplicity $m \neq 1$ are contracted by $X \to X'$. Thus condition (E0) is satisfied.

Recall that the *Tjurina number* of an isolated hypersurface singularity given by a power series equation $f(t_1, \ldots, t_n) = 0$ is the colength of the ideal $\mathfrak{a} \subset k[[t_1, \ldots, t_n]]$ generated by f and its partial derivatives $\partial f/\partial t_i$, compare the discussion in [21], Section 1. Each A_1-singularity is formally isomorphic to $z^2 + xy = 0$, whence its Tjurina number is $\tau = 2$. According to Artin's computations, there are exactly two formal isomorphism classes of D_4-singularities, one of which is simply-connected ([3], Section 3). In light of the universal homeomorphism $\mathbb{P}^1 \times \mathbb{P}^1 \to (C \times C)/G$, our rational double point must be simply-connected, hence is formally given by $z^2+x^2y+xy^2 = 0$, with Tjurina number $\tau = 8$. We see that the Tjurina numbers of the singularities add up to $n = 24$, so (E2) holds. Moreover, all singular local rings $\mathscr{O}_{X',a}$ are Zariski singularities, hence (E6) is true. In particular the stalks $\Theta_{X',a}$ of the tangent sheaf are isomorphic to $\mathscr{O}_{X',a}^{\oplus 2}$, see [21], Corollary 1.6.

The configuration $C_1 + E_{13} + C_3' + E_{43} + C_4 + E_{42} + C_2' + E_{12}$ defines another elliptic fibration $g : X \to \mathbb{P}^1$, which descends to an elliptic fibration $g' : X' \to \mathbb{P}^1$. It follows that there is an elliptic curve $F \subset X'$ that is horizontal for $f' : X' \to \mathbb{P}^1$. This establishes condition (E5).

Summing up, [21], Theorem 6.4 applies, and we conclude that $\Theta_{X'/k} = \mathscr{O}_{X'}^{\oplus 2}$, all members of the restricted Lie algebra $\mathfrak{g} = H^0(X, \Theta_{X'/k})$ are p-closed, and for almost all non-zero vectors $\delta \in \mathfrak{g}$ the corresponding local group scheme $H \subset \mathrm{Aut}_{X'/k}$ of order p acts freely, such that the quotient $Y = X'/H$ is an Enriques surface, with X' as the K3-like covering. ☐

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