UNIRATIONALITY AND GEOMETRIC UNIRATIONALITY FOR HYPERSURFACES IN POSITIVE CHARACTERISTICS

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Abstract. Building on work of Segre and Kollár on cubic hypersurfaces, we construct over imperfect fields of characteristic $p \geq 3$ particular hypersurfaces of degree p, which show that geometrically rational schemes that are regular and whose rational points are Zariski dense are not necessarily unirational. A likewise behavior holds for certain cubic surfaces in characteristic $p = 2$.

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Introduction

Let F be a ground field of arbitrary characteristic $p \geq 0$, and X be a geometrically integral scheme of dimension $n \geq 0$. One says that X is *rational* or *unirational* if there is a rational map $\mathbb{P}^n \dashrightarrow X$ that is birational or dominant, respectively. If this condition holds after base-change with respect to some finite field extension $F \subset E$, one says that X is *geometrically rational* or *geometrically unirational*.

Let $X \subset \mathbb{P}^{n+1}$ be an integral cubic hypersurface of dimension $n \geq 2$ that is not a cone. Generalizing earlier results of Segre [18], Manin [14] and Colliot-Thélène, Sansuc and Swinnerton-Dyer [3], Kollár showed over perfect fields F that the following three conditions are equivalent [11]:

(i) The scheme X is unirational.
(ii) The set of rational points $X(F)$ is non-empty.
(iii) There is a rational point $a \in X$ whose local ring $\mathscr{O}_{X,a}$ is regular.

For smooth cubic hypersurfaces $X \subset \mathbb{P}^{n+1}$, this actually holds over arbitrary ground fields F. Furthermore, the result carries over to imperfect fields of characteristic

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$p \geq 5$, and it is asserted that the same holds for the remaining primes under certain technical conditions.

Indeed, Kollár gave the explicit equation $y^3 - yz^2 + \sum t_i x_i^3 = 0$ over the function field $F = k(t_1, \ldots, t_n)$ in characteristic three, which yields a cubic hypersurface that is regular, geometrically rational and contains exactly three rational points, and is thus not unirational. He asks whether a similar equation exists for characteristic two, and raises for geometrically unirational schemes X the question in what situations the implications

$$X \text{ is unirational} \quad \Longrightarrow \quad X(F) \text{ is Zariski dense} \quad \Longrightarrow \quad X(F) \text{ is non-empty}$$

might admit reverse implications, say with X smooth and F infinite.

The goal of this paper is to analyze certain hypersurfaces $X \subset \mathbb{P}^{n+1}$ of degree p over imperfect fields F that show that *none of these reverse implications hold, at least with X regular*. Generalizing Kollár's equation to arbitrary $p \geq 3$, we study

$$y^p - yz^{p-1} + \sum_{i=1}^{n} t_i x_i^p = 0,$$

where $x_1, \ldots, x_n, y, z$ are indeterminates and $t_1, \ldots, t_n \in F$ are scalars, with $n \geq 1$. Here our main result is:

Theorem. (see Thm. 2.7) *Suppose the scalars $t_1, \ldots, t_n \in F$ are algebraically independent over some subfield k of characteristic $p \geq 3$, and that F is separable over the rational function field $k(t_1, \ldots, t_n)$. Let $F \subset E$ be the extension obtained by adjoining the roots $t_1^{1/p}, \ldots, t_n^{1/p}$. Then our hypersurface $X \subset \mathbb{P}^{n+1}$ has the following properties:*

- (i) *The scheme X is regular.*
- (ii) *There is no dominant rational map $\mathbb{P}^n \dashrightarrow X$ over F.*
- (iii) *The base-change $X \otimes_F E$ is birational to $\mathbb{P}^n \otimes_F E$.*
- (iv) *The set of rational points $X(F)$ is non-empty.*
- (v) *If the field F is separably closed, the rational points are Zariski dense.*
- (vi) *If F is contained in the field $k((t_1, \ldots, t_n))$, then $X(F)$ is finite.*

Properties (i) and (ii) already hold if the differentials $dt_1, \ldots, dt_n$ in the F-vector space of absolute Kähler differentials Ω^1_F are linearly independent, in other words, if the scalars $t_1, \ldots, t_n \in F$ are *p-independent*, a notion going back to Teichmüller [19]. Apparently, this is the correct framework to treat questions of regularity and unirationality over imperfect fields.

In characteristic $p = 2$, we consider the cubic surface $X \subset \mathbb{P}^3$ defined by the equation

$$y_1^3 + t_1 x_1^2 y_1 + y_2^3 + t_2 x_2^2 y_2 = 0$$

and obtain in Theorem 4.4 analogous results. Here the set of rational points $X(F)$ is always infinite, because the cubic surface contains a line, but we could not determine whether or not $X(F)$ is Zariski dense. As remarked after Proposition 4.5, this cubic surface also shows that, for regular cubic hypersurfaces over of characteristic two, the implication

$$\begin{matrix} \exists a \in X(F) \text{ with } \mathscr{O}_{X,a} \text{ regular} \\ \text{and } \pi_a : X \dashrightarrow \mathbb{P}^n \text{ separable} \end{matrix} \quad \Longrightarrow \quad X \text{ is unirational}$$

formulated in the remark on imperfect ground fields in [11], page 468, does not hold without an additional assumption. The problem seems to be that all tangent plane intersections $C_a = X \cap T_a(X)$, which are non-regular cubic curves, are actually non-integral. Note that for rational points $a \in X$, the local ring $\mathscr{O}_{X,a}$ is regular if and only if the scheme X is smooth at the point.

The non-unirationality of our cubic surface depends on the following criterion, which is of independent interest:

Theorem. (see Thm. 3.1) *Let X be unirational over some infinite ground field F of characteristic $p > 0$. Suppose there a fibration $f : X \to \mathbb{P}^1$ such that the fibers over almost all rational points $a \in \mathbb{P}^1$ contain no rational curve. Then the reduced base-change along the relative Frobenius map $\mathbb{P}^1 \to \mathbb{P}^1$ remains unirational.*

After the completion of the paper, Olivier Benoist kindly informed us that he recently studied related questions with Olivier Wittenberg [2]. In particular, they show that for certain quadrics $Q_1, Q_2 \subset \mathbb{P}^5$ over $F = k((t))$, the intersection $X = Q_1 \cap Q_2$ is a smooth threefold that contains rational points, is unirational but not rational, yet becomes rational over $E = k((t^{1/2}))$.

The paper is organized as follows: In Section 1 we recall basic facts on p-independence of scalars $t_1, \ldots, t_n \in F$, and discuss some implications concerning regularity of schemes and Zariski density of rational points. In Section 2 we study hypersurfaces $X \subset \mathbb{P}^{n+1}$ defined by the equation $y^p - yz^{p-1} + \sum t_i x_i^p = 0$ at odd primes. In Section 3 we relate unirationality with Frobenius base-change. This is used in Section 4 for the analysis of the cubic surface $X \subset \mathbb{P}^3$ defined by the equation $x_1^3 + t_1 x_1 y_1^2 + x_2^3 + t_2 x_2 y_2^2 = 0$ in characteristic two.

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1. Generalities

Here we recall some general facts that will be used throughout, concerning Kähler differentials, p-independence, regularity, and Zariski density of rational points. Let F be a field of characteristic $p > 0$, and $\Omega^1_F = \Omega^1_{F/\mathbb{Z}} = \Omega^1_{F/F^p}$ be the F-vector space of absolute Kähler differentials. The scalars $t \in F$ yield differentials $dt \in \Omega^1_F$, which form a generating set. Let us say that a family of scalars $t_i \in F$, $i \in I$ is *p-independent* if the vectors $dt_i \in \Omega^1_F$ are linearly independent. We need the following facts:

Proposition 1.1. *Consider the following conditions:*

(i) *The $t_i \in F$ form a separable transcendence basis over a subfield k.*
(ii) *The $t_i \in F$ are p-independent.*
(iii) *The $t_i \in F$ are linearly independent over the subfield F^p.*

Then the implications (i)⇒(ii)⇒(iii) hold. Moreover, for each $t \in F$ the condition $dt = 0$ is equivalent to $t \in F^p$.

Proof. The first implication follows from [13], Lemma 3 on page 382. The second is a consequence of the characterization of p-independence ([15], Theorem 86 or [16], Theorem 26.5), which is frequently taken as a definition: The monomials $\prod_{i\in I} t_i^{d_i} \in F$ are linearly independent over the subfield F^p, where the exponents satisfy $0 \leq d_i \leq p-1$ and almost all vanish. In particular, the $t_i \in F$ are linearly independent.

Clearly, each $t \in F^p$ has $dt = 0$. Conversely, suppose that $t \in F$ is not a p-th power. The extension $F^p \subset F$ is purely inseparable of height one, so the minimal polynomial of t must be of the form $T^p - \lambda$ for some $\lambda \in F^p$. In turn, the powers $1, t, \ldots, t^{p-1} \in F$ are linearly independent over the subfield F^p, and the above characterization shows $dt \neq 0$. □

Let us list several elementary but useful permanence properties for p-independent scalars:

Proposition 1.2. *Let $F \subset E$ be a separable extension. If $t_i \in F$, $i \in I$ are p-independent, so are the $t_i \in E$.*

Proof. According to [15], Theorem 88 or [16], Theorem 26.6, the canonical map $\Omega^1_F \otimes_F E \to \Omega^1_E$ given by $dt \otimes \lambda \mapsto \lambda dt$ is injective. It follows that F-linearly independent subsets are mapped to E-linearly independent subsets. □

Proposition 1.3. *If $t_1, \ldots, t_n \in F$ are p-independent, then the same holds for the $t_1, \ldots, t_{n-1}, t'_n \in F$ with the new element $t'_n = t_n/t_{n-1}$.*

Proof. First note that all scalars t_i are non-zero. Set $f = t_{n-1}$ and $g = t_n$. Inside the vector space Ω^1_F, the product rule gives $g^2 d(f/g) = gdf - fdg$, and the assertion follows from the exchange property for linear independent sets. □

Proposition 1.4. *Suppose that $t_1, \ldots, t_n \in F$ are p-independent. Then the purely inseparable extension $E = F(t_n^{1/p})$ has degree p, and the $t_1, \ldots, t_{n-1} \in E$ remain p-independent.*

Proof. We have $t_n \notin F^p$, and whence $[E : F] = p$. Clearly, the monomials $t_n^{j/p}$, $0 \leq j \leq p-1$ are linearly independent over the subfield F, hence also over $E^p \subset F$, and we infer that $\Omega^1_{E/F}$ is one-dimensional, with basis $dt_n^{1/p}$. The field extensions $F^p \subset F \subset E$ gives an exact sequence

$$0 \longrightarrow \Upsilon_{E/F/F^p} \longrightarrow \Omega^1_F \otimes_F E \longrightarrow \Omega^1_E \longrightarrow \Omega^1_{E/F} \longrightarrow 0 \tag{1}$$

Here the term on the left is called the *module of imperfection*, and is defined by the above exact sequence; here we follow the notation from [7], Definition 20.6.1. Cartier's Equality ([15], Theorem 92 or [16], Theorem 26.10)

$$\dim_E(\Omega^1_{E/F}) = \operatorname{trdeg}_F(E) + \dim_E(\Upsilon_{E/F/F^p})$$

for the finitely generated field extension $F \subset E$ shows that our module of imperfection is one-dimensional. The non-zero vector $dt_n \otimes 1$ clearly belongs to the kernel, whence can be regarded as a basis for $\Upsilon_{E/F/F^p}$. It follows that the remaining vectors $dt_1, \ldots, dt_{n-1}$ remain linearly independent in Ω^1_E. □

Now let $x_0,\ldots,x_n$ be indeterminates for some $n\geq 0$, and regard $\mathbb{P}^n$ as the homogeneous spectrum of the polynomial ring $F[x_0,\ldots,x_n]$. Given a sequence of scalars $t_0,\ldots,t_n\in F$, not all of which vanish, we consider the Fermat hypersurface $D\subset\mathbb{P}^n$ defined by the equation $t_0x_0^p+\ldots+t_nx_n^p=0$. Note that D is irreducible but geometrically non-reduced, and becomes a p-fold hyperplane after base-changing to the perfect closure.

Proposition 1.5. *Suppose $t_0=1$. Then the scheme D is regular if and only if the $t_1,\ldots,t_n\in F$ are p-independent.*

Proof. The extension $F'=F^p(t_1,\ldots,t_n)$ defines an intermediate field $F^p\subset F'\subset F$. The *p-degree* $d=\operatorname{pdeg}(F'/F^p)$ is defined as the vector space dimension of Ω^1_{F'/F^p}, and is also characterized by the degree formula $[F':F^p]=p^d$. We have $d\leq n$, because the differentials $dt_1,\ldots,dt_n\in\Omega^1_{F'/F^p}$ form a generating set. According to [17], Theorem 3.3 the scheme D is regular if and only if $d=n$. Hence we have to show the equality

$$\dim_{F'}(\Omega^1_{F'/F^p})=\dim_F(Fdt_1+\ldots+Fdt_n) \tag{2}$$

of vector space dimensions. Taking p-th roots, we see that the left hand side equals the dimension of $\Omega^1_{E/F}$. Here $F\subset E$ denotes the extension generated by $t_1^{1/p},\ldots,t_n^{1/p}$, to avoid confusion with F'. Using induction on $n\geq 0$ with Proposition 1.4, one sees that the right hand side $r=\dim_F(Fdt_1+\ldots+Fdt_n)$ obeys the formula $[E:F]=p^r$, hence also coincides with the dimension of $\Omega^1_{E/F}$. This gives the desired equality (2). □

Now suppose that X is an F-scheme of finite type. One says that X is *geometrically reduced* if for some algebraically closed field extension E, the base-change $X'=X\otimes_F E$ is reduced.

Lemma 1.6. *If the scheme X is geometrically reduced and the field F is separably closed, then the set of rational points $X(F)$ is Zariski dense.*

Proof. We have to verify that each non-empty open set contains a rational point, so it suffices to check that $X(F)$ is non-empty, and we may assume that X is affine. By Bertini's Theorem ([9], Theorem 6.3) there is a hyperplane $H\subset X$ that remains geometrically reduced. By induction on the dimension, this reduces us to the case $\dim(X)=0$. Hence our scheme is the spectrum of a product $E_1\times\ldots\times E_r$ of $r\geq 1$ separable field extensions. Since F is separably closed, we must have $E_i=F$.

The following more direct argument was suggested to us by János Kollár: According to [13], Theorem 15 the function field of X has a a separating transcendence basis over F. In turn, we may assume that X is étale over $\mathbb{A}^n$. For each rational point $a\in\mathbb{A}^n$ lying in the image of X, the preimage is the spectrum of a product $E_1\times\ldots\times E_r$ as above. □

Suppose now that X is equidimensional of dimension $n\geq 0$. Then the *locus of non-smoothness* $\operatorname{Sing}(X/F)$ is the set of points $a\in X$ where $\Omega^1_{X/F}\otimes\kappa(a)$ has vector space dimension $d>n$. It has a natural scheme structure, defined via Fitting ideals for the coherent sheaf $\Omega^1_{X/F}$, compare the discussion in [5], Section 2. Depending on the context, we also call $\operatorname{Sing}(X/F)$ the *scheme of non-smoothness*.

Lemma 1.7. *Suppose that* $\operatorname{Sing}(X/F)$ *and some effective Cartier divisor* $D \subset X$ *have the same support, and that* X *contains no embedded components. Then* X *is geometrically reduced but geometrically non-normal. Furthermore, the scheme* X *is regular provided that* D *is regular.*

Proof. The open set $X \smallsetminus D$ is smooth. The base-change $X' = X \otimes_F E$ to the perfect closure $E = F^{\mathrm{perf}}$ also contains no embedded component, and is generically smooth. In turn, the structure sheaf $\mathscr{O}_{X'}$ has no non-zero nilpotent elements, so X is geometrically reduced. Let ζ be some generic point in $D' = D \otimes_F E$. Then the local ring $\mathscr{O}_{X',\zeta}$ is one-dimensional and not regular. Now recall that by Serre's Criterion ([8], Theorem 5.8.6), a noetherian scheme is normal if and only it satisfies (R_1) and (S_2), hence X is not geometrically normal.

Suppose now that the scheme D is regular. Fix a point $a \in D$, and let $f \in \mathscr{O}_{X,a}$ be an element defining the Cartier divisor in some neighborhood. This element is regular and contained in the maximal ideal. Since the local ring $\mathscr{O}_{D,a} = \mathscr{O}_{X,a}/(f)$ is regular, the same must hold for $\mathscr{O}_{X,a}$. □

2. Hypersurfaces of p-degree

Let F be a ground field of characteristic $p \geq 3$. Fix some integer $n \geq 1$ and scalars $t_1, \ldots, t_n \in F$, only subject to the condition $t_1 \neq 0$. Regard $\mathbb{P}^{n+1}$ as the homogeneous spectrum of the polynomial ring $F[x_1, \ldots, x_n, y, z]$. We now consider the hypersurface $X \subset \mathbb{P}^{n+1}$ of dimension $\dim(X) = n$ and degree $\deg(X) = p$ defined by the equation

$$y^p - yz^{p-1} + \sum_{i=1}^{n} t_i x_i^p = 0. \tag{3}$$

For function fields $F = k(t_1, \ldots, t_n)$ in characteristic three, this is the cubic hypersurface studied by Kollár in [11], Section 4. Here we work over arbitrary characteristics $p \geq 3$ and more general ground fields F.

Proposition 2.1. *The scheme* X *is geometrically integral.*

Proof. Replacing F by some algebraic closure, we have to show that the left-hand side of (3) is an irreducible polynomial. Set $x = \sum t_i^{1/p} x_i$ and $v = x+y$. Now our task is to verify that $P(v) = v^p - yz^{p-1}$ is irreducible as polynomial over $R = k[y, z]$. This follows immediately with the Eisenstein Criterion with the prime element $y \in R$. □

Proposition 2.2. *If* $t_1, \ldots, t_n \in F^p$*, then the scheme* X *is birational to* $\mathbb{P}^n$*.*

Proof. As in the previous proof, we may assume that our hypersurface $X \subset \mathbb{P}^{n+1}$ is given by the equation $y^p - yz^{p-1} + x_1^p = 0$. This does not involve the variables $x_2, \ldots, x_n$, hence X is a cone with respect to the $(n-2)$-dimensional linear subspace $V \subset \mathbb{P}^{n+1}$ given by $x_1 = y = z = 0$ as apex, over the plane curve $C \subset \mathbb{P}^2$ defined by the equation $x^p - yz^{p-1} = 0$, where we have made the substitution $x = y + x_1$.

Geometrically, this means that X is birational to $C \times \mathbb{P}^{n-1}$, and it remains to check that the integral curve C is rational. On the affine chart given by $z \neq 0$, the coordinate ring for the curve becomes the polynomial ring $F[x/z]$, hence C must be rational. □

Proposition 2.3. *The scheme of non-smoothness* $\mathrm{Sing}(X/F) \subset X$ *and the effective Cartier divisor* $D \subset X$ *defined by the equation* $z = 0$ *have the same support. Moreover,* X *is regular provided that* $t_1, \ldots, t_n \in F$ *are* p*-independent.*

Proof. For our hypersurface $X \subset \mathbb{P}^{n+1}$, the scheme of non-smoothness $\mathrm{Sing}(X/F)$ is defined by the additional equations coming from the partial derivatives of (3). These partial derivatives are z^{p-1} and $-yz^{p-2}$. It follows that D and $\mathrm{Sing}(X/F)$ have the same support.

Now suppose that $t_1, \ldots, t_n \in F$ are p-independent. We may regard D as the divisor in $\mathbb{P}^n$ defined by the Fermat equation $y^p + t_1x^p + \ldots + t_nx_n^p$. According to Proposition 1.5, the hypersurface D is regular. By Lemma 1.7, the scheme X is regular as well. □

In order to apply induction, we will relate our hypersurface in dimension n with one in dimension $n-1$. This is based on the following observation:

Lemma 2.4. *Suppose* $n \geq 2$*, that* $t_{n-1} \neq 0$ *and that* $t_n/t_{n-1} \in F^p$*. Then the hypersurface* $X \subset \mathbb{P}^{n+1}$ *is projectively equivalent to the hypersurface* $X' \subset \mathbb{P}^{n+1}$ *defined by another equation of the form* (3)*, with coefficients* $t'_i = t_i$ *for* $i \leq n-1$ *and* $t'_n = 0$*.*

Proof. Let $\lambda \in F$ be the scalar with $\lambda^p = t_n/t_{n-1}$, rewrite the equation (3) as

$$y^p - yz^{p-1} + t_1x_1^p + \ldots + t_{n-2}x_{n-2}^p + t_{n-1}(x_{n-1} + \lambda x_n)^p = 0,$$

and use the coordinate change $x'_{n-1} = x_{n-1} + \lambda x_n$. □

Proposition 2.5. *If* $t_1, \ldots, t_n \in F$ *are* p*-independent, then the scheme* X *is not unirational.*

Proof. We proceed by induction on $n = \dim(X)$. Suppose first that $n = 1$. Seeking a contradiction, we assume that there is a rational dominant map $\mathbb{P}^1 \dashrightarrow X$. In other words, the function field of X becomes a subfield of the function field of $\mathbb{P}^1$. By Lüroth's Theorem ([20], §73), X is birational to $\mathbb{P}^1$. According to Proposition 2.3, the curve X is regular, so by [6], Proposition 7.4.9 we actually have an isomorphism $X \simeq \mathbb{P}^1$. In particular X is smooth. On the other hand, the scheme of non-smoothness $\mathrm{Sing}(X/F)$ is non-empty, contradiction.

Suppose now that $n \geq 2$, and that the assertion is true for $n-1$. Seeking a contradiction, we assume that there is a rational dominant map $\mathbb{P}^n \dashrightarrow X$. Let us write $X = X_F(t_1, \ldots, t_n)$ to indicate the dependence of our hypersurface $X \subset \mathbb{P}^n$ on the ground field F and the scalars $t_1, \ldots, t_n \in F$. Consider its base-change $\mathbb{P}^n_E \dashrightarrow X_E(t_1, \ldots, t_n)$ for the field extension $E = F(t_n^{1/p})$. According to Lemma 2.4 there is linear isomorphism $X_E(t_1, \ldots, t_n) \to X_E(t_1, \ldots, t_{n-1}, 0)$. The latter becomes a cone in $\mathbb{P}^{n+1}_E$, because its equation no longer involves the indeterminate x_n, whence there is a dominant rational map

$$X \otimes_F E = X_E(t_1, \ldots, t_{n-1}, 0) \dashrightarrow X_E(t_1, \ldots, t_{n-1}) = X'.$$

Composing these maps we get a dominant rational map $\mathbb{P}^n_E \dashrightarrow X'$. According to [11], Lemma 2.3 the hypersurface Y is unirational. On the other hand, the scalars $t_1, \ldots, t_{n-1} \in E$ are p-independent according to Proposition 1.4. By induction hypothesis, the hypersurface $X' \subset \mathbb{P}^n_E$ is not unirational, contradiction. □

The hypersurface $X \subset \mathbb{P}^{n+1}$ contains the obvious rational points

$$(0 : \ldots : 0 : \lambda : 1), \quad \lambda \in \mathbb{F}_p. \tag{4}$$

Under suitable assumptions on the ground field F, there are no further rational points:

Proposition 2.6. *Suppose that F is contained in the field $k((t_1, \ldots, t_n))$ of formal Laurent series with respect to indeterminates $t_1, \ldots, t_n$ and some subfield k. Then $X(F)$ consists of the p rational points listed in (4).*

Proof. This is essentially Kollár's argument from [11], Section 4, which we repeat for the convenience of the reader. It suffices to treat the case that F equals the field of formal Laurent series over an infinite field k. This means $F = \mathrm{Frac}(R)$ for the ring $R = k[[t_1, \ldots, t_n]]$. Let $a \in X(F)$ be a rational point, and write it as $a = (h_1 : \ldots : h_n : f : g)$ with some relatively prime power series $h_i, f, g \in R$. This is indeed possible because the ring R is factorial by [16], Theorem 20.8. Our task is to show that the h_i vanish. Seeking a contradiction, we assume that this is not the case. Given some exponents $u_i \geq 1$, we obtain a homomorphism $\varphi : R \to k[[t]]$ defined by $t_i \mapsto t^{u_i}$, inducing an equation $f^p - fg^{p-1} + th^p = 0$, now with $f, g, h \in k[[t]]$. According to [1], §3, No. 7, Lemma 2 we may choose the exponents so that $h \neq 0$. Then also $f \neq 0$.

Dividing by some common factor, we may assume that $\gcd(f, g, h) = 1$. Each irreducible factor d of $\gcd(f, g)$ has the property $d^p | th^p$. Since t is a prime element, we must have $d|h$, contradiction. Thus $\gcd(f, g) = 1$. Rewrite our equation as $th^p = \prod_{j=0}^{p-1}(f - jg)$. The factors $P_j = f - jg$ on the right are pairwise coprime, because this holds for f, g. Hence we can write $f - jg = Q_j^p$ for all j with one exception i, which has $f - ig = tQ_i^p$. Then

$$tQ_i^p + (\sum_{j \neq i} Q_j)^p = \sum_{j=0}^{p-1}(f - jg) = pf - p\frac{p-1}{2}g = 0.$$

We conclude that in the prime factorization of tQ_i^p, all exponents are divisible by p. This contradicts the fact that t is a prime element in the ring $k[[t]]$. □

We now summarize our results in the following form:

Theorem 2.7. *Suppose the scalars $t_1, \ldots, t_n \in F$ are algebraically independent over some subfield k of characteristic $p \geq 3$, and that F is separable over the rational function field $k(t_1, \ldots, t_n)$. Let $F \subset E$ be the extension obtained by adjoining the roots $t_1^{1/p}, \ldots, t_n^{1/p}$. Then the hypersurface $X \subset \mathbb{P}^{n+1}$ that is defined by the equation $y^p - yz^{p-1} + \sum_{i=1}^n t_i x_i^p = 0$ has the following properties:*

(i) *The scheme X is regular.*
(ii) *There is no dominant rational map $\mathbb{P}^n \dashrightarrow X$ over F.*
(iii) *The base-change $X \otimes_F E$ is birational to $\mathbb{P}^n \otimes_F E$.*
(iv) *The set of rational points $X(F)$ is non-empty.*
(v) *If the field F is separably closed, the rational points are Zariski dense.*
(vi) *If F is contained in the field $k((t_1, \ldots, t_n))$, then $X(F)$ is finite.*

Proof. According to Proposition 1.1, the scalars $t_1, \ldots, t_n \in F$ are p-independent, so the scheme X must by regular by Proposition 2.3. Furthermore, it is not unirational according to Proposition 2.5. The base-change $X \otimes_F E$ becomes rational, in light of Proposition 2.2. If F is separably closed, the rational points must be dense by Lemma 1.6. If F is contained in the field of formal Laurent series, we saw in Proposition 2.6 that there are only p rational points. □

With the setting of the above theorem, our regular scheme X is geometrically unirational but not unirational. Furthermore, no separable extension achieves unirationality. *As one of the main insights of this paper, we conclude that none of the implications*

$$X \text{ is unirational} \implies X(F) \text{ is Zariski dense} \implies X(F) \text{ is non-empty}$$

does admit a converse valid for geometrically unirational regular schemes X over infinite fields F; compare the discussion by Kollár ([11], Question 1.3).

3. Unirationality and Frobenius base-change

Let F be an infinite ground field of characteristic $p > 0$. Suppose X is an integral proper scheme endowed with a surjective morphism $f : X \to \mathbb{P}^1$. Write the projective line as the homogeneous spectrum of $F[T_0, T_1]$, and regard the indeterminates T_i as global sections of the ample sheaf $\mathscr{O}_{\mathbb{P}^1}(1)$. Fix an integer $\nu \geq 1$. The resulting global sections $T_i^{p^\nu}$ of $\mathscr{O}_{\mathbb{P}^1}(p^\nu)$ define a purely inseparable morphism $h : \mathbb{P}^1 \to \mathbb{P}^1$ of degree p^ν. This map can also be described by the inclusion of coordinate rings $F[s^{p^\nu}] \subset F[s]$, where we set $s = T_1/T_0$. This reveals that $h : \mathbb{P}^1 \to \mathbb{P}^1$ coincides with the iterated relative Frobenius map for the projective line. Let us write $X' = (X \times_{\mathbb{P}^1} \mathbb{P}^1)_{\text{red}}$ for the ensuing base-change, endowed with the reduced scheme structure.

In what follows, a *rational curve* denotes an integral proper scheme C birational to $\mathbb{P}^1$ over our ground field F, and *almost every* means all but finitely many.

Theorem 3.1. *Suppose the scheme X is unirational, and that for almost every rational point $a \in \mathbb{P}^1$, the fiber $f^{-1}(a)$ contains no rational curve. Then the reduced base-change $X' = (X \times_{\mathbb{P}^1} \mathbb{P}^1)_{\text{red}}$ is unirational as well.*

Proof. Set $n = \dim(X)$, and choose a dominant rational map $\mathbb{P}^1 \times \mathbb{P}^{n-1} \dashrightarrow X$. By the Valuative Criterion for properness, the domain of definition contains $\mathbb{P}^1_U = \mathbb{P}^1 \times U$ for some open dense set $U \subset \mathbb{P}^{n-1}$, so we have a dominant morphism $g : \mathbb{P}^1_U \to X$.

We now write $B = \mathbb{P}^1$ for the base of the given surjection $f : X \to \mathbb{P}^1 = B$. Let $b_1, \ldots, b_r \in B$ be the finitely many rational points whose fibers contain rational curves. The preimages of $f^{-1}(b_i)$ on $\mathbb{P}^1_U$ are closed sets not containing the generic point. Since the projection $\mathbb{P}^1_U \to U$ is proper, we may shrink U and suppose that the image of $g : \mathbb{P}^1_U \to X$ is disjoint from the fibers $f^{-1}(b_i)$. This means that for every rational point $u \in U$, the image $g(\mathbb{P}^1_u) \subset X$ is not contained in any of the fibers of $f : X \to B$, and thus dominates B. It follows that for the generic point $\eta \in U$, the induced projection $\mathbb{P}^1_E = \mathbb{P}^1_\eta \to B = \mathbb{P}^1$ is surjective, where $E = \kappa(\eta)$ denotes the function field of the open set $U \subset \mathbb{P}^{n-1}$.

Consider the composite morphism $\mathbb{P}^1_U \to B$ and the ensuing base-change $(\mathbb{P}^1_U) \times_B B$ with respect to the purely inseparable morphism $h : B = \mathbb{P}^1 \to \mathbb{P}^1 = B$ of degree $\deg(h) = p^\nu$. It comes with a projection $\text{pr} : (\mathbb{P}^1_U) \times_B B \to U$ and a dominant morphism $(\mathbb{P}^1_U) \times_B B \to X \times_B B$. To check that X' is unirational, it thus suffices to verify that the reduction of the generic fiber $\mathrm{pr}^{-1}(\eta)$ is a rational curve over the function field $E = \kappa(\eta)$ of the open set $U \subset \mathbb{P}^{n-1}$.

This is a consequence of the following property of the iterated relative Frobenius map $h : \mathbb{P}^1 \to \mathbb{P}^1$: We claim that for each field extension $F \subset E$ and each surjective F-morphism $\varphi : \mathbb{P}^1_E \to \mathbb{P}^1_F$, there is a commutative diagram

$$\begin{array}{ccc} \mathbb{P}^1_E & \xleftarrow{h_E} & \mathbb{P}^1_E \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \mathbb{P}^1_F & \xleftarrow[h]{} & \mathbb{P}^1_F \end{array}$$

for some ψ. Indeed, the morphism φ is defined via some invertible sheaf $\mathscr{L} = \mathscr{O}_{\mathbb{P}^1_E}(n)$ and two global sections without common zeros, which can be viewed as homogeneous polynomials $Q_0, Q_1 \in E[T_0, T_1]$ of degree n that are relatively prime. Set $q = p^\nu$. Then the morphism ψ defined by the polynomials $Q_0^q, Q_1^q \in E[T_0^q, T_1^q]$ makes the diagram commutative.

The above diagram yields a surjection $\mathbb{P}^1_E \to \mathbb{P}^1_E \times_{\mathbb{P}^1_F} \mathbb{P}^1_F$. This is an E-morphism, because the iterated relative Frobenius map h_E is an E-morphism. Lüroth's Theorem ([20], §73) ensures that the reduction of the fiber product is a rational curve over E. □

The following consequence will later play an important role:

Corollary 3.2. *Suppose that for almost every rational point $a \in \mathbb{P}^1$, the fiber $f^{-1}(a)$ contains no rational curves, and that X' is birational to $Z \times \mathbb{P}^{n-1}$, where Z is not a rational curve. Then X is not unirational.*

Proof. Seeking a contradiction, we assume that X is unirational. By the theorem, X' is unirational and hence Z are rational, contradiction. □

4. A cubic surface in characteristic two

Let F be a ground field of characteristic $p = 2$. Regard $\mathbb{P}^3$ as the homogeneous spectrum of the polynomial ring $F[x_1, x_2, y_1, y_2]$, and let $t_1, t_2 \in F$ be scalars, subject only to the condition $t_1 \neq 0$ and $t_2 \neq 0$. The goal of this section is to study the cubic surface $X \subset \mathbb{P}^3$ defined by the equation

$$y_1^3 + t_1 x_1^2 y_1 + y_2^3 + t_2 x_2^2 y_2 = 0. \tag{5}$$

The defining polynomial is irreducible, which can be seen by setting $x_2 = 0$ and observing that $y_1(y_1^2 + t_1 x_1^2)$ is not a cube in $F[x_1, y_1]$. Thus X is a geometrically integral.

The scheme X is equidimensional of dimension two, has $h^0(\mathscr{O}_X) = 1$, all local rings $\mathscr{O}_{X,x}$ are Gorenstein, and the dualizing sheaf $\omega_X = \mathscr{O}_X(-1)$ is anti-ample. In other words, X is a *del Pezzo surface*. Moreover, we have $h^1(\mathscr{O}_X) = h^2(\mathscr{O}_X) = 0$, and the degree of the del Pezzo surface is $K_X^2 = 3$.

We shall see in Theorem 4.4 that if the scalars are p-independent, the scheme X is regular and geometrically rational, yet not unirational. The Picard group with its intersection form will be determined in Proposition 4.6. We do not now whether or not $X(F)$ is Zariski dense.

Proposition 4.1. *The scheme of non-smoothness $D = \operatorname{Sing}(X/F)$ is an irreducible curve defined inside $\mathbb{P}^3$ by the two equations $y_1^2 + t_1x_1^2 = 0$ and $y_2^2 + t_2x_2^2 = 0$. Moreover, the inclusion $D \subset X$ is Cartier.*

Proof. The partial derivatives of the defining polynomial $P = y_1^3 + t_1x_1^2y_1 + y_2^3 + t_2x_2^2y_2$ with respect to y_i are $P_i = y_i^2 + t_ix_i^2$, whereas $\partial P/\partial x_i = 0$. Moreover, the Jacobian ideal $\mathfrak{a} = (P, P_1, P_2)$ is already generated by the two partial derivatives, which yields the assertion on the embedding $D \subset \mathbb{P}^3$. If $t_i \in F$ are squares, a change of coordinate reveals that D is the intersection of two double planes, which shows that D is an irreducible curve.

From (5), one sees that on the open set given by $y_2 \neq 0$, the inclusion $D \subset X$ is already defined by the single equation $y_1^2 + t_1x_1^2 = 0$. An analogous statement holds on the open set given by $y_1 \neq 0$. It follows that $D \subset X$ is Cartier outside the closed set $L \subset X$ defined by $y_1 = 0$ and $y_2 = 0$. From the equations for $D \subset \mathbb{P}^3$ one sees it is disjoint from L, hence $D \subset X$ must be Cartier. □

As usual, an effective Cartier divisor $C \subset X$ with $C \simeq \mathbb{P}^1$ and $C^2 = -1$ is called a (-1)-*curve*. The line $L \subset \mathbb{P}^3$ given by the equations $y_1 = 0$ and $y_2 = 0$ is contained in X and actually lies in the smooth locus. The adjunction formula for the inclusions $X \subset \mathbb{P}^3$ and $L \subset X$ gives $\omega_X = \mathscr{O}_X(-1)$ and $-2 = (L + K_X) \cdot L = L^2 - 1$. Hence:

Proposition 4.2. *The selfintersection number of the line L on the cubic surface X is given by $L^2 = -1$. In other words, $L \subset X$ is a (-1)-curve.*

Now consider the plane $H_1 \subset \mathbb{P}^3$ given by the equation $y_1 = 0$. Then the plane section $H_1 \cap X$ is defined by $y_1 = 0$ and $y_2(y_2^2 + t_2x_2^2) = 0$, thus decomposes as $L + C_1$, where C_1 is the irreducible conic defined by $y_1 = 0$ and $y_2^2 + t_2x_2^2 = 0$. Likewise, the plane $H_2 \subset \mathbb{P}^3$ defined by $y_2 = 0$ has $H_2 \cap X = L + C_2$, where the irreducible conic C_2 is defined by $y_2 = 0$ and $y_1^2 + t_1x_1^2 = 0$.

The equations reveal that $C_1 \cap C_2 = \varnothing$. Moreover, the curves $C_i \subset X$ are Cartier, because the intersections $C_i \cap L$ lies in the smooth locus. Since $H_1, H_2 \subset \mathbb{P}^2$ are linearly equivalent, the same holds for $C_1, C_2 \subset X$. In turn, the invertible sheaf $\mathscr{L} = \mathscr{O}_X(C_1)$ is globally generated, and the two-dimensional linear system inside $H^0(X, \mathscr{L})$ generated by global sections defining $C_i \subset X$ yield a morphism $f : X \to \mathbb{P}^1$ with $\mathscr{L} = f^*\mathscr{O}_{\mathbb{P}^1}(1)$.

Now it is convenient to use the term *double line* for a curve isomorphic to the first infinitesimal neighborhood of a line $\mathbb{P}^1$ in $\mathbb{P}^2$. Note that the *twisted forms of the double line* are precisely the conics that are geometrically non-reduced.

Proposition 4.3. *The morphism $f : X \to \mathbb{P}^1$ extends the rational map $X \dashrightarrow \mathbb{P}^1$ given by $(x_1 : y_1 : x_2 : y_2) \mapsto (y_1 : y_2)$. All fibers are twisted forms of the double line. The induced finite morphisms*

$$f : L \longrightarrow \mathbb{P}^1 \quad \textit{and} \quad f : D = \operatorname{Sing}(X/F) \longrightarrow \mathbb{P}^1$$

are purely inseparable of degree two and four, respectively.

Proof. Let s_1, s_2 be sections of $\mathscr{L}$ defining $C_1, C_2 \subset X$, and $E \subset H^0(X, \mathscr{L})$ the resulting linear system. By construction, we have $\mathscr{L} = \mathscr{O}_X(1) \otimes \mathscr{O}_X(-L)$. Under the canonical inclusion $\mathscr{L} \subset \mathscr{O}_X(1)$ and up to scalars, the sections s_i become the restrictions of $y_i \in H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(1))$, and L is the fixed part of the y_1, y_2. The rational map $\varphi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ given by $(x_1 : y_1 : x_2 : y_2) \mapsto (y_1 : y_2)$ has the open set $U = \mathbb{P}^3 \smallsetminus L$ as domain of definition, and it also can be described by the two-dimensional linear system generated by $y_1, y_2 \in H^0(\mathbb{P}^3, \mathscr{O}_{\mathbb{P}^3}(1))$. Thus the map $\varphi|X$ coincides with the morphism $f : X \to \mathbb{P}^1$ on the open set $X \cap U$.

Now let $a \in \mathbb{P}^1$ be a point. To check that the fiber is a twisted form of the double line, it suffices to treat the case that $a = (\lambda_1 : \lambda_2)$ is a rational point. Then the fiber $Z = f^{-1}(a)$ is the zero-scheme for $\lambda_1 s_1 + \lambda_2 s_2$, and is contained in the zero-scheme $Z' \subset X$ for $\lambda_1 y_1 + \lambda_2 y_2$, which is a plane section. In turn, $Z' = Z \cup L$ is a reducible cubic curve, thus decomposes into the union of a conic Z and a line L. This shows that the fiber $Z = f^{-1}(a)$ is isomorphic to a conic. To proceed, it suffices by symmetry to treat the case that $\lambda_2 = 1$, and we write $\lambda = \lambda_1$. Then $f^{-1}(a) \subset X$ is defined inside $\mathbb{P}^3$ by the homogeneous equations

$$\lambda y_1 + y_2 = 0 \quad \text{and} \quad (1 + \lambda^3) y_1^2 + t_1 x_1^2 + \lambda t_2 x_2^2 = 0, \tag{6}$$

which indeed is a twisted form of the double line. Taking intersections with L and $D = \operatorname{Sing}(X/F)$, one sees that the induced projections are purely inseparable of degree $d = 2$ and $d = 4$, respectively. □

Recall that $p = 2$. We now come to the main result on our cubic surface:

Theorem 4.4. *Suppose the scalars $t_1, t_2 \in F$ are p-independent. Let $F \subset E$ be the purely inseparable field extension obtained by adjoining the root $\sqrt{t_1}$. Then the cubic surface $X \subset \mathbb{P}^3$ defined by the equation $y_1^3 + t_1 x_1^2 y_1 + y_2^3 + t_2 x_2^2 y_2 = 0$ has the following properties:*

- (i) *The scheme X is regular.*
- (ii) *There is no dominant rational map $\mathbb{P}^2 \dashrightarrow X$ over F.*
- (iii) *The base-change $X \otimes_F E$ is birational to $\mathbb{P}^2 \otimes_F E$.*
- (iv) *The set of rational points $X(F)$ is infinite.*
- (v) *If F is separably closed, the rational points are Zariski dense.*

Proof. The assertion (iv) is a consequence of Proposition 1.6, and (iv) follows from the existence of the line $L \subset X$. Over the field extension E, we set $x_1' = y_1 + \sqrt{t_1} x_1$. In the new indeterminates x_1', y_1, x_2, y_2 our cubic surface is given by the equation $y_1 x_1^2 + y_2^3 + t_2 x_2^2 y_2 = 0$. Localizing with respect to x_1 we see that y_1 can be expressed by the other three indeterminates. This ensures that the base-change $X \otimes_F E$ is a rational surface, hence (iii).

We next verify that the scheme X is regular. Recall that the scheme of non-smoothness $D = \operatorname{Sing}(X/F)$ was described in Proposition 4.1. Consider first the non-rational closed point $a = (1 : 0 : \sqrt{t_1} : 0) \in D$. On the open set given by $x_1 \neq 0$, the cubic surface is defined by the inhomogeneous equation

$$\frac{y_1}{x_1}\left(\left(\frac{y_1}{x_1}\right)^2 + t_1\right) + \left(\frac{y_2}{x_1}\right)^3 + t_2\left(\frac{x_2}{x_1}\right)^2 \frac{y_2}{x_1} = 0,$$

and the polynomial on the left lies in the maximal ideal of $\mathfrak{m}_R$ of the local ring $R = \mathscr{O}_{\mathbb{A}^3,a}$, but not in $\mathfrak{m}_R^2$. In turn, $\mathscr{O}_{X,a}$ is regular. By symmetry, the same holds at the closed point $b = (0 : 1 : 0 : \sqrt{t_2})$. According to Lemma 1.7, it suffices to verify that the scheme $D \smallsetminus \{a, b\}$ is regular. This lies in the open set given by $y_1, y_2 \neq 0$, hence equals the spectrum of the ring

$$F[u,v,w^{\pm 1}]/(1+t_1u_1^2, 1+t_2u_2^2)$$

where we set $u_1 = x_1/y_1$ and $u_2 = x_2/y_2$ and $w = y_1/y_2$. Clearly, this ring is isomorphic to the ring of Laurent polynomials in w over the tensor product $A = F(\sqrt{t_1})\otimes_F F(\sqrt{t_2})$. The latter is a field, because $t_1, t_2 \in F$ are p-independent, hence $D \smallsetminus \{a,b\}$ is indeed regular. This establishes (i).

It remains to verify (ii), which is the most interesting part. For this we apply Corollary 3.2 to our fibration $f: X\to\mathbb{P}^1$. Let us examine the fiber $f^{-1}(a)$ over the rational points $a = (\lambda : 1)$ with $\lambda^3\neq 1$, which means $a\notin\mathbb{P}^1(\mathbb{F}_4)$. According to (6) this is a conic $C\subset\mathbb{P}^2_F$ given by the equation

$$(1+\lambda^3)u_0^2 + t_1u_1^2 + \lambda t_2 u_2^2 = 0 \tag{7}$$

in some indeterminates u_0, u_1, u_2. Base-changing to the field extension $F' = F(\sqrt{\lambda})$, and making a linear change of variables, the equation can be rewritten as

$$v_0^2 + t_1v_1^2 + t_2v_2^2 = 0. \tag{8}$$

The short exact sequence (1) and Cartier's Equality ([15], Theorem 92 or [16], Theorem 26.10) reveal that the kernel for $\Omega^1_F\otimes F'\to\Omega^1_{F'}$ is at most one-dimensional. So without loss of generality, we may assume that $dt_1\in\Omega^1_{F'}$ remains non-zero. According to [17], Theorem 3.3 the conic $C\otimes_F F'$ is reduced, hence the same holds for C. Since the latter is geometrically non-reduced, it is not rational. Summing up, for almost all rational points $a\in\mathbb{P}^1$, the fiber $f^{-1}(a)$ is not rational.

We proceed with a similar computation for the generic fiber of $f: X\to\mathbb{P}^1$ and its Frobenius base-change. Regard $\mathbb{P}^1$ as the homogeneous spectrum of $F[y_1,y_2]$, and now write $\lambda = y_2/y_1$ for the transcendental generator of the function field. Then the generic fiber for $f: X\to\mathbb{P}^1$ is the conic given by (7) over $F(\lambda)$, and the generic fiber of the Frobenius base-change is given by the same equation over $F(\sqrt{\lambda})$. This is already defined over the subfield F, and we conclude that the Frobenius base-change $X\times_{\mathbb{P}^1}\mathbb{P}^1$ is birational to $C\times\mathbb{P}^1$, where $C\subset\mathbb{P}^2_F$ is the conic defined by the above equation. According to Proposition 1.5, the curve C is regular. Being geometrically non-reduced, it is not rational. Thus Corollary 3.2 applies, and we conclude that X is not unirational. □

Each rational point $a\in X\subset\mathbb{P}^3$ comes from a linear surjection $\varphi: F^4\to F$. Then the kernel $\mathrm{Ker}(\varphi)$ is three-dimensional; choosing a basis we obtain a rational map $\pi_a : X \dashrightarrow \mathbb{P}^2$. If moreover $\mathscr{O}_{X,a}$ is regular, the intersection $C_a = X\cap T_a(X)$ is a singular cubic curve in the tangent plane $T_a(X)\subset\mathbb{P}^3$. Note that these $C_a\subset T_a(X)$ are crucial in the work of Segre [18], Manin [14] and Kollár [11].

Proposition 4.5. *The rational map $\pi_a : X \dashrightarrow \mathbb{P}^2$ is purely inseparable if and only if $a\in L$. Moreover, the intersection C_a is not integral for every $a\in\mathrm{Reg}(X)$.*

Proof. Clearly each rational point $a\in L$ yields a purely inseparable map. Let $V\subset\mathbb{P}^3$ be the linear span of all rational points $a\in X$ with purely inseparable projection $\pi_a : X\dashrightarrow\mathbb{P}^2$. Seeking a contradiction, we assume $L\subsetneq V$. According to [11], Lemma 5.1 our cubic surface $X\subset\mathbb{P}^3$ can be described after some change of coordinates by an equation $\lambda y^3 + \sum_{i=1}^3 \lambda_i y x_i^2 = 0$ in certain new variables x_i, y

for some scalars $\lambda_i, \lambda \in F$. It follows that the scheme X is reducible, contradiction. This proves the first assertion.

Now suppose that $\mathscr{O}_{X,a}$ is regular, and write $a = (\alpha_1 : \beta_1 : \alpha_2 : \beta_2)$. Taking partial derivatives in (5), we see that the tangent plane $T_a(X) \subset \mathbb{P}^3$ is given by the equation $(\alpha_1^2 + t_1\beta_1^2)y_1 + (\alpha_2^2 + t_2\beta_2^2)y_2 = 0$. Without loss of generality, we may assume that the second coefficient does not vanish. In turn, the cubic curve $C_a \subset \mathbb{P}^2$ becomes the zero-locus of a polynomial $P(x_1, y_1, x_2)$ divisible by y_1. Thus C_a is not integral. □

The $a = (0 : 0 : 1 : \lambda) \in X$ with $\lambda \in \mathbb{F}_4^\times$ show that there are indeed rational points with $\mathscr{O}_{X,a}$ regular and $\pi_a : X \dashrightarrow \mathbb{P}^2$ separable. This reveals that, for regular cubic hypersurfaces in characteristic two, the implication

$$\begin{array}{c}\exists a \in X(F) \text{ with } \mathscr{O}_{X,a} \text{ regular} \\ \text{and } \pi_a : X \dashrightarrow \mathbb{P}^n \text{ separable}\end{array} \quad \Longrightarrow \quad X \text{ is unirational}$$

formulated in the remark on imperfect ground fields in [11], page 468, does not hold without an additional assumption. The problem seems to be that all C_a fail to be integral.

Let us close the paper with the following observations:

Proposition 4.6. *The Picard group* $\operatorname{Pic}(X)$ *is freely generated by the classes of the invertible sheaves* $\mathscr{O}_X(C_1)$ *and* $\mathscr{O}_X(L)$. *The resulting Gram matrix is* $\left(\begin{smallmatrix} 0 & 2 \\ 2 & -1\end{smallmatrix}\right)$, *and the anticanonical class is given by* $-K_X = L + C_1$.

Proof. Let $S \subset \operatorname{Pic}(X)$ be the subgroup generated by the effective Cartier divisors $C_1, L \subset X$. From the intersection numbers $L^2 = -1$, $C_1^2 = 0$ and $(C_1 \cdot L) = 2$ we see that $C_1, L \in S$ form a basis, with the Gram matrix from the assertion. Furthermore, we have $-K_X = L + C_1$. Our task is to show that $S \subset \operatorname{Pic}(X)$ is an equality.

Recall that we have a fibration $f : X \to \mathbb{P}^1$. The generic fiber is a twisted form of the double line, and its Picard group is generated by $\mathscr{O}_{X_\eta}(L)$. Likewise, all closed fibers are irreducible, and we conclude that $S \subset \operatorname{Pic}(X)$ has finite index.

Using $\operatorname{disc}(S) = -4$, we see that the discriminant group S^*/S has order four. Write $e_1, e_2 \in S$ for the basis corresponding to the Cartier divisors $C_1, L \subset X$, and $e_1^*, e_2^* \in S^*$ be the dual basis. One easily checks that $e_2^* = \frac{1}{2}e_1$ generates the discriminant group. Seeking a contradiction, we assume that this generator comes from an invertible sheaf $\mathscr{N}$. Then $(\mathscr{N} \cdot \mathscr{N}) - (\mathscr{N} \cdot \omega_X) = \frac{1}{2}(L \cdot C_1) = 1$ is odd. However, this number must be even by Riemann–Roch, contradiction. Thus $S = \operatorname{Pic}(X)$. □

The scheme of non-smoothness $D = \operatorname{Sing}(X/F)$ is disjoint from L and has $\deg(D/\mathbb{P}^1) = 4$. With the description of $\operatorname{Pic}(X)$ one infers that D is linearly equivalent to $C_1 + 2L$. Using this information, we can clarify the occurrence of singularities:

Proposition 4.7. *Let* $0 \leq n \leq 2$ *be the dimension of the subvector space generated by the* $dt_1, dt_2 \in \Omega^1_F$. *Then the scheme* X *satisfies the regularity condition* (R_n), *and we have the following implications:*

(i) *If* $n = 2$ *then the cubic surface* X *is regular.*
(ii) *If* $n = 1$ *then* X *is normal, and* $\mathscr{O}_{X,b}$ *is singular for some closed* $b \in D$.
(iii) *If* $n = 0$ *then the scheme* X *is non-normal, with singular locus* $\operatorname{Sing}(X) = D$.

Proof. Assertion (i) already appeared in Theorem 4.4. Now suppose that $n = 0$, such that both $t_1, t_2 \in F$ are squares. After a change of coordinates, we may assume that $t_1 = t_2 = 1$. Then for each rational point of the form $a = (\lambda : \lambda : \mu : \mu)$ the defining polynomial $P = x_1(x_1 - y_1)^2 + x_2(x_2 - y_2)^2$ lies in the square of the maximal ideal in $\mathscr{O}_{\mathbb{P}^3,a}$ and it follows that all the local rings $\mathscr{O}_{X,a}$, $a \in D$ are singular. Thus X is singular in codimension one, hence non-normal. This gives (iii).

Finally, assume that $n = 1$. Without restriction, we may assume that $dt_1 \neq 0$. Then $t_1 \in F$ is not a square, so the closed point $a = (\sqrt{t_1} : 0 : 1 : 0) \in X$ is non-rational. Consider the resulting local ring $R = \mathscr{O}_{\mathbb{P}^3,a}$. The defining polynomial (5) for the cubic surface obviously lies in $\mathfrak{m}_R$ but not in $\mathfrak{m}_R^2$, hence $\mathscr{O}_{X,a}$ is regular. If follows that the localization $\mathscr{O}_{X,\zeta}$ is regular as well, where ζ is the generic point of the scheme of non-smoothness $D = \mathrm{Sing}(X/F)$. Hence X satisfies (R_1), thus our cubic surface is normal.

It remains to verify that $\mathscr{O}_{X,b}$ is singular for some closed point $b \in X$. Seeking a contradiction, we assume that this does not hold. Then suppose for a moment that $D = \mathrm{Sing}(X/F)$ is non-reduced. Since $D \subset X$ is Cartier, the reduction $E = D_{\mathrm{red}}$ is another effective Cartier divisor, and we have $D = nE$ for some integer $n \geq 2$. However, D is linearly equivalent to $C_1 + 2E$. This is primitive in the Picard group, contradiction. Thus we merely have to check that D is non-reduced. Its homogeneous coordinate ring is the tensor product $A = A_1 \otimes_F A_2$ with factors

$$A_i = F[x_i, y_i]/(x_i^2 - t_i y_i^2),$$

according to Proposition 4.1. Consider the field extension $E_1 = F(\sqrt{t_1})$. Then the map $A_1 \subset E_1[x_1]$ given by $y_1 \mapsto t_1 x_1$ is a finite ring extension inside the field of fractions. Since dt_2 is a multiple of dt_1, the scalar $t_2 \in E_1$ becomes a square, and we conclude that the rings $A \subset E_1[x_1, x_2, y_2]/(x_2 - \sqrt{t_2} y_2)^2$ are non-reduced. In turn, the scheme $D = \mathrm{Proj}(A)$ is non-reduced. □

Suppose that $t_1, t_2 \in F$ are p-independent, such that X is a *regular del Pezzo surface that is not geometrically normal*. The cone of curves $\mathrm{Eff}(X)$ is the real cone generated by the irreducible curves in the real vector space $N^1(X)_{\mathbb{R}} = \mathrm{Num}(X) \otimes \mathbb{R}$. In our situation the vector space has rank $\rho = 2$, and contains two extremal rays, which are generated by the fiber C_1 and the negative-definite curve L, compare [10], Lemma 4.12.

In turn there is precisely one minimal model $X \to Y$, which is the contraction of L. This is another regular del Pezzo surface that is not geometrically normal. Now the degree is $K_X^2 = 2$, and the anticanonical class generates $\mathrm{Pic}(X) = \mathbb{Z}$. Such examples are interesting, because they may occur as generic fibers in *Mori fiber spaces*. Note that over fields of p-degree $\mathrm{pdeg}(F) \leq 1$ there are no regular del Pezzo surfaces that are not geometrically normal, according to [5], Theorem 14.1. For more information on del Pezzo surfaces of degree two, we refer to the monographs of Manin [14], Dolgachev [4] and Kollár, Smith and Corti [12].

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