WHAT IS MISSING IN CANONICAL MODELS FOR PROPER NORMAL ALGEBRAIC SURFACES?

STEFAN SCHRÖER

ABSTRACT. Smooth surfaces have finitely generated canonical rings and projective canonical models. For normal surfaces, however, the graded ring of multicanonical sections is possibly nonnoetherian, such that the corresponding homogeneous spectrum is noncompact. I construct a canonical compactification by adding finitely many non-$\mathbb{Q}$-Gorenstein points at infinity, provided that each Weil divisor is numerically equivalent to a $\mathbb{Q}$-Cartier divisor. Similar results hold for arbitrary Weil divisors instead of the canonical class.

INTRODUCTION

Each proper normal algebraic surface X comes along with the graded ring $R(K_X) = \bigoplus_{n \geq 0} H^0(nK_X)$ and the homogeneous spectrum $P(K_X) = \operatorname{Proj} R(K_X)$. If X has canonical singularities (that is, rational Gorenstein singularities), these are called the *canonical ring* and the *canonical model*. It is then a classical theorem, and Mori theory reassures us, that the ring $R(K_X)$ is finitely generated, such that the scheme $P(K_X)$ is projective.

A natural question to ask: What happens for proper surfaces with arbitrary normal singularities? The usual approach is to pass to smooth models, but I want so see what happens on the singular surface itself. To my knowledge, neither canonical rings nor canonical models were studied from this viewpoint, possibly because Zariski [10] observed that the ring $R(K_X)$ might be nonnoetherian. However, $P(K_X)$ is always a scheme of finite type [9], so it makes perfect sense to analyze it from a geometric point of view.

According to the Nagata Compactification Theorem, we can embed $P(K_X)$ into a proper normal scheme by adding points and curves at infinity. I prefer to add just points, because such compactifications are minimal and therefore unique. In this paper, we shall construct such a compactification $P(K_X) \subset \overline{P}(K_X)$ with discrete boundary by adding finitely many non-$\mathbb{Q}$-Gorenstein points at infinity. I have, however, to assume that each Weil divisor on X is numerically equivalent to a $\mathbb{Q}$-Cartier divisor. This holds, for example, for surfaces with geometric genus $p_g = 0$, or for surfaces with rational singularities. Much of this generalizes to $P(D)$, where $D \in Z^1(X)$ is an arbitrary Weil divisor.

The paper is organized as follows. In Section 1, we collect some facts on the intersection of the base loci $\bigcap \operatorname{Bs}(nD)$. In Section 2, we determine the structure of the rational map $r_D : X \dashrightarrow P(D)$. In the next section, we use this rational map and construct a compactification $P(D) \subset \overline{P}(D)$ as an algebraic space. Section 4

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contains some results on the multiplicities occurring in the base loci $\mathrm{Bs}(nD)$. In the last section, we apply our results to the canonical model $P(K_X)$.

1. Stable base locus

Throughout the paper, X is a proper normal algebraic surface. In other words, a normal 2-dimensional scheme proper over an arbitrary base field k. Each Weil divisors $D \in Z^1(X)$ yields a graded algebra $R(D) = \bigoplus_{n \geq 0} H^0(nD)$, which in turn defines the homogeneous spectrum $P(D) = \mathrm{Proj}\, R(D)$. Here we use $H^0(nD)$ as a shorthand for $H^0(X, \mathcal{O}_X(nD))$. Let me quote the following fact ([9], Section 6).

Proposition 1.1. *The homogeneous spectrum $P(D)$ is a normal algebraic scheme of dimension ≤ 2. Furthermore, if $\dim P(D) \leq 1$, then $P(D)$ is projective.*

The interesting point is that $P(D)$ is of finite type over k (which does not necessarily hold in higher dimensions). By definition, there is an affine open covering

$$P(D) = \bigcup \mathrm{Spec}(R(D)_{(s)})$$

where the union runs over all homogeneous $s \in R(D)$ of positive degree. Note that, if $X_s \subset X$ is the open subset where $s : \mathcal{O}_X \to \mathcal{O}_X(nD)$ is bijective, we have $\Gamma(X_s, \mathcal{O}_X) = R(D)_{(s)}$. Thus we can write

$$P(D) = \bigcup X_s^{\mathrm{aff}},$$

where $X_s^{\mathrm{aff}} = \mathrm{Spec}\,\Gamma(X_s, \mathcal{O}_X)$ is the *affine envelope*. Gluing the canonical morphisms $U \to U^{\mathrm{aff}}$ gives a rational map $r_D : X \dashrightarrow P(D)$.

The *base locus* $\mathrm{Bs}(nD) \subset X$ is the intersection of all curves $C \sim D$ (linear equivalence). Following Fujita [3], Definition 1.17, we define the *stable base locus*

$$\mathrm{SBs}(D) = \bigcap_{n > 0} \mathrm{Bs}(nD) \subset X.$$

Obviously, the rational map $r_D : X \dashrightarrow P(D)$ is defined on $X - \mathrm{SBs}(D)$. Let us collect some facts on stable base loci. Clearly, $x \in \mathrm{SBs}(D)$ if the divisor germ $D_x \in Z^1(\mathcal{O}_{X,x})$ is not $\mathbb{Q}$-Cartier. There is a partial converse as follows. Define

$$\mathrm{SBs}^0(D) \subset \mathrm{SBs}(D)$$

to be the 0-dimensional part of the stable base locus.

Proposition 1.2. *For each $x \in \mathrm{SBs}^0(D)$, the divisor germ $D_x \in Z^1(\mathcal{O}_{X,x})$ is not $\mathbb{Q}$-Cartier.*

Proof. Clearly $\mathrm{SBs}(D) = \mathrm{SBs}(nD)$ for all integers $n > 0$, so we may assume that D is a curve. Seeking a contradiction, we assume that D_x is $\mathbb{Q}$-Cartier. Passing to a suitable multiple, we may assume that it is Cartier. Set $\mathcal{I} = \mathcal{O}_X(-D)$. The blowing up $Y = \mathrm{Proj}(\bigoplus \mathcal{I}^n)$ and the invertible sheaf $\mathcal{O}_Y(1)$ depend only on the linear equivalence class of D. Let $\tilde{Y}$ be the normalization of the blowing-up. The induced morphism $f : \tilde{Y} \to Y$ is bijective near x, and $y = f^{-1}(x)$ is an isolated

base point for the Cartier divisor $f^{-1}(D)$. The latter contradicts the Fujita–Zariski Theorem ([3], Theorem 1.19). □

Next, we consider the 1-dimensional part of the stable base locus.

Proposition 1.3. *Suppose $D \in Z^1(X)$ is a Weil divisor with $H^0(nD) \neq 0$ for some $n > 0$. Let $E \subset X$ be a curve supported by $\mathrm{SBs}(D)$. Then the canonical map $H^0(D) \to H^0(D + E)$ is bijective.*

Proof. Inducting on the number of irreducible components of E, we may assume that E is irreducible. Assuming that E is also reduced, we have to check that the canonical inclusion $H^0(D) \subset H^0(D + mE)$ is bijective for all $m > 0$. Suppose to the contrary that $D + mE \sim A + m'E$ for some $0 \leq m' < m$ and a curve $A \subset X$ not containing E. Subtracting $m'E$, we may assume $D + mE \sim A$. Decompose $nD = B + lE$ for some integer $l \geq 1$ and a curve $B \subset X$ not containing E. Then

$$(mn + l)D \sim mB + mlE + lD \sim mB + lA,$$

contradicting $E \subset \mathrm{SBs}(D)$. □

A curve $E \subset X$ is called *negative definite* if the intersection matrix $(E_i \cdot E_j)$ is negative definite, where $E_i \subset E$ are the irreducible components. Here we use Mumford's rational intersection numbers [8].

Proposition 1.4. *Let $D \in Z^1(X)$ be a Weil divisor, and $E \subset X$ a negative definite curve. If $D \cdot E_i = 0$ for all irreducible components $E_i \subset E$, then the canonical map $H^0(D) \to H^0(D + E)$ is bijective.*

Proof. We may assume that E is reduced and have to check that the inclusion $H^0(D) \to H^0(D + \sum \mu_i E_i)$ is surjective for all integers $\mu_i \geq 0$. Suppose there is a linear equivalence $D + \sum \mu_i E_i \sim A + \sum \lambda_i E_i$ for some curve $A \subset X$ not containing any E_i, and certain integers $\lambda_i \geq 0$. Since E is negative definite, there is a unique $\mathbb{Q}$-divisor $\sum \gamma_i E_i$ with

$$\sum \gamma_i E_i \cdot E_j = A \cdot E_j \quad \text{for all } E_j \subset E.$$

By [4], Equation 7, we have $\gamma_i \leq 0$ because $A \cdot E_j \geq 0$. Consequently, $\sum \mu_i E_i \cdot E_j = \sum (\gamma_i + \lambda_i) E_j \cdot E_j$, therefore $\mu_i \leq \lambda_i$. In other words, $\sum \mu_i E \subset A + \sum \lambda_i E_i$, hence $H^0(D) \to H^0(D + E)$ is bijective. □

Remark 1.5. A curve $E \subset X$ is called *contractible* if there is a proper birational morphism $f : X \to Y$ of proper normal algebraic surfaces so that $X - R$ is the isomorphism locus. The preceding results imply that the graded ring $R(D)$ does not change under certain contractions. Namely, if either $E \subset \mathrm{SBs}(D)$, or $D \cdot E_i = 0$ for all irreducible components $E_i \subset E$, then the canonical injection $R(D) \subset R(f_*(D))$ is bijective.

2. Rational maps defined by Weil divisors

Fix a proper normal algebraic surface X and a Weil divisor $D \in Z^1(X)$. The task now is to describe, in geometric terms, the rational map $r_D : X \dashrightarrow P(D)$. The following decomposition is useful for this. For each $n \geq 0$ so that $H^0(nD) \neq 0$, write

$$nD \sim M_n + F_n,$$

where $F_n \subset X$ is the *fixed part*, and $M_n = nD - F_n$ is the *movable part* of nD. Note that $H^0(M_n) = H^0(nD)$ and that $M_n \cdot C \geq 0$ for all curves $C \subset X$. Furthermore, decompose

$$F_n = F_n' + F_n'',$$

where $F_n' \subset F_n$ is the part of F_n consisting of all connected components $C \subset F_n$ with $C \cdot M_n > 0$. For n sufficiently divisible, the support of F_n is the 1-dimensional part of $\mathrm{SBs}(D)$.

Proposition 2.1. *Suppose that $P(D)$ is a surface and assume $\mathrm{Bs}(nD) = \mathrm{SBs}(D)$. Then there is a maximal reduced curve $R \subset X$ with $M_n \cdot R = 0$ and $F_n' \cap R = \emptyset$. Furthermore, R is negative definite and contractible.*

Proof. Fix a connected component $C \subset F_n'$ and decompose $C = C_1 + \ldots + C_r$ into irreducible components. Rearranging indices and allowing repetitions, we may assume that $M_n \cdot C_1 > 0$ and that the intersections $C_i \cap C_{i+1}$ are 0-dimensional. Choose a rational $\lambda_1 > 0$ so that $A_1 = \lambda_1 C_1$ satisfies $(M_n + A_1) \cdot C_1 > 0$. Inductively, define $A_i = A_{i-1} + \lambda_i C_i$ for some rational number $\lambda_i > 0$ so that $(M_n + A_i) \cdot C_j > 0$ for $1 \leq j \leq i$. We end up with an effective $\mathbb{Q}$-divisor $A = A_r$ with support C.

Repeating this for the other connected components of F_n', we see that there is an effective $\mathbb{Q}$-divisor A with support F_n' so that $(M_n + A) \cdot C_j > 0$ for all irreducible components $C_j \subset F_n'$. Then

$$(M_n \cdot R = 0 \quad \text{and} \quad F_n' \cap R = \emptyset) \quad \Longleftrightarrow \quad (M_n + A) \cdot R = 0$$

holds for each curve $R \subset X$. By the Hodge Index Theorem, there is a maximal reduced curve $R \subset X$ satisfying $(M_n + A) \cdot R = 0$, and this curve is negative definite.

It remains to check that each connected component $R_i \subset R$ is contractible. Choose a curve $A \sim M_n$. Since M_n is movable, each R_i is either disjoint from $A \cup F_n'$ or entirely contained in A. Now the contraction criterion [9], Proposition 3.3 applies, and we deduce that R_i is contractible. $\square$

The rational map $r_D : X \dashrightarrow P(D)$ has a maximal open subset $\mathrm{dom}(r_D) \subset X$ on which it is definable. This open subset is called the *domain of definition*. Clearly, $X - \mathrm{SBs}(D) \subset \mathrm{dom}(r_D)$. This inclusion, however, might be strict. To explain this, let $n > 0$ be an integer satisfying $\mathrm{Bs}(nD) = \mathrm{SBs}(D)$. Write

$$\mathrm{SBs}(D) = \mathrm{SBs}'(D) \cup \mathrm{SBs}''(D) \cup \mathrm{SBs}^0(D),$$

where $\mathrm{SBs}^0(D)$ is the 0-dimensional part, $\mathrm{SBs}'(D)$ is the part corresponding to F_n', and $\mathrm{SBs}''(D)$ is the part corresponding to F_n''. The next result tells us that this decomposition depends only on the Weil divisor D, and not on the integer $n > 0$:

Theorem 2.2. *The open subset $X - (\mathrm{SBs}(D)' \cup \mathrm{SBs}^0(D))$ is the domain of definition for the rational map $r_D : X \dashrightarrow P(D)$. Furthermore, the induced morphism $r_D : \mathrm{dom}(r_D) \to P(D)$ is proper.*

Proof. First suppose that $P(D)$ is a curve. Then $M_n^2 = 0$ and $F_n' = 0$ by [9], Proposition 6.5. So M_n is a globally generated Cartier divisor, and $P(D) = P(M_n)$. Consequently, the rational map r_D is everywhere defined. Furthermore $\mathrm{SBs}^0(D) = \emptyset$, and the assertion follows.

Now assume that $P(D)$ is a surface. Choose a curve $C \sim M_n$ and set $U = X - (C \cup F_n)$. Then $P(D)$ is covered by affine open subsets of the form U^{aff}. Let $A \subset X$ and $R \subset X$ be the curves from the proof of Proposition 2.1, and set

$$V = X - (C \cup F_n') = X - (C \cup A).$$

By [9], Proposition 3.2, the morphism $V \to V^{\mathrm{aff}}$ is proper, hence its exceptional curve is $R \cap V$. Consequently, $U^{\mathrm{aff}} = V^{\mathrm{aff}}$, and $P(D)$ is nothing but the contraction of $R \subset X - (F'_n \cup \mathrm{SBs}^0(D))$. This implies that $X - (F'_n \cup \mathrm{SBs}^0(D))$ is the domain of definition for the rational map $r_D : X \dashrightarrow P(D)$. $\qquad\square$

The preceding proof shows more, namely:

Corollary 2.3. *Suppose that $P(D)$ is a surface. Then the curve $R \subset X$ from Proposition 2.1 is the exceptional set for the proper morphism $\mathrm{dom}(r_D) \to P(D)$.*

Finally, we mention that the ring $R(D)$ already lives on the algebraic surface $P(D)$. In some sense, this reduced the study of Weil divisors on proper surfaces to the study of $\mathbb{Q}$-Cartier divisors on algebraic surfaces.

Proposition 2.4. *Suppose that $P(D)$ is a surface, and set $D' = (r_D)_*(D|_{\mathrm{dom}(r_D)})$. Then the canonical map $R(D) \to R(D')$ is bijective.*

Proof. Given an integer $n > 0$, we have to check that the inclusion $H^0(nD) \subset H^0(nD + E)$ is bijective for each curve $E \subset X$ supported by $\mathrm{SBs}(D)$. But this follows from Proposition 1.3. $\qquad\square$

3. Compactification

Let D be a Weil divisor on a proper normal algebraic surface X, and assume that $P(D)$ is 2-dimensional. This algebraic surfaces is not necessarily proper. The task now is to construct *compactifications* of $P(D)$, that is, proper normal surfaces containing $P(D)$ as an open dense subset.

We start with a rather simple compactification. Let $R \subset X$ be the contractible curve from Proposition 2.1 and $f : X \to Y$ be its contractions.

Proposition 3.1. *There is an open embedding $P(D) \subset Y$, and the complement is isomorphic to $\mathrm{SBs}'(D) \cup \mathrm{SBs}^0(D)$.*

Proof. According to Corollary 2.3, we have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{dom}(r_D) & \longrightarrow & X \\
\downarrow & & \downarrow \\
P(D) & \longrightarrow & Y,
\end{array}
$$

where both vertical arrows are the contractions of R. Hence the lower horizontal arrow exists and gives the desired open embedding. By Theorem 2.2, its complement equals $\mathrm{SBs}'(D) \cup \mathrm{SBs}^0(D)$. $\qquad\square$

The boundary at infinity for $P(D) \subset Y$ contains the 1-dimensional part $\mathrm{SBs}'(X)$, so this compactification might not be minimal. The following is useful for the construction of smaller compactifications:

Lemma 3.2. *Each 1-dimensional connected component of $C \subset \mathrm{SBs}(D)$ contains an irreducible component C_i with $D \cdot C_i \leq 0$.*

Proof. Suppose to the contrary that $D \cdot C_i > 0$ for all irreducible components $C_i \subset C$. Let $f : Y \to X$ be a resolution of the singularities contained in C, and $E \subset Y$ the exceptional divisor. Since E is negative definite, there is a divisor $A \in Z^1(Y)$ supported by E and ample on E. For $n > 0$ sufficiently large, the divisor $D' = A + f^*(nD)$ is effective and ample on $f^{-1}(C)$. On the other hand, since

$nD = f_*(D')$), the preimage $f^{-1}(C)$ contains a connected component of $\mathrm{SBs}(D')$. This contradicts the Fujita–Zariski Theorem ([3], Theorem 1.19). □

Proposition 3.3. *The 1-dimensional part of* $\mathrm{SBs}(D)$ *is negative definite.*

Proof. We saw in Proposition 2.1 that $\mathrm{SBs}''(D)$ is negative definite, so it remains to see that each connected component $C \subset \mathrm{SBs}'(D)$ is negative definite. To do so, choose $n > 0$ so that $\mathrm{Bs}(nD) = \mathrm{SBs}(D)$, decompose $C = C_1 + \ldots + C_r$ into irreducible components, and let $B \subset F_n'$ be the part disjoint to C.

First, assume that some linear combination $A = \sum \lambda_i C_i$ has $A^2 > 0$. As in the proof of [9], Proposition 3.2, we may assume that $\lambda_i > 0$ and $A \cdot C_i > 0$ for $1 \leq i \leq r$. Replacing A by a suitable multiple, we obtain $F_n' \subset A + B$. By Lemma 3.2, there is some component $C_i \subset C$ not contained in the stable base locus of $M_n + A + B$. On the other hand, the inclusions

$$H^0(mD) \supset H^0(m(M_n + F_n')) \subset H^0(m(M_n + A + B))$$

are equalities by Proposition 1.3, contradiction.

Second, assume that the intersection matrix $(C_i \cdot C_j)$ is *negative semidefinite*. By the Hodge Theorem, its radical has rank 1. This implies that there is a linear combination $A = \sum \lambda_i C_i$ with $A \cdot C_i = 0$ and $\lambda_i > 0$ for all i. Up to multiples, such a divisor is unique. Rearranging indices, we may assume $M_n \cdot C_1 > 0$. The remaining curve $C_2 + \ldots + C_r$ is negative definite, so there is a linear combination $E = \sum \mu_i C_i$ with $E \cdot C_i > 0$ for $2 \leq i \leq r$. Enlarging n if necessary, we have $(M_n + E) \cdot C_i > 0$ for $1 \leq i \leq r$. Furthermore, for some $k > 0$, the divisor $M_n + E + kA + B$ is effective and contains F_m'. We conclude the proof as in the first case. □

This implies that the homogeneous spectrum $P(D) = \mathrm{Proj}(R(D))$ is not too far from being proper:

Corollary 3.4. *The algebraic surface* $P(D)$ *is not quasiaffine.*

Proof. Suppose to the contrary that $P(D)$ is quasiaffine, and consider the open embedding $P(D) \subset Y$ from Proposition 3.1. By Proposition 3.3, the 1-dimensional part of the boundary is negative definite. On the other hand, [9], Proposition 3.2 ensures that the boundary supports a Weil divisor $C \in Z^1(Y)$ with $C^2 > 0$, contradiction. □

A negative definite curve is not necessarily contractible. However, it can be contracted in the category of *algebraic spaces*. Roughly speaking, an algebraic space is an object that admits an étale covering by a scheme [7]. Over the complex numbers, 2-dimensional algebraic spaces correspond to *Moishezon surfaces*.

Theorem 3.5. *There is a proper normal 2-dimensional algebraic space* $\overline{P}(D)$ *containing* $P(D)$ *as an open dense subscheme with 0-dimensional complement.*

Proof. Since the 1-dimensional part $E \subset \mathrm{SBs}(D)$ is negative definite, a result of Artin ([2], Corollary 6.12) implies that there is a contraction $f : X \to \overline{P}(D)$ of $E \subset X$ in the category of algebraic spaces. In light of Proposition 2.3, we obtain the desired open embedding $P(D) \subset \overline{P}(D)$. □

Remark 3.6. We have constructed a commutative diagram

$$
\begin{array}{ccc}
\mathrm{dom}(r_D) & \longrightarrow & X \\
{\scriptstyle r_D}\downarrow & & \downarrow{\scriptstyle f} \\
P(D) & \longrightarrow & \overline{P}(D).
\end{array}
$$

The boundary at infinity $\overline{P}(D) - P(D)$ comprises the image of $\mathrm{SBs}'(D) \cup \mathrm{SBs}^0(D)$. Clearly, the points $b \in \overline{P}(D)$ corresponding to $\mathrm{SBs}^0(D)$ are *scheme-like* [6], page 131. In contrast, the points corresponding to $\mathrm{SBs}'(D)$ might be nonscheme-like. The algebraic space $\overline{P}(D)$ is a scheme if and only if the negative definite curve $\mathrm{SBs}'(D)$ is contractible.

Let us collect some properties of the compactification.

Proposition 3.7. *Suppose that the algebraic space $\overline{P}(D)$ is a scheme. Then the open subset $P(D) \subset \overline{P}(D)$ is the $\mathbb{Q}$-Cartier locus for the Weil divisor $f_*(D)$.*

Proof. Clearly, $f_*(D)$ is $\mathbb{Q}$-Cartier on $P(D)$. According to Proposition 1.2, it is not $\mathbb{Q}$-Cartier on each point of $\mathrm{SBs}^0(D)$. Suppose there is a connected component $C \subset \mathrm{SBs}'(D)$ so that $f_*(D)$ is $\mathbb{Q}$-Cartier on the image $y = f(C)$. For $n > 0$ sufficiently divisible, $f_*(nD) = f_*(M_n)$ is movable. Again by Proposition 1.2, we find a curve $E \sim f_*(nD)$ disjoint from y. Since $R(D) = R(f_*(D))$, the corresponding curve $E' \sim nD$ contains C as a connected component, contradicting the definition of $\mathrm{SBs}'(D)$. $\square$

In light of Nakai's ampleness criterion, we call a Weil divisor $D \in Z^1(X)$ *numerically ample* if $D^2 > 0$ and $D \cdot C > 0$ for all curves $C \subset X$.

Proposition 3.8. *Suppose that the algebraic space $\overline{P}(D)$ is a scheme. Then the Weil divisor $f_*(D)$ is numerically ample.*

Proof. Choose $n > 0$ so that $\mathrm{Bs}(nD) = \mathrm{SBs}(D)$. Then $f_*(nD) = f_*(M_n)$ is movable, so $f_*(D) \cdot C \geq 0$ for all integral curves $C \subset \overline{P}(D)$. Suppose $f_*(D) \cdot C = 0$. Then C is disjoint from the non-$\mathbb{Q}$-Cartier locus of $f_*(D)$. Consequently, the effective $\mathbb{Q}$-divisor $f^*(C)$ is disjoint from F_n'. The projection formula gives

$$
M_n \cdot f^*(C) = f_*(nD) \cdot C = 0.
$$

By Proposition 2.1, the curve $f^*(C)$ is contracted by $f : X \to \overline{P}(D)$, which is absurd. Therefore, $f_*(D) \cdot C > 0$ for all curves C. Then $f_*(D)^2 > 0$ holds as well, because $f_*(nD)$ is effective. $\square$

Remark 3.9. The preceding two Propositions hold true without the assumption that $\overline{P}(D)$ is a scheme. I leave it to the interested reader to formulate the corresponding results.

4. MULTIPLICITIES INSIDE THE STABLE BASE LOCUS

We keep our proper normal algebraic surface X and Weil divisor $D \in Z^1(X)$. Decompose $nD = M_n + F_n$ and $F_n = F_n' + F_n''$ as in Section 2. Zariski [10] pointed out that multiplicities in F_n play role for the structure of the graded ring $R(D)$. First, we start with the part $\mathrm{SBs}''(D)$ of the stable base locus.

Proposition 4.1. *Suppose $\mathrm{Bs}(nD) = \mathrm{SBs}(D)$. Then $F_{mn}'' = mF_n''$ for all $m > 0$. In particular, the multiplicities in F_{mn}'' are unbounded.*

Proof. Since $F_n'' = \mathrm{SBs}(D)$ is negative definite and $mnD \cdot C = mF_n'' \cdot C$ for all curves $C \subset \mathrm{SBs}''(D)$, we infer $F_{mn}'' = mF_n''$. $\square$

Next, we turn to the part $\mathrm{SBs}'(D)$ of the stable base locus. Suppose $\mathrm{Bs}(nD) = \mathrm{SBs}(D)$. Decompose $\mathrm{SBs}'(D) = C_1 + \ldots + C_r$ into integral components. For each such component, define

$$\lambda_i = \inf \left\{ \mathrm{mult}_{C_i}(F_n' - A) \mid A \text{ as below} \right\},$$

where A runs through all effective $\mathbb{Q}$-divisors contained in F_n' such that that the condition $(M_n + A) \cdot C_j > 0$ holds for $1 \leq j \leq r$.

Proposition 4.2. *With the preceding notation,* $\liminf_{m \to \infty} \mathrm{mult}_{C_i}(F_{mn}'/m) \leq \lambda_i$.

Proof. Let A be an effective $\mathbb{Q}$-divisor as above. By Lemma 3.2, the curve C_i is not contained in $\mathrm{SBs}(M_n + A)$. Now the decomposition into effective summands $mnD = m(M_n + A) + m(F_n - A)$ gives

$$\mathrm{mult}_{C_i}(F_{mn}/m) \leq \mathrm{mult}_{C_i}(F_n - A)$$

for m sufficiently divisible, hence the assertion. $\square$

5. The canonical model

Fix a proper normal algebraic surface X. In this section, we shall apply the results of the preceding Sections to the *canonical ring* $R(K_X) = \bigoplus H^0(nK_X)$ and the corresponding *canonical model*

$$P(K_X) = \mathrm{Proj}(R(K_X)).$$

Note that our canonical model is defined on the surface X itself, and not on a resolution of singularities. The algebraic space $\overline{P}(K_X)$ constructed in Proposition 3.5 is called the *compactified canonical model.* The task now is to determine whether or not $\overline{P}(D)$ is a scheme.

To do this, the following notions are useful. Two Weil divisors $A, B \in Z^1(X)$ on a proper normal algebraic surfaces X are called *numerically equivalent* if $A \cdot C = B \cdot C$ for all curves $C \subset X$. Given a subset $S \subset X$, we say that X is *numerically $\mathbb{Q}$-factorial* with respect to S if each Weil divisor on X is numerically equivalent to a $\mathbb{Q}$-divisor that is $\mathbb{Q}$-Cartier near S. If this holds for $S = X$, we call X numerically $\mathbb{Q}$-factorial. For example, $\mathbb{Q}$-factorial surfaces or surface of geometric genus $p_g = 0$ are numerically $\mathbb{Q}$-factorial.

Theorem 5.1. *If X is numerically $\mathbb{Q}$-factorial with respect to $\mathrm{SBs}'(K_X)$, then the proper algebraic space $\overline{P}(K_X)$ is a scheme.*

Proof. We have to check that the negative definite curve $C = \mathrm{SBs}'(K_X)$ is contractible. According to Proposition 1.3, the inclusion $H^0(K_K) \subset H^0(K_X + C)$ is bijective. But now the contraction criterion [9], Theorem 5.1 applies, and we deduce that $C \subset X$ is contractible. $\square$

To obtain examples, we shall relate numerically $\mathbb{Q}$-factorial surfaces to surfaces with rational singularities.

Proposition 5.2. *A proper normal algebraic surface is $\mathbb{Q}$-factorial on the locus of rational singularities.*

Proof. Over an algebraically closed ground field, this easily follows from [1], Theorem 1.7. The following argument using group schemes works over arbitrary ground fields k. Let $f : Y \to X$ be a resolution of the rational singularities, $E \subset Y$ the reduced exceptional curve, and $\mathfrak{Y} \subset Y$ the corresponding formal completion. Then $H^1(E, \mathcal{O}_{nE}) = 0$ for all $n > 0$. Hence the group schemes $\mathrm{Pic}^0_{nE/k}$ vanish, and $\mathrm{Pic}^0(\mathfrak{Y}) = 0$. Given a Weil divisor $D \in Z^1(X)$, there is an integer $n > 0$ so that the $\mathbb{Q}$-divisor $C = f^*(nD)$ has integral coefficients. Beeing numerically trivial, the formal line bundle $\mathcal{O}_{\mathfrak{Y}}(C)$ is trivial. Hence $f_*\mathcal{O}_Y(C)$ is invertible and $f_*(C) = nD$ is Cartier on the locus of rational singularities. $\square$

Together with Theorem 5.1, this gives a condition for schematicity.

Corollary 5.3. *Suppose X has rational singularities along* $\mathrm{SBs}'(K_X)$. *Then the algebraic space* $\overline{P}(K_X)$ *is a proper scheme.*

We also have a condition for projectivity:

Corollary 5.4. *Suppose X has rational singularities along* $\mathrm{SBs}'(K_X) \cup \mathrm{SBs}^0(K_X)$. *Then the algebraic space* $\overline{P}(K_X)$ *is a projective scheme.*

Proof. By Proposition 3.8, the canonical class K of $\overline{P}(K_X)$ is numerically ample. The locus where K is not $\mathbb{Q}$-Cartier is nothing but the image of $\mathrm{SBs}'(K_X) \cup \mathrm{SBs}^0(K_X)$. Hence K is numerically equivalent to a $\mathbb{Q}$-Cartier divisor, so the compactified canonical model is projective. $\square$

The canonical model of a regular surfaces is not necessarily regular. Rather, it has rational Gorenstein singularities. However, canonical models of surfaces with rational Gorenstein singularities have rational Gorenstein singularities, so this class of surfaces is closed under passing to minimal models. Here is a similar result.

Proposition 5.5. *If the surface X is numerically $\mathbb{Q}$-factorial, then the scheme* $\overline{P}(K_X)$ *is numerically $\mathbb{Q}$-factorial as well.*

Proof. I claim that the negative definite curve $C = \mathrm{SBs}'(K_X) \cup \mathrm{SBs}''(K_X)$ is contractible. Indeed: By Proposition 1.3, the curve C is contained in the fixed curve of $K_X + mC$ for all $m > 0$. Furthermore, we assume that X is numerically $\mathbb{Q}$-factorial. Hence [9], Theorem 5.1 ensures that $C \subset X$ is contractible.

Next, we check that the corresponding contraction $h : X \to Y$ yields a numerically $\mathbb{Q}$-factorial surface. To see this, consider for each $m > 0$ the exact sequence

$$H^1(X, \mathcal{O}_X) \longrightarrow H^1(mF, \mathcal{O}_{mF}) \longrightarrow H^2(X, \mathcal{O}_X(-mF)) \longrightarrow H^2(X, \mathcal{O}_X).$$

The map on the right is Serre dual to $H^0(K_X) \to H^0(K_X + mF)$, which is surjective by Proposition 1.3. Hence the map on the left is surjective. By Grothendieck's Existence Theorem, the map $H^1(X, \mathcal{O}_X) \to H^1(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is surjective as well, where $\mathfrak{X} = X_{/F}$ is the formal completion. Now [9], Proposition 4.2 ensures that Y is numerically $\mathbb{Q}$-factorial.

Recall that the contraction $f : X \to \overline{P}(K_X)$ admits a factorization $g : Y \to \overline{P}(K_X)$. Furthermore, K_Y has trivial intersection number on each irreducible component of the exceptional curve $E \subset Y$. Now Proposition 1.4 ensures that $H^0(K_Y) = H^0(K_Y + mE)$, and we deduce as in the preceding paragraph that the compactified canonical model $\overline{P}(K_X)$ is numerically $\mathbb{Q}$-factorial. $\square$

Remark 5.6. The numerical criterion for ampleness implies that numerically $\mathbb{Q}$-factorial surfaces are projective. Therefore, their compactified canonical model is projective as well. More precisely, a multiple of the canonical class of $\overline{P}(K_X)$ deforms to an ample invertible sheaf.

Let me record the following special case of Propositions 3.7 and 3.8.

Proposition 5.7. *Suppose that $\overline{P}(K_X)$ is a scheme. Then the open subset $P(K_X)$ is its $\mathbb{Q}$-Gorenstein locus, and the canonical class of $\overline{P}(K_X)$ is numerically ample.*

A Weil divisor $A \in Z^1(X)$ is called *nef* if $A \cdot R \geq 0$ holds for all curves $R \subset C$. We have the following characterization for the compactified canonical model to be a scheme.

Theorem 5.8. *The algebraic space $\overline{P}(K_X)$ is a scheme if and only if for each connected component $R \subset \mathrm{SBs}'(K_X)$, there is a nef Weil divisor $A \in Z^1(X)$ that is Cartier near R, so that for each integral curve $C \subset X$, the condition $A \cdot C = 0$ holds if and only $C \subset R$.*

Proof. You easily check that the condition is necessary. For sufficiency, we shall apply the characterization of contractible curves in [9], Theorem 3.4 to each connected component $R \subset \mathrm{SBs}'(D)$. Let R be as in the Theorem. Fix an integer $m > 0$ and consider the exact sequence

$$H^1(X, \mathcal{O}_X) \longrightarrow H^1(mR, \mathcal{O}_{mR}) \longrightarrow H^2(X, \mathcal{O}_X(-mR)) \longrightarrow H^2(X, \mathcal{O}_X).$$

The map on the right is Serre dual to $H^0(K_X) \to H^0(K_X + mR)$. The latter is surjective according to Proposition 1.3. Consequently, $H^1(X, \mathcal{O}_X) \to H^1(mR, \mathcal{O}_{mR})$ is surjective, so $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(mR)$ is surjective up to torsion. So, for some $n > 0$, the invertible sheaf $\mathcal{O}_{mR}(nA)$ is the restriction of some numerically trivial invertible $\mathcal{O}_X$-module. Thus the conditions of the before mentioned characterization of contractible curves applies, and we conclude that C is contractible. $\square$

Remark 5.9. One might say that a proper normal algebraic surface X with rational singularities has *two* canonical models: First the canonical model $P(K_Y) = \overline{P}(K_Y)$ in the sense of Mori theory defined using a resolution of singularities $Y \to X$, and second the compactification $P(K_X) \subset \overline{P}(K_X)$ defined on the normal surface X itself. I wish to know what happens in higher dimensions.

Question 5.10. Does there exist a proper normal surface whose compactified canonical model is not a scheme?

<div align="center">REFERENCES</div>

[1] M. Artin: Some numerical criteria for contractability of curves on algebraic surfaces. Am. J. Math. 84 (1962), 485–496.

[2] M. Artin: Algebraization of formal moduli II: Existence of modifications. Annals of Math. 91, 88–135 (1970).

[3] T. Fujita: Semipositive line bundles. J. Fac. Sci. Univ. Tokyo 30, 353–378 (1983).

[4] J. Giraud: Improvement of Grauert–Riemenschneider's theorem for a normal surface. Ann. Inst. Fourier 32, 13–23 (1982).

[5] A. Grothendieck: Éléments de géométrie algébrique III: Étude cohomologique des faiscaux cohérent. Publ. Math., Inst. Hautes Étud. Sci. 11 (1961).

[6] D. Knutson: Algebraic spaces. Lect. Notes Math. 203. Springer, Berlin, 1971.

[7] G. Laumon, L. Moret-Bailly: Champs algebriques. Ergeb. Math. Grenzgebiete 39, Springer, Berlin, 2000.

[8] D. Mumford: The topology of an normal surface singularity of an algebraic variety and criterion for simplicity. Publ. Math., Inst. Hautes Étud. Sci. 9, 5–22 (1961).

[9] S. Schröer: On contractible curves on normal surfaces. J. Reine Angew. Math. 524, 1–15 (2000).

[10] O. Zariski: The theorem of Riemann–Roch for high multiples of an effective divisor on an algebraic surface. Ann. Math. (2) 76, 560–615 (1962).

MATHEMATISCHE FAKULTÄT, RUHR-UNIVERSITÄT, 44780 BOCHUM, GERMANY

Current address: Massachusetts Institute of Technology, Department of Mathematics, Building 2 Room 155, 77 Massachusetts Avenue, Cambridge MA 02139-4307, USA

E-mail address: `s.schroeer@ruhr-uni-bochum.de`