



THE p -RADICAL CLOSURE OF LOCAL NOETHERIAN RINGS

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Given a local noetherian ring R whose formal completion is integral, we introduce and study the p -radical closure R^{prc} . Roughly speaking, this is the largest purely inseparable R -subalgebra inside the formal completion \hat{R} . It turns out that the finitely generated intermediate rings $R \subset A \subset R^{\text{prc}}$ have rather peculiar properties. They can be used in a systematic way to provide examples of integral local rings whose normalization is nonfinite, that do not admit a resolution of singularities, and whose formal completion is nonreduced.

Introduction

In commutative algebra and algebraic geometry, one frequently considers *integral closures* $R \subset R'$ of an integral noetherian domain R with respect to finite extensions $F \subset F'$ of the field of fractions $F = \text{Frac}(R)$. The whole theory of number fields and their rings of integers hinges on this process. Moreover, the induced morphism $\text{Spec}(R') \rightarrow \text{Spec}(R)$ can be regarded as a higher-dimensional analogue of branched coverings of Riemann surfaces.

Indeed, under very general assumptions, the R -algebras R' are finite, as in the case of number rings and Riemann surfaces. This holds, for example, if the field extension $F \subset F'$ is separable, or if the ring R is essentially of finite type over a field, or a complete local noetherian ring. This finiteness is also a consequence of the defining conditions for *excellent rings*, a class introduced by Grothendieck [12] that is stable under forming algebras of finite type, localizations and completions. A recent overview was given by Raynaud and Laszlo in [22, exposé I].

However, the finiteness property does not hold for each and every noetherian ring, not even for all discrete valuation rings. To my knowledge, the first counterexamples were devised by Akizuki [1] in characteristic zero, see also Reid's discussion [34, Section 9.5], and Schmidt [23] in characteristic $p > 0$. A counterexample that is a discrete valuation ring was given by Nagata [29, Example (E3.3) on page 207]: the tensor product ring $R = k^p[[T]] \otimes_{k^p} k$, which is a discrete valuation ring with $\hat{R} = k[[T]]$. Here k is a field of positive characteristic $p > 0$ with infinite p -degree, that is, $[k : k^p] = \infty$. He also found intermediate discrete valuation rings $R \subset A \subset \hat{R}$ with $\hat{A} = k[[T]]$ so that $A \subset \hat{A}$ is an infinite integral extension of finite degree p . The study of intermediate rings $R \subset A \subset \hat{R}$ initiated by Nagata has a long tradition: Bennett [2] gave a detailed study, via birational geometry, of bad one-dimensional local noetherian rings. Olberding [32] analyzed which discrete valuation rings appear as normalization of such rings. Heinzer, Rotthaus and Wiegand studied various aspects of intermediate rings $R \subset A \subset \hat{R}$ in a series of papers, among others in [17; 18; 19; 20].

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Recall the following terminology: An integral domain R is called *japanese* if for every finite extension $F \subset F'$ of its field of fractions, the integral closure R' is a finite R -algebra. Now let us restrict our consideration to local noetherian domains R . Then R' is a finite R -algebra provided the *generic formal fiber* $\hat{R} \otimes_R F$ is geometrically reduced [4, chapitre IX, §4, no. 3, proposition 3]. In fact, Rees [33] showed that the generic formal fiber is reduced if and only if all the intermediate rings $R \subset R' \subset \text{Frac}(R)$ that are finitely generated R -algebras have finite normalization. A comprehensive account on rings without finite normalization is given by Olberding [31]. Recently, Kollár [24] reformulated the theory of normalizations in terms of pairs (X, Z) consisting of noetherian scheme X and a closed subscheme $Z \subset X$, which yields notions that are preserved under formal completion and work well even for schemes with nonreduced formal fibers.

The goal of this paper is to introduce and study the *p -radical closure* $R \subset R^{\text{prc}}$ for a local noetherian ring R whose formal completion \hat{R} is integral. If \hat{R} is normal, this equals the integral closure of R with respect to the relative p -radical closure of $\text{Frac}(R) \subset \text{Frac}(\hat{R})$, where $p \geq 1$ is the characteristic exponent of the fields of fractions. In some sense, it might be viewed as the purely inseparable analogue of the henselization $R^h \subset \hat{R}$. It would be interesting to compute the p -radical closure in concrete examples. Here, however, our main goal is to uncover interesting general facts and formal consequences for rings having nontrivial $R \subsetneq R^{\text{prc}}$.

It turns out that the *intermediate rings* between R and its p -radical closure R^{prc} have amazing properties, and yield, in a rather systematic way, examples of local noetherian rings with bad behavior. Key features are as follows: If $R \subsetneq A \subset R^{\text{prc}}$ is an intermediate ring that is finite as an R -algebra, then its formal completion \hat{A} contains nilpotent elements, and we actually have $(\hat{A})_{\text{red}} = \hat{R}$. The latter is our key observation, and most of the paper hinges on this result. These rings A are also examples of local noetherian rings that do not admit resolutions of singularities. In the one-dimensional situation, they must be nonnormal, and the normalization map is nonfinite.

However, if $R \subsetneq B \subset R^{\text{prc}}$ is an intermediate ring that is noetherian, with reduced generic formal fibers, then $\hat{B} = \hat{R}$ holds and $R \subset B$ is not finite; it follows that the extension $R \subset B$ is faithfully flat, and B inherits regularity and Cohen–Macaulay properties from R . It would be interesting to know under what circumstances the family of such intermediate rings is cofinal. From this it would follow that the whole p -radical closure R^{prc} belongs to this family, at least if R is regular.

In the case where R is a discrete valuation ring, the theory simplifies a lot: This is due to the Krull–Akizuki theorem, and the fact that there is only one relevant formal fiber $\hat{R} \otimes_R \kappa(\mathfrak{p})$, namely the generic formal fiber.

The paper is organized as follows: In the first section, we review several well-known facts pertaining to extensions and formal completions of local rings used throughout. Since we have to cope with non-noetherian rings, we do not restrict our discussion to the noetherian case. In [Section 2](#) we introduce the p -radical closure for local noetherian rings R that are formally integral, and establish its basic properties. The heart of the paper is [Section 3](#): here we study the intermediate rings between R and its p -radical closure R^{prc} . In the final [Section 4](#), we specialize our findings to the case of discrete valuation rings.

1. Extensions and completions of local rings

Here we recall some well-known basic facts on local rings and their formal completions that will be used throughout the paper. Even though we are mainly interested in noetherian rings, the nonnoetherian rings

are included in our discussion, because they may show up naturally when it comes to taking integral closures.

A homomorphism $\varphi : R \rightarrow A$ between local rings is *local* if $\varphi^{-1}(\mathfrak{m}_A) = \mathfrak{m}_R$. An *extension of local rings* is an injective local homomorphism $\varphi : R \rightarrow A$ between local rings, and we then usually write $R \subset A$. In general, one has the following two *numerical invariants*

$$e = \text{length}_A(A/\mathfrak{m}_R A), \quad \text{and} \quad f = [\kappa(A) : \kappa(R)],$$

which we refer to as the *ramification index* $e \geq 1$ and the *residual degree* $f \geq 1$ for $\varphi : R \rightarrow A$. Here $\kappa(R) = R/\mathfrak{m}_R$ and $\kappa(A) = A/\mathfrak{m}_A$ are the residue fields, and the numerical invariants are regarded as elements from $\{1, 2, \dots\} \cup \{\infty\}$. If R is integral, with field of fractions $F = \text{Frac}(A)$, there is another invariant

$$n = [A \otimes_R F : F]$$

called the *degree* $n \geq 0$ of the homomorphism $R \rightarrow A$. In case that the R -algebra A flat, finitely presented and finite, that is, the underlying R -module is free of finite rank, the famous formula $n = ef$ holds. These numerical invariants originate from the theory of number fields and Riemann surfaces, and I find them quite useful in the general context. Throughout, we are particularly interested in extensions of local rings with invariants $e = f = 1$. In the context of valuation theory, such extension are called *immediate extensions*.

The *formal completion* of the local ring R is written as

$$\hat{R} = \varprojlim_{n \geq 0} R/\mathfrak{m}_R^n.$$

This is another local ring, with maximal ideal $\mathfrak{m}_{\hat{R}} = \mathfrak{m}_R \hat{R}$ and residue field $\kappa(\hat{R}) = \kappa(R)$ [5, chapitre III, §2, no. 13, proposition 19]. The canonical map $R \rightarrow \hat{R}$, $a \mapsto (a, a, \dots)$ is local, with numerical invariants $e = f = 1$. The local ring R is called *complete* if $R \rightarrow \hat{R}$ is bijective. Clearly, the kernel of the canonical map is the intersection $\bigcap_{n \geq 0} \mathfrak{m}_R^n$. It is trivial if R is noetherian, or more generally if R , viewed as a topological ring with respect to the \mathfrak{m}_R -adic topology, is Hausdorff. On the other hand, the kernel coincides with the maximal ideal if R is an fppf-local ring, as studied by Gabber and Kelly in [10, Section 3] and myself in [37, Section 4]. Using [5, chapitre III, §2, no. 10, corollaire 5], we immediately get:

Proposition 1.1. *The complete local ring \hat{R} is noetherian if and only if the cotangent space $\mathfrak{m}_R/\mathfrak{m}_R^2$ has finite dimension as vector space over the residue field $\kappa(R) = R/\mathfrak{m}_R$.*

If $\varphi : R \rightarrow A$ is a local homomorphism between local rings, we get an induced local map $\hat{\varphi} : \hat{R} \rightarrow \hat{A}$, making the diagram

$$\begin{array}{ccc} \hat{R} & \xrightarrow{\hat{\varphi}} & \hat{A} \\ \text{can} \uparrow & & \uparrow \text{can} \\ R & \xrightarrow{\varphi} & A \end{array}$$

commutative. Consequently, the residual degree $f \geq 1$ for $R \rightarrow A$ and $\hat{R} \rightarrow \hat{A}$ coincide. Furthermore:

Proposition 1.2. *If the fiber ring $A/\mathfrak{m}_R A$ is noetherian, then the ramification indices $e \geq 1$ for the local maps $R \rightarrow A$ and $\hat{R} \rightarrow \hat{A}$ coincide.*

Proof. The ramification indices in question are the lengths of the rings $B = A/\mathfrak{m}_R A$ and $B' = \hat{A}/\mathfrak{m}_{\hat{R}} \hat{A} = \hat{A}/\mathfrak{m}_R \hat{A}$. The latter is complete, so the canonical map $B \rightarrow B'$ factors over \hat{B} , and the resulting map $\hat{B} \rightarrow B'$ is bijective. Replacing A by B , we thus reduce to the situation where A is noetherian.

Now consider the complements $U \subset \text{Spec}(A)$ and $U' \subset \text{Spec}(\hat{A})$ of the closed point. The resulting morphism $f : U' \rightarrow U$ is quasicompact and faithfully flat. Let M be a finitely generated A -module, and \mathcal{F} be the quasicoherent sheaf on U obtained by restricting the quasicoherent sheaf \tilde{M} on $\text{Spec}(A)$. By fpqc-descent, the preimage $f^*(\mathcal{F})$ vanishes if and only if \mathcal{F} vanishes [14, exposé VIII, §1, corollaire 1.3]. It follows that M has finite length if and only if $\hat{M} = M \otimes_A \hat{A}$ has finite length. In this situation, $\mathfrak{m}_A^n M = 0$ for some $n \geq 0$, whence $\hat{M} = M$, and one easily infers that $\text{length}_A(M) = \text{length}_{\hat{A}}(\hat{M})$. Applying this for $M = A/\mathfrak{m}_R A$ yields the assertion. \square

The schematic fibers of the morphism $\text{Spec}(\hat{R}) \rightarrow \text{Spec}(R)$ are called the *formal fibers* of R . The corresponding rings $\hat{R} \otimes_R \kappa(\mathfrak{p})$, where $\mathfrak{p} \subset R$ are the prime ideals, are called the *formal fiber rings*. Note that these are noetherian rings, provided that \hat{R} is noetherian. They are endowed with the structure of an algebra over the fields $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$, but these algebras are usually not finitely generated. The closed fiber equals the spectrum of the residue field $k = \hat{R}/\mathfrak{m}_{\hat{R}} = R/\mathfrak{m}_R$, whence is of little interest. Rather important are the *generic formal fibers*, which are given by the rings $\hat{R} \otimes_R \kappa(\mathfrak{p})$, where $\mathfrak{p} \subset R$ are the minimal prime ideals. Let us recall:

Proposition 1.3. *Let R be a local noetherian ring. Then the complete local scheme $\text{Spec}(\hat{R})$ is reduced or irreducible if and only if the respective property holds for the local scheme $\text{Spec}(R)$ and its generic formal fiber.*

Proof. Suppose \hat{R} is reduced. Then so is the subring $R \subset \hat{R}$. Let $\mathfrak{p} \subset R$ be a minimal prime ideal, and write $S = R \setminus \mathfrak{p}$ for the ensuing multiplicative system. Being a localization of a reduced ring, the generic formal $\hat{R} \otimes_R \kappa(\mathfrak{p}) = S^{-1} \hat{R}$ is reduced.

Conversely, suppose that R and its generic formal fiber is reduced. We have to show that \hat{R} contains no embedded prime, and is regular in codimension zero. According to [27, Corollary in (9.B)], each associated point $\eta \in \text{Spec}(\hat{R})$ lies over a generic point of $\text{Spec}(R)$, and is generic in its formal fiber. Thus $\eta \in \text{Spec}(\hat{R})$ is generic. Moreover, the local ring corresponding to $\eta \in \text{Spec}(\hat{R})$ is regular, according to [12, corollaire 6.5.2]. Thus \hat{R} is reduced. The arguments for irreducibility are analogous, and left to the reader. \square

Note that it may happen that R is integral, yet there are embedded primes $\mathfrak{p} \subset \hat{R}$, as examples of Ferrand and Raynaud reveal [9]. Such behavior is ubiquitous, according to Lech [26]. In this situation, however, we must have $\mathfrak{p} \cap R = 0$, that is, $\text{Ass}(\hat{R})$ maps to the generic point of $\text{Spec}(R)$.

We say that R is *formally reduced* if the equivalent conditions for reducedness of Proposition 1.3 hold. In [29], the term *analytically unramified* was used. We prefer the former locution, because “unramified” now is used in algebraic geometry in a different way [13, §17]. We say that R is *formally integral* if the equivalent conditions for integrality hold. In [29], such rings are called *analytically irreducible*. The following observation, which is a consequence of [29], (18.3) and (18.4), applies in particular to the formal completion $R \subset \hat{R}$ of formally integral local noetherian rings:

Lemma 1.4. *Let $B \subset C$ be a faithfully flat extension of domains. Then we have $B = C \cap \text{Frac}(B)$ as subsets of $\text{Frac}(C)$.*

If $R \subset A$ is an extension of local noetherian rings, the induced map $\hat{R} \rightarrow \hat{A}$ stays injective provided that the R -algebra A is finite, by [5, chapitre III, §3, no. 4, théorème 3]. In general, however, the map is not injective, for example if A but not R is formally reduced. We shall encounter such behavior later. The following is a situation in which injectivity is preserved without any finiteness assumptions. The second assertion can be seen as a generalization of [31, Lemma 3.1], which dealt with discrete valuation rings:

Proposition 1.5. *Let $R \subset A$ be an extension of local noetherian rings with numerical invariants $e = f = 1$. Assume that $\dim(A) = \dim(R)$, and that R is formally integral. Then the induced map $\hat{R} \rightarrow \hat{A}$ is bijective, and the R -algebra A is faithfully flat. Moreover, for any intermediate ring $R \subset R' \subsetneq A$, the R' -algebra A is not finite.*

Proof. The local map $\hat{R} \rightarrow \hat{A}$ has invariants $e = f = 1$, by Proposition 1.2. Hence the map is surjective, according to [5, chapitre III, §2, no. 9, proposition 11]. Write $\hat{A} = \hat{R}/\mathfrak{a}$ for some ideal $\mathfrak{a} \subset \hat{R}$. Using

$$\dim(A) = \dim(R) = \dim(\hat{R}) \geq \dim(\hat{A}) = \dim(A),$$

we infer that $\dim(\hat{R}) = \dim(\hat{A})$. Since \hat{R} is integral, we can apply Krull's principal ideal theorem and conclude that $\mathfrak{a} = 0$, whence the local map $\hat{R} \rightarrow \hat{A}$ is bijective. This ensures, by [5, chapitre III, §3, no. 5, proposition 10], that the R -algebra A is faithfully flat.

Now assume that for some intermediate ring $R \subset R' \subset A$ the R' -algebra A is finite. We have to verify that $R' = A$. The map $\text{Spec}(A) \rightarrow \text{Spec}(R')$ is closed since it is finite. It is also dominant, whence surjective, because the homomorphism $R' \rightarrow A$ is injective. Consequently, R' is local, and $R' \subset A$ and whence $R \subset R'$ are extensions of local rings. By the Eakin–Nagata theorem ([8] or [30]), the local ring R' is noetherian. The extension of local rings $R' \subset A$ has invariants $e = f = 1$, because this holds for $R \subset A$. Taking formal completions thus gives a finite extension of complete local rings $\hat{R}' \subset \hat{A} = A \otimes_{R'} \hat{R}'$ with invariants $e = f = 1$, whence the inclusion is an equality. By faithfully flat descent, the inclusion $R' \subset A$ is an equality [14, exposé VIII, corollaire 1.3]. \square

Now suppose that our local noetherian ring R is integral, with field of fractions $F = \text{Frac}(R)$. Then R is called *japanese* if for every finite extension $F \subset F'$, the integral closure $R \subset R'$ inside F' is a finite R -algebra. This automatically holds if the ring R is normal and the field F has characteristic zero (see [27, Proposition 31.B]).

A ring R is called *universally japanese* if each integral R -algebra of finite type is japanese. Such rings are also known as *pseudogeometric*, *J-rings* or *Nagata rings*. A local ring R is called *excellent* if it is noetherian, universally catenary, and has geometrically regular formal fibers; such rings are universally japanese (see [12, scholie 7.8.3(i) et (vi)]).

2. The p -radical closure

Throughout this section, R denotes a local noetherian ring that is formally integral. The extension of local rings $R \subset \hat{R}$ induces an extension of fields of fractions $\text{Frac}(R) \subset \text{Frac}(\hat{R})$. We write $p \geq 1$ for the *characteristic exponent* of these fields. Recall that $p = 1$ if the prime field is \mathbb{Q} , and equals the

characteristic $p \geq 2$ otherwise. In the latter case, the canonical map $\mathbb{Z} \rightarrow R \subset \text{Frac}(R)$ yields an inclusion $\mathbb{F}_p \subset R$. Consider the subset

$$R^{\text{prc}} = \{b \in \hat{R} \mid b^{p^\nu} \in R \text{ for some exponent } \nu \geq 0\}.$$

Clearly, this subset is a subring $R^{\text{prc}} \subset \hat{R}$ containing R ; we call it the p -radical closure of R . Note that if $p = 1$, that is, the fields of fractions have characteristic zero, we have $R^{\text{prc}} = R$, and the situation is of little interest. In general, set $F = \text{Frac}(R)$ and let $F \subset F_\infty \subset \text{Frac}(\hat{R})$ be the relative p -radical closure in the sense of field theory.

Proposition 2.1. *We have $R^{\text{prc}} = \hat{R} \cap F_\infty$ inside the field of fractions $\text{Frac}(\hat{R})$. This is also the integral closure of R with respect to $R \subset F_\infty$, provided that \hat{R} is normal.*

Proof. The inclusion $R^{\text{prc}} \subset \hat{R} \cap F_\infty$ is obvious. For the reverse inclusion, suppose $b \in \hat{R}$ is a root of a polynomial $T^{p^\nu} - a$ with $a \in F$. The task is to verify $a \in R$. Indeed, we have $a = b^{p^\nu} \in \hat{R}$, and [Lemma 1.4](#) tells us that $a \in \hat{R} \cap F = R$. Now suppose that \hat{R} is normal, and let $R \subset R' \subset F_\infty$ be the integral closure. Then $R' \subset \hat{R}$, because the latter is normal, and consequently $R^{\text{prc}} \subset R' \subset \hat{R} \cap F_\infty$. Since the right-hand side is contained in R^{prc} , the assertion follows. \square

The p -radical closure is closely related to the generic formal fiber:

Proposition 2.2. *Consider the following three conditions:*

- (i) *The inclusion $R \subset R^{\text{prc}}$ is an equality.*
- (ii) *The generic formal fiber of R is geometrically reduced.*
- (iii) *The local ring R is japanese.*

Then (ii) \Rightarrow (iii) holds. If R is normal, we also have (i) \Leftarrow (ii). The converse (i) \Rightarrow (ii) holds provided that \hat{R} is normal and the field extension $F \subset \text{Frac}(\hat{R})$ can be written as an purely inseparable extension followed by a separable extension.

Proof. The implication (ii) \Rightarrow (iii) is [[4](#), chapitre IX, §4, no. 3, proposition 3]. Now suppose that R is normal, and that (ii) holds. Set $F = \text{Frac}(R)$, and write $B = \hat{R} \otimes_R F$ for the generic formal fiber ring. Suppose that $R^{\text{prc}} \neq R$. Then there is some $b \in R^{\text{prc}}$ with $b \notin R$ but $a = b^p \in R$. Since R is normal, we also have $b \notin F$, and get a field extension $F \subsetneq F(a^{1/p})$. Hence the element $c = b \otimes 1 - 1 \otimes a^{1/p}$ from the ring $B \otimes_F F(a^{1/p})$ is nonzero and satisfies $c^p = 0$, thus B is not geometrically reduced.

Finally, suppose that \hat{R} is normal, that the field extension $F \subset \text{Frac}(\hat{R})$ can be written as a purely inseparable extension $F \subset F'$ followed by a separable extension $F' \subset \text{Frac}(\hat{R})$, and that condition (i) holds, that is, $R = R^{\text{prc}}$. We shall verify (ii). Clearly, the intermediate field F' coincides with the relative p -radical closure F_∞ . First, we check that $F = F_\infty$: Suppose $b/b' \in \text{Frac}(\hat{R})$ is an element with $(b/b')^p = a/a'$ for some $a, a' \in R$. Then $(a'b/b')^p = aa'^{p-1} \in R \subset \hat{R}$. Since \hat{R} is normal, we must have $a'b/b' \in \hat{R}$, and thus $a'b/b' \in R^{\text{prc}} = R$. Consequently $b/b' = (a'b/b')/a' \in \text{Frac}(R) = F$. This shows $F = F_\infty$, and it follows that the field extension $F \subset \text{Frac}(\hat{R})$ is separable.

Being a localization of the integral ring \hat{R} , the formal fiber ring $B = \hat{R} \otimes_R F$ is integral, with $\text{Frac}(B) = \text{Frac}(\hat{R})$. For any field extension $F \subset F'$, the inclusion $B \subset \text{Frac}(\hat{R})$ induces an inclusion $B \otimes_F F' \subset \text{Frac}(\hat{R}) \otimes_F F'$. The right-hand side is reduced, because the field extension $F \subset \text{Frac}(\hat{R})$ is separable, and thus the left-hand side is reduced as well. Hence B is geometrically reduced. \square

Note that there are nonseparable field extension $F \subset E$ whose relative p -radical closure is $F_\infty = F$. There are examples where $F \subset E$ is finite (the *exceptional field extensions* from [6, Chapter V, Exercises 1 and 2 for §7]), or where $F \subset E$ is relatively algebraically closed (related to *quasifibrations* $X \rightarrow B$ of proper normal schemes with geometrically nonreduced generic fiber, compare the discussion in [36]).

The following functoriality property is immediate from the definition of the p -radical closure:

Proposition 2.3. *Let $R \subset A$ be an extension of local noetherian rings, both of which are formally integral. Then the induced map $\hat{R} \rightarrow \hat{A}$ sends R^{prc} to A^{prc} .*

Note that the induced map $\hat{R} \rightarrow \hat{A}$ is not necessarily injective; this phenomenon was analyzed by Hübl [21]. If the map is injective, one may view both R^{prc} and A as subrings inside \hat{A} . Combining with Proposition 2.2, we get the following property of the p -radical closure:

Corollary 2.4. *Assumptions as in Proposition 2.3. Additionally suppose that the induced map $\hat{R} \rightarrow \hat{A}$ is injective. If the generic formal fiber of A is geometrically reduced and A is normal, then we have $R^{\text{prc}} \subset A$ as subrings inside \hat{A} .*

Let us call a polynomial over a field *purely inseparable* if it has exactly one root in the algebraic closure. With an additional hypothesis, we may regard the p -radical closure as a *purely inseparable algebraic closure*:

Proposition 2.5. *Suppose the extension of local rings $R^{\text{prc}} \subset \hat{R}$ is flat. Write $F = \text{Frac}(R)$. Then $R^{\text{prc}} \subset \hat{R}$ is the set of all elements $b \in \hat{R}$ that satisfy an algebraic equation $f(b) = 0$ for some nonzero polynomial $f \in R[T]$ that is purely inseparable as polynomial over F .*

Proof. Clearly, every $b \in R^{\text{prc}}$ satisfies the condition. Conversely, suppose that $b \in \hat{R}$ is the root of such a polynomial $f \in R[T]$. Since this polynomial is purely inseparable it must be of the form $f(T) = c(T - b)^n$ for some nonzero $c \in R$ and some integer $n \geq 1$. Write $n = p^\nu m$ for some exponent $\nu \geq 0$ and $p \nmid m$. Then

$$f(T) = c(T^{p^\nu} - b^{p^\nu})^m = cT^{p^\nu m} + cmb^{p^\nu} T^{p^\nu(m-1)} + \dots + cb^{p^\nu m}.$$

Comparing coefficients and using $m \in R^\times$, we see $cb^{p^\nu} \in R$. The element $cb \in \hat{R}$ thus has $(cb)^{p^\nu} \in R$, hence $cb \in R^{\text{prc}}$, so $b = cb/c \in \hat{R} \cap \text{Frac}(R^{\text{prc}})$. Now we use that assumption that $R^{\text{prc}} \subset \hat{R}$ is flat. It follows from Proposition 3.1 below that it is faithfully flat; whence Lemma 1.4 ensures that $\hat{R} \cap \text{Frac}(R^{\text{prc}}) = R^{\text{prc}}$. \square

Note that this interpretation makes no reference to the characteristic exponent; perhaps this would lead to a meaningful definition of p -radical closure for arbitrary local noetherian rings, which need not be formally integral.

Moreover, it reveals a striking analogy between p -radical closure and *henselization*: if A is any local noetherian ring, with henselization A^h , the formal completion \hat{A} is also henselian, and the universal property of henselization gives inclusions $A \subset A^h \subset \hat{A}$, see [13, théorème 18.6.6]. Note that if A is formally irreducible, then A^h is irreducible, which in turn means that A is unibranch.

If A is formally normal and universally japanese, then the henselization A^h coincides with the *algebraic closure* of A inside the formal completion \hat{A} , according to [29, Theorem 44.1]. In other words, A^h is the set of elements $b \in \hat{A}$ that satisfy an algebraic equation $f(b) = 0$ for some nonzero $f \in A[T]$; note that here f is not necessarily monic. This even holds if A is merely integral, and its formal fibers are

reduced, and the generic formal fiber is normal, according to [25, Proposition 2.10.2]. This was further generalized to Hensel couples in [11, Theorem 3, see also Remark 2].

We have the following variant, which says that for our formally integral local noetherian ring R the henselization is the *separable algebraic closure* inside the formal completion, under a rather mild assumption:

Theorem 2.6. *Suppose the henselization R^h has geometrically connected generic formal fiber. Write $F = \text{Frac}(R)$. Then $R^h \subset \hat{R}$ is the set of all elements $b \in \hat{R}$ that satisfy an algebraic equation $f(b) = 0$ for some nonzero polynomial $f \in R[T]$ that is separable as polynomial over F .*

Proof. First, we check that each $b \in R^h$ satisfies the conditions. By definition of the henselization in [13, Section 18.6] we have $b \in B_{\mathfrak{p}}$, where B is an étale R -algebra, and $\mathfrak{p} \subset B$ is a prime ideal lying over the maximal ideal $\mathfrak{m}_R \subset R$. After localization, we may assume that B is integral, because R is unibranch. Let $g \in F[T]$ be the minimal polynomial for the element $b \in \text{Frac}(B)$. Then g is separable, and multiplying with a suitable nonzero element $r \in R$ yields a polynomial $f(T) = rg(T)$ with coefficients in R . The canonical inclusions $B_{\mathfrak{p}} \subset R^h \subset \hat{R}$ reveal that b satisfies the desired conditions. Note that this holds without the assumption on the generic formal fiber of R^h .

Conversely, suppose that we have $b \in \hat{R}$ satisfying $f(b) = 0$ for some polynomial

$$f(T) = cT^m + \lambda_{m-1}T^{m-1} + \dots + \lambda_0$$

with coefficients in R and leading coefficient $c \neq 0$, such that $f(T)$ is separable over F . Our goal is to show $b \in R^h$. Let us first assume that the algebraic equation is integral, that is, $c = 1$. Consider the R^h -subalgebra $B = R^h[b]$ inside \hat{R} . The ring B is integral, and the R^h -algebra B is finite. Let $n = [B : R^h]$. It suffices to verify $n = 1$, because then $b \in \hat{R} \cap \text{Frac}(R^h) = R^h$, by Lemma 1.4. Indeed, it follows from [13, théorème 18.6.6] that the canonical map $\widehat{R^h} \rightarrow \hat{R}$ is bijective, so we may regard the inclusion $R^h \subset \hat{R}$ as the formal completion. Seeking a contradiction, we now suppose $n \geq 2$.

By assumption, the generic formal fiber of R^h is geometrically connected. It follows that $B \subset B \otimes_{R^h} \hat{R}$, $x \mapsto x \otimes 1$ corresponds to a morphism on schemes whose generic fiber is connected. On the other hand, using that $B \subset \hat{R}$, we get a bijection

$$(B \otimes_{R^h} B) \otimes_B \hat{R} \rightarrow B \otimes_{R^h} \hat{R}, \quad x \otimes y \otimes z \mapsto xy \otimes z.$$

Since $\text{Frac}(R^h) \subset \text{Frac}(B)$ is a separable extension of degree $n \geq 2$, it follows from the Galois correspondence that the inclusion $B \subset B \otimes_{R^h} B$, $y \mapsto 1 \otimes y$ corresponds to a morphism of schemes whose generic fiber is disconnected. In turn, the composite map

$$B \rightarrow B \otimes_{R^h} B \rightarrow (B \otimes_{R^h} B) \otimes_B \hat{R} \rightarrow B \otimes_{R^h} \hat{R},$$

which is nothing but the canonical inclusion $B \subset B \otimes_{R^h} \hat{R}$, induces a morphism of schemes with disconnected generic fiber, contradiction.

We finally treat the general case, where the leading coefficient $c \in R$ an arbitrary nonzero element. Clearly, the element $b' = cb$ from \hat{R} is a root for the monic polynomial

$$T^m + c^0 \lambda_{m-1} T^{m-1} + \dots + c^{m-1} \lambda_0 = c^{m-1} f(T/c),$$

which is separable over F . By the preceding paragraph, we have $b' \in R^h$, and in turn $b = b'/c \in \hat{R} \cap \text{Frac}(R^h)$. Since $R^h \subset \hat{R}$ is faithfully flat, [Lemma 1.4](#) applies, which gives $\hat{R} \cap \text{Frac}(R^h) = R^h$. \square

Note that the formal fibers of R^h indeed are geometrically connected provided they are geometrically normal, according to [\[13, théorème 18.9.1\]](#). I do not know an example where the generic formal fiber of R^h is geometrically disconnected. This does not happen if R^h has the approximation property, and the latter implies, according to [\[35\]](#), that R^h and whence R are excellent.

Let me also make a comment on étale cohomology: Given a scheme X , we denote by $\text{Et}(X)$ the site of all étale morphisms $U \rightarrow X$, endowed with the Grothendieck topology whose covering families $(U_\alpha \rightarrow U)_{\alpha \in I}$ are those with $\bigcup_{\alpha \in I} U_\alpha \rightarrow U$ is surjective. Let X_{et} be the resulting topos of sheaves. For any abelian étale sheaf F on X , that is, an abelian sheaf on the site $\text{Et}(X)$, in other words an object in the category X_{et} , one gets the cohomology groups $H^i(X_{\text{et}}, F)$. Now set $X = \text{Spec}(R)$ and $X^{\text{prc}} = \text{Spec}(R^{\text{prc}})$. According to [\[14, exposé IX, théorème 4.10\]](#), together with footnote (5), the pullback functor $\text{Et}(X) \rightarrow \text{Et}(X^{\text{prc}})$, $U \mapsto U \times_X X^{\text{prc}}$ is an equivalence of categories. It induces a continuous map of topoi

$$\epsilon = (\epsilon_*, \epsilon^*) : (X^{\text{prc}})_{\text{et}} \rightarrow X_{\text{et}},$$

where the adjoint functors ϵ_*, ϵ^* are equivalences of categories. In particular, computing étale cohomology on X amounts to the same as computing it on X^{prc} .

3. Intermediate rings

Let R be a local noetherian ring that is formally integral. We now want to study intermediate rings between R and its p -radical closure R^{prc} . Let us first record:

Proposition 3.1. *For every intermediate ring $R \subset A \subset R^{\text{prc}}$, the induced morphisms*

$$\text{Spec}(R^{\text{prc}}) \rightarrow \text{Spec}(A) \quad \text{and} \quad \text{Spec}(A) \rightarrow \text{Spec}(R)$$

are integral universal homeomorphisms. The extensions of local rings $R \subset A \subset R^{\text{prc}}$ have numerical invariant $f = 1$. Moreover, the local rings A are separated with respect to the \mathfrak{m}_A -adic topology.

Proof. By definition, the morphisms are integral. By the going-up theorem, they are universally closed. Since the ring homomorphisms are injective, the induced maps on affine schemes are dominant, whence surjective. For each point $x \in \text{Spec}(R)$, the fibers of $\text{Spec}(R^{\text{prc}}) \rightarrow \text{Spec}(R)$ and $\text{Spec}(A) \rightarrow \text{Spec}(R)$ over x are affine, and given by algebras C over $K = \kappa(x)$ where each $c \in C$ has some $c^{p^n} \in K$. It follows that $C \otimes_K K^{1/p^\infty}$ is a ring containing precisely one prime ideal, with trivial residue field extension. In light of [\[16, proposition 3.7.1\]](#), the morphisms $\text{Spec}(R^{\text{prc}}) \rightarrow \text{Spec}(A)$ and $\text{Spec}(A) \rightarrow \text{Spec}(R)$ are universally bijective. Being universally closed and universally bijective, they are universal homeomorphisms.

The assertion on residue fields holds, because all rings are contained in \hat{A} , and the extension of local rings $A \subset \hat{A}$ has trivial residue field extension. Finally, suppose we have some $f \in \bigcap_{n \geq 0} \mathfrak{m}_A^n$. Using the extension of local rings $A \subset \hat{R}$, we see that $f \in \bigcap_{n \geq 0} \mathfrak{m}_{\hat{R}}^n = 0$, thus A is separated. \square

In particular, such A are integral local rings, the topological space $\text{Spec}(A)$ is noetherian, with $\dim(A) = \dim(R)$. Moreover, $R \subset A$ is an extension of local rings, and we thus get an induced map $\hat{R} \rightarrow \hat{A}$ on formal completions. Note, however, that there is no a priori reason that A should be noetherian.

Now consider the direct system $A_\lambda \subset R^{\text{prc}}$, $\lambda \in L$ of all finite R -subalgebras, that is, the underlying R -module is finitely generated. The rings A_λ are noetherian by Hilbert's basis theorem. Clearly, the direct system is filtered, and we have

$$R^{\text{prc}} = \bigcup_{\lambda \in L} A_\lambda = \varinjlim_{\lambda \in L} A_\lambda.$$

The index set, viewed as an ordered set, has a smallest element $\lambda_{\min} \in L$. To simplify notation, we denote this smallest element as $\lambda_{\min} = 0$, such that $A_0 = R$. Let us write $\hat{A}_\lambda = \widehat{A_\lambda}$ for the formal completions of the local noetherian rings A_λ , and consider the *reduced formal completions*

$$(\hat{A}_\lambda)_{\text{red}} = \hat{A}_\lambda / \text{Nil}(\hat{A}_\lambda)$$

and the composite map $\hat{R} \rightarrow \hat{A}_\lambda \rightarrow (\hat{A}_\lambda)_{\text{red}}$. The following is our key result:

Theorem 3.2. *For each $\lambda \in L$, the composite map $\hat{R} \rightarrow (\hat{A}_\lambda)_{\text{red}}$ is bijective.*

Proof. The assertion is trivial if $R^{\text{prc}} = R$. In light of [Proposition 2.2](#), it thus suffices to treat the case that $F = \text{Frac}(R)$ has positive characteristic $p > 0$, such that R contains the prime field \mathbb{F}_p . First note that since $R \subset A_\lambda$ is finite, the canonical map

$$A_\lambda \otimes_R \hat{R} \rightarrow \hat{A}_\lambda, \quad x \otimes y \mapsto xy$$

is bijective [[27](#), Theorem 55 on page 170]. Now we use that the R -algebra A_λ is contained in \hat{R} , and observe that the resulting map

$$(A_\lambda \otimes_R A_\lambda) \otimes_{A_\lambda} \hat{R} \rightarrow A_\lambda \otimes_R \hat{R}, \quad (x \otimes y) \otimes z \mapsto xy \otimes z$$

is bijective. Clearly, the composite surjection

$$(A_\lambda \otimes_R A_\lambda) \otimes_{A_\lambda} \hat{R} \rightarrow A_\lambda \otimes_R \hat{R} \rightarrow \hat{A}_\lambda \rightarrow (\hat{A}_\lambda)_{\text{red}}$$

factors over $\tilde{R} = (A_\lambda \otimes_R A_\lambda)_{\text{red}} \otimes_{A_\lambda} \hat{R}$. Applying [Lemma 3.11](#) below with $B = R$ and $C = A_\lambda$, we infer that the mapping $A_\lambda \rightarrow (A_\lambda \otimes_R A_\lambda)_{\text{red}}$, $a \mapsto a \otimes 1$ is bijective. Consequently, we get a surjection

$$\hat{R} \rightarrow A_\lambda \otimes_{A_\lambda} \hat{R} \rightarrow (A_\lambda \otimes_R A_\lambda)_{\text{red}} \otimes_{A_\lambda} \hat{R} \rightarrow (\hat{A}_\lambda)_{\text{red}}.$$

This map is given by $a \mapsto 1 \otimes a \mapsto (1 \otimes 1) \otimes a \mapsto a$, whence coincides with the canonical map $\hat{R} \rightarrow (\hat{A}_\lambda)_{\text{red}}$. Therefore, the latter is surjective.

Let $d \geq 0$ be the Krull dimension of the local noetherian ring R . In light of [Proposition 3.1](#), this is also the dimension of A_λ , and thus of $(\hat{A}_\lambda)_{\text{red}}$. By assumption, the ring \hat{R} is integral, whence the surjective map $\hat{R} \rightarrow (\hat{A}_\lambda)_{\text{red}}$ has trivial kernel, by Krull's principal ideal theorem. \square

Corollary 3.3. *For each $\lambda \neq 0$, the intermediate rings A_λ is an integral local noetherian ring whose generic formal fiber is nonreduced, and whose spectrum $\text{Spec}(A_\lambda)$ does not admit a resolution of singularities.*

Proof. Suppose the generic formal fiber of A_λ is reduced. By [Proposition 1.3](#), this means that \hat{A}_λ is reduced. In light of the Theorem, the map $\hat{A} \rightarrow \hat{A}_\lambda$ is bijective. By faithfully flat descent [[14](#), exposé VIII, corollaire 1.3], the inclusion $A \subset A_\lambda$ is an equality. But this means $\lambda = 0$, contradiction.

Thus the generic formal fiber of A_λ is nonreduced, and in particular nonregular. According to [12, proposition 7.9.3] this implies that the scheme $\text{Spec}(A_\lambda)$ admits no resolution of singularities. \square

We see that formally integral local noetherian rings R with $R \subsetneq R^{\text{prc}}$ give rise, in a rather systematic way, to families of integral local noetherian rings without resolutions of singularities. In dimension one, this means:

Corollary 3.4. *Suppose $\dim(R) = 1$. Then for each $\lambda \neq 0$, the intermediate rings A_λ is an integral local noetherian ring with $\dim(A_\lambda) = 1$ whose normalization $A_\lambda \subset A'_\lambda$ is not a finite extension.*

Proof. If the normalization $A_\lambda \subset A'_\lambda$ is finite, then the normal one-dimensional ring A'_λ is noetherian, thus a Dedekind domain. Hence $\text{Spec}(A'_\lambda) \rightarrow \text{Spec}(A_\lambda)$ is a resolution of singularities, in contradiction with Corollary 3.3. \square

Now let $R \subset B \subset R^{\text{prc}}$ be an arbitrary intermediate ring. If B is not a finite R -algebra, it may or may not be noetherian. Indeed, Nagata [29] describes on page 207 examples constructed from regular local rings of the form $k^p[[x_1, \dots, x_n]] \otimes_{k^p} k$, which yield nonnoetherian B : In Example 4 the ring R is integral of dimension two, and in Example 5 we have R regular of dimension three. In any case, $R \subset B$ is an extension of local rings, hence we get an induced map on formal completions $\hat{R} \rightarrow \hat{B}$. Here $\hat{B} = \varprojlim B/\mathfrak{m}_B^n$.

Corollary 3.5. *The image of the canonical map $B \rightarrow (\hat{B})_{\text{red}}$ is contained in the image of $\hat{R} \rightarrow (\hat{B})_{\text{red}}$.*

Proof. Let $b \in B$, and write $\bar{b} \in (\hat{B})_{\text{red}}$ for its image. Choose some index $\lambda \in L$ with $b \in A_\lambda \subset B$. Consider the commutative diagram

$$\begin{array}{ccccc} \hat{R} & \longrightarrow & (\hat{A}_\lambda)_{\text{red}} & \longrightarrow & (\hat{B})_{\text{red}} \\ \uparrow & & \uparrow & & \uparrow \\ R & \longrightarrow & A_\lambda & \longrightarrow & B \end{array}$$

According to the theorem, the upper left vertical map is surjective. By construction, the class \bar{b} lies in the image of the upper right vertical map. It follows that \bar{b} is also in the image of $\hat{R} \rightarrow (\hat{B})_{\text{red}}$. \square

Corollary 3.6. *The reduced local ring $(\hat{B})_{\text{red}}$ is noetherian. The local ring \hat{B} is noetherian if and only if its nilradical is finitely generated.*

Proof. Since the local ring \hat{B} is complete, so is the residue class ring $C = (\hat{B})_{\text{red}}$. It suffices to check that the latter has a finite cotangent space, according to Proposition 1.1. Since each vector from $\mathfrak{m}_C/\mathfrak{m}_C^2$ comes from some element in B , Corollary 3.5 reveals that it also comes from some element of R . This implies that the canonical map $\mathfrak{m}_{\hat{R}}/\mathfrak{m}_{\hat{R}}^2 \rightarrow \mathfrak{m}_C/\mathfrak{m}_C^2$ is surjective. These vector spaces are finite-dimensional, because \hat{R} is noetherian.

The condition in the second assertion is trivially necessary. It is also sufficient: Suppose the nilradical $N \subset C$ is finitely generated. Tensoring the exact sequence $N \rightarrow \mathfrak{m}_{\hat{B}} \rightarrow \mathfrak{m}_C \rightarrow 0$ of \hat{B} -modules with the residue field k gives an exact sequence

$$N \otimes_{\hat{B}} k \rightarrow \mathfrak{m}_{\hat{B}}/\mathfrak{m}_{\hat{B}}^2 \rightarrow \mathfrak{m}_C/\mathfrak{m}_C^2 \rightarrow 0.$$

Consequently, the k -vector space in the middle is finite-dimensional. Using Proposition 1.1, we deduce that the complete local ring \hat{B} is noetherian. \square

Our second main result deals with intermediate rings that are well-behaved:

Theorem 3.7. *Suppose the intermediate ring $R \subset B \subset R^{\text{ptc}}$ is noetherian, and that its generic formal fiber is reduced. Then the extension of local rings $R \subset B$ has invariants $e = f = 1$, is faithfully flat, and the induced map on formal completions $\hat{R} \rightarrow \hat{B}$ is bijective. Furthermore, if $B \neq R$ then the R -algebra B is not finite.*

Proof. We have $f = 1$ by [Proposition 3.1](#). In order to see $e = 1$, we first check that the induced map on cotangent spaces

$$(1) \quad \mathfrak{m}_R/\mathfrak{m}_R^2 \rightarrow \mathfrak{m}_{\hat{B}}/\mathfrak{m}_{\hat{B}}^2$$

is surjective. Let $\bar{b} \in \mathfrak{m}_{\hat{B}}/\mathfrak{m}_{\hat{B}}^2$. Since B is noetherian, the map $\mathfrak{m}_B/\mathfrak{m}_B^2 \rightarrow \mathfrak{m}_{\hat{B}}/\mathfrak{m}_{\hat{B}}^2$ is bijective, and we may choose an element $b \in \mathfrak{m}_B$ mapping to \bar{b} . Since the generic formal fiber of B is reduced, the formal completion \hat{B} is reduced, by [Proposition 1.3](#). According to [Corollary 3.5](#), there is an element $a' \in \mathfrak{m}_{\hat{R}}$ mapping to \bar{b} . Its class in the cotangent space $\bar{a} \in \mathfrak{m}_{\hat{R}}/\mathfrak{m}_{\hat{R}}^2$ is the image of some element $a \in \mathfrak{m}_R$. Whence the map (1) on cotangent spaces is surjective.

Now choose some $a_1, \dots, a_r \in \mathfrak{m}_R$ so that their images in the cotangent space form a vector space basis. According to the Nakayama Lemma, the elements $a_1, \dots, a_r \in \mathfrak{m}_B$ generate this maximal ideal. It follows that the extension of local rings $R \subset B$ has invariant $e = 1$.

According to [Proposition 1.5](#), the map $\hat{R} \rightarrow \hat{B}$ is bijective and $R \subset B$ is faithfully flat. Furthermore, if the inclusion $R \subset B$ is not an equality, then B is not a finite R -algebra. □

Corollary 3.8. *Suppose the intermediate local ring $R \subset B \subset R^{\text{ptc}}$ is noetherian, and its generic formal fiber is reduced. If the local ring R is regular, or Cohen–Macaulay, the same holds for B .*

Proof. A local noetherian ring is regular or Cohen–Macaulay if and only if the respective property holds for its formal completion [[15](#), propositions 17.1.5 et 16.5.2]. If R is regular, or Cohen–Macaulay, the same holds for $\hat{R} = \hat{B}$, and thus for B . □

The assumption that the local ring B is noetherian, with reduced generic formal fiber, can be rephrased as a flatness condition:

Proposition 3.9. *The intermediate ring $R \subset B \subset R^{\text{ptc}}$ is noetherian, with reduced generic formal fiber if and only if the extension of local rings $B \subset \hat{R}$ is flat, and the induced map $\hat{B} \rightarrow \hat{R}$ is injective.*

Proof. The condition is necessary: The composite map

$$\hat{R} \rightarrow \hat{B} \rightarrow \hat{R}$$

is the identity, the map $\hat{R} \rightarrow \hat{B}$ on the left is bijective by [Theorem 3.7](#). Whence the map $\hat{B} \rightarrow \hat{R}$ on the right is even bijective. Since $B \rightarrow \hat{B}$ is flat, so must be the composition $B \rightarrow \hat{B} \rightarrow \hat{R}$.

The condition is also sufficient: The composite mapping

$$\text{Spec}(\hat{R}) \rightarrow \text{Spec}(B) \rightarrow \text{Spec}(R)$$

is surjective, the map $\text{Spec}(B) \rightarrow \text{Spec}(R)$ on the right is bijective according to [Proposition 3.1](#), whence the map $\text{Spec}(\hat{R}) \rightarrow \text{Spec}(B)$ on the left is surjective. Therefore, the inclusion $B \subset \hat{R}$ is faithfully flat. Now let $\mathfrak{b}_0 \subset \mathfrak{b}_1 \subset \dots$ be an ascending chain of ideals in B . The ascending chain $\mathfrak{b}_0\hat{R} \subset \mathfrak{b}_1\hat{R} \subset \dots$ of ideals in the noetherian ring \hat{R} is stationary. By flatness, the canonical map $\mathfrak{b}_i \otimes_B \hat{R} \rightarrow \mathfrak{b}_i\hat{R}$ is bijective,

and by faithfully flat descent, the original ascending chain is stationary as well. Thus the local ring B is noetherian. Since $\hat{B} \rightarrow \hat{R}$ is injective and \hat{R} is reduced, the ring \hat{B} is reduced. By Proposition 1.3, the integral local ring B has reduced generic formal fiber. \square

Suppose from now on that the complete local noetherian ring \hat{R} is normal. According to Proposition 2.1, also R^{prc} is normal. Write $A_\lambda \subset R_\lambda \subset R^{\text{prc}}$ for the normalization of the finite R -algebras A_λ . This gives another filtered direct system $R_\lambda \subset R^{\text{prc}}$, $\lambda \in L$ of subrings containing R , with $R^{\text{prc}} = \bigcup_{\lambda \in L} R_\lambda$. By the Mori–Nagata theorem the R_λ are Krull domains (see [3, chapitre VII, §1, no. 8, proposition 12]). Such rings are not necessarily noetherian. Note, however, that if R is one-dimensional, all the R_λ are discrete valuation rings, by the Krull–Akizuki theorem (see [3, §2, no. 5, corollaire 2 de proposition 5]). If R is two-dimensional, the local rings R_λ at least remain noetherian, by a result of Nagata [28] attributed to Mori.

We now want to give a sufficient condition for the p -radical closure R^{prc} to be noetherian. To proceed, consider the subset $L' \subset L$ of all indices λ so that the local Krull domain R_λ is noetherian, and its generic formal fiber is reduced. The latter happens, for example, if the R_λ are regular.

Theorem 3.10. *Suppose that all formal fibers of the local noetherian ring R are reduced, and that the subset $L' \subset L$ is cofinal. Then the local ring R^{prc} is noetherian, its generic formal fiber is reduced, the extension of local rings $R \subset R^{\text{prc}}$ is faithfully flat, with invariants $e = f = 1$, and the canonical map $\hat{R} \rightarrow \widehat{R^{\text{prc}}}$ is bijective.*

Proof. First note that the order set L' is filtered, because it is cofinal in the filtered ordered set L . We thus have a filtered direct system R_λ , $\lambda \in L'$ of local noetherian rings. All transition maps are faithfully flat, and local with invariants $e = f = 1$. Clearly, $R^{\text{prc}} = \bigcup_{\lambda \in L'} R_\lambda$. It follows that the extension of local rings $R \subset R^{\text{prc}}$ also has invariants $e = f = 1$. Consequently, the maximal ideal $\mathfrak{m}_{R^{\text{prc}}} = \mathfrak{m}_R R^{\text{prc}}$ is finitely generated.

The main issue is to verify that the ring R^{prc} is noetherian. This means that every ideal is finitely generated. According to Cohen’s theorem [7], it suffices to check this merely for the prime ideals $\mathfrak{p} \subset R^{\text{prc}}$. Set $\mathfrak{p}_\lambda = \mathfrak{p} \cap R_\lambda$. For each $\lambda \leq \mu$, we have an inclusion $\mathfrak{p}_\lambda R_\mu \subset \mathfrak{p}_\mu$, and $\mathfrak{p} = \varprojlim \mathfrak{p}_\lambda$. Since $R_0 = R$ is noetherian, it suffices to check that the inclusion $\mathfrak{p}_0 R_\mu \subset \mathfrak{p}_\mu$ is an equality for all $\mu \in L'$. By Proposition 3.1, the corresponding closed embedding $\text{Spec}(R_\mu/\mathfrak{p}_\mu) \subset \text{Spec}(R_\mu/\mathfrak{p}_0 R_\mu)$ is a bijection. It remains to verify that the scheme $\text{Spec}(R_\mu/\mathfrak{p}_0 R_\mu)$ is reduced.

Set $Y = \text{Spec}(R_0)$ and $X = \text{Spec}(R_\mu)$, write $f : X \rightarrow Y$ for the canonical morphism, with induced map $\hat{f} : \hat{X} \rightarrow \hat{Y}$ on formal completions, and consider the resulting commutative diagram of schemes

$$\begin{array}{ccc} \hat{Y} & \xleftarrow{\hat{f}} & \hat{X} \\ q \downarrow & & \downarrow p \\ Y & \xleftarrow{f} & X \end{array}$$

According to Theorem 3.7, the upper vertical arrow is an isomorphism. Let $C \subset Y$ be the integral closed subscheme defined by the ideal $\mathfrak{p}_0 \subset R_0 = R$. Since the formal fibers of Y are reduced, the scheme $q^{-1}(C)$ is reduced, whence also

$$p^{-1}(f^{-1}(C)) = \hat{f}^{-1}(q^{-1}(C)).$$

Since the morphism $p : \hat{X} \rightarrow X$ is faithfully flat, the scheme $f^{-1}(C) = \text{Spec}(R_\mu/\mathfrak{p}_0 R_\mu)$ must be reduced.

We next check that the extension of local rings $R \subset R^{\text{prc}}$ has numerical invariants $e = f = 1$. The residual degree is $f = 1$, which holds for all intermediate rings by [Proposition 3.1](#). Since each extension of local rings $R \subset R_\lambda, \lambda \in L'$ has invariants $e = 1$, the same holds for the union $R^{\text{prc}} = \bigcup_{\lambda \in L'} R_\lambda$.

Finally, we examine the generic formal fiber. By assumption, the ring R is formally integral, so [Proposition 3.1](#) ensures that $\dim(R) = \dim(R^{\text{prc}})$. We thus may apply [Proposition 1.5](#) and infer that $\hat{R} \rightarrow \widehat{R^{\text{prc}}}$ is bijective. In particular, the local noetherian ring R^{prc} is formally reduced. Thus its generic formal fiber is reduced, according to [Proposition 1.3](#). \square

It is perhaps worthwhile to point out that there is a canonical exhaustive ascending chain of intermediate rings

$$R = B_0 \subset B_1 \subset \dots \subset R^{\text{prc}}$$

given by *heights*: Recall that F_∞ denotes the relative p -radical closure of $F = \text{Frac}(R) \subset \text{Frac}(\hat{R})$. Let $F_n \subset F_\infty$ be the set of elements $a \in F_\infty$ of height $\leq n$, that is, of degree $[F(a) : F] \leq p^n$, and define B_n to be the integral closure of R with respect to the field extension $F \subset F_n$. Then we have $R^{\text{prc}} = \bigcup_{n \geq 0} B_n$; it would be interesting to know whether these B_n are noetherian, with reduced generic formal fiber.

In the proof of [Theorem 3.2](#), we have used the following basic fact:

Lemma 3.11. *Let B be a reduced \mathbb{F}_p -algebra, and $B \subset C$ be a ring extension such that every element $c \in C$ has $c^{p^v} \in B$ for some exponent $v \geq 0$. Then the homomorphism*

$$C \rightarrow (C \otimes_B C)_{\text{red}}, \quad c \mapsto (c \otimes 1 \text{ modulo nilradical})$$

is bijective.

Proof. Let $f : C \rightarrow C \otimes_B C$ be the homomorphism given by $c \mapsto c \otimes 1$, and write $\bar{f} : C \rightarrow (C \otimes_B C)_{\text{red}}$ for the composite map. The multiplication map $g : C \otimes_B C \rightarrow C$ given by $c \otimes c' \mapsto cc'$ satisfies $g \circ f = \text{id}_C$. It factors over $(C \otimes_B C)_{\text{red}}$, because C is reduced. Thus \bar{f} is injective.

To see that \bar{f} is surjective, it suffices to show that the class of each $1 \otimes c$ lies in the image of f , up to nilpotent elements. Choose an exponent $v \geq 0$ with $c^{p^v} \in B$. Using $\mathbb{F}_p \subset B$, we have $(c \otimes 1 - 1 \otimes c)^{p^v} = 0$. Thus $c \otimes 1 - 1 \otimes c$ is nilpotent. From

$$1 \otimes c = f(c) - (c \otimes 1 - 1 \otimes c)$$

we see that \bar{f} is surjective. \square

4. Discrete valuation rings

We now specialize our results to the case that our local noetherian ring R is a discrete valuation ring. The field of fractions is given by $F = \text{Frac}(R) = R[1/a]$, where $a \in R$ is any nonzero nonunit. Its formal completion \hat{R} is an excellent discrete valuation ring. Obviously, there are only two formal fibers: the closed formal fiber, which is the spectrum of the residue field $k = \hat{R}/\mathfrak{m}_{\hat{R}} = R/\mathfrak{m}_R$ and of little interest, and the generic formal fiber, which is

$$(2) \quad \hat{R} \otimes_R F = \hat{R}[1/a] = \text{Frac}(\hat{R}).$$

From this one deduces the following well-known fact:

Proposition 4.1. *Consider the following three conditions:*

- (i) *The inclusion $R \subset R^{\text{prc}}$ is bijective.*
- (ii) *The field extension $F \subset \text{Frac}(\hat{R})$ is separable.*
- (iii) *The integral ring R is japanese.*
- (iv) *The discrete valuation ring R is excellent.*

Then the implications (i) \Leftarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv) hold. The four conditions are equivalent provided that the field extension $F \subset \text{Frac}(\hat{R})$ can be written as a purely inseparable extension followed by a separable extension.

Proof. The implications (i) \Leftarrow (ii) \Rightarrow (iii) are special cases of [Proposition 2.2](#). If R is japanese, then for every prime ideal $\mathfrak{p} \subset R$ the residue class ring R/\mathfrak{p} is japanese; hence the implication (ii) \Leftarrow (iii) follows from the Zariski–Nagata theorem [[12](#), théorème 7.6.4]. The implication (iv) \Rightarrow (ii) is trivial. Conversely, if $F \subset \text{Frac}(R)$ is separable, then the two formal fibers of R are geometrically regular. The other two conditions for excellence also hold: The ring is obviously noetherian, and since it is one-dimensional, it must be universally catenary, compare [[12](#), remarque 7.1.2]. Thus (ii) \Rightarrow (iv) holds. Under the additional assumption on the field extension $F \subset \text{Frac}(\hat{R})$, the equivalence of all four conditions again follows from [Proposition 2.2](#). □

Let us examine the p -radical closure $R \subset R^{\text{prc}} \subset \hat{R}$ of our discrete valuation ring in more detail. Recall that $R \subset A_\lambda \subset R^{\text{prc}}$, $\lambda \in L$ denotes the filtered direct system of all finite R -subalgebras. Each A_λ is a one-dimensional integral local noetherian ring, and the extensions of local rings $R \subset A_\lambda$ is finite and flat. We write $n_\lambda = \text{rank}_R(A_\lambda)$ for the degrees.

Proposition 4.2. *The local extension $R \subset A_\lambda$ has numerical invariants $f = 1$ and $e = n_\lambda$. If $n_\lambda \neq 1$, then the rings R_λ are not normal.*

Proof. The first assertion follows from [Proposition 3.1](#). Since A_λ is finite and flat, the formula $n = ef$ holds, and thus the ramification index coincides with the degree. The last assertion follows from [Corollary 3.4](#). □

Let $A_\lambda \subset R_\lambda$ be the normalization inside the field $\text{Frac}(\hat{R})$. According to [Proposition 2.1](#), we have $R_\lambda \subset R^{\text{prc}}$. This gives another filtered direct system of R -subalgebras $R_\lambda \subset R^{\text{prc}}$, with $R^{\text{prc}} = \bigcup_{\lambda \in L} R_\lambda$. Clearly, we have

$$n_\lambda = \text{rank}_R(A_\lambda) = [R_\lambda \otimes_R F : F],$$

where $F = \text{Frac}(R)$ is the field of fractions.

Proposition 4.3. *Each R_λ is a discrete valuation ring, and the extension of local rings $R \subset R_\lambda$ has numerical invariants $e = f = 1$ and $n = n_\lambda$. If $n_\lambda \neq 1$, then R_λ is not a finite R -algebra.*

Proof. By the Krull–Akizuki theorem (see [[3](#), §2, no. 5, corollaire 2 de proposition 5]), the normalization R_λ is a discrete valuation ring, an in particular noetherian, with regular formal fibers. The other assertions thus follow from [Theorem 3.7](#). □

Theorem 4.4. *Suppose that the field extension $F \subset \text{Frac}(\hat{R})$ can be written as a purely inseparable extension followed by a separable extension. Then the p -radical closure R^{prc} is an excellent discrete valuation ring. The extension of local rings $R \subset R^{\text{prc}}$ has invariants $e = f = 1$ and $n = \sup_{\lambda \in L} \{n_\lambda\}$, and the induced map on formal completions $\hat{R} \rightarrow \hat{R}^{\text{prc}}$ is bijective. If R is not excellent, then the ring extension $R \subset R^{\text{prc}}$ is integral but not finite.*

Proof. Obviously, for every index $\lambda \in L$, the local Krull domain $A'_\lambda = R_\lambda$ is noetherian and regular, in particular its formal fibers are reduced. Thus [Theorem 3.10](#) applies, and the result follows. \square

We now can give the following universal property:

Corollary 4.5. *Assumptions as in the theorem. Let $R \subset A$ be an extension of local rings. If A is an excellent discrete valuation ring, or more generally any formally integral local noetherian ring with geometrically reduced generic formal fiber, then we have $R^{\text{prc}} \subset A$ inside the field of fractions $\text{Frac}(\hat{A})$.*

Proof. Since R is a discrete valuation ring, the induced map $\hat{R} \rightarrow \hat{A}$ is injective: Otherwise it would factor over $\hat{R}/\mathfrak{m}_{\hat{R}}$, and the image of $\text{Spec}(\hat{A}) \rightarrow \text{Spec}(\hat{R})$ consists of the closed points. The same then holds for composite map $\text{Spec}(\hat{A}) \rightarrow \text{Spec}(A) \rightarrow \text{Spec}(R)$. The latter maps, however, are dominant, contradiction. We thus may apply [Corollary 2.4](#), and the assertion follows. \square

Let me close this paper with the following question: do similar results as for discrete valuation rings hold for regular local noetherian rings?

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