RATIONAL POINTS IN COARSE MODULI SPACES AND TWISTED REPRESENTATIONS

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Abstract. We study moduli spaces and moduli stacks for representations of associative algebras in Azumaya algebras, in rather general settings. We do not impose any stability condition and work over arbitrary ground rings, but restrict attention to the so-called Schur representations, where the only automorphisms are scalar multiplications. The stack comprises twisted representations, which are representations that live on the gerbe of splittings for the Azumaya algebra. Such generalized spaces and stacks appear naturally: For any rational point on the classical coarse moduli space of matrix representations, the machinery of non-abelian cohomology produces a modified moduli problem for which the point acquires geometric origin. The latter are given by representations in Azumaya algebras.

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Introduction

In algebraic geometry, the discrepancy between a *coarse moduli space* and a *fine moduli stack* is an endless source of puzzlement, wonder and research. Loosely speaking, a "moduli space" is a geometric object that parameterizes a given class of other geometric objects, in an "optimal way". For example, the Hilbert scheme $\mathrm{Hilb}_{X/k}$ parameterizes the closed subschemes Z of a given projective scheme X, say over a ground field k. This is the prime example of a *fine moduli space*, which represents the corresponding functor of families of closed subschemes. If X is merely proper, not necessarily having an ample invertible sheaf, $\mathrm{Hilb}_{X/k}$ still exists as an

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algebraic space, which are certain sheaves $(\mathrm{Aff}/k) \to (\mathrm{Set})$ on the category of affine schemes over k that are closely related to schemes. Despite this additional layer of technicalities, $\mathrm{Hilb}_{X/k}$ remains a fine moduli space.

In most other cases of interest, the situation is more difficult, due to non-trivial automorphisms of objects. Here the prime example is the coarse moduli space M_g that parameterizes smooth curves of genus $g \geq 2$. Although it is true that $M_g(\Omega)$ corresponds to isomorphism classes of such curves defined over algebraically closed fields Ω, this does not hold true for general fields k, let alone rings R. Similar things can be said about the Picard scheme $\mathrm{Pic}_{X/k}$. In fact, in presence of non-trivial automorphisms a fine moduli space *never exists*, a fact that cannot be over-emphasized.

To cope with such issues, one has to replace schemes $(X, \mathscr{O}_X)$ or algebraic spaces $X : (\mathrm{Aff}/k) \to (\mathrm{Set})$ by stacks $\mathscr{X} \to (\mathrm{Aff}/k)$, which are certain categories fibered in groupoids. This turns all our players into categories. In particular, a scheme or an algebraic space X becomes the stack $\mathscr{X} = (\mathrm{Aff}/X)$ via the comma construction. A fundamental insight of Deligne and Mumford [14] and Artin [4] was that the geometric intuition stemming from schemes still works very well in the realm of *Deligne–Mumford stacks* or even *Artin stacks*. For comprehensive general presentations, see [34], [43] and [50].

Our initial motivation was to generalize a beautiful work of Hoskins and Schaffhauser [28] on rational points on moduli of quiver representations into an abstract setting. Recall that a *quiver* Δ is a finite directed graph, with loops and multiple arrows allowed. A *quiver representation* attaches to each vertex $i \in \Delta$ a finite-dimensional vector space E_i, and to each arrow $i \stackrel{\alpha}{\to} j$ in Δ a linear map $f_\alpha : E_i \to E_j$. Let $M_{\Delta/k}$ be the resulting coarse moduli space of stable quiver representations over some ground field k, with fixed dimension vector and with respect to some stability condition, both omitted from notation. The construction involves the formation of a GIT quotient [42], and consequently the rational points $x \in M_{\Delta/k}(k)$ do not necessarily come from an actual quiver representation $(E_i \mid f_\alpha)_{i,\alpha\in\Delta}$. In [28] the obstructions are analyzed in an explicit, down-to-earth way, and a geometric interpretation in terms of modified data are given, of course involving the *Brauer group* $\mathrm{Br}(k)$.

We put ourselves into a general, abstract and categorified situation, which encompasses all sorts of applications: Let us work in a topos $\mathscr{T}$ with final object S, and a central extension of group objects $1 \to N \to G \to H \to 1$, with H acting freely on some object X, with ensuing quotient $Q = X/H^{\mathrm{op}}$. Intuitively, we think of X as a space that "over-parameterizes" a certain class of geometric objects. The H-action corresponds to the isomorphism relation, whereas N reflects automorphisms. In turn, one may say that Q "under-parameterizes" our class of geometric objects.

We now say that an S-valued point $g : S \to Q$ is *of geometric origin* if it admits a lifting $\tilde{g} : S \to X$. One obtains an obstruction map $\mathrm{ob} : Q(S) \to H^2(S, N)$. By non-abelian cohomology and twisted forms we construct, with fixed $g \in Q(S)$ and in a canonical way, a "modified moduli problem" $\tilde{X}$ and $1 \to N \to \tilde{G} \to \tilde{H} \to 1$, without changing the quotient Q. Our first result is:

Theorem. *(See Thm. 1.2) With respect to the quotient $Q = \tilde{X}/\tilde{H}^{\mathrm{op}}$ for the modified moduli problem, the given S-valued point $g \in Q(S)$ acquires geometric origin.*

In one form or another, this appears to be well-known (compare the monograph of Skorobogatov [49], Section 2.2). The moduli spaces $\mathcal{M}^{\theta-gs}_{\Delta,d',D} = \mathrm{Rep}^{\theta-gs}_{\Delta,d',D}/\mathbf{G}_{\Delta,d',D}$ constructed by Hoskins and Schaffhauser ([28], Theorem 1.3), which comprise representations of a quiver Δ over a perfect ground field k that are geometrically stable with respect to a chosen stability condition θ and twisted by a central division algebra D, can be seen as a special case of the above general procedure.

As a basic application we also give a sufficient criterion at the end of Section 2 for all S-valued points of Q to be of geometric origin for $S = \mathrm{Spec}(K)$ the spectrum of a field, which could be imperfect. Of course, every such point is of geometric origin if S is the spectrum of an algebraically closed field K. However, in an arithmetic setting, and already over the field $K = \mathbb{R}$, this usually fails.

We apply our theory to *representations of associative algebras* Λ over ground rings R. Note that this encompasses representations of groups Γ via the *group algebra* $\Lambda = R[\Gamma]$, representations of quivers Δ via the *path algebra* $\Lambda = R[\Delta]$. This apparently applies to coherent sheaves on projective schemes as well, by an ingenious construction of Álvarez-Cónsul and King [1] relying on *Kronecker modules*; it would be very interesting to pursue this further.

For simplicity we assume that the associative algebra Λ is finitely presented. To work with $H = \mathrm{PGL}_n$ and $G = \mathrm{GL}_n$, we have to restrict to representations whose endomorphism ring comprises only scalars. Note that there is no consistent designation for this important property. Following Derksen and Weyman ([16], Section 2.3) we like to call them *Schur representations*. One also finds the terminology "brick" ([6], Chapter VII, Definition 2.4) and "stably indecomposable" ([33], Section 2.6). In all other respects we work in far greater generality than customary in geometric invariant theory, *without any stability conditions and over arbitrary ground rings.* Note that over fields, the Schur representations contain all geometrically stable representations.

According to Grothendieck's fundamental insight [25], the twisted forms $\tilde{H}$ of the group scheme $H = \mathrm{PGL}_n$ correspond to *Azumaya algebras*. Consequently, it becomes imperative to consider representations of Λ not only in matrix algebras $\mathrm{Mat}_n(R)$, but in general Azumaya algebras Λ^{azu} of degree $n \geq 1$. Our second main result is:

Theorem. *(See Thm. 3.5) Suppose Λ is finitely presented as an associative R-algebra, and Λ^{azu} is an Azumaya algebra of degree $n \geq 1$. Then the functor $X = X^{\Lambda^{\mathrm{azu}}}_{\Lambda/R}$ of Schur representations is representable by a quasiaffine scheme of finite presentation. Moreover, the group scheme $H = \mathrm{Aut}_{\Lambda^{\mathrm{azu}}/R}$ acts freely, and the quotient $Q = X/H^{\mathrm{op}}$ is an algebraic space that is of finite presentation.*

For the path algebra $\Lambda = R[\Delta]$ of a quiver we also give an independent argument using Grassmann varieties and vector bundles. Note that *algebraic spaces* are generalizations of schemes introduced by Artin ([2], [32], [34], [43]) that allow the formation of quotients which usually do no exist as schemes. Moreover, they are indispensable in the very definition of Artin stacks. Note that algebraic spaces like $Q = X/H^{\mathrm{op}}$ are frequently non-separated, and easily become non-schematic (compare [48]).

In general, such quotients are non-separated, and this seems to be the reason why they have received comparatively little attention even for $\Lambda^{\mathrm{azu}} = \mathrm{Mat}_n(R)$ in the literature so far. In geometric representation theory and in particular in quiver representation theory it is customary to instead choose a stability condition θ, restrict to the locus $\mathrm{Rep}^{\theta-gs}$ of geometrically θ-stable representations and form the quotient $\mathcal{M}^{\theta-gs} = \mathrm{Rep}^{\theta-gs} / \mathrm{PGL}_n^{\mathrm{op}}$ by means of GIT, where PGL_n coincides with $H = \mathrm{Aut}_{\mathrm{Mat}_n(R)/R}$. Since every geometrically θ-stable representation is in particular Schur, $\mathrm{Rep}^{\theta-gs} \subset X = X_{\Lambda/R}^{\mathrm{Mat}_n(R)}$ is an open H-stable subset. Therefore the quotients $\mathcal{M}^{\theta-gs}$ arise as quasiprojective open subschemes in the above $Q = X/H^{\mathrm{op}}$. The latter then should be crucial to understand how these GIT quotients change when the stability conditions are changed, a phenomenon often called "wall crossing".

The union of the quotients $\mathcal{M}^{\theta-gs}$ forms the open subspace $Q' \subset Q$ of representations which are geometrically stable for some stability condition θ. It should be pointed out that Q' is usually not equal to Q, meaning that there are Schur representations which are not geometrically stable for any θ, already over the field $K = \mathbb{C}$ of complex numbers. We will give examples of such at the end of Section 3.

Our third main result is a description of the moduli stack $[X/G/Q]$, where $G = U_{\Lambda^{\mathrm{azu}}/R}$ is the group scheme of units of the Azumaya algebra Λ^{azu}. Quotient stacks are constructed in the geometric language of principal homogeneous spaces. Our description is in a representation-theoretic way, much closer to the moduli problem at hand: We define a *twisted Schur representation* of Λ in Λ^{azu} over an affine scheme V as a pair $(\mathscr{E}', \rho')$, where $\mathscr{E}'$ is a locally free sheaf of rank $n = \deg(\Lambda^{\mathrm{azu}})$ and weight $w = 1$ on the gerbe V' of splittings for $\Lambda^{\mathrm{azu}} \otimes_R \mathscr{O}_{V'}$, and $\rho' : \Lambda \otimes_R \mathscr{O}_{V'} \to \underline{\mathrm{End}}(\mathscr{E}')$ is a Schur representation. This of course relies on the notion of *twisted sheaves*, which where introduced by Căldăraru ([10], [11]), de Jong [13] and Lieblich ([37] and [38]), and have attracted tremendous interest in the past decades. Earlier, Edidin, Hassett, Kresch and Vistoli [17] already established a deep link between the existence of Azumaya algebras representing a given Brauer class and internal properties of the Artin stacks of splittings. Our third main result is:

Theorem. *(See Thm. 6.4) The stack of twisted Schur representations*

$$\mathscr{M}_{\Lambda/R}^{\Lambda^{\mathrm{azu}}} = \{(V, \mathscr{E}', \rho')\}$$

is equivalent to the quotient stack $[X/G^{\mathrm{op}}/Q]$, where $X = X_{\Lambda/R}^{\Lambda^{\mathrm{azu}}}$ is the quasiaffine scheme that represents the functor of Schur representations of Λ in Λ^{azu}.

In other words, $\mathscr{M}_{\Lambda/R}^{\Lambda^{\mathrm{azu}}}$ is the "true" moduli stack for representations of associative rings Λ in Azumaya algebras Λ^{azu}. The algebraic space $Q = X/H^{\mathrm{op}}$ is the coarse moduli space over $S = \mathrm{Spec}(R)$. In particular, the above answers the question of Hoskins and Schaffhauser to describe the ring-valued points in the quotient stack of quiver representations ([28], end of Section 4.3), which was also considered by Le Bruyn [36].

Given some $g \in Q(S)$, we now can determine when it has geometric origin, and describe the modified moduli problem $Q = \tilde{X}/\tilde{H}$, for which g has acquired geometric origin:

Theorem. *(See Thm. 7.2 and 7.1) If the non-abelian cohomology set $H^1(S, \mathrm{GL}_n)$ is a singleton and the category $\mathscr{M}^{\Lambda^{\mathrm{azu}}}_{\Lambda/R}(S)$ is non-empty, then g has geometric origin for $Q = X/H^{\mathrm{op}}$. In any case, the twisted form $\tilde{X}$ is the scheme of Schur representations of Λ in the Azumaya algebra $\tilde{\Lambda}^{\mathrm{azu}}$ obtained by twisting Λ^{azu}.*

Our fourth main result deals with the *tautological sheaf* $\mathscr{T}_{\mathscr{M}}$ on the moduli stack $\mathscr{M} = \mathscr{M}^{\Lambda^{\mathrm{azu}}}_{\Lambda/R}$. Locally, this is a Hom sheaf between two locally free sheaves on the gerbe of splittings of weight one. Its endomorphism algebra has weight zero, and thus yields an Azumaya algebra $\mathscr{A}_Q$ over the algebraic space $Q = X/H^{\mathrm{op}}$, which contains a smaller Azumaya algebra $\mathscr{A}^0_Q$ as the commutant of Λ^{azu}. Their classes in the Brauer group $\mathrm{Br}(Q) \subset H^2(Q, \mathbb{G}_m)$ have the following meaning:

Theorem. *(See Thm. 8.3) In the cohomology group $H^2(Q, \mathbb{G}_m)$ of the algebraic space $Q = X/H^{\mathrm{op}}$, we have $[\mathscr{A}_Q] = \partial[X]$ and $[\mathscr{A}^0_Q] = [\Lambda^{\mathrm{azu}} \otimes_R \mathscr{O}_Q]$.*

Here $[X] \in H^1(Q, H)$ is the torsor class for the quotient map $X \to Q$, and $\partial[X]$ is its Brauer class stemming from the non-abelian coboundary map.

The paper is structured as follows: In Section 1 we collect some generalities on torsors, introduce the points of geometric origin, and the technique of modifying moduli problems. Section 2 contains generalities on quotient stacks and gerbes. In Section 3 we introduce the notion of Schur representations of an associative algebra Λ in another associative algebra Λ' over arbitrary ground rings R, and establish representability results for the functor $X = X^{\Lambda^{\mathrm{azu}}}_{\Lambda/R}$ and quotient $Q = X/H^{\mathrm{op}}$. In Section 4 we give an alternative approach for quiver representations. Section 5 contains a detailed and explicit discussion of splitting gerbes for Azumaya algebras. The central part of the paper is Section 6, where we introduce the Artin stack $\mathscr{M} = \mathscr{M}^{\Lambda^{\mathrm{azu}}}_{\Lambda/R}$ of twisted Schur representations, and show that it is equivalent to a quotient stack $[X/G/Q]$. In Section 7, we apply our findings to understand the modification of the moduli problem of Schur representations. In Section 8 we introduce the tautological sheaf $\mathscr{T}_{\mathscr{M}}$ on the moduli stack $\mathscr{M}$, and use it to explain various Azumaya algebras and Brauer classes.

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1. Points of geometric origin

In this section we review the theory of torsors and twisting, and introduce the notation of S-valued points of geometric origin. We will freely use the language of sites, topoi, stacks, torsors, and gerbes, but sometimes recall and comment on some relevant issues along the way. For details we refer to [20], [5], [18], [34], [43], [49].

Let $\mathcal{C}$ be a site and $\mathscr{T} = \mathrm{Sh}(\mathcal{C})$ be the ensuing topos of sheaves. To simplify exposition, we assume that $\mathcal{C}$ contains a terminal object S, all fiber products exist, and all representable functors $\mathcal{C} \to (\mathrm{Set})$ satisfy the sheaf axiom. The latter gives, via the Yoneda Lemma, a fully faithful embedding $\mathcal{C} \subset \mathscr{T}$, and in particular we can regard S as terminal object in $\mathscr{T}$. One then also says that the Grothendieck topology is *subcanonical*. Throughout, all objects belong to this $\mathscr{T}$ if not said otherwise.

Let G be a group object, acting on another object Z. For each G-torsor P, we thus obtain a *twisted form*

$$^{P}Z = P \wedge^{G} Z = G\backslash(P \times Z) = (P \times Z)/G^{\mathrm{op}}$$

of Z. Here G acts diagonally on the product $P \times Z$, and G^{op} denotes the opposite group, which indeed acts from the right rather than the left. Note that the construction is functorial in Z, and commutes with products. In particular, if Z is endowed with some algebraic structure respected by the G-action, the twisted form ^{P}Z inherits the algebraic structure. In the special case $Z = G$, on which G acts via conjugation, we thus get a new group object ^{P}G. More generally, if Z is endowed with an object of operators Ω (in the sense of [8], Chapter I, §3, No. 1), and G acts on Z and Ω in a compatible way, that is, $\Omega \times Z \to Z$ is G-equivariant, then ^{P}Z is endowed with operators $^{P}\Omega$.

Suppose now that $Z = P$, and write P_0 for the underlying object where the G-action is omitted, giving an inclusion $\mathrm{Aut}_{P/S} \subset \mathrm{Aut}_{P_0/S}$. Combining [29], Section 9 and [47], Lemma 3.1, we have a canonical homomorphism

$$^{P}G \times {}^{P}G^{\mathrm{op}} \longrightarrow {}^{P}\mathrm{Aut}_{G_0/S} = \mathrm{Aut}_{P_0/S}$$

stemming from the action of $G \times G^{\mathrm{op}}$ on G by left-right multiplication $(g_1, g_2) \cdot g = g_1 g g_2$. This endows the underlying object P_0 of the G-torsor P with the additional structures of a ^{P}G-torsor and a $^{P}G^{\mathrm{op}}$-torsor. In fact, $^{P}G^{\mathrm{op}}$ becomes the automorphism group object for the G-torsor P, in other words

$$\mathrm{Aut}_{P/S} = {}^{P}G^{\mathrm{op}}. \tag{1}$$

Write $(G\text{-Tors}/S)$ and $(^{P}G\text{-Tors}/S)$ for the groupoids of torsors for G and ^{P}G, respectively. We then get the *torsor translation functor*

$$(^{P}G\text{-Tors}/S) \longrightarrow (G\text{-Tors}/S), \quad T \longmapsto P \wedge^{^{P}G} T,$$

where the G-action is via P. This is indeed well-defined, because in light of (1) the left G-action on P commutes with the right ^{P}G-action. It sends the trivial torsor $T = {}^{P}G$ to the torsor P, which usually is non-trivial. Likewise, we have a functor in the other direction

$$(G\text{-Tors}/S) \longrightarrow (^{P}G\text{-Tors}/S), \quad T \longmapsto \mathrm{Isom}_G(P, T),$$

where the ^{P}G-action is via P, again given by (1). This sends $T = P$ to the trivial torsor $\mathrm{Isom}_G(P, P) = G \cdot \mathrm{id}_P$. The above functors are mutually inverse equivalences, where the adjunction maps

$$T \longrightarrow \mathrm{Isom}_G(P, P \wedge^{^{P}G} T) \quad \text{and} \quad P \wedge^{^{P}G} \mathrm{Isom}_G(P, T) \longrightarrow T$$

are given by $t \mapsto (p \mapsto p \wedge t)$ and $p \wedge f \mapsto f(p)$, respectively. On the sets of isomorphism classes, we get mutually inverse *torsor translation maps*

$$H^1(S, {}^PG) \longrightarrow H^1(S, G) \quad \text{and} \quad H^1(S, G) \longrightarrow H^1(S, {}^PG) \tag{2}$$

of sets. Note that in general these maps do not respect the distinguished points.

Suppose now that $G \to H$ is a homomorphism of group objects. For each G-torsor P we get an induced H-torsor $H \wedge^G P$. If $G \to H$ is an epimorphism with kernel N, this becomes $N\backslash P = P/N^{\text{op}}$. Conversely, for general $G \to H$, we say that an H-torsor T admits a *reduction of structure group* if there is some G-torsor P with $H \wedge^G P \simeq T$. If the homomorphism is a monomorphism $G \subset H$, this simply means that T/G^{op} admits a section. Reduction to $G = \{e\}$ boils down to the triviality of the H-torsor.

Now let X be an object, and H be a group object acting freely on X, and write

$$Q = H\backslash X = X/H^{\text{op}}$$

for the resulting quotient. Note that the canonical map $X(S) \to Q(S)$ is not surjective in general, because the formation of quotients involves sheafification. The objects $H_Q = H \times Q$ and X are endowed with canonical morphisms to Q, via the second projection and the quotient map, respectively. In turn, we may view them as objects in the *comma category* $\mathscr{T}_{/Q}$, comprising pairs (U, g) with U an object and $g : U \to Q$ a morphism from $\mathscr{T}$. This is again a topos ([5], Éxpose IV, Theorem 1.2). Furthermore, H_Q has the structure of a group object in $\mathscr{T}_{/Q}$, with X as an H_Q-torsor.

Given an S-valued point $g : S \to Q$ of the quotient $Q = X/H^{\text{op}}$, we get an induced torsor $g^*(X) = X \times_Q S$ with respect to $H = g^*(H_Q) = (H \times Q) \times_Q S$. The following locution will be useful throughout:

Definition 1.1. In the above setting, we say that the S-valued point $g : S \to Q$ of the quotient $Q = X/H^{\text{op}}$ is of *geometric origin* if the induced H-torsor $g^*(X)$ is trivial.

The condition simply means that the projection $g^*(X) \to S$ admits a section. By the cartesian diagram

$$\begin{tikzcd} g^*(X) \arrow[r] \arrow[d] & X \arrow[d] \\ S \arrow[r, "g"'] & Q, \end{tikzcd}$$

this also means that $g \in Q(S)$ lies in the image of $X(S)$. This of course is automatic if $X \to Q$ admits a section, but in general the map $X(S) \to Q(S)$ is far from surjective.

A first instructive example arises from the polynomial ring $R = \mathbb{R}[u]$ and the extension $A = R[t]/(f)$ given by the polynomial $f(t) = t^2 + ut + 1$. Write X and Q for the respective spectra of the localizations A_Δ and R_Δ with respect to the discriminant $\Delta = u^2 - 4$. Then $X \to Q$ is a finite étale Galois covering with Galois group $H = \mathbb{Z}/2\mathbb{Z}$, and we see that an $\mathbb{R}$-valued point $s \in Q$ given by $u = \lambda$ is of geometric origin if and only if $|\lambda| > 2$.

Our motivation for the terminology stems from the following: Think of X as a space that over-parameterizes *geometric objects* stemming from some *moduli problem*, in such a way that the H-orbits correspond to the *isomorphism classes*. One may regard Q as a *coarse moduli space*, the S-valued points $g : S \to Q$ as S-families of isomorphism classes, and the S-valued points $\tilde{g} : S \to X$ as the more significant S-families of geometric objects. The map $X(S) \to Q(S)$ is not necessarily surjective, and coping with this failure is the main topic of this paper. In praxis, H often sits in an exact sequence $1 \to N \to G \to H \to 1$, where N reflects the *automorphisms* in the moduli problem, a situation studied in the next section.

The following example elucidates what we have in mind: Fix a ground ring R, endow the category $\mathcal{C} = (\mathrm{Aff}/R)$ of affine schemes with the fppf topology, and consider the functor $Y(A) = \mathrm{Mat}_n(A) \times \mathrm{Mat}_n(A)$ of matrix pairs. The projective linear group $H = \mathrm{PGL}_n$ acts via simultaneous conjugation, and we consider the subfunctor $X \subset Y$ of matrix pairs whose stabilizer is trivial. Note that $G = \mathrm{GL}_n$ and $N = \mathbb{G}_m$ give the short exact sequence $1 \to N \to G \to H \to 1$, that the functor X is a quasiaffine scheme (see Theorem 3.5 below), and that the quotient X/H^{op} can be seen as the coarse moduli space of certain n-dimensional representations of the associative algebra $\Lambda_0 = R\langle t_1, t_2\rangle$, namely those with only scalars as automorphisms. Also note that if $R = K$ is a field, Λ_0 is the proto-typical example of a *wild algebra*. In other words, its representation theory "contains" the representation theory of every finite-dimensional K-algebra Λ (compare [31], Section 7.1).

Back to the general setting. We now fix an S-valued point $g : S \to Q$, and consider the resulting H-torsor $P = g^*(X) = X \times_Q S$. With respect to the conjugation action of H on itself, we obtain a twisted form $\tilde{H} = P \wedge^H H$. Recall that X is a torsor with respect to H_Q, and the same holds for the pull-back $P_Q = P \times Q$. We now apply the above considerations in the comma category $\mathscr{T}_{/Q}$, and regard the twisted form

$$\tilde{X} = \mathrm{Isom}_{H_Q}(P_Q, X)$$

as a torsor for $\tilde{H}_Q = ({}^P H)_Q = {}^{P_Q}(H_Q)$. By construction, both quotients X/H^{op} and $\tilde{X}/\tilde{H}^{\mathrm{op}}$ coincide with the terminal object Q from the comma category, and thus

$$X/H^{\mathrm{op}} = Q = \tilde{X}/\tilde{H}^{\mathrm{op}}.$$

Our first main result is the following foundational fact, now almost a triviality:

Theorem 1.2. *With respect to the quotient $Q = \tilde{X}/\tilde{H}^{\mathrm{op}}$, the given S-valued point $g : S \to Q$ has acquired geometric origin.*

Proof. Making a base-change, it suffices to treat the case that $S = Q$, $g = \mathrm{id}_Q$ and $P = X$. Then $\tilde{X}$ is just the image of $T = P$ under the torsor translation map

$$H^1(S, H) \longrightarrow H^1(S, {}^P H), \quad T \longmapsto \mathrm{Isom}_H(P, T).$$

We already remarked that the object $\mathrm{Isom}_H(P, P)$ admits the identity as section. □

Intuitively, one should regard the passage from X to $\tilde{X}$ as *modifying* the moduli problem *without* changing the coarse moduli space. Thus $g : S \to Q$, which is just an S-family of isomorphism classes for the initial moduli problem, is promoted to an S-family of geometric objects for the modified moduli problem. *For* *a given concrete moduli problem X, this raises the question to understand the geometric meaning of the new moduli problem $\tilde{X}$.* In the following sections we will obtain answers for certain moduli of representations. Note that the moduli spaces $\mathcal{M}^{\theta-gs}_{\Delta,d',D} = \operatorname{Rep}^{\theta-gs}_{\Delta,d',D}/\mathbf{G}_{\Delta,d',D}$ constructed by Hoskins and Schaffhauser ([28], Theorem 1.3), which comprise representations of a quiver Δ over a perfect ground field k that are geometrically stable with respect to a chosen stability condition θ and twisted by a central division algebra D, can be seen as a special case of the above general procedure.

Also note that the set $Q(S)$ comes with a partition

$$Q(S) = \dot{\bigcup_{[P]\in H^1(S,H)}} Q(S)_P$$

where the $Q(S)_P$ comprise the $g : S \to Q$ for which $g^*(X)$ is isomorphic to P. Note that these are exactly the S-valued points which acquire geometric origin with respect to the quotient $Q = \tilde{X}/\tilde{H}^{\mathrm{op}}$, where $\tilde{X}$ and $\tilde{H}$ are obtained by twisting X and H with P. In the context of quiver moduli, the above partition was already obtained by Hoskins and Schaffhauser, where the quotient map $Q(S) \to H^1(S,H)$ was called the *type map* ([28], Theorem 1.1 and Remark 3.5).

2. Quotient stacks

We keep the set-up of the preceding section. Let X be an object from the topos $\mathscr{T} = \operatorname{Sh}(\mathcal{C})$, and H be a group object, acting freely on X with quotient $Q = X/H^{\mathrm{op}}$. In this section, we additionally assume that H sits in some short exact sequence

$$1 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 1 \tag{3}$$

of group objects, where we assume that the epimorphism $G \to H$ locally admits sections, not necessarily respecting the group laws. Note that this holds with $G = \mathrm{GL}_n$ and $H = \mathrm{PGL}_n$, but fails for $G = H = \mathbb{G}_m$ and taking powers by $d \geq 2$. We now ask whether the H-torsor $g^*(X) = X \times_Q S$, for fixed $g \in Q(S)$, admits a reduction of structure group with respect to the epimorphism $G \to H$. This of course holds if the torsor is trivial, but is in general a much weaker condition.

If N is non-trivial, the G-action on X is non-free. Instead of forming the quotient $X/G^{\mathrm{op}} = X/H^{\mathrm{op}} = Q$, which loses too much information, we consider the category

$$[X/G^{\mathrm{op}}/Q] = \{(U, g, P, f) \mid f : P \to X_U\}$$

whose objects are quadruples where U is an object from $\mathcal{C}$, and $g : U \to Q$ is a morphism in $\mathscr{T}$, and P is a G_U-torsor, and $f : P \to X_U$ is a morphism that is G_U-equivariant. Note that the latter condition holds if and only if the composite map $P \stackrel{f}{\to} X_U \stackrel{\mathrm{pr}_1}{\to} X$ is G-equivariant. Also note that g is determined as the map induced from $\mathrm{pr}_1 \circ f$.

Morphisms $(U', g', P', f') \to (U, g, P, f)$ are defined as pairs (u, p) of morphisms $u : U' \to U$ and $p : P' \to P$ such that the diagram

$$\begin{tikzcd} X_{U'} \arrow[d, "\mathrm{id}\times u"'] & P' \arrow[l, "f'"'] \arrow[r] \arrow[d, "p"] & U' \arrow[r, "g'"] \arrow[d, "u"] & Q \arrow[d, "\mathrm{id}"] \\ X_U & P \arrow[l, "f"] \arrow[r] & U \arrow[r, "g"'] & Q \end{tikzcd}$$

is commutative, and the resulting $P' \to P \times_U U'$ is $G_{U'}$-equivariant. The category comes with a forgetful functor

$$[X/G^{\mathrm{op}}/Q] \longrightarrow \mathcal{C}_{/Q}, \quad (U, g, P, f) \longmapsto (U, g)$$

to the comma category of objects $U \in \mathcal{C}$ endowed with a structure morphism to $Q \in \mathscr{T}$.

Proposition 2.1. *The above functor gives $[X/G^{\mathrm{op}}/Q]$ the structure of a category fibered in groupoids over $\mathcal{C}_{/Q}$.*

Proof. For each $(U, g) \in \mathcal{C}_{/Q}$, a morphism $(u, p) : (U, g, P', f') \to (U, g, P, f)$ in the fiber category has $u = \mathrm{id}_U$, and is thus determined by the morphism $p : P' \to P$ of G_U-torsors. Any morphism of torsors is necessarily an isomorphism, so the fiber categories are groupoids.

It remains to check that every morphism $(u, p) : (U', g', P', f') \to (U, g, P, f)$ is cartesian. The induced morphism in $\mathcal{C}_{/Q}$ is $u : U' \to U$, and p is already determined by the isomorphism $P' \to P \times_U U'$. The universal property of fiber products in $\mathscr{T}$ reveals that (u, p) is cartesian. $\square$

We now regard the comma category $\mathcal{C}_{/Q}$ as a site: A family $((U_\lambda, g_\lambda) \to (U, g))$ is a *covering* if and only if $(U_\lambda \to U)$ is a covering family for the site $\mathcal{C}$.

Proposition 2.2. *$[X/G^{\mathrm{op}}/Q] \to \mathcal{C}_{/Q}$ satisfies the stack axioms, and is actually a gerbe.*

Proof. Using that for $\mathrm{Sh}(\mathcal{C})$ all descent data are effective, one easily shows that every descent datum for $[X/G^{\mathrm{op}}/Q] \to \mathcal{C}_{/Q}$ is effective. Now let (U, g, P, f) be an object over some (U, g). For each morphism $(U', g') \to (U, g)$ in the comma category, we choose pullbacks (U', g', P', f') in $[X/G^{\mathrm{op}}/Q]$. Using the sheaf property of $P \in \mathscr{T}$, one easily checks that the resulting presheaf

$$(U', g') \longmapsto \mathrm{Aut}\left((U', g', P', f')/(U', g')\right),$$

satisfies the sheaf axiom. Note that the automorphism group is formed in the fiber category. Summing up, $[X/G^{\mathrm{op}}/Q] \to \mathcal{C}_{/Q}$ is a stack.

It remains to check that all objects in the stack over a fixed (U, g) are locally isomorphic. Fix two objects (U, g, P_1, f_1) and (U, g, P_2, f_2), and choose a covering family $(U_\lambda \to U)$ that trivializes the G_U-torsors P_1 and P_2. This reduces us to the special case $P_i = G_U$. Write $f_1(e_G) = \sigma \cdot f_2(e_G)$ for some section $\sigma : U \to H_U$. Choose another covering family $(U_\mu \to U)$ such that $\sigma | U_\mu$ admits a lift to G_{U_μ}. Note that here we use our standing assumption that the epimorphism $G \to H$ locally admits sections, not necessarily respecting the group laws. This reduces our problem to the case that $\sigma : U \to H_U$ arises from some $\tilde{\sigma} : U \to G_U$. Then $(\mathrm{id}_U, \tilde{\sigma})$ defines the desired isomorphism between (U, g, P_1, f_1) and (U, g, P_2, f_2). $\square$

For each $(U, g) \in \mathcal{C}_{/Q}$, we have a canonical identification of comma categories $(\mathcal{C}_{/Q})_{/(U,g)} = \mathcal{C}_{/U}$. We now seek to compute the sheaf of groups

$$\underline{\mathrm{Aut}}_{(U,g,P,f)/(U,g)} \subset \underline{\mathrm{Aut}}_{P/U} = {}^P G_U^{\mathrm{op}},$$

where the last identification comes from (1). On the other hand, twisting the exact sequence (3) yields another exact sequence $1 \to {}^P N_U \to {}^P G_U \to {}^P H_U \to 1$, which gives an inclusion ${}^P N_U^{\mathrm{op}} \subset {}^P G_U^{\mathrm{op}} = \underline{\mathrm{Aut}}_{P/U}$.

Proposition 2.3. *We have* $\underline{\mathrm{Aut}}_{(U,g,P,f)/(U,g)} = {}^P N_U^{\mathrm{op}}$ *as subsheaves inside* $\underline{\mathrm{Aut}}_{P/U}$.

Proof. The problem is local in U, so it suffices to treat the case that $P = G_U$, and thus ${}^P N_U^{\mathrm{op}} = N_U^{\mathrm{op}}$. Given a section $a : U \to N_U$, we write $p : G_U \to G_U$ for the right-multiplication with a. Then (id_U, p) is the desired automorphism of (U, g, P, f). Conversely, each automorphism (id_U, p) yields a commutative diagram

$$\begin{array}{ccc} X_U & \longleftarrow & G_U \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle p} \\ X_U & \longleftarrow & G_U. \end{array}$$

The map on the right takes the form $p(x) = xg$ for some $g \in G(U)$. Since the H-action on X is free, the above diagram ensures $g \in N(U)$. $\square$

Let us say that N is *central* if the subgroup object $N \subset G$ is contained in the center $Z(G)$. In turn, N is abelian, the conjugation action of G on N is trivial, and (3) is a central extension. In particular we have identifications ${}^P N_U^{\mathrm{op}} = N_U^{\mathrm{op}} = N_U$. Thus for each object $(U, g, P, f) \in [X/G^{\mathrm{op}}/Q]$ we get an isomorphism

$$\varphi_{(U,g,P,f)} : N_U \longrightarrow \underline{\mathrm{Aut}}_{(U,g,P,f)/(U,g)},$$

which are compatible with respect to morphisms. In other words:

Corollary 2.4. *If N is central, the datum of the above maps endows the stack $[X/G^{\mathrm{op}}/Q]$ over $\mathcal{C}_{/Q}$ with the structure of an N_Q-gerbe.*

Suppose N is central. For each $g : U \to Q$, the resulting N_U-gerbe yields a *cohomology class*

$$\mathrm{cl}([g^* X/G^{\mathrm{op}}/U]) = g^* \,\mathrm{cl}([X/G^{\mathrm{op}}/Q]) \in H^2(U, N_U),$$

which are compatible with respect to pullbacks. We refer to [18], Chapter IV, Section 3.4 for more details. In fact, by the geometric interpretation of low-degree cohomology, one may view $H^1(U, N_U)$ as the set of isomorphism classes of N_U-torsors, and $H^2(U, N_U)$ as the set of equivalence classes of N_U-gerbes. Together with $H^0(U, N_U) = \Gamma(U, N_U)$, these indeed form a delta functor in degree $i \leq 2$, in the sense of [21].

For $U = S$, we obtain the so-called *obstruction map*

$$\mathrm{ob} : Q(S) \longrightarrow H^2(S, N), \quad g \longmapsto g^* \,\mathrm{cl}([X/G^{\mathrm{op}}/Q]).$$

Under an additional assumption, this is *precisely* the obstruction against having geometric origin:

Theorem 2.5. *Suppose N is central, and that the canonical map $H^1(S,N) \to H^1(S,G)$ is surjective. Then $g \in Q(S)$ is of geometric origin if and only if the obstruction* $\mathrm{ob}(g) \in H^2(S,N)$ *is zero.*

Proof. Saying that $g : S \to Q$ has geometric origin means that the H-torsor $g^*(X)$ is trivial. Then it obviously admits a reduction of structure group with respect to $G \to H$, and thus $\mathrm{ob}(g)$ vanishes. Conversely, suppose that $\mathrm{ob}(g) = 0$. Then $g^*(X)$ arises from a G-torsor P. The latter comes from an N-torsor T, by assumption. Thus $g^*(X)$ is induced from an N-torsor T with respect to the composite map $N \to G \to H$, which is trivial, and thus $g^*(X)$ is trivial. □

Let us now specialize the above result to the case that $G = \mathrm{GL}_n$ and $N = \mathbb{G}_m$, and $\mathcal{C} = (\mathrm{Aff}/R)$ is the category of affine schemes over a ground ring R, endowed with the fppf topology. The final object then is $S = \mathrm{Spec}(R)$.

Corollary 2.6. *In the above setting, suppose that the ground ring R is*

(i) *semilocal,*
(ii) *or a polynomial ring in finitely many indeterminates, either over a field or a principal ideal domain,*
(iii) *or a factorial Dedekind domain.*

Then $g \in Q(S)$ is of geometric origin if and only if $\mathrm{ob}(g) \in H^2(S, \mathbb{G}_m)$ *vanishes.*

Proof. According to Hilbert 90, every GL_n-torsor over a scheme is trivial in the Zariski topology. If R is semilocal, the set $H^1(S, \mathrm{GL}_n)$ is thus a singleton, and the assumption of the Theorem obviously holds.

If R is a Dedekind ring, the structure theory of locally free modules of finite rank gives that the determinant map $H^1(S, \mathrm{GL}_n) \to H^1(S, \mathbb{G}_m)$ is bijective. If $\mathrm{Pic}(S) = 0$, in other words R is factorial, the set $H^1(S, \mathrm{GL}_n)$ must be a singleton. By the Quillen–Suslin Theorem ([44] and [51]), the same is true if $R = k[T_1, \ldots, T_r]$ or $R = D[T_1, \ldots, T_r]$, where k is a field and D is a principal ideal domain. □

If $R = K$ is a field, the cohomology group $H^2(S, \mathbb{G}_m)$ coincides with the Brauer group $\mathrm{Br}(K)$. If the latter vanishes, we conclude with the above result that every rational point $g \in Q(K)$ is of geometric origin. This generalizes [28], Example 3.8, which dealt with quiver moduli spaces. Recall that the Brauer group vanishes for separably closed fields or C_1-fields, and in particular for function fields over algebraically closed fields (Tsen's Theorem). Note that this includes imperfect fields.

3. Spaces of Schur representations

Throughout, an *associative ring* is an abelian group Λ, endowed with a bilinear multiplication that possesses a unit element and satisfies the axiom of associativity. If additionally the axiom of commutativity holds it is simply called *ring*. An *associative algebra* over a ring R is an associative ring Λ, together with a homomorphism $\varphi : R \to \Lambda$ whose image $\varphi(R)$ belongs to the center $Z(\Lambda)$. Then for each ring homomorphism $R \to A$, the tensor product $\Lambda \otimes_R A$ carries in a canonical way the structure of an A-algebra.

Let R be a ground ring, and Λ and Λ' be two associative R-algebras. Important special cases arise when $\Lambda = R[\Gamma]$ is a group algebra of some group Γ, or the path algebra $\Lambda = R[\Delta]$ of some quiver Δ, and Λ' is a matrix algebra or more generally an Azumaya algebra. For each R-algebra A we can form the set

$$F(A) = F_{\Lambda/R}^{\Lambda'}(A) = \mathrm{Hom}_{A\text{-Alg}}(\Lambda \otimes_R A, \Lambda' \otimes_R A).$$

The construction is functorial in A, and defines a set-valued contravariant functor on the category (Aff/R) of affine R-schemes, which is opposite to the category of R-algebras. The elements $\rho \in F(A)$ are called *A-valued representations of Λ in Λ'.*

Proposition 3.1. *The above contravariant functor $F : (\mathrm{Aff}/R) \to (\mathrm{Set})$ satisfies the sheaf axiom with respect to the fpqc topology.*

Proof. Let A be an R-algebra, $A \subset A_0$ be a faithfully flat ring extension, and $A_1 = A_0 \otimes_A A_0$. We have to check that

$$F(A) \longrightarrow F(A_0) \rightrightarrows F(A_1)$$

is an equalizer diagram of sets. In other words, the arrow on the left is injective, and its image comprises the $\rho_0 \in F(A_0)$ whose two images in $F(A_1)$ coincide. Consider the larger functor $G(A) = \mathrm{Hom}_{A\text{-Mod}}(\Lambda \otimes_R A, \Lambda' \otimes_R A)$ given by linear maps, not necessarily preserving multiplication and unit element. By fpqc descent, the above diagram with G instead of F is an equalizer ([26], Exposé VIII, Corollary 1.2). It follows that $F(A) \to F(A_0)$ is injective, and each $\rho_0 \in F(A_0)$ whose images in $F(A_1)$ coincide descends to some $\rho \in G(A)$. It remains to check that $\rho(xy) = \rho(x)\rho(y)$ for all $x, y \in \Lambda \otimes_R A$, and $\rho(1) = 1$. This follows from the commutative diagram

$$\begin{array}{ccc}
\Lambda \otimes_R A & \xrightarrow{\iota} & \Lambda \otimes_R A_0 \\
{\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho_0} \\
\Lambda' \otimes_R A & \xrightarrow[\iota']{} & \Lambda' \otimes_R A_0,
\end{array}$$

because the canonical map ι' is injective, and ι, ρ_0 and ι' respect multiplications and unit element. □

Suppose from now on that the structure map $R \to \Lambda'$ is locally a direct summand of R-modules; this ensures that all induced maps $A \to \Lambda' \otimes A$ remain injective. We then consider the subsets

$$F^0(A) \subset F(A)$$

comprising the homomorphisms $\rho : \Lambda \otimes_R A \to \Lambda' \otimes_R A$ such that for each prime ideal $\mathfrak{p} \subset A$, the only elements in $\Lambda' \otimes_R \kappa(\mathfrak{p})$ that commute with all $\rho(x)$, $x \in \Lambda \otimes_R \kappa(\mathfrak{p})$ are the members of $R \otimes_R \kappa(\mathfrak{p}) = \kappa(\mathfrak{p})$, in other words, the scalars. Clearly, the formation of $F^0(A) \subset F(A)$ is functorial in A, and thus defines a subfunctor $F^0 \subset F$. The sheaf axiom with respect to the fpqc topology holds for F^0, because it holds for F, and the formation of commutants commutes with field extensions. The elements $\rho \in F^0(A)$ are called A-valued *Schur representations* of Λ in Λ'.

Proposition 3.2. *Suppose Λ' is locally free of finite rank as R-module. Then the inclusion $F^0 \subset F$ is relatively representable by open embeddings.*

Proof. Let $V = \operatorname{Spec}(A)$ be an affine R-scheme, and $\rho : \Lambda \otimes_R A \to \Lambda' \otimes_R A$ be some homomorphism. By the Yoneda Lemma, we may regard it as a natural transformation $\rho : V \to F$, and have to check that the ensuing fiber product $F^0 \times_F V$ is representable by a scheme, and that the projection to V is an open embedding. Clearly, the projection $F^0 \times_F V \to V$ is a monomorphism.

Without loss of generality we may assume $A = R$, and that the R-module Λ' is free of rank $r \geq 1$. Let $U \subset V$ be the set of points $a \in V$ where the canonical morphism $\operatorname{Spec} \kappa(a) \to V$ factors over $F^0 \times_F V$. Suppose for the moment that this is an open set, and thus defines an open subscheme. By the very definition of F^0 and U, the two monomorphisms $F^0 \times_F V \to V$ and $U \to V$ factor over each other, hence the former is represented by the latter.

It remains to verify that the subset U is indeed open. Fix a point $a \in U$, in other words, $\kappa(a) \subset \Lambda' \otimes \kappa(a)$ is the commutant for $\rho \otimes \kappa(a)$. Using that the vector space $\Lambda' \otimes \kappa(a)$ is finite-dimensional, we find finitely many $g_1, \ldots, g_n \in \Lambda$ such that $\kappa(a)$ is the commutant for the $\rho(g_i) \otimes 1$, $1 \leq i \leq n$. Consider the R-linear map

$$\Psi : \Lambda' \longrightarrow \bigoplus_{i=1}^{n} \Lambda', \quad f \longmapsto (f\rho(g_i) - \rho(g_i)f)_{1 \leq i \leq n}$$

between free R-modules of finite rank. For each point $v \in V$, the kernel for $\Psi \otimes \kappa(v)$ contains $\kappa(v)$, hence $\operatorname{rank}(\Psi \otimes \kappa(v)) \leq r - 1$. For $v = a$ this becomes an equality. Viewing Ψ as an $rn \times r$-matrix, we see that some $(r-1)$-minor $h \in R$ does not vanish in $\kappa(a)$. Replacing R by the localization $R[h^{-1}]$ we may assume that the minor is a unit. This ensures that the function $v \mapsto \operatorname{rank}(\Psi \otimes \kappa(v))$ takes constant value $r - 1$. By [22], Chapter 0, Proposition 5.5.4 the image of Ψ is locally free of rank $r - 1$, hence the kernel is invertible. For each point $v \in V$ the unit element $1 \in \Lambda'$ generates $\operatorname{Ker}(\Psi) \otimes \kappa(v)$, so the inclusion $R \subset \operatorname{Ker}(\Psi)$ is an equality. It follows that $\kappa(v) \subset \Lambda \otimes \kappa(v)$ is the commutant for the $\rho(g_i) \otimes 1$, and thus for $\rho \otimes \kappa(v)$, for all $v \in V$. This shows $U = V$. □

We also have an absolute representability statement:

Proposition 3.3. *Suppose Λ' is locally free of finite rank as R-module. Then $F : (\mathrm{Aff}/R) \to (\mathrm{Set})$ is representable by an affine scheme. It is of finite type provided that Λ is finitely generated.*

Proof. Suppose first that the associative algebra $\Lambda = R\langle T_i \rangle_{i \in I}$ is free. One easily checks that F is represented by the spectrum of

$$B = \bigotimes_{i \in I} \operatorname{Sym}^\bullet(\Lambda'^\vee) = \operatorname{Sym}^\bullet(\bigoplus_{i \in I} \Lambda'^\vee),$$

by using various universal properties and biduality $\Lambda' = \Lambda'^{\vee\vee}$.

For the general case, express the associative algebra Λ in terms of generators $g_i \in \Lambda$, $i \in I$ and relations $r_j \in R\langle T_i \rangle_{i \in I}$, $j \in J$. The preceding paragraph gives a monomorphism $F \subset \operatorname{Spec}(B)$, and we have to check that this is relatively representable by closed embeddings. This is a local problem, so we may assume that Λ'

admits a basis $e_1, \ldots, e_r$ as R-module. For each $P \in R\langle T_i\rangle_{i\in I}$, the equation

$$P(\ldots, x_i, \ldots) = \sum_{k=1}^{r} P_k(\ldots, x_i, \ldots) \cdot e_k$$

inside Λ' defines polynomial maps $P_k : \bigoplus_{i\in I} \Lambda' \to R$. These maps stem from elements $P_k \in \mathrm{Sym}^\bullet(\bigoplus_{i\in I} \Lambda'^\vee) = B$, because multiplication in Λ' is bilinear. One easily checks that $F \subset \mathrm{Spec}(B)$ is the closed subscheme defined by the ideal generated by the $P_1, \ldots, P_r$, where $P = r_j$, $j \in J$ ranges over the relations. $\square$

Suppose now that Λ' is an *Azumaya algebra* of degree $n \geq 1$, that is, a twisted form of $\mathrm{Mat}_n(R)$. Note that $\Lambda' = \mathrm{Mat}_n(R)$ is indeed an important special case. Write $U_{\Lambda'/R}$ and $\mathrm{Aut}_{\Lambda'/R}$ for the group-valued sheaves on (Aff/R) defined by

$$U_{\Lambda'/R}(A) = (\Lambda' \otimes_R A)^\times \quad \text{and} \quad \mathrm{Aut}_{\Lambda'/R}(A) = \mathrm{Aut}(\Lambda' \otimes_R A).$$

These are twisted forms of GL_n and PGL_n, confer the discussion in [47], Lemma 3.1. In particular, $U_{\Lambda'/R}$ and $\mathrm{Aut}_{\Lambda'/R}$ are smooth affine group schemes of finite type. The conjugacy map defines an exact sequence

$$1 \longrightarrow \mathbb{G}_m \longrightarrow U_{\Lambda'/R} \xrightarrow{\mathrm{conj}} \mathrm{Aut}_{\Lambda'/R} \longrightarrow 1,$$

where the inclusion on the left comes from the structure inclusion $R \subset \Lambda'$.

The group scheme $G = \mathrm{Aut}_{\Lambda'/R}$ acts on the sheaf $F : (\mathrm{Aff}/R) \to (\mathrm{Set})$ via composition $(g, \rho) \mapsto g \circ \rho$. Now let $\tilde{\Lambda}'$ be a twisted form of Λ', that is, another Azumaya algebra of the same degree n. Write P for the corresponding G-torsor, and let $\tilde{F} : (\mathrm{Aff}/R) \to (\mathrm{Set})$ be the sheaf defined with Λ and $\tilde{\Lambda}'$.

Proposition 3.4. *Notation as above. Then $F^0 \subset F$ is G-stable, the induced G-action on F^0 is free, and we have canonical identifications*

$$P \wedge^G F = \tilde{F} \quad \text{and} \quad P \wedge^G F^0 = \tilde{F}^0$$

of contravariant functors on (Aff/R).

Proof. The open subfunctor F^0 is clearly G-stable. Suppose we have an R-algebra A, and elements $g \in G(A)$ and $\rho \in F^0(A)$ with $g \circ \rho = \rho$. Choose some fpqc extension $A \subset A_0$ so that $\Lambda' \otimes_R A_0 \simeq \mathrm{Mat}_n(R_0)$. Then $g_0 = g \otimes A_0$ becomes an element in $\mathrm{PGL}_n(A_0)$. Replacing A_0 by some further fpqc extension, we may assume that it stems from some $S \in \mathrm{GL}_n(A_0)$. Setting $\rho_0 = \rho \otimes A_0$, we obtain $g_0 \circ \rho_0 = S \cdot \rho_0 \cdot S^{-1}$. In other words, $S \cdot \rho_0 = \rho_0 \cdot S$. Using that ρ is a Schur representation, we see that S must be a scalar matrix. Thus g_0 and hence also g are trivial. Consequently, the G-action on F^0 is free.

For the last assertion we first assume that there is an isomorphism $\psi : \Lambda' \to \tilde{\Lambda}'$ of Azumaya algebras. This corresponds to an equivariant map $\psi : G \to P$, and the arrows in

$$P \wedge^G F \xleftarrow{\psi} G \wedge^G F = F \xrightarrow{\psi} \tilde{F}$$

yield the desired canonical identification $P \wedge^G F = \tilde{F}$.

In the general case, we find an fpqc extension $R \subset R_0$ and an isomorphism $\psi_0 : \Lambda' \otimes_R R_0 \to \tilde{\Lambda}' \otimes_R R_0$, with corresponding equivariant $\psi_0 : G \otimes_R R_0 \to P \otimes_R R_0$. This yields $P \wedge^G F = \tilde{F}$ if we restrict the contravariant sheaves along the forgetful functor $(\mathrm{Aff}/R_0) \to (\mathrm{Aff}/R)$. Since both $P \wedge^G F$ and $\tilde{F}$ satisfy the sheaf axiom with respect to the fpqc topology, and since every fpqc extension $A \subset A'$ of R-algebras can be refined to an fpqc covering $A \otimes_R R_0 \subset A' \otimes_R R_0$ of R_0-algebras, the identification $P \wedge^G F = \tilde{F}$ already holds over (Aff/R). Since the G-action on F stabilizes F^0, we get an induced identification $P \wedge^G F^0 = \tilde{F}^0$. □

Let us summarize our findings:

Theorem 3.5. *Suppose Λ is finitely presented as an associative R-algebra, and Λ^{azu} is an Azumaya algebra of degree $n \geq 1$. Then the functor $X = X_{\Lambda/R}^{\Lambda^{\mathrm{azu}}}$ of Schur representations is representable by a quasiaffine scheme of finite presentation. Moreover, the group scheme $H = \mathrm{Aut}_{\Lambda^{\mathrm{azu}}/R}$ acts freely, and the quotient $Q = X/H^{\mathrm{op}}$ is an algebraic space that is of finite presentation.*

Proof. By Proposition 3.2 and 3.3, the functor X is representable by some open set in some affine scheme of finite type. Obviously, the structure morphism $H \to S$ is flat and locally of finite presentation. Since the H-action on X is free, the quotient $Q = X/H^{\mathrm{op}}$ exists as an algebraic space (see for example [35], Lemma 1.1). It follows from [24], Proposition 2.7.1 that Q is locally of finite type. Moreover, Q is of finite presentation if this holds for X.

It remains to check that X is quasicompact and locally of finite presentation. In light of loc. cit., we may replace R by some fppf extension, and assume that $\Lambda^{\mathrm{azu}} = \mathrm{Mat}_n(R)$. Since the associative algebra Λ involves only finitely many structure constants, we may furthermore assume that the ground ring R is finitely generated over $\mathbb{Z}$. Using that X is open in some affine scheme of finite type, we see that it is noetherian. □

To close this section, let us relate Schur representations to other notions from representation theory, for simplicity when $R = K$ is a field. Let M be a Λ-module that is finite-dimensional as K-vector space, and $E = \mathrm{End}_\Lambda(M)$ be its endomorphism ring. One says that M is *geometrically simple* if for all field extensions $K \subset L$, the base change $M_L = M \otimes_K L$ is simple as representation of $\Lambda_L = \Lambda \otimes_K L$. The notion of *geometrically indecomposable* is formed in a similar way. The alternative terms *absolutely simple* and *absolutely indecomposable* are also frequently in use. Recall that the associative ring E is *local* if the non-units form a left ideal $\mathfrak{m}$. This is indeed maximal and two-sided, giving the residue skew field $\kappa = E/\mathfrak{m}$.

Lemma 3.6. *In the above situation, the following holds:*

(i) *If the module M is simple, then E is a skew field.*
(ii) *The module M is indecomposable if and only if E is local.*
(iii) *If M is geometrically simple, then M is Schur.*

Moreover, if M is indecomposable and L is purely inseparable, then the Λ_L-module M_L is indecomposable.

Proof. The first assertion is Schur's Lemma. Statement (ii) can be found in [12], Proposition 6.10. For (iii) choose a separably closed extension L. Since M_L is simple and the Brauer group $\mathrm{Br}(L)$ vanishes, the inclusion $L \subset E_L$ must be an equality. Hence the same holds for $K \subset E$, and M is Schur.

Finally, let M be indecomposable and L purely inseparable. To check that M_L is indecomposable it suffices to treat the case that $L \subset K^{1/p}$, where $p > 0$ is the characteristic. Let $f, g \in E_L$ be two non-units. Then f^p, g^p are non-units that belong to $E \subset E_L$. They vanish in the residue skew field κ, so the same holds for f, g. It follows that $f + g$ vanishes in the residue class ring κ_L of E_L, so $f + g \in E_L$ is a non-unit. Consequently, the non-units form a left ideal. □

Lemma 3.6(ii) shows in particular that Schur representations are geometrically indecomposable, because Schur representations are preserved by base change. Note that Kraft and Riedtmann ([33], Theorem in 2.6) showed that a quiver representation M is Schur if and only if the point M in the representation space admits an open neighborhood where all points are geometrically indecomposable.

Also note that moduli spaces of representations are traditionally formed in the realm of *Geometric Invariant Theory* [42], which relies on the choice of *stability conditions*. The latter ensure, by design, that the *geometrically stable objects* are Schur representations. In fact the geometrically stable objects for any given stability condition form an open subset of X. In turn, all possible GIT quotients are simultaneously contained as schematic open subsets in our algebraic space $Q = X/H^{\mathrm{op}}$. However, the semistable objects, which frequently appear on the boundary in GIT quotients, often fail to be Schur and are thus beyond the scope of this paper.

We want to conclude this section by giving examples of Schur representations which are not geometrically stable. All of our examples are from the realm of quiver representations, which will also be the topic of Section 4 below. See for example [28], Section 2 for a discussion of geometrically stable representations in this context.

Example 3.7. Let $\Lambda = K\langle t_1, \ldots, t_m\rangle$ be the free associative algebra in $m \geq 2$ generators over K. Note that Λ can also be viewed as the path algebra of the quiver Δ consisting of a single vertex and m loops (see (4) below). For every stability parameter $\theta \in \mathbb{Z}^{\Delta_0} = \mathbb{Z}$ the associated slope function μ_θ is constant, hence, a representation of Λ is θ-stable if and only if it is simple.

A K-valued representation of Λ is given by an m-tuple $(x_1, \ldots, x_m)$ of square matrices $x_i \in \mathrm{Mat}_n(K)$, an endomorphism is a square matrix $f \in \mathrm{Mat}_n(K)$ which commutes with each x_i and a subrepresentation is a K-linear subspace $V \subset K^n$ which is x_i-stable for all i.

Assume that $n \geq 2$ and fix scalars $\lambda, \mu, \lambda', \mu' \in K$ satisfying $\lambda \neq \mu$ and $\lambda' \neq \mu'$. We define

$$x_1 = \begin{pmatrix} J_2(\lambda) & 0 \\ 0 & J_{n-2}(\mu) \end{pmatrix}, \quad x_2 = \begin{pmatrix} \lambda' & 0 \\ 0 & J_{n-1}(\mu') \end{pmatrix}, \quad \text{and } x_i = 0 \text{ for } i \geq 3$$

where $J_l(\lambda)$ denotes the lower triangular Jordan block of size l. By computing the centralizers of Jordan blocks one sees that the only matrices commuting which x_1 and x_2 are the scalar matrices, so the representation is Schur. Conversely the n-th standard basis vector is a simultaneous eigenvector of all x_i and therefore spans a proper non-trivial subrepresentation showing that the representation is not stable, so in particular not geometrically stable.

Example 3.8. Let Λ be the associative K-algebra on five generators $e_1, e_2, \alpha_1, \alpha_2, \alpha_3$ with relations

$$e_1^2 = e_1,\ e_2^2 = e_2,\ e_1e_2 = 0 = e_2e_1$$

and

$$e_2\alpha_k = \alpha_k = \alpha_k e_1 \quad \text{and} \quad \alpha_k e_2 = 0 = e_1\alpha_k \ \text{ for } \ k = 1,2,3$$

Λ arises as the path algebra of the quiver Δ consisting of two vertices i_1 and i_2 and three arrows α_k each with source i_1 and target i_2.

We consider the representation $\rho : \Lambda \to \mathrm{Mat}_4(K)$ given by

$$\rho(e_1) = \begin{pmatrix} 0 & 0 \\ 0 & E_2 \end{pmatrix}, \quad \rho(e_2) = \begin{pmatrix} E_2 & 0 \\ 0 & 0 \end{pmatrix}$$

where $E_2 \in \mathrm{Mat}_2(K)$ denotes the unity matrix, and

$$\rho(\alpha_1) = \begin{pmatrix} 0&0&1&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \quad \rho(\alpha_2) = \begin{pmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \quad \rho(\alpha_3) = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$$

The representation ρ can also be viewed as a quiver representation of Δ with dimension vector $v = (2,2)$, while the span of the first and third standard basis vector form a subrepresentation of dimension $w = (1,1)$ (see the next Section for more on the correspondence between quiver representations and representations of path algebras). If $\theta \in \mathbb{Z}^{\Delta_0} = \mathbb{Z}^2$ is any stability parameter, then $\mu_\theta(v) = \frac{1}{2}(\theta_1 + \theta_2) = \mu_\theta(w)$ showing that ρ is not stable for any θ. However a straightforward computation shows that ρ is a Schur representation:

An endomorphism of ρ is a square matrix $f \in \mathrm{Mat}_4(K)$ which commutes with $\rho(x)$ for every $x \in \Lambda$. Such an f in particular preserves the eigenspaces of $\rho(e_1)$, so it has a block form $f = \left(\begin{smallmatrix} f_1 & 0 \\ 0 & f_2 \end{smallmatrix}\right)$ with $f_1, f_2 \in \mathrm{Mat}_2(K)$. Now the condition $f\rho(\alpha_k) = \rho(\alpha_k)f$ for $k = 1,2,3$ shows that f is scalar matrix.

4. Quiver representations

Our initial motivation for this paper was to understand twisting of quiver representations, as studied by Hoskins and Schaffhauser [28]. In this section we indeed take a closer look at representations of quivers in Azumaya algebras, as a special case of the general theory developed so far, and show that the resulting space of Schur representations embeds into a product of Grassmann varieties and vector bundles.

Recall that a *quiver* is a finite directed graph Δ, with loops and multiple arrows allowed. Formally, it comprises a set of vertices Δ_0 and a set of arrows Δ_1, together with source and target maps $\Delta_1 \rightrightarrows \Delta_0$. An arrow α starting at a vertex i and ending at a vertex j thus has $i = \mathrm{source}(\alpha)$ and $j = \mathrm{target}(\alpha)$. By abuse of notation, we write $i, \alpha \in \Delta$ instead of $i \in \Delta_0, \alpha \in \Delta_1$ to denote both vertices and arrows from the quiver. No confusion is possible, since we always use Latin letters for vertices, and Greek letters for arrows. A *quiver representation* over a field k is a collection of vector spaces V_i for the vertices $i \in \Delta$, together with linear maps $f_\alpha : V_{\mathrm{source}(\alpha)} \to V_{\mathrm{target}(\alpha)}$ for the arrows $\alpha \in \Delta$. We usually write this datum in the form $(V_i \mid f_\alpha)_{i,\alpha\in\Delta}$. With the obvious notion of homomorphisms, the quiver representations form a k-linear abelian category.

Now fix a ground ring R and write $\Lambda = R[\Delta]$ for the resulting *path algebra*, which is the associative R-algebra generated by formal symbols e_i and f_α, for the vertices and arrows $i, \alpha \in \Delta$, subject to the relations

$$e_ie_j = \delta_{i,j}e_i \quad \text{and} \quad f_\alpha e_i = \delta_{\mathrm{source}(\alpha),i}f_\alpha \quad \text{and} \quad e_jf_\beta = \delta_{j,\mathrm{target}(\beta)}f_\beta, \tag{4}$$

where $\delta_{i,j}$ denotes the Kronecker delta. From the relations one immediately sees that multiplication with $\sum_{i\in\Delta} e_i$, from either the left or the right, fixes each generator, and thus $\sum_{i\in\Delta} e_i = 1$. One also says that the $e_i \in \Lambda$, $i \in \Delta$ form a *partition of unity into orthogonal idempotents*. Recall that quiver representations over R can be identified with left modules over the path algebra $R[\Delta]$ (compare [31], Theorem 1.7).

We will now generalize this correspondence to representations of $R[\Delta]$ in an Azumaya algebra Λ^{azu}. Let U be a scheme and $\mathscr{A}$ be an associative $\mathscr{O}_U$-algebra whose underlying $\mathscr{O}_U$-module is quasicoherent. We call a datum of the form $(\mathscr{V}_i \mid f_\alpha)_{i,\alpha\in\Delta}$, where $\mathscr{V}_i$ is a sheaf of right $\mathscr{A}$-modules and $f_\alpha : \mathscr{V}_{\mathrm{source}(\alpha)} \to \mathscr{V}_{\mathrm{target}(\alpha)}$ is a morphism of $\mathscr{A}$-modules, a *quiver representation of* Δ *in right* $\mathscr{A}$*-modules*. Note that this is an incarnation of the more general setup of quiver representations in abelian categories studied in [19] and [41]. We will discuss at the end of this section that this indeed generalizes the ordinary notion of quiver representations.

Now let $\rho : \mathscr{O}_U[\Delta] = \Lambda \otimes_R \mathscr{O}_U \to \mathscr{A}$ be a homomorphism of associative $\mathscr{O}_U$-algebras. The subsheaves

$$e_i \cdot \mathscr{A} = \rho(e_i \otimes 1) \cdot \mathscr{A} \subset \mathscr{A}$$

yield a direct sum decomposition $\bigoplus_{i\in\Delta}(e_i \cdot \mathscr{A}) = \mathscr{A}$ of right $\mathscr{A}$-modules, in light of $\sum_{i\in\Delta} e_i = 1$. By the relations (4), each arrow $\alpha \in \Delta$ defines $\mathscr{A}$-linear maps between the summands via left multiplication with $\rho(f_\alpha \otimes 1)$. By abuse of notation these maps are denoted by $f_\alpha : e_i \cdot \mathscr{A} \to e_j \cdot \mathscr{A}$. All of them vanish, except for $i = \mathrm{source}(\alpha)$ and $j = \mathrm{target}(\alpha)$. Hence, $(e_i \cdot \mathscr{A} \mid f_\alpha)_{i,\alpha\in\Delta}$ is a quiver representation of Δ in right $\mathscr{A}$-modules and we will see in the proof of Proposition 4.1 below that $\rho : \mathscr{O}_U[\Delta] \to \mathscr{A}$ is already determined by $(e_i \cdot \mathscr{A} \mid f_\alpha)_{i,\alpha\in\Delta}$. Note that if $\mathscr{A}$ is locally free of finite rank as an $\mathscr{O}_U$-module, the same holds for the summands $e_i \cdot \mathscr{A}$.

Fix an Azumaya algebra Λ^{azu} of degree $n \geq 1$, and set $H = \mathrm{Aut}_{\Lambda^{\mathrm{azu}}/R}$. We now examine the quasiaffine scheme $X = X^{\Lambda^{\mathrm{azu}}}_{R[\Delta]/R}$ of Schur representations in the Azumaya algebra Λ^{azu}. Our observation here is that X can also be described in terms of Grassmann varieties and vector bundles: Given a projective R-module E of finite rank $r \geq 0$, we write

$$\mathbb{A}^1 \otimes_R E = \mathrm{Spec}\,\mathrm{Sym}^\bullet(E^\vee) \quad \text{and} \quad \mathrm{Grass}^d_{E^\vee/R}$$

for the corresponding vector bundle of rank r and Grassmann varieties of relative dimension $d(r-d)$. The elements in the bidual $E = E^{\vee\vee}$ and the locally free quotients $\psi : E^\vee \to M$ of rank d give the respective sets of R-valued points. The latter can also be seen as the locally direct summands $N \subset E$ of corank d, via

$N = M^\vee$. By the discussion above, we have a canonical morphism

$$X = X^{\Lambda^{\mathrm{azu}}}_{R[\Delta]/R} \subset F^{\Lambda^{\mathrm{azu}}}_{R[\Delta]/R} \longrightarrow \dot{\bigcup_m} \left(\prod_{i\in\Delta} \mathrm{Grass}^{n^2-m_i}_{(\Lambda^{\mathrm{azu}})^\vee/R} \times \prod_{\alpha\in\Delta} \mathbb{A}^1 \otimes_R \Lambda^{\mathrm{azu}} \right), \tag{5}$$

where the disjoint union is indexed by tuples $m = (m_i)_{i\in\Delta}$ of natural numbers subject to $\sum_{i\in\Delta} m_i = n^2$. It sends an A-valued path algebra representation $\rho : A[\Delta] \to \Lambda^{\mathrm{azu}} \otimes_R A$ to the quiver representation $(e_i \cdot \Lambda^{\mathrm{azu}} \otimes_R A \mid f_\alpha)_{i,\alpha\in\Delta}$. The group scheme $H = \mathrm{Aut}_{\Lambda^{\mathrm{azu}}/R}$ acts in a canonical way on X, and also on the disjoint union on the right. Note that by our choice of conventions, the action is in both cases indeed from the left.

Proposition 4.1. *The above morphism is H-equivariant, and an embedding of quasiprojective schemes.*

Proof. Write Y for the right-hand side of (5). The group elements $\sigma \in H(A)$ act via

$$\sigma \cdot \varphi = \sigma \circ \varphi \quad \text{and} \quad \sigma \cdot (e_i \cdot \Lambda^{\mathrm{azu}} \otimes_R A \mid f_\alpha) = (\sigma(e_i \cdot \Lambda^{\mathrm{azu}} \otimes_R A) \mid \sigma \circ f_\alpha \circ \sigma^{-1}),$$

so equivariance of $X \to Y$ is clear. We now actually prove that this morphism is relatively representable by embeddings. Note that this gives, for quiver representations, a proof for representability independent of Theorem 3.5.

Let us start by checking that $X \to Y$ is a monomorphism. Suppose we have an A-valued representation $\rho : A[\Delta] \to \Lambda^{\mathrm{azu}} \otimes_R A$ of the path algebra, and set $\mathfrak{a}_i = e_i \cdot \Lambda^{\mathrm{azu}} \otimes_R A$. We need to show that ρ can be reconstructed from the datum $(\mathfrak{a}_i \mid f_\alpha)_{i,\alpha\in\Delta}$, and for this it suffices to treat the case $A = R$. As explained in [9], Chapter VIII, §8, No. 4 the decomposition $\Lambda^{\mathrm{azu}} = \bigoplus_{i\in\Delta} \mathfrak{a}_i$ into right ideals corresponds to a partition of unity into orthogonal idempotents $e_i \in \Lambda^{\mathrm{azu}}$, $i \in \Delta$. Together with the $f_\alpha \in \Lambda^{\mathrm{azu}}$, $\alpha \in \Delta$ we recover the path algebra representation $\rho : R[\Delta] \to \Lambda^{\mathrm{azu}}$.

Given a morphism $S \to Y$ from some affine scheme $S = \mathrm{Spec}(A)$, it remains to verify that $X' = X \times_Y S$ is representable by a scheme, and that the projection to S is an embedding. In other words, it factors as a closed embedding inside some open set. Again it suffices to treat the case $A = R$. Let $(\mathfrak{a}_i \mid f_\alpha)_{i,\alpha\in\Delta}$ be the datum defining $S \to Y$. So $\mathfrak{a}_i \subseteq \Lambda^{\mathrm{azu}}$ are locally free R-submodules and $f_\alpha \in \Lambda^{\mathrm{azu}}$ are arbitrary elements.

The cokernel M for the canonical mapping $\bigoplus_{i\in\Delta} \mathfrak{a}_i \to \Lambda^{\mathrm{azu}}$ is of finite presentation, so its support defines a closed set $Z \subset S$, and $X' \to S$ factors over the complementary open set. Replacing R by suitable localizations, we thus may assume that $\bigoplus_{i\in\Delta} \mathfrak{a}_i \to \Lambda^{\mathrm{azu}}$ is surjective. Since both R-modules are locally free of the same rank, the map is actually bijective.

Next we fix a vertex $i \in \Delta$, and choose algebra generators $g_1, \ldots, g_r \in \Lambda^{\mathrm{azu}}$. The R-submodule $\mathfrak{a}_i \subset \Lambda^{\mathrm{azu}}$ is a right ideal if and only if the cokernels M_{ij} for the right multiplications $\mathfrak{a}_i \xrightarrow{g_j} \Lambda^{\mathrm{azu}}/\mathfrak{a}_i$ are zero. The support of M_{ij} defines a closed set and $X' \to S$ factors over the complementary open set. As in the preceding paragraph, we may replace R by suitable localizations, and assume that the $\mathfrak{a}_i$ are right ideals.

Now we fix an arrow $\alpha \in \Delta$, and write $i, j \in \Delta$ for its source and target. The right multiplication $f_\alpha : \Lambda^{\mathrm{azu}} \to \Lambda^{\mathrm{azu}}$ sends $\mathfrak{a}_i$ to $\mathfrak{a}_j$ and vanishes on $\mathfrak{a}_r$, $r \neq i$ if and only if the induced maps $\mathfrak{a}_i \to \Lambda^{\mathrm{azu}}/\mathfrak{a}_j$ and $\mathfrak{a}_r \to \Lambda^{\mathrm{azu}}$ vanish. These are closed conditions.

This shows that $F = F^{\Lambda^{\mathrm{azu}}}_{R[\Delta]/R} \to Y$ is relatively representable by embeddings. Since $X = F^0 \subset F$ is relatively representable by open embeddings by Proposition 3.2, we obtain that $X \to Y$ is relatively representable by embeddings as well. □

To close this section, let us discuss the special case of matrix algebras, and verify that the quiver representations in right modules over $\Lambda^{\mathrm{azu}} = \mathrm{Mat}_n(R)$ corresponds to the classical notation of quiver representations over R. Recall that the functors

$$M \longmapsto M \otimes_{\mathrm{Mat}_n(R)} \mathrm{Mat}_{n\times 1}(R) \quad \text{and} \quad N \longmapsto N \otimes_R \mathrm{Mat}_{1\times n}(R)$$

are quasi-inverse equivalences between the category of right modules M over the matrix algebra $\mathrm{Mat}_n(R)$ and modules N over the commutative ring R (Morita equivalence, compare also the result of Watts [53]). Set $V = \mathrm{Mat}_{n\times 1}(R)$. Under the above equivalences, the decomposition $\mathrm{Mat}_n(R) = \bigoplus_{i\in\Delta} \mathfrak{a}_i$ into right ideals corresponds to the decomposition $V = \bigoplus_{i\in\Delta} V_i$ into submodules V_i, where the summands are locally free of some rank v_i, subject to $\sum v_i = n$. From $\mathfrak{a}_i = V_i \otimes_R \mathrm{Mat}_{1\times n}(R)$ we see $v_i = m_i/n$.

In the theory of quiver representations, the tuple $(v_i)_{i\in\Delta}$ is called *dimension vector*. Analogously to Proposition 4.1 one obtains an embedding

$$X = X^{\mathrm{Mat}_n(R)}_{R[\Delta]/R} \longrightarrow \dot{\bigcup_v} \left(\prod_{i\in\Delta} \mathrm{Grass}^{n-v_i}_{\mathrm{Mat}_{n\times 1}(R)^\vee/R} \times \prod_{\alpha\in\Delta} \mathbb{A}^1 \otimes_R \mathrm{Mat}_n(R) \right), \tag{6}$$

where the disjoint union now runs over all tuples $v = (v_i)_{i\in\Delta}$ subject to $\sum v_i = n$. In the theory of quiver representations, it is customary to fix the dimension vector v, and a decomposition $\mathrm{Mat}_{n\times 1}(R) = \bigoplus_{i\in\Delta} V_i$ with $\mathrm{rank}(V_i) = v_i$. The corresponding closed subscheme $X' \subset X$ becomes an open subscheme of the vector bundle $\prod_{\alpha\in\Delta} \mathbb{A}^1 \otimes_R \mathrm{Hom}(V_{\mathrm{source}(\alpha)}, V_{\mathrm{target}(\alpha)})$, which can be seen as a subbundle in $\prod_{\alpha\in\Delta} \mathbb{A}^1 \otimes_R \mathrm{Mat}_n(R)$. Note that this also shows that X is smooth, a fact that could also be proven by using that path algebras of quivers are formally smooth, in the sense of associative algebras. This shows more generally that $X^{\Lambda^{\mathrm{azu}}}_{R[\Delta]/R}$ is smooth over R for any Azumaya algebra Λ^{azu}, because smoothness can be checked locally and Λ^{azu} is locally isomorphic to $\mathrm{Mat}_n(R)$.

5. The gerbe of splittings for an Azumaya algebra

In this section we develop some facts on splittings of Azumaya algebras in the context of stacks. The material is well-known (confer the work of Lieblich [37], [38], [39]), but for our purposes we need it in a form that makes the involved fibered categories, the resulting topoi of sheaves, and the continuous maps between them explicit.

Fix a ground ring R, and some Azumaya algebra Λ^{azu} of degree $n \geq 1$. Note that now all rings, schemes etc. are considered as objects over the ground ring R. For any scheme V, let us write V' for the $\mathbb{G}_{m,V}$-*gerbe of splittings* for the Azumaya algebra $\Lambda^{\mathrm{azu}} \otimes_R \mathscr{O}_V$. This is the category

$$V' = \{(U, h, \mathscr{F}, \psi) \mid \ \psi : \underline{\mathrm{End}}(\mathscr{F}) \to \Lambda^{\mathrm{azu}} \otimes_R \mathscr{O}_U\}$$

whose objects are quadruples $(U, h, \mathscr{F}, \psi)$ where U is an affine scheme, $h : U \to V$ is a morphism of schemes, $\mathscr{F}$ is a locally free sheaf of rank n over U, and $\psi : \underline{\mathrm{End}}(\mathscr{F}) \to \Lambda^{\mathrm{azu}} \otimes_R \mathscr{O}_U$ is an isomorphism of sheaves of associative algebras.

Arrows $(U_1, h_1, \mathscr{F}_1, \psi_1) \to (U_2, h_2, \mathscr{F}_2, \psi_2)$ in the category V' are pairs (g, Ψ) where $g : U_1 \to U_2$ is a morphism of schemes with $h_2 \circ g = h_1$, and $\Psi : \mathscr{F}_2 \to g_*(\mathscr{F}_1)$ is a homomorphism of quasicoherent sheaves such that the adjoint $\Phi : g^*(\mathscr{F}_2) \to \mathscr{F}_1$ is an isomorphism making the diagram

$$\begin{tikzcd} g^*(\mathscr{F}_2)^\vee \otimes g^*(\mathscr{F}_2) \arrow[rr, "\Phi^{\vee -1} \otimes \Phi"] \arrow[dr, "g^*(\psi_2)"'] & & \mathscr{F}_1^\vee \otimes \mathscr{F}_1 \arrow[dl, "\psi_1"] \\ & \Lambda^{\mathrm{azu}} \otimes_R \mathscr{O}_{U_1} & \end{tikzcd}$$

commutative. Here we employ the identifications $\mathscr{F}_i^\vee \otimes \mathscr{F}_i = \mathrm{End}(\mathscr{F}_i)$ discussed in the proof for Lemma 6.3 below. Using tensor products and fppf descent for quasicoherent sheaves, one easily checks that the forgetful functor

$$V' \longrightarrow (\mathrm{Aff}/V), \quad (U, h, \mathscr{F}, \psi) \longmapsto (U, h)$$

endows V' with the structure of a fibered category satisfying the stack axioms. The Skolem–Noether Theorem ensures that the fiber categories are groupoids. Note that despite our notation, V' is a stack rather than a scheme.

The category V' carries a Grothendieck topology, where $(U_\lambda, h_\lambda, \mathscr{F}_\lambda, \psi_\lambda)_{\lambda \in L} \to (U, h, \mathscr{F}, \psi)$ is a covering family if $(U_\lambda \to U)_{\lambda \in L}$ is an fppf covering of affine schemes. In this way we regard V' as a site, with the ensuing notion of sheaves F' on V'. We remark in passing that there is the Zariski topology, the étale topology, and the fpqc topology as well. The site V' comes with a *structure sheaf* $\mathscr{O}_{V'}$ and a *tautological sheaf* $\mathscr{F}_{V'}^{\mathrm{taut}}$, defined for objects $\tilde{U} = (U, h, \mathscr{F}, \psi)$ by

$$\Gamma(\tilde{U}, \mathscr{O}_{V'}) = \Gamma(U, \mathscr{O}_U) \quad \text{and} \quad \Gamma(\tilde{U}, \mathscr{F}_{V'}^{\mathrm{taut}}) = \Gamma(U, \mathscr{F}), \tag{7}$$

with obvious restriction maps. We also have a *tautological splitting*, which is the isomorphism

$$\psi_{V'}^{\mathrm{taut}} : \underline{\mathrm{End}}(\mathscr{F}_{V'}^{\mathrm{taut}}) \longrightarrow \Lambda^{\mathrm{azu}} \otimes_R \mathscr{O}_{V'}$$

constructed as follows: For each endomorphism α of $\Gamma(\tilde{U}, \mathscr{F}_{V'}^{\mathrm{taut}}) = \Gamma(U, \mathscr{F})$ that is linear with respect to $\Gamma(\tilde{U}, \mathscr{O}_{V'}) = \Gamma(U, \mathscr{O}_U)$, we set $\psi_{V'}^{\mathrm{taut}}(\alpha) = \psi(\alpha)$ as elements in

$$\Gamma(U, \Lambda^{\mathrm{azu}} \otimes_R \mathscr{O}_U) = \Lambda^{\mathrm{azu}} \otimes_R \Gamma(U, \mathscr{O}_U) = \Gamma(\tilde{U}, \Lambda^{\mathrm{azu}} \otimes_R \mathscr{O}_{V'}).$$

Obviously, this is compatible with restrictions.

Given a morphism $f : V_1 \to V_2$ of schemes, we get an induced functor

$$f' : V_1' \longrightarrow V_2', \quad (U_1, h_1, \mathscr{F}_1, \psi_1) \longmapsto (U_1, f \circ h_1, \mathscr{F}_1, \psi_1),$$

and the formation $f \mapsto f'$ is covariant functorial, in the strict sense. From this one obtains a pair of adjoint functors

$$f'_* : \mathrm{Sh}(V_1') \longrightarrow \mathrm{Sh}(V_2') \quad \text{and} \quad f'^{-1} : \mathrm{Sh}(V_2') \longrightarrow \mathrm{Sh}(V_1'), \tag{8}$$

which define a *continuous map of topoi* $\mathrm{Sh}(V_1') \to \mathrm{Sh}(V_2')$, in other words, f'^{-1} commutes with finite inverse limits ([5], Exposé IV, Definition 3.1). To be explicit, the inverse image is given by

$$\Gamma((U,h,\mathscr{F},\psi), f'^{-1}F_2) = \Gamma((U, f\circ h,\mathscr{F},\psi), F_2), \tag{9}$$

with $(U,h,\mathscr{F},\psi)\in V_1'$, whereas the direct image takes the form

$$\Gamma((U,h,\mathscr{F},\psi), f'_*F_1) = \Gamma\left((U\times_{V_2}V_1, \mathrm{pr}_2, \mathrm{pr}_1^*(\mathscr{F}), \mathrm{pr}_1^*(\psi)), F_1\right), \tag{10}$$

now with $(U,h,\mathscr{F},\psi)\in V_2'$. It is straightforward to determine the effect on structure sheaves and tautological sheaves:

Proposition 5.1. *In the above setting we have identifications*

$$f'^{-1}(\mathscr{O}_{V_2'}) = \mathscr{O}_{V_1'} \quad \textit{and} \quad f'^{-1}(\mathscr{F}^{\mathrm{taut}}_{V_2'}) = \mathscr{F}^{\mathrm{taut}}_{V_1'} \quad \textit{and} \quad f'_*(\mathscr{F}^{\mathrm{taut}}_{V_1'}) = \mathscr{F}^{\mathrm{taut}}_{V_2'}\otimes f'_*(\mathscr{O}_{V_1'}).$$

Proof. Using (9) with $\tilde U=(U,h,\mathscr{F},\psi)$ from V_1' we immediately get

$$\Gamma(\tilde U, f^{-1}(\mathscr{O}_{V_2'})) = \Gamma((U,f\circ h,\mathscr{F},\psi),\mathscr{O}_{V_2'}) = \Gamma(U,\mathscr{O}_U) = \Gamma(\tilde U,\mathscr{O}_{V_1'}),$$

and likewise for the tautological sheaves. Suppose now that $\tilde U=(U,h,\mathscr{F},\psi)$ is from V_2'. Consider the fiber product $U_1=U\times_{V_2}V_1$ with respect to $f:V_1\to V_2$, and write $h_1:U_1\to V_1$ for the second projection. Using (10) we obtain

$$\Gamma(\tilde U, f'_*(\mathscr{F}^{\mathrm{taut}}_{V_1'})) = \Gamma((U_1,h_1,\mathrm{pr}_1^*(\mathscr{F}),\mathrm{pr}_1^*(\psi)),\mathscr{F}^{\mathrm{taut}}_{V_1'}) = \Gamma(U_1,\mathrm{pr}_1^*(\mathscr{F})).$$

The latter equals $\Gamma(U,\mathscr{F}\otimes\mathrm{pr}_{1*}(\mathscr{O}_{U_1}))$, as one sees by applying the projection formula for $\mathrm{pr}_1:U_1\to U$ and the locally free sheaf $\mathscr{F}$. By definition, this coincides with the group of local sections of $\mathscr{F}^{\mathrm{taut}}_{V_2'}\otimes f'_*(\mathscr{O}_{V_1'})$ over $\tilde U$. □

For each $\tilde U=(U,h,\mathscr{F},\psi)$, we get an inclusion $\mathbb{G}_m\subset\underline{\mathrm{Aut}}_{\tilde U/(U,h)}$ by sending an invertible scalar λ to the automorphism (f,Ψ) with $f=\mathrm{id}_U$ and $\Psi=\lambda\cdot\mathrm{id}_{\mathscr{F}}$. One easily checks that this gives $V'\to(\mathrm{Aff}/V)$ the structure of a $\mathbb{G}_m$-gerbe. In particular, the two projections $V'\to(\mathrm{Aff}/V)$ and $B(\mathbb{G}_{m,V})\to(\mathrm{Aff}/V)$ are étale locally equivalent. It follows that V' is an Artin stack ([34], Example 4.6.1).

Moreover, given an object $\tilde U=(U,h,\mathscr{F},\psi)$ from V' as well as an open set $U_0\subset U$ we get a new object $\tilde U_0=(U_0,h|U_0,\mathscr{F}|U_0,\psi|U_0)$ by restricting the additional data. In turn, for each sheaf F' on the stack V' we get sheaves $F_{\tilde U}$ on the schemes U. An $\mathscr{O}_{V'}$-module $\mathscr{E}'$ is called *quasicoherent* if the $\mathscr{E}_{\tilde U}$ are quasicoherent sheaves on U in the usual sense, for all objects $\tilde U\in V'$. Particular examples are the locally free sheaves of finite rank.

For each quasicoherent sheaf $\mathscr{E}'$ on V', and each object $\tilde U=(U,h,\mathscr{F},\psi)$, we get an action of the group $\Gamma(U,\mathbb{G}_m)$ on the quasicoherent sheaf $\mathscr{E}'_{\tilde U}$ on U. By compatibility with restrictions, this yields a *linearization* with respect to the group scheme $\mathbb{G}_{m,U}$, and thus a weight decomposition $\mathscr{E}'_{\tilde U}=\bigoplus_{w\in\mathbb{Z}}\mathscr{E}'_{\tilde U,w}$, as explained in [15], Éxpose I, Proposition 4.7.3. This defines the *weight decomposition* $\mathscr{E}'=\bigoplus_{w\in\mathbb{Z}}\mathscr{E}'_w$ for the sheaf on the stack V'. If for a given $w\in\mathbb{Z}$ the inclusion $\mathscr{E}'_w\subset\mathscr{E}'$ is an equality, one says that $\mathscr{E}'$ is *pure of weight w*.

Proposition 5.2. *The structure sheaf $\mathscr{O}_{V'}$ and the tautological sheaf $\mathscr{F}^{\mathrm{taut}}_{V'}$ are pure, of respective weights $w=0$ and $w=1$.*

Proof. For each object $\tilde{U} = (U, h, \mathscr{F}, \psi)$ and each $\lambda \in \Gamma(U, \mathbb{G}_m)$, the action on $\Gamma(\tilde{U}, \mathscr{F}_{V'}^{\mathrm{taut}})$ is multiplication by λ^w with $w = 1$, which one sees by making the restriction maps in (7) explicit. Similarly, one sees that the action on $\Gamma(\tilde{U}, \mathscr{O}_{V'})$ is trivial, that is, has weight $w = 0$. □

Write $\mathrm{QCoh}(V')$ for the abelian category of quasicoherent sheaves $\mathscr{E}'$ and linear maps. Let $f : V_1 \to V_2$ be a morphism. Note that the linear pullback

$$f'^*(\mathscr{E}_2') = f'^{-1}(\mathscr{E}_2') \otimes_{f'^{-1}(\mathscr{O}_{V_2'})} \mathscr{O}_{V_1'}$$

can be identified with the set-theoretic pullback $f'^{-1}(\mathscr{E}_2')$, in light of Proposition 5.1.

Proposition 5.3. *For each morphism $f : V_1 \to V_2$, the adjoint functors (8) respect quasicoherence, and also weight decompositions.*

Proof. We start with the preimage functor. With $\tilde{U} = (U, h, \mathscr{F}, \psi)$ from V_1' we see from (9) that

$$\Gamma(\tilde{U}, f^{-1}\mathscr{E}_2) = \Gamma((U, f \circ h, \mathscr{F}, \psi), \mathscr{E}_2).$$

It follows that $(f^{-1}\mathscr{E}_2)_{\tilde{U}} = (\mathscr{E}_2)_{(U, f\circ h, \mathscr{F}, \psi)}$ as sheaves on U, so quasicoherence is preserved. Moreover, if $\mathscr{E}_2$ is pure of weight w, the action of $\lambda \in \Gamma(U, \mathbb{G}_m)$ on the group of local sections is via λ^w. By the above equality, this also holds for $f^{-1}(\mathscr{E}_2)$, which is therefore pure of weight w. Being left adjoint, the preimage functor respects all direct limits, and thus direct sum decompositions. Summing up, the weight decompositions are preserved.

We come to the direct image functor. Now let $\tilde{U} = (U, h, \mathscr{F}, \psi)$ be an object from V_2', and let $U_1 = U \times_{V_2} V_1$ be the base-change with respect to $f : V_1 \to V_2$. According to (10) we have

$$\Gamma(\tilde{U}, f_*\mathscr{E}_1) = \Gamma((U_1, \mathrm{pr}_2, \mathrm{pr}_1^*(\mathscr{F}), \mathrm{pr}_1^*(\psi)), \mathscr{E}_1).$$

It follows that $(f_*\mathscr{E}_1)_{\tilde{U}} = \mathrm{pr}_{1,*}((\mathscr{E}_1)_{(U_1, \mathrm{pr}_2, \mathrm{pr}_1^*(\mathscr{F}), \mathrm{pr}_1^*(\psi))})$ as sheaves on U. Since the schemes U, V_1, V_2 are affine, the projection $\mathrm{pr}_1 : U_1 = U \times_{V_2} V_1 \to U$ is affine, so the direct image functor $\mathrm{pr}_{1,*}$ preserves quasicoherence. Thus the sheaf $(f_*\mathscr{E}_1)_{\tilde{U}}$ on U is quasicoherent. Moreover, for quasicoherent sheaves the functor $\mathrm{pr}_{1,*}$ preserves direct sum decompositions. If $\mathscr{E}_1$ is pure of weight w, we argue as in the preceding paragraph to see that $f_*(\mathscr{E}_1)$ is pure of the same weight. In turn, the weight decompositions are preserved. □

6. The stack of twisted Schur representations

Let R be a ground ring, Λ be an associative algebra, and $n \geq 1$ be some integer. We call a pair (E, ρ), where E is a locally free R-module of rank n, and $\rho : \Lambda \to \mathrm{End}(E)$ is a homomorphism of associative algebras, a *linear representation of degree n*. For some suitable Zariski open covering $A_i = R[1/f_i]$ and choices of bases for $E \otimes_R A_i$, this yields a collection of *matrix representations* $\rho_i : \Lambda \otimes_R A_i \to \mathrm{Mat}_n(R) \otimes_R A_i$, not necessarily compatible. The goal of this section is to clarify these seemingly innocuous facts by using stacks, and generalize it from matrix algebras to Azumaya algebras. Furthermore, we relate it to our results on modifying moduli problems and spaces of representations in Sections 1 and 3. Indeed, Azumaya algebras are needed to perform and explain such modifications.

Throughout, we fix an Azumaya algebra Λ^{azu} of degree $n \geq 1$, with $\Lambda^{\mathrm{azu}} = \mathrm{Mat}_n(R)$ as important special case. To apply the results of Section 3, we also assume that the given associative algebra Λ is finitely presented. Consider the resulting functor of representations

$$F : (\mathrm{Aff}/R) \longrightarrow (\mathrm{Set}), \quad A \longmapsto \mathrm{Hom}_{A\text{-}\mathrm{Alg}}(\Lambda \otimes_R A, \Lambda^{\mathrm{azu}} \otimes_R A).$$

From Theorem 3.5 we see that the subfunctor of Schur representations is representable by a quasiaffine scheme of finite presentation, which we denote by $X = X_{\Lambda/R}^{\Lambda^{\mathrm{azu}}}$.

Recall that we have a short exact sequence

$$1 \longrightarrow \mathbb{G}_m \longrightarrow U_{\Lambda^{\mathrm{azu}}/R} \xrightarrow{\mathrm{conj}} \mathrm{Aut}_{\Lambda^{\mathrm{azu}}/R} \longrightarrow 1,$$

where the terms on the right are the respective group schemes of units and automorphisms for Λ^{azu}. According to Proposition 3.4, the canonical action of $\mathrm{Aut}_{\Lambda^{\mathrm{azu}}/R}$ on the sheaf F stabilizes the subsheaf of Schur representations, and the induced action on the quasiaffine scheme X is free. To conform with the notation in Section 2, we now set $G = U_{\Lambda^{\mathrm{azu}}/R}$ and $H = \mathrm{Aut}_{\Lambda^{\mathrm{azu}}/R}$. The quotient $Q = X/H^{\mathrm{op}}$ is representable by an algebraic space, and its formation commutes with base-change along arbitrary $\tilde{Q} \to Q$, see for example [35], Lemma 1.1. Note that such quotients may easily be *non-separated*, the simplest example stemming from the action $\lambda \cdot (x_1, x_2) = (\lambda x_1, \lambda^{-1} x_2)$ of the multiplicative group on the pointed affine plane, whose quotient is the affine line with double origin.

Also note that such quotients are usually *non-schematic*. Examples are abundant, but at the same time far from obvious: Working over a ground field $k = k^{\mathrm{alg}}$, we start with some non-schematic surface Q that is proper, integral and normal ([3], Example 4.4). This has the resolution property ([40], Theorem 6.8; the arguments in [46], which hold true for algebraic spaces, already suffice). According to Totaro's result ([52], Theorem 1.1), there is a locally free sheaf $\mathscr{E}$ such that the corresponding principal bundle $P \to Q$ with respect to $G = \mathrm{GL}_n$, $n = \mathrm{rank}(\mathscr{E})$ has quasiaffine total space. The arguments in loc. cit. (below the proof of Corollary 5.2) reveal that we may replace the sheaf by $\mathscr{E} \oplus \mathscr{E}^\vee$, and thus may assume that the principal bundle admits a reduction of structure $P_0 \subset P$ to $G_0 = \mathrm{SL}_n$, which contains μ_n as a finite normal subgroup scheme. Setting $X = P_0/\mu_n$, we get a principal bundle $X \to Q$ with respect to $H = \mathrm{PGL}_n$. By construction, P_0 is quasiaffine, X is normal and schematic, and the quotient map $P_0 \to X$ is finite and flat. Viewing $\mathscr{O}_X$ as the norm of $\mathscr{O}_{P_0}$, we see that $\mathscr{O}_X$ is ample ([23], Proposition 6.6.1). Summing up, X is quasiaffine with a free action of $H = \mathrm{PGL}_n$ such that the quotient Q is non-schematic.

There are also examples taken from the realm of moduli spaces: Weißmann and Zheng showed that for every smooth proper curve of genus $g \geq 4$ the coarse moduli space of simple sheaves that are locally free of rank $n \geq 2$ is a non-schematic algebraic space ([54], Corollary 2.8).

Back to our $X = X_{\Lambda/R}^{\Lambda^{\mathrm{azu}}}$. The algebraic space $Q = X/H^{\mathrm{op}}$ can be regarded as a "moduli space" $M = M_{\Lambda/R}^{\Lambda^{\mathrm{azu}}}$ of Schur representations. However, we regard such locutions as "dangerous", and seek to define the "true" *moduli stack* $\mathscr{M} = \mathscr{M}_{\Lambda/R}^{\Lambda^{\mathrm{azu}}}$ of

Schur representations, in terms of "concrete" representation-theoretic data. We then construct a *comparison functor* $\Phi : \mathscr{M} \to [X/G^{\mathrm{op}}/Q]$ to the quotient stack, which is an equivalence relating representation-theoretic and algebro-geometric data. The quotient space $Q = X/H^{\mathrm{op}}$ turns out to be the *coarse moduli space*.

Throughout, we find it psychologically helpful to regard schemes, which by definition are certain ringed spaces $V = (|V|, \mathscr{O}_V)$, not only as functors $(\mathrm{Aff}/R) \to (\mathrm{Set})$ via the Yoneda embedding, but actually as *categories fibered in groupoids* $(\mathrm{Aff}/V) \to (\mathrm{Aff}/R)$ via the comma construction. Recall that $V' \to (\mathrm{Aff}/V)$ denotes the splitting gerbe for the Azumaya algebra $\Lambda^{\mathrm{azu}} \otimes_R \mathscr{O}_V$. We come to a central notion:

Definition 6.1. A *twisted representation* of the associative algebra Λ in the Azumaya algebra Λ^{azu} over a scheme V is a pair $(\mathscr{E}', \rho')$ where $\mathscr{E}'$ is a locally free sheaf on the $\mathbb{G}_{m,V}$-gerbe V' that has rank $n = \deg(\Lambda^{\mathrm{azu}})$ and is pure of weight $w = 1$, and $\rho' : \Lambda \otimes_R \mathscr{O}_{V'} \to \underline{\mathrm{End}}(\mathscr{E}')$ is a homomorphism of $\mathscr{O}_{V'}$-algebras.

If for all objects $\tilde{U} = (U, h, \mathscr{F}, \psi)$ from V' the $\rho'_{\tilde{U}} : \Lambda \otimes_R \mathscr{O}_U \to \underline{\mathrm{End}}(\mathscr{E}'_{\tilde{U}})$ are Schur representations, we say that $(\mathscr{E}', \rho')$ is a *twisted Schur representation.*

Let us emphasize that we regard "twisted" representations as "true" representations, albeit defined on a stack rather than a scheme. In other words, we favor the approach of de Jong and Lieblich ([13], [37], [38]) over Căldăraru's point of view ([10], [11]). Consider now the category

$$\mathscr{M} = \mathscr{M}_{\Lambda/R}^{\Lambda^{\mathrm{azu}}} = \{(V, \mathscr{E}', \rho') \mid \rho' : \Lambda \otimes_R \mathscr{O}_{V'} \to \underline{\mathrm{End}}(\mathscr{E}')\}$$

whose objects are triples $(V, \mathscr{E}', \rho')$ where V is an affine scheme and $(\mathscr{E}', \rho')$ is a twisted Schur representation over the scheme V, in other words, a weight-one Schur representation over the stack V'. Morphisms $(V_1, \mathscr{E}'_1, \rho'_1) \to (V_2, \mathscr{E}'_2, \rho'_2)$ are pairs (f, Ψ') where $f : V_1 \to V_2$ is a morphism of schemes, and $\Psi' : f'^{-1}(\mathscr{E}'_2) \to \mathscr{E}'_1$ is an isomorphism of modules over $\mathscr{O}_{V'_1} = f^{-1}(\mathscr{O}_{V'_2})$ making the diagram

$$\begin{tikzcd} & \Lambda \otimes_R \mathscr{O}_{V'_1} \arrow[dl, "f'^{-1}(\rho'_2)"'] \arrow[dr, "\rho'_1"] & \\ \underline{\mathrm{End}}(f'^{-1}(\mathscr{E}'_2)) \arrow[rr, "\Psi'^{\vee -1} \otimes \Psi'"'] & & \underline{\mathrm{End}}(\mathscr{E}'_1) \end{tikzcd}$$

commutative.

Proposition 6.2. *The forgetful functor $\mathscr{M} \to (\mathrm{Aff}/R)$ given by $(V, \mathscr{E}', \rho') \mapsto V$ endows $\mathscr{M}$ with the structure of a category fibered in groupoids. Moreover, the stack axioms hold with respect to the fppf topology.*

Proof. By definition, in morphisms $(f, \Psi') : (V_1, \mathscr{E}'_1, \rho'_1) \to (V_2, \mathscr{E}'_2, \rho'_2)$ the linear map Ψ' is an isomorphism. From this one immediately infers that all morphisms in $\mathscr{M}$ are cartesian. In particular, all fiber categories are groupoids ([26], Exposé VI, Remark after Definition 6.1; but note that the very existence of cartesian maps was overlooked there). We next check that arrows lift with respect to the forgetful functor, with prescribed target. This is immediate: Suppose that $f : V_1 \to V_2$ is a morphism of affine schemes, and $(V_2, \mathscr{E}'_2, \rho'_2)$ is an object over V_2. Setting $\mathscr{E}'_1 = f^{-1}(\mathscr{E}'_2)$ and $\rho'_1 = f^{-1}(\rho'_2)$ and $\Psi' = \mathrm{id}_{\mathscr{E}'_1}$, we get the desired morphism (f, Ψ') over f.

It remains to verify that $\mathscr{M}$ is a stack. We first check that the automorphism presheaves satisfy the sheaf axioms. Let A be an R-algebra, $A \subset A_0$ be an fppf extension, and write $A_1 = A_0 \otimes_A A_0$. Set $V_i = \operatorname{Spec}(A_i)$ and fix an object $(V, \rho', \mathscr{E}')$ over $V = \operatorname{Spec}(A)$. We have to check that $\operatorname{Aut}((V, \rho', \mathscr{E}')/V)$ is an equalizer of the diagram

$$\operatorname{Aut}((V_0, h_0^{-1}(\rho'), h_0^{-1}(\mathscr{E}'))/V_0) \rightrightarrows \operatorname{Aut}((V_1, h_1^{-1}(\rho'), h_1^{-1}(\mathscr{E}'))/V_1)$$

where $h_i : V_i \to V$ denote the canonical morphisms. First suppose (id_V, Ψ') is an automorphism over V that becomes the identity over V_0. Hence Ψ' is an automorphism of $\mathscr{E}'$ compatible with ρ'. Consider the quasicoherent sheaves $\mathscr{E}'_{\tilde{U}}$ on the affine scheme U, which are indexed by the objects $\tilde{U} = (U, h, \mathscr{F}, \psi)$ of the splitting gerbe V'. Our Ψ' is determined by the compatible collection of automorphisms $\Psi'_{\tilde{U}} : \mathscr{E}'_{\tilde{U}} \to \mathscr{E}'_{\tilde{U}}$. The latter become identities on $U \times_{V_0} V_1$, so by [26], Exposé VIII, Theorem 1.1 the $\Psi'_{\tilde{U}}$ must be identities. Likewise, one checks that every automorphism over V_0 whose two inverse images on V_1 coincide comes from an automorphism over V.

Finally we need to check that every descent datum is effective. Using the above notation, we now fix an object $\zeta_0 = (V_0, \mathscr{E}'_0, \rho'_0)$ over $V_0 = \operatorname{Spec}(A_0)$. Consider the two inclusions $A_0 \rightrightarrows A_1$ given by $x \mapsto 1 \otimes x$ and $x \mapsto x \otimes 1$. As customary, we write $\zeta_0 \otimes_{A_0} A_1$ and $A_1 \otimes_{A_0} \zeta_0$ for the resulting pullbacks, say formed as in the first paragraph above. Suppose now that we have an isomorphism

$$(f_1, \Psi'_1) : \zeta_0 \otimes_{A_0} A_1 \to A_1 \otimes_{A_0} \zeta_0$$

that satisfies the cocycle condition over $V_2 = \operatorname{Spec}(A_2)$, with $A_2 = A_0 \otimes_A A_0 \otimes_A A_0$. Consider the collection of locally free sheaves $\mathscr{E}'_{\tilde{U}_0}$ arising from $\mathscr{E}'_0$, indexed by the objects $\tilde{U}_0 = (U_0, h_0, \mathscr{F}_0, \psi_0)$ of the splitting gerbe V'_0. These live on the affine schemes U_0, and are endowed with comparison morphisms with respect to the morphisms in V'_0. Applying [26], Exposé VIII, Theorem 1.1 for each affine scheme U appearing in the objects $\tilde{U} = (U, h, \mathscr{F}, \psi)$ of V', we get a collection $\mathscr{E}'_{\tilde{U}}$ together with comparison maps. The desired sheaf on V' is now defined via $\Gamma(\tilde{U}, \mathscr{E}') = \Gamma(U, \mathscr{E}'_{\tilde{U}})$, where the restriction maps stem from the comparison maps. $\square$

We call $\mathscr{M} = \mathscr{M}_{\Lambda/R}^{\Lambda^{\mathrm{azu}}}$ the *stack of twisted Schur representations* of the associative algebra Λ in the Azumaya algebra Λ^{azu}. The task now is to construct a comparison functor $\Phi : \mathscr{M} \to [X/G^{\mathrm{op}}/Q]$ that relates representation-theoretic with algebro-geometric data. Recall that the quotient stack comprises tuples (U, g, P, f) where U is an affine scheme, $g : U \to Q$ is a morphism of algebraic spaces, P is a G_U-torsor, and $f : P \to X_U$ is a G_U-equivariant morphism that induces g on quotients.

Let $(V, \mathscr{E}', \rho')$ be an object from $\mathscr{M}$. On the gerbe of splittings V' we then have two locally free sheaves $\mathscr{E}'$ and $\mathscr{F}_{V'}^{\mathrm{taut}}$, both having rank $n = \deg(\Lambda^{\mathrm{azu}})$ and weight $w = 1$. Composition endows the sheaf $\underline{\operatorname{Hom}}(\mathscr{E}', \mathscr{F}_{V'}^{\mathrm{taut}})$ with a module structure over $\underline{\operatorname{End}}(\mathscr{F}_{V'}^{\mathrm{taut}})$, and the latter comes with an isomorphism $\psi_{V'}^{\mathrm{taut}} : \underline{\operatorname{End}}(\mathscr{F}_{V'}^{\mathrm{taut}}) \to \Lambda^{\mathrm{azu}} \otimes_R \mathscr{O}_{V'}$. These sheaves are locally free of rank n^2 and weight $w = 0$, and thus correspond to locally free sheaves of likewise rank on the affine scheme V. Let us write

$$P_{V,\mathscr{E}'} \longrightarrow V$$

for the resulting torsor with respect to the group scheme $G_V = U_{\Lambda^{\mathrm{azu}}/R}|V$; it corresponds to the locally free module of rank one over the sheaf of Azumaya algebras $\Lambda^{\mathrm{azu}}\otimes_R \mathscr{O}_{V'}$ given by $\underline{\mathrm{Hom}}_{\mathscr{O}_{V'}}(\mathscr{E}', \mathscr{F}^{\mathrm{taut}}_{V'})$. Note that the base and the total space of the torsor are affine. One easily checks that the formation of $P_{V,\mathscr{E}'}$ is functorial in $(V,\mathscr{E}',\rho')$.

Our next task is to construct a morphism of schemes $f = f_{V,\mathscr{E}',\rho'}$ from $P = P_{V,\mathscr{E}'}$ to the quasiaffine scheme $X = X^{\Lambda^{\mathrm{azu}}}_{\Lambda/R}$ representing the functor F^0 of Schur representations of Λ in Λ^{azu}. Note that the former is affine, whereas the latter is quasiaffine. We shall specify the morphism as a functor $(\mathrm{Aff}/P)\to(\mathrm{Aff}/X)$. Let U be an affine scheme, together with a morphism $U\to P$. By the universal property of fiber products, this can be seen as a morphism $f : U\to V$, together with a section of the induced G_U-torsor $P_U = P\times_V U$. By the very definition of the torsor, the latter is nothing but an isomorphism $\psi' : \mathscr{E}'|U' \to \mathscr{F}^{\mathrm{taut}}_{U'}$ of locally free sheaves on the gerbe of splittings U'. Now the given linear representation $\rho' : \Lambda\otimes_R\mathscr{O}_{V'}\to\underline{\mathrm{End}}(\mathscr{E}')$ enters the picture: The composite map

$$\Lambda\otimes_R\mathscr{O}_{U'} \xrightarrow{f'^{-1}(\rho')} \underline{\mathrm{End}}(f'^{-1}\mathscr{E}') \xrightarrow{\psi'^{\vee-1}\otimes\psi'} \underline{\mathrm{End}}(\mathscr{F}^{\mathrm{taut}}_{U'}) \xrightarrow{\psi^{\mathrm{taut}}_{U'}} \Lambda^{\mathrm{azu}}\otimes_R\mathscr{O}_{U'}$$

is a Schur representation of Λ in Λ^{azu} over the splitting gerbe U'. It corresponds to a Schur representation over the scheme U, because the involved sheaves have weight $w=0$. Since $X = X^{\Lambda^{\mathrm{azu}}}_{\Lambda/R}$ represents the functor F^0, this can be seen as a morphism $U\to X$. One easily checks that the formation is functorial, and thus defines the desired morphism

$$f_{V,\mathscr{E}',\rho'} : P_{V,\mathscr{E}'}\longrightarrow X.$$

Recall that the group scheme $G = U_{\Lambda^{\mathrm{azu}}/R}$ acts on X via conjugation, and on the G_V-torsor $P_{V,\mathscr{E}'}$ in an obvious way. The following technical fact is a crucial observation:

Lemma 6.3. *The above morphism $f_{V,\mathscr{E}',\rho'} : P_{V,\mathscr{E}'}\to X$ is equivariant with respect to the G-actions.*

Proof. We have to check that the map $P_{V,\mathscr{E}'}(A)\to X(A)$ is equivariant for the abstract groups $G(A)$, for each R-algebra A. The morphism $P_{V,\mathscr{E}'}\to V$ endows A with an algebra structure over $A_0 = \Gamma(V,\mathscr{O}_V)$. Since our constructions commute with base-change, it suffices to treat the case $A = A_0 = R$.

Fix some $\sigma\in G(R) = (\Lambda^{\mathrm{azu}})^\times$. Recall that its effect on the set $P_{V,\mathscr{E}'}(R) = \mathrm{Hom}(\mathscr{E}',\mathscr{F}^{\mathrm{taut}}_{V'})$ is via composition with the automorphism $\eta' = (\psi^{\mathrm{taut}}_{V'})^{-1}(\sigma)$ from $\mathrm{End}(\mathscr{F}^{\mathrm{taut}}_{V'})$. By definition, our map $f_{V,\mathscr{E}',\rho'}$ sends the element of $P_{V,\mathscr{E}'}(R)$ corresponding to a linear map $\psi' : \mathscr{E}'\to\mathscr{F}^{\mathrm{taut}}_{V'}$ to the element $X(R)$ stemming from the composite map

$$\Lambda\otimes_R\mathscr{O}_{V'} \xrightarrow{\rho'} \underline{\mathrm{End}}(\mathscr{E}') \xrightarrow{\psi'^{\vee-1}\otimes\psi'} \underline{\mathrm{End}}(\mathscr{F}^{\mathrm{taut}}_{V'}) \xrightarrow{\psi^{\mathrm{taut}}_{V'}} \Lambda^{\mathrm{azu}}\otimes_R\mathscr{O}_{V'}.$$

So $f_{V,\mathscr{E}',\rho'}(\sigma\cdot\psi')$ is given by a likewise composition, formed with

$$(\eta'\circ\psi')^{\vee-1}\otimes(\eta'\circ\psi') = (\eta'^{\vee-1}\otimes\eta')\circ(\psi'^{\vee-1}\otimes\psi')$$

instead of $\psi'^{\vee-1}\otimes\psi'$. The task is to verify that composition of endomorphisms of $\mathscr{F}^{\mathrm{taut}}_{V'}$ with $\eta'^{\vee-1}\otimes\eta'$ is the same as conjugation with η'. This has to be checked over the objects $(U, h, \mathscr{F}, \varphi)$ of the splitting gerbe V', thus becomes a statement about sheaves on the affine schemes U, hence, a problem in commutative algebra:

Suppose E is a locally free R-module of rank n. Recall that the canonical map

$$E^\vee \otimes E \longrightarrow \mathrm{End}(E), \quad f \otimes a \longmapsto (x \mapsto f(x) \cdot a). \tag{11}$$

is bijective, and that under this identification composition of endomorphisms corresponds to the pairing of tensors $(f \otimes a) \otimes (g \otimes b) \mapsto f(b) \cdot g \otimes a$, see [8], Chapter II, §4, No. 2. Each $h \in \mathrm{GL}(E)$ yields bijective linear maps

$$E^\vee \otimes E \longrightarrow E^\vee \otimes E \quad \text{and} \quad \mathrm{End}(E) \longrightarrow \mathrm{End}(E)$$

given by $h^{\vee -1} \otimes h$ and $g \mapsto h \circ g \circ h^{-1}$, respectively. We have to verify that these bijections coincide under (11). The problem is local, so we may assume that there is a basis $e_1, \ldots, e_n \in E$, with ensuing identification $\mathrm{End}(E) = \mathrm{Mat}_n(R)$, standard basis $E_{ij} \in \mathrm{Mat}_n(R)$, and dual basis $e_1^\vee, \ldots, e_n^\vee \in E^\vee$. Write (α_{ij}) and (β_{ij}) for the matrices for h and h^{-1}, respectively. Note that the tensor $e_s^\vee \otimes e_r$ corresponds to the matrix E_{rs}, and that $h^{\vee -1} = h^{-1\vee}$ has matrix (β_{ji}). One computes

$$(h^{\vee -1} \otimes h)(e_s^\vee \otimes e_r) = (\sum_j \beta_{sj} e_j^\vee) \otimes (\sum_i \alpha_{ir} e_i) = \sum_{i,j} (\beta_{sj}\alpha_{ir} \cdot e_j^\vee \otimes e_i).$$

Using Kronecker deltas, we write $E_{rs} = (\delta_{ir}\delta_{sj})$ and see that the (i,j)-entry of the triple matrix product $(\alpha_{ik})(\delta_{ir}\delta_{sj})(\beta_{kj})$ is given by $\sum_{k,l} \alpha_{ik} \cdot \delta_{kr}\delta_{sl} \cdot \beta_{lj} = \alpha_{ir}\beta_{sj}$. Thus $h \circ E_{rs} \circ h^{-1} = \sum_{i,j} \alpha_{ir}\beta_{sj} \cdot E_{ij}$, which indeed corresponds to $(h^{\vee -1} \otimes h)(e_s^\vee \otimes e_r)$. $\square$

Since the G_V-action on $P_{V,\mathscr{E}'}$ is free, our equivariant $f_{V,\mathscr{E}',\rho'}$ induces a morphism

$$g_{V,\mathscr{E}',\rho'} : V = P/G_V^{\mathrm{op}} \longrightarrow X/G^{\mathrm{op}} = X/H^{\mathrm{op}} = Q.$$

Summing up, we have attached to every object $(V, \mathscr{E}', \rho') \in \mathscr{M}$ from the stack of twisted Schur representations an object

$$(V, g_{V,\mathscr{E}',\rho'}, P_{V,\mathscr{E}'}, f_{V,\mathscr{E}',\rho'}) \in [X/G^{\mathrm{op}}/Q]$$

from the quotient stack. One easily checks that the formation is functorial. This defines the desired *comparison functor*

$$\Phi : \mathscr{M} = \mathscr{M}_{\Lambda/R}^{\Lambda^{\mathrm{azu}}} \longrightarrow [X/G^{\mathrm{op}}/Q].$$

Obviously, this is compatible with the forgetful functor to (Aff/R). One main result of this paper is that this relation between representation-theoretic and algebro-geometric data is essentially an identification:

Theorem 6.4. *The comparison functor $\Phi : \mathscr{M} \to [X/G^{\mathrm{op}}/Q]$ is an equivalence of categories.*

Proof. We have to check that Φ is fully faithful and essentially surjective. This relies to a large degree on internal properties of the category $\mathscr{M}$. As a preparation, note that we also have a functor

$$\mathscr{M} \longrightarrow (\mathrm{Aff}/Q), \quad (V, \mathscr{E}', \rho') \longmapsto (V, g_{V,\mathscr{E}',\rho'}). \tag{12}$$

So both $\mathscr{M}$ and $[X/G^{\mathrm{op}}/Q]$ are categories over (Aff/Q), and Φ is a functor relative to this. We now proceed in three steps.

Step 1: *The above functor* (12) *endows $\mathscr{M}$ with the structure of a category fibered in groupoids. Moreover, the stack axioms hold with respect to the fppf topology on* (Aff/Q). The arguments are exactly as in the proof for Proposition 6.2.

Step 2: *The comparison functor Φ is fully faithful.* In light of [26], Exposé VI, Proposition 6.10 and step 1, it suffices to verify this for the fiber categories over (Aff/Q). Fix an affine scheme V and a morphism of algebraic spaces $g : V \to Q$. Consider an object $(V, \mathscr{E}', \rho')$ in the stack of Schur representations $\mathscr{M}$ over (V, g), in other words $g = g_{V,\mathscr{E}',\rho'}$. Setting $P = P_{V,\mathscr{E}'}$ and $f = f_{V,\mathscr{E}',\rho'}$ we get an object (V, g, P, f) in the quotient stack $[X/G^{\mathrm{op}}/Q]$ over (V, g). According to Corollary 2.4, the structure map $R \to \Lambda^{\mathrm{azu}}$ induces a bijection $\Gamma(V, \mathscr{O}_V^\times) \to \mathrm{Aut}_{(V,g)}(V, g, P, f)$. Scalar multiplication of the structure sheaf on $\mathscr{E}'$ gives an injective group homomorphism $\Gamma(V, \mathscr{O}_V^\times) \to \mathrm{Aut}_V(V, \mathscr{E}', \rho')$. Since the representation ρ' is Schur, the injection $\Gamma(V, \mathscr{O}_V^\times) \to \mathrm{Aut}_V(V, \mathscr{E}', \rho')$ is actually bijective. Consider the diagram

$$\begin{tikzcd} & \mathrm{Aut}_V(V, \mathscr{E}', \rho') \arrow[d, "\Phi"] \\ \Gamma(V, \mathscr{O}_V^\times) \arrow[ur] \arrow[dr] & \\ & \mathrm{Aut}_{(V,g)}(V, g, P, f), \end{tikzcd}$$

where the diagonal arrows are as specified above. Using that the G_V-torsor P stems from the locally free sheaf $\underline{\mathrm{Hom}}(\mathscr{E}', \mathscr{F}_{V'}^{\mathrm{taut}})$, we infer that the diagram is commutative. It follows that the vertical arrow is bijective, and moreover that every automorphism of $(V, \mathscr{E}', \rho')$ over V is actually over (V, g). So Φ yields bijections on automorphism groups in fiber categories. Since the fibers are groupoids, Φ is fully faithful on the fibers.

Step 3: *The functor Φ is essentially surjective.* In both $\mathscr{M}$ and $[X/G^{\mathrm{op}}/Q]$, viewed as fibered categories over (Aff/Q), every descent datum is effective. It thus suffices to check that every descent datum of the latter arises from a descent datum on the former. But this is immediate from step 2. $\square$

Recall that for any category $\mathscr{G}$ fibered in groupoids over (Aff/R), a morphism $\mathscr{G} \to Z$ to an algebraic space Z that is universal for morphisms into algebraic spaces is called *coarse moduli space*.

Corollary 6.5. *The category fibered in groupoids $\mathscr{M} \to (\mathrm{Aff}/R)$ is an Artin stack, and the morphism $\Phi^{\mathrm{crs}} : \mathscr{M} \to Q = X/H^{\mathrm{op}}$ is the coarse moduli space. Moreover, $\mathscr{M}$ carries the structure of a $\mathbb{G}_m$-gerbe over* (Aff/Q).

Proof. It suffices to check these statements for the quotient stack $[X/G^{\mathrm{op}}/Q]$, which by the theorem is equivalent to $\mathscr{M}$, as categories over (Aff/Q). We saw in Section 2 that the quotient stack is a category fibered in groupoids, satisfies the stack axioms, and is a $\mathbb{G}_m$-gerbe over (Aff/Q). According to [34], Example 4.6.1 it is indeed an Artin stack. It remains to check the statement on the coarse moduli space. Obviously, the transformation

$$[X/G^{\mathrm{op}}/Q] \longrightarrow [X/H^{\mathrm{op}}/Q] = Q, \quad (U, g, P, f) \longmapsto (U, g, \bar{P}, \bar{f}),$$

where $\bar{P} = H \wedge^G P = P/\mathbb{G}_m$ is the induced H-torsor, and $\bar{f} : \bar{P} \to X_U$ is the induced H_U-equivariant map, yields the sheafification of the functor of isomorphism classes for the $\mathbb{G}_m$-gerbe. It follows that $\mathscr{M} \to Q$ is the universal morphism from the stack to an algebraic space. $\square$

7. Modifying moduli of representations

We keep the set-up of the preceding section: Let R be a ground ring, Λ be a finitely presented associative algebra, and Λ^{azu} be an Azumaya algebra of degree $n \geq 1$, with resulting group schemes $G = U_{\Lambda^{\text{azu}}/R}$ and $H = \text{Aut}_{\Lambda^{\text{azu}}/R}$ of units and automorphisms, respectively.

In order to deal with moduli of representations of Λ in Λ^{azu}, a first attempt is to consider the quasiaffine scheme $X = X^{\Lambda^{\text{azu}}}_{\Lambda/R}$ of Schur representations. This "over-represents" our moduli problem, because it does not take into account the isomorphism relation, and is therefore unsatisfactory. One may next form the algebraic space $Q = X/H^{\text{op}}$. But this "under-represents" the moduli problem, since it neglects automorphisms, and is still unsatisfactory. The correct framework is the quotient stack $[X/G^{\text{op}}/Q]$. This general construction of algebraic geometry, however, obscures the representation-theoretic content of the given moduli problem. This is rectified by Theorem 6.4, which tells us that one can work with the Artin stack $\mathscr{M} = \mathscr{M}^{\Lambda^{\text{azu}}}_{\Lambda/R}$ of twisted Schur representations instead.

The quasiaffine scheme X, the algebraic space Q, and the Artin stack $\mathscr{M}$ are related by a commutative diagram of functors

$$\begin{tikzcd} & (\text{Aff}/X) \arrow[dl, "\text{can}"'] \arrow[dr, "\text{can}"] & \\ \mathscr{M} \arrow[rr, "\Phi^{\text{crs}}"'] & & (\text{Aff}/Q). \end{tikzcd} \tag{13}$$

The diagonal arrow on the right stems from the quotient map for $Q = X/H^{\text{op}}$, the diagonal arrow on the left is the classifying map for the universal Schur representation $\Lambda \otimes_R \mathscr{O}_X \to \Lambda^{\text{azu}} \otimes_R \mathscr{O}_X$ pulled back to the splitting gerbe X', and the horizontal functor is given by (12), which stems from the comparison morphism $\Phi : \mathscr{M} \to [X/G^{\text{op}}/Q]$.

We can now easily answer the question raised at the end of Section 1 on modified moduli problems for the situation at hand: Let $g \in Q(R)$ be an R-valued point, not necessarily of geometric origin, and consider the H-torsor $g^*(X)$ and the resulting twisted forms

$$\tilde{H} \quad \text{and} \quad \tilde{X} \quad \text{and} \quad \tilde{\Lambda}^{\text{azu}}$$

of H and X and Λ^{azu}, respectively. Then $\tilde{\Lambda}^{\text{azu}}$ is another Azumaya algebra of degree n. According to Theorem 1.2, our R-valued point $g \in Q(R)$ has acquired geometric origin with respect to $\tilde{X}/\tilde{H}^{\text{op}} = Q$. The following unravels the representation-theoretic content of this statement:

Theorem 7.1. *The twisted form $\tilde{X}$ is the scheme of Schur representations of Λ in the Azumaya algebra $\tilde{\Lambda}^{\text{azu}}$, and there is a Schur representation $\tilde{\rho} : \Lambda \to \tilde{\Lambda}^{\text{azu}}$ inducing $g \in Q(R)$.*

Proof. The first statement is a consequence of Proposition 3.4, and the second follows from Theorem 1.2. □

Note that the above result does not rely whatsoever on the Artin stacks

$$\mathscr{M} = \mathscr{M}_{\Lambda/R}^{\Lambda^{\mathrm{azu}}} \quad \text{and} \quad \tilde{\mathscr{M}} = \mathscr{M}_{\Lambda/R}^{\tilde{\Lambda}^{\mathrm{azu}}}$$

of twisted Schur representations. However, it raises the question whether the existence of an R-valued object in $\mathscr{M}$ already implies the existence of an R-valued point in X, and likewise for $\tilde{\mathscr{M}}$ and $\tilde{X}$.

Recall that S' denotes the gerbe of splittings for the Azumaya algebra Λ^{azu} over the base scheme $S = \mathrm{Spec}(R)$. We say that a twisted Schur representation $\rho' : \Lambda \otimes_R \mathscr{O}_{S'} \to \underline{\mathrm{End}}(\mathscr{E}')$ *induces a given R-valued point* $g \in Q(R)$ if the object $(S, \mathscr{E}', \rho') \in \mathscr{M}(R)$ maps to $g \in Q(R)$ in the diagram (13). Note that this generalizes our terminology from Schur representation $\rho : \Lambda \to \Lambda^{\mathrm{azu}}$ to the twisted case.

Theorem 7.2. *Let $g \in Q(R)$, and assume that the non-abelian cohomology set $H^1(S, \mathrm{GL}_n)$ is a singleton. Then the following three conditions are equivalent:*

(i) *The R-valued point $g \in Q(R)$ has geometric origin.*
(ii) *There is a Schur representation $\Lambda \to \Lambda^{\mathrm{azu}}$ inducing g.*
(iii) *There is a twisted Schur representation $\Lambda \otimes_R \mathscr{O}_{S'} \to \underline{\mathrm{End}}(\mathscr{E}')$ inducing g.*

Moreover, (i)⇔(ii)⇒(iii) hold without the assumption on $H^1(S, \mathrm{GL}_n)$.

Proof. For the equivalence (i)⇔(ii) recall that g is of geometric origin if and only if it is in the image of $X(R) \to Q(R)$. The implication (ii)⇒(iii) is trivial. Suppose now that the set $H^1(S, \mathrm{GL}_n)$ is a singleton, and that $\rho' : \Lambda \otimes_R \mathscr{O}_{S'} \to \underline{\mathrm{End}}(\mathscr{E}')$ is a twisted representation inducing g. This gives an object $(S, \mathscr{E}', \rho') \in \mathscr{M}$, and via the comparison functor $\Phi : \mathscr{M} \to [X/G^{\mathrm{op}}/Q]$ a G-torsor $P \to S$ together with a G-equivariant morphism $f : P \to X$ making the diagram

$$\begin{array}{ccc} P & \xrightarrow{f} & X \\ \downarrow & & \downarrow \\ S & \xrightarrow[g]{} & Q \end{array}$$

commutative. The identifications $H^1(S, \mathrm{GL}_n) = H^1(S, G)$ stemming from the torsor translation maps (2) reveals that our torsor admits a section $s : S \to P$. The composite $f \circ s : S \to X$ corresponds to a Schur representation $\rho : \Lambda \to \Lambda^{\mathrm{azu}}$ inducing g, which therefore has geometric origin. □

8. Tautological sheaves

We keep the set-up of the preceding sections: R is a ground ring, Λ is a finitely presented associative algebra, Λ^{azu} is an Azumaya algebra of degree $n \geq 1$, and $H = \mathrm{Aut}_{\Lambda^{\mathrm{azu}}/R}$. We then have the quasiaffine scheme $X = X_{\Lambda/R}^{\Lambda^{\mathrm{azu}}}$ of Schur representations, and the Artin stack $\mathscr{M} = \mathscr{M}_{\Lambda/R}^{\Lambda^{\mathrm{azu}}}$ of twisted Schur representations, which is a $\mathbb{G}_m$-gerbe over the algebraic space $Q = X/H^{\mathrm{op}}$. The goal of this section is to define on the stack the *tautological sheaf* $\mathscr{T}_{\mathscr{M}}$, which will act as a "tilting object" and relate various Azumaya algebras and Brauer classes.

Recall that the objects $\hat{V}=(V,\mathscr{E}',\rho')$ from $\mathscr{M}$ are triples $(V,\mathscr{E}',\rho')$ where V is an affine scheme, $\mathscr{E}'$ is a locally free sheaf of rank $n=\deg(\Lambda^{\text{azu}})$ and weight $w=1$ on the gerbe V' of splittings for $\Lambda^{\text{azu}}\otimes_R\mathscr{O}_V$, and $\rho':\Lambda\otimes_R\mathscr{O}_{V'}\to\underline{\text{End}}(\mathscr{E}')$ is a Schur representation. Also recall that the objects of V' are quadruples $(U,h,\mathscr{F},\psi)$ where U is an affine scheme, $h:U\to V$ is a morphism, $\mathscr{F}$ is a locally free sheaf of rank n over U, and $\psi:\underline{\text{End}}(\mathscr{F})\to\Lambda^{\text{azu}}\otimes_R\mathscr{O}_U$ is an isomorphism.

We regard $\mathscr{M}$ as a site, where the covering families $(V_\lambda,\mathscr{E}'_\lambda,\rho'_\lambda)_{\lambda\in L}\to(V,\mathscr{E}',\rho')$ are those where $V_\lambda\to V$, $\lambda\in L$ is a covering family of schemes with respect to the fppf topology. The *structure sheaf* $\mathscr{O}_{\mathscr{M}}$ is given by the formula

$$\Gamma(\hat{V},\mathscr{O}_{\mathscr{M}})=\Gamma(V',\mathscr{O}_{V'})=\Gamma(V,\mathscr{O}_V) \tag{14}$$

for $\hat{V}=(V,\mathscr{E}',\rho')$, with obvious notion of restriction maps.

To define the *tautological sheaf* $\mathscr{T}_{\mathscr{M}}$ on the stack, first note that for each $\hat{V}=(V,\mathscr{E}',\rho')$, the splitting gerbe V' comes with two locally free sheaves, namely $\mathscr{E}'$ and the tautological sheaf $\mathscr{F}_{V'}^{\text{taut}}$, both of rank $n=\deg(\Lambda^{\text{azu}})$ and weight one. In turn, the Hom sheaf $\underline{\text{Hom}}(\mathscr{F}_{V'}^{\text{taut}},\mathscr{E}')$ has weight zero, and thus can be seen as a locally free sheaf $\mathscr{T}_{\mathscr{M},\hat{V}}$ on V of rank n^2. This said, we define

$$\Gamma(\hat{V},\mathscr{T}_{\mathscr{M}})=\text{Hom}(\mathscr{F}_{V'}^{\text{taut}},\mathscr{E}')=\Gamma(V,\mathscr{T}_{\mathscr{M},\hat{V}}), \tag{15}$$

with obvious notion of restriction maps. The rings (14) act by scalar multiplication, which turns $\mathscr{T}_{\mathscr{M}}$ into a presheaf of $\mathscr{O}_{\mathscr{M}}$-modules.

Proposition 8.1. *The presheaf $\mathscr{T}_{\mathscr{M}}$ on the Artin stack $\mathscr{M}$ satisfies the sheaf axiom. As $\mathscr{O}_{\mathscr{M}}$-module, it is locally free of rank n^2 and pure of weight one.*

Proof. First note that $\mathbb{G}_m$ acts on the objects $\hat{V}=(V,\mathscr{E}',\psi')$ via scalar multiplication on the sheaf $\mathscr{E}'$. The induced action on (15) is again via scalar multiplication, which has weight $w=1$. The formation of the Hom sheaves $\underline{\text{Hom}}(\mathscr{F}_{V'}^{\text{taut}},\mathscr{E}')$ is compatible with morphisms $\hat{V}_1\to\hat{V}_2$ in the category $\mathscr{M}$, and this implies the sheaf axiom for $\mathscr{T}_{\mathscr{M}}$. Since $\underline{\text{Hom}}(\mathscr{F}_{V'}^{\text{taut}},\mathscr{E}')$ are locally free of rank n^2, the same holds for $\mathscr{T}_{\mathscr{M}}$. □

We call $\mathscr{T}_{\mathscr{M}}$ the *tautological sheaf* on the Artin stack $\mathscr{M}$. The crucial observation now is that the local sections (15) also carry the structure of an Λ^{azu}-module: For each object $\tilde{U}=(U,h,\mathscr{F},\varphi)$ from the splitting gerbe V', we get

$$\Lambda^{\text{azu}}\otimes\mathscr{O}_U\xrightarrow{\varphi^{-1}}\underline{\text{End}}(\mathscr{F})\xrightarrow{\circ}\underline{\text{End}}(\mathscr{T}_{\mathscr{M}}|U').$$

The map on the right arises from $\mathscr{F}=\mathscr{F}_{V'}^{\text{taut}}|U$, and is given by left composition with respect to the Hom sheaf $\mathscr{T}_{\mathscr{M}}|U'=\underline{\text{Hom}}(\mathscr{F}_{V'}^{\text{taut}}|U,\mathscr{E}'|U)$. The above is compatible with restrictions, and turns $\mathscr{T}_{\mathscr{M}}$ into a sheaf of $\Lambda^{\text{azu}}\otimes_R\mathscr{O}_{\mathscr{M}}$-modules. The latter can be seen as inclusion $\Lambda^{\text{azu}}\otimes_R\mathscr{O}_{\mathscr{M}}\subset\underline{\text{End}}_{\mathscr{O}_{\mathscr{M}}}(\mathscr{T}_{\mathscr{M}})$ of Azumaya algebras over $\mathscr{M}$, both of weight zero. We now set

$$\mathscr{A}_{\mathscr{M}}=\underline{\text{End}}_{\mathscr{O}_{\mathscr{M}}}(\mathscr{T}_{\mathscr{M}})\quad\text{and}\quad\mathscr{A}^0_{\mathscr{M}}=\underline{\text{End}}_{\Lambda^{\text{azu}}\otimes\mathscr{O}_{\mathscr{M}}}(\mathscr{T}_{\mathscr{M}}).$$

In other words, $\mathscr{A}^0_{\mathscr{M}} \subset \mathscr{A}_{\mathscr{M}}$ is the *commutant* of $\Lambda^{\mathrm{azu}} \otimes \mathscr{O}_{\mathscr{M}} \subset \mathscr{A}_{\mathscr{M}}$. Taking the *bi-commutant* $\underline{\mathrm{End}}_{\mathscr{A}^0_{\mathscr{M}}}(\mathscr{T}_{\mathscr{M}})$, we arrive at a commutative diagram

$$\begin{tikzcd}
\Lambda^{\mathrm{azu}} \otimes_R \mathscr{O}_{\mathscr{M}} \arrow[r] \arrow[d] & \mathscr{A}_{\mathscr{M}} \arrow[d, "\mathrm{id}"] & \mathscr{A}^0_{\mathscr{M}} \arrow[l] \arrow[d, "\mathrm{id}"] \\
\underline{\mathrm{End}}_{\mathscr{A}^0_{\mathscr{M}}}(\mathscr{T}_{\mathscr{M}}) \arrow[r] & \underline{\mathrm{End}}_{\mathscr{O}_{\mathscr{M}}}(\mathscr{T}_{\mathscr{M}}) & \underline{\mathrm{End}}_{\Lambda^{\mathrm{azu}}\otimes\mathscr{O}_{\mathscr{M}}}(\mathscr{T}_{\mathscr{M}}), \arrow[l]
\end{tikzcd} \tag{16}$$

where all maps are injective. The following complements Theorem 6.4, and will elucidate the significance of the tautological sheaf $\mathscr{T}_{\mathscr{M}}$:

Lemma 8.2. *In the above diagram, all terms are Azumaya algebras of weight zero, and the vertical arrow on the left is bijective.*

Proof. Obviously, $\Lambda^{\mathrm{azu}} \otimes_R \mathscr{O}_{\mathscr{M}}$ and $\mathscr{A}_{\mathscr{M}} = \underline{\mathrm{End}}_{\mathscr{O}_{\mathscr{M}}}(\mathscr{T}_{\mathscr{M}})$ are Azumaya algebras over $\mathscr{O}_{\mathscr{M}}$. It then follows from [7], Theorem 3.3 that this carries over to the commutant $\mathscr{A}^0_{\mathscr{M}}$, and that the inclusion $\Lambda^{\mathrm{azu}} \otimes_R \mathscr{O}_{\mathscr{M}} \subset \mathscr{A}^0_{\mathscr{M}}$ is an equality. Since $\mathscr{A}_{\mathscr{M}}$ has weight zero, the same holds for the quasicoherent subsheaves. □

Since the terms have weight zero, the upper horizontal row in (16) descends along the $\mathbb{G}_m$-gerbe $\mathscr{M} \to (\mathrm{Aff}/Q)$ to sheaves of locally free algebras

$$\Lambda^{\mathrm{azu}} \otimes_R \mathscr{O}_Q \longrightarrow \mathscr{A}_Q \longleftarrow \mathscr{A}^0_Q$$

on the algebraic space Q. Now recall that an associative R-algebra A is an Azumaya algebra if and only if the underlying R-module is locally free of finite rank, and the canonical map $A \otimes_R A^{\mathrm{op}} \to \mathrm{End}_R(A)$ given by left-right multiplication is bijective. Using this characterization, we infer that the above are Azumaya algebras over Q, and give rise to classes $[\Lambda^{\mathrm{azu}} \otimes_R \mathscr{O}_Q]$ and $[\mathscr{A}_Q]$ and $[\mathscr{A}^0_Q]$ in the Brauer group $\mathrm{Br}(Q)$. By construction, $\mathscr{A}_{\mathscr{M}} = \mathscr{A}_Q \otimes \mathscr{O}_{\mathscr{M}}$ becomes the endomorphism algebra for the tautological sheaf $\mathscr{T}_{\mathscr{M}}$, and thus has trivial class in $\mathrm{Br}(\mathscr{M})$. Note that this observation does not carry over to $\mathscr{A}_Q$, because the tautological sheaf $\mathscr{T}_{\mathscr{M}}$ has weight one and therefore does not descend.

Now recall that the quotient map $X \to X/H^{\mathrm{op}} = Q$ is a torsor with respect to H_Q for the twisted form $H = \mathrm{Aut}_{\Lambda^{\mathrm{azu}}/R}$ of PGL_n, and that $G = U_{\Lambda^{\mathrm{azu}}/R}$ is a twisted form of GL_n. We have a short exact sequence $1 \to \mathbb{G}_{m,Q} \to G_Q \to H_Q \to 1$, and the torsor class $[X] \in H^1(Q, H_Q)$ yields via the non-abelian coboundary some $\partial[X] \in H^2(Q, \mathbb{G}_m)$. As a direct consequence of the preceding result, we obtain:

Theorem 8.3. *In the Brauer group* $\mathrm{Br}(Q) \subset H^2(Q, \mathbb{G}_m)$ *of the algebraic space* $Q = X/H^{\mathrm{op}}$*, we have*

$$[\mathscr{A}_Q] = \partial[X] \quad \textit{and} \quad [\mathscr{A}^0_Q] = [\Lambda^{\mathrm{azu}} \otimes_R \mathscr{O}_Q].$$

Proof. By definition of the non-abelian coboundary, the class $\partial[X]$ is represented by the $\mathbb{G}_m$-gerbe $[X/G_Q/Q]$. In light of Theorem 6.4, the $\mathbb{G}_m$-gerbe $\mathscr{M} = \mathscr{M}^{\Lambda^{\mathrm{azu}}}_{\Lambda/R}$ is another representative. The tautological sheaf $\mathscr{T}_{\mathscr{M}}$ is locally free, of rank n^2 and weight one, and by construction $\mathscr{A}_Q \otimes \mathscr{O}_{\mathscr{M}} = \underline{\mathrm{End}}(\mathscr{T}_{\mathscr{M}})$. Now de Jong's observation ([13], Lemma 2.14) gives $[\mathscr{A}_Q] = [\mathscr{M}]$.

For the second assertion, we first work on the stack $\mathscr{M}$. Set $T = \mathscr{T}_{\mathscr{M}}$. The sheaf of Azumaya algebras $A = \mathrm{End}(T)$ contains both $B = \Lambda^{\mathrm{azu}} \otimes_R \mathscr{O}_{\mathscr{M}}$ and $C = \mathscr{A}^0_{\mathscr{M}}$, and thus becomes a sheaf of modules over $B \otimes_A C^{\mathrm{op}}$. According to Lemma 8.2, the canonical maps $B \to \mathrm{End}_C(T)$ and $C \to \mathrm{End}_B(T)$ are bijective. Since these sheaves have weight zero, the very same statements hold on the algebraic space Q: Changing notation, we set

$$T = \mathscr{T}_Q \quad \text{and} \quad A = \mathrm{End}(T) \quad \text{and} \quad B = \Lambda^{\mathrm{azu}} \otimes_R \mathscr{O}_Q \quad \text{and} \quad C = \mathscr{A}^0_Q.$$

Again the canonical maps $B \to \mathrm{End}_C(T)$ and $C \to \mathrm{End}_B(T)$ are bijective. The equality $[B] = [C]$ in the Brauer group $\mathrm{Br}(Q)$ follows (see for example [12], Theorem 3.54). □

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