PARA-ABELIAN VARIETIES AND ALBANESE MAPS

BRUNO LAURENT AND STEFAN SCHRÖER

12 January 2023

Abstract. We construct for every proper algebraic space over a ground field an Albanese map to a para-abelian variety, which is unique up to unique isomorphism. This holds in the absence of rational points or ample sheaves, and also for reducible or non-reduced spaces, under the mere assumption that the structure morphism is in Stein factorization. It also works under suitable assumptions in families. In fact the treatment of the relative setting is crucial, even to understand the situation over ground fields. This also ensures that Albanese maps are equivariant with respect to actions of group schemes. Our approach depends on the notion of families of para-abelian varieties, where each geometric fiber admits the structure of an abelian variety, and representability of tau-parts in relative Picard groups, together with structure results on algebraic groups.

Contents

Introduction

The *Albanese variety* and the *Albanese map* are truly fundamental constructions in algebraic geometry. The terms were coined by André Weil ([59], page 438 and commentary on page 570, compare also [36], page 428), in reference to work of Giacomo Albanese on zero-cycles and correspondences for surfaces ([2] and [3]). Originally, it was a purely *transcendental construction*, depending on path integrals over closed holomorphic one-forms. A general formulation for compact complex

2010 *Mathematics Subject Classification.* 14K30, 14A20, 14L30, 14D10.

spaces X was given by Blanchard [8]: The Albanese variety is the complex torus V^*/Δ, constructed with the dual of the vector space $V \subset H^0(X, \Omega^1_{X/\mathbb{C}})$ of closed forms, and the Lie group closure Δ of the subgroup given by integrals over loops. The resulting Albanese map

$$X \longrightarrow V^*/\Delta, \quad x \longmapsto (\sigma \mapsto \int_{x_0}^{x} \sigma),$$

is defined by integrals over paths, and depends on the choice of a base point x_0. For compact connected Riemann surfaces, this simplifies to $J = H^0(X, \Omega^1_X)^*/H_1(X, \mathbb{Z})$. Together with the principal polarization stemming from the intersection form this is called the *jacobian variety*. For genus $g \geq 1$ the Albanese map is a closed embedding $X \subset J$, and for genus $g \geq 2$ one obtains an embedding of the Deligne–Mumford stacks $\mathscr{M}_g \subset \mathscr{A}_g$, which comprises families of curves and principally polarized abelian varieties, of genus and dimension g, respectively.

It is natural to ask for *algebraic constructions* of Albanese varieties and Albanese maps, say for projective varieties, that extend to general proper schemes X over arbitrary ground fields k, and beyond. It quickly became clear that such a generalization is possible: Matsusaka [40] and Serre [55] constructed the Albanese map in the classical language of algebraic varieties, by regarding them as universal maps to abelian varieties, compare also the discussion of Esnault, Srinivas and Viehweg [19]. Grothendieck used Picard schemes and Poincaré sheaves to obtain Albanese maps, as outlined in [22], and [23], Section 3. The representability of the Picard functor was extended to arbitrary proper schemes by Murre [44].

In the absence of rational points, however, notorious complications arise and Albanese maps take values in *principal homogeneous spaces* rather than abelian varieties. These problems become even more pronounced over imperfect fields of characteristic $p > 0$: To our best knowledge, the existence of Albanese map, and its base-change behavior, is only established for geometrically integral proper schemes, or more generally for geometrically reduced and geometrically connected proper schemes, compare the discussions of Conrad [14] and Wittenberg [60]. Also see Achter, Casalaina-Martin and Vial [1] for non-proper X. After the completion of the present work, we learned that Albanese maps where also constructed in the setting of algebraic stacks by Brochard ([13], Section 7 and 8). They take values in commutative group stacks that combine abelian varieties and finite group schemes, and exist under the condition that $\mathrm{Pic}^{\tau}_{X/k}$ is proper (loc. cit. Theorem 8.1).

The goal of this paper is to settle such issues, by systematically working in the *relative setting* over some base scheme S, and also to use *algebraic spaces* rather than schemes. Recall that algebraic spaces are important generalization of schemes introduced by Michael Artin. Roughly speaking, they take over the role of Moishezon spaces from complex geometry. However, their definition is much more indirect: algebraic spaces are contravariant functors $X : (\mathrm{Aff}/S) \to (\mathrm{Set})$ that satisfy the sheaf axiom with respect to the étale topology, and are also otherwise closely related to schemes. Notions like the underlying topological space $|X|$ or the structure sheaf $\mathscr{O}_X$ still exist but lose much of their immediate significance, in comparison to the schematic situation.

Let X be an algebraic space over S such that the structure morphism is proper, flat, of finite presentation and cohomologically flat in degree $d = 0$, with $h^0(\mathscr{O}_{X_s}) = 1$

for all points $s \in S$. In particular, the structure morphism $X \to S$ is in Stein factorization. Our main result is:

Theorem. (see Thm. 10.2) *Assumptions as above. If* $\mathrm{Pic}^\tau_{X/S}$ *admits a family of maximal abelian subvarieties, then there is an Albanese map* $f : X \to \mathrm{Alb}_{X/S}$. *Moreover, it is universal for morphisms into families of para-abelian varieties, equivariant with respect to actions of group spaces, and commutes with base-change.*

If the base scheme S is integral, and the generic fiber of $\mathrm{Pic}^\tau_{X/S} \to S$ is proper, all assumptions are satisfied at least on some dense open set U, over which the Albanese map thus exists. If S is the spectrum of a field k we get for each proper algebraic space X with $h^0(\mathscr{O}_X) = 1$ unconditional results. Note that X may be reducible, non-reduced, or non-schematic, and the ground field k is arbitrary. Building on this, the second author has treated the case of non-proper algebraic spaces over ground fields [54]. Furthermore, the existence of sign involutions on para-abelian varieties are studied in [6].

To establish the above result we develop a theory of *para-abelian varieties*, which is of independent interest. Apparently, the name was coined by Grothendieck ([23], Theorem 3.3), but did not gain widespread use. Again it is crucial to work in the relative setting: A *family of para-abelian varieties* is a smooth proper morphism $P \to S$ such that for each point $s \in S$, there is a field extension k of the residue field $\kappa(s)$ such that $P \otimes k$ admits the structure of an abelian variety. *Note that our definition does not involve any a priori torsor structure.* However, we show in Section 5 that a certain inertia subsheaf in $\mathrm{Aut}_{P/S}$ is a family of abelian varieties, which a posteriori yields a torsor structure.

The second main ingredient is a systematic study for the *tau-part* $\mathrm{Pic}^\tau_{X/S}$ of the relative Picard functor. Roughly speaking, it parameterizes invertible sheaves that are fiberwise numerically trivial. We provide a new point of view, by directly using Artin's representability criteria [4]. Here the representation is via algebraic spaces rather than schemes, even if X is schematic, which highlights that the correct framework for our goals is given by algebraic spaces. As already pointed out by Grothendieck [23], the tau-parts are much better behaved than the connected components $\mathrm{Pic}^0_{X_s/\kappa(s)}$ for the fiberwise Picard groups: The latter form an abelian sheaf that is usually not representable by algebraic spaces, and we give in Proposition 2.2 an explicit example with families of Enriques surfaces.

An important technical issue are the *separation properties* of the ensuing quotient $N = \mathrm{Num}_{X/S}$ of the Picard space by its tau-part, which we call the *numerical sheaf*. These are algebraic spaces endowed with a group law whose fibers are étale group schemes. In general, *they are neither schematic nor separated, but at least locally separated*, which means that the diagonal monomorphism $N \to N \times N$ is an embedding. This very weak separation axiom is enough to carry out some crucial constructions that lead to Albanese maps.

An important insight of this paper is to *re-define the notion of Albanese maps* for algebraic spaces X over S: Here it denotes pairs (P, f) where P is an S-family of para-abelian varieties, and $f : X \to P$ is an S-morphism such that for each point $s \in S$, the induced homomorphism f^* identifies the abelian variety $\mathrm{Pic}^\tau_{P_s/\kappa(s)}$ with the *maximal abelian subvariety* for the group scheme $\mathrm{Pic}^\tau_{X_s/\kappa(s)}$. In some sense, this definition is rather close to the original approach with path integrals, and its base-change properties follow very easily. We establish a posteriori that such Albanese maps have a universal property, which ensures uniqueness and equivariance.

In this approach, the *maximal abelian subvariety* plays a crucial role. Extending some results of Brion [11] on the structure of group schemes G of finite type over a base field k, we introduce a *three-step filtration* $G_3 \subset G_2 \subset G_1$ and describe the maximal abelian subvariety $G_{\mathrm{ab}} \subset G$ in terms of the filtration and extensions of abelian varieties by multiplicative groups, which appears to be of independent interest.

The paper is organized as follows: In Section 1 we briefly review algebraic spaces and local separatedness, and give an existence result for quotients in a relative setting. In Section 2 we thoroughly study the tau-part $\mathrm{Pic}^{\tau}_{X/S}$ for the relative Picard functor, and collect criteria for separatedness, properness, flatness and smoothness. This leads to the *numerical sheaf* $\mathrm{Num}_{X/S}$, which is studied in Section 3. In Section 4 we develop a theory of para-abelian varieties and their families. Using actions of automorphism group schemes, we uncover in Section 5 the relation between families of para-abelian varieties and families of abelian varieties. In Section 6 and 7 we work over ground fields k and obtain general results on the structure of group schemes and the maximal abelian subvariety. We then combine our results so far and state the definition of Albanese maps in Section 8. Here we already establish some partial results concerning its uniqueness and existence. To proceed, we have to analyze the notion of Poincaré sheaves in Section 9. In the final Section 10, we obtain existence and uniqueness for Albanese maps and Albanese varieties in full generality. In the Appendix, we collect some facts concerning embeddings of algebraic spaces.

Acknowledgement. We wish to thank Jeff Achter, Brian Conrad, Laurent Moret-Bailly, Olivier Wittenberg and the referees for valuable comments that helped to improve the paper and correct mistakes. This research was conducted in the framework of the research training group *GRK 2240: Algebro-geometric Methods in Algebra, Arithmetic and Topology*, which is funded by the Deutsche Forschungsgemeinschaft.

1. Algebraic spaces

Throughout the paper, S usually denotes a base scheme. We write (Aff/S) for the category of affine schemes $T = \mathrm{Spec}(R)$ endowed with a structure morphism $T \to S$. Recall that an *algebraic space* is a contravariant functor $X : (\mathrm{Aff}/S) \to (\mathrm{Set})$ satisfying the sheaf axiom with respect to the étale topology, such that the diagonal $X \to X \times X$ is relatively representable by schemes, and that there is an étale surjection $U \to X$ from some scheme U. These are important generalizations of schemes, because modifications, quotients, families, or moduli spaces of schemes are frequently algebraic spaces rather than schemes. We refer to the monographs of Olsson [46], Laumon and Moret-Bailly [38], Artin [5], Knutson [37], and to the stacks project [57], Part 0ELT.

Note that there is an important novel separation axiom for algebraic spaces, namely *local separatedness*, which means that the diagonal $X \to X \times X$ is an embedding (compare the Appendix for more details). This is automatic for schemes, but may fail for algebraic spaces (see the end of this section for examples). Throughout we use the term *embedding* in the sense of [24], Definition 4.2.1, where the word *immersion* is used instead. Note that it is not necessarily closed, open, or quasicompact. Also note that some authors impose an additional condition, besides representability, on diagonals for algebraic spaces, but we make no separation assumptions whatsoever.

By definition, an algebraic space $X : (\mathrm{Aff}/S) \to (\mathrm{Set})$ satisfies the sheaf axiom with respect to the étale topology. According to [57], Lemma 076M it then also satisfies the sheaf axiom with respect to the fppf topology. Throughout, we shall regard (Aff/S) as a site with this Grothendieck topology, if not stated otherwise. It is obtained by the procedure explained in [16], Exposé IV, Section 6 from the families $(U_i \to T)_{i\in I}$, where $U_i \to T$ are open embeddings, the index set I is arbitrary, and $\bigcup U_i \to T$ is surjective, together with the families $(V_j \to T')_{j\in J}$, where the index set J is finite, the $V_j \to T'$ are flat and of finite presentation, and $\bigcup V_j \to T'$ is surjective. Note that by dropping the assumption that T and U_i belong to (Aff/S), the procedure allows to pass to the larger category (Sch/S).

Let G be an algebraic space such that the $G(R)$, with $T = \mathrm{Spec}(R)$ from (Aff/S), are endowed with functorial group structures. For simplicity, we say that G is an *algebraic space with group structure*. Let X be an algebraic space endowed with a G-action. One may form the quotient X/G as a sheaf on (Aff/S). Note that since the action is from the left, it would be more appropriate to write $G\backslash X$. This common inconsistency should not cause confusion, because in most of our applications G will be commutative.

We want to understand when the quotient X/G is representable by an algebraic space. Let us say that the action $\mu : G \times X \to X$ is *free* if

$$(\mu, \mathrm{pr}_2) : G \times X \longrightarrow X \times X, \quad (\sigma, x) \longmapsto (\sigma x, x)$$

is a *monomorphism*. In other words, the groups $G(R)$ act freely on the sets $X(R)$, for all $T = \mathrm{Spec}(R)$. Note that this deviates from other terminology that is sometimes used, where (μ, pr_2) is assumed to be a *closed embedding* (for example [43], Definition 0.8). For convenience of the reader, we state the following result ([57], Tag 06PH):

Lemma 1.1. *Suppose the structure morphism $G \to S$ is flat and locally of finite presentation, and that the action on X is free. Then the quotient X/G in the category of sheaves is representable by an algebraic space. Moreover, the quotient map $X \to X/G$ is flat and locally of finite presentation, and the formation of X/G commutes with base-change.*

In the situation of the Lemma, the quotient morphism $q : X \to Y = X/G$ yields a cartesian diagram

$$\begin{tikzcd} G \times X \arrow[r, "(\mu,\mathrm{pr}_2)"] \arrow[d, "q\circ\mu"'] & X \times X \arrow[d, "q\times q"] \\ Y \arrow[r, "\Delta"'] & Y \times Y. \end{tikzcd}$$

The vertical map on the right is a torsor with respect to $(G \times G)_{Y\times Y}$, hence flat and locally of finite presentation.

Proposition 1.2. *Assumptions as in Lemma 1.1. Then the monomorphism* $(\mu, \mathrm{pr}_2) : G \times X \to X \times X$ *is quasicompact, or an embedding, or a closed embedding if and only if the quotient* $Y = X/G$ *is quasiseparated, or locally separated, or separated, respectively.*

Proof. If Δ has one of the properties in question, so does the base-change (μ, pr_2). For the converse, recall that for a morphism being quasicompact, an embedding, or a closed embedding is local in the range. Choose an étale surjection $\amalg_{\lambda \in L} U_\lambda \to X \times X$ for some affine schemes U_λ. The compositions $U_\lambda \to Y \times Y$ are flat and locally of finite presentation, hence universally open ([28], Theorem 2.4.6). Their images define open embeddings $V_\lambda \subset Y \times Y$ that cover $Y \times Y$, together with an fppf morphism $U_\lambda \to V_\lambda$. So if (μ, pr_2) has one of the properties in questions, so does Δ, by fppf descent. □

Let us close this section with two examples of quotients that are not locally separated, and in particular not schematic:

First suppose that $S = \mathrm{Spec}(k)$ is the spectrum of a field, that H is a group scheme of finite type, and $\Lambda \subset H(k)$ is an infinite subgroup. Regard the latter as a constant group scheme $G = \amalg_{\sigma \in \Lambda} \mathrm{Spec}(k)$. The canonical homomorphism $G \to H$ is a monomorphism, hence the translation action is free, and the quotient $X = H/G$ exists as an algebraic space. It is not locally separated, because the monomorphism $G \to H$ is not an embedding. Note that the section $S \to X$ coming from the neutral section in H is not an embedding, because this does not hold for the base-change $G \to H$, so Corollary 11.3 below also ensures that X is not locally separated.

The second example lives over the spectrum $S = \mathrm{Spec}(R)$ of a discrete valuation ring. Write U for the open set comprising the generic point, and Z for the complementary closed set. Suppose H is a relative group scheme, and N is a closed subgroup scheme. Assume that the structure morphisms of $H \to S$ and $N \to S$ are fppf. Assume that $N_U \subset H_U$ is also open, and that $N_Z \subset H_Z$ is bijective. Then the *disjoint* union $G = N \amalg (H_U \smallsetminus N_U)$ is a group scheme, and the resulting $G \to H$ is a monomorphism. Obviously, the translation action of G on H is free, and the structure morphism $G \to S$ is fppf. It follows that the quotient $X = H/G$ exists as an algebraic space. Now suppose there are $a, b \in H(S)$ such that ab^{-1} does not lie in G, but $ab^{-1}|Z$ is contained in $N(Z)$. Then the restrictions of a, b to both Z and U are congruent modulo G. Now suppose that the diagonal $\Delta : X \to X \times X$ would be an embedding. Then the cartesian diagram

$$\begin{tikzcd} S' \arrow[r] \arrow[d] & S \arrow[d, "(\bar{a}, \bar{b})"] \\ X \arrow[r, "\Delta"'] & X \times X \end{tikzcd}$$

defines a bijective embedding $S' \subset S$, whence $S' = S$. This implies that $ab^{-1} \in G$, contradiction. To make the situation concrete, one may choose $H = \mu_{p,R}$ and $N = \{e\}_R$ over the ring $R = \mathbb{Z}_p[e^{2\pi i/p}]$.

2. Pic and Pic-tau for families

Let S be a base scheme, X be an algebraic space, and suppose that the structure morphism $X \to S$ is flat, proper, of finite presentation, and cohomologically flat in degree $d = 0$. The latter ensures that the direct image of the structure sheaf $\mathscr{O}_X$ is locally free on S, and its formation commutes with base-change. According to [4], Theorem 7.3 the sheafification of $R \mapsto \operatorname{Pic}(X \otimes R)$ with respect to the fppf topology is representable by an algebraic space $\operatorname{Pic}_{X/S}$, which is locally of finite presentation, quasiseparated and locally separated. By abuse of notation we here write $R \mapsto \operatorname{Pic}(X \otimes R)$ instead of the contravariant functor $T \mapsto \operatorname{Pic}(X \times_S T)$. Note that in Artin's formulation of Theorem 7.3, local separatedness is not stated explicitly because only algebraic spaces satisfying this separation axiom were considered. Also note that $\operatorname{Pic}_{X/S}$ frequently fails to be schematic, and this already happens for certain families of curves over discrete valuation rings ([23], Section 0).

If S is the spectrum of a field k, the component of the origin $\operatorname{Pic}^0_{X/k}$ is quasicompact, and the resulting quotient $\operatorname{NS}_{X/k}$ is a local system of finitely generated abelian groups, called the *Néron–Severi group scheme*. Likewise, the preimage $\operatorname{Pic}^\tau_{X/k}$ of the torsion part in $\operatorname{NS}_{X/k}$ is quasicompact. According to [7], Exposé XIII, Theorem 4.6 the geometric points on $\operatorname{Pic}^\tau_{X/k}$ correspond to invertible sheaves that are numerically trivial, in other words the *intersection number* $(\mathscr{L} \cdot C) = \chi(\mathscr{L}) - \chi(\mathscr{O}_C)$ vanishes for every curve $C \subset X$. Note that the argument depends on Chow's Lemma, which holds in our context ([37], Theorem 3.1 or [50], Theorem 8.8), and the proof immediately carries over from schemes to algebraic spaces.

For a general base S, we now define $\operatorname{Pic}^\tau(X \otimes R)$ *as the subgroup of the Picard group comprising all invertible sheaves that are fiberwise numerically trivial.* Clearly, this is functorial in R. We need the following result, which is already implicit in Artin's paper [4], and was formulated by Brochard in the realm of stacks ([12], Theorem 3.3.3):

Theorem 2.1. *The sheafification of $R \mapsto \operatorname{Pic}^\tau(X \otimes R)$ is representable by an algebraic space $\operatorname{Pic}^\tau_{X/S}$ whose structure morphism is of finite presentation (hence quasiseparated) and locally separated. Moreover, the inclusion into $\operatorname{Pic}_{X/S}$ is an open embedding.*

Proof. Let us give a prove using Artin's representability criterion. The problems are local in the base, so we may assume that $S = \operatorname{Spec}(A)$ is affine. Let $A_\lambda \subset A$ be the direct system of subrings that are finitely generated over the ring $\mathbb{Z}$. Using the results from [29], §8 there is an index μ such that $X = X_\mu \otimes_{A_\mu} A$ for some proper flat X_μ over A_μ; compare [50] Proposition B.2 and B.3 for statements entirely in the realm of algebraic spaces. We also have to ensure cohomological flatness: Forming the Čech complex for some étale surjection $U_\mu \to X_\mu$ with an affine scheme U_μ and arguing as in [42], Section 5 we find a bounded complex $K^\bullet$ of finitely generated projective A_μ-modules giving an identification

$$H^r(X_\mu \otimes B, \mathscr{O}_{X_\mu \otimes B}) = H^r(K^\bullet \otimes B), \quad r \geq 0$$

that is functorial in the A_μ-algebras B. Clearly, the image $M = \operatorname{Im}(K^0 \to K^1)$ is finitely generated and its formation commutes with base-change. With the long exact Tor sequence for $0 \to H^0 \to K^0 \to M \to 0$, we infer that the formation of

$H^0(X_\mu, \mathscr{O}_{X_\mu})$ commutes with base-change if and only if M is locally free. So by assumption, $M\otimes_{A_\mu} A$ is locally free. Hence there is some $\lambda\geq\mu$ such that $M\otimes_{A_\mu} A_\lambda$ is locally free. Summing up, we may replace $X\to \operatorname{Spec}(A)$ with the base change of $X_\mu\to\operatorname{Spec}(A_\mu)$ to A_λ, and assume that A is finitely generated over the excellent Dedekind ring $\mathbb{Z}$.

Artin established the representability for the sheafification of $R\mapsto \operatorname{Pic}(X\otimes R)$ in [4], Theorem 7.3 by an application of his Theorem 5.3, which involves checking certain conditions [0']–[5']. The reasoning for $\operatorname{Pic}^\tau(X\otimes R)$ is virtually the same. Only condition [1'] requires additional arguments: Let R be a complete local noetherian ring, with residue field $k=R/\mathfrak{m}_R$. We have to verify that the canonical map

$$\operatorname{Pic}^\tau(X\otimes R)\longrightarrow \varprojlim_n \operatorname{Pic}^\tau(X\otimes R/\mathfrak{m}_R^{n+1})$$

is bijective. It is injective, by Grothendieck's Existence Theorem ([26], Theorem 5.1.4 for schemes and [57], Theorem 08BE for algebraic spaces). Moreover, given invertible sheaves $\mathscr{L}_n$ on $X\otimes R/\mathfrak{m}_R^{n+1}$ such that the restriction of $\mathscr{L}_{n+1}$ to $X\otimes R/\mathfrak{m}_R^{n+1}$ is isomorphic to $\mathscr{L}_n$, and that $(\mathscr{L}_0\cdot C_0)=0$ for every integral curve C_0 in $X_0=X\otimes k$, the isomorphism classes come from some invertible sheaf $\mathscr{L}$ on X. Our task is to check that the restriction to $X\otimes\kappa(\mathfrak{p})$ is numerically trivial, for each prime ideal $\mathfrak{p}\subset R$. By [25], Proposition 7.1.4 it suffices to treat the case that R is a discrete valuation ring, $\mathfrak{p}$ is the zero ideal, and that $S=\operatorname{Spec}(R)$. Let $\eta\in S$ be the generic point, and suppose that there is some integral curve $C_\eta\subset X_\eta$ with $(\mathscr{L}_\eta\cdot C_\eta)\neq 0$. The closure $C\subset X$ of this curve is flat, whence the closed fiber $C_0\subset X_0$ is a curve. Note that by [30], Proposition 21.9.11 the total space C carries an ample sheaf, hence the algebraic space C is a scheme. Since Euler characteristics are constant in families ([27], Theorem 7.9.4), we see $(\mathscr{L}_0\cdot C_0)=(\mathscr{L}_\eta\cdot C_\eta)\neq 0$, contradiction.

This shows that $\operatorname{Pic}^\tau_{X/S}$ is an algebraic space. According to [7], Exposé XIII, Theorem 4.7 the monomorphism to $\operatorname{Pic}_{X/S}$ is an open embedding, and the structure morphism $\operatorname{Pic}^\tau_{X/S}\to S$ is quasicompact. The proof relies on approximation and Chow's Lemma as above, and carries over from schemes to algebraic spaces. Since $\operatorname{Pic}_{X/S}$ is locally of finite presentation, quasiseparated and locally separated, the same holds for the open subspace $\operatorname{Pic}^\tau_{X/S}$. $\square$

In [23], Grothendieck defined $\operatorname{Pic}^\tau_{X/S}$ as the subsheaf comprising those R-valued points of $\operatorname{Pic}_{X/S}$ that are torsion in the fiberwise Néron–Severi groups, and showed in his Theorem 1.1 that it is open, under the assumption that $\operatorname{Pic}_{X/S}$ is a scheme. The above approach seems to be more adequate in the realm of algebraic spaces.

Similarly, one defines $\operatorname{Pic}^0_{X/S}$ as the *abelian subsheaf* comprising the R-valued points of $\operatorname{Pic}_{X/S}$ that are trivial in the fiberwise Néron–Severi groups. This indeed seems the only possible approach, because the subsheaf is not representable in general, as the following counterexample shows:

Suppose S is the spectrum of $\mathbb{F}_2[[t]]$, and let $X\to S$ be a family of Enriques surfaces whose generic fiber is classical but whose closed fiber is ordinary or supersingular (for details see [9], Section 3). Then $G=\operatorname{Pic}^\tau_{X/S}$ is finite and flat of degree $n=2$. It is the union $G=G'\cup G''$ of two sections intersecting in the closed fiber, which is a singleton.

Proposition 2.2. *In the above situation, the subsheaf* $\mathrm{Pic}^0_{X/S}$ *inside* $\mathrm{Pic}^\tau_{X/S}$ *is not representable by an algebraic space.*

Proof. Suppose $H = \mathrm{Pic}^0_{X/S}$ is representable. By definition, the canonical morphism to $G = \mathrm{Pic}^\tau_{X/S}$ is a monomorphism, and the set of image points is the zero-section. It follows that the structure morphism $H \to S$ is a universal homeomorphism. In particular, it is universally closed, separated, and has affine fibers. According to [50], Theorem 8.5 it must be integral, and in particular affine. Write $H = \mathrm{Spec}(A)$ and $G = \mathrm{Spec}(R)$, and consider the canonical homomorphism $R \to A$. The induced map $R/t^nR \to A/t^nA$ is bijective for each $n \geq 0$. We have $\bigcap_{n\geq 0} t^nA = 0$ by Krull's Intersection Theorem, thus $R \to A$ is injective. It follows that the image of $H \to G$ contains both generic points, contradiction. □

Let us finally collect the basic properties of the tau-part:

Proposition 2.3. *The algebraic space* $\mathrm{Pic}^\tau_{X/S}$ *has the following properties:*

(i) *If the geometric fibers of* $f : X \to S$ *are integral, then the structure morphism* $\mathrm{Pic}^\tau_{X/S} \to S$ *is separated.*
(ii) *If the geometric fibers of* $f : X \to S$ *are integral and locally factorial, then* $\mathrm{Pic}^\tau_{X/S} \to S$ *is proper.*
(iii) *If for all points* $s \in S$ *we have* $h^1(\mathscr{O}_{X_s}) - h^2(\mathscr{O}_{X_s}) = b_1(X_s)/2$ *then the morphism* $\mathrm{Pic}^\tau_{X/S} \to S$ *is flat.*
(iv) *Suppose for each Artin local ring* B *with residue field* $k = B/\mathfrak{m}_B$*, and each* $\mathrm{Spec}(B) \to S$*, there is an algebraic space* Z*, some proper* B*-morphisms* $h_1, \ldots, h_r : Z \to X \otimes B$ *and integers* $n_1, \ldots, n_r$ *such that the induced map*
$$\sum n_i h_i^* : H^2(X \otimes k, \mathscr{O}_{X\otimes k}) \longrightarrow H^2(Z \otimes k, \mathscr{O}_{Z\otimes k})$$
is injective, whereas $\sum n_i h_i^* : \mathrm{Pic}^\tau_{X\otimes B/B} \to \mathrm{Pic}^\tau_{Z/B}$ *is zero. Then* $\mathrm{Pic}^\tau_{X/S} \to S$ *is smooth.*

Proof. It suffices to treat the case that $S = \mathrm{Spec}(R)$ is the spectrum of a noetherian ring. Set $G = \mathrm{Pic}^\tau_{X/S}$. We start with assertion (i) and (ii), which follow from Valuative Criteria, similar to [23], Theorem 2.1. Suppose the fibers are geometrically integral. Then the map $\mathscr{O}_S \to f_*(\mathscr{O}_X)$ is bijective. Let V be a discrete valuation ring with field of fractions $F = \mathrm{Frac}(V)$, such that the residue field $k = V/\mathfrak{m}_V$ is algebraically closed and that V is complete.

For (i) we have to verify that the map $G(V) \to G(F)$ is injective ([38], Proposition 7.8 or [57], Lemma 03KV). Let $l_1, l_2 \in G(V)$ be two elements that coincide in $G(F)$. Without restriction, we may assume $V = R$. Then $H^2(S, \mathbb{G}_m) = 0$, for example by [41], Chapter IV, Corollary 1.7 combined with Corollary 2.12. Now the Leray–Serre spectral sequence for the structure morphism $X \to S$ shows that the canonical map $\mathrm{Pic}^\tau(X) \to G(R)$ is bijective and that the canonical map $\mathrm{Pic}^\tau(X \otimes F) \to G(F)$ is injective. So the sections l_i come from invertible sheaves $\mathscr{L}_i$ that are isomorphic on the complement U of the integral Cartier divisor $D = X \otimes k$. Choose an isomorphism $\varphi : \mathscr{L}_1|U \to \mathscr{L}_2|U$. It extends to a homomorphism $\varphi : \mathscr{L}_1(-nD) \to \mathscr{L}_2$ for some integer $n \geq 0$, as in [24], Theorem 6.8.1. This gives a short exact sequence $0 \to \mathscr{L}_1(-nD) \to \mathscr{L}_2 \to \mathscr{F} \to 0$. The cokernel is an invertible sheaf on some effective Cartier divisor $E \subset X$ supported by D. Since the latter is integral, we have $E = mD$ for some integer $m \geq 0$. Setting $\mathscr{N} = \mathscr{L}_2 \otimes \mathscr{L}_1^{\otimes -1} \otimes \mathscr{O}_X(rD)$ with $r = m - n$, we get another short exact sequence $0 \to \mathscr{N}(-mD) \to \mathscr{N} \to \mathscr{F} \to 0$. Computing the determinant of $\mathscr{F}$ with the second sequence and using $\mathscr{O}_X(D) \simeq \mathscr{O}_X$ gives $\det(\mathscr{F}) \simeq \mathscr{O}_X$. Using the first sequence then reveals $\mathscr{L}_1 \simeq \mathscr{L}_2$. This shows (i). Note that this generalizes a result of Ischebeck [34], who considered affine schemes.

For assertion (ii), suppose we have a point $l \in G(F)$. Choose a finite separable extension $F \subset F'$ so that the image $l' \in G(F')$ comes from an invertible sheaf on $X \otimes F'$, and let V' be the integral closure of V. Our task is to check that l' lies in the subset $G(V') \subset G(F')$ ([38], Theorem 7.3 or [57], Lemma 0A3Z). Without loss of generality, we may assume that $R = V = V'$. Let $\mathscr{L}_F$ be the invertible sheaf on $X \otimes F$ corresponding to the point l, and $\mathscr{F}$ be any coherent extension to X. We now check that its bi-dual $\mathscr{L}$ is invertible. The total space X is integral, because this holds for the closed fiber $X \otimes k$. Since $X \otimes F$ is locally factorial, the same holds for X, by Nagata's result ([45], Lemma 1). In particular, X is normal, and $\mathscr{L}$ is reflexive of rank one, hence invertible. This settles (ii).

Assertion (iii) is due to Ekedahl, Hyland and Shepherd-Barron ([18], Proposition 4.2). Note the Betti numbers $b_i(X_s)$ are vector space dimensions for the étale cohomology groups $H^i(X_{\bar{s}}, \mathbb{Q}_l(i)) = \varprojlim_\nu H^i(X_{\bar{s}}, \mu_{l^\nu}^{\otimes i}) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$, which by definition are computed over the algebraic closure of the residue field $\kappa(s)$.

We finally come to statement (iv), which is an abstraction of Mumford's arguments for abelian varieties ([43], Proposition 6.7). We saw in Theorem 2.1 that $G = \mathrm{Pic}^\tau_{X/S}$ is of finite presentation over S. By definition of smoothness, we have to check that the canonical map $G(A) \to G(A/J)$ is surjective, for each R-algebra A and each ideal J with $J^2 = 0$. In light of [30], Remark 17.5.4 it suffices to consider the case where A is a local Artin ring. We may further assume that the residue field $A/\mathfrak{m}_A$ is algebraically closed, that the ideal J has length one, and that $A = R$.

Let $h : X \to S$ be the structure morphisms and $X^{\mathrm{aff}} = \mathrm{Spec}\,\Gamma(X, \mathscr{O}_X)$. The proof for [53], Lemma 1.4 reveals that we have an identification $H^2(S, h_*(\mathbb{G}_{m,X})) = H^2(X^{\mathrm{aff}}, \mathbb{G}_m)$. The Brauer group $H^2(X^{\mathrm{aff}}, \mathbb{G}_m)$ vanishes, because X^{aff} is an Artin scheme with algebraically closed residue fields. Consequently, the Leray–Serre spectral sequence for h gives an exact sequence

$$\mathrm{Pic}(X) \longrightarrow G(R) \longrightarrow H^2(S, h_*(\mathbb{G}_{m,X})) = 0.$$

We infer that the canonical map $\mathrm{Pic}^\tau(X) \to G(R)$ and also $\mathrm{Pic}^\tau(X \otimes B) \to G(B)$ is bijective, with $B = A/J$. The short exact sequence $0 \to \mathscr{O}_{X\otimes k} \to \mathscr{O}_X^\times \to \mathscr{O}_{X\otimes B}^\times \to 1$ and the corresponding sequence for Z induce a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Pic}^\tau(X) & \longrightarrow & \mathrm{Pic}^\tau(X\otimes B) & \longrightarrow & H^2(X\otimes k, \mathscr{O}_{X\otimes k}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Pic}^\tau(Z) & \longrightarrow & \mathrm{Pic}^\tau(Z\otimes B) & \longrightarrow & H^2(Z\otimes k, \mathscr{O}_{Z\otimes k}).
\end{array}$$

Here the vertical arrows are given by the linear combination $\sum n_i h_i^*$. By assumption, the vertical map on the right is injective, but the vertical map in the middle is zero. So by exactness of the upper row, each invertible sheaf on $X \otimes B$ that is fiberwise numerically trivial extends to X. $\square$

Note that assertion (i) actually hold for the structure morphisms $\mathrm{Pic}_{X/S} \to S$ of the whole Picard scheme, with the same proof.

3. The numerical sheaf

Let S be a base scheme, and X be an algebraic space whose structure morphism $X \to S$ is proper, flat, of finite presentation and cohomologically flat in degree $d = 0$. We then define an abelian sheaf $\mathrm{Num}_{X/S}$ on the site (Aff/S) by the short exact sequence

$$0 \longrightarrow \mathrm{Pic}^{\tau}_{X/S} \longrightarrow \mathrm{Pic}_{X/S} \longrightarrow \mathrm{Num}_{X/S} \longrightarrow 0,$$

and call it the *numerical sheaf*. Note that the translation action of $\mathrm{Pic}^{\tau}_{X/S}$ on $\mathrm{Pic}_{X/S}$ is free, so the formation of the quotient commutes with base-change. In particular, for each point s and each field k' containing the separable closure $\kappa(s)^{\mathrm{sep}}$, the abelian group $\mathrm{Num}_{X/S}(k')$ is free, and its rank is the Picard number $\rho \geq 0$ of $X \otimes \kappa(s)^{\mathrm{alg}}$, by [7], Exposé XIII, Theorem 5.1. We actually have:

Theorem 3.1. *Suppose $\mathrm{Pic}^{\tau}_{X/S} \to S$ is flat. Then $\mathrm{Num}_{X/S}$ is representable by an algebraic space. The structure morphism $\mathrm{Num}_{X/S} \to S$ is locally of finite presentation, quasiseparated and locally separated. Moreover, all fibers are separated, schematic and étale.*

Proof. The problem is local in S, and we may assume that S is the spectrum of a noetherian ring R. The first assertion follows from Lemma 1.1. Since $\mathrm{Pic}^{\tau}_{X/S}$ is noetherian, the open embedding into $\mathrm{Pic}_{X/S}$ is quasicompact, and the assertion on the structure morphism follows from Proposition 1.2. To establish the statement on the fibers, we may assume that $R = k$ is a field. Then $G = \mathrm{Pic}_{X/k}$ is a scheme, and $H = \mathrm{Pic}^{\tau}_{X/k}$ is an open subgroup scheme. Hence it is also closed, and the zero element $e \in G/H$ is a closed point. In particular, the algebraic space with group structure G/H is separated. It must be étale and hence zero-dimensional, because $H \subset G$ is open. Summing up, the algebraic space G/H is locally of finite type over the field k, and zero-dimensional. It the must be schematic by [57], Lemma 06LZ. $\square$

Suppose now that $\mathrm{Pic}^{\tau}_{X/S}$ is flat over S, so that the numerical sheaf $\mathrm{Num}_{X/S}$ is an algebraic space, for example by the criteria in Proposition 2.3. We need the following fact, which is a variant of Mumford's Rigidity Lemma ([43], Proposition 6.1):

Lemma 3.2. *Let A be an algebraic space whose structure morphism $\alpha : A \to S$ is proper, flat, of finite presentation, cohomologically flat in degree $d = 0$, and with $h^0(\mathscr{O}_{A_s}) = 1$ for all $s \in S$. Then for each $\varphi \in \mathrm{Num}_{X/S}(A)$, there is a unique $\sigma \in \mathrm{Num}_{X/S}(S)$ making the following diagram commutative:*

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr, "\alpha"'] & & \mathrm{Num}_{X/S} \arrow[dl] \\ & S \arrow[ur, bend right, "\sigma"'] & \end{tikzcd} \tag{1}$$

Proof. Write $N = \mathrm{Num}_{X/S}$. In light of the uniqueness assertion, the problem is local, and as usual we may assume that S is the spectrum of a noetherian ring R. Again by uniqueness, together with fppf descent ([32], Exposé VIII, Theorem 5.2 for schemes and [57], Lemma 0ADV for algebraic spaces), it suffices to treat the case $\alpha : A \to S$ admits a section $\tilde{\sigma}$. So the diagram (1) reveals that if σ exists, it must coincide with the section $\varphi \circ \tilde{\sigma}$, which thus already settles the uniqueness assertion. It remains to verify that $\sigma = \varphi \circ \tilde{\sigma}$ indeed makes the diagram commutative.

To start with, suppose that $R = k$ is a field, so that N is a separated scheme whose underlying topological space is discrete. Since A is proper, the set-theoretical image $\varphi(A) \subset N$ is closed and quasicompact. It carries exactly one scheme structure, because N is étale, and must be the spectrum of an étale k-algebra $L = k_1 \times \ldots \times k_r$. In particular, the schematic image is affine. Using $h^0(\mathscr{O}_A) = 1$ we conclude $L = k$, and $\varphi = \sigma \circ \alpha$ follows.

Next suppose that R is artinian. Still N is a separated scheme, and the schematic image $\varphi(A) \subset N$ is affine. Since A is separated, the section $\tilde{\sigma}$ is a closed embedding. Write $A_0 \subset A$ for the corresponding closed subspace. Both maps φ and $\varphi|A_0$ factor over the affine hulls of A and A_0, respectively. It thus suffices to check that $H^0(A, \mathscr{O}_A) \to H^0(A_0, \mathscr{O}_{A_0})$ is bijective. Since $\alpha : A_0 \to S$ is an isomorphism, the composite map $R \to H^0(A, \mathscr{O}_A) \to H^0(A_0, \mathscr{O}_{A_0})$ is bijective. The first map is also bijective, by the assumptions on cohomological flatness, hence the second map is bijective, too.

We now come to the general case. First note that $\sigma : S \to N$ is not necessarily a closed embedding. However, since $N \to S$ is locally separated, the section σ must be an embedding, according to Corollary 11.3 below. It is also quasicompact, because S is noetherian and $N \to S$ is quasiseparated. Thus σ factors over some Zariski open $U \subset N$ via some closed embedding $\sigma : S \to N$, by Lemma 11.1 below. We next verify that $\varphi : A \to N$ also factors over U. Since the latter is open, this can be checked fiberwise, and was established above. It remains to show that $\varphi : A \to U$ factors over the closed embedding $\sigma : S \to U$. The latter corresponds to some quasicoherent $\mathscr{I} \subset \mathscr{O}_U$, and we have to check that $\mathscr{J} = \varphi^{-1}(\mathscr{I})\mathscr{O}_A$ is the zero ideal. The problem is local in S, and it suffices to treat the case that the noetherian ring R is complete and local. Let $X_n = X \otimes R/\mathfrak{m}_R^{n+1}$, and write $X' \subset X$ and $X'_n \subset X_n$ for the closed subspaces coming from $\mathscr{J}$. We saw above that $X'_n \subset X_n$ is an equality. By Grothendieck's Comparison Theorem ([26], Theorem 5.4.1 for schemes and [57], Lemma 0A4Z for algebraic spaces) the same holds for $X' \subset X$, thus $\mathscr{J} = 0$. □

4. Para-abelian varieties

Let us call an algebraic space P over some field k a *para-abelian variety* if there is a field extension $k \subset k'$ such that the base-change $P' = P \otimes_k k'$ admits the structure of an abelian variety. The terminology goes back to Grothendieck, who introduced it in [23], Theorem 3.3 by a different condition. A posteriori, we shall see that the notions are equivalent. By fpqc descent, our P is proper and smooth over k, with $h^0(\mathscr{O}_P) = 1$. Moreover:

Lemma 4.1. *The algebraic space P is a projective scheme.*

Proof. Choose an algebraically closed field extension $k' \subset \Omega$. According to [42], page 62 there is an ample sheaf $\mathscr{L}$ on the base-change $P \otimes_k \Omega$. This defines a morphism $\mathrm{Spec}(\Omega) \to \mathrm{Pic}_{P/k}$, which factors over some connected component $\mathrm{Pic}^l_{P/k}$. Fix a closed point a in this component. Then the field extension $k \subset \kappa(a)$ is finite, and there is a finite extension $\kappa(a) \subset k''$ and some invertible sheaf $\mathscr{A}$ on $P \otimes k''$ mapping to $a \in \mathrm{Pic}_{P/k}$. Choose an embedding $k'' \subset \Omega$. By the numerical criterion for ampleness the base-change of $\mathscr{A}$ to $P \otimes \Omega$ is ample ([35], page 343, Theorem 1). It follows that $P \otimes k''$ admits an ample invertible sheaf $\mathscr{N}''$. Let $\mathscr{N} = N(\mathscr{N}'')$ be the norm with respect to the finite locally free morphism $P \otimes k'' \to P$. According to [25], Proposition 6.6.1 the base-change of $\mathscr{N}$ to $P \otimes k''$ is ample. Thus $\mathscr{N}$ is ample on P. $\square$

For our purposes it is crucial to work in the relative setting:

Definition 4.2. A *family of para-abelian varieties* over some scheme S is an algebraic space P, together with a morphism $P \to S$ that is proper, flat and of finite presentation, such that the fibers P_s are para-abelian varieties over the residue field $\kappa(s)$, for every $s \in S$.

Particular examples are the *families of abelian varieties*. By this we mean an algebraic space A, together with a proper flat morphism of finite presentation $A \to S$ endowed with a group structure, such that all fibers are abelian varieties. These are often called *abelian schemes* in the literature. According to Raynaud's result (see [20], Theorem 1.9) the total space A indeed must be a scheme. It actually satisfies the *AF-property*, that is, each finite set of points admits a common affine open neighborhood, provided the base scheme S is affine. Actually, the total space is quasiprojective if the base is affine and normal ([20], Remark 1.10). Note, however, that there are examples without ample sheaves ([47], Chapter XII, 4.2), and this happens already over the spectrum of the ring of dual numbers $R = \mathbb{C}[\epsilon]$.

In what follows, we fix a family of para-abelian varieties $f : P \to S$. Note that the structure morphism is surjective and smooth. Moreover, the canonical map $\mathscr{O}_S \to f_*(\mathscr{O}_P)$ is bijective ([27], Proposition 7.8.6). In other words, f is cohomologically flat in degree $d = 0$, and $h^0(\mathscr{O}_{P_s}) = 1$ for all points $s \in S$. In contrast to families of abelian varieties, the total space P often fails to be a scheme. This already happens in relative dimension $g = 1$ over local schemes S of dimension $n = 2$, see [47], Chapter XIII, Section 3.2 and also [61].

We seek to relate the sheaves $\mathrm{Aut}_{P/S}$ and $\mathrm{Pic}_{P/S}$ to $P \to S$. Let us start with some useful observations, which generalize [43], Theorem 6.14:

Proposition 4.3. *For each $e \in P(S)$, there is a unique group law $\mu : P \times_S P \to P$ that turns $P \to S$ into a family of abelian varieties, with $e : S \to P$ as the zero section.*

Proof. Uniqueness and hence also existence are local problems, so we may assume that $S = \mathrm{Spec}(R)$ is affine. Suppose there are two group laws μ_1 and μ_2 with e as zero section. Recall that the algebraic space P then must be schematic. Since P and hence $P \times_S P$ are of finite presentation, the scheme P and the morphisms μ_i are already defined over some noetherian subring $R_0 \subset R$. Now [43], Corollary 6.6 ensures that $\mu_1 = \mu_2$. This settles uniqueness.

It remains to verify existence. For this it also suffices to treat the case that R is noetherian. Suppose first that there is an fpqc extension $R \subset R'$ such that a group law μ' exists for $P' = P \otimes_R R'$, with some origin $e' \in P(R')$. Using translation by $e \otimes 1 - e'$, we may assume that e' is the base-change of e. Consider the ring $R'' = R' \otimes_R R'$. By fpqc descent, we have to verify that the two pull-backs $\mu' \otimes 1$ and $1 \otimes \mu'$ to R'' coincide. Both are group laws, and in both cases the origin is the pull-back of e. Uniqueness ensures $\mu' \otimes 1 = 1 \otimes \mu'$. Note that this settles the assertion if $R = k$ is a field.

Suppose next that S is the spectrum of some local Artin ring R. Then P_{red} is the closed fiber, which is schematic according to Proposition 4.1. By [50], Corollary 8.2 the total space P is schematic as well. (This result already appears in [37], Theorem 3.3, at least for quasiseparated algebraic spaces.) Hence the group law μ exists by [43], Proposition 6.15.

Now suppose that R is a general noetherian ring. Fix a closed point $a \in S$, corresponding to a maximal ideal $\mathfrak{m} \subset R$. The preceding paragraph gives a formal group law over the completion $\hat{R} = \varprojlim R/\mathfrak{m}_a^n$. It comes from a group law over $\hat{R}$, according to Grothendieck's Existence Theorem ([26], Theorem 5.4.1 for schemes and [57], Lemma 0A4Z for algebraic spaces). Since $R_{\mathfrak{m}} \subset \hat{R}$ is an fpqc extension, the group law already exists over $R_{\mathfrak{m}}$. Thus it is already defined over some open neighborhood $U \subset S$ of a ([29], Theorem 8.8.2 for schemes and [50], Proposition B.2 for algebraic spaces). Applying this for all closed points $a \in S$, we obtain an open covering $S = U_1 \cup \ldots \cup U_r$ such that the group law exists over each U_i. By uniqueness, these local group laws glue and yield the desired global group law. $\square$

Corollary 4.4. *Suppose $S = \mathrm{Spec}(R)$ is henselian, with closed point $a \in S$. Then $P \to S$ admits the structure of a family of abelian varieties if and only if the closed fiber P_a contains a rational point.*

Proof. The condition is obviously necessary. Conversely, suppose there is a rational point $e_a \in P_a$. It follows from [30], Corollary 17.16.3 that there is a subscheme $Z \subset P$ containing e_a, and such that $Z \to S$ is étale and quasi-finite. In turn, the singleton $\{e_a\}$ is a connected component of the closed fiber Z_a. It corresponds to a connected component $U \subset Z$, because R is henselian. Moreover, $U \to S$ is étale and finite, and thus defines a section $e \in P(S)$ extending e_a. By the Proposition, $P \to S$ becomes a family of abelian varieties. $\square$

Corollary 4.5. *There is an étale surjection $S' \to S$ such that the base-change $P' = P \times_S S'$ admits the structure of a family of abelian varieties over S'.*

Proof. According to [30], Corollary 17.16.3 there is an étale surjection $S' \to S$ such that $P(S')$ is non-empty. The Proposition ensures the existence of a group law. $\square$

Recall that $B = \mathrm{Pic}^{\tau}_{P/S}$ is an algebraic space, and the morphism $B \to S$ is of finite presentation. For each integer $n \geq 1$, the kernel $B[n]$ for multiplication by n is another algebraic space. Write $G_n = \mathrm{Aut}_{B[n]/S}$ for the ensuing automorphism sheaf.

Corollary 4.6. *In the above situation, the structure morphism $B \to S$ is a family of abelian varieties. Moreover, for each integer $n \geq 1$ the morphism $B[n] \to S$ is finite and locally free, and $G_n \to S$ is relatively representable by affine schemes.*

Proof. We first check that $B \to S$ is a family of abelian varieties. The morphism is proper according to Proposition 2.3, part (ii). To check smoothness, it suffices to treat the case that $P \to S$ admits a section ([30], Proposition 17.7.1). So $P \to S$ becomes a family of abelian varieties, by our Proposition. Mumford then showed that $B = \mathrm{Pic}^{\tau}_{P/S}$ is smooth ([43], Proposition 6.7). He actually assumed that $P \to S$ is projective to have the existence of $\mathrm{Pic}^{\tau}_{B/S}$ as projective scheme. However, his arguments carry over without change to our situation (relying on Theorem 2.1 and Proposition 2.3, part (iv)). It remains to verify that $B \to S$ has geometrically connected fibers. For this it suffices to treat the case that S is the spectrum of an algebraically closed field k, and that P is an abelian variety. Then the Néron–Severi group $\mathrm{NS}(P)$ is torsion free, according to [42], Corollary 2 on page 178. Thus $B \to S$ is a family of abelian varieties.

For the remaining assertions, it suffices to treat the case that S is the spectrum of a noetherian ring R, and $P \to S$ is a family of abelian varieties. Since $B \to S$ is separated, the inclusion $B[n] \subset B$ is closed, hence the structure morphism $B[n] \to S$ is proper. It is also quasi-finite ([42], Appendix to §6). By [30], Corollary 18.12.4 it must be finite. For each point $s \in S$, the fiberwise multiplication $n : B_s \to B_s$ is finite and surjective, and hence flat ([56], Proposition 22 on page IV-37). In light of [29], Proposition 11.3.11 this holds true for $n : B \to B$. Consequently $B[n] \to S$ is finite and locally free. It follows that $G_n \to S$ is affine, for example by [39], Lemma 4.1. ☐

Let G be an algebraic space endowed with a group structure, and assume that the structure morphism $g : G \to S$ is flat and of finite presentation, and that $\mathscr{O}_S \to g_*(\mathscr{O}_G)$ is bijective. Suppose that we have a relative G-action on P. By functoriality, it induces a relative action on $B = \mathrm{Pic}^{\tau}_{P/S}$.

Corollary 4.7. *In the above situation, the G-action on B is trivial.*

Proof. It suffices to treat the case that S is the spectrum of a noetherian ring R. The induced action on $B[n]$ is trivial, because the latter are affine, whereas $R = \Gamma(G, \mathscr{O}_G)$. According to [29], Theorem 11.10.9 the collection of closed subgroup schemes $B[n] \subset B$, $n \geq 1$ is schematically dense, and it follows by from loc. cit., Proposition 11.10.1 that the action on B must be trivial as well. ☐

5. Equivariance

Fix a base scheme S, and let X be an algebraic space whose structure morphism $X \to S$ is locally of finite presentation and separated. According to [4], Theorem 6.1 the Hilbert functor $\mathrm{Hilb}_{X/S}$ is representable by an algebraic space that is locally of finite presentation and separated. Recall that its R-valued points are the closed subspaces $Z \subset X \otimes R$ such that the projection $Z \to \mathrm{Spec}(R)$ is proper, flat and of finite presentation. If furthermore $X \to S$ itself is proper and flat, one sees that $\mathrm{Aut}_{X/S}$ is an open subspace of $\mathrm{Hilb}_{(X \times X)/S}$, by interpreting automorphisms via their graphs.

Now let $P \to S$ be a family of para-abelian varieties. The action from the left of the algebraic space $\mathrm{Aut}_{P/S}$ on P induces an action from the right on the family of abelian varieties $B = \mathrm{Pic}^{\tau}_{P/S}$, via pull-back of invertible sheaves. Write $G \subset \mathrm{Aut}_{P/S}$ for the ensuing *inertia subgroup sheaf*; its group of R-valued points comprises the

R-isomorphisms $f : P \otimes R \to P \otimes R$ where the induced map $f^* : B \otimes R \to B \otimes R$ is the identity.

Proposition 5.1. *The inclusion $G \subset \mathrm{Aut}_{P/S}$ is representable by open-and-closed embeddings, the structure morphism $G \to S$ is a family of abelian varieties, and the total space G is a scheme.*

Proof. First observe that once we know that $G \to S$ is a family of abelian varieties, the total space must be a scheme by Raynaud's result ([20], Theorem 1.9). To verify the statements on $G \subset \mathrm{Aut}_{P/S}$ and $G \to S$, it suffices to treat the case that $P \to S$ is a family of abelian varieties, and S is the spectrum of a noetherian ring R. Note that then the translation action gives a monomorphism $P \to \mathrm{Aut}_{P/S}$. This is a closed embedding, because P is proper and $\mathrm{Aut}_{P/S}$ is separated ([30], Corollary 18.12.6).

By Corollary 4.7 we have $P \subset G$ as subsheaves inside $\mathrm{Aut}_{P/S}$. We claim that this inclusion is an equality. This is a statement on R'-valued points; by making a base-change it suffices to treat the case $R = R'$. Suppose we have some automorphism $f : P \to P$ that lies in $G(R)$. Write $f = t_a \circ h$, where h respects the chosen group law and t_a is the translation map with respect to the section $a = f(e)$. The induced homomorphism h^* on $\mathrm{Pic}^\tau_{P/S}$ is the identity, because this holds for the translation t_a^* by Corollary 4.7, and for f^* by assumption. Our task is to verify $h = \mathrm{id}_P$. In light of Grothendieck's Comparison Theorem ([26], Theorem 5.4.1 for schemes and [57], Lemma 0A4Z for algebraic spaces) it suffices to treat the situation when R is artinian, and we also may assume that the residue field $k = R/\mathfrak{m}_R$ is algebraically closed.

Suppose first that $R = k$ is a field. According to [42], Corollary on page 131 the classifying map

$$P \longrightarrow \mathrm{Pic}^\tau_{B/k}, \quad x \longmapsto [\mathscr{P}|\{x\} \times B]$$

stemming from a normalized Poincaré sheaf $\mathscr{P}$ on $P \times B$ is an isomorphism of abelian varieties. Moreover, one easily sees that it is natural with respect to P. In particular, the homomorphism $h : P \to P$ coincides with $(h^*)^*$. Since $h^* = \mathrm{id}_B$, it follows that $h = \mathrm{id}_P$. We refer to Section 8 for a detailed discussion of Poincaré sheaves.

In the general situation, choose a composition series $\mathfrak{m}_R = \mathfrak{a}_0 \supset \ldots \supset \mathfrak{a}_r = 0$ whose quotients have length one, and write $R_i = R/\mathfrak{a}_i$. We now show by induction on $i \geq 0$ that $h_i = h \otimes R_i$ are identities. We just checked this for $i = 0$. Now suppose $i > 0$, and that h_{i-1} is the identity. Recall that $H^0(P_0, \Theta_{P_0/k})$ is the Lie algebra for the group scheme $\mathrm{Aut}_{P/S} \otimes k$, where $\Theta_{P_0/k} = \underline{\mathrm{Hom}}(\Omega^1_{P_0/k}, \mathscr{O}_{P_0})$ is the tangent sheaf. It follows that each $g \in \mathrm{Aut}_{P/S}(k[\epsilon])$ with $g \otimes k = e \otimes k$ is the translation by some $a \in P(k[\epsilon])$. Now both h_i and the identity on $P \otimes R_i$ are extensions of h_{i-1}. As explained in [58], Corollary 4.4 we may view the difference as an element in $H^0(P_0, \Theta_{P_0/k}) \otimes \mathfrak{a}_i/\mathfrak{a}_{i-1}$, and conclude that h_i is translation by some $a_i \in P(R_i)$. The latter must be the neutral element $e_i \in P(R_i)$, because h_i is a homomorphism. This shows that $G = P$ as subsheaves of $\mathrm{Aut}_{P/S}$.

Summing up, we have shown that $G \subset \mathrm{Aut}_{P/S}$ is representable by closed embeddings, and that the structure morphism $G \to S$ is a family of abelian varieties. It remains to verify that the inclusion $G \subset \mathrm{Aut}_{P/S}$ is open. For this it suffices to check that for a given automorphism $f : P \to P$, the set

$$U = \{s \in S \mid f_s : P_s \to P_s \text{ is a translation}\}$$

is open. Since this is a closed set in a noetherian topological space, the task is to verify that it is stable under generization. So we may assume that R is local and f_s is a translation for the closed point $s \in S$, and have to show that $U = S$. Using Grothendieck's Comparison Theorem ([26], Theorem 5.4.1 for schemes and [57], Lemma 0A4Z for algebraic spaces), it suffices to treat the case that R is artinian. We then argue as in the preceding paragraph. □

Proposition 5.2. *The canonical action of G on P is free and transitive.*

Proof. We have to show that the morphism $(\mu, \mathrm{pr}_2) : G \times_S P \to P \times_S P$ is an isomorphism, where $\mu : G \times P \to P$ denotes the action. It suffices to treat the case that $P \to S$ is a family of abelian varieties (Proposition 4.3), and that S is the spectrum of a noetherian ring R. Then we saw in the proof for Proposition 5.1 that the G-action coincides with the P-action via translations. The latter is obviously free and transitive. □

Let us sum up the content of the preceding propositions:

Theorem 5.3. *The family of para-abelian varieties $P \to S$ induces, in the above canonical way, a family of abelian varieties $G \to S$ inside $\mathrm{Aut}_{P/S}$, and P becomes a principal homogeneous G-space, in other words a representable G-torsor.*

In turn, we get a cohomology class $[P] \in H^1(S, G)$, where the cohomology is taken with respect to the fppf topology. Since the structure morphism $G \to S$ is smooth, the cohomology remains unchanged if computed with the étale topology ([31], Theorem 11.7). Indeed, we already saw in Corollary 4.5 that $P \to S$ admits sections locally in the étale topology. Note, however, that the order of the cohomology class $[P]$ may be infinite ([47], Chapter XIII, Section 3.2 and also [61]).

Now let $f : P_1 \to P_2$ be a morphism between families of para-abelian varieties, and $G_i \subset \mathrm{Aut}_{P_i/S}$ be the resulting families of abelian varieties as above, such that P_i is a principal homogeneous G_i-space.

Proposition 5.4. *In the above situation, there is a unique homomorphism $f_* : G_1 \to G_2$ between families of abelian varieties such that $f : P_1 \to P_2$ is equivariant with respect to the action of G_1.*

Proof. Uniqueness is clear: For each R-valued point $\sigma \in G_1(R)$, there is an fppf extension $R \subset R'$ such that there is some $a' \in P_1(R')$. Then

$$f(\sigma_{R'} \cdot a') = f_*(\sigma_{R'}) \cdot f(a') = f_*(\sigma)_{R'} \cdot f(a').$$

Consequently $f_*(\sigma) \in G_2(R)$ is uniquely determined by $f : P_1 \to P_2$.

We now verify existence. In light of the uniqueness and fppf descent, it suffices to treat the case that there is a section $e_1 \in P_1(S)$. Composition with f yields a section $e_2 \in P_2(S)$. In turn, we obtain identifications $G_i = G_i \cdot e_i = P_i$. With respect to these identifications, we can regard $f : P_1 \to P_2$ as a morphism $f_* : G_1 \to G_2$ between families of abelian varieties that respect the zero sections. This is a homomorphism by [43], Corollary 6.4. Given R-valued points $\sigma \in G_1(R)$ and $a \in P_1(R)$, we write $a = \eta \cdot e_1$ for some $\eta \in G_1(R)$ and obtain

$$f(\sigma \cdot a) = f(\sigma\eta \cdot e_1) = f_*(\sigma\eta) \cdot e_2 = f_*(\sigma)f_*(\eta) \cdot e_2 = f_*(\sigma) \cdot f(\eta \cdot e_1) = f_*(\sigma) \cdot f(a).$$

So $f : P_1 \to P_2$ is equivariant with respect to the G_1-actions stemming from the inclusion $G_1 \subset \mathrm{Aut}_{P_1/S}$ and the homomorphism $G_1 \xrightarrow{f_*} G_2 \subset \mathrm{Aut}_{P_2/S}$. $\square$

Each R-valued point $a \in P(R)$ can be seen as an isomorphism $\xi : G_R \to P_R$, which comes with an inverse $\varphi = \xi^{-1}$, and each $l \in \mathrm{Pic}_{G/S}(R)$ yields some $\varphi^*(l) \in \mathrm{Pic}_{P/S}$. As explained in [47], Chapter XIII, Proposition 1.1 this induces a canonical isomorphism

$$\mathrm{Pic}_{G/S} \wedge^G P \longrightarrow \mathrm{Pic}_{P/S}, \quad (l, a) \longmapsto \varphi^*(l). \tag{2}$$

Here the wedge symbol denotes the quotient of $\mathrm{Pic}_{G/S} \times X$ by the diagonal left action $g \cdot (l, a) = (lg^{-1}, ga)$. The G-action on P is free and transitive (Proposition 5.2), whereas the G-action on the invariant open subspace $\mathrm{Pic}^\tau_{G/S}$ is trivial (Corollary 4.7). It follows that the projection

$$\mathrm{pr}_1 : \mathrm{Pic}^\tau_{G/S} \wedge^G P \longrightarrow (\mathrm{Pic}^\tau_{G/S})/G = \mathrm{Pic}^\tau_{G/S} \tag{3}$$

is an isomorphism. As observed by Raynaud in loc. cit., composing (2) with the inverse of (3) yields:

Proposition 5.5. *The above maps gives an identification* $\mathrm{Pic}^\tau_{G/S} = \mathrm{Pic}^\tau_{P/S}$ *of families of abelian varieties.*

For each family $A \to S$ of abelian varieties, the family of abelian varieties $\mathrm{Pic}^\tau_{A/S}$ is called the *dual family*. The above observation identifies the family $B = \mathrm{Pic}^\tau_{P/S}$ of abelian varieties coming from our family $P \to S$ of para-abelian varieties with the dual family for $G \to S$, where the latter is defined via the inclusion $G \subset \mathrm{Aut}_{P/S}$.

This has a remarkable consequence: Let $P \to S$ and $P' \to S$ be two families of para-abelian varieties, and consider the canonical map

$$\mathrm{Hom}_S(P', P) \longrightarrow \mathrm{Hom}_{\mathrm{Gr}/S}(\mathrm{Pic}^\tau_{P/S}, \mathrm{Pic}^\tau_{P'/S}), \quad f \longmapsto f^*.$$

Note that the term on the left is a set, which might be empty, whereas the term on the right is an abelian group. Moreover, the G-action on P induces an action of the group $G(S)$ on the set $\mathrm{Hom}_{\mathrm{Sch}}(P', P)$.

Lemma 5.6. *In the above situation, the action of* $G(S)$ *on the set* $\mathrm{Hom}_S(P', P)$ *is free, and the fibers of the map* $f \mapsto f^*$ *are precisely the orbits.*

Proof. Suppose some $\sigma \in G(S)$ fixes a morphism $f : P' \to P$. We have to verify that $\sigma = e$. By descent, we may replace S with P' and assume that there is a section $a \in P'(S)$. Then $f(a) \in P(S)$. The action of $G(S)$ on $P(S)$ is free, and from $\sigma + f(a) = f(a)$ it follows $\sigma = e$.

Corollary 4.7 ensures that the orbits are contained in the fibers. Conversely, suppose that $f, g : P' \to P$ are two morphisms with $f^* = g^*$. We have to produce some $\sigma \in G(S)$ with $f = g + \sigma$. It must be unique, if it exists, according to the preceding paragraph. By descent, our problem is local, so we may assume that $P' \to S$ and hence also $P \to S$ is a family of abelian varieties, that S is the spectrum of a noetherian ring R, and that we have an identification $G = P$. Write $f = t_a \circ f_0$

and $g = t_b \circ g_0$, where $f_0, g_0 : P' \to P$ are homomorphisms, and t_a, t_b are translations by some elements $a, b \in P(S) = G(S)$. Our task is to verify that $g_0 = f_0$. In light of Grothendieck's Comparison Theorem ([26], Theorem 5.4.1 for schemes and [57], Lemma 0A4Z for algebraic spaces) it suffices to check this if R is a local Artin ring. From Corollary 4.7 we have $f_0^* = g_0^*$. We now argue as in Proposition 5.1, by choosing a composition series $\mathfrak{m}_R = \mathfrak{a}_0 \supset \ldots \supset \mathfrak{a}_r = 0$ whose quotients have length one, and applying induction on $i \geq 0$ with $R_i = R/\mathfrak{a}_i$. □

6. Some extensions of group schemes

Let k be a ground field of characteristic $p \geq 0$. Recall that an *abelian variety* A is a group scheme that is smooth, connected and proper. By abuse of notation, we simply say that A is *abelian*. A group scheme N is called *of multiplicative type* if there is a field extension $k \subset k'$ such that $N \otimes k'$ is isomorphic to the spectrum of the Hopf algebra $k'[\Lambda]$ for some commutative group Λ. We are mainly interested in the case that the scheme N is of finite type; then the group Λ is finitely generated, and one may choose $k \subset k'$ finite and separable. Moreover, N is a twisted form, already in the étale topology, of $\mathbb{G}_m[n_1] \oplus \ldots \oplus \mathbb{G}_m[n_r]$, with certain invariant factors $n_r | \ldots | n_1$. Here the summands are the kernels of the multiplicative group $\mathbb{G}_m$ with respect to multiplication by $n_i \geq 0$. For brevity, we call such N *multiplicative*.

Throughout this section, we study extensions of group schemes

$$0 \longrightarrow N \longrightarrow E \longrightarrow A \longrightarrow 0, \tag{4}$$

where A is abelian, N is multiplicative, and the middle term E is commutative, and analyze their splittings. Some of the assertions below are valid over general base schemes, but for the sake of exposition we stick to a ground field k. First note that $\mathrm{Hom}(A, N) = 0$, because N is affine and $h^0(\mathscr{O}_A) = 1$. Hence a splitting is unique, if it exists.

Recall that *exactness* means that for each $T = \mathrm{Spec}(R)$, the sequence of groups $0 \to N(R) \to E(R) \to A(R)$ is exact, and for each $a \in A(R)$ there is an fppf extension $R \subset R'$ such that the base-change a' is in the image of $E(R') \to A(R')$.

For any commutative group scheme G, we can consider the resulting abelian sheaves $\underline{\mathrm{Ext}}^i(G, \mathbb{G}_m)$ on the category (Aff/k) of affine schemes, endowed with the fppf topology. These are defined with injective resolutions of $\mathbb{G}_m$, but can also be seen as sheafifications of the presheaves that assign to each ring R the groups $\mathrm{Ext}^i(G_R, \mathbb{G}_{m,R})$. Its elements can also be interpreted as equivalence classes of Yoneda extensions, formed with sheaves of abelian groups. Note that the $\underline{\mathrm{Ext}}^i(G, \mathbb{G}_m)$ may or may not be representable by group schemes. Let us recall the following facts:

Proposition 6.1. *We have* $\underline{\mathrm{Ext}}^1(N, \mathbb{G}_m) = 0$. *Moreover,* $\underline{\mathrm{Hom}}(A, H) = 0$ *for any affine group scheme* H.

Proof. The first is contained in [33], Exposé VIII, Proposition 3.3.1. For the second assertion, suppose $f : A_R \to H_R$ is a homomorphism over some ring R. Since the formation of $\Gamma(A, \mathscr{O}_A) = k$ commutes with flat ring extensions, we have $R = \Gamma(A_R, \mathscr{O}_{A_R})$, and conclude that f factors over the zero section. □

We infer that the short exact sequence (4) yields an exact sequence

$$0 \longrightarrow \underline{\mathrm{Hom}}(E, \mathbb{G}_m) \longrightarrow \underline{\mathrm{Hom}}(N, \mathbb{G}_m) \longrightarrow \underline{\mathrm{Ext}}^1(A, \mathbb{G}_m) \longrightarrow \underline{\mathrm{Ext}}^1(E, \mathbb{G}_m) \longrightarrow 0$$

of abelian sheaves. The sheaf $\underline{\mathrm{Ext}}^1(A, \mathbb{G}_m)$ is representable by an abelian variety, and the theory of bi-extension gives an identification with the *dual abelian variety* $B = \mathrm{Pic}^\tau_{A/k}$, see the discussion in [49], Section 2. Actually, $A \mapsto B$ is an anti-equivalence of the category of abelian varieties with itself, coming with the *biduality identification* $A = \underline{\mathrm{Ext}}^1(B, \mathbb{G}_m)$.

Let us call a group scheme L a *local system* if it is a twisted form of the constant group scheme $(\Lambda)_k$, where Λ is a finitely generated abelian group. Equivalently, L is étale, and the group $L(k^{\mathrm{sep}})$ is finitely generated. For each multiplicative group scheme G, the sheaf $L = \underline{\mathrm{Hom}}(G, \mathbb{G}_m)$ is a local system. Actually, the functor $G \mapsto L$ is an anti-equivalence between the category of multiplicative group schemes and the category of local systems, which follows from [17], Exposé IX, Corollary 1.2. Summing up, our extension (4) gives a coboundary map

$$L = \underline{\mathrm{Hom}}(N, \mathbb{G}_m) \xrightarrow{\partial} \underline{\mathrm{Ext}}^1(A, \mathbb{G}_m) = B$$

from the local system L to the dual abelian variety B.

Proposition 6.2. *Our extension* (4) *splits if and only if* $\partial : L \to B$ *vanishes.*

Proof. The condition is obviously necessary. Suppose now that the coboundary map vanishes. We already remarked that the section is unique, if it exists, so with Galois descent it suffices to treat the case that k is separably closed. Then N is a direct sum of copies of the multiplicative group $\mathbb{G}_m$ and the kernels $\mu_n = \mathbb{G}_m[n]$, hence it is enough to consider the cases $N = \mathbb{G}_m$ and $N = \mu_n$. In the former case, the extension class of (4) is the image of the identity map $\mathrm{id} : N \to \mathbb{G}_m$ under the coboundary map, whence the extension splits. It remains to treat the case $N = \mu_n$. The canonical inclusion $\mu_n \subset \mathbb{G}_m$ yields a push-out extension $E' = (E \oplus \mathbb{G}_m)/\mu_n$, which yields a commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}(\mathbb{G}_m, \mathbb{G}_m) \arrow[r, "\partial"] \arrow[d] & \underline{\mathrm{Ext}}^1(A, \mathbb{G}_m) \arrow[d, "\mathrm{id}"] \\
\underline{\mathrm{Hom}}(\mu_n, \mathbb{G}_m) \arrow[r, "\partial"'] & \underline{\mathrm{Ext}}^1(A, \mathbb{G}_m).
\end{tikzcd}$$

On the right we have the dual abelian variety $B = \underline{\mathrm{Ext}}^1(A, \mathbb{G}_m)$. The image of the identity on $\mathbb{G}_m$ is a rational point $b \in B$, and we have $b = 0$ by assumption. Thus E' splits. The Kummer sequence gives an exact sequence

$$\mathrm{Hom}(A, \mathbb{G}_m) \longrightarrow \mathrm{Ext}^1(A, \mu_n) \longrightarrow \mathrm{Ext}^1(A, \mathbb{G}_m).$$

The term on the left vanishes. Thus the map on the right is injective, and E splits as well. $\square$

The coboundary map is functorial in the extension (4), by the very definition of delta functors ([21], Section 2.1). Consequently, the pull-back along some homomorphism of abelian varieties $A' \to A$ splits if and only if the composition $L \to B \to B'$ vanishes. We exploit this as follows: Let $Z \subset B$ be the Zariski closure of the set-theoretical image for $L \to B$. This is a smooth subgroup scheme, and its formation commutes with ground field extensions ([17], Exposé VIb, Proposition 7.1). The short exact sequence $0 \to Z \to B \to B' \to 0$ defines an abelian variety $B' = B/Z$, and we now consider the dual abelian variety A'. It comes with a homomorphism

$$A' = \underline{\mathrm{Ext}}^1(B', \mathbb{G}_m) \xrightarrow{f} \underline{\mathrm{Ext}}^1(B, \mathbb{G}_m) = A.$$

Theorem 6.3. *The homomorphism $f : A' \to A$ has the following properties:*

(i) *The formation of A' and f commutes with ground field extensions.*
(ii) *The pull-back of our extension (4) along $f : A' \to A$ splits.*
(iii) *The induced homomorphism $A' \to E$ is a closed embedding, and its image contains every abelian subvariety inside E.*
(iv) *For every homomorphism $g : A'' \to A$ from an abelian variety A'' such that the pullback of (4) along g splits, there is a unique factorization over $f : A' \to A$.*

Proof. Assertion (i) follows from the fact that the formation of B' commutes with ground field extensions, whereas (ii) is a consequence of Proposition 6.2. We next verify (iv). Suppose our extension is split by some $g : A'' \to A$. Let $B \to B''$ be the dual homomorphism. Its kernel $K \subset B$ is a closed subscheme. We already observed that the composite map $L \to B \to B''$ vanishes. Thus the set-theoretical image of $L \to B$ is contained in K, hence the Zariski closure Z is contained in K. The Isomorphism Theorem gives a unique factorization of $B \to B' \to B''$. Dualizing gives the desired factorization $A'' \to A' \to A$. This factorization is unique, by biduality.

It remains to establish (iii). The splitting for $E' = E \times_A A'$ is unique, as we already observed below (4). In turn, there is a unique lift $g : A' \to E$ for $f : A' \to A$. Consider the schematic image $A_0 = g(A')$ inside E, and the induced factorization $A' \to A_0 \to A$, and the dual factorization $B \to B_0 \to B'$. Obviously, the pull-back of (4) along $A_0 \to A$ splits. From (iv) we get a decomposition $A' = A_0 \oplus A_1$. This ensures that $B_0 \to B'$ is a closed embedding. Using that $B \to B'$ is surjective, we infer that $B_0 = B'$ is bijective, and it follows that $A' \to A_0$ is an isomorphism. □

The sheaf kernel $\mathrm{Ker}(\partial)$ and the sheaf image $\mathrm{Im}(\partial) = L/\mathrm{Ker}(\partial)$ for the homomorphism $\partial : L \to B$ are group schemes that are locally of finite type, and in fact local systems. According to Lemma 1.1, the sheaf cokernel $\mathrm{Coker}(\partial) = B/\mathrm{Im}(\partial)$ is an algebraic space that is not necessarily locally separated.

7. Maximal abelian subvarieties

Let k be a ground field of characteristic $p \geq 0$, and G be a group scheme of finite type. The goal of this section is to describe the maximal abelian subvariety $G_{\mathrm{ab}} \subset G$, making evident that its formation commutes with ground field extensions. We start by defining a three-step filtration $G = G_0 \supset G_1 \supset G_2 \supset G_3$, which is of independent interest.

Let G^{aff} be the spectrum of the ring $\Gamma(G, \mathscr{O}_G)$. Then the group law on G induces a group law on G^{aff}, and the canonical map $G \to G^{\mathrm{aff}}$ is a homomorphism. We denote by G_1 its kernel. Then $G \to G^{\mathrm{aff}}$ is flat and surjective, so that $G/G_1 = G^{\mathrm{aff}}$, and furthermore $h^0(\mathscr{O}_{G_1}) = 1$ ([15], Chapter III, §3, Theorem 8.2). The latter condition means that G_1 is *anti-affine*. According to [11], Proposition 3.3.4 this ensures that G_1 is smooth and commutative. We then define $G_2 \subset G_1$ as the largest subgroup scheme that is smooth, connected and affine ([11], Lemma 3.1.4). Finally, write $G_3 \subset G_2$ for the largest subgroup scheme that is multiplicative, so that G_2/G_3 is unipotent ([15], Chapter IV, §3, Theorem 1.1). This defines the desired three-step filtration on G.

Proposition 7.1. *The three-step filtration on G has the following properties:*

(i) *Each homomorphism $f : G \to G'$ respects the filtrations.*
(ii) *The formation of $G_i \subset G$ commutes with ground field extensions.*
(iii) *The group scheme G_1 is anti-affine, and G_1/G_2 is an abelian variety.*
(iv) *The extension $0 \to G_3 \to G_2 \to G_2/G_3 \to 0$ has a unique splitting.*
(v) *We have $G_2 = G_3$ in characteristic $p > 0$, whereas $G_2/G_3 \simeq \mathbb{G}_a^{\oplus r}$ for some $r \geq 0$ in characteristic zero.*

Proof. We start with assertion (iii). We already observed above that G_1 is anti-affine. In characteristic $p > 0$ it is also semi-abelian, so that there is a short exact sequence

$$0 \longrightarrow T \longrightarrow G_1 \longrightarrow A \longrightarrow 0, \tag{5}$$

for some torus T and some abelian variety A, according to [10], Proposition 2.2. The torus is contained in G_2, by maximality of the latter. The resulting surjection $G_1/T \to G_1/G_2$ reveals that G_1/G_2 is abelian. In characteristic zero we use the smallest subgroup scheme $N \subset G_1$ such that the quotient G_1/N is proper. This exists in all characteristics, and is affine and connected, according to [11], Theorem 2. For $p = 0$ the group schemes G_1/N and N are automatically smooth. Consequently G_1/N is abelian and $N \subset G_2$, and we conclude again that G_1/G_2 is abelian.

Next we consider assertion (i). Let $f : G \to G'$ be a homomorphism. The composite morphism $G_1 \to G'^{\mathrm{aff}}$ vanishes, because $h^0(\mathscr{O}_{G_1}) = 1$. The next composite $G_2 \to G'_1/G'_2$ also vanishes: Its image is a quotient of G_2 and thus smooth, connected and affine. It is also a closed subgroup scheme in G'_1/G'_2, hence proper, and therefor trivial. Finally, $G_3 \to G'_2/G'_3$ is zero, because the domain is multiplicative and the range is unipotent ([15], Chapter IV, §3, Proposition 1.3).

We now come to (iv). The extension in question has at most one splitting, because there are no non-trivial homomorphisms from the unipotent group scheme G_2/G_3 to the multiplicative group scheme G_2. If k is perfect, such a splitting indeed exists, by [15], Chapter IV, §3, Theorem 1.1. This also ensures that G_3 is smooth and connected. Suppose now $p > 0$. Then we have an exact sequence (5). The canonical projection $G_3 \to A$ vanishes, and we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & G_3 \arrow[r] \arrow[d] & G_2 \arrow[r] \arrow[d] & G_2/G_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T \arrow[r] & G_1 \arrow[r] & A \arrow[r] & 0.
\end{tikzcd}$$

The induced map $G_2/G_3 \to A$ is zero, so the Snake Lemma gives an inclusion $G_2/G_3 \subset T/G_3$. Consequently, G_2/G_3 is both unipotent and multiplicative, hence zero. In turn, the extension splits for trivial reasons. This also establishes the first part of (iv). In characteristic zero, the unipotent scheme G_2/G_3 must be smooth. It follows from [15], Chapter IV, §2 that it is isomorphic to a sum of additive groups $\mathbb{G}_a$.

It remains to establish (ii). For each quasicompact and quasiseparated scheme X, the formation of $H^0(X, \mathscr{O}_X)$ commutes with ground field extensions $k \subset k'$. In turn, the same holds for the kernel $G_1 \subset G$. Suppose N' is a smooth connected affine subgroup scheme in the base-change $G_1' = G_1 \otimes k'$ that contains $G_2' = G_2 \otimes k'$. Then the quotient $H' = N'/G_2'$ is smooth, connected and affine, and comes with an embedding into the base-change of the abelian variety $A = G_1/G_2$. It follows that $H' = 0$. Thus $G_2 \subset G_1$ commutes with base-change. Finally, the inclusion $G_3 \subset G_2$ commutes with base-change by [15], Chapter IV, §3, Proposition 1.3. ☐

Let $H \subset G$ be an abelian subvariety of largest dimension. Then $H \subset G_1$, because G/G_1 is affine and $h^0(\mathscr{O}_H) = 1$. Since G_1 is commutative, and sums and quotients of abelian varieties remain abelian, every other abelian subvariety $H' \subset G_1$ must be contained in H. Therefore, $G_{\mathrm{ab}} = H$ is the largest abelian subvariety, and we also call it the *maximal abelian subvariety* of G. Clearly, this is functorial in G. However, it is not immediately evident that its formation commutes with ground field extensions.

In characteristic $p > 0$, we have $G_2 = G_3$ and an extension $0 \to T \to G_1 \to A \to 0$ of the abelian variety $A = G_1/G_2$ by the torus $T = G_2$. Let $A' \subset G_1$ be the abelian subvariety constructed in Section 6 as the dual of $B' = B/Z$, and regard A' as a subgroup scheme of G.

Theorem 7.2. *The formation of the maximal abelian subvariety $G_{\mathrm{ab}} \subset G$ commutes with ground field extensions $k \subset k'$. In characteristic $p > 0$ we have $G_{\mathrm{ab}} = A'$.*

Proof. Let $k \subset \Omega$ be a field extension and write $G_\Omega = G \otimes \Omega$. It suffices to check that the maximal abelian subvarieties inside G and G_Ω have the same dimension. Seeking a contradiction, we assume that there is an abelian subvariety $N \subset G_\Omega$ with $\dim(N) > \dim(G_{\mathrm{ab}})$. Replacing G by the subgroup scheme G_1 we reduce to the case that G is commutative, hence G_{ab} must be normal. By passing to G/G_{ab} we may also assume that $G_{\mathrm{ab}} = 0$. We now have a non-trivial abelian subvariety $N \subset G_\Omega$, and will reach a contradiction by producing a non-trivial abelian subvariety $H \subset G$. Note that for along the way we may enlarge Ω.

We now argue as in [54], proof for Theorem 6.1: Using [29], Theorem 8.8.2 we reduce to the case that the field extension $k \subset \Omega$ is finitely generated. By considering suitable intermediate field and enlarging Ω if necessary, it suffices to treat the cases that $k \subset \Omega$ is either purely transcendental, or a finite Galois extension, or a finite radical extension in characteristic $p > 0$. In the first case, we extend $N \subset G_\Omega$ to a family of abelian varieties in G_R over some localization $R = k[T_1, \ldots, T_n]_f$, and obtain a contradiction by specializing to a rational point in $\mathrm{Spec}(R)$. In the second case we use that the maximal abelian subvariety of G_Ω is stabilized by the elements of the Galois group $\Gamma = \mathrm{Gal}(\Omega/k)$, hence descends to an abelian subvariety in G. In the third case we are in characteristic $p > 0$, and this was essentially solved above: From Theorem 6.3 we get $G_{\mathrm{ab}} = A'$, and its formation commutes with ground field extensions. ☐

8. The notion of Albanese maps

Let S be a base scheme, and X be an algebraic space where the structure morphism $X \to S$ is proper, flat, of finite presentation, and cohomologically flat in degree $d = 0$. Then $\operatorname{Pic}^{\tau}_{X/S}$ exists as an algebraic space whose structure morphism is of finite presentation (hence quasiseparated) and locally separated, according to Theorem 2.1. We now come to the central topic of this paper, where we re-define and generalize classical notions of Albanese varieties and Albanese maps:

Definition 8.1. An *Albanese map* for X is a pair (P, f) where $P \to S$ is a family of para-abelian varieties and $f : X \to P$ is a morphism satisfying the following condition: For each $s \in S$ the homomorphism f^* identifies $\operatorname{Pic}^{\tau}_{P/S} \otimes \kappa(s)$ with the maximal abelian subvariety of $\operatorname{Pic}^{\tau}_{X/S} \otimes \kappa(s)$.

By abuse of notation, *we simply say that $f : X \to P$ is an Albanese map.* From Lemma 11.4 we see that $f^* : \operatorname{Pic}^{\tau}_{P/S} \to \operatorname{Pic}^{\tau}_{X/S}$ is a monomorphism. It is actually a closed embedding provided that $\operatorname{Pic}^{\tau}_{X/S}$ is separated, and the latter indeed holds if S is artinian.

A priori, our notion of Albanese maps has good base-change properties: Let $f : X \to P$ be any morphism to a family P of para-abelian varieties. Given $S' \to S$ we write $X' = X \times_S S'$ and $P' = P \times_S S'$ for the base-changes, and $f' : X' \to S'$ for the induced morphism.

Proposition 8.2. *In the above situation, if $f : X \to P$ is an Albanese map, the same holds for the base-change $f' : X' \to P'$. The converse remains true if $S' \to S$ is surjective.*

Proof. This follows from the fact that the maximal abelian subvarieties are stable under field extensions (Theorem 7.2). ☐

The next observation is an important intermediate step towards existence, uniqueness and universal property of Albanese maps:

Proposition 8.3. *Suppose $f : X \to P$ is an Albanese map. Let $g : X \to Q$ be a morphism to some other family Q of para-abelian varieties. Then the homomorphism $g^* : \operatorname{Pic}^{\tau}_{Q/S} \to \operatorname{Pic}^{\tau}_{X/S}$ admits a unique factorization over $f^* : \operatorname{Pic}^{\tau}_{P/S} \to \operatorname{Pic}^{\tau}_{X/S}$. Moreover, there is at most one morphism $h : P \to Q$ such that the diagram*

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ P \arrow[rr, "h"'] & & Q \end{tikzcd} \tag{6}$$

is commutative.

Proof. Since f^* is a monomorphism, there is at most one factorization for g^*. Consequently, the existence of a factorization is local in S for the fpqc topology, and it suffices to treat the case that S is the spectrum of a ring R. Since X is of finite presentation, we moreover may assume that R is noetherian. Set

$$G = \operatorname{Pic}^{\tau}_{X/S} \quad \text{and} \quad N = \operatorname{Pic}^{\tau}_{P/S} \quad \text{and} \quad H = \operatorname{Pic}^{\tau}_{Q/S}.$$

The structure morphism $N \to S$ is fppf, and the translation action of N on G is free, hence the quotient G/N exists as an algebraic space, and its formation commutes with base-change, according to Lemma 1.1. We see that $g^* : H \to G$ factors over N if and only if the composite homomorphism $c : H \to G/N$ is trivial. Note that the structure morphism $G/N \to S$ is of finite type, because the same holds for $G \to S$ and the projection $G \to G/N$ is fppf. Moreover, for each point $s \in S$, the inclusion $N_s \subset G_s$ is the maximal abelian subvariety, so the quotient $(G/N)_s = G_s/N_s$ does not contain any non-zero abelian subvarieties.

Now suppose that R is a local Artin ring, such that $S = \{s\}$. The closed fiber $(G/N)_s$ is a separated scheme. Hence also G/N is separated, and it is schematic by [50], Corollary 8.2. The set-theoretic image of $H \to G/N$ is the origin $e_s \in (G/N)_s$ in the closed fiber. By the Rigidity Lemma ([43], Proposition 6.1), the morphism $H \to G/N$ must factor over some section $\sigma : S \to G/N$. We must have $\sigma = e$, because the image of the map $H(R) \to G/N(R)$ is a subgroup.

Using Grothendieck's Comparison Theorem ([26], Theorem 5.4.1 for schemes and [57], Lemma 0A4Z for algebraic spaces), we infer that $H \to G/N$ is trivial if R is a local noetherian ring that is complete. For general local noetherian rings R, the formal completion $\widehat{R} = \varprojlim R/\mathfrak{m}_R^n$ is an fpqc extension, and it follows from uniqueness applied over $\widehat{R} \otimes_R \widehat{R}$ and fpqc descent that $H \to G/N$ then is trivial as well. For general noetherian rings R, one sees that for each prime $\mathfrak{p} \subset R$, there is an element $f \in R \smallsetminus \mathfrak{p}$ such that $H \to G/N$ is trivial over the localization R_f. Hence $H \to G/N$ is trivial over the whole spectrum $S = \operatorname{Spec}(R)$.

Suppose there are two morphisms $h_1, h_2 : P \to Q$ making the above diagram commutative. It remains to show that $h_1 = h_2$. We just saw that the induced homomorphisms $h_i^* : \operatorname{Pic}^\tau_{Q/S} \to \operatorname{Pic}^\tau_{P/S}$ coincide. Let $I \subset \operatorname{Aut}_{Q/S}$ be the inertia subgroup scheme for the numerically trivial sheaves. We showed in Section 5 that this is a family of abelian varieties, and its action on Q is free and transitive. By Lemma 5.6 there is a unique section $\sigma \in I(S)$ with $h_2 = \sigma + h_1$. By fppf descent, we may replace S with X and assume that the structure morphism $X \to S$ admits a section $\tau \in X(S)$. The commutativity of the diagram (6) reveals that the h_1 and h_2 coincide on the section $f \circ \tau \in P(S)$, thus $\sigma \in I(S)$ must be trivial. This shows $h_1 = h_2$. $\square$

In the situation of the Proposition, let $S' \to S$ be some morphism of schemes, and write X', P', Q' for the base-changes, and $f' : X' \to P'$ and $g' : X' \to Q'$ for the induced maps.

Corollary 8.4. *Suppose that $S' \to S$ is fpqc, and that there is some $h' : P' \to Q'$ with $g' = h' \circ f'$. Then there is also a morphism $h : P \to Q$ with $g = h \circ f$, and h' equals the base-change of h.*

Proof. Consider the fiber product $S'' = S' \times_S S'$, which comes with two projections $\mathrm{pr}_i : S'' \to S'$. By the uniqueness in the proposition applied over S'', we have an equality $\mathrm{pr}_1^*(h') = \mathrm{pr}_2^*(h')$. From fpqc descent ([32], Exposé VIII, Theorem 5.2 for schemes and [57], Lemma 0ADV for algebraic spaces) one deduces that there is a unique S-morphism $h : P \to Q$ inducing the S'-morphism $h' : P' \to Q'$. The property $g = h \circ f$ can be checked after base-changing to S', where it holds by assumption. $\square$

9. Poincaré sheaves

We now review the notion of Poincaré sheaves, which will be used in the next section to construct Albanese maps. Fix a base scheme S, and let X be an algebraic space whose structure morphism $\varphi : X \to S$ is proper, flat, of finite presentation, and cohomologically flat in degree $d = 0$, with $h^0(\mathscr{O}_{X_s}) = 1$ for all points $s \in S$. Thus $\varphi_*(\mathscr{O}_X) = \mathscr{O}_S$, and this commutes with base-change. Consequently $\varphi_*(\mathbb{G}_{m,X}) = \mathbb{G}_{m,S}$, and this also commutes with base-change. By the very definition, the algebraic space $\operatorname{Pic}_{X/S}$ represents the higher direct image $R^1\varphi_*(\mathbb{G}_{m,X})$. For each T, the Leray–Serre spectral sequence for the projection $\operatorname{pr}_2 : X_T \to T$, together with Hilbert 90, gives a short exact sequence

$$(7) \quad 0 \longrightarrow \operatorname{Pic}(T) \longrightarrow \operatorname{Pic}(X_T) \longrightarrow \operatorname{Pic}_{X/S}(T) \longrightarrow H^2(T, \mathbb{G}_m) \xrightarrow{\operatorname{pr}_2^*} H^2(X_T, \mathbb{G}_m).$$

We are particularly interested in the case that T coincides with $\operatorname{Pic}_{X/S}$, or at least comes with a morphism to $\operatorname{Pic}_{X/S}$:

Definition 9.1. Suppose we have a morphism $f : T \to \operatorname{Pic}_{X/S}$. An invertible sheaf $\mathscr{P}$ on $X \times T$ is called a *Poincaré sheaf with respect to* $f : T \to \operatorname{Pic}_{X/S}$ if its class in $\operatorname{Pic}(X_T)$ maps to the element $f \in \operatorname{Pic}_{X/S}(T)$ in the above sequence.

By abuse of notation, we simply say that $\mathscr{P}$ is a Poincaré sheaf on $X \times T$. Note that as customary, the product is formed over the base scheme S, such that $X \times T = X \times_S T$. Poincaré sheaves exist if the map $H^2(T, \mathbb{G}_m) \to H^2(X_T, \mathbb{G}_m)$ is injective, by exactness of (7). This obviously holds if the structure map $\varphi : X \to S$ has a section, or if the cohomology group $H^2(T, \mathbb{G}_m)$ vanishes. In any case, Poincaré sheaves are unique up to preimages of invertible sheaves on T.

Now let $\mathscr{L}$ be any invertible sheaf on the product $X \times T$, for some algebraic space T. This gives an element in $\operatorname{Pic}(X_T)$, which induces a T-valued point for $\operatorname{Pic}_{X/S}$. Let $f : T \to \operatorname{Pic}_{X/S}$ be the resulting classifying map. From the exact sequence (7) we immediately get:

Proposition 9.2. *Suppose $\mathscr{P}$ is a Poincaré sheaf $\mathscr{P}$ on $X \times T$. Up to isomorphism, there is a unique invertible sheaf $\mathscr{N}$ on T such that $(\mathrm{id}_X \times f)^*(\mathscr{L})$ is isomorphic to $(\mathscr{P}|X_T) \otimes \operatorname{pr}_2^*(\mathscr{N})$.*

In particular, for every field k and every $l : \operatorname{Spec}(k) \to T$, the isomorphism class of the pull-back $\mathscr{P}|X \otimes k$ corresponds to the induced k-valued point $l \in \operatorname{Pic}_{X/S}(k)$. We also see that in general, Poincaré sheaves do not exist: For example, a smooth curve C of genus $g = 0$ over a ground field k has constant Picard scheme $\operatorname{Pic}_{C/k} = (\mathbb{Z})_k$. By Riemann–Roch, if there is a Poincaré sheaf on $T = \operatorname{Pic}_{C/k}$, or even on $T = \{1\}$ then C contains a rational point, and is thus isomorphic to the projective line. This leads to the following criterion:

Proposition 9.3. *Suppose there are morphisms $g_i : Z_i \to X$, $1 \leq i \leq r$ such that each structure morphism $h_i : Z_i \to S$ is locally free of degree $d_i \geq 1$, with $\gcd(d_1, \ldots, d_r) = 1$. Then there is a Poincaré sheaf on $X \times \operatorname{Pic}_{X/S}$.*

Proof. Set $T = \operatorname{Pic}_{X/S}$. Fix an index $1 \leq i \leq r$, write $Z = Z_i$ and $g = g_i$. The map on the right in (7) sits in a sequence

$$H^2(T, \mathbb{G}_m) \xrightarrow{\operatorname{pr}_2^*} H^2(X_T, \mathbb{G}_m) \xrightarrow{g^*} H^2(Z_T, \mathbb{G}_m).$$

Write $\psi = \operatorname{pr}_2 \circ g$ for the composite morphism $Z_T \to T$. One easily checks that the direct image sheaves $R^i\psi_*(\mathbb{G}_{m,Z_T})$ vanish for all degrees $i \geq 1$ (compare [53], proof for Lemma 1.4). Consequently, the Leray–Serre spectral sequence gives an identification $H^2(Z_T,\mathbb{G}_m) = H^2(T,\psi_*(\mathbb{G}_{m,Z_T}))$. The composition of the inclusion $\mathbb{G}_{m,T}\subset\psi_*(\mathbb{G}_{m,Z_T})$ with the norm map $N:\psi_*(\mathbb{G}_{m,Z_T})\to\mathbb{G}_{m,T}$ is multiplication by $d=d_i$. So for each $\alpha\in H^2(T,\mathbb{G}_m)$ we have $d\cdot\alpha = N_*(g^*(\mathrm{pr}_2^*(\alpha)))$. We conclude that the kernel for the map on the right in (7) is annihilated by $\gcd(d_1,\ldots,d_r)=1$. Thus $\mathrm{Pic}(X_T)\to\mathrm{Pic}_{X/S}(T)$ is surjective, hence a Poincaré sheaf exists. $\square$

We also have a non-existence result:

Proposition 9.4. *Suppose S is the spectrum of a field k, and that there is a quasicompact scheme Z so that the pull-back map $H^2(Z,\mathbb{G}_m)\to H^2(X_Z,\mathbb{G}_m)$ is not injective. Then there is some point $l\in\mathrm{Num}_{X/k}$ and some connected component $T\subset\mathrm{Pic}^l_{X/k}$ such that there is no Poincaré sheaf with respect to T.*

Proof. Choose some section $s_Z\in H^0(Z,R^1\mathrm{pr}_{2,*}(\mathbb{G}_{m,X_Z}))$ mapping to some nontrivial element of the kernel for $H^2(Z,\mathbb{G}_m)\to H^2(X_Z,\mathbb{G}_m)$. The resulting classifying map $h:Z\to\mathrm{Pic}_{X/k}$ factors over a finite union of connected components, corresponding to points $l_1,\ldots,l_r\in\mathrm{Num}_{X/k}$. Moreover, s_Z is the pull-back of the universal section. The decomposition of $\mathrm{Pic}_{X/k}$ into connected components gives a decomposition of the quasicompact scheme Z into open-and-closed subschemes $Z_1,\ldots,Z_r$. By passing to one of them, we may assume $r=1$ and let $T\subset\mathrm{Pic}^{l_1}_{X/k}$ be the connected component containing the image. Suppose that there is a Poincaré sheaf $\mathscr{P}_{X\times T}$. Then the restriction s_T of the universal section maps to zero under the coboundary. In turn, its pull-back s_Z also maps to zero under the coboundary, contradiction. $\square$

10. Existence and universal property

In this section we establish existence and uniqueness results for Albanese maps. These are consequences of a general criterion, for which we have to generalize maximal abelian subvarieties to a relative setting. We start by doing this. Let S be a base scheme, and G be an algebraic space endowed with a group structure. Assume that the structure morphism $G\to S$ is of finite type.

Definition 10.1. A *family of maximal abelian subvarieties* for G is a family of abelian varieties $A\to S$, together with a homomorphism $i:A\to G$ such that for each point $s\in S$, the map identifies the fiber A_s with the maximal abelian subvariety inside the group scheme G_s.

Note that $i:A\to G$ is a monomorphism, provided that G is locally separated and S is noetherian, according to Lemma 11.4. Then the translation action of A on G is free, so the quotient G/A exists as an algebraic space by Lemma 1.1.

Now suppose that X is an algebraic space whose structure morphism $X\to S$ is proper, flat, of finite presentation and cohomologically flat in degree $d=0$, with $h^0(\mathscr{O}_{X_s})=1$ for all points $s\in S$. Then $\mathrm{Pic}^\tau_{X/S}$ is an algebraic space endowed with a group structure whose structure morphism is of finite presentation (hence quasiseparated) and locally separated. Note that it is neither flat nor separated in general. We can formulate the main result of this paper:

Theorem 10.2. *Assumptions as above. Then $G=\mathrm{Pic}^\tau_{X/S}$ admits a family of maximal abelian subvarieties if and only if there is an Albanese map $f:X\to P$.*

Moreover, it is universal for morphisms to families of para-abelian varieties, and commutes with base-change.

Note that in the relative setting, the assumption is restrictive: For example, a Weierstraß equation over a discrete valuation ring R whose discriminant Δ is non-zero and belongs to $\mathfrak{m}_R$ defines a family $X \subset \mathbb{P}^2_R$ of cubics where $G = \operatorname{Pic}^\tau_{X/S}$ does not admit a family of maximal abelian subvarieties: The generic fiber G_η is a one-dimensional abelian variety isomorphic to X_η, whereas the closed fiber of G is isomorphic to the multiplicative or the additive group. Also note that for the family $X \to S$ of Enriques surfaces considered in Proposition 2.2 the $\operatorname{Pic}^\tau_{X/S}$ admits a family of maximal abelian subvarieties, namely the zero family.

The proof of the theorem is given at the end of this section. The universal property ensures that the Albanese map is *unique up to unique isomorphism.* This justifies to write $\operatorname{Alb}_{X/S} = P$, and we call it the *family of Albanese varieties* for X. The Albanese map becomes

$$f : X \longrightarrow \operatorname{Alb}_{X/S}.$$

This is equivariant with respect to actions of algebraic spaces endowed with a group structure, an observation that seems to be new for infinitesimal actions, even over ground fields:

Corollary 10.3. *Assumptions as in the theorem. Then there is a unique action of* $\operatorname{Aut}_{X/S}$ *on the family of Albanese varieties* $\operatorname{Alb}_{X/S}$ *that makes the Albanese map* $f : X \to \operatorname{Alb}_{X/S}$ *equivariant.*

Proof. Let $\sigma \in \operatorname{Aut}(X)$ be an S-automorphism. By the universal property in the theorem, there is a unique morphism σ_* completing the following diagram:

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ \operatorname{Alb}_{X/S} \arrow[r, dashed, "\sigma_*"'] & \operatorname{Alb}_{X/S} \end{tikzcd}$$

The uniqueness ensures that $\sigma \mapsto \sigma^*$ respects compositions and identities. In turn, there is a unique action of the group $\operatorname{Aut}(X)$ making the Albanese map equivariant.

Since the Albanese map commutes with base-change, the same holds for the σ_*. Applying the above reasoning with R-valued points of $\operatorname{Aut}_{X/S}$ and using the Yoneda Lemma, we obtain the desired action of the group scheme $\operatorname{Aut}_{X/S}$ making the Albanese map equivariant. $\square$

One may reformulate the theorem in categorical language: Let $\mathcal{C}$ be the category of all algebraic spaces X whose structure morphism $X \to S$ is proper, flat, of finite presentation, cohomologically flat in degree $d = 0$, with $h^0(\mathscr{O}_{X_s}) = 1$ for all $s \in S$, and such that $G = \operatorname{Pic}^\tau_{X/S}$ admits a family of maximal abelian subvarieties. Write $\mathcal{C}' \subset \mathcal{C}$ for the full subcategory comprising all families of para-abelian varieties. The arguments for the following, which are purely formal and analogous to the preceding proof, are left to the reader:

Corollary 10.4. *For each morphism $\varphi : X \to X'$ in the category $\mathcal{C}$, there is a unique morphism $\varphi_* : \mathrm{Alb}_{X/S} \to \mathrm{Alb}_{X'/S}$ making the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ \mathrm{Alb}_{X/S} & \xrightarrow[\varphi_*]{} & \mathrm{Alb}_{X'/S} \end{array}$$

commutative. Moreover, $\varphi \mapsto \varphi_*$ respects composition and identities, and the resulting functor $\mathcal{C} \to \mathcal{C}'$ given by $X \mapsto \mathrm{Alb}_{X/S}$ is left adjoint to the inclusion functor $\mathcal{C}' \to \mathcal{C}$.

Over ground fields, the assumption in the theorem is vacuous, which gives:

Corollary 10.5. *Suppose that S is the spectrum of field k. Then there is an Albanese map $f : X \to \mathrm{Alb}_{X/k}$. It is universal for morphisms to para-abelian varieties, commutes with field extensions, is equivariant with respect to group scheme actions, and functorial in X.*

This generalizes previous results for perfect ground fields, or for geometrically integral schemes, or geometrically connected and geometrically reduced schemes, compare the appendix in [60]. See also [1] when X is non-proper. We can actually formulate an unconditional result in the relative setting, in the spirit of [7], Exposé XII, Section 1:

Corollary 10.6. *Suppose that the base scheme S is integral, and that the generic fiber of $\mathrm{Pic}^\tau_{X/S} \to S$ is proper. Then after replacing S with some dense open set U, there is an Albanese map $f : X \to \mathrm{Alb}_{X/S}$. It is universal for morphisms into families of para-abelian varieties, commutes with base-change, and is equivariant with respect to actions of relative group spaces.*

Proof. Without restriction we may assume that S is the spectrum of an integral noetherian ring R. We have to find some U such that $G = \mathrm{Pic}^\tau_{X/S}$ admits a family of maximal abelian subvarieties over U. Let $\eta \in S$ be the generic point. Applying Proposition 7.1 to the proper group scheme G_η over the field of fractions $F = \mathrm{Frac}(R)$, we see that the maximal abelian variety $A_\eta \subset G_\eta$ is the kernel for the affinization map.

It suffices to treat the case that the algebraic space G is schematic, and that A_η extends to a family of abelian varieties $A \to S$, by shrinking S. In the same way we may assume that the inclusion $A_\eta \subset P_\eta$ extends to some homomorphism $i : A \to G$, and that the diagonal embedding $G \to G \times G$ is closed. Now the kernel $N = \mathrm{Ker}(i)$ is a closed subgroup scheme of A, and in particular proper, with $N_\eta = 0$. By Chevalley's Semicontinuity Theorem ([29], Corollary 13.1.5) we may shrink S further making the fibers N_s finite. By generic flatness ([28], Theorem 6.9.1), we even achieve $N = 0$.

It remains to verify that $A_s \subset G_s$ are maximal abelian subvarieties for a dense open set of points $s \in S$. Let $f : G \to S$ be the structure morphism, and set $g = \dim(G_\eta)$. Again by Chevalley's Semicontinuity Theorem ([29], Theorem 13.1.3), the set $Z \subset G$ of all points $x \in G$ with $\dim_x(G_{f(x)}) \geq g + 1$ is closed. Its image $f(Z) \subset S$ is constructible, and disjoint from η. So after shrinking S, we may assume that all fibers of $f : G \to S$ are g-dimensional. For dimension reasons, the inclusion $A_s \subset G_s$ must be the maximal abelian subvariety, for all points $s \in S$. $\square$

Moret-Bailly and one referee alerted us that the conclusion does not hold without suitable assumption on the generic fiber, as the following example shows: Let $X_\mathbb{Q}$ be the denormalization of an elliptic curve $E_\mathbb{Q}$ that identifies the origin $e \in E_\mathbb{Q}$ with a rational point $a \in E_\mathbb{Q}$ of infinite order. Any morphism $X_\mathbb{Q} \to P_\mathbb{Q}$ to an abelian variety induces a homomorphism $E_\mathbb{Q} \to P_\mathbb{Q}$ with a in the kernel. It follows that these morphisms are constant, hence $\mathrm{Alb}_{X_\mathbb{Q}/\mathbb{Q}}$ is trivial. On the other hand, an extension X over some suitable $R = \mathbb{Z}[1/n]$ yields a $\mathrm{Pic}^\tau_{X/S}$ that is an extension of a family $E \to S$ of elliptic curves by the multiplicative group. Over each closed point $s \in S$, the residue field $\kappa(s) = \mathbb{F}_p$ is finite, hence the corresponding Ext group is finite, and the class of the extension has finite order. It follows that $\mathrm{Alb}_{X_s/\mathbb{F}_p}$ is one-dimensional.

Proof of Theorem 10.2. If there is an Albanese map $f : X \to P$, the image of the monomorphism $f^* : \mathrm{Pic}^\tau_{P/S} \to \mathrm{Pic}^\tau_{X/S}$ is a family maximal abelian subvarieties. Our task is to establish the converse: Suppose there is a family $A \subset \mathrm{Pic}_{X/S}$ of maximal abelian subvarieties. We already saw in Proposition 8.2 that the Albanese map commutes with base-change, once it exists. Our task here is to establish existence and universal property. We proceed in five intertwined steps, with various temporary assumptions:

Step 1: *We show existence, assuming that there is a Poincaré sheaf $\mathscr{P}$ on the product $X \times \mathrm{Pic}_{X/S}$.* Since the family of abelian varieties A has a section, there is also a Poincaré sheaf $\mathscr{F}$ on $\mathrm{Pic}_{A/S} \times A$. Tensoring with the preimage of some invertible sheaf on $\mathrm{Pic}^\tau_{A/S}$ we may assume that $\mathscr{F}$ becomes trivial on both $\{e\} \times A$ and $\mathrm{Pic}^\tau_{A/S} \times \{e\}$. Such Poincaré sheaves are called *normalized*. Now regard the restriction $\mathscr{P}|_{X\times A}$ as a family of invertible sheaves on A parameterized by X. Let $f : X \to \mathrm{Pic}_{A/S}$ be the classifying map. Then the sheaves $\mathscr{P}|_{X\times A}$ and $(f \times \mathrm{id}_A)^*(\mathscr{F})$ define the same elements in $\mathrm{Pic}_{A/S}(X)$. Like (7) we have an exact sequence

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X \times A) \longrightarrow \mathrm{Pic}_{A/S}(X),$$

so $\mathscr{P}|_{X\times A}$ and $(f \times \mathrm{id}_A)^*(\mathscr{F})$ differ only by the pull-back of some invertible sheaf on X. Form the relative numerical group $\mathrm{Num}_{A/S}$ and consider the composite map $\bar{f} : X \to \mathrm{Num}_{A/S}$. This factors over some section $\delta : S \to \mathrm{Num}_{A/S}$, by Lemma 3.2. The preimage $P = \mathrm{Pic}^\delta_{A/S}$ is a family of para-abelian varieties. It comes with the structure of a principal homogeneous space with respect to $\mathrm{Pic}^\tau_{A/S}$, stemming from tensor product of invertible sheaves. By construction, the classifying map factors as $f : X \to P$. We claim that this is an Albanese map.

To see this, it suffices by our very definition to treat the case that S is the spectrum of an algebraically closed field k. In light of Hilbert's Nullstellensatz, there is a rational point $a \in X$. This induces a rational point $e = f(a)$ on P, and the latter becomes an abelian variety. Moreover, the rational point $\delta \in \mathrm{Num}_{A/S}$ comes from an invertible sheaf $\mathscr{N}$ on A. Tensor products with $\mathscr{N}$ give an isomorphism $\mathrm{Pic}^\tau_{A/S} \to \mathrm{Pic}^\delta_{A/S}$. Composing f with its inverse, we may assume that $\delta = e$. The normalized Poincaré sheaf $\mathscr{F}$ induces identifications $A = \mathrm{Pic}^\tau_{P/k}$ and $P = \mathrm{Pic}^\tau_{A/k}$. By construction, the abelian varieties P and A have the same dimension, so we merely have to check that the kernel N of $f^* : \mathrm{Pic}^\tau_{P/k} \to \mathrm{Pic}^\tau_{X/k}$ is trivial. Suppose this is not the case. Then N contains either a non-zero rational point or a tangent vector supported by the origin. In any case, there is a k-algebra R of degree $[R : k] = 2$ with an embedding $\mathrm{Spec}(R) \subset N \subset \mathrm{Pic}^\tau_{P/k} = A$ and some non-trivial invertible sheaf $\mathscr{L}$ on $P \otimes_k R$ such that $(f \otimes \mathrm{id}_R)^*(\mathscr{L})$ becomes trivial on $X \otimes_k R$. In particular, $\mathscr{L}$ is numerically trivial. Let $l : \mathrm{Spec}(R) \to \mathrm{Pic}^\tau_{P/k} = A$ be the classifying map, such that $\mathscr{L} \simeq \mathscr{F}|_{P\otimes_k R}$. It follows that

$$(f \otimes \mathrm{id}_R)^*(\mathscr{F}|_{P\otimes_k R}) \simeq (f \otimes \mathrm{id}_R)^*(\mathscr{L}) \simeq \mathscr{O}_{X\otimes_k R}.$$

Now recall that $\mathscr{P}|_{X\times A}$ and $(f \times \mathrm{id}_A)^*(\mathscr{F})$ differ by the preimage of some invertible sheaf $\mathscr{M}$ on X. We infer that $\mathscr{P}|_{X\otimes_k R} \simeq \mathscr{M} \otimes_k R$. Regarding $\mathscr{P}|_{X\otimes_k R}$ as a family of invertible sheaves on X parameterized by $\mathrm{Spec}(R)$, we conclude that the classifying map $\mathrm{Spec}(R) \to \mathrm{Pic}_{X/k}$ factors over a closed point. On the other hand, since $\mathscr{P}$ is the Poincaré sheaf, the classifying map is the composition of the embeddings $\mathrm{Spec}(R) \subset A$ and $A \subset \mathrm{Pic}_{X/k}$, contradiction.

Step 2: *We verify the universal property for the particular Albanese map above, assuming that X and all the algebraic spaces $\mathrm{Pic}^\delta_{X/S}$, $\delta \in \mathrm{Num}_{X/S}(S)$ admit sections.* Since X has a section, there is a Poincaré sheaf $\mathscr{P}$ on $X \times \mathrm{Pic}_{X/S}$. Let $f : X \to P$ be the resulting Albanese map constructed in step 1, and $g : X \to Q$ be a morphism into another family of para-abelian varieties. We have to verify that g factors over f, via some $h : P \to Q$. Note that we already saw in Proposition 8.3 that such a factorization is unique, once it exists. Fix a section $s : S \to X$. This induces a section for $Q \to S$, which therefore becomes a family of abelian varieties. Choose a normalized Poincaré sheaf $\mathscr{G}$ on $\mathrm{Pic}^\tau_{Q/S} \times Q$. Viewing this as a family of invertible sheaves on $\mathrm{Pic}^\tau_{Q/S}$ parameterized by Q, we see that $g : X \to Q$ is the classifying map for each of the sheaves

$$\mathscr{M} = (\mathrm{id} \times g)^*(\mathscr{G}) \otimes \mathrm{pr}_X^*(\mathscr{L}) \tag{8}$$

on $\mathrm{Pic}^\tau_{Q/S} \times X$, where $\mathscr{L}$ is any invertible sheaf on X. Our task is to find $\mathscr{L}$ such that $\mathscr{M} \simeq (\mathrm{id}\times f)^*(\mathscr{N})$ for some invertible sheaf $\mathscr{N}$ on $\mathrm{Pic}^\tau_{Q/S} \times P$, which then gives the desired factorization $g = h \circ f$. We shall achieve this by successively constructing the dashed arrows in the following commutative diagram, starting with $\varphi \times \mathrm{id}$ and proceeding clockwise. For increased clarity, we write Pic_Q rather than $\mathrm{Pic}_{Q/S}$ etc.:

$$\begin{tikzcd}
 & \mathrm{Pic}^\tau_Q \times P \arrow[rr, dashed, "\mathrm{id}\times h"] \arrow[dl, dashed, "\psi\times\mathrm{id}"'] & & \mathrm{Pic}^\tau_Q \times Q \\
\mathrm{Pic}^\tau_P \times P & & \mathrm{Pic}^\tau_Q \times X \arrow[ul, "\mathrm{id}\times f"'] \arrow[ur, "\mathrm{id}\times g"] \arrow[dl, dashed, "\psi\times\mathrm{id}"'] \arrow[dd, dashed, "\varphi\times\mathrm{id}"] \arrow[dr, "g^*\times\mathrm{id}"] & \\
 & \mathrm{Pic}^\tau_P \times X \arrow[ul, "\mathrm{id}\times f"] \arrow[dr, "f^*\times\mathrm{id}"'] & & \mathrm{Pic}_X \times X \\
 & & \mathrm{Pic}^\tau_X \times X \arrow[ur, hook, "\mathrm{can}\times\mathrm{id}"'] &
\end{tikzcd} \tag{9}$$

Fix any invertible sheaf $\mathscr{L}$ on X, regard the resulting sheaf $\mathscr{M}$ in (8) as a family of invertible sheaves on X parameterized by $\mathrm{Pic}^\tau_{Q/S}$, and consider the classifying map

$\varphi : \mathrm{Pic}^\tau_{Q/S} \to \mathrm{Pic}_{X/S}$. In light of Lemma 3.2, the composite map $\overline{\varphi} : \mathrm{Pic}^\tau_{Q/S} \to \mathrm{Num}_{X/S}$ factors over some section $\delta : S \to \mathrm{Num}_{X/S}$. By assumption $P = \mathrm{Pic}^\delta_{X/S}$ admits a section. The latter comes from an invertible sheaf $\mathscr{L}_0$ on X, because X has a section, too. Replacing $\mathscr{L}$ with $\mathscr{L} \otimes \mathscr{L}_0^{\otimes -1}$, we may assume that δ is the zero section. This yields a factorization $\varphi : \mathrm{Pic}^\tau_{Q/S} \to \mathrm{Pic}^\tau_{X/S}$ and gives the vertical dashed arrow in (9). By Proposition 8.3, it comes with a factorization $\psi : \mathrm{Pic}^\tau_{Q/S} \to \mathrm{Pic}^\tau_{P/S}$, yielding the two diagonal dashed arrows. By construction, (8) is the pullback of the normalized Poincaré sheaf $\mathscr{F}$ on $\mathrm{Pic}^\tau_{P/S} \times P$. Hence it is the pullback of $\mathscr{N} = (\psi \times \mathrm{id}_P)^*(\mathscr{F})$, giving the desired factorization $g = h \circ f$ of classifying maps and the upper dashed arrow in (9).

Step 3: *We verify the universal property for the particular Albanese map above, now only assuming that X admits a section.* Keep the notation from the previous step. There the arguments only relied on the existence of sections in $\mathrm{Pic}^\delta_{X/S}$ for one particular $\delta \in \mathrm{Num}_{X/S}(S)$ occurring along the way. Choose an fppf morphism $S' \to S$ so that this $\mathrm{Pic}^\delta_{X/S}$ acquires an S'-valued point. Then step 2 gives the desired factorization $g' = h' \circ f'$ for the base-changes $X' = X \times_S S'$, $P' = P \times_S S'$ and $Q' = Q \times_S S'$. The two pullbacks of h' to $S' \times_S S'$ coincide, by the uniqueness in Proposition 8.3. Now fppf descent ([32], Exposé VIII, Theorem 5.2 for schemes and [57], Lemma 0ADV for algebraic spaces) gives the desired factorization $h : P \to Q$.

Step 4: *We prove the universal property for general Albanese maps $f : X \to P$.* Let $g : X \to Q$ be another morphism into some family of para-abelian varieties. As above we have to find a factorization $g = h \circ f$, and we shall achieve this by descent. Note that uniqueness was already established in Proposition 8.3. Choose some fppf morphism $S' \to S$ so that the base-change $X' = X \times_S S'$ admits a section. Set $P' = P \times_S S'$, and let $X' \to P'_1$ be the Albanese map over S' constructed in step 1 with Poincaré sheaves. We then have a factorization $P'_1 \to P'$, according to step 3. By our definition of Albanese maps, the induced proper homomorphism $\mathrm{Pic}^\tau_{P'_1/S'} \to \mathrm{Pic}^\tau_{P'/S'}$ is fiberwise an isomorphism, thus a monomorphism (Lemma 11.4), hence a bijective closed embedding ([30], Corollary 18.12.6). With the Nakayama Lemma one infers that $\mathrm{Pic}^\tau_{P'_1/S'} \to \mathrm{Pic}^\tau_{P'/S'}$ is an isomorphism. By biduality, the original homomorphism $P'_1 \to P'$ is an isomorphism. Regard it as an identification $P' = P'_1$. Using step 3 again, we get a factorization $g' = h' \circ f'$. The two pullbacks of $h' : P' \to Q'$ to $S'' = S' \times_S S'$ coincide, by uniqueness. Now [57], Lemma 0ADV gives the desired factorization $h : P \to Q$.

Step 5: *We establish existence in general.* Choose some fppf morphism $S' \to S$ so that the base-change $X' = X \times_S S'$ acquires a section. This happens, for example, with $S' = X$. Now step 1 gives an Albanese map $f' : X' \to P'$. Set $S'' = S' \times_S S'$, and consider the two base-changes $\mathrm{pr}_1^*(P')$ and $\mathrm{pr}_2^*(P')$. The two induced morphisms from $X' = X \times_S S''$ are related by a morphism $\varphi : \mathrm{pr}_1^*(P') \to \mathrm{pr}_2^*(P')$, according to the existence part in step 4, applied over S'', and uniqueness ensures that φ is an isomorphism. This satisfies the cocycle condition over $S' \times_S S' \times_S S'$, in light of the uniqueness part in step 4. In turn, the scheme P' over S' descends to an algebraic space P over S'. By construction, $P' \to S'$ is a family of abelian varieties, hence $P \to S$ is a family of para-abelian varieties. In the same way, the Albanese map

$f' : X' \to P'$ descends to a morphism $f : X \to P$. The latter is an Albanese map, by Proposition 8.2. ☐

11. Appendix: Embeddings for algebraic spaces

Here we collect some general facts about embeddings of algebraic spaces that were used throughout, and seem to be of independent interest. The topic indeed requires attention, because schemes are locally separated, whereas not all algebraic spaces share this property. Recall that a morphism of schemes $f : Z \to Y$ is an *embedding* if it factors into a closed embedding $Z \to U$ followed by an open embedding $U \to Y$. Then the set-theoretical image $C = f(Z)$ is locally closed, and Z is determined, up to unique isomorphism, by an open set $U \subset Y$ and a quasicoherent $\mathscr{I} \subset \mathscr{O}_U$. Note that such U are not unique, but there is a maximal one, namely the complement of the closed set $\bar{C} \smallsetminus C = \bar{C} \cap (Y \smallsetminus U)$.

A morphism of algebraic spaces $f : Z \to Y$ is called an *embedding* if for each affine étale neighborhood $V \to Y$ the fiber product $Z \times_Y V$ is a scheme, and the projection $Z \times_Y V \to V$ is an embedding.

Lemma 11.1. *Let $f : Z \to Y$ be an embedding of algebraic spaces. Suppose that the morphism is quasicompact. Then it admits a factorization into a closed embedding $Z \to U$ followed by an open embedding $U \to Y$.*

Proof. Let $V_\lambda \to Y$, $\lambda \in I$ be the affine étale neighborhoods, and $C_\lambda \subset V_\lambda$ be the set-theoretical image of the projection $Z \times_Y V_\lambda \to V_\lambda$. This is locally closed, so $\bar{C}_\lambda \smallsetminus C_\lambda$ is a closed set in V_λ. Moreover, the scheme $Z \times_Y V_\lambda$ is quasicompact, by the corresponding assumption on f. For each morphism of neighborhoods $i : V_\lambda \to V_\mu$ we have $i^{-1}(C_\mu) = C_\lambda$. Since the morphism is étale and hence flat, and the scheme $Z \times_Y V_\lambda$ is quasicompact, we also have $i^{-1}(\bar{C}_\mu) = \bar{C}_\lambda$ by [32], Exposé VIII, Theorem 4.1, and thus $i^{-1}(\bar{C}_\mu \smallsetminus C_\mu) = \bar{C}_\lambda \smallsetminus C_\lambda$. Thus the complementary open sets $U_\lambda \subset V_\lambda$ are compatible, and thus define an open embedding $U \to Y$, according to [57], Lemma 0ADV. Note that in the latter result, we indeed can ignore the condition on the cardinality of the index set I by working in a fixed Grothendieck universe, confer the discussion in [51], Section 2. By construction, the morphism $Z \to Y$ factors over U, and the resulting $Z \to U$ is a closed embedding. ☐

Proposition 11.2. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of algebraic spaces. If the composition $g \circ f$ is an embedding, and g is locally separated, then f is an embedding.*

Proof. This follows as in [24], Section 5.2. We recall the argument for convenience: In the diagram

$$\begin{tikzcd}
 & Y \times_Z Y & & Z \times_Z Y & \\
Y \arrow[ur, "\Delta"] & & X \times_Z Y \arrow[ul, "f\times\mathrm{id}"'] \arrow[ur, "gf\times\mathrm{id}"] & & Y \arrow[ul, "\simeq"'] \\
 & X \times_Y Y \arrow[ul, "f\times\mathrm{id}"] \arrow[ur, "\mathrm{id}\times\mathrm{id}"] \arrow[rr, "\simeq"'] & & X \arrow[ul, "\Gamma_f"'] \arrow[ur, "f"'] &
\end{tikzcd}$$

the square to the left is cartesian. Since the arrows Δ and gf are embeddings, the same holds for $\mathrm{id}\times\mathrm{id}$ and $gf\times\mathrm{id}$, and thus also for f. □

The following application was a crucial technical step in the proof for Lemma 3.2:

Corollary 11.3. *Let N be a locally separated algebraic space. For each $\sigma\in N(S)$ the morphism $\sigma: S\to N$ is an embedding.*

Proof. The structure morphism $\varphi: N\to S$ is locally separated, and the composition $\varphi\circ\sigma=\mathrm{id}_S$ is an embedding. By the Proposition, also σ is an embedding. □

Let us record the following useful consequence for homomorphisms $f: G\to N$ between algebraic spaces G, N endowed with group structures:

Lemma 11.4. *In the above situation, suppose that the structure morphism $N\to S$ is locally separated, and that $G\to S$ is separated and of finite type. If $G_s\to N_s$ has trivial kernel for all points $s\in S$, then $f: G\to N$ is a monomorphism.*

Proof. We have to show that $H=\mathrm{Ker}(f)$ is trivial. This kernel is given by a fiber product $H=G\times_N\{e_S\}$. By Corollary 11.3, the neutral section $e: S\to N$ is an embedding, so the same holds for the base-change $H\to G$. It follows that the structure morphism $H\to S$ is separated and of finite type. Obviously, the formation of kernels commutes with base-change, and the fibers H_s are trivial. Thus $H\to S$ is universally bijective, and in particular quasi-finite. This ensures that the algebraic space H is schematic ([38], Theorem A.2). The relative group scheme $H\to S$ is thus trivial, by [17], Exposé VI_B, Corollary 2.10. □

References

[1] J. Achter, S. Casalaina-Martin, C. Vial: A functorial approach to regular homomorphisms. Preprint, arXiv:1911.09911.

[2] G. Albanese: Corrispondenze algebriche fra i punti di due superficie algebriche (memoria prima). Ann. Scuola Norm. Super. Pisa Cl. Sci. 3 (1934), 1–26.

[3] G. Albanese: Corrispondenze algebriche fra i punti di due superficie algebriche (memoria seconda). Ann. Scuola Norm. Super. Pisa Cl. Sci. 3 (1934), 149–182.

[4] M. Artin: Algebraization of formal moduli I. In: D. Spencer, S. Iyanaga (eds.), Global Analysis, pp. 21–71. Univ. Tokyo Press, Tokyo, 1969.

[5] M. Artin: Algebraic spaces. Yale University Press, New Haven, Conn.-London, 1971.

[6] J. Bergqvist, T. Dang, S. Schröer: Sign involutions on para-abelian varieties. Preprint, arXiv:2301.03314.

[7] P. Berthelot, A. Grothendieck, L. Illusie (eds.): Théorie des intersections et théorème de Riemann–Roch (SGA 6). Springer, Berlin, 1971.

[8] A. Blanchard: Sur les variétés analytiques complexes. Ann. Sci. Ecole Norm. Sup. 73 (1956), 157–202.

[9] E. Bombieri, D. Mumford: Enriques' classification of surfaces in char. p, III. Invent. Math. 35 (1976), 197–232.

[10] M. Brion: Anti-affine algebraic groups. J. Algebra 321 (2009), 934–952.

[11] M. Brion: Some structure theorems for algebraic groups. In: M. Can (ed.), Algebraic groups: structure and actions, pp. 53–126. Amer. Math. Soc., Providence, RI, 2017.

[12] S. Brochard: Finiteness theorems for the Picard objects of an algebraic stack. Adv. Math. 229 (2012), 1555–1585.

[13] S. Brochard: Duality for commutative group stacks. Int. Math. Res. Not. IMRN 2021, 2321–2388.

[14] B. Conrad: Answer to question "Albanese variety over non-perfect fields". MathOverflow, https://mathoverflow.net/q/261032 (2017).
[15] M. Demazure, P. Gabriel: Groupes algébriques. Masson, Paris, 1970.
[16] M. Demazure, A. Grothendieck (eds.): Schémas en groupes I (SGA 3 Tome 1). Springer, Berlin, 1970.
[17] M. Demazure, A. Grothendieck (eds.): Schémas en groupes II (SGA 3 Tome 2). Springer, Berlin, 1970.
[18] T. Ekedahl, J. Hyland, N. Shepherd-Barron, Moduli and periods of simply connected Enriques surfaces. Preprint, math.AG/1210.0342.
[19] H. Esnault, V. Srinivas, E. Viehweg: The universal regular quotient of the Chow group of points on projective varieties. Invent. Math. 135 (1999), 595–664.
[20] G. Faltings, C.-L. Chai: Degeneration of abelian varieties. Ergeb. Math. Grenzgebiete (3) 22. Springer, Berlin, 1990
[21] A. Grothendieck: Sur quelques points d'algèbre homologique. Tohoku Math. J. 9 (1957), 119–221.
[22] A. Grothendieck: Technique de descente et théorèmes d'existence en géométrie algébrique. V. Les schémas de Picard: théorèmes d'existence. Séminaire Bourbaki, Vol. 7, Exp. 232, 143–161. Soc. Math. France, Paris, 1995.
[23] A. Grothendieck: Technique de descente et théorèmes d'existence en géométrie algébrique. VI. Les schémas de Picard: propriétés générales. Séminaire Bourbaki, Vol. 7, Exp. 236, 221–243. Soc. Math. France, Paris, 1995.
[24] A. Grothendieck, J. Dieudonné: Éléments de géométrie algébrique I: Le langage des schémas. Springer, Berlin, 1970.
[25] A. Grothendieck: Éléments de géométrie algébrique II: Étude globale élémentaire de quelques classes de morphismes. Publ. Math., Inst. Hautes Étud. Sci. 8 (1961).
[26] A. Grothendieck: Éléments de géométrie algébrique III: Étude cohomologique des faisceaux cohérents. Publ. Math., Inst. Hautes Étud. Sci. 11 (1961).
[27] A. Grothendieck: Éléments de géométrie algébrique III: Étude cohomologique des faisceaux cohérents. Publ. Math., Inst. Hautes Étud. Sci. 17 (1963).
[28] A. Grothendieck: Éléments de géométrie algébrique IV: Étude locale des schémas et des morphismes de schémas. Publ. Math., Inst. Hautes Étud. Sci. 24 (1965).
[29] A. Grothendieck: Éléments de géométrie algébrique IV: Étude locale des schémas et des morphismes de schémas. Publ. Math., Inst. Hautes Étud. Sci. 28 (1966).
[30] A. Grothendieck: Éléments de géométrie algébrique IV: Étude locale des schémas et des morphismes de schémas. Publ. Math., Inst. Hautes Étud. Sci. 32 (1967).
[31] A. Grothendieck: Le groupe de Brauer III. In: J. Giraud (ed.) et al.: Dix exposés sur la cohomologie des schémas, pp. 88–189. North-Holland, Amsterdam, 1968.
[32] A. Grothendieck: Revêtements étales et groupe fondamental (SGA 1). Springer, Berlin, 1971.
[33] A. Grothendieck (ed.): Groupe de monodromie en géométrie algébrique (SGA 7 I). Springer, Berlin, 1972.
[34] F. Ischebeck: Über die Abbildung $\mathrm{Pic}(A) \to \mathrm{Pic}(A_f)$. Math. Ann. 243 (1979), 237–245.
[35] S. Kleiman: Toward a numerical theory of ampleness. Ann. Math. 84 (1966), 293–344.
[36] S. Kleiman: What is Abel's theorem anyway? In: O. Laudal and R. Piene (eds.), The legacy of Niels Henrik Abel, pp. 395–440. Springer, Berlin, 2004.
[37] D. Knutson: Algebraic spaces. Springer, Berlin, 1971.
[38] G. Laumon, L. Moret-Bailly: Champs algebriques. Springer, Berlin, 2000.
[39] D. Lorenzini, S. Schröer: Moderately ramified actions in positive characteristic. Math. Z. 295 (2020), 1095–1142.
[40] T. Matsusaka: On the algebraic construction of the Picard variety. II. Jpn. J. Math. 22 (1952), 51–62 (1953).
[41] J. Milne: Étale cohomology. Princeton University Press, Princeton, 1980.

[42] D. Mumford: Abelian varieties. Tata Institute of Fundamental Research Studies in Mathematics 5. Oxford University Press, London, 1970.
[43] D. Mumford, J. Fogarty, F. Kirwan: Geometric invariant theory. Third edition. Springer, Berlin, 1993.
[44] J. Murre: On contravariant functors from the category of pre-schemes over a field into the category of abelian groups (with an application to the Picard functor). Inst. Hautes Études Sci. Publ. Math. 23 (1964), 5–43.
[45] M. Nagata: A remark on the unique factorization theorem. J. Math. Soc. Japan 9 (1957), 143–145.
[46] M. Olsson: Algebraic spaces and stacks. American Mathematical Society, Providence, RI, 2016.
[47] M. Raynaud: Faisceaux amples sur les schémas en groupes et les espaces homogènes. Springer, Berlin, 1970.
[48] M. Romagny, D. Rydh, G. Zalamansky: The complexity of a flat groupoid. Doc. Math. 23 (2018), 1157–1196.
[49] D. Rössler, S. Schröer: Moret-Bailly families and non-liftable schemes. Algebr. Geom. 9 (2022), 93–121.
[50] D. Rydh: Noetherian approximation of algebraic spaces and stacks. J. Algebra 422 (2015), 105–147.
[51] S. Schröer: Points in the fppf topology. Ann. Sc. Norm. Super. Pisa Cl. Sci. 17 (2017), 419–447.
[52] S. Schröer: The Deligne–Illusie Theorem and exceptional Enriques surfaces. Eur. J. Math. 7 (2021), 489–525.
[53] S. Schröer: There is no Enriques surface over the integers Ann. of Math. 197 (2023), 1–63.
[54] S. Schröer: Albanese maps for open algebraic spaces. Preprint, arXiv:2204.02613.
[55] J.-P. Serre: Morphismes universels et variétés d'Albanese. Sém. C. Chevalley 3, 1960.
[56] J.-P. Serre: Algèbre locale. Multiplicités. Springer, Berlin, 1965.
[57] Stacks project authors: Stacks project. https://stacks.math.columbia.edu, 2018.
[58] M. Talpo, A. Vistoli: Deformation theory from the point of view of fibered categories. In: G. Farkas and I. Morrison (eds.), Handbook of moduli. Vol. III. International Press, Somerville, MA; Higher Education Press, Beijing, 2013.
[59] A. Weil: Scientific works. Collected papers. Vol. I (1926–1951). Springer, New York, 1979.
[60] O. Wittenberg: On Albanese torsors and the elementary obstruction. Math. Ann. 340 (2008), 805–838.
[61] W. Zomervrucht: Ineffective descent of genus one curves. Preprint, arXiv:1501.04304, 2015.

Mathematisches Institut, Heinrich-Heine-Universität, 40204 Düsseldorf, Germany

Email address: bruno.laurent@hhu.de

Mathematisches Institut, Heinrich-Heine-Universität, 40204 Düsseldorf, Germany

Email address: schroeer@math.uni-duesseldorf.de