VARIETIES WITH FREE TANGENT SHEAVES

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Abstract. We coin the term *T-trivial varieties* to denote smooth proper schemes over ground fields k whose tangent sheaf is free. Over the complex numbers, this are precisely the abelian varieties. However, Igusa observed that in characteristic $p \leq 3$ certain bielliptic surfaces are T-trivial. We show that T-trivial varieties X separably dominated by abelian varieties A can exist only for $p \leq 3$. Furthermore, we prove that every T-trivial variety, after passing to a finite étale covering, is fibered in T-trivial varieties with Betti number $b_1 = 0$. We also show that if some n-dimensional T-trivial X lifts to characteristic zero and $p \geq 2n+2$ holds, it admits a finite étale covering by an abelian variety. Along the way, we establish several results about the automorphism group of abelian varieties, and the existence of relative Albanese maps.

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Introduction

Let k be a ground field of characteristic $p \geq 0$, and X be a smooth proper scheme with $h^0(\mathscr{O}_X) = 1$, of dimension $n \geq 0$. Recall that the term *K-trivial variety* often refers to those X where he dualizing sheaf $\omega_X = \det(\Omega^1_{X/k})$ is isomorphic to the structure sheaf. In this paper we are interested in the much stronger condition, where the sheaf of Kähler differentials $\Omega^1_{X/k}$ or equivalently the tangent sheaf $\Theta_{X/k} = \underline{\mathrm{Hom}}(\Omega^1_{X/k}, \mathscr{O}_X)$ itself are isomorphic to $\mathscr{O}_X^{\oplus n}$, in other words, these locally free sheaves are *free*. We find it convenient to coin the term *T-trivial varieties* for such X.

To simplify exposition, we assume that k is algebraically closed in the following discussion. For every abelian variety A, the group law gives an identification of the tangent sheaf with $\mathrm{Lie}(A) \otimes_k \mathscr{O}_A$, and consequently such schemes are T-trivial. In

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characteristic zero, the theory of Albanese maps easily shows that every T-trivial variety arises in this way. However, Igusa [34] noted that for $p \leq 3$ certain bielliptic surfaces $X = (E \times E')/G$ become T-trivial. Building on this, Li [42] conjectured that the T-trivial varieties in characteristic $p \geq 5$ are precisely the T-trivial varieties. Our paper is concerned with the following natural questions:

(i) Suppose a T-trivial variety X that is not an abelian variety arises as a quotient $X = A/G$ of an abelian variety. Does this imply $p \leq 3$?
(ii) What can be said about the Albanese map $X \to \mathrm{Alb}_{X/k}$ for T-trivial varieties X in characteristic $p > 0$?
(iii) Suppose a T-trivial variety X in characteristic $p > 0$ admits a lifting to characteristic zero. Does this imply that X is an abelian variety?

In their important work [45], Mehta and Srinivas answered (iii) affirmatively, under the assumption that X is ordinary and projective. This was refined by Joshi [36], who checked in characteristic $p \geq 5$ that every T-trivial surface is an abelian surface, and by Li [42], who established that for $p \geq 3$ that every ordinary T-trivial variety is an abelian variety.

Roughly speaking, our contributions are as follows: First, we establish that (i) indeed is true. Second, we show that every T-trivial variety X admits a finite étale covering $\tilde{X}$ where the fibers of the Albanese map are T-trivial varieties with Betti number $b_1 = 0$. Third, we show that (iii) holds if characteristic and dimension satisfy $p \geq 2\dim(X) + 2$.

Let us now describe these results in more detail. Our first main results clarifies if and in which ways Igusa's construction $X = (E \times E')/G$ might be carried out at other primes or in higher dimensions.

Theorem A. *(See Thm. 3.8) Let X be a T-trivial variety that is not an abelian variety, but has a finite surjection $A \to X$ from an abelian variety, with $k(X) \subset k(A)$ separable. Then the following holds:*

(i) *The characteristic satisfies $p \leq 3$.*
(ii) *The abelian variety A is not simple and contains a point of order p.*
(iii) *For $p = 3$ the T-trivial variety X is not ordinary, and the abelian variety A has a supersingular quotient.*

Indeed, for Igusa's bielliptic surface $X = (E \times E')/G$ one needs on $A = E \times E'$ a point of order p, and for $p = 3$ it turns out that the factor E must be supersingular. Our result also answers a question of Joshi ([36], Section 6). Furthermore, one obtains a new proof for Li's Theorem mentioned above ([42], Theorem 4.2).

Note that quotients $X = A/G$ as above are sometimes called *hyperelliptic varieties* [40]. Over the complex numbers, their classification problem boils down to understand discontinuous groups of affine transformations inside the semidirect product $\mathbb{C}^n \rtimes \mathrm{GL}_n(\mathbb{C})$, a topic that received considerable attention in dimension three ([59], [13], [14], [15], [3]).

The proof for the above theorem relies on a statement on automorphism groups of abelian varieties, which seems to be of independent interest:

Theorem B. *(See Thm. 2.1) Let A be an abelian variety in characteristic $p > 0$. Suppose the kernel of $\mathrm{Aut}(A) \to \mathrm{GL}(\mathrm{Lie}(A))$ contains a non-trivial element h of finite order. Then the following holds:*

(i) *The characteristic must be $p \leq 3$.*
(ii) *The order of $h \in \mathrm{Aut}(A)$ is a p-power.*
(iii) *For $p = 3$ the abelian variety $A/\operatorname{Ker}(\mathrm{id} - h)$ is supersingular.*

This can be seen as a variant of Minkowski's Theorem ([46], Section 1), which states that an integral matrix $H \in \mathrm{Mat}_n(\mathbb{Z})$ with $H \equiv E$ modulo some integer $r \geq 3$ is already the identity matrix, and Serre's result ([24], Appendix), which asserts that an automorphism $h \in \mathrm{Aut}(A)$ that is the identity on the group scheme $A[r]$ for some $r \geq 3$ is already the identity. Our proof relies on the *Weil Conjectures* for abelian varieties over prime fields, and number-theoretical properties of the eigenvalues of Frobenius, the so-called *Weil numbers*.

Our third main results implies that if Li's Conjecture fails, it must fail in a spectacular way:

Theorem C. *(See Thm. 5.1) For every T-trivial variety X, there is a finite étale covering $X' \to X$ such that the fibers of the Albanese map $X' \to \mathrm{Alb}_{X'/k'}$ are T-trivial varieties with Betti number $b_1 = 0$.*

The T-trivial varieties with $b_1 = 0$ would be extremely remote from the situation over the complex numbers, and from any Igusa-type construction in positive characteristics as well. Except for singletons, we have no clue so far whether or not such bizarre objects exist.

For the proof for the above result we use the theory of *relative Albanese map*. It is now high time to dismiss the assumption that the ground field k is algebraically closed. Recall that a smooth proper scheme P with $h^0(\mathscr{O}_P) = 1$ is called *para-abelian variety* if, for some field extension $k \subset k'$, the base-change $P' = P \otimes k'$ can be endowed with a group law, and thus becomes an abelian variety. This notion was developed in [41] and [56], and actually goes back to Grothendieck [25]. It turns out that $G = \mathrm{Aut}^0_{P/k}$ is an abelian variety, acting freely and transitively on P. Summing up, para-abelian varieties allow for an intrinsic way to handle torsors with respect to abelian varieties. In connection to T-trivial varieties, which in the first place have no distinguished point, it is indeed preferable to work with para-abelian varieties instead of abelian varieties.

For proper flat morphism $f : X \to S$ of finite presentation, the *relative Albanese variety* is a family of para-abelian varieties $\mathrm{Alb}_{X/S}$, and the *relative Albanese map* is a universal arrow $X \to \mathrm{Alb}_{X/S}$ to such families. Their existence depends on particular properties of $\mathrm{Pic}_{X/S}$, which do not always hold ([41], Theorem 10.2). In Corollary 4.3, we provide a new unconditional statement in characteristic zero. For T-trivial varieties X with $p > 0$, we seek to form the relative Albanese variety with respect to the absolute Albanese map, but in this setting existence is unclear. The following work-around, which relies on the *Weil Extension Theorem*, seems to be of independent interest:

Theorem D. *(See Thm. 4.6) Suppose that S is normal, and that the generic fiber X_η contains a rational point. After removing a closed set $Z \subset S$ of codimension at least two, $P_\eta = \mathrm{Alb}_{X_\eta/\kappa(\eta)}$ extends to a family of abelian varieties P over S, and the Albanese map $g_\eta : X_\eta \to P_\eta$ extends to a morphism $g : X \to P$.*

Our last main result connects the theory of T-trivial varieties with lifting properties. Suppose X is smooth and projective over an algebraically closed ground field of characteristic $p \geq 0$. For $k = \mathbb{C}$ it follows from Yau's proof of the Calabi Conjecture that X admits a finite étale covering by an abelian variety if and only the Chern classes c_1 and c_2 vanish. It is unclear to what extend the reverse implication holds true for $p > 0$. We show:

Theorem E. *(See Thm. 6.1) In the above situation, suppose the following holds:*

(i) *For some $\ell \neq p$, the ℓ-adic Chern classes c_1 and c_2 both vanish.*
(ii) *The scheme X projectively lifts to characteristic zero.*
(iii) *Characteristic and dimension satisfy $p \geq 2n + 2$.*

Then there is a finite étale covering $A \to X$ by some abelian variety A.

Note that condition (i) automatically holds if X is T-trivial. In dimension $n = 2$, we see that every T-trivial surface in characteristic $p \geq 7$ arises from an abelian surface. Note that this already actually holds for $p \geq 5$, by the Bombieri–Mumford generalization of the Bagnera–de Francis classification for bielliptic surface ([7], Section 3).

The paper is organized as follows: In Section 1 we collect generalities on Weil restriction, free sheaves, abelian varieties, ℓ-adic cohomology, and algebraic fundamental groups. Section 2 contains results on automorphisms of abelian varieties that act trivially on the Lie algebra. We introduce the notion of T-trivial varieties X and establish their basic properties in Section 3. There we also establish some structure results if it is dominated by some abelian variety. In Section 4 the theory of relative Albanese maps is developed further. This is used in Section 5 to obtain a splitting result where T-trivial varieties with $b_1 = 0$ appear. In the final Section 6, we connect the theory of T-trivial varieties with liftings to characteristic zero.

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1. Generalities

In this section we collect some general facts on Weil restriction, free sheaves, abelian varieties, ℓ-adic cohomology and algebraic fundamental groups that are relevant throughout, and perhaps of independent interest. For simplicity, we work over a ground field k, of characteristic $p \geq 0$.

Let us start with *Weil restrictions*. Suppose $k_0 \subset k$ is a subfield such that the degree of the extension is finite. For each scheme or algebraic space X over k, the functor

$$(\mathrm{Aff}/k_0) \longrightarrow (\mathrm{Set}), \quad R_0 \longmapsto X(R_0 \otimes_{k_0} k)$$

is called *Weil restriction* $X_0 = \mathrm{Res}_{k/k_0}(X)$. It is representable by an algebraic space ([35], Theorem 6.5.2), which we denote by the same symbol. Note that X_0 is schematic provided that X has the *AF-property*, which means that every finite set of points admits a common affine neighborhood. In any case, Weil restriction is right adjoint to base-change, such that $\mathrm{Hom}(Y, \mathrm{Res}_{k/k_0}(X)) = \mathrm{Hom}(Y \otimes_{k_0} k, X)$. The following property is most useful (loc. cit., Theorem 6.1.5):

Lemma 1.1. *Suppose $k_0 \subset k$ is separable, with Galois closure k'. Then*

$$\operatorname{Res}_{k/k_0}(X) \otimes_{k_0} k' = \prod_{\iota : k \to k'} (X \otimes_k k'), \tag{1}$$

where the product runs over all k_0-embeddings $\iota : k \to k'$.

If $k_0 \subset k$ is already Galois, the indices ι become the elements from the Galois group. In any case, the ι form a set of cardinality $[k : k_0]$, according to [8], Chapter V, §6, No. 3, Corollary to Proposition 1, giving the dimension formula

$$\dim(X_0) = [k : k_0] \cdot \dim(X).$$

Moreover, $X_0 = \operatorname{Res}_{k/k_0}(X)$ is the quotient of (1) by the canonical action of $G' = \operatorname{Gal}(k'/k_0)$, which is free. Furthermore, we see by descent that if X is smooth and proper with $h^0(\mathscr{O}_X) = 1$, the same holds for the Weil restriction $X_0 = \operatorname{Res}_{k/k_0}(X)$. Also note that the identification (1) is natural: Given $f : X \to Y$, the induced morphism $f_0 = \operatorname{Res}_{k/k_0}(f)$ on Weil restrictions has the property

$$f_0 \otimes_{k_0} k' = \prod_{\iota : k \to k'} (f \otimes_k k') \tag{2}$$

We next turn to *free sheaves*. Recall that a locally free sheaf $\mathscr{E}$ of rank $r \geq 0$ on a scheme X is called *free* if it is isomorphic to $\bigoplus_{i=1}^r \mathscr{O}_X$. The following categorical observation is useful:

Lemma 1.2. *Suppose $h^0(\mathscr{O}_X) = 1$. Then the functor $V \mapsto V \otimes_k \mathscr{O}_X$ from the abelian category finite-dimensional k-vector spaces V to the abelian category of quasicoherent sheaves $\mathscr{E}$ on X is exact and fully faithful, and its essential image is the category of free sheaves of finite rank.*

Proof. The statement on the essential image is obvious. The functor is fully faithful, because both $\operatorname{Hom}(k^n, k^m)$ and $\operatorname{Hom}(\mathscr{O}_X^{\oplus n}, \mathscr{O}_X^{\oplus m})$ are given by $m \times n$-matrices (φ_{ij}) with entries from $\operatorname{Hom}(k, k) = k = \Gamma(X, \mathscr{O}_X) = \operatorname{Hom}(\mathscr{O}_X, \mathscr{O}_X)$. The functor is exact since the structure map $X \to \operatorname{Spec}(k)$ is flat. $\square$

It follows that all short exact sequences $0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$ with free sheaves of finite rank are split. Moreover, a map $\varphi : \mathscr{E} \to \mathscr{F}$ of free sheaves of finite rank is injective or surjective provided that corresponding property holds after tensoring with the residue field of some point $a \in X$. The following useful observation is essentially contained in [50], Lemma 4.2:

Proposition 1.3. *Suppose $h^0(\mathscr{O}_X) = 1$, and let $\mathscr{E}$ be quasicoherent sheaf on X. If for some field extension $k \subset k'$, the base-change $\mathscr{E}' = \mathscr{E} \otimes k'$ to $X' = X \otimes k'$ becomes free of rank $r \geq 0$, the same already holds for $\mathscr{E}$.*

Proof. We have $h^0(\mathscr{E}) = \dim_{k'} H^0(X', \mathscr{E}') = \dim_{k'} H^0(X', \mathscr{O}_{X'}) \cdot r = r$. Choose a basis $s_1, \ldots, s_r \in H^0(X, \mathscr{E})$, and consider the resulting homomorphism of quasicoherent sheaves

$$k^r \otimes_k \mathscr{O}_X \xrightarrow{s_1, \ldots, s_r} H^0(X, \mathscr{E}) \otimes_k \mathscr{O}_X \xrightarrow{\text{can}} \mathscr{E}.$$

The map on the left is bijective by construction, and the map on the right becomes bijective after base-change to k'. It follows that both maps are bijective, hence $\mathscr{E}$ is free of rank r. $\square$

Let us also record:

Proposition 1.4. *Suppose $h^0(\mathscr{O}_X)=1$. Let $\mathscr{E}$ be locally free sheaf of finite rank on X that is globally generated, and assume that there is a scheme X' with $h^0(\mathscr{O}_{X'})=1$, and a surjection $f:X'\to X$ such that the pullback $\mathscr{E}'=f^*(\mathscr{E})$ is free. Then $\mathscr{E}$ is free.*

Proof. Set $r=\mathrm{rank}(\mathscr{E})$ and $V=H^0(X,\mathscr{E})$. The canonical map $V\otimes_k\mathscr{O}_X\to\mathscr{E}$ is surjective, and the same holds for its pullback to X'. By Lemma 1.2, there is a vector subspace $U\subset V$ such that the pullback of $\varphi:U\otimes_k\mathscr{O}_X\to\mathscr{E}$ is bijective. We may regard $s=\det(\varphi)$ as a section for the invertible sheaf $\mathscr{L}=\det(\mathscr{E})$, and the task is to verify that is has no zeros. Let $a\in X$ be a point and $a'\in X'$ be some preimage. By construction $s(a')=s(a)\otimes\kappa(a')$ does not vanish, hence $s(a)\neq 0$. □

We next turn to *abelian varieties* A in positive characteristics $p>0$. Then $\mathrm{Lie}(A)$ is a *restricted Lie algebra*, having zero brackets $[x,y]=0$ and some *p-map* $x\mapsto x^{[p]}$. One says that A is *superspecial* if $\mathrm{Lie}(A)$ is isomorphic to k^g, the restricted Lie algebra where both bracket and p-map are zero. In dimension $g=1$, this are precisely the supersingular elliptic curves. Also note that up to twists, there is but one superspecial abelian variety in each dimension $g\geq 2$, namely the product of supersingular elliptic curves, a result attributed to Deligne ([58], Theorem 3.5, see also [51], Theorem 6.2). One calls A *supersingular* if and only if it is isogeneous to such a product $E_1\times\ldots\times E_g$ of supersingular elliptic curves. This condition can be rephrased in terms of Dieudonné modules, and holds over k if and only if it holds over k^{alg} ([52], Theorem 4.2 and [62], Theorem 1.2). The following observation will be useful:

Lemma 1.5. *For the g-dimensional abelian variety A, the following are equivalent:*

(i) *There is a non-zero supersingular quotient A/N.*

(ii) *Some abelian subvariety in $A\otimes k^{\mathrm{alg}}$ has a non-zero supersingular quotient.*

Under these equivalent conditions, the map $H^g(A^{(p)},\mathscr{O}_{A^{(p)}})\to H^g(A,\mathscr{O}_A)$ induced by the relative Frobenius $F:A\to A^{(p)}$ is zero.

Proof. The implications (i)⇒(ii) is trivial. For the converse, let $k\subset k'$ be any field extension, set $A'=A\otimes k'$, and consider the ordered set of abelian subvarieties $N'_\lambda\subset A'$, $\lambda\in L$ such that A'/N'_λ is supersingular. For any two members N'_λ and N'_μ, the abelian variety $(N'_\lambda\cap N'_\mu)^0_{\mathrm{red}}$ belongs to the family, because the class of supersingular abelian varieties are stable under products and subvarieties. In turn, our collection contains a smallest member $N'_0\subset A'$. Obviously, this is stable under the action of $G=\mathrm{Aut}(k'/k)$.

For $k'=k^{\mathrm{alg}}$, the Poincaré Irreducibility Theorem easily implies that A'/N'_0 is non-zero. Via Galois descent we see that the base-change to the perfect closure k^{perf} admits a non-zero supersingular quotient.

Changing notation, we find a finite purely inseparable field extension $k\subset k'$ with a non-zero supersingular quotient $f':A\otimes k'\to B'$. The corresponding homomorphism $f:A\to\mathrm{Res}_{k'/k}(B')$ factors over the abelian variety $\bar{A}=\mathrm{Im}(f)$. Pulling back we obtain

$$A\otimes k'\longrightarrow\bar{A}\otimes k'\longrightarrow\mathrm{Res}_{k'/k}(B)\otimes k'\longrightarrow B.$$

The map from $\bar{A}\otimes k'$ to B is surjective, because $A\otimes k' \to B$ is surjective, whereas the map to $\mathrm{Res}_{k'/k}(B)\otimes k'$ is a closed embedding, because this holds for $\bar{A} \to \mathrm{Res}_{k'/k}(B)$. Using that the kernel for the adjunction $\mathrm{Res}_{k'/k}(B)\otimes k' \to B$ is affine ([17], Appendix A, Proposition 5.11), we conclude that $\bar{A} \otimes k' \to B$ is an isogeny, and infer that $\bar{A}$ is supersingular.

For the remaining statement, suppose that $B = A/N$ is a supersingular quotient. Passing to a further quotient, we may assume that B is supersingular elliptic curve, so the canonical map $H^1(B^{(p)}, \mathscr{O}_{B^{(p)}}) \to H^1(B, \mathscr{O}_B)$ induced by the relative Frobenius, which is identical to the Verschiebung on $\mathrm{Pic}^0_{B/k} = B$, vanishes. The cup product $\Lambda^g H^1(A, \mathscr{O}_A) \to H^g(A, \mathscr{O}_A)$ is bijective (see for example [55], Proposition 2.3). Passing to the Stein factorization for the projection $A \to B$, we may furthermore assume that the canonical map $H^1(B, \mathscr{O}_B) \to H^1(A, \mathscr{O}_A)$ is injective, and the same for Frobenius pullbacks. Choose a non-zero $\epsilon_1 \in H^1(B^{(p)}, \mathscr{O}_{B^{(p)}})$ and extend it to a basis $\epsilon_1, \ldots, \epsilon_g \in H^1(A^{(p)}, \mathscr{O}_{A^{(p)}})$. The relative Frobenius vanishes on the generator $\epsilon_1 \cup \ldots \cup \epsilon_g$ of $H^g(A^{(p)}, \mathscr{O}_{A^{(p)}})$, because it vanishes the first factor. $\square$

If the equivalent conditions of Lemma 1.5 hold, we say that A *has a supersingular quotient*. This property played a crucial role in [53], where the term *has a supersingular factor* was used.

Over finite ground fields $k = \mathbb{F}_q$, supersingularity can be characterized in terms of Frobenius eigenvalues: Write $q = p^\nu$, and let $\Phi : A \to A$ be the ν-th power of the absolute Frobenius map. Fix a prime $\ell \neq p$ and consider the induced $\mathbb{Q}_\ell$-linear endomorphisms on $H^i(A\otimes k^{\mathrm{alg}}, \mathbb{Q}_\ell)$. For each embedding $\mathbb{Q}_\ell \subset \mathbb{C}$ the resulting eigenvalues $\alpha_1, \ldots, \alpha_{2g} \in \mathbb{C}$ are algebraic integers and have absolute value $|\alpha_j| = p^{i/2}$, according to [19], Corollary 3.3.9. An algebraic integer $\alpha \in \mathbb{C}$ all whose conjugates α' have $|\alpha'| = p^{i/2}$ are called *Weil numbers of weight i*. Those of the particular simple form $\alpha = \zeta \cdot p^{i/2}$ for some root of unity ζ are called *supersingular*. By [61], Theorem 2.9 the abelian variety A is a supersingular if and only if the Weil numbers $\alpha_1, \ldots, \alpha_{2g} \in \mathbb{C}$ are supersingular.

We next come to *ℓ-adic cohomology*, where $\ell > 0$ is a fixed prime that is invertible in the ground field k. Let X be a scheme, for the sake of exposition assumed to be proper. Recall that $\mu_{\ell^\nu} = \mu_{X,\ell^\nu}$, $\nu \geq 0$ denotes the sheaf of ℓ^ν-th roots of unity on the site (Et/X) of étale X-schemes, endowed with the étale topology. The *étale cohomology groups* $H^i(X, \mu_{\ell^\nu}^{\otimes j})$, $\nu \geq 0$ form an inverse system of $\mathbb{Z}_\ell$-modules, and one defines the *ℓ-adic cohomology* as

$$H^i(X, \mathbb{Z}_\ell(j)) = \varprojlim_\nu H^i(X, \mu_{\ell^\nu}^{\otimes j}) \quad \text{and} \quad H^i(X, \mathbb{Q}_\ell(j)) = H^i(X, \mathbb{Z}_\ell(j)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

To discard arithmetical contributions, one frequently considers the cohomology groups $H^i(X \otimes k^{\mathrm{alg}}, \mathbb{Z}_\ell(j))$ and $H^i(X \otimes k^{\mathrm{alg}}, \mathbb{Q}_\ell(j))$, which are finitely generated. The resulting *Betti numbers* are

$$b_i(X) = \dim_{\mathbb{Q}_\ell} H^i(X \otimes k^{\mathrm{alg}}, \mathbb{Q}_\ell(j))$$

Here both ℓ and j are irrelevant, the latter because $H^i(X \otimes k^{\mathrm{alg}}, \mathbb{Z}_\ell(j))$ is obtained from $H^i(X \otimes k^{\mathrm{alg}}, \mathbb{Z}_\ell)$ by tensoring with the invertible $\mathbb{Z}_\ell$-module $\mathbb{Z}_\ell(j) = \varprojlim_\nu (\mu_{\ell^\nu}(k^{\mathrm{alg}})^{\otimes j})$. Also note that in all this one might use k^{sep} instead of k^{alg}.

Although not via cycle class maps, cohomology in degree one has the following well-known significance:

Lemma 1.6. *Suppose X is geometrically normal with $h^0(\mathscr{O}_X) = 1$. Then the first Betti number is given by $b_1(X) = 2\dim(\mathrm{Pic}_{X/k})$.*

Proof. First note that the Kummer sequence $0 \to \mu_{\ell^\nu} \to \mathbb{G}_m \xrightarrow{\ell^\nu} \mathbb{G}_m \to 0$ yields an exact sequence

$$0 \longrightarrow k^\times/k^{\times\ell^\nu} \longrightarrow H^1(X, \mathbb{Z}/\ell^\nu\mathbb{Z}(1)) \longrightarrow \mathrm{Pic}(X)[\ell^\nu] \longrightarrow 0. \tag{3}$$

Next recall that the connected component $P = \mathrm{Pic}^\tau_{X/k}$ is a group scheme of finite type. It is actually proper (see for example [41], Proposition 2.3) hence an extension of a finite group scheme G by an abelian variety A. We thus have exact sequences $0 \to A[\ell^\nu] \to P[\ell^\nu] \to G[\ell^\nu]$ of group schemes, and obtain exact sequences

$$0 \longrightarrow \varprojlim_\nu A[\ell^\nu](k) \longrightarrow \varprojlim_\nu P[\ell^\nu](k) \longrightarrow \varprojlim_\nu G[\ell^\nu](k)$$

of groups, where the terms on the right are finite. To proceed, we may assume that k is algebraically closed. Then the term on the left is a free $\mathbb{Z}_\ell$ module of rank $2\dim(A)$. The term in the middle can be identified with $H^i(X, \mathbb{Z}_\ell(1))$ by (3), and the assertion follows. □

Suppose now that X is connected. Let $a : \mathrm{Spec}(\Omega) \to X$ be a geometric point, and $\pi_1(X, a)$ be the ensuing *algebraic fundamental group*, introduced by Grothendieck ([30], Exposé V, Section 7) as automorphism group of the fiber functor $V \mapsto V(\Omega)$, which sends a finite étale covering V to its fiber with respect to a. This functor indeed yields an equivalence between the category of finite étale coverings $V \to X$ and the category of finite sets endowed with a continuous $\pi_1(X, a)$-action.

An abelian sheaf F_ν on the étale site (Et/X) is called *ℓ^ν-local system* if it is a twisted form of the constant sheaf $(\mathbb{Z}/\ell^\nu\mathbb{Z})^{\oplus r}_X$, for some rank $r \geq 0$. An *ℓ-adic local system* is an inverse system $F = (F_\nu)$, where the entries are ℓ^ν-local systems, and the transition maps yield identifications $F_\nu = F_\mu \otimes (\mathbb{Z}/\ell^\nu\mathbb{Z})_X$ whenever $\mu \geq \nu$. Our geometric point $a : \mathrm{Spec}(\Omega) \to X$ yields the *monodromy representation* $\pi_1(X, a) \to \mathrm{GL}(F_\nu \mid \Omega)$. This actually gives an equivalence between the additive category of ℓ^ν-local systems of rank $r \geq 0$ and the additive category of continuous representations $\pi_1(X, a) \to \mathrm{GL}_r(\mathbb{Z}/\ell^\nu\mathbb{Z})$. For ℓ-adic local systems $F = (F_\nu)$, one obtains an equivalence to the additive category of continuous representations

$$\pi_1(X, a) \longrightarrow \varprojlim_\nu \mathrm{GL}_r(\mathbb{Z}/\ell^\nu\mathbb{Z}) = \mathrm{GL}_r(\mathbb{Z}_\ell).$$

Localizing the category of ℓ-adic sheaves by tensoring the Hom sets with $\mathbb{Q}_\ell$, one obtains an equivalence to the category of continuous representations in $\mathrm{GL}_r(\mathbb{Q}_\ell)$. By abuse of notation, we also writes $\mathbb{Q}_{\ell,X}$ for the ℓ-adic local system $(\mathbb{Z}/\ell^\nu\mathbb{Z})_X$, $\nu \geq 0$ in the localized category.

For each smooth proper morphism $f : X \to Y$ and each ℓ^ν-local system F_ν on X, the higher direct images $R^if_*(F_\nu)$ are ℓ^ν-local systems on Y. Moreover, for each cartesian diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

the Proper Base-Change Theorem yields an identification $R^if_*(F_\nu)|Y' = R^if'_*(F_\nu|X')$, with likewise statements for ℓ-adic local systems $F = (F_\nu)$.

Finally, suppose that X is proper over the ground field k, with $h^0(\mathscr{O}_X) = 1$. Choose an algebraic closure k^{alg}, fix a closed point x_0 on the base-change $X \otimes k^{\text{alg}}$, and set $S = \operatorname{Spec}(k)$. The structure morphism $X \to S$ induces a short exact sequence

$$1 \longrightarrow \pi_1(X \otimes k^{\text{alg}}, x_0) \longrightarrow \pi_1(X, x_0) \longrightarrow \pi_1(S, x_0) \longrightarrow 1, \tag{4}$$

where the term on the right becomes the Galois group $\operatorname{Gal}(k^{\text{sep}}/k) = \operatorname{Aut}(k^{\text{alg}}/k)$. To simplify notation we set

$$\Pi^{\text{alg}} = \pi_1(X \otimes k^{\text{alg}}, x_0) \quad \text{and} \quad \Pi = \pi_1(X \otimes k^{\text{alg}}, x_0) \quad \text{and} \quad \Gamma = \pi_1(S, x_0).$$

Conjugacy defines a homomorphism $\Gamma \to \operatorname{Out}(\Pi^{\text{alg}})$. As explained in [12], Chapter IV, Section 6, the isomorphism classes of extensions with such an outer representation become a principal homogeneous space with respect to the cohomology group $H^2(\Gamma, Z(\Pi^{\text{alg}}))$, formed with respect to the center.

Recall that the open subgroups $H \subset \Pi$ are precisely the closed subgroups of finite index. By Galois theory, the transitive Π-set Π/H for such subgroups correspond to the finite étale covering $X' \to X$ with connected total space. Moreover,

$$\Gamma/H \quad \text{and} \quad \bigcup_{\Pi^{\text{alg}}\cdot\sigma\cdot H} \Pi^{\text{alg}}/(\Pi^{\text{alg}} \cap \sigma H \sigma^{-1})$$

correspond to the étale k-algebra $k' = H^0(X', \mathscr{O}_{X'})$ and the base-change $X' \otimes k^{\text{alg}}$, respectively. The kernel $N \subset \Pi$ for the permutation representation on Π/H is the largest normal subgroup contained in H, which is also open, and gives the Galois closure X'' for X'. Set $k' = H^0(X', \mathscr{O}_{X'})$ and $k'' = H^0(X'', \mathscr{O}_{X''})$. Note that we may easily have $h^0(\mathscr{O}_{X''}) > h^0(\mathscr{O}_{X'})$, compare the discussion [18], Section 2.8. To get rid of such constant field extensions one may use the following observation:

Lemma 1.7. *The canonical morphism $X'' \to X \otimes_k k''$ is a finite étale Galois covering, the k''-vector spaces $H^0(X'', \mathscr{O}_{X''})$ and $H^0(X \otimes k'', \mathscr{O}_{X \otimes k''})$ are one-dimensional*

Proof. Both schemes are étale over X, hence the morphism is étale by [29], Proposition 17.3.4. Let $N \subset \Pi$ be the open normal subgroup corresponding to the composite map $X'' \to X$, and $\Gamma'' \subset \Gamma$ be its image. Then $X \otimes k''$ corresponds to the subgroup $\Pi'' = \Pi \times_\Gamma \Gamma''$ of Π, and $X'' \to X \otimes_k k''$ corresponds to $N \subset \Pi''$. The first statement follows. The statement on $H^0(X'', \mathscr{O}_{X''})$ is trivial, and the statement on $H^0(X \otimes k'', \mathscr{O}_{X \otimes k''})$ follows from $h^0(\mathscr{O}_X) = 1$. ∎

2. Automorphisms of abelian varieties

The goal of this section is to establish several results on automorphisms of abelian varieties, which will play a crucial role in the next section, and appears to be of independent interest.

Let k be a ground field of characteristic $p \geq 0$, and A be an abelian variety of dimension $g \geq 0$. It comes with an associative algebra $\operatorname{End}(A)$, and its unit group $\operatorname{Aut}(A)$ is a countable group. Each element fixes the neutral element $e \in A$, and thus stabilizes all infinitesimal neighborhood $\operatorname{Spec}(\mathscr{O}_{A,e}/\mathfrak{m}^{n+1})$. According to [44], Lemma in Section 3 the resulting linear representations

$$\operatorname{Aut}(A) \longrightarrow \operatorname{GL}(\mathscr{O}_{A,e}/\mathfrak{m}^{n+1})$$

are injective for $n \geq 0$ sufficiently large. Moreover, the kernels of the above maps define a series of normal subgroups. Throughout, we are interested in the case $n = 1$, where the above can also be seen as the canonical map $\operatorname{Aut}(A) \to \operatorname{GL}(\operatorname{Lie}(A))$. Our first main result reveals that the torsion inside the kernel is rather restricted:

Theorem 2.1. *Suppose $p > 0$, and that the kernel of $\operatorname{Aut}(A) \to \operatorname{GL}(\operatorname{Lie}(A))$ contains a non-trivial element h of finite order. Then the following holds:*

(i) *The characteristic must be $p \leq 3$.*
(ii) *The order of $h \in \operatorname{Aut}(A)$ is a p-power.*
(iii) *For $p = 3$ the abelian variety $A/\operatorname{Ker}(h - \mathrm{id})$ must be supersingular.*

Proof. The key idea is to understand the case where the ground field is a prime field, and then reduce to this situation via standard arguments. We proceed in three steps.

Step 1: *Suppose that $k = \mathbb{F}_p$ is the prime field.* First note that $A^{(p)} = A$ and that the relative and absolute Frobenius maps for A coincide. We thus have a short exact sequence $0 \to A[F] \to A \xrightarrow{F} A \to 0$. Also note that this sequence is functorial. Thus F belongs to the center of the associative algebra $\operatorname{End}(A)$, and every endomorphism stabilizes the Frobenius kernel $A[F]$.

Replacing our group element $h \in \operatorname{Aut}(A)$ of finite order by a suitable power, we may assume that $r = \operatorname{ord}(h)$ is prime. The difference $\mathrm{id} - h$ induces the zero map on $\operatorname{Lie}(A)$. In light of the Demazure–Gabriel Correspondence ([20], Chapter II, §7, Theorem 3.5), this actually means $A[F] \subset \operatorname{Ker}(\mathrm{id} - h)$. The Isomorphism Theorem gives an endomorphism $f : A \to A$ with $\mathrm{id} - h = f \circ F$, or equivalently

$$h = \mathrm{id} - f \circ F.$$

Fix a complex embedding $\mathbb{Q}_\ell^{\mathrm{alg}} \subset \mathbb{C}$ and consider the effect of f and F on the vector space $V = H^1(A \otimes k^{\mathrm{alg}}, \mathbb{Q}_\ell) \otimes \mathbb{C}$, which has dimension $2g$. Since f and F commute, there is a basis $e_1, \ldots, e_{2g} \in V$ such that the resulting matrices for f^* and F^* are both lower triangular ([8], Chapter VII, §5, No. 9, Proposition 19). Let $\alpha_1, \ldots, \alpha_{2g}$ and $\beta_1, \ldots, \beta_{2g}$ be the matrix entries on the diagonal for f^* and F^*, respectively. In turn, the matrix for h^* is lower triangular as well, and $1 - \alpha_i\beta_i$ are its diagonal entries. These are the eigenvalues for h^*, all of which are r-th roots of unity. If all of them are trivial, then h^* is the identity, because it is diagonalizable, and hence h is the identity ([48], Section 18, Theorem 3), contradiction.

Fix an index $1 \leq d \leq 2g$ for which the r-th root of unity $\zeta = 1 - \alpha_d\beta_d$ primitive. Since the characteristic polynomial for h^* belongs to $\mathbb{Z}[T]$, all conjugates of ζ appear among the $1 - \alpha_i\beta_i$. Choose indices $i_1, \ldots, i_{r-1}$ so that

$$\{1 - \alpha_{i_1}\beta_{i_1}, \ldots, 1 - \alpha_{i_{r-1}}\beta_{i_{r-1}}\} = \{\zeta^1, \ldots, \zeta^{r-1}\}.$$

Using this and the cyclotomic polynomial $P(T) = T^{r-1} + \ldots + 1 = \prod_{j=1}^{r-1}(T - \zeta^j)$ for substitutions, we obtain

$$r = P(1) = \prod_{j=1}^{r-1}(1-\zeta^j) = \prod_{j=1}^{r-1}\alpha_{i_j} \cdot \prod_{j=1}^{r-1}\beta_{i_j}.$$

Now recall that the β_{i_j} are Weil numbers, having $|\beta_{i_j}| = p^{1/2}$. Taking absolute values from the above equation gives $r = sp^{(r-1)/2}$, with the factor $s = \prod_{j=1}^{r-1}|\alpha_{i_j}|$. Each $\alpha = \alpha_{i_j}$ is an algebraic integer, so the same holds for the conjugate $\bar{\alpha}$ and the absolute value $|\alpha| = (\alpha \cdot \bar{\alpha})^{1/2}$. Consequently, the real number $s = rp^{(1-r)/2}$ is also an algebraic integer.

In the special case $r = 2$ the factor becomes $s = 2p^{-1/2}$, which is a root for $T^2 - 4/p \in \mathbb{Q}[T]$. This polynomial is irreducible regardless of p, hence must be the minimal polynomial for s. The latter is an algebraic integer, so $4/p$ must be an integer, and thus $p = 2 = r$. In the general case $r \geq 3$ our algebraic integer $s = rp^{(1-r)/2}$ already belongs to $\mathbb{Q}$, and thus is contained in $\mathbb{Z}$. From the uniqueness of prime factorization, we infer $r = p$ and $s = 1$ and $(r-1)/2 = 1$. The latter ensure $p = r = 3$. This establishes (i) and (ii).

It remains to verify (iii). Set $N = \operatorname{Ker}(h - \mathrm{id})$ and $A' = A/N$. Recall that by [61], Theorem 2.9 we have to verify that, with respect to all complex embeddings $\mathbb{Q}_\ell \subset \mathbb{C}$, the eigenvalues of Frobenius on $H^1(A' \otimes k^{\mathrm{alg}}, \mathbb{Q}_\ell) \otimes \mathbb{C}$ take the form $\xi \cdot p^{1/2}$ for some root of unity $\xi \in \mathbb{C}^\times$, and thus are supersingular Weil numbers.

Let us first reduce to the case that N is finite. Write $q : A \to A/N = A'$ for the quotient map. Since the commutative group schemes of finite type form an abelian category, there is a unique monomorphism $i : A' \to A$ such that $i \circ q = h - \mathrm{id}$. Our $h \in \mathrm{Aut}(A)$ induces by construction an automorphism $h' \in \mathrm{Aut}(A')$ of finite order, which satisfies the equation $q \circ (h - \mathrm{id}) = (h' - \mathrm{id}) \circ q$. We claim that $h' - \mathrm{id}$ has a finite kernel. Suppose this is not the case. To produce a contradiction, we may replace k by its algebraic closure. Using the surjectivity of $q : A \to A'$, we find a closed point $a \in A$ such that $q(a)$ has infinite order and belongs to $\operatorname{Ker}(h' - \mathrm{id})$. Then $(h - \mathrm{id})(a) \in N$, and thus $(h - \mathrm{id})^2(a) = 0$. On the other hand, the gcd of $(T-1)^2$ and $T^r - 1$ in $\mathbb{Q}[T]$ is $T - 1$, so there is an integer $m \geq 1$ and polynomials $Q(T), R(T) \in \mathbb{Z}[T]$ such that $Q(T) \cdot (T-1)^2 + R(T) \cdot (T^r - 1) = m(T-1)$. We conclude

$$m \cdot (h - \mathrm{id})(a) = (h - \mathrm{id})(m \cdot a) = (i \circ q)(m \cdot a) = i(m \cdot q(a)) = 0.$$

Since i is a closed immersion, we see that $q(a) \in A$ is a torsion point, contradiction. We may thus replace A (resp. h) by A' (resp. h') and suppose that N is finite.

From $0 = (h - \mathrm{id}) \circ (h^{p-1} + h^{p-2} + \ldots + \mathrm{id})$ we infer that the all eigenvalues $\zeta_i = 1 - \alpha_i\beta_i$ for h^* are primitive p-th roots of unity. Now we can exploit $p = 3$: For every complex embedding $\mathbb{Q}_\ell \subset \mathbb{C}$ we have $\zeta_i = e^{\pm 2\pi i/3}$. Set $\omega = e^{2\pi i/6}$. Using $-e^{2\pi i/3} = \omega^{-1}$ and $-e^{-2\pi i/3} = \omega$ and $\omega^{-1} + \omega = 1$ we compute

$$|1 - \zeta_i|^2 = (1 - e^{2\pi i/3})(1 - e^{-2\pi i/3}) = 3,$$

and see $1 - \zeta_i = \omega^{\pm 1} \cdot p^{1/2}$. In particular $|1 - \zeta_i| = p^{1/2}$, which coincides with $|\beta_i| = p^{1/2}$. From $\alpha_i\beta_i = 1 - \zeta_i$ we get $|\alpha_i| = 1$. Since this applies to all complex embeddings, the $\alpha_i \in \mathbb{C}$ are algebraic integers all whose conjugates lie on the unit circle, so they actually must be roots of unity ([60], Chapter I, Lemma 1.6). Setting $\xi_i = \alpha_i^{-1}\omega^{\pm 1}$, we get $\beta_i = (1-\zeta_i)/\alpha_i = \xi_i \cdot p^{1/2}$, as desired.

Step 2: *Suppose that the ground field k is finite.* Let $k_0 = \mathbb{F}_p$ be the prime field. Then $k_0 \subset k$ is a finite Galois extension, with cyclic $G = \mathrm{Gal}(k/k_0)$, say of order $m \geq 1$. We now form the Weil restriction $A_0 = \mathrm{Res}_{k/k_0}(A)$, which comes with an induced automorphism $h_0 = \mathrm{Res}_{k/k_0}(h)$. By Lemma 1.1 we have

$$A_0 \otimes_{k_0} k = \prod_{\sigma\in G} A_\sigma \quad \text{and} \quad h_0 \otimes_{k_0} k = \prod_{\sigma \in G} h_\sigma, \tag{5}$$

where A_σ denotes the abelian variety $A \otimes_k k$, with base-change via $\sigma : k \to k$. This shows that A_0 is para-abelian of dimension $g_0 = mg$. Using the image $e_0 \in A_0$ of the origin $e \in A$, it becomes an abelian variety, with $h_0 \in \mathrm{Aut}(A_0)$. Then $h_0 \neq \mathrm{id}$, while the induced action on $\mathrm{Lie}(A_0)$ is trivial, because this holds after base-change to k. Using step 1 with $h_0 \in \mathrm{Aut}(A_0)$, we immediate see $p \leq 3$. Furthermore, the order of h_0 must be a p-power, so by (5) this also holds for h. Finally, $A_0' = A_0/\mathrm{Ker}(h_0 - \mathrm{id})$ is supersingular, which carries over to

$$A_0' \otimes_{k_0} k = \prod_{\sigma \in G} A_\sigma / \mathrm{Ker}(h - \mathrm{id}).$$

Then each factor, and in particular $A/\mathrm{Ker}(h-\mathrm{id})$, is supersingular.

Step 3: *Now the ground field k is general.* Let $R_\lambda \subset k$, $\lambda \in L$ be the ordered set of all subrings that are finitely generated over the prime field $\mathbb{F}_p$. By [28], Theorem 8.8.2, we find some member $R = R_\lambda$ such that A and h arise from a family of abelian varieties $\mathfrak{A} \to \mathrm{Spec}(R)$ and some relative automorphism $h : \mathfrak{A} \to \mathfrak{A}$, which we denote by the same letter. Set $r = \mathrm{ord}(h_\eta)$. Since the generic fiber $\mathfrak{A}_\eta$ is schematically dense, we actually have $h^r = \mathrm{id}_{\mathfrak{A}}$. Localizing further, we may assume that $\mathrm{Ker}(h^i - \mathrm{id}_{\mathfrak{A}})$, $0 \leq i \leq r-1$ are flat. This ensures that for all $s \in S$ the element $h_s \in \mathrm{Aut}(\mathfrak{A}_s)$ also has order r. Moreover, the quotient $\mathfrak{B} = \mathfrak{A}/\mathrm{Ker}(h - \mathrm{id}_{\mathfrak{A}})$ exists as an algebraic space (see for example [41], Lemma 1.1), which is actually a family of abelian varieties.

The closed points $s \in S$ are Zariski dense, because S is a Jacobson space, and the residue fields $\kappa(s)$ must by finite, by Hilbert's Nullstellensatz. Applying step 2 with $h_s \in \mathrm{Aut}(\mathfrak{A}_s)$, we see that the characteristic satisfies $p \leq 3$, that the common order r is a p-power, and that the closed fiber $\mathfrak{B}_s$ is supersingular. In turn, for the finite flat group scheme $\mathfrak{B}[p]$ the closed fibers are geometrically connected. By [28], Theorem 9.7.7 the generic fiber is geometrically connected as well, which implies that $\mathfrak{B}_\eta$ is supersingular. □

We next study automorphisms of the smooth proper scheme P obtained from the abelian variety A by forgetting the group law, and express various phenomena in terms of group cohomology. First note that

$$\mathrm{Aut}(P) = A(k) \rtimes \mathrm{Aut}(A),$$

where the semidirect product is formed with respect to the canonical action of $\mathrm{Aut}(A)$ on the abelian group $A(k)$. We actually have $\mathrm{Aut}_{P/k} = A \rtimes \mathrm{Aut}_{A/k}$ as group schemes. Note that the group law for $\mathrm{Aut}(P)$ and its action on P are given by

$$(a,h)\cdot(a',h') = (a + h(a'), hh') \quad \text{and} \quad (a,h)\cdot x = a + h(x).$$

We now fix some element $h \in \mathrm{Aut}(A)$ of finite order. Each $a \in A(k)$ gives an element $g = (a, h)$ from $\mathrm{Aut}(P)$. Then $r = \mathrm{ord}(h)$ divides $\mathrm{ord}(g)$, and one immediately computes

$$g^i = (a + h(a) + \ldots + h^{i-1}(a), h^i) \qquad \text{and} \qquad g \cdot x = x \Longleftrightarrow a = x - h(x).$$

This shows:

Lemma 2.2. *The element $g = (a, h)$ from $\mathrm{Aut}(P)$ has order $r = \mathrm{ord}(h)$ if and only if $a \in \mathrm{Ker}(\mathrm{id} + h + \ldots + h^{r-1})$. In this situation, the scheme of fixed points for $g : P \to P$ is empty if and only if $a \notin \mathrm{Im}(\mathrm{id} - h)$.*

More precisely, $g : P \to P$ has no rational fixed point if and only if a is not the image of $\mathrm{id} - h : A(k) \to A(k)$.

Consider the cyclic group $G = \mathbb{Z}/r\mathbb{Z}$, and recall that for each G-module M the first group cohomology can be expressed as

$$H^1(G, M) = \mathrm{Ker}(1 + \sigma + \ldots + \sigma^{r-1}) / \mathrm{Im}(1 - \sigma),$$

where $\sigma : M \to M$ is the effect of the canonical generator in G. Using the G-module $M = A(k)$, we rephrase and refine the above lemma as follows:

Proposition 2.3. *The map $(a, h) \mapsto [a]$ gives a bijection between the set of $A(k)$-conjugacy classes in*

$$\{g \in \mathrm{Aut}(P) \mid \mathrm{ord}(g) = r, \text{ and } g = (a, h) \text{ for some } a \in A(k)\}, \tag{6}$$

and the cohomology group $H^1(G, A(k))$. Moreover, the automorphism $g : P \to P$ has no rational fixed points if and only if $[a] \neq 0$.

Proof. By Lemma 2.2, for each element $g = (a, h)$ from the set (6) the entry a belongs to the kernel of $1 + h + \ldots + h^{r-1}$, and thus gives a cohomology class $[a] \in H^1(G, A(k))$. Moreover, each cohomology class arises in this way. The conjugacy action in question is given by $(b, \mathrm{id}) \cdot (a, h) \cdot (b, \mathrm{id})^{-1} = (b + a - h(b), h)$, which shows that the map $(a, h) \mapsto [a]$ becomes injective when passing to $A(k)$-conjugacy classes. The statement on the fixed points also follows from the lemma. □

Corollary 2.4. *Suppose k is algebraically closed. Then each $g = (a, h)$ of order $r = \mathrm{ord}(h)$ is conjugate in $\mathrm{Aut}(P)$ to some $g' = (a', h)$ where $a' \in A(k)$ is annihilated by an r-power.*

Proof. Set $M = A(k)$, which is a divisible group. Let M' be the torsion submodule, and $M'' = M/M'$. This gives a short exact sequence $0 \to M' \to M \to M'' \to 0$ of G-modules, and thus an exact sequence

$$H^1(G, M') \longrightarrow H^1(G, M) \longrightarrow H^1(G, M'').$$

The torsion-free divisible group M'' carries a unique structure of a vector space over $\mathbb{Q}$, necessarily respected by the G-action. Thus $H^1(G, M)$ vanishes, because it is both a $\mathbb{Q}$-vector space and annihilated by $|G| = r$. Let $M'_0 \subset M'$ be the subgroup of elements annihilated by some r-power, and $M'_1 = M'/M'_0$. Arguing as above we see that $H^1(G, M'_0) \to H^1(G, M')$ is surjective, and the statement follows from the proposition. □

The following assertion will be useful as well:

Proposition 2.5. *The addition map of commutative group schemes*

$$\mathrm{Ker}(\mathrm{id}+h+\ldots+h^{r-1})\times\mathrm{Ker}(\mathrm{id}-h)\longrightarrow A$$

is surjective, and its kernel is isomorphic to a subgroup scheme inside $A[r]$.

Proof. Set $N=\mathrm{Ker}(\mathrm{id}+h+\ldots+h^{r-1})$ and $N'=\mathrm{Ker}(\mathrm{id}-h)$. The kernel of the the addition map is isomorphic to $N\cap N'$. This intersection is h-stable, and the induced map satisfies $h=\mathrm{id}$ and $\mathrm{id}+h+\ldots+h^{r-1}=0$, hence $r\cdot\mathrm{id}=0$, and thus $N\cap N'\subset A[r]$.

For the first assertion it suffices to verify $\dim(A)\leq\dim(N)+\dim(N')$, in light of the preceding paragraph. Set $B=\mathrm{Im}(\mathrm{id}+h+\ldots+h^{r-1})$. From the relation $(\mathrm{id}+h+\ldots+h^{r-1})(\mathrm{id}-h)=0$ we see $B\subset N'$. This gives $\dim(A)-\dim(N)=\dim(B)\leq\dim(N')$, as desired. □

The following consequence will be important in the next section:

Proposition 2.6. *Suppose* $g=(a,h)$ *has order* $r=\mathrm{ord}(h)$*, and that the fixed scheme for* $g:P\to P$ *is empty. Then the abelian variety* A *is not simple, and the finite group scheme* $A[r]$ *is disconnected.*

Proof. We have $r\geq 2$ because g has no fixed points. Thus $h\neq\mathrm{id}$, and furthermore the abelian variety $A'=\mathrm{Im}(\mathrm{id}-h)$ does not contain a by Lemma 2.2. Thus $0\subsetneqq A'\subsetneqq A$, hence A is not simple.

Seeking a contradiction, we assume that $A[r]$ is connected. It is then a local Artin scheme with residue field k, so we may also assume that k is algebraically closed. Being successive extensions, the kernels $A[r^\nu]$, $\nu\geq 0$ are connected as well. By Proposition 2.4 we may assume that some of these kernels contain a. Thus $a=0$, hence $g=(0,h)$ fixes the origin $0\in P$, contradiction. □

3. T-trivial varieties

Let k be a ground field of characteristic $p\geq 0$. Recall that a locally free sheaf $\mathscr{E}$ of rank $n\geq 0$ on some scheme X is called *free* if it is isomorphic to $\bigoplus_{i=1}^n\mathscr{O}_X$. The central topic of this paper are smooth schemes with free tangent sheaf. In the literature, one often finds the locution "trivial tangent bundle". We find the following terminology useful:

Definition 3.1. A T*-trivial variety* is a smooth proper scheme X with $h^0(\mathscr{O}_X)=1$ such that the tangent sheaf $\Theta_{X/k}=\underline{\mathrm{Hom}}(\Omega^1_{X/k},\mathscr{O}_X)$ is free.

Equivalently, the sheaf of Kähler differentials $\Omega^1_{X/k}$ is free. Note that T-trivial varieties are very special cases of K*-trivial varieties*, where $\omega_X=\det(\Omega^1_{X/k})$ is isomorphic to the structure sheaf. Abelian varieties A have $\Theta_{A/k}=\mathrm{Lie}(A)\otimes_k\mathscr{O}_A$, and are thus examples of T-trivial varieties. Let us collect some basic property:

Proposition 3.2. (i) *If X and Y are T-trivial, so is their product $X\times Y$.*

(ii) *If $f:X\to Y$ is a finite étale covering of a T-trivial variety Y, then every irreducible component $X'\subset X$ is a T-trivial variety over the field of constants $k'=\Gamma(X',\mathscr{O}_{X'})$.*

(iii) *Let $k\subset k'$ be a field extension. Then X is T-trivial if and only if the base-change $X'=X\otimes k'$ is T-trivial.*

(iv) *The Weil restriction $X_0 = \mathrm{Res}_{k/k_0}(X)$ of a T-trivial variety X over k along a finite separable extension $k_0 \subset k$ is a T-trivial variety over k_0.*

Proof. The first statement follows from $\Omega^1_{(X\times Y)/k} = \mathrm{pr}_1^*(\Omega^1_{X/k}) \otimes \mathrm{pr}_2^*(\Omega^1_{Y/k})$. For (ii) it suffices to treat the case that X is irreducible. Since X is smooth over k, the finite extension $k \subset k'$ must be separable. The structure maps $X \to \mathrm{Spec}(k') \to \mathrm{Spec}(k)$ induces an exact sequence $\Omega^1_{k'/k} \otimes_k \mathscr{O}_X \to \Omega^1_{X/k} \to \Omega^1_{X/k'} \to 0$, where the term on the left vanishes. In the exact sequence $f^*(\Omega^1_{Y/k}) \to \Omega^1_{X/k} \to \Omega^1_{X/Y} \to 0$, the term on the right vanishes, and the other terms are locally free of the same rank. Thus the arrow on the left is bijective, so $\Omega^1_{X/k'}$ is free, and (ii) follows.

Statement (iii) follows from Lemma 1.3. It remains to check (iv). In light of (iii) it suffices to verify that $X_0 \otimes_{k_0} k$ is T-trivial. This indeed follows from Lemma 1.1, together with (i). ☐

As for abelian varieties, morphisms between T-trivial varieties have remarkable rigidity properties. Let $h : X \to Z$ be a surjective morphism between T-trivial varieties, with Stein factorization $Y = \mathrm{Spec}\, h_*(\mathscr{O}_X)$. The resulting morphisms are denoted by

$$X \xrightarrow{\ g\ } Y \xrightarrow{\ f\ } Z, \qquad h = f\circ g : X \to Z.$$

Proposition 3.3. *Suppose in the above setting that the function field extension $k(Z) \subset k(X)$ is separable. Then the following holds:*

(i) *The morphisms h and g are smooth, and f is étale.*
(ii) *The scheme Y and all fibers $g^{-1}(y)$, $y \in Y$ are T-trivial varieties.*
(iii) *If X and Z have the same dimension, then $h : X \to Z$ is étale.*
(iv) *The canonical sequence $0 \to h^*(\Omega^1_{Z/k}) \to \Omega^1_{X/k} \to \Omega^1_{X/Y} \to 0$ is split exact, and all terms are free.*

Proof. Set $m = \dim(X)$ and $n = \dim(Z)$, and consider the exact sequence

$$h^*(\Omega^1_{Z/k}) \longrightarrow \Omega^1_{X/k} \longrightarrow \Omega^1_{X/Z} \longrightarrow 0.$$

By assumption, the terms on the left are free, and the map on the left is injective at the generic point of X. According to Lemma 1.2, the term on the right is free, the map on the left is injective, and the short exact sequence splits, which already gives (iv). Furthermore, $\mathrm{rank}(\Omega^1_{X/Z}) = m-n$. It then follows from [33], Chapter III, Proposition 10.4 that $h : X \to Z$ is smooth. If X and Z have the same dimension, the generic fiber is zero-dimensional, hence h is finite, and thus étale, giving (iii).

The smoothness of h ensures that the Stein factorization $f : Y \to Z$ is étale ([27], Remark 7.8.10). So in the exact sequence $f^*(\Omega^1_{Z/k}) \to \Omega^1_{Y/k} \to \Omega^1_{Y/Z} \to 0$, the term on the right vanishes, and the map on the left is injective ([29], Theorem 17.11.1). Thus $\Omega^1_{Y/k}$ is free, and it follows that Y is a T-trivial variety. Using that $\Omega^1_{X/Y}$ is free, we infer that the $g^{-1}(y)$ are T-trivial varieties, which gives (ii). Applying the preceding paragraph with $g : X \to Y$ instead of $h : X \to Z$, we see that the former is smooth, which establishes (i). ☐

A *para-abelian variety* is a smooth proper scheme P such that for some field extension $k \subset k'$, the base-change $P' = P \otimes k'$ admits the structure of an abelian variety. Such P are T-trivial according to Proposition 3.2. As explained in [41], Section 5 the subgroup scheme $G \subset \operatorname{Aut}_{P/k}$ that acts trivial on $\operatorname{Pic}^\tau_{P/k}$ is an abelian variety, and its action on P is free and transitive. Moreover, we have an identification $\operatorname{Pic}^\tau_{P/k} = \operatorname{Pic}^\tau_{G/k}$. The *Serre–Lang Theorem* on abelian varieties ([48], Section 18) takes the following form:

Proposition 3.4. *Let P be a para-abelian variety, and $Q \to P$ be a finite étale covering with connected total space. Then Q is a para-abelian variety over the field $k' = H^0(Q, \mathscr{O}_Q)$.*

For every proper scheme X with $h^0(\mathscr{O}_X) = 1$, there is morphisms $f : X \to P$ to a para-abelian variety P that is universal for morphisms to para-abelian varieties ([41], Corollary 10.5). Note that $\operatorname{Pic}^\tau_{P/k}$ coincides with the maximal abelian subvariety in $\operatorname{Pic}^\tau_{X/k}$. One calls $P = \operatorname{Alb}_{X/k}$ the *Albanese variety*, and f the *Albanese map*. Its formation functorial, stable under base-change, and equivariant for the action of the group scheme $\operatorname{Aut}_{X/k}$. The following observation is due to Mehta and Srinivas ([45], Lemma 1.4):

Proposition 3.5. *Let X be a T-trivial variety and $P = \operatorname{Alb}_{X/k}$ be its Albanese variety. Then the Albanese map $h : X \to P$ is smooth and $\mathscr{O}_P \to h_*(\mathscr{O}_X)$ is bijective.*

Proof. For the sake of completeness we provide an argument: By the proposition, $h : X \to P$ must be smooth, and its Stein factorization $f : Q \to P$ is étale. Using Proposition 3.4 we infer that Q is para-abelian. From the universal property of the Albanese variety one sees that f admits a section $s : P \to Q$. Thus the field extension $k(P) \subset k(Q)$ is an equality, and it follows that $f : Q \to P$ is an isomorphism. □

The discrepancy between T-trivial varieties and para-abelian varieties now can be seen as a question about Betti numbers:

Proposition 3.6. *For each T-trivial variety X we have $b_1(X) \leq 2\dim(X)$. Moreover, equality holds if and only if X is para-abelian.*

Proof. The Albanese map $h : X \to P$ is smooth, according to the proposition, hence $\dim(X) \geq \dim(P)$. The group scheme $\operatorname{Pic}^0_{X/k}$ is proper ([41], Proposition 2.3), and thus $\dim(P) = \dim(\operatorname{Pic}_{X/k})$. Finally, we have $2\dim(\operatorname{Pic}_{X/k}) = b_1(X)$ by Lemma 1.6. Thus

$$b_1(X) = 2\dim(P) \leq 2\dim(X).$$

The outer terms are equal if and only if X and P have the same dimension. In this case, the Albanese map $h : X \to P$ is étale, by Proposition 3.3, and thus an isomorphism, in light of Proposition 3.5. □

Corollary 3.7. *In characteristic zero, every T-trivial variety is para-abelian.*

Proof. Hodge theory gives $H^1(X(\mathbb{C}), \mathbb{Z}) \otimes \mathbb{C} = H^{1,0}(X) \oplus H^{0,1}(X)$, where the summands on the right have the same dimension. Using $h^0(\Omega^1_{X/k}) = \dim(X)$ one gets $b_1(X) = 2\dim(X)$, and the assertion follows from the Proposition. □

Let X be an n-dimensional K-trivial variety. Serre duality ensures $h^n(\mathscr{O}_X) = 1$, and the relative Frobenius map $F : X \to X^{(p)}$ yields a linear map

$$F^* : H^n(X^{(p)}, \mathscr{O}_{X^{(p)}}) \longrightarrow H^n(X, \mathscr{O}_X)$$

between one-dimensional vector spaces. Choosing a non-zero vector $a \in H^n(X, \mathscr{O}_X)$ yields via $F^*(a^{(p)}) = \lambda a$ some scalar $\lambda \in k$, whose class modulo $k^{\times p-1}$ does not depend on the vector. If $\lambda \neq 0$, it is customary to call X *ordinary*. Note that there are several other, more refined versions, involving sheaves of cocycles and coboundaries in the de Rham complex $\Omega^\bullet_{X/k}$. For T-trivial varieties, however, all these notions coincide, and are also equivalent to the condition that X is *Frobenius split*, that is, $\mathscr{O}_{X^{(p)}} \to F_*(\mathscr{O}_X)$ admits a retraction ([45], Lemma 1.1).

Let X be a T-trivial variety. Mehta and Srinivas suggested that the collection of all finite étale coverings $X_\lambda \to X$, $\lambda \in L$ with connected total space should contain abelian varieties, at least if k is algebraically closed ([45], page 191). Our second main result pertains to this, where we also take into account the possible fields of constants $k_\lambda = H^0(X_\lambda, \mathscr{O}_{X_\lambda})$ over general ground fields:

Theorem 3.8. *Let X be a T-trivial variety that is not para-abelian, but has a finite surjection $f : P \to X$ where P is para-abelian over its fields of constants $k' = H^0(P, \mathscr{O}_P)$, and $k(X) \subset k(P)$ is separable. Then the following holds:*

- (i) *The characteristic satisfies $p \leq 3$.*
- (ii) *The abelian variety $A = \operatorname{Aut}^0_{P/k'}$ is not geometrically simple and the finite group scheme $A[p]$ is disconnected.*
- (iii) *For $p = 3$ the T-trivial variety X is not ordinary, and the abelian variety A has a supersingular quotient.*

Proof. The base-change $X' = X \otimes_k k'$ is a T-trivial variety over k', and the induced morphism $f' : P \to X'$ is finite and surjective, and respects the k'-structures. By Lemma 1.7, we may replace X by $X \otimes k'$ and k by k', and thus may assume P is a para-abelian variety over k. In light of Lemma 1.5, it now suffices to treat the case that k is algebraically closed.

Choose a finite étale covering $Q \to P$ with connected total space such that the composite map $Q \to X$ is Galois. Then Q is para-abelian, by the Serre–Lang Theorem. Fix a rational point $e_Q \in Q$, and write $e_P \in P$ for the image point. One may regard the pairs (Q, e_Q) and (P, e_P) as abelian varieties, and the morphism $Q \to P$ as an isogeny. According to Lemma 1.5, we may replace P by Q, and thus may assume that $P \to X$ is Galois. Write $G = \operatorname{Aut}(P/X)$ for the Galois group, such that $X = P/G$. Using Grothendieck's spectral sequence for equivariant cohomology ([23], Theorem 5.2.1), we see that the canonical inclusion $H^0(X, \Omega^1_{X/k}) \subset H^0(P, \Omega^1_{P/k})^G$ is an equality. From $h^0(\Omega^1_{X/k}) = h^0(\Omega^1_{P/k})$ we infer that the induced G-action on $H^0(P, \Omega^1_{P/k})$ is trivial.

Fix a rational point $e \in P$. The resulting map $A \to P$ is an isomorphism, so one may regard A as the pair (P, e), and obtains $\operatorname{Aut}(P) = A(k) \rtimes \operatorname{Aut}(A)$. The translational part $N = G \cap A(k)$ is normal in G, and the quotient P/N remains para-abelian. So without restriction, we may assume that the projection $G \to \operatorname{Aut}(A)$ is injective. Note that the group G is non-trivial, because X is not para-abelian.

Fix a non-trivial element $g \in G$, and write $g = (a, h)$ with $a \in A(k)$ and $h \in \operatorname{Aut}(A)$. Let $r \geq 2$ be the common order for g and h. Since g and the translation a act trivially on $H^0(A, \Omega^1_{A/k})$, the same holds for h. In turn, it also acts trivially on the dual vector space $H^0(A, \Theta_{A/k})$, and thus belongs to the kernel of the canonical map $\operatorname{Aut}(A) \to \operatorname{GL}(\operatorname{Lie}(A))$. By Theorem 2.1, we must have $p \leq 3$, and $r = \operatorname{ord}(h)$ is some p-power. Using Proposition 2.6, we see that A is not simple and contains a point of order p. For $p = 3$ we also have a supersingular quotient, again by Theorem 2.1. □

Let us recall Igusa's construction [34] of T-trivial surfaces, and observe that the key features carry over to higher dimensions: Let E and E' be elliptic curves, $h \in \operatorname{Aut}(E)$ and $a \in E'(k)$ be group elements both of order p. The resulting diagonal action of the cyclic group $G = C_p$ on the abelian surface $A = E \times E'$ is free. It turns out that the induced action on $H^0(A, \Omega^1_{X/k})$ is trivial, and the quotient $X = A/G$ must be T-trivial. Note that this is a *bielliptic surfaces*, which have Betti numbers are $b_1 = b_2 = 2$. Also note that for $k = k^{\text{alg}}$, the possibilities for the group $\operatorname{Aut}(E)$ are

$$C_2, \quad C_4, \quad C_6, \quad C_3 \rtimes C_4, \quad \text{and} \quad Q \rtimes C_3,$$

formed with cyclic groups C_i and the quaternion group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$. We see that Igusa's construction is only possible in characteristic $p \leq 3$, requires an abelian variety $A = E \times E'$ that is not simple and contains a point of order p, and for $p = 3$ has a supersingular quotient.

We also record the following consequence, which was already established by Li ([42], Theorem 0.3):

Corollary 3.9. *Every ordinary T-trivial variety X in characteristic $p \geq 3$ is para-abelian.*

Proof. We may assume that k is algebraically closed. According to the results of Mehta and Srinivas ([45], Theorem 1), there is a finite étale covering $A \to X$ by some abelian variety A, which must be ordinary ([45], Theorem 1 and Lemma 1.2). Seeking a contradiction, we assume that X is not para-abelian. The theorem ensure the characteristic is $p = 3$, and the abelian variety A fails to be ordinary, contradiction. □

4. Relative Albanese maps

In this section we investigate the existence of relative Albanese maps, and what to do if they fail to exist. The results, which appears to be of independent interest, will be used in the next section to reduce the study of general T-trivial varieties to those with Betti number $b_1 = 0$. Throughout this section, we work over a fixed base scheme S.

The theory of relative Albanese maps was developed in [25], [11], [41] and [56]. Let us recall some basic facts. Suppose $f : X \to S$ is flat proper morphism of finite presentation, and with $\mathscr{O}_S = f_*(\mathscr{O}_X)$. Then the abelian sheaf $R^1 f_*(\mathbb{G}_m)$, formed with respect to the fppf topology, is representable by an algebraic space $\operatorname{Pic}_{X/S}$, and the abelian subsheaf given by the numerically trivial invertible sheaves is representable by an open subspace $\operatorname{Pic}^\tau_{X/S}$, which is of finite presentation over S. Note that the formation of these algebraic spaces commutes with arbitrary base-change.

A *family of para-abelian variety* is a proper morphism $P \to S$ of finite presentation all whose fibers are para-abelian, that is, admit the structure of an abelian variety after some ground field extension. It then follows that the subgroup space $G \subset \mathrm{Aut}_{P/S}$ that fixes $\mathrm{Pic}^\tau_{P/S}$ is a family of abelian varieties, that its action on P is free and transitive, and that $\mathrm{Pic}^\tau_{P/S}$ is dual to G.

A *relative Albanese map* is a morphism $g : X \to P$ to a family of para-abelian varieties that is universal for arrows to families of para-abelian varieties. Equivalently, the homomorphism $\mathrm{Pic}^\tau_{P/S} \to \mathrm{Pic}^\tau_{X/S}$ induces, for each $s \in S$, an identifications of the abelian variety $\mathrm{Pic}^\tau_{P/S} \otimes \kappa(s)$ with the *maximal abelian subvariety* $\mathrm{Pic}^\alpha_{X_s/\kappa(s)}$ inside $\mathrm{Pic}^\tau_{X_s/\kappa(s)}$, compare [41], Definition 8.1 and Theorem 10.2. If these conditions hold, we also write $P = \mathrm{Alb}_{X/S}$. Note that despite uniqueness, the existence of relative Albanese maps in specific situations is often unclear.

If S is integral, with generic point $\eta \in S$, we write $A_\eta \subset \mathrm{Pic}^\tau_{X_\eta/\kappa(\eta)}$ for the maximal abelian subvariety, and $A \subset \mathrm{Pic}^\tau_{X/S}$ for its schematic closure. It is unclear whether this is a family of subgroup schemes, let alone a family of abelian varieties. We start with some easy observations:

Lemma 4.1. *Suppose S is integral and noetherian, and $f : X \to S$ is smooth. Then the algebraic spaces $\mathrm{Pic}^\tau_{X/S}$ and A are proper, and A is equi-dimensional over S. Moreover, for each $s \in S$ the following holds:*

(i) *The affinization $\mathrm{Pic}^{\mathrm{aff}}_{X_s/\kappa(s)}$ of the group scheme $\mathrm{Pic}^\tau_{X_s/\kappa(s)}$ is finite.*
(ii) *The kernel of the affinization map is the maximal abelian subvariety $\mathrm{Pic}^\alpha_{X_s/\kappa(s)}$.*
(iii) *We have $\mathrm{Pic}^\alpha_{X_s/\kappa(s)} = (A_s)_{\mathrm{red}}$ as closed subschemes inside $\mathrm{Pic}^\tau_{X_s/\kappa(s)}$.*
(iv) *The fiber A_s is reduced provided that the group scheme $\mathrm{Pic}^{\mathrm{aff}}_{X_s/\kappa(s)}$ is reduced.*

Proof. The algebraic space $\mathrm{Pic}^\tau_{X/S}$ is proper by [41], Proposition 2.3, and the same holds for the closed subspace A. For the remaining statements it suffices to treat the case that S is the spectrum of a complete local noetherian ring R, with closed point $s \in S$ and separably closed residue field.

To see (i) and (ii) we consider the group scheme $G = \mathrm{Pic}^\tau_{X_s/\kappa(s)}$, which is of finite type over the residue field $k = R/\mathfrak{m}_R$. By [20], Chapter III, Theorem 8.2 it sits in a short exact sequence $0 \to N \to G \to G^{\mathrm{aff}} \to 0$ where $G^{\mathrm{aff}} = \mathrm{Spec}\,\Gamma(G, \mathscr{O}_G)$ is the affinization, and the kernel N is *anti-affine*, which means $h^0(\mathscr{O}_N) = 1$. The latter is an extension of an abelian variety N/H by some smooth connected affine group scheme H, according to [10], Lemma 3.1.4. In our situation, G is proper, whence G^{aff} is finite and H is trivial. This gives (i) and (ii).

Since A is irreducible, the closed fiber A_s must be connected, according to Hensel's Lemma ([29], Theorem 18.5.11). Since A contains the zero section of P, we thus get $(A_s)_{\mathrm{red}} \subset \mathrm{Pic}^\alpha_{X_s/\kappa(s)}$, by (i) and (ii). Set $P = \mathrm{Pic}^\tau_{X/S}$, suppose for the moment that the function $s \mapsto \dim(P_s)$ is constant, and write $g \geq 0$ for the common value. With Chevalley's Semicontinuity Theorem ([28], Corollary 13.1.5) we get $g \geq \dim(A_s) \geq \dim(A_\eta) = g$, so A is equi-dimensional. Moreover, the inclusion $(A_s)_{\mathrm{red}} \subset \mathrm{Pic}^\alpha_{X_s/\kappa(s)}$ is an equality, because both schemes are g-dimensional, and the right hand side is irreducible. This yields (iii). To see (iv), suppose that P_s^{aff} is reduced. Then P_s^α is a connected component of P_s, and thus the inclusion $P_s^\alpha \subset A_s$ must be an equality. This gives (iv).

It remains to check that $s \mapsto \dim(P_s)$ is constant, which by Chevalley's Semicontinuity Theorem boils down to $\dim(P_s) \leq \dim(P_\eta)$. Fix a prime ℓ that does not divide the order of the finite group scheme P_s^{aff} and P_η^{aff}, and also differs from the characteristic $p \geq 0$ of the residue field $R/\mathfrak{m}_R$. Consider the infinitesimal neighborhoods $X_n = X \otimes R/\mathfrak{m}_R^{n+1}$. The short exact sequence $0 \to \mathscr{O}_{X_0} \to \mathscr{O}_{X_{n+1}}^\times \to \mathscr{O}_{X_n}^\times \to 1$ induces an exact sequence

$$H^1(X_0, \mathscr{O}_{X_0}) \longrightarrow \operatorname{Pic}(X_{n+1}) \longrightarrow \operatorname{Pic}(X_n) \longrightarrow H^2(X_0, \mathscr{O}_{X_0})$$

Since ℓ is prime to the characteristic of the residue field $R/\mathfrak{m}_R$, the map in the middle induces a bijection on ℓ-torsion elements. It then follows from Grothendieck's Existence Theorem ([26], Theorem 5.4.1) that the restriction map $\operatorname{Pic}(X) \to \operatorname{Pic}(X_s)$ indeed induces a bijection on ℓ-torsion. So $\dim_{\mathbb{F}_\ell} \operatorname{Pic}(X_s)[\ell] \leq \dim_{\mathbb{F}_\ell} \operatorname{Pic}(X_\eta)[\ell]$. In turn, we have $\dim(P_s) \leq \dim(P_\eta)$. ☐

This already gives a sufficient condition for the existence of relative Albanese maps:

Theorem 4.2. *Suppose S is integral, normal and excellent, $f : X \to S$ is smooth, and the fibers for the structure morphism $A \to S$ are reduced. Then the relative Albanese map $g : X \to \operatorname{Alb}_{X/S}$ exists.*

Proof. We first check that $A \to S$ is a family of para-abelian varieties. Being the closure of the regular scheme A_η, the total space A contains no embedded components. By Lemma 4.1, the structure morphism $A \to S$ is proper and equi-dimensional, and the fiber-wise inclusions $\operatorname{Pic}^\alpha_{X_s/\kappa(s)} \subset A_s$ are equalities. According to Kollár's generalization of Hironaka's Flatness Lemma ([39], Corollary 11), the morphism $f : A \to S$ must be flat, and thus is a family of para-abelian varieties.

Obviously, the zero section $e : S \to \operatorname{Pic}^\tau_{X/S}$ factors over A. By [41], Proposition 4.3, there is a unique group law that turns A into a family of abelian varieties. The inclusion $A \subset \operatorname{Pic}^\tau_{X/S}$ respects the zero section. Using [49], Corollary 6.4 one infers that it actually respects the group laws. We already observed that each A_s is the maximal abelian subvariety inside $\operatorname{Pic}^\tau_{X_s/\kappa(s)}$, so by [41], Theorem 10.2 the relative Albanese map $g : X \to \operatorname{Alb}_{X/S}$ exists. ☐

This basically settles the case of characteristic zero:

Corollary 4.3. *Suppose S is a $\mathbb{Q}$-scheme that is integral, normal and excellent, and that $f : X \to S$ is smooth. Then the relative Albanese map $g : X \to \operatorname{Alb}_{X/S}$ exists.*

Proof. By Cartier's Theorem ([20], Chapter II, Theorem 1.1), group schemes of finite type over the points $s \in S$ are automatically reduced. Lemma 4.1 ensures that the fiber A_s is reduced, so the theorem applies. ☐

Without the assumption on the characteristic, the conclusions holds true if for all points $s \in S$ we have $H^2(X_s, \mathscr{O}_{X_s}) = 0$, because then $\operatorname{Pic}^\tau_{X_s/\kappa(s)}$ and hence also its affinization are smooth ([47], Lecture 27).

All the above results, however, are still insufficient for the applications we have in mind. For lack of better existence criteria, we seek to weaken the very notion of relative Albanese maps. Our first observation in this direction is:

Lemma 4.4. *Suppose S is integral and noetherian, and $f : X \to S$ is smooth. Then the abelian variety A_η extends to a family of abelian varieties over some open set $U \subset X$ containing all points $s \in S$ where $\mathscr{O}_{S,s}$ is regular of dimension one.*

Proof. Since A_η extends over some dense open subset, it suffices to treat the case that S is the spectrum of a discrete valuation ring R, and $s \in S$ is the closed point. The task is to check that A_η has *good reduction*. For this we further may assume that R is complete, with separably closed residue field ([9], Section 7.2, Theorem 1).

Fix a prime $\ell > 0$ that does not divide the order of the finite group scheme $\operatorname{Pic}^{\text{aff}}_{X_\eta/\kappa(\eta)}$, and also differs from the characteristic $p \geq 0$ of the residue field $R/\mathfrak{m}_R$. The former ensures that the inclusion $A_\eta[\ell] \subset \operatorname{Pic}^\tau_{X_\eta/\kappa(\eta)}[\ell]$ is an equality. In light of the Néron–Ogg–Shafarevich Criterion ([57], Theorem 1), our task is to verify that the finite étale group scheme $\operatorname{Pic}^\tau_{X_\eta/\kappa(\eta)}[\ell]$ is constant. Equivalently, the $\mathbb{F}_\ell$-vector space $\operatorname{Pic}(X_\eta)[\ell]$ has dimension $2g$, where $g \geq 0$ is the relative dimension of $\operatorname{Pic}^\tau_{X/S}$. It thus suffices to verify that the restriction map $\operatorname{Pic}(X)[\ell] \to \operatorname{Pic}(X_0)[\ell]$ is surjective, where X_0 is the closed fiber. This indeed follows as in the proof for Lemma 4.1. $\square$

According to [41], Corollary 10.6 there is an open neighborhood V of the generic point $\eta \in S$ such that $X_V \to \operatorname{Alb}_{X_V/V}$ exists. A necessary condition for $V = S$ is that the para-abelian variety $P_\eta = \operatorname{Alb}_{X_\eta/\kappa(\eta)}$ extends to a family of para-abelian varieties P over S. In this situation, the Albanese map $g_\eta : X_\eta \to P_\eta$ can be seen as rational map $g : X \dashrightarrow P$ between integral noetherian schemes. Its *domain of definition* $\operatorname{Dom}(g)$ is an open set in X, so its image $U = f(\operatorname{Dom}(g))$ is an open set in S. In fact, U comprises all $s \in S$ where our the rational map induces a rational map $g_s : X_s \dashrightarrow P_s$ on the fiber. One then says that g is a *U-rational map*.

Proposition 4.5. *Suppose that S is integral, normal and noetherian, $f : X \to S$ is smooth, and that the Albanese variety $P_\eta = \operatorname{Alb}_{X_\eta/\kappa(\eta)}$ extends to a family of para-abelian varieties P over S. Then the open set $U = f(\operatorname{Dom}(g))$ contains every codimension-one point $s \in S$, and the Albanese map $g_\eta : X_\eta \to P_\eta$ extends to a morphism $g_U : X_U \to P_U$.*

Proof. For the statement on U, it suffices to treat the case that S is the spectrum of a discrete valuation ring R, with closed point $s \in S$. Our task is to verify that the inclusion $X_\eta \subset \operatorname{Dom}(g)$ is strict. For this we argue as in [9], Section 2.5, Proposition 5: Let $\Gamma \subset X \times P$ be the closure of the graph for the Albanese map $g_\eta : X_\eta \to P_\eta$. We have to check that the projection $\operatorname{pr}_1 : \Gamma \to X$ is an isomorphism. By fppf descent, we may replace S by X, and assume that the structure morphism $f : X \to S$ has a section. The induced section for P endows it with the structure of a family of abelian varieties. This is actually the *Néron model* of P_η. Since $X \to S$ is smooth, the Albanese map $g_\eta : X_\eta \to P_\eta$ extends to a morphism over $S = \operatorname{Spec}(R)$.

For the remaining statement, we may assume that $\operatorname{Dom}(g)$ surjects onto the normal scheme S, such that $g : X \dashrightarrow P$ is an S-rational map. Each codimension-one point $\zeta \in X$ either belongs to the generic fiber X_η, or maps to a codimension-one point $s \in S$. By the previous paragraph, the rational map $g : X \dashrightarrow P$ is defined at all such $\zeta \in X$. By the Weil Extension Theorem ([9], Section 4.4, Theorem 1), the S-rational map g is defined everywhere. $\square$

Our main result on relative Albanese maps is a weak form of existence:

Theorem 4.6. *Suppose that S is integral and normal, $f: X \to S$ is smooth, and that the generic fiber X_η contains a rational point. After removing a closed set $Z \subset S$ of codimension at least two, $P_\eta = \mathrm{Alb}_{X_\eta/\kappa(\eta)}$ extends to a family of abelian varieties P over S, and the Albanese map $g_\eta : X_\eta \to P_\eta$ extends to a morphism $g: X \to P$.*

Proof. Let $G_\eta = \mathrm{Pic}^\tau_{A_\eta/\kappa(\eta)}$ be the dual for the abelian variety $A_\eta = \mathrm{Pic}^\alpha_{X_\eta/\kappa(\eta)}$. By Lemma 4.4 we may assume that it extends to a family of abelian varieties $G \to S$. Fix a rational point on X_η. The resulting rational point on the Albanese variety yields an identification $P_\eta = \mathrm{Alb}_{X_\eta/\kappa(\eta)}$ with G_η. Thus P_η extends to a family $P \to S$. By Proposition 4.5, the Albanese map $g_\eta : X_\eta \to P_\eta$ extends to a morphism $g: X \to P$, at least after removing a closed subset $Z \subset S$ of codimension at least two. □

Although it is unclear whether the above $g: X \to P$ enjoys a universal property, it can serve as a useful substitute for the relative Albanese map, as we shall see in the next section.

5. The first Betti number

Let k be a ground field of characteristic $p \geq 0$. Throughout this section, X denotes an n-dimensional T-trivial variety. To unravel its geometry, the chief tool is the Albanese variety $B = \mathrm{Alb}_{X/k}$ and the Albanese map $f: X \to B$. Recall that the former has $\dim(B) = 2b_1(X)$, and that the latter is a family of T-trivial varieties, usually of smaller dimension $n' < n$. Our third main result is:

Theorem 5.1. *Assumptions as above. Then there is a finite étale covering $X' \to X$ whose total space is a T-trivial variety X' over the field $k' = H^0(X', \mathscr{O}_{X'})$ where the fibers of the Albanese map $X' \to \mathrm{Alb}_{X'/k'}$ are T-trivial varieties with Betti number $b_1 = 0$.*

In characteristic zero, this is a consequence of the Beauville–Bogomolov Decomposition for K-trivial varieties ([6] and [4]), and then actually $X' = \mathrm{Alb}_{X'/k}$. By the result of Mehta and Srinivas ([45], Theorem 1), this carries over to positive characteristics provided X is ordinary. *Our theorem above raises the question whether or not T-trivial varieties with $b_1 = 0$ exists, besides the obvious example of the singleton in dimension $n = 0$. And if so, for which primes $p > 0$ do they occur?* At present, we are unable to offer further insights on this. It also would be interesting to know if one may choose X' without constant field extension.

The proof requires some preparation, and will be given at the end of the section. We would like to use the *relative* Albanese variety $\mathrm{Alb}_{X/B}$ over the *absolute* Albanese variety $B = \mathrm{Alb}_{X/k}$. As discussed in the previous section, the unconditional existence of such a relative construction is in doubt. To circumvent this issue, consider commutative diagrams

$$\begin{array}{ccc} C' & \longrightarrow & X \\ \downarrow & & \downarrow f \\ C & \longrightarrow & B, \end{array} \tag{7}$$

where C and C' are regular curves, the vertical maps are surjective, and the horizontal maps are finite. Let $\eta \in C$ and $\eta' \in C'$ be the generic points. For simplicity, we also assume $h^0(\mathscr{O}_{C'}) = 1$. By Lemma 4.4, the abelian variety G_η dual to the maximal abelian subvariety $A_\eta = \operatorname{Pic}^\alpha_{X_\eta/\kappa(\eta)}$ extends to a family of abelian varieties $G \to C$. Set

$$G_{\eta'} = G_\eta \otimes_{\kappa(\eta)} \kappa(\eta') \quad \text{and} \quad G_{C'} = G \times_C C'.$$

The commutative diagram (7) provides a $\kappa(\eta')$-valued point for X_η, which gives an identification $\operatorname{Alb}_{X_{\eta'}/\kappa(\eta')} = G_{\eta'}$. By Theorem 4.6, the Albanese map $X_{\eta'} \to G_{\eta'}$ extends to a morphism $X_{C'} \to G_{C'}$.

Proposition 5.2. *The family $G \to C$ of abelian varieties is isotrivial.*

Proof. It suffices to treat the case that k is separably closed. To start with, we also assume that k is algebraically closed. Consider the invertible sheaf $\omega_{G/C} = e^*(\Omega^g_{G/C})$ on the smooth curve C, where $e : C \to G$ denotes the zero section and $g = \dim(G/C)$. According to [22], Chapter V, Proposition 2.2 it suffices to verify $\deg(\omega_{G/C}) \leq 0$. For this, we may replace C by any finite covering, and assume $C = C'$.

As observed above, the Albanese map $g_\eta : X_\eta \to G_\eta$ extends to a morphism $g : X \to G$. This yields an exact sequence

$$g^*(\Omega^1_{G/C}) \longrightarrow \Omega^1_{X_C/C} \longrightarrow \Omega^1_{X_C/G} \longrightarrow 0. \tag{8}$$

The term on the left is locally free, and the term in the middle is free, because $\Omega^1_{X/B}$ is free. Let $d \geq 0$ be its rank. The generic fiber X_η is a T-trivial variety, hence its Albanese map g_η is smooth. So in (8) the map on the left is locally a direct summand over X_η, and in particular injective. Pulling back along the section $s : C \to X$ yields an inclusion $e^*(\Omega^1_{G/C}) \subset \mathscr{O}_C^{\oplus d}$, and thus $\deg(\omega_{G/C}) \leq 0$.

It remains to cope with the case that k is merely separably closed. Fix a prime $\ell \geq 3$ different from the characteristic $p \geq 0$. Let D be the normalization of the reduction for $C^{\text{alg}} = C \otimes k^{\text{alg}}$. The canonical map $D \to C^{\text{alg}}$ is a finite universal homeomorphism. Using [30], Exposé IX, Theorem 4.10, we find a symplectic level structure $(\mathbb{Z}/\ell\mathbb{Z})^{2g}_C \to G$. According to [22], Corollary 2 for Theorem 6.7, the stack $\mathscr{A}_{g,\ell}$ of g-dimensional abelian varieties with such a level structure is an algebraic space. Let $C \to \mathscr{A}_{g,\ell}$ be the classifying map for G. By the previous paragraph, it factors over a singleton after base-change to k^{alg}, so the same holds over k. ☐

Now back to the absolute Albanese variety $B = \operatorname{Alb}_{X/k}$. Fix a prime $\ell > 0$ different from the characteristic $p \geq 0$. Since the Albanese map $f : X \to B$ is smooth and proper, the higher direct images $R^i f_*(\mathbb{Q}_\ell)$ are ℓ-adic local systems, of some rank $d_i \geq 0$. Taking fibers over the geometric point $a \in X^{\text{alg}}$ turns them into representations $\pi_1(B, a) \to \operatorname{GL}_{d_i}(\mathbb{Q}_\ell)$, and the local system is *isotrivial* if and only if the image of the representation is finite.

Proposition 5.3. *The ℓ-adic local system $R^1 f_*(\mathbb{Q}_\ell)$ is isotrivial.*

Proof. It suffices to treat the case that the ground field k is algebraically closed. Let $d \geq 0$ be the rank of the local system $R^1 f_*(\mathbb{Q}_\ell)$. Our task is to show that the corresponding representation $\pi_1(B, a) \to \operatorname{GL}_d(\mathbb{Q}_\ell)$ has finite image. This is trivial for $n = 0$, because then $d = 0$, so we assume $n \geq 1$. We now introduce particular curves C and C' to form the diagram (7).

By Bertini's Theorem together with the Lefschetz Theorem for algebraic fundamental groups ([37], Chapter I, Theorem 6.3 and [31], Exposé XII, Corollary 3.5), there is a smooth curve $C \subset B$ containing the image of $a \in X$ such that the induced map $\pi_1(C, a) \to \pi_1(B, a)$ is surjective. Choose an algebraic closure of the function field $\kappa(\eta) = k(C)$. Working with the resulting geometric point $\bar{\eta}$ rather than a as base point, we have to show that the composite map $\rho : \pi_1(C, \bar{\eta}) \to \mathrm{GL}_d(\mathbb{Q}_\ell)$ has finite image.

By Proposition 5.2, the family $g : G \to C$ is isotrivial. Moreover, the projection $f_C : X_C \to C$ is smooth. We thus find a finite branched covering $C' \to C$ such that $G_{C'} = G_0 \times C'$ for some abelian variety G_0 over k, and that $f_{C'} : X_{C'} \to C'$ has a section. Choose a section to get the diagram (7). Now the Albanese map $X_{\eta'} \to G_{\eta'}$ extends to a morphism $X_{C'} \to G_{C'}$, and $g_{C'} : G_{C'} \to C'$ is given by the projection $G_0 \times C' \to C'$.

To restate our findings in terms of fundamental groups, we lift the geometric point $\bar{\eta}$ to C'. The Proper Base-Change Theorem ([2], Exposé XII, Theorem 5.1) gives $R^1(f_C)_*(\mathbb{Q}_\ell) = R^1 f_*(\mathbb{Q}_l)|C'$, so $\rho : \pi_1(C, \bar{\eta}) \to \mathrm{GL}_d(\mathbb{Q}_\ell)$ is the representation corresponding to the local system $R^1(f_C)_*(\mathbb{Q}_\ell)$. By Lemma 5.6, the local systems $R^1(f_{C'})_*(\mathbb{Q}_\ell)$ and $R^1(g_{C'})_*(\mathbb{Q}_\ell)$ are isomorphic at the generic point η', and by construction $G \otimes_{\kappa(\eta)} \kappa(\eta') = G_0 \otimes_k \kappa(\eta')$. Thus the diagram

$$\begin{array}{ccccc} \pi_1(\eta', \bar{\eta}) & \longrightarrow & \pi_1(\eta, \bar{\eta}) & \longrightarrow & \pi_1(C, \bar{\eta}) \\ \downarrow & & & & \downarrow \rho \\ \pi_1(S, \bar{\eta}) & & \xrightarrow{\quad e \quad} & & \mathrm{GL}_d(\mathbb{Q}_l), \end{array} \tag{9}$$

is commutative, where $S = \mathrm{Spec}(k)$. Of course, the latter is simply-connected and the lower horizontal map is trivial. In the upper row, the map on the left is injective with image of finite index, and the map on the right is surjective by [30], Exposé V, Proposition 8.2. It follows that the representation $\rho : \pi_1(B, \bar{\eta}) \to \mathrm{GL}_d(\mathbb{Q}_\ell)$ vanishes on $\pi_1(\eta', \bar{\eta})$, and thus has finite image. □

Proposition 5.4. *Suppose that the ℓ-adic local system $R^1 f_*(\mathbb{Q}_\ell)$ is constant. Then this system vanishes, and the fibers of the Albanese map $f : X \to \mathrm{Alb}_{X/k}$ are T-trivial varieties with first Betti number $b_1 = 0$.*

Proof. It suffices to treat the case that k is algebraically closed. The Leray–Serre spectral sequence for the Albanese map gives an exact sequence

$$H^1(B, \mathbb{Q}_\ell) \longrightarrow H^1(X, \mathbb{Q}_\ell) \longrightarrow H^0(B, R^1 f_*(\mathbb{Q}_\ell)) \longrightarrow H^2(B, \mathbb{Q}_\ell) \longrightarrow H^2(X, \mathbb{Q}_\ell),$$

where $B = \mathrm{Alb}_{X/k}$. By assumption, the term in the middle is a $\mathbb{Q}_\ell$-vector space of dimension $d = \mathrm{rank}(R^1 f_*(\mathbb{Q}_\ell))$. The outer maps are injective by Lemma 5.5 below. The map on the left is actually bijective, which follows from Lemma 1.6. Thus $d = 0$. According to the Proper Base Change Theorem, all geometric fibers of $f : X \to B$ have $b_1 = 0$. □

Proof for Theorem 5.1. Let X be an n-dimensional T-trivial variety, $B = \mathrm{Alb}_{X/k}$ its Albanese variety, and $f : X \to B$ be the Albanese map. In light of Lemma 1.7, it suffices to treat the case that k is algebraically closed.

The case $n = 0$ is trivial, so we assume $n \geq 1$. Consider the collection of all finite étale cover $X_\lambda \to X$, $\lambda \in L$ with connected total space. Each X_λ is an n-dimensional T-trivial variety, and thus has $b_1(X_\lambda) \leq 2n$. Replacing X by some X_μ whose first Betti number attains the largest value, we may assume $b_1(X) = b_1(X_\lambda)$ for all $\lambda \in L$.

Set $B = \mathrm{Alb}_{X/k}$, and consider the Albanese map $f : X \to B$ and the ensuing local system $R^1 f_*(\mathbb{Q}_\ell)$. The latter is isotrivial, according to Proposition 5.3. Choose some finite étale cover $B' \to B$ on which it becomes constant. Without loss of generality we may assume that B' is connected. Then B' can be seen as an abelian variety, and $X' = X \times_B B'$ is an n-dimensional T-trivial variety. Let $\mathrm{pr}_2 : X' \to B'$ be the induced projection, and $f' : X' \to \mathrm{Alb}_{X'/k}$ be the Albanese map. Then $\mathrm{pr}_2 = g \circ f'$ for some unique $g : \mathrm{Alb}_{X'/k} \to B'$. Since pr_2 and f' are smooth, the same holds for g. By construction

$$\dim(\mathrm{Alb}_{X'/k}) = 2b_1(X') = 2b_1(X) = \dim(B) = \dim(B').$$

Thus the smooth proper morphism $g : \mathrm{Alb}_{X'/k} \to B'$ is finite. Since both pr_2 and g are in Stein factorization, the same holds for g. Thus g yields an identification $\mathrm{Alb}_{X'/k} = B'$. By construction, $R^1 f'_*(\mathbb{Q}_\ell) = R^1 f_*(\mathbb{Q}_l)|B'$ is constant. The assertion now follows from Proposition 5.4. □

The following two observations where used in the proof for Proposition 5.4:

Lemma 5.5. *Let Z be a smooth proper scheme, and $g : Y \to Z$ be a proper surjective morphism. Then the induced maps $H^i(Z^{\mathrm{alg}}, \mathbb{Q}_\ell) \to H^i(Y^{\mathrm{alg}}, \mathbb{Q}_\ell)$ are injective, for all $i \geq 0$.*

Proof. Without loss of generality we may assume $k = k^{\mathrm{alg}}$ and $h^0(\mathscr{O}_Z) = 1$. Set $n = \dim(Z)$. Choose a closed point ζ in the generic fiber for $g : Y \to Z$. By functoriality of cohomology, we may replace Y be the closure of ζ, and thus may assume that Y is integral and n-dimensional. Let $\alpha \in H^i(B, \mathbb{Q}_\ell)$ be a non-zero class. By Poincaré Duality there is a class $\beta \in H^{2n-i}(B, \mathbb{Q}_\ell)$ such that $\alpha \cup \beta \neq 0$. It thus suffices to treat the case $i = 2n$.

Set $\Lambda = \mathbb{Z}/\ell^\nu\mathbb{Z}$. Given an open set U in Y, we can consider cohomology with compact support $H^{2n}_c(U, \Lambda)$. The inclusion map $i : U \to Y$ yields a short exact sequence $0 \to i_!(\Lambda_U) \to \Lambda_Y \to \Lambda_D \to 0$, where $D = Y \smallsetminus U$. In the ensuing long exact sequence

$$H^{2n-1}(D, \Lambda) \to H^{2n}(Y, i_!\Lambda) \longrightarrow H^{2n}(Y, \Lambda) \longrightarrow H^{2n}(D, \Lambda),$$

the outer terms vanish for dimension reasons, and the second terms coincides with $H^{2n}_c(U, \Lambda)$. This gives identification $H^{2n}(Y, \Lambda) = H^{2n}_c(U, \Lambda)$, compatible with respect to inclusions of open sets.

We now use the trace maps $\mathrm{Tr}_g : R^{2d} g_! \Lambda_U(d) \to \Lambda_V$ constructed in [2], Exposé XVIII, Theorem 2.9. These satisfy various naturality conditions, and are defined for certain morphisms $g : U \to V$ between separated schemes of finite type. The condition is that $g|U_0$ is flat of relative dimension d on some open set U_0, and the fibers for $g|U \smallsetminus U_0$ have dimension $< d$.

For some dense open set $V \subset Z$ the preimage $U = f^{-1}(V)$ is smooth and $g = f|U$ is finite and flat. This yields a commutative diagram

$$\begin{array}{ccc} H^{2n}_c(U, \Lambda(n)) & & \\ \downarrow \scriptstyle{\mathrm{Tr}_g} & \searrow & \\ & & \Lambda \\ & \nearrow & \\ H^{2n}_c(V, \Lambda(n)) & & \end{array}$$

of trace maps. Note that by [2], Exposé XVIII, Theorem 2.14 the diagonal arrows are bijective, so the vertical map is bijective as well. The composition $\mathrm{Tr}_g \circ g^*$ with the canonical map $g^* : H^{2n}_c(V, \Lambda(n)) \to H^{2n}_c(U, \Lambda(n))$ is multiplication by $\deg(U/V)$, according to loc. cit., Theorem 2.9. Passing to the limit with respect to ν in $\Lambda = \mathbb{Z}/\ell^\nu\mathbb{Z}$ and tensoring with $\mathbb{Q}_\ell$, we infer that $g^* : H^{2n}(Z, \mathbb{Q}_\ell(n)) \to H^{2n}(Y, \mathbb{Q}_\ell(n))$ is a bijection of one-dimensional vector spaces over $\mathbb{Q}_\ell$. □

Lemma 5.6. *Let Y be a geometrically normal proper scheme with $h^0(\mathscr{O}_Y) = 1$, and set $P = \mathrm{Alb}_{Y/k}$ and $S = \mathrm{Spec}(k)$. Write $g : Y \to S$ and $h : P \to S$ for the structure maps, and $f : Y \to P$ for the Albanese map. Then $f^* : R^1h_*(\mathbb{Q}_\ell) \to R^1g_*(\mathbb{Q}_\ell)$ is a bijection of ℓ-adic local systems over S.*

Proof. It suffices to treat the case that k is algebraically closed, and to verify that $f^* : H^1(P, \mathbb{Z}_\ell) \to H^1(Y, \mathbb{Z}_\ell)$ becomes bijective after tensoring with $\mathbb{Q}_\ell$, and we may replace the coefficient sheaf $\mathbb{Z}_\ell$ by the Tate twist $\mathbb{Z}_\ell(1)$. Let A be the maximal abelian subvariety in $\mathrm{Pic}^\tau_{Y/k}$. One may identify A with the maximal abelian subvariety in $\mathrm{Pic}^\tau_{P/k}$, compare [41], Definition 8.1 and Proposition 8.3. Arguing as in the proof for Lemma 1.6, we obtain a commutative diagram

$$\begin{array}{ccc} & & H^1(Y, \mathbb{Z}_\ell) \\ & \nearrow & \uparrow \scriptstyle{f^*} \\ \varprojlim_\nu A[\ell^\nu] & & \\ & \searrow & \\ & & H^1(P, \mathbb{Z}_\ell) \end{array}$$

where the maps from the left to the right are injective, with finite cokernel. Thus $f^* \otimes \mathbb{Q}_\ell$ is bijective. □

6. Liftability

Let k be a ground field of characteristic $p \geq 0$, and X be a smooth proper scheme with $h^0(\mathscr{O}_X) = 1$, and of dimension $n = \dim(X)$. An important invariant are the *ℓ-adic Chern classes*

$$c_i = c_i(X) = c_1(\Omega^1_{X/k}) \in H^{2i}(X, \mathbb{Q}_\ell(i)),$$

where $\ell > 0$ is a prime different from p. For $k = \mathbb{C}$ and $c_1 = 0$, the *Beauville–Bogomolov Splitting Theorem* tells us that there is a finite étale covering $X' \to X$ such that $X' = A \times Y \times Z$, where A is an abelian variety, Y is a hyperkähler manifold, and Z is Calabi–Yau variety, at least if X is projective ([6] and [4]). This actually holds for compact Kähler manifolds, and the arguments are entirely transcendental. Under the additional assumption $c_2 = 0$ we necessarily have $X' = A$, which is already a direct consequence of Yau's proof of the Calabi Conjecture (see the discussion in [38]). Summing up, X admits a finite étale covering by some abelian variety if and only if $c_1 = 0$ and $c_2 = 0$.

From now on we assume $p > 0$. If X admits a finite étale covering by some para-abelian variety P, one of course has $c_1 = 0$ and $c_2 = 0$. We now seek to understand to what extend the converse holds true. Let us say that X *projectively lifts to characteristic zero* if there is a discrete valuation ring R with residue field $R/\mathfrak{m}_R = k$ and field of fractions $F = \mathrm{Frac}(R)$ of characteristic zero, together with scheme $\mathfrak{X}$ and a projective flat morphism $\mathfrak{X} \to \mathrm{Spec}(R)$ with closed fiber $\mathfrak{X}\otimes_R k = X$.

Theorem 6.1. *In the above situation, suppose the following holds:*

(i) *For some $\ell \neq p$, the ℓ-adic Chern classes c_1 and c_2 both vanish.*
(ii) *The scheme X projectively lifts to characteristic zero.*
(iii) *Characteristic and dimension satisfy $p \geq 2n + 2$.*
(iv) *The ground field k is separably closed.*

Then there is a finite étale covering $A \to X$ by some abelian variety A.

The proof requires some preparation, and will be given at the end of the section. To start with, we consider for each integer $n \geq 0$ the expression

$$\mathrm{MN}(n) = \lim_{t\to\infty} \gcd\Big\{ \prod_{i=1}^{n} (\ell^{2i} - 1) \mid \ell \geq t \text{ prime}\Big\}.$$

Note that the integer sequence defining the limit is increasing, and actually stabilizes. The eventual value turns out to be

$$\mathrm{MN}(n) = 2^{3n+\mathrm{val}_2(n!)} \cdot \prod_{3\leq \ell\leq 2n+1} \ell^{\nu+\mathrm{val}_\ell(\nu!)}, \tag{10}$$

where the product runs over all primes ℓ in the indicated range, $\nu = \lfloor \frac{2n}{\ell-1} \rfloor$ is a Gauß bracket, and $\mathrm{val}_\ell(m)$ denotes the ℓ-adic valuation. For all this, see [16], Theorem 6.4 and its proof. Also note that $\mathrm{val}_\ell(\nu!) = \frac{n-s}{\ell-1}$, where $s = \sum s_i$ is the digit sum in $\nu = \sum s_i \ell^i$, see [54], Chapter 5, Section 3.1. Let us tabulate the first five values of the above function:

n	0	1	2	3	4
$\mathrm{MN}(n)$	1	$2^3 \cdot 3$	$2^7 \cdot 3^2 \cdot 5$	$2^{10} \cdot 3^3 \cdot 5 \cdot 7$	$2^{15} \cdot 3^4 \cdot 5^2 \cdot 7$

Note that $\mathrm{MN}(1) = 24$ is the order of the group $\{\pm 1, \pm i, \pm j\} \rtimes C_3$, which plays a special role for elliptic curves. Indeed, our interest in $\mathrm{MN}(n)$ stems from the following general fact, which should be well-known:

Lemma 6.2. *For each n-dimensional abelian variety A and each finite subgroup $G \subset \mathrm{Aut}(A)$, the order $|G|$ divides the integer $\mathrm{MN}(n)$.*

Proof. Fix an ample invertible sheaf $\mathscr{L}_0$ on A. Then $\mathscr{L} = \bigotimes_{\sigma\in G} \sigma^*(\mathscr{L}_0)$ is ample, and its class in $\mathrm{Pic}(A)$ is G-fixed. Write $A^\vee = \mathrm{Pic}^0_{A/k}$ for the dual abelian variety, and consider the homomorphism

$$f : A \longrightarrow A^\vee, \quad a \longmapsto \tau_a^*(\mathscr{L}) \otimes \mathscr{L}^\vee,$$

where $\tau_a(x) = x + a$ denotes translation. By contravariance, the G-action on A induces an action of the opposite group G^{op} on the dual $A^\vee$, which is converted back to a G-action via $\sigma \cdot \mathscr{N} = (\sigma^{-1})^*(\mathscr{N})$. The above map is equivariant with respect to these G-actions: First observe

$$(\sigma^{-1})^* \circ \tau_a^* = (\tau_a \circ \sigma^{-1})^* = (\sigma^{-1} \circ \tau_{\sigma(a)})^* = \tau_{\sigma(a)}^* \circ (\sigma^{-1})^*$$

for $\sigma \in G$ and $a \in A(k)$. Together with $(\sigma^{-1})^*(\mathscr{L}) \simeq \mathscr{L}$ this gives

$$\sigma \cdot f(a) = (\sigma^{-1})^*(\tau_a^*\mathscr{L}) \otimes (\sigma^{-1})^*(\mathscr{L}^\vee) = \tau_{\sigma(a)}^*(\mathscr{L}) \otimes \mathscr{L}^\vee = f(\sigma \cdot a).$$

Furthermore, the homomorphism $f : A \to A^\vee$ has finite kernel, because the invertible sheaf $\mathscr{L}$ is ample.

Now fix a prime $\ell > 0$ not dividing $2p \cdot \deg(f) \cdot |G|$. This has three consequences: First, the induced G-equivariant map $f : A[\ell] \to A^\vee[\ell]$ is an isomorphism. With this identification, the Weil pairing $A[\ell] \times A^\vee[\ell] \to \mu_\ell$ becomes a G-fixed symplectic form on $A[\ell]$. Second, Serre's result ensures that the homomorphism $G \to \operatorname{Aut}_{A[\ell]}$ is injective ([24], Appendix or [48], Section 21, Theorem 5). Choosing a symplectic basis of k^{sep}-valued points in the finite étale group scheme $A[\ell]$, we obtain an inclusion $G \subset \operatorname{Sp}_{2n}(\mathbb{F}_\ell)$. Now recall that the finite symplectic group has order

$$|\operatorname{Sp}_{2n}(\mathbb{F}_\ell)| = \ell^{n^2} \cdot \prod_{i=1}^{n} (\ell^{2i} - 1),$$

see for example [32], Theorem 3.1. As a third consequence, we see that the order $|G|$ must be a divisor of $\prod_{i=1}^n (\ell^{2i} - 1)$. This holds for almost all primes $\ell > 0$, and the assertion follows from the very definition of $\operatorname{MN}(n)$. □

The relation to T-trivial varieties arises as follows:

Proposition 6.3. *Suppose the ground field k is separably closed. Let X be a proper scheme of dimension $n \geq 0$, and $f : A \to X$ be a finite étale covering by some abelian variety A. Then there is a finite étale covering $g : B \to X$ by another abelian variety B such that the degree $\deg(B/X)$ divides the integer $\operatorname{MN}(n)$.*

Proof. Without loss of generality we may assume that $f : A \to X$ is Galois. Write P for the underlying para-abelian variety of A. The finite group $H = \operatorname{Aut}(P/X)$ is a subgroup of the semi-direct product $\operatorname{Aut}(P) = A(k) \rtimes \operatorname{Aut}(A)$. Set $N = H \cap A(k)$. The quotient $B = A/N$ is another abelian variety, and the induced map $g : B \to X$ is a finite étale Galois covering, with relative automorphism group $G = H/N$. By construction, we have $G \subset \operatorname{Aut}(B)$, and Lemma 6.2 ensures that $\deg(B/X) = |G|$ divides the integer $\operatorname{MN}(n)$. □

Proof of Theorem 6.1. By assumption, we have a discrete valuation ring R whose residue field $k = R/\mathfrak{m}_R$ is separably closed of characteristic $p > 0$, and whose field of fraction $F = \operatorname{Frac}(R)$ contains the rational numbers, together with an algebraic space $\mathfrak{X}$ and a proper flat morphism to $S = \operatorname{Spec}(R)$ whose closed fiber $X = \mathfrak{X} \otimes k$ is smooth of dimension $n \geq 0$, with $h^0(\mathscr{O}_X) = 1$, and Chern classes $c_1 = 0$ and $c_2 = 0$. Furthermore $p \geq 2n + 2$. Our task is to find a finite étale covering $A \to X$ by some abelian variety A.

We may assume that R is henselian, and contained in the field of complex numbers. Furthermore, by [30], , Exposé IV, Theorem 4.10 it suffices to construct the covering after passing to a finite extension k' of our separably closed field k, and we are thus free to replace R by the normalization in some finite extension $F \subset F'$.

The ℓ-adic local systems $R^i h_*(\mathbb{Q}_\ell)$ are constant, because R is strictly henselian, and it follows that the relative Chern classes $c_i(\Omega^1_{\mathfrak{X}/S})$ vanishes for $i \leq 2$. So the generic fiber $U = \mathfrak{X} \otimes_R F$ and the complex fiber $V = \mathfrak{X} \otimes_R \mathbb{C}$ both satisfy $c_1 = 0$ and $c_2 = 0$. As discussed at the beginning of the section, there is a finite étale covering $V' \to V$ by some abelian variety V'. In light of the preceding paragraph and [30], Exposé X, Corollary 1.8 we may assume that it arises from U by base-change. With Proposition 6.3 we obtain a finite étale covering $U' \to U$ by some abelian variety U' with the additional property that $\deg(U'/U)$ divides the integer $\mathrm{MN}(n)$. The latter is relatively prime to p, by Formula (10) and our assumption $p \geq 2n + 2$. Hence, by the theory of specialization of algebraic fundamental group ([30], Exposé X, Corollary 3.9), we see that $U' \to U$ extends to some finite étale covering $\mathfrak{A} \to \mathfrak{X}$. The schematic closure of the origin $e \in U'(F)$ defines a section $e : S \to \mathfrak{A}$. This turns $\mathscr{A} \to S$ into a family of abelian varieties, by [49], Theorem 6.14, and the closed fiber yields the desired $A \to X$. □

Note that the projectivity assumptions in Theorem 6.1 are superflous, and the above arguments carry over if X and the structure map $\mathfrak{X} \to \mathrm{Spec}(R)$ are merely proper, and $\mathfrak{X}$ is an algebraic space, such that the complex fiber $V = \mathfrak{X} \otimes_R \mathbb{C}$ is a proper algebraic space over $\mathbb{C}$. Indeed: By Artin's result ([1], Theorem 7.3), the category of algebraic spaces proper over $\mathbb{C}$ is equivalent to the category of compact Moishezon spaces, and the Beauville–Bogomolov Splitting Theorem was recently extended to compact Moishezon manifolds by Biswas, Cao, Dumitrecu and Guenancia [5]. However, the finite étale covering $A \to X$ by an abelian variety a posteriori reveals that X must be projective.

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