THE BRAUER GROUP OF ANALYTIC K3 SURFACES

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Abstract. We show that for complex analytic K3 surfaces any torsion class in $H^2(X, \mathcal{O}_X^*)$ comes from an Azumaya algebra. In other words, the Brauer group equals the cohomological Brauer group. For algebraic surfaces, such results go back to Grothendieck. In our situation, we use twistor spaces to deform a given analytic K3 surface to suitable projective K3 surfaces, and then stable bundles and hyperholomorphy conditions to pass back and forth between the members of the twistor family.

In analogy to the isomorphism $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^*)$, Grothendieck investigated in [8] the possibility of interpreting classes in $H^2(X, \mathcal{O}_X^*)$ as geometric objects. He observed that the Brauer group $\mathrm{Br}(X)$, parameterizing equivalence classes of sheaves of Azumaya algebras on X, naturally injects into $H^2(X, \mathcal{O}_X^*)$. It is not difficult to see that $\mathrm{Br}(X) \subset H^2(X, \mathcal{O}_X^*)$ is contained in the torsion part of $H^2(X, \mathcal{O}_X^*)$ and Grothendieck asked:

Is the natural injection $\mathrm{Br}(X) \subset H^2(X, \mathcal{O}_X^*)_{\mathrm{tor}}$ *an isomorphism?*

This question is of interest in various geometric categories, e.g. X might be a scheme, a complex space, a complex manifold, etc. It is also related to more recent developments in the application of complex algebraic geometry to conformal field theory. Certain elements in $H^2(X, \mathcal{O}_X^*)$ have been interpreted as so-called B-fields, and those are used to construct super conformal field theories associated to Ricci-flat manifolds. Thus, understanding the geometric meaning of the cohomological Brauer group $\mathrm{Br}'(X) := H^2(X, \mathcal{O}_X^*)_{\mathrm{tor}}$ is also of interest for the mathematical interpretation of string theory and mirror symmetry.

An affirmative answer to Grothendieck's question has been given only in very few special cases:

• If X is a complex curve, then $H^2(X, \mathcal{O}_X^*) = 0$. Hence, $\mathrm{Br}(X) = \mathrm{Br}'(X) = H^2(X, \mathcal{O}_X^*) = 0$ (see [8, Cor.2.2] for the general case of a curve).
• For smooth algebraic surfaces the surjectivity has been proved by Grothendieck [8, Cor.2.2] and for normal algebraic surfaces a proof was given more recently by Schröer [14].
• Hoobler [9] and Berkovich [3] gave an affirmative answer for abelian varieties of any dimension and Elencwajg and Narasimhan gave another proof for complex tori [6].

• Demeyer and Ford showed equality of the Brauer group and the cohomological Brauer group for smooth toric varieties [5].

• The case of smooth affine algebraic varieties was settled by Hoobler in [10]. Gabber proved the surjectivity in the case of arbitrary affine schemes and also for the union of two affine schemes glued over an affine scheme [7].

• Bogomolov and Landia [4] showed that for a class $\alpha \in \mathrm{Br}'(X)$ on a scheme X, there is a blowing-up $f : Y \to X$ with $f^*(\alpha) \in \mathrm{Br}(Y)$.

Thus, besides the case of complex tori the question has not been answered for any challenging class of compact complex varieties in dimension at least three. In fact, even in dimension two, i.e. for surfaces, a complete answer is still missing for non-algebraic surfaces.

In this note we give a complete answer to Grothendieck's question for analytic K3 surfaces.

Theorem 0.1. *Let X be any K3 surface. Then* $\mathrm{Br}(X) = \mathrm{Br}'(X) = H^2(X, \mathcal{O}_X^*)_{\mathrm{tor}}$.

If $0 \leq \rho \leq 20$ is the Picard number of X then one has $\mathrm{Br}(X) = (\mathbb{Q}/\mathbb{Z})^{22-\rho}$.

For algebraic K3 surfaces this was known due to the aforementioned result of Grothendieck. The proof we give depends essentially on non-algebraic K3 surfaces and their deformation theory. Maybe the most interesting aspect of our approach is that the existence of the Ricci-flat metric is crucial for the whole argument.

Using standard techniques one easily shows that our result implies an affirmative answer to Grothendieck's question for all compact Ricci-flat Kähler surfaces (Cor. 4.1).

1. Recollection: Brauer group

In this first section we recall the basic facts concerning Brauer groups. The standard reference for this is [8]. As we will mainly be interested in complex manifolds, we let X be a connected complex manifold endowed with the analytic topology.

Let us begin with the definition of the Brauer group.

Definition 1.1. *An* Azumaya algebra *on the complex manifold X is an associative (non-commutative) $\mathcal{O}_X$-algebra $\mathcal{A}$ which is locally (in the analytic topology) isomorphic to a matrix algebra* $\mathrm{M}_r(\mathcal{O}_X)$ *for some $r > 0$.*

Thus, any Azumaya algebra $\mathcal{A}$ is locally free of constant rank r^2. Two Azumaya algebras are isomorphic if they are isomorphic as $\mathcal{O}_X$-algebras. By the Skolem-Noether theorem, $\mathrm{Aut}(\mathrm{M}_r(\mathbb{C})) \cong \mathrm{PGL}_r(\mathbb{C})$ (acting by conjugation). Hence, the set of isomorphism classes of Azumaya algebras $\mathcal{A}$ of rank r^2 is in bijection with the set $H^1(X, \mathrm{PGL}_r(\mathcal{O}_X))$.

Note that any vector bundle E of rank r induces an Azumaya algebra $\mathcal{A} = \mathcal{E}nd(E)$ of rank r^2. Moreover, the associated projective bundle $\mathbb{P}(E)$ also defines a class in $H^1(X, \mathrm{PGL}_r(\mathcal{O}_X))$ which actually coincides with the class defined by $\mathcal{A}$. Azumaya algebras arising this way will be declared 'trivial' by means of the following equivalence relation.

Definition 1.2. *Two Azumaya algebras $\mathcal{A}$ and $\mathcal{A}'$ are called* equivalent *if there exist non-zero vector bundles E and E' such that $\mathcal{A}\otimes\mathcal{E}nd(E)$ and $\mathcal{A}'\otimes\mathcal{E}nd(E')$ are isomorphic Azumaya algebras.*

Definition 1.3. *The* Brauer group $\mathrm{Br}(X)$ *is the set of isomorphism classes of Azumaya algebras modulo the above equivalence relation.*

That $\mathrm{Br}(X)$ is indeed a group stems from the fact that $\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}}\cong\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{A})$, i.e. $\mathcal{A}^{\mathrm{op}}$ defines an inverse (with respect to "$\otimes$") of $\mathcal{A}$ as an element in $\mathrm{Br}(X)$. Clearly, any Azumaya algebra of the form $\mathcal{A}\cong\mathcal{E}nd(E)$ defines the unit element in $\mathrm{Br}(X)$. In fact, due to the following result, the converse holds also true.

Lemma 1.4. *An Azumaya algebra $\mathcal{A}$ of rank r^2 is trivial if and only if its class in $H^1(X,\mathrm{PGL}_r(\mathcal{O}_X))$ is contained in the image of the natural map $H^1(X,\mathrm{GL}_r(\mathcal{O}_X))\to H^1(X,\mathrm{PGL}_r(\mathcal{O}_X))$, i.e. $\mathcal{A}=\mathcal{E}nd(E)$.*

In order to prove this lemma one considers the long exact cohomology sequence induced by

$$1\to\mathcal{O}_X^*\to\mathrm{GL}_r(\mathcal{O}_X)\to\mathrm{PGL}_r(\mathcal{O}_X)\to 1.$$

Since GL and PGL are not abelian, one only has a long exact sequence up to $H^2(X,\mathcal{O}_X^*)$:

$$\to H^1(X,\mathcal{O}_X^*)\to H^1(X,\mathrm{GL}_r(\mathcal{O}_X))\to H^1(X,\mathrm{PGL}_r(\mathcal{O}_X))\to H^2(X,\mathcal{O}_X^*)$$

In particular, the kernel of the boundary map $H^1(X,\mathrm{PGL}_r(\mathcal{O}_X))\to H^2(X,\mathcal{O}_X^*)$ is the set of isomorphism classes of Azumaya algebras $\mathcal{E}nd(E)$ with E locally free of rank r. One then shows that all these maps factor over a group homomorphism

$$\delta:\mathrm{Br}(X)\to H^2(X,\mathcal{O}_X^*),$$

i.e. for any r one has a commutative diagram

$$\begin{array}{ccc} H^1(X,\mathrm{PGL}_r(\mathcal{O}_X)) & \to & H^2(X,\mathcal{O}_X^*) \\ \downarrow & & \| \\ \mathrm{Br}(X) & \to & H^2(X,\mathcal{O}_X^*) \end{array}$$

Together with the exact sequence above one finds that $\delta:\mathrm{Br}(X)\to H^2(X,\mathcal{O}_X^*)$ is injective.

In order to see that the image of δ is contained in the torsion part one uses the commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & \mu_r & \to & \mathrm{SL}_r & \to & \mathrm{PGL}_r & \to & 1 \\ & & \cap & & \cap & & \| & & \\ 1 & \to & \mathcal{O}_X^* & \to & \mathrm{GL}_r & \to & \mathrm{PGL}_r & \to & 1 \end{array}$$

The cohomology sequence induced by the short exact sequence on the bottom provides us with a boundary map $\eta_r : H^1(X, \mathrm{PGL}_r(\mathcal{O}_X)) \to H^2(X, \mu_r)$. Since the map $H^1(X, \mathrm{PGL}_r(\mathcal{O}_X)) \to H^2(X, \mathcal{O}_X^*)$ factorizes over η_r the image of it is r-torsion. Thus, one obtains the natural injection

$$\delta : \mathrm{Br}(X) \hookrightarrow H^2(X, \mathcal{O}_X^*)_{\mathrm{tor}}$$

and $H^2(X, \mathcal{O}_X^*)_{\mathrm{tor}}$ is called the *cohomological Brauer group* $\mathrm{Br}'(X)$.

Later we will also make use of the Kummer sequence

$$1 \to \mu_r \to \mathcal{O}_X^* \xrightarrow{(\ )^r} \mathcal{O}_X^* \to 1$$

and the induced short exact sequence

$$1 \to H^1(X, \mathcal{O}_X^*)/r \cdot H^1(X, \mathcal{O}_X^*) \to H^2(X, \mu_r) \to H^2(X, \mathcal{O}_X^*)_{r-\mathrm{tor}} \to 1. \tag{1}$$

Passing to the direct limits we obtain a surjection

$$H^2(X, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\exp} \lim H^2(X, \mu_r) \twoheadrightarrow H^2(X, \mathcal{O}_X^*)_{\mathrm{tor}}.$$

The sheaf μ_r viewed as $\mathbb{Z}/r\mathbb{Z}$ also sits in the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot r} \mathbb{Z} \to \mu_r \to 0.$$

The induced boundary morphism $\beta : H^2(X, \mu_r) \to H^3(X, \mathbb{Z})$ is called the Bockstein. The Bockstein is compatible with the boundary map $H^2(X, \mathcal{O}_X^*) \to H^3(X, \mathbb{Z})$ induced by the exponential sequence. This is due to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z} & \xrightarrow{\cdot r} & \mathbb{Z} & \to & \mu_r & \to & 0 \\ & & \| & & \downarrow \cdot 1/r & & \cap & & \\ 0 & \to & \mathbb{Z} & \to & \mathcal{O}_X & \xrightarrow{\exp} & \mathcal{O}_X^* & \to & 0 \end{array}$$

Thus, a class in $\alpha \in H^2(X, \mu_r)$ induces a topologically trivial Brauer class, i.e. its image in $H^3(X, \mathbb{Z})$ is trivial, if and only if its Bockstein $\beta(\alpha)$ is trivial.

We will also need the following easy

Lemma 1.5. *Let E be a vector bundle of rank r on the complex manifold X. Then the image of $\det(E)^*$ under the natural mapping $H^1(X, \mathcal{O}_X^*) \to H^2(X, \mu_r)$ equals $\eta_r(\mathbb{P}(E))$.*

Proof. Indeed, if E is given by the cocycle $\{\varphi_{ij} \in \Gamma(U_{ij}, \mathrm{GL}_r)\}$ such that there exist $\lambda_{ij} \in \Gamma(U_{ij}, \mathcal{O}_X^*)$ with $\lambda_{ij}^r = \det(\varphi_{ij})$, then $\det(E)^* \in H^1(X, \mathcal{O}_X^*)$ is given by $\{\det(\varphi_{ij})^{-1}\}$ and its image in $H^2(X, \mu_r)$ by $\{\lambda_{ij}^{-1}\lambda_{jk}^{-1}\lambda_{ik}\}$.

On the other hand, $[\mathbb{P}(E)] = \{\overline{\varphi_{ij}}\} = \{\overline{\varphi_{ij} \cdot \lambda_{ij}^{-1}}\} \in H^1(X, \mathrm{PGL}_r(\mathcal{O}_X))$ and $\varphi_{ij} \cdot \lambda_{ij}^{-1} \in \mathrm{SL}_r$. This shows that the image of $[\mathbb{P}(E)]$ in $H^2(X, \mu_r)$ is given by the cocycle $\{\varphi_{ij}\lambda_{ij}^{-1}\varphi_{jk}\lambda_{jk}^{-1}(\varphi_{ik}\lambda_{ik}^{-1})^{-1}\}$. $\square$

In case that X is projective one may as well work with the étale topology. The Kummer sequence immediately shows that the two cohomological Brauer groups coincide. Eventually, note that any Brauer class on a K3 surface is automatically topologically trivial, for $H^3(X,\mathbb{Z})=0$.

2. Recollection: K3 surfaces

As a reference for the theory of K3 surfaces we recommend [1]. By definition a K3 surface is a compact complex surface X such that the canonical bundle K_X is trivial and $H^1(X,\mathcal{O}_X)=0$. It has been shown by Siu that any K3 surface is Kähler. Thus, the Kähler cone $\mathcal{K}_X\subset H^{1,1}(X,\mathbb{R})$ of all Kähler classes is a non-empty open convex cone. Moreover, due to Yau's solution of the Calabi-conjecture any class in $\omega\in\mathcal{K}_X$ can uniquely be represented by a Ricci-flat Kähler form, which we will also call ω.

For holomorphic symplectic (also called hyperkähler) manifolds in general and for K3 surfaces in particular any Ricci-flat Kähler metric g is actually hyperkähler. More precisely this means that in addition to the given complex structure I defining X there exist complex structures J and $K=I\circ J=-J\circ I$ all making g a Kähler metric. The induced sphere of complex structures $\{aI+bJ+cK\mid a^2+b^2+c^2=1\}$ is isomorphic to a projective line called $\mathbb{P}(\omega)\cong\mathbb{P}^1$.

Twistor theory of K3 surfaces shows that there exists a universal complex structure on $X\times\mathbb{P}(\omega)$. This complex manifold will be called $\mathcal{X}(\omega)$. Moreover, it is known that the projection $\mathcal{X}(\omega)\to\mathbb{P}(\omega)$ is holomorphic, and by definition the fiber over a complex structure λ is isomorphic to X endowed with this structure. Usually we will denote by $0\in\mathbb{P}(\omega)$ the point that corresponds to the original complex structure.

Although the twistor space $\mathcal{X}$ itself is not kähler, each fiber $\mathcal{X}_t$ comes with a natural Kähler form ω_t. If t corresponds to the complex structure λ_t then $\omega_t=g(\lambda_t(\ ),\ )$. In particular, $\omega_0=\omega$.

Lemma 2.1. *Let $\alpha\in H^2(X,\mathbb{Z})$ with $\alpha^2>0$. Then there exists $t\in\mathbb{P}(\omega)$ such that α is of type $(1,1)$ on $\mathcal{X}_t$ and $\mathcal{X}_t$ is projective.*

Proof. We denote the holomorphic two-form (or rather its cohomology class) on $\mathcal{X}_t$ by σ_t. Thus, $Q_\omega:=\{\sigma_t\}\subset\mathbb{P}(\mathbb{C}\omega_I\oplus\mathbb{C}\omega_J\oplus\mathbb{C}\omega_K)$ is the quadric defined by the intersection form restricted to the three-dimensional space spanned by $\omega_I,\omega_J,\omega_K$. The class α is of type $(1,1)$ with respect to the complex structure corresponding to $t\in\mathbb{P}(\omega)$ if and only if $\alpha\wedge\sigma_t=0$. Since the intersection of the quadric Q_ω with the hyperplane defined by $\alpha\wedge x=0$ is non-empty, one finds a point t as required.

The second assertion follows from the general fact that a compact complex surface is projective if and only if there exists a line bundle L with $\mathrm{c}_1(L)^2>0$. This is applied to the line bundle L whose first Chern class is α. □

We next discuss stable and hyperholomorphic bundles. Stability will always mean slope-stability and if we write ω_t-stable it means slope-stable with respect to the Kähler class ω_t on $\mathcal{X}_t$.

Definition 2.2. *A vector bundle E (or, projective bundle P) on $\mathcal{X}_t$ is called* hyperholomorphic *(with respect to the twistor space $\mathcal{X}(\omega)$) if there exists a vector bundle $\mathcal{E}$ (respectively, projective bundle $\mathcal{P}$) on $\mathcal{X}(\omega)$ such that $\mathcal{E}|_{\mathcal{X}_t} \cong E$ (respectively, $\mathcal{P}|_{\mathcal{X}_t} \cong P$).*

Since the twistor space $\mathcal{X}(\omega)$ comes with the natural $\mathcal{C}^\infty$-trivialization $\mathcal{X}(\omega) \cong X \times \mathbb{P}(\omega)$, any holomorphic vector bundle E on a fiber $\mathcal{X}_t$ lives naturally as a complex (!) vector bundle on any other fiber $\mathcal{X}_s$. Thus, E is hyperholomorphic if and only if the complex vector bundle E admits a $\bar{\partial}$-operator on any $\mathcal{X}_s$ depending holomorphically on s.

The following result is due to Verbitsky. For the convenience of the reader we include a sketch of its proof.

Proposition 2.3. *Let E be an ω_t-stable vector bundle on $\mathcal{X}_t$. Then the associated projective bundle $\mathbb{P}(E)$ is hyperholomorphic with respect to $\mathcal{X}(\omega)$ (see* [16]*).*

Proof. To give an idea of the proof we first assume that the determinant of E is trivial. Then we show that the vector bundle E itself is hyperholomorphic and hence also $\mathbb{P}(E)$. Due to Donaldson's result on the existence of Hermite-Einstein metrics there exists a hermitian metric on E such that the curvature $F_\nabla \in \mathcal{A}^2(X, \mathrm{End}(E))$ of the induced Chern connection ∇ on E satisfies $\Lambda_{\omega_t} F_\nabla = 0$ (this is the Hermite-Einstein condition). Since we are working on a surface, this condition is equivalent to $F \wedge \omega = 0$. We are going to show that the connection ∇ is hyperholomorphic, i.e. with respect to any other complex structure given by a point $s \in \mathbb{P}(\omega)$ the $(0,1)_s$-part $\nabla^{(0,1)_s}$ of ∇ is a $\bar{\partial}$-operator for the complex bundle E on $\mathcal{X}_s$. Indeed, $(\nabla^{(0,1)_s})^2$ is the $(0,2)_s$-part of the curvature F_∇. By definition of the Chern connection, F_∇ is of type $(1,1)$ on $\mathcal{X}_t$. Thus, its suffices to show that it stays of type $(1,1)$ when $\mathcal{X}_t$ is deformed to $\mathcal{X}_s$.

In fact, it suffices to show that F is of type $(1,1)$ for J and K, because then it will automatically be of type $(1,1)$ with respect to any complex structure parameterized by $\mathbb{P}(\omega)$. Since the curvature F is real, it is of type $(1,1)$ with respect to J if and only if $F \wedge \sigma_J = 0$, where σ_J is the holomorphic two-form with respect to the complex structure J. Using $\sigma_J = \omega_K + i\omega_I$ the Hermite-Einstein condition immediately yields $F \wedge \mathrm{Im}(\sigma_J) = 0$. On the other hand, $\mathrm{Re}(\sigma_J) = \omega_K = \mathrm{Im}(\sigma_I)$. Hence, $F \wedge \mathrm{Re}(\sigma_J) = F \wedge \mathrm{Im}(\sigma_I) = 0$, since F is of type $(1,1)$ with respect to I. Analogously one proves $F \wedge \sigma_K = 0$.

If the determinant of E is no longer trivial, then the Hermite-Einstein condition for E reads $F \wedge \omega_t = \omega^2 \cdot \mu \cdot \mathrm{id}_E$, where μ is a constant measuring the degree of $\det(E)$. The given Chern connection ∇ for E on $\mathcal{X}_t$ defines a holomorphic structure on the induced projective bundle $\mathbb{P}(E)$ with respect to another complex structure $s \in \mathbb{P}(\omega)$ if and only if the tracefree part F_0 of F, given by $F = F_0 + \frac{\mathrm{tr}(F)}{r}\mathrm{id}_E$, is of type $(1,1)$ with respect to s. In order to see this we replace F by F_0 in the argument above and use the Hermite-Einstein condition. $\square$

3. Proof

The rough idea of the proof of Theorem 0.1 is as follows. In order to show the assertion it suffices to realize any element $\bar{\alpha} \in H^2(X, \mu_r)$ as the class $\eta_r(\mathcal{A})$ of some Azumaya algebra. Here we use the surjectivity of the map $H^2(X, \mu_r) \to H^2(X, \mathcal{O}_X^*)_{r-\mathrm{tor}}$. Clearly, if $\mathcal{A}_t$ is a flat family of Azumaya algebras over a deformation $\mathcal{X}_t$ of $\mathcal{X}_0 = X$, then $\eta_r(\mathcal{A}_t)$ is constant. Thus, we may try to first deform X to some $\mathcal{X}_t$ and prove the existence of the required Azumaya algebra there. In order to be able to deform back to X we have to find a 'hyperholomorphic' Azumaya algebra on $\mathcal{X}_t$. The reason why it might be easier to find an Azumaya algebra on some deformation is the fact that the Picard group changes when X is deformed and one can arrange things such that the given class in $H^2(X, \mu_r)$ is actually in the image of $H^1(X, \mathcal{O}_X^*) \to H^2(X, \mu_r)$ induced by the Kummer sequence. Thus, the complex structure is deformed in a way that the Brauer class induced by $\bar{\alpha}$ becomes trivial. It is a curious fact that by making the assertion trivial on some deformation of X one can in the end conclude the existence of a non-trivial Azumaya algebra on X itself. Note that in his article [13], de Jong also used the fact that under deformation of the surface a trivial Azumaya algebra sometimes becomes non-trivial.

Let us now pass on to the details of the proof. Let $\beta \in \mathrm{Br}'(X)$ be r-torsion and choose a lift $\bar{\alpha} \in H^2(X, \mu_r)$.

Lemma 3.1. *There exists an element $\alpha \in H^2(X, \mathbb{Z})$ inducing $\bar{\alpha}$ such that $\alpha^2 > 0$.*

Proof. This is elementary: Choose any class α that lifts $\bar{\alpha}$ and add $k \cdot r \cdot \beta$ with $k \gg 0$ and $\beta \in H^2(X, \mathbb{Z})$ an arbitrary class with $\beta^2 > 0$. □

The next lemma is a refinement of Lemma 2.1

Lemma 3.2. *If $\omega \in \mathcal{K}_X$ is a very general Kähler class on X then there exists a point t in the twistor base $\mathbb{P}(\omega)$ such that the fiber $\mathcal{X}_t$ is projective with Picard number one and α is of type $(1,1)$ on $\mathcal{X}_t$.*

Proof. This is proved by standard techniques. We nevertheless provide all details. Let $V_{\mathbb{R}} = (\mathcal{K}_X - \{0\})/\mathbb{R}^*$ be the real projectivization of the Kähler cone $\mathcal{K}_X$. It is an open subset of the real projective space $(H^{1,1}(X, \mathbb{R}) - \{0\})/\mathbb{R}^*$ and is contained in the complex manifold $V = (\mathbb{C}\mathcal{K}_X - \{0\})/\mathbb{C}^*$, the complex projectivization of the open convex cone $\mathbb{C}\mathcal{K}_X = \mathcal{K}_X + \sqrt{-1}H^{1,1}(X, \mathbb{R}) \subset H^{1,1}(X, \mathbb{C})$. Locally $V_{\mathbb{R}} \subset V$ looks like $\mathbb{R}^{20} \subset \mathbb{C}^{20}$.

Fix some Kähler class $[\omega_0] \in \mathcal{K}_X$. We now construct an open neighborhood $U \subset V$ of $[\omega_0]$ and an open embedding $\varphi : U \to S_\alpha$, where S_α is the hypersurface in the moduli space $\mathcal{M}$ of marked K3 surfaces $(Y, \psi : H^2(Y, \mathbb{Z}) \cong H^2(X, \mathbb{Z}))$ such that $\psi^{-1}(\alpha)$ is of type $(1,1)$ on Y.

By definition S_α is a hyperplane section of the moduli space $\mathcal{M}$ and thus intersects any 'generalized twistor line' $\mathbb{P}(\omega) := \mathcal{P}^{-1}(\mathbb{P}(\mathbb{C}\omega \oplus \mathbb{C}\sigma \oplus \mathbb{C}\bar{\sigma}))$ for any $\omega \in V$ close to

ω_0 either in a non-reduced point or in two different points. Here, $\mathcal{P}$ is the period map $\mathcal{P} : \mathcal{M} \to \mathbb{P}(H^2(X, \mathbb{C}))$. Let $D \subset V$ be the subset of all $[\omega]$ such that $\mathbb{P}(\omega) \cap S_\beta$ is non-reduced. This is a closed complex subspace which does not intersect $V_\mathbb{R}$. Thus, we may find an open neighborhood $U \subset V \setminus D$ of $[\omega_0]$. Choosing locally one of the two intersection points yields an open embedding $\varphi : U \to S_\beta$.

Now set $S := \cup S_\beta$, where the union, which is countable, runs over all classes $\beta \in H^2(X, \mathbb{Z})$ that are $\mathbb{Q}$-linear independent from $\alpha \in H^2(X, \mathbb{Z})$. Set $u = [\omega_0] \in U$ and $s = \varphi(u) \in S_\beta$. Suppose that $\varphi(U_\mathbb{R}) \subset S$. It then follows for the real tangent spaces that

$$\varphi_*(T_u U_\mathbb{R}) \subset \bigcup T_s(S_\alpha \cap S_\beta)$$

and hence $\varphi_*(T_u U_\mathbb{R}) \subset T_s(S_\alpha \cap S_\beta)$ for some β. Clearly, $T_s(S_\alpha \cap S_\beta) \subset T_s S_\alpha$ is a complex subspace of complex codimension ≥ 1, and moreover the induced map $\varphi_* : T_u U = T_u U_\mathbb{R} \otimes \mathbb{C} \to T_s S_\alpha$ factors over $T_s(S_\alpha \cap S_\beta)$. On the other hand, the $\mathbb{C}$-linear map $\varphi_* : T_u U \to T_s S_\alpha$ is bijective, and this yields a contradiction.

Consequently, for the very general $[\omega] \in U_\mathbb{R}$ the image $\varphi([\omega]) \in S_\alpha$ is not contained in any S_β. Hence, ω and $t := \varphi([\omega])$ satisfy the assertion. □

In particular the K3 surface $\mathcal{X}_t$ as in the previous lemma will be projective and $\alpha = \mathrm{c}_1(L)$ for some holomorphic line bundle L. Furthermore, $\mathcal{X}_t$ is endowed with a natural Kähler structure ω_t.

Proposition 3.3. *There exists an ω_t-stable vector bundle E on $\mathcal{X}_t$ of rank r and with* $\det(E) = L^*$.

Proof. Since the Picard number of $\mathcal{X}_t$ is one, there exists only one polarization H (up to scaling). It is a standard result (eg. [11]) that for a given polarization H and determinant there exists a H-stable vector bundle (for high enough second Chern number). Since the Picard number is one, a H-stable vector bundle will in fact be stable with respect to any Kähler class. This way we find plenty of ω_t-stable vector bundles E with $\det(E) \cong L^*$. □

End of the proof of Theorem 0.1. Due to Proposition 2.3 we have found a holomorphic projective bundle $\mathcal{P}$ on $\mathcal{X}(\omega)$ such that $\mathcal{P}|_{\mathcal{X}_t} \cong \mathbb{P}(E)$, where E is a holomorphic vector bundle on $\mathcal{X}_t$ of rank r and with $\mathrm{c}_1(E) = -\alpha$.

Due to Lemma 1.5 the class $\eta_r(\mathbb{P}(E))$ equals $\bar{\alpha}$. Hence, for any $s \in \mathbb{P}(\omega)$ one has $\eta_r(\mathcal{P}_s) = \bar{\alpha}$. In particular, $\bar{\alpha}$ is realized by a projective bundle and hence by an Azumaya algebra on $X = \mathcal{X}_0$.

This eventually yields that the given cohomological Brauer class β on X is the class of an Azumaya algebra on X, and the proof of Theorem 0.1 is complete.

In order to see that $\mathrm{Br}(X) \cong (\mathbb{Q}/\mathbb{Z})^{22-\rho}$ use the exponential sequence. The kernel of $H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X) \cong \mathbb{C}$ is the Picard group and $H^2(X, \mathbb{Z})$ is of rank 22. Hence, the torsion part of $H^2(X, \mathcal{O}_X)/H^2(X, \mathbb{Z})$ is isomorphic to $(\mathbb{Q}/\mathbb{Z})^{22-\rho}$. Actually, the latter argument has not much to do with K3 surfaces, and the same reasoning shows that the topologically trivial part of $\mathrm{Br}'(X)$ equals $(\mathbb{Q}/\mathbb{Z})^{b_2-\rho}$ for any compact complex space.

4. Further comments

1. The result for K3 surfaces is enough to prove the following

Corollary 4.1. *Let X be a Ricci-flat compact Kähler surface, i.e.* $0 = \mathrm{c}_1(X) \in H^2(X, \mathbb{R})$. *Then* $\mathrm{Br}(X) = \mathrm{Br}'(X)$.

Proof. In fact any compact Ricci-flat Kähler manifold X admits a finite étale cover $\tilde{X} \to X$ such that $\tilde{X}$ is isomorphic to a product of complex tori, hyperkähler, and Calabi-Yau manifolds (cf. [2]). Thus, our surface X admits a finite étale cover $\pi : \tilde{X} \to X$ such that $\tilde{X}$ is either a K3 surface or a complex torus. For complex tori due to [9, 3, 6] and for K3 surfaces due to Thm. 0.1 one knows that $\mathrm{Br}(\tilde{X}) \cong \mathrm{Br}'(\tilde{X})$. Using a result of Gabber for finite maps [7, Ch.II, Lemma 4] one concludes from this $\mathrm{Br}(X) = \mathrm{Br}'(X)$. □

2. It is tempting to try to extend the above proof to the case of compact hyperkähler manifolds, which generalize the notion of K3 surfaces in many respects. In fact, most of the arguments go through. E.g. Lemma 3.2 can be proved in arbitrary dimension by using results of [12], in particular the projectivity criterion. One could hope to prove this way that any topologically trivial Brauer class on a compact Ricci-flat Kähler manifold is contained in the Brauer group. The point were the proof as presented is insufficient for higher-dimensional hyperkähler manifolds is in Proposition 2.3. In the higher-dimensional version of this result, also due to Verbitsky, one needs to assume that the stable vector bundle E with first Chern class α has a discriminant $2r\mathrm{c}_2(E) - (r-1)\mathrm{c}_1(E)^2$ of type $(2,2)$ with respect to any complex structure parametrized by the twistor space in question. Of course, in complex dimension two, the latter condition is automatic, since $H^{2,2}(X) = H^4(X)$. In higher dimension however, $H^{2,2}(X)$ is strictly smaller than $H^4(X)$ and, moreover, depends on the complex structure. To show the existence of the required stable bundle seems highly non-trivial. In fact, the only hyperholomorphic projective bundles on higher-dimensional hyperkähler manifolds known to us are all descendants of the tangent bundle (which is stable!) and we were not able to produce a bundle that would serve our purpose.

As before, the result for topologically trivial Brauer classes on hyperkähler manifolds together with the result of Hoobler, Berkovich, Elencwajg, and Narasimhan would show that any topologically trivial Brauer class on a compact Ricci-flat Kähler variety is contained in the Brauer group. (One uses that by definition $H^2(X, \mathcal{O}_X) = 0$ for a true Calabi-Yau manifold.) This would certainly be strong evidence for an affirmative answer of Grothendieck's question. Note however that topologically non-trivial classes on Calabi-Yau threefolds are of particular interest in M-theory and for those the question remains open.

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