NORMAL DEL PEZZO SURFACES CONTAINING A NONRATIONAL SINGULARITY

STEFAN SCHRÖER

Revised version, 28 Oktober 2000

Abstract. Working over perfect ground fields of arbitrary characteristic, I classify minimal normal del Pezzo surfaces containing a nonrational singularity. As an application, I determine the structure of 2-dimensional anticanonical models for proper normal algebraic surfaces. The anticanonical ring may be non-finitely generated. However, the anticanonical model is either a proper surface, or a proper surface minus a point.

Introduction

Proper smooth algebraic surfaces with ample anticanonical divisor are called del Pezzo surfaces. Prominent examples are $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$. We call a proper *normal* algebraic surface X del Pezzo if $(-K_X)^2 > 0$ and $(-K_X) \cdot C > 0$ holds for all curves $C \subset X$. Here we use Mumford's rational intersection numbers. Note that X might be non-$\mathbb{Q}$-Gorenstein, such that the antipluricanonical sections do not define a closed embedding into any projective space.

Several authors studied normal del Pezzo surfaces. For example, Hidaka and Watanabe [11] considered the Gorenstein case. Sakai [20] studied complex rational surfaces. Bădescu [2] analyzed complex ruled surfaces. Fujisawa [6] and Cheltsov [5] classified complex normal del Pezzo surfaces.

Here we shall study normal del Pezzo surfaces over perfect ground fields of arbitrary characteristic. Contractions of sections on suitable $\mathbb{P}^1$-bundles give plenty of examples. The main result is that, starting from such contracted bundles, we can reach any normal del Pezzo surface with base number $\rho(X) = 1$ containing a nonrational singularity via a sequence of *generalized elementary transformations*. Such elementary transformations are closely related to the monoid $\mathrm{SL}_2(\mathbb{N})$ and continued fractions.

We can view such del Pezzo surfaces as "minimal models" in the category of normal surfaces. As an application of our results, we determine the structure of anticanonical models $P(-K_X) = \mathrm{Proj}(\bigoplus_{n\geq 0} H^0(X, -nK_X))$ for normal surfaces. It turns out that each 2-dimensional anticanonical model is either a proper normal $\mathbb{Q}$-Gorenstein surface, or there is a canonical compactification $P(-K_X) \subset \overline{P}$ by adding a single non-$\mathbb{Q}$-Gorenstein point at infinity. In the latter case, the anticanonical ring $R(-K_X) = \bigoplus_{n\geq 0} H^0(X, -nK_X)$ is not finitely generated. This happens, for example, if the stable base locus of $-K_X$ contains isolated non-$\mathbb{Q}$-Gorenstein singularities.

1991 *Mathematics Subject Classification.* 14J17, 14J20, 14J25, 14J26 .

Acknowledgement. I wish to thank the referee for his remarks and suggestions, which helped to clarify the paper.

1. Surfaces with ineffective canonical class

We fix some notation. Throughout the paper, we work over a perfect ground field k of arbitrary characteristic $p \geq 0$. Suppose X is a proper normal algebraic surface with $k = \Gamma(X, \mathcal{O}_X)$. Let $p_g(X) = h^2(\mathcal{O}_X)$ be the *geometric genus* and $q(X) = \dim(\operatorname{Pic}^0_X)$ the *irregularity.* We are mainly interested in surfaces with $p_g(X) = 0$. For such surfaces, the Picard scheme is smooth [17] p. 198, so the irregularity is also $q(X) = h^1(\mathcal{O}_X)$.

Two Weil divisors $A, B \in Z^1(X)$ are *numerically equivalent*, written $A \equiv B$, if $A \cdot C = B \cdot C$ for all curves $C \subset X$. Here we use Mumford's rational intersection numbers [16]. Numerical equivalence yields two quotients $Z^1(X) \to N(X)$ and $\operatorname{Pic}(X) \to N^1(X)$, together with an inclusion $N^1(X) \subset N(X)$. The rank of $N^1(X)$ is called the *base number* $\rho(X)$. Note that the base number might be zero [22]. However, the following holds.

Proposition 1.1. *Suppose $p_g(X) = 0$. Then each Weil divisor on X is numerically equivalent to a $\mathbb{Q}$-Cartier divisor. Moreover, X is projective.*

Proof. See [23] Corollary 4.4. □

Let $f\colon X \to Y$ be a proper morphism with $\mathcal{O}_Y = f_*(\mathcal{O}_X)$ onto another proper normal algebraic scheme Y. If Y is a curve, we say that $f\colon X \to Y$ is a *fibration.* The arithmetic genus $p_a(X_\eta) = h^1(\mathcal{O}_{X_\eta})$ of the generic fiber is called the *genus of the fibration.* If Y is a surface, we say that $f\colon X \to Y$ is a (birational) *contraction.* A curve $R \subset X$ is called *contractible* if there is a contraction $f\colon X \to Y$ such that R is the exceptional curve.

Proposition 1.2. *Assume that $p\colon X \to B$ is a fibration of genus zero. Then $R^1p_*(\mathcal{O}_X) = 0$. Moreover, for all $b \in B$, each curve $R \subsetneq p^{-1}(b)$ is contractible.*

Proof. Passing to a resolution of singularities and a relatively minimal model, we easily deduce $R^1p_*(\mathcal{O}_X) = 0$. Now the contractibility of R follows from Artin's contraction criterion ([1] Thm. 2.3). □

The *pseudoeffective cone* $\overline{\mathrm{NE}}(X) \subset N(X)_\mathbb{R}$ is the closed convex cone generated by all curves $C \subset X$. The pseudoeffective cone has full dimension and is generated by its extremal rays. According to Kollár [14], Lemma 4.12, each extremal ray $\mathbb{R}_+E \subset \overline{\mathrm{NE}}(X)$ with $E^2 < 0$ is generated by an integral curve $R \subset X$.

Suppose the pseudoeffective cone is simplicial, in other words, the convex hull of $n = \operatorname{rank} N(X)$ linearly independent extremal rays $\mathbb{R}_+E_i \subset \overline{\mathrm{NE}}(X)$. This holds for example if $\operatorname{rank} N(X) = 2$. Decompose the canonical class

$$K_X \equiv \lambda_1 E_1 + \lambda_2 E_2 + \ldots + \lambda_n E_n,$$

for certain unique coefficients $\lambda_i \in \mathbb{R}$.

Proposition 1.3. *With the preceding assumptions, suppose $E_i^2 < 0$ and $\lambda_j < 0$ for some pair $j \neq i$. Then the curve $R_i \subset Y$ generating $\mathbb{R}_+E_i$ is contractible. Moreover, if $f\colon X \to Y$ is the corresponding contraction, then K_Y is not pseudoeffective.*

Proof. The conditions ensure that $K_X + nR_i$ is not pseudoeffective for all integers $n \geq 0$. According to [23], Corollary 5.2, the contraction $f\colon X \to Y$ of R exists. Seeking a contradiction, assume that K_Y is pseudoeffective. Then $f^*(K_Y)$ is also pseudoeffective. Write $K_X = f^*(K_Y) + \lambda R_i$ for some coefficient $\lambda \in \mathbb{Q}$. Then $K_X + nR_i$ is pseudoeffective for some $n \gg 0$, a contradiction. Thus K_Y is not pseudoeffective. □

A proper normal algebraic surface with $(-K_X)^2 > 0$ and $(-K_X) \cdot C > 0$ for all curves $C \subset X$ is called a *del Pezzo surface*. This holds, for example, if $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor.

Proposition 1.4. *The contraction of a del Pezzo surface is a del Pezzo surface.*

Proof. Let $f\colon X \to Y$ be such a contraction. According to the projection formula, $K_Y \cdot C < 0$ for all curves $C \subset Y$. Writing $K_X = f^*(K_Y) + K_{X/Y}$ with $K_{X/Y}$ supported on the exceptional curve, we obtain

$$0 < K_X^2 = K_Y^2 + 2f^*(K_Y) \cdot K_{X/Y} + K_{X/Y}^2 = K_Y^2 + K_{X/Y}^2 \leq K_Y^2.$$

Therefore Y is del Pezzo. □

Recall that a curve $R \subset X$ is called *negative definite* if the intersection matrix $(R_i \cdot R_j)$ is negative definite, where $R_i \subset R$ are the irreducible components. Contractible curves are negative definite. For surfaces with many antipluricanonical sections, the converse holds as well:

Proposition 1.5. *Suppose there is an integer $m > 0$ with $h^0(-mK_X) > 1$. Then each negative definite curve $R \subset X$ is contractible.*

Proof. By induction on the number of irreducible components of R, it suffices to treat the case that R is integral. Suppose $D = K_X + nR$ is effective for some integer $n \geq 0$. Then $-mK_X = mnR - mD$, so we have an inclusion $H^0(-mK_X) \subset H^0(mnR)$. Using $R^2 < 0$, you easily see that $h^0(mnR) = 1$, contradiction. Consequently, $K_X + nR$ is not effective for all $n \geq 0$. Now [23], Corollary 5.2 ensures that $R \subset X$ is contractible. □

The next result reduces all surfaces with ineffective canonical class to a special situation:

Theorem 1.6. *Suppose X is a proper normal algebraic surface with $-K_X$ not pseudoeffective. Then there is a contraction $h\colon X \to Y$ such that one of the following holds:*

(i) *The contraction Y is a del Pezzo surface with base number $\rho(Y) = 1$.*
(ii) *There is a genus zero fibration $p\colon Y \to B$ with irreducible fibers, and the base number is $\rho(Y) = 2$.*

Proof. For $\rho(X) = 1$ there is nothing to prove. Suppose $\rho(X) \geq 2$. Choose a pseudoample class $A \in N(X)_\mathbb{R}$ with $K_X \cdot A < 0$, and let $\mathbb{R}_+ E \subset \overline{\mathrm{NE}}(X)_\mathbb{R} \cap A^\perp$ be an extremal ray.

First, suppose $E^2 < 0$. Then $\mathbb{R}_+ E$ is generated by an integral curve $E \subset X$. Since $(K_X + nE) \cdot A < 0$ holds for all $n \geq 0$, the class $K_X + nE$ is not pseudoeffective, so the contraction $X \to X'$ of B exist by [23], Corollary 5.2. Since A comes from an ample class on X', the class $K_{X'}$ is not pseudoeffective, and we can proceed by induction on the base number $\rho(X)$.

Second, suppose $E^2 = 0$. Then the existence of a genus zero fibration $p\colon X \to B$ easily follows from Mori [15], Theorem 2.3. Since $\mathbb{R}_+E$ is extremal, each fiber X_b, $b \in B$ is irreducible. Using $\operatorname{Pic}^0(Y_\eta) = 0$, we directly deduce $\rho(Y) = 2$. □

Remark 1.7. If $p\colon Y \to B$ is a genus zero fibration with irreducible fibers, it may or may not be possible to contract Y, and the result may or may not be del Pezzo. We take up this issues in Sections 3 and 5.

2. Surfaces containing a nonrational singularity

The aim of this section is to determine the structure of proper normal algebraic surfaces Z with ineffective pluricanonical classes containing a nonrational singularity. The following observation will be useful:

Lemma 2.1. *Suppose Z is a proper normal algebraic surface with irregularity $q(Z) = 0$. If no multiple nK_Z with $n > 0$ is effective, then $\operatorname{Pic}(Z)$ is a free $\mathbb{Z}$-module of finite rank.*

Proof. Assume there is an invertible $\mathcal{O}_Z$-module $\mathcal{L} \in \operatorname{Pic}(X)$ of order $n > 0$. The Riemann–Roch theorem tells us $\chi(\mathcal{L}) = \chi(\mathcal{O}_Z) > 0$. On the other hand,

$$\chi(\mathcal{L}) \leq h^0(Z, \mathcal{L}) + h^0(Z, \mathcal{L}^{\otimes -1} \otimes \omega_Z).$$

Since Z is integral and $\mathcal{L}$ is nontrivial, the first summand is zero. Because the invertible sheaf $\mathcal{L}^{\otimes -n}(nK_Z) = \mathcal{O}_Z(nK_Z)$ is not effective, the second summand vanishes as well. Thus $\chi(\mathcal{L}) \leq 0$, a contradiction. Hence $\operatorname{Pic}(Z)$ is torsion free.

The group scheme Pic^0_Z vanishes because its Lie algebra $H^1(Z, \mathcal{O}_Z) = 0$ does, so the map $\operatorname{Pic}(Z) \to \operatorname{NS}(Z)$ onto the Néron–Severi group is bijective. Since the Néron–Severi group is finitely generated, $\operatorname{Pic}(Z)$ is free and finitely generated. □

Suppose Z is a proper normal algebraic surface. Let $f\colon X \to Z$ be a resolution of singularities, and $R \subset X$ the exceptional divisor. Choose a maximal subcurve $R' \subset R$ whose formal completion $X_{/R'}$ satisfies $H^1(\mathcal{O}_{X_{/R'}}) = 0$. According to Artin [1], Theorem 2.3, the contraction $h : X \to Y$ of $R' \subset X$ exists. We call the corresponding partial resolution $g : Y \to Z$ a *minimal resolution of nonrational singularities.*

Theorem 2.2. *Suppose Z is a proper normal algebraic surface containing a nonrational singularity, and that no multiple nK_Z with $n > 0$ is effective. Let $: Y \to Z$ be a minimal resolution of nonrational singularities and $S \subset Y$ the exceptional curve. Then the following hold:*

(i) *There is a genus zero fibration $p\colon Y \to B$ over a curve B of genus $g > 0$.*
(ii) *The curve $S \subset Y$ is a section over B, and each singularity on Z is rational except for $z = g(S)$.*
(iii) *The irregularity is $q(Z) = 0$, and $\operatorname{Pic}(Z)$ is a free $\mathbb{Z}$-module of finite rank.*

Proof. Let $f\colon X \to Z$ be the minimal resolution of singularities. Clearly, $nK_X \neq 0$ for all $n > 0$. Suppose there is a nonzero section $s \in H^0(X, nK_X)$. The corresponding curve is supported on R, and this implies $K_X \cdot R_i < 0$ for some irreducible component $R_i \subset R$, contradicting the minimality of the resolution. Therefore X has Kodaira dimension $\kappa(X) = -\infty$. The Enriques–Kodaira classification ensures the existence of a fibration $\overline{X} \to \overline{B}$ of genus zero over the algebraic closure $k \subset \bar{k}$ (see Mumford [18] Sect. 1). Since $p_g(Z) = 0$, the sequence

$$0 \longrightarrow H^1(Z, \mathcal{O}_Z) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X_{/R}, \mathcal{O}_{X_{/R}}) \longrightarrow 0$$

is exact, where $R \subset X$ is the exceptional divisor and $X_{/R}$ is the corresponding formal completion. As $H^1(\mathcal{O}_{\overline{B}}) \to H^1(\mathcal{O}_{\overline{X}})$ is bijective, the curve $\overline{B}$ has genus $g > 0$. Thus each horizontal curve on $\overline{X}$ has positive arithmetic genus. Consequently, the Galois group $\mathrm{Aut}(\bar{k}/k)$ permutes the vertical curves, so the fibration descends to a fibration $X \to B$. The exceptional curve $R' \subset X$ for the contraction $h\colon X \to Y$ must be vertical, hence there is an induced fibration $p\colon Y \to B$. This proves (i).

By Proposition 1.2, the exceptional divisor $S \subset Y$ is horizontal. For each integer $n \geq 0$, the kernel of the restriction homomorphism $\mathrm{Pic}^0_{nS} \to \mathrm{Pic}^0_S$ is a smooth unipotent group scheme. On the other hand, Pic^0_B is an Abelian variety. The restriction map $\mathrm{Pic}^0_B \to \mathrm{Pic}^0_{nS}$ is an epimorphism, since the induced map $H^1(\mathcal{O}_B) \to H^1(\mathcal{O}_{nS})$ on Lie algebras is surjective. We deduce $\mathrm{Pic}^0_{nS} = \mathrm{Pic}^0_S$ and $H^1(\mathcal{O}_{Y_{/S}}) = H^1(\mathcal{O}_S)$, where $Y_{/S}$ is the formal completion of $S \subset Y$. This gives an exact sequence

$$0 \longrightarrow \mathrm{Pic}^\tau_Z \longrightarrow \mathrm{Pic}^0_B \longrightarrow \mathrm{Pic}^0_S \longrightarrow 0$$

of group schemes. Here Pic^τ_Z is the group scheme of numerically trivial invertible sheaves. Since Pic^0_B is an abelian variety, the kernel Pic^τ_Z is proper. According to [9] Sect. XII, Cor. 1.5, it must be affine, because $S \to B$ is proper. Since $H^2(\mathcal{O}_Z) = 0$, the group scheme Pic^τ_Z is smooth (see [17] p. 198). Being smooth, proper, and affine, Pic^τ_Z is étale. This implies $q(Z) = 0$. According to Lemma 2.1, $\mathrm{Pic}(Z)$ is free and finitely generated, so $\mathrm{Pic}^\tau_Z = 0$. This proves (iii).

It remains to verify (ii). Since $S \subset Y$ is horizontal, each integral component of S has genus $g > 0$. Hence S is geometrically integral, and in particular $k \subset \Gamma(\mathcal{O}_S)$ is bijective. We also see that $z = g(S)$ is the only nonrational singularity on the surface Z. Assume that $S \subset Y$ is not a section. Using that $\mathrm{Pic}^0_B \to \mathrm{Pic}^0_S$ is an isomorphism, we derive a contradiction as follows.

First, assume that the normalization $\tilde{S} \to S$ is not an isomorphism. Since S is irreducible, the kernel of $\mathrm{Pic}^0_S \to \mathrm{Pic}^0_{\tilde{S}}$ is a nonfinite unipotent group scheme, contradicting that $\mathrm{Pic}^0_S = \mathrm{Pic}^0_B$ is abelian. Consequently, S is normal. Next, decompose $S \to B$ into a purely inseparable morphism $S \to A$ and a separable morphism $A \to B$. Using $\mathrm{Pic}^0_B = \mathrm{Pic}^0_S$, we deduce that $\mathrm{Pic}^0_B \to \mathrm{Pic}^0_A$ and $\mathrm{Pic}^0_A \to \mathrm{Pic}^0_S$ are isomorphisms. Set $n = \deg(A/B)$. The Hurwitz formula gives

$$2g - 2 = n(2g - 2) + \deg(K_{A/B}) \geq n(2g - 2).$$

If $n > 1$, then $g = 1$, and $A \to B$ is an isogeny of degree n between elliptic curves. So the dual isogeny $\mathrm{Pic}^0_B \to \mathrm{Pic}^0_A$ has a kernel of length n, hence $n = 1$. Consequently, the projection $S \to B$ is purely inseparable. Finally, set $p^m = \deg(S/B)$. The two curves A and B are isomorphic without k-structures, and the projection $S \to B$ is just the iterated linear Frobenius map $\mathrm{Fr}^m\colon B \to B$. The induced morphism $\mathrm{Fr}^*\colon \mathrm{Pic}^0_B \to \mathrm{Pic}^0_B$ is multiplication by p, which has a kernel of length p^{2g} (see Mumford [19] p.64). Hence $m = 0$, and $S \subset Y$ is a section. □

Remark 2.3. In the preceding result, the generic fiber of the genus zero fibration $Y \to B$ is a form of the projective line. However, since $S \subset Y$ is a section, Y_η contains a rational point and is therefore a projective line over the function field $\kappa(B)$.

Remark 2.4. The arguments in this section do not work over nonperfect ground fields. Curves with "genus change" cause problems. Compare Tate [25].

3. Contractions of $\mathbb{P}^1$-bundles

The simplest del Pezzo surfaces containing a nonrational singularity are contractions of $\mathbb{P}^1$-bundles. The task now is to discuss such contractions. Fix a smooth curve B of genus $g > 0$ and a $\mathbb{P}^1$-bundle $q\colon Y \to B$. Then the pseudoeffective cone $\overline{\mathrm{NE}}(Y) \subset \mathrm{N}(Y)_{\mathbb{R}}$ is 2-dimensional and generated by two extremal rays. The first extremal ray is the *fiber class* $F \in N(Y)_{\mathbb{Q}}$, defined by the equations $Y_b = \dim_k \kappa(b) \cdot F$ for each $b \in B$. The second extremal ray may or may not be generated by a curve.

Lemma 3.1. *Let $R \subset Y$ be a horizontal curve, say of degree n over B. Then n divides R^2, and $nK_{Y/B} \equiv -2R + R^2/n \cdot F$.*

Proof. Write $nK_{Y/B} \equiv -2R + \lambda F$ for some $\lambda \in \mathbb{Z}$. Using $K^2_{Y/B} = 0$, we obtain $0 = (nK_{Y/B})^2 = 4R^2 - 4\lambda n$, hence the result. $\square$

Set $e = -\inf\left\{S^2 \mid S \subset Y \text{ a section}\right\}$. Note that there is a section $S \subset Y$ with $S^2 < 0$ if and only if $e > 0$.

Proposition 3.2. *The surface Y contracts to a normal del Pezzo surface if and only if $2g - 2 - e < 0$.*

Proof. The condition is sufficient: Let $S \subset Y$ be the section with $S^2 = -e < 0$. Since $K_Y \equiv -2S + (2g - 2 - e)F$, the contraction exists by Proposition 1.3. Since $K_Z = (2g - 2 - e) \cdot g_*(F)$, the resulting surface is del Pezzo. The condition is necessary: By Theorem 2.2, the exceptional curve $S \subset Y$ is a section. Since $2g - 2 - e$ is the multiplicity of K_Z, the condition follows. $\square$

Proposition 3.3. *Suppose $R \subset Y$ is an integral curve with $R^2 < 0$. Then the projection $q\colon R \to B$ is purely inseparable.*

Proof. The separable closure of the field extension $\kappa(B) \subset \kappa(R)$ defines a proper normal algebraic curve A, together with a purely inseparable morphism $R \to A$ and a separable morphism $A \to B$. Set $n = \deg(A/B)$. On the induced $\mathbb{P}^1$-bundle Y_A, the preimage of R splits into n irreducible components $R_A = R_1 + \ldots + R_n$, which are permuted by the Galois action. This implies $R_1^2 = \ldots = R_n^2$, hence all irreducible components have $R_i^2 < 0$. Since the pseudoeffective cone $\overline{\mathrm{NE}}(Y_A)$ has at most one extremal ray with negative self intersection, we conclude $n = 1$. Thus $A = B$, and $R \to B$ is purely inseparable. $\square$

In characteristic zero, we conclude that each integral curve $R \subset Y$ with $R^2 < 0$ is necessarily a section. Suppose there is such a section. Setting $\mathcal{E} = p_*(\mathcal{O}_Y(R))$ and $\mathcal{L} = p_*(\mathcal{O}_S(R))$, we obtain an extension

$$0 \longrightarrow \mathcal{O}_B \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0$$

with $X = \mathbb{P}(\mathcal{E})$ and $R = \mathbb{P}(\mathcal{L})$.

Proposition 3.4. *Suppose the characteristic is $p = 0$, and let $R \subset Y$ be as above. Then R is contractible if and only if the extension $\mathcal{E}$ splits.*

Proof. If there is a splitting $\mathcal{E} \to \mathcal{O}_B$, then $A = \mathbb{P}(\mathcal{O}_B)$ defines another section disjoint from S with $A^2 > 0$. By the Fujita–Zariski Theorem ([7] Thm. 1.10), $\mathcal{O}_Y(A)$ is pseudoample, hence a suitable multiple is base point free and defines the desired contraction.

Conversely, assume that the contraction $g\colon Y \to Z$ exists. A direct calculation gives

$$\mathrm{Pic}(2R) = \mathrm{Pic}(R) \oplus H^1(S, \mathcal{O}_R(-R))$$

Since Z contains only one singularity, it is projective. This is because the complement of an affine neighborhood of the singularity supports an ample Cartier divisor, as explained in [10] p. 69. Let $\mathcal{L}$ be the preimage of an ample invertible $\mathcal{O}_Z$-module. We calculate the restriction $\mathcal{L}|_{2R}$ in two ways. Since $\mathcal{L}$ comes from Z, the restriction is trivial. On the other hand, $\mathcal{L}|_{2R}$ defines a class $(0, \alpha) \in \mathrm{Pic}(2R)$ according to the decomposition of $\mathrm{Pic}(2R)$. A straightforward cocycle computation left to the reader reveals that $\alpha \in H^1(R, \mathcal{O}_R(-R))$ coincides with a multiple of the extension class of $\mathcal{E}$ in $H^1(B, \mathcal{L}^\vee)$. Since the characteristic is zero, this extension class is zero. □

The situation in positive characteristic is different:

Theorem 3.5. *Suppose the characteristic is $p > 0$. Let $\mathcal{E}$ be a rank two vector bundle on B with $X = \mathbb{P}(\mathcal{E})$. Then the following are equivalent:*

(i) *For some integer $m \geq 0$, there is a splitting $(\mathrm{Fr}^m)^*\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2$ with invertible summands satisfying $\deg(\mathcal{L}_1) \neq \deg(\mathcal{L}_2)$.*
(ii) *There is a contractible curve $R \subset Y$.*
(iii) *There is a curve $R \subset Y$ with negative self-intersection.*

Proof. First, note that $\mathbb{P}(\mathcal{F}) \simeq \mathbb{P}(\mathcal{E})$ if and only if there is an invertible sheaf $\mathcal{L}$ with $\mathcal{F} \simeq \mathcal{E} \otimes \mathcal{L}$. Hence the condition in (i) does not depend on the choice of $\mathcal{E}$.

Suppose (i) holds, say with $\deg(\mathcal{L}_1) < \deg(\mathcal{L}_2)$. Set $Y_m = \mathbb{P}((\mathrm{Fr}^m)^*\mathcal{E})$. Then there is a cartesian diagram

$$\begin{array}{ccc} Y_m & \longrightarrow & Y \\ \downarrow & & \downarrow \\ B & \xrightarrow[\mathrm{Fr}^m]{} & B \end{array}$$

whose horizontal maps are bijective. The section $R_m = \mathbb{P}(\mathcal{L}_1)$ of Y_m satisfies $(R_m)^2 = \deg(\mathcal{L}_1) - \deg(\mathcal{L}_2) < 0$. Since $\mathbb{P}(\mathcal{L}_2)$ is another disjoint section, we conclude as in the proof of Proposition 3.4 that $R_m \subset Y_m$ is contractible. Similarly, the image $R \subset Y$ of R_m is contractible as well. Hence (ii) holds.

The implication (ii) $\Rightarrow$ (iii) is well known. Finally, suppose an integral curve $R \subset Y$ has $R^2 < 0$. By Proposition 3.3, the projection $R \to B$ is purely inseparable, say of degree p^m. Hence its normalization $\tilde{R}$ is isomorphic to B, and the induced morphism $\tilde{R} \to B$ it the iterated linear Frobenius $\mathrm{Fr}^m\colon B \to B$. Making base change along this morphism, we can assume that $R \subset Y$ is a section. The section defines an extension

$$0 \longrightarrow \mathcal{O}_B \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0$$

with $Y = \mathbb{P}(\mathcal{E})$ and $\mathcal{L} = p_*(\mathcal{O}_R(R))$. The extension class for $\mathcal{E}$ lies in $H^1(B, \mathcal{L}^{-1})$. Since $\deg(\mathcal{L}^{-1}) = -R^2 > 0$, the sheaf $\mathcal{L}^{-1}$ is ample, so $H^1(B, \mathcal{L}^{-p^m}) = 0$ for all m sufficiently large. Since $\mathcal{L}^{-p^m} = (\mathrm{Fr}^m)^*\mathcal{L}^{-1}$, the induced extension

$$0 \longrightarrow \mathcal{O}_B \longrightarrow (\mathrm{Fr}^m)^*\mathcal{E} \longrightarrow (\mathrm{Fr}^m)^*\mathcal{L} \longrightarrow 0$$

splits. Thus condition (i) holds. □

Remark 3.6. The existence of nonsplit rank two vector bundles that split on taking a purely inseparable cover follows from Tango's work. In [24], Theorem 15, he showed that on smooth prper curves B satisfying certain conditions related to the Hasse–Witt matrix, there is an ample invertible $\mathcal{O}_X$-module $\mathcal{M}$ so that the Frobenius map $\mathrm{Fr}\colon H^1(B,\mathcal{M}^\vee) \to H^1(B,\mathrm{Fr}^*\mathcal{M}^\vee)$ is not injective. Then the corresponding extension $0 \to \mathcal{O}_B \to \mathcal{E} \to \mathcal{M} \to 0$ gives the desired rank two vector bundle $\mathcal{E}$. For explicit examples, see [24], Section 5.

4. Elementary transformations and continued fractions

In this section, we calculate the behavior of multiplicities on fibrations under birational transformations. The idea is to use the monoid $\mathrm{SL}_2(\mathbb{N})$ in its various disguises. In the next section, they shall apply the results for the classification of singular del Pezzo surfaces.

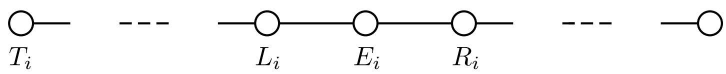
Rather than working over a ground field, we shall use the following set-up: Suppose A is a henselian discrete valuation ring with residue field k and fraction field K, and let X be a proper normal 2-dimensional A-scheme with $\Gamma(X,\mathcal{O}_X) = A$. We call such schemes *$A$-surfaces.*

The generic fiber $X \otimes K$ is a normal curve over the function field K, degenerating into the possibly singular closed fiber $X \otimes k$. Call X is *minimal* if the closed fiber is irreducible. If X is not minimal, a result of Bosch, Lütkebohmert and Raynaud ([4] Cor. 3, p. 169) ensures that each irreducible component of $X \otimes k$ is contractible (here we need the assumption that A is henselian).

Suppose Y is a minimal A-surface as above. We seek to describe the passage to birational minimal A-surface algorithmically. Let $y \in Y$ be a closed point in $\mathrm{Reg}(Y)$ and $f_1\colon X_1 \to Y$ be its blowing-up. In the following, we consider sequences of blowing-ups

$$X_{n+1} \xrightarrow{f_{n+1}} X_n \longrightarrow \ldots \longrightarrow X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} Y,$$

subject to the condition that each center $x_i \in X_i$ lies on both the exceptional divisor $E_i = f_i^{-1}(x_{i-1})$ of the preceding blowing-up and another irreducible component of $X_i \otimes k$. This implies that the intersection graph of $X_i \otimes k$ is a chain



Here T_i is the strict transform of $Y \otimes k$, and E_i is the exceptional divisor of the blowing-up $f_i\colon X_i \to X_{i-1}$. The two neighbors of E_i are labeled L_i and R_i; they are distinguished by the fact that L_i lies between S_i and E_i. In the special case $i = 1$ we have $T_1 = L_1$ and $R_1 = \emptyset$.

Observe that there are two alternatives for the blowing-up $f_{i+1}\colon X_{i+1} \to X_i$: Either the center $x_i \in X_i$ is $L_i \cap E_i$ or $E_i \cap R_i$. In other words: The sequence $X_n, \ldots, X_1$ is uniquely determined by the initial center $y \in Y$, together with a word $w = L^{l_1} R^{r_1} L^{l_2} \ldots$ of length $n \geq 0$ in two letters L, R not starting with R. The rule is: If the ith letter in w is L, the center $x_i \in X_i$ is $L_i \cap E_i$, otherwise $E_i \cap R_i$.

Set $X = X_{n+1}$, let $h\colon X \to Y$ be the composition of the blowing-ups, and $\hat{h}\colon X \to \widehat{Y}$ be the contraction of all irreducible components in $X_{n+1} \otimes k$ except for

E_{n+1}. We call the resulting minimal A-surface $\widehat{Y}$ the *elementary transformation* of Y with respect to the word $w = L^{l_1}R^{r_1}\ldots$ and the initial center $y \in Y$.

Starting with a regular minimal A-surface, it is easy to see that each birational equivalent minimal A-surface can be reached by a sequence of elementary transformations. If Y is a $\mathbb{P}^1$-bundle, and y is a k-rational point, and $w = \emptyset$ is the empty word, then the construction coincides with the classical notion of elementary transformation.

Let $\langle L, R\rangle$ be the free monoid on two letters and $\mathrm{SL}_2(\mathbb{N})$ the monoid of unimodular 2×2-matrices whose entries are natural numbers. By inspection, the map

$$\langle L, R\rangle \longrightarrow \mathrm{SL}_2(\mathbb{N}), \quad L \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad R \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

is bijective. Similarly, the map

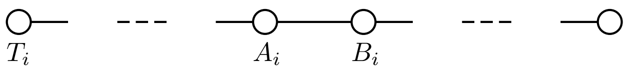
$$\mathrm{SL}_2(\mathbb{N}) \longrightarrow \mathbb{Q}_+, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{a+b}{c+d}$$

is bijective. Composing both maps, we associate to each word $w = L^{l_1}R^{r_1}\ldots$ a fraction $r/s > 0$.

Now let $\widehat{Y}$ be the elementary transformation of Y with respect to the word $w = L^{l_1}R^{r_1}\ldots$ and the initial center $y \in Y$. The multiplicities $\mu, \hat{\mu} > 0$ of the closed fibers are defined by $Y \otimes k = \mu \cdot (Y \otimes k)^{\mathrm{red}}$ and $\widehat{Y} \otimes k = \hat{\mu} \cdot (\widehat{Y} \otimes k)^{\mathrm{red}}$.

Proposition 4.1. *Let r/s be the fraction associated to the word w. Then the multiplicities $\mu, \hat{\mu}$ of the closed fibers of $Y, \widehat{Y}$ are related by $\hat{\mu} = r \cdot \mu$.*

Proof. The trick is to introduce alternative names for the curves on the blowing-ups. Let



be the intersection graph of $X_i \otimes k$. Here T_i is the strict transform of $Y \otimes k$ as above, but A_i, B_i are the curves with $A_i \cap B_i = \{x_i\}$. They can be distinguished by the fact that A_i lies between T_i and B_i. Let a_i, b_i be the multiplicities of A_i, B_i in the cycle $X_i \otimes k$. Then $\hat{\mu} = a_n + b_n$.

We proceed to calculate the pairs (a_i, b_i). Let $a_0 = \mu$ and $b_0 = 0$ be dummy variables. Suppose $f_{i+1}\colon X_{i+1} \to X_i$ is defined by the letter L. Then $A_i = L_i$, $B_i = E_i$, and

$$(a_i, b_i) = (a_{i-1}, a_{i-1} + b_{i-1}) = (a_{i-1}, b_{i-1}) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Otherwise $A_i = E_i$, $B_i = R_i$, and

$$(a_i, b_i) = (a_{i-1} + b_{i-1}, b_{i-1}) = (a_{i-1}, b_{i-1}) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Writing

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{l_1} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{r_1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{l_2} \ldots,$$

we have $r/s = (a+b)/(c+d)$. Inductively,

$$(a_n, b_n) = (a_0, b_0) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (\mu, 0) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mu(a, b),$$

hence $\hat{\mu} = a_n + b_n = r \cdot \mu$. $\square$

The next task is to calculate the contribution of the relative canonical divisor $K_{X/Y}$ on the elementary transformation $\widehat{Y}$.

Proposition 4.2. *Let r/s be the fraction associated to the word w, and μ be the multiplicity of the closed fiber $Y \otimes k$. Then*

$$\hat{h}_*(K_{X/Y}) = (r+s-1) \cdot (\widehat{Y} \otimes k)^{\text{red}} = \frac{r+s-1}{r\mu} \cdot (\widehat{Y} \otimes k).$$

Proof. We use the same notation as in the preceding proof, except that a_i, b_i denote the multiplicities of A_i, B_i in $K_{X_i/Y}$. We introduce dummy multiplicities $a_0 = 0$ and $b_0 = 0$. By the adjunction formula, the multiplicity of E_{n+1} in $K_{X_{n+1}/Y}$ is $a_n + b_n + 1$. It turns out that the pairs (a_i+1, b_i+1) are easy to compute.

Suppose $f_{i+1}\colon X_{i+1} \to X_i$ is defined by the letter L. Then $A_i = L_i$, $B_i = E_i$, and

$$(a_i+1, b_i+1) = (a_{i-1}+1, a_{i-1}+1+b_{i-1}+1) = (a_{i-1}+1, b_{i-1}+1)\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Otherwise $A_i = E_i$, $B_i = R_i$, and

$$(a_i+1, b_i+1) = (a_{i-1}+1+b_{i-1}+1, b_{i-1}+1) = (a_{i-1}+1, b_{i-1}+1)\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Writing

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{l_1} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{r_1} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{l_2} \ldots,$$

we have $r/s = (a+b)/(c+d)$. Inductively,

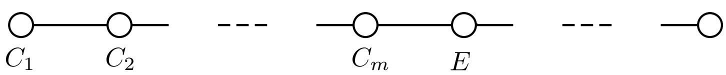
$$(a_n+1, b_n+1) = (a_0+1, b_0+1)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (1,1)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+c, b+d).$$

Hence the multiplicity of $\hat{h}_*(K_{X/Y})$ is

$$a_n + b_n + 1 = a + c + b + d - 1 = r + s - 1.$$

Together with Proposition 4.1, this concludes the proof. $\square$

For your convenience, I recall how self intersection numbers are related to continued fractions. Let



be the intersection graph of $X \otimes k$, labeled in such a way that C_1 is the reduced strict transform of $Y \otimes k$, and E is the reduced strict transform of $\widehat{Y} \otimes k$.

Let $r/s > 0$ be the fraction associated to the word $w = L^{l_1} R^{r_1} \ldots$ describing the elementary transformation, and

$$r/s = [s_1, s_2, \ldots, s_m] = s_1 - \cfrac{1}{s_2 - \cfrac{1}{s_3 - \cfrac{1}{\ldots}}}$$

its unique continued fraction development with $m \geq 1$, $s_1 > 0$, and $s_i > 1$ for $i \geq 2$.

Proposition 4.3. *Let $r/s = [s_1, s_2, \ldots, s_m]$ be the continued fraction as above. Then $C_i^2 = -s_i$. Here the intersection numbers are computed as degrees over $\kappa(y)$.*

Proof. The set $\mathfrak{S}$ of all finite sequences $(s_1, s_2, \ldots, s_m)$ with $m \geq 1$, $s_1 > 0$, and $s_i > 1$ for $i \geq 2$ becomes a monoid with respect to the composition law

$$(s_1, \ldots, s_{m-1}, s_m) \circ (t_1, t_2, \ldots, t_n) = (s_1, \ldots, s_{m-1}, s_m - 1 + t_1, t_2, \ldots, t_n).$$

By inspection, the map $\langle L, R\rangle \to \mathfrak{S}$ defined by $L \mapsto (2)$ and $R \mapsto (1, 2)$ is bijective. We also have a bijective map $\mathfrak{S} \to \mathbb{Q}_+$ by $(s_1, \ldots, s_m) \mapsto [s_1, \ldots, s_m]$. The maps obtained so far fit into a commutative diagram

$$\begin{array}{ccc} \langle L, R\rangle & \longrightarrow & \mathrm{SL}_2(\mathbb{N}) \\ \big\downarrow & & \big\downarrow \\ \mathfrak{S} & \longrightarrow & \mathbb{Q}_+. \end{array}$$

The rest of the proof is much alike the preceding proofs and left as an exercise. $\square$

Let $S \subset Y$ be a section disjoint from the initial center $y \in Y$, and $R \subset X$, $\widehat{S} \subset \widehat{Y}$ its strict transforms. With the notation introduced before the last Proposition, Mumford's rational pullback takes the form

$$\hat{h}^*(\widehat{S}) = R + \gamma_1 C_1 + \gamma_2 C_2 + \ldots + \gamma_m C_m$$

for certain rational coefficients $\gamma_i \geq 0$. For future reference, we determine the first coefficient:

Proposition 4.4. *Let r/s be the fraction associated to the word w, and μ the multiplicity of $Y \otimes k$, and $l = \dim_k \kappa(y)$. Then $\gamma_1 = s/(rl\mu)$.*

Proof. By definition of Mumford's pullback, the coefficients γ_i solve the equations

$$(R + \gamma_1 C_1 + \gamma_2 C_2 + \ldots + \gamma_m C_m) \cdot C_i = 0, \quad i = 1, \ldots, m.$$

Observe that $C_1 \cdot R = 1/\mu$ and $R \cap C_i = \emptyset$ for $i > 1$, since $S \subset Y$ is a section disjoint from the initial center $y \in Y$. Using Kronecker's δ-function, we can rewrite the equations as $(\gamma_1 C_1 + \ldots + \gamma_m C_m) \cdot C_i = -\delta_{i,1}/\mu$. Introducing dummy variables $\gamma_0 = 1/(l\mu)$ and $\gamma_{m+1} = 0$, we put the equations into simplified form

$$\gamma_{i-1} - s_i\gamma_i + \gamma_{i+1} = 0, \quad i = 1, \ldots, m.$$

Here the correction factor $l = \dim_k \kappa(y)$ is necessary since we calculate intersection numbers over $\kappa(y)$ and not over k. We have $\gamma_0/\gamma_1 = 1/(l\mu\gamma_1)$. On the other hand, an easy inductive argument gives

$$\gamma_0/\gamma_1 = [s_1, \ldots, s_m] = r/s,$$

and the result follows. $\square$

5. Classification of del Pezzo surfaces

In this section, Z is a proper normal del Pezzo surface with base number $\rho(Z) = 1$ containing a nonrational singularity. The issue is to understand the geometry of such surfaces. We show that these surfaces are arranged in an infinite hierarchy; moving inside the hierarchy involves elementary transformations. Let $f\colon X \to Z$ be the minimal resolution of singularities and

$$X \xrightarrow{h} Y \xrightarrow{g} Z$$

the factorization over the the minimal resolution of nonrational singularities. According to Theorem 2.2, there is a fibration $p\colon Y \to B$ of genus zero over a curve of genus $g > 0$ with irreducible fibers, and the exceptional curve $S \subset Y$ is a section.

In the simplest case, Y is already a $\mathbb{P}^1$-bundle. Otherwise, we seek to relate Y with a $\mathbb{P}^1$-bundle. Let $R' \subset X$ be the strict transform of $S \subset Y$, and $X \to Y'$ be the contraction of all vertical curves disjoint to R'. Then Y' is smooth, and the induced fibration $p'\colon Y' \to B$ is a $\mathbb{P}^1$-bundle. Hence the normal algebraic surface Y is obtained from the smooth surface Y' by a sequence

$$\begin{tikzcd}
& X_1 \arrow[dl, "h_1"'] \arrow[dr, "\hat{h}_1"] & & X_2 \arrow[dl, "h_2"'] & \cdots & X_{n-2} \arrow[dr, "\hat{h}_{n-2}"] & & X_{n-1} \arrow[dl, "h_{n-1}"'] \arrow[dr, "\hat{h}_{n-1}"] & \\
Y' = Y_1 & & Y_2 & & & & Y_{n-1} & & Y_n = Y
\end{tikzcd}$$

of elementary transformations. Note that each $h_i\colon X_i \to Y_i$ decomposes into a sequence of blowing-ups

$$X_i = X_{i,n_i+1} \longrightarrow X_{i,n_i} \longrightarrow \ldots \longrightarrow X_{i,2} \longrightarrow X_{i,1} \longrightarrow Y_i$$

as in Section 4. We have to fix more notation. Let $R_i \subset X_i$ and $S_i \subset Y_i$ be the strict transforms of $S \subset Y$. Let $y_i \in Y_i$ be the center of the elementary transformation $Y_i \leftarrow X_i \to Y_{i+1}$, and $w_i \in \langle L, R\rangle$ the corresponding word. Set $b_i = p_i(y_i)$, let d_i be the k-dimension of $\kappa(b_i)$, and let r_i/s_i be the fraction associated to the word w_i. Denote by μ_i the multiplicity of $Y_i \otimes \kappa(b_i)$, and by λ_i the coefficient in $K_{Y_i} \equiv -2S_i + \lambda_i F_i$. Here $F_i \in N(Y_i)$ is the fiber class. These numbers obey the following rule:

Lemma 5.1. $\lambda_{i+1} = \lambda_i + d_i(r_i + s_i - 1)/(\mu_i r_i)$.

Proof. By Proposition 4.1, the multiplicity of the fiber $Y_{i+1} \otimes \kappa(b_i)$ is the numerator $\mu_i r_i$. Moreover, we have $Y_{i+1} \otimes \kappa(b_i) \equiv d_i F_{i+1}$. According to Proposition 4.2, the denominator $r_i + s_i - 1$ is the multiplicity in $(\hat{h}_i)_*(K_{X_i/Y_i})$. Since

$$-2S_{i+1} + \lambda_{i+1} F_{i+1} \equiv K_{Y_{i+1}} \equiv -2S_{i+1} + \lambda_i F_{i+1} + (\hat{h}_i)_*(K_{X_i/Y_i}),$$

the claim is true. $\square$

Theorem 5.2. *For each surface Y_i, the curve $S_i \subset Y_i$ is contractible; the resulting contraction $g_i\colon Y_i \to Z_i$ yields a normal del Pezzo surface Z_i with base number $\rho(Z_i) = 1$ containing a nonrational singularity.*

Proof. We verify the assertion by descending induction on i. For $i = n$, there is nothing to prove. Suppose that the statement is true for some $i + 1 \leq n$. It follows from the definition of Mumford's rational intersection numbers that $R_i^2 \leq S_{i+1}^2 < 0$. Since $X_i \to Y_i$ is locally an isomorphism near S_i, we have $S_i^2 = R_i^2$. Thus S_i has negative selfintersection. By induction, we have $\lambda_{i+1} < 0$. According to Lemma 5.1, $\lambda_i \leq \lambda_{i+1}$. Hence Proposition 1.3 applies, and the contraction $g_i\colon Y_i \to Z_i$ of $R_i \subset Y_i$ exists. Since the canonical class is given by $K_{Z_i} = \lambda_i \cdot (g_i)_*(F_i)$, the resulting surface is del Pezzo. $\square$

We see that our del Pezzo surface $Z = Z_n$ is the last in a sequence of del Pezzo surface $Z_1, \ldots, Z_n$ underlying a sequence of elementary transformations

$$\begin{tikzcd}[column sep=small]
 & X_1 \arrow[dl, "h_1"'] \arrow[dr, "\hat{h}_1"] & & X_2 \arrow[dl, "h_2"'] & \cdots & X_{n-2} \arrow[dr, "\hat{h}_{n-2}"] & & X_{n-1} \arrow[dl, "h_{n-1}"'] \arrow[dr, "\hat{h}_{n-1}"] & \\
Y' = Y_1 \arrow[d, "g_1"] & & Y_2 \arrow[d, "g_2"] & & & & Y_{n-1} \arrow[d, "g_{n-1}"] & & Y_n = Y \arrow[d, "g_n"] \\
Z' = Z_1 & & Z_2 & & \cdots & & Z_{n-1} & & Z_n = Z.
\end{tikzcd}$$

The initial term $Z' = Z_1$ is a contraction of the $\mathbb{P}^1$-bundle $Y' = Y_1$. To obtain complete command of the situation, we have to describe how such sequence comes about. The question is: can we prolong the sequence one step further?

Suppose $Y_n \leftarrow X_n \rightarrow Y_{n+1}$ is a elementary transformation with center $y_n \in Y_n$ in $\mathrm{Reg}(Y_n)$ disjoint from $S_n \subset Y_n$. Introduce the same notation laid down in front of Theorem 5.2 as for the preceding elementary transformations. The issue is to determine whether or not the strict transform $S_{n+1} \subset Y_{n+1}$ contracts to a del Pezzo surface.

Theorem 5.3. *Notation as above. Then the curve $S_{n+1} \subset Y_{n+1}$ contracts to a del Pezzo surface Z_{n+1} if and only if*

$$\lambda_n + d_n \frac{r_n + s_n - 1}{\mu_n r_n} < 0.$$

Proof. According to Lemma 5.1, $\lambda_{n+1} = \lambda_n + d_n(r_n + s_n - 1)/(\mu_n r_n)$. This being the coefficient in $K_{Y_{n+1}} \equiv -2S_{n+1} + \lambda_{n+1} F_{n+1}$, the condition is necessary.

Conversely, assume that the condition holds. Then $\lambda_{n+1} < 0$. The main problem is to check that $S_{n+1} \subset Y_{n+1}$ has negative selfintersection. I claim that $S_i^2 \leq \lambda_i$ holds for $1 \leq i \leq n+1$. Suppose for a moment that this is true. Then S_{n+1} has negative selfintersection, and the desired contraction exists by to Proposition 1.3.

It remains to prove the claim. We proceed by induction on i. For $i = 1$, the fibration $p_1 \colon Y_1 \rightarrow B$ is a $\mathbb{P}^1$-bundle. The adjunction formula gives

$$K_{Y_1} \equiv -2S_1 + (S_1^2 + 2g - 2)F_1,$$

hence $S_1^2 \leq S_1^2 + 2g - 2 = \lambda_1$.

Suppose the claim is correct for some $i \geq 1$. Let $T_i \subset X_i$ be the reduced strict transform of the fiber $Y_{i+1} \otimes \kappa(b_{i+1})$. Write

$$(\hat{h}_i)^*(S_{i+1}) = R_i + \gamma_i \cdot T_i + \ldots$$

for some rational coefficient $\lambda_i \geq 0$. With $l_i = \dim_{\kappa(b_i)} \kappa(y_i)$, Proposition 4.4 gives $\gamma_i = s_i/(r_i l_i \mu_i)$. Note that $R_i \subset X_i$ hits the fiber over b_i only in T_i, since the center $y_i \in Y_i$ is disjoint from S_i. Using Mumford's definition of rational pullback, we obtain

$$\begin{aligned} S_{i+1}^2 &= R_i \cdot (R_i + \gamma_i T_i + \ldots) \\ &= R_i^2 + d_i/\mu_i \cdot s_i/(r_i l_i \mu_i) \\ &= S_i^2 + d_i s_i/(r_i l_i \mu_i^2) \\ &\leq \lambda_i + d_i(s_i + r_i - 1)/(r_i \mu_i). \end{aligned}$$

By Lemma 5.1, the latter expression equals λ_{i+1}. This completes the induction, and therefore the whole proof. □

6. Anticanonical rings and anticanonical models

Each proper normal algebraic surface X comes along with the *anticanonical ring*

$$R(-K_X) = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(-nK_X)),$$

which in turn defines the *anticanonical model* $P(-K_X) = \operatorname{Proj} R(-K_X)$. In this section we shall study the scheme $P(-K_X)$. Zariski ([26] p. 562) observed that the ring $R(-K_X)$ is possible non-Noetherian. Surprisingly, the scheme $P(-K_X)$ is of finite type ([22] Thm. 6.2). It is either empty, a point, a proper normal curve, or a normal algebraic surface (possibly nonproper).

There is a rational map $r\colon X \dashrightarrow P(-K_X)$ defined as follows: By definition, the homogeneous spectrum $P(-K_X)$ is covered by affine open subsets $D_+(s) = \operatorname{Spec} R_{(s)}$ with $s \in H^0(-nK_X)$ and $n > 0$. Let $X_s \subset X$ be the open subset where $s\colon \mathcal{O}_X \to \mathcal{O}_X(-nK_X)$ is bijective. Then the affine hull $X_s^{\text{aff}} = \operatorname{Spec} \Gamma(X_s, \mathcal{O}_X)$ and $D_+(s)$ are canonically isomorphic, and we obtain a morphism $r\colon U \to P(-K_X)$ defined on $U = \bigcup X_s$. We call $\operatorname{SBs}(-K_X) = X \setminus U$ the *stable base locus.*

Theorem 6.1. *Let X be a proper normal algebraic surface with 2-dimensional anticanonical model $P = P(-K_X)$. Then there is a unique open dense embedding $P \subset \overline{P}$ into a proper normal algebraic surface $\overline{P}$ such that the following holds:*

(i) *The open subset $P \subset \overline{P}$ is the $\mathbb{Q}$-Gorenstein locus of $\overline{P}$.*
(ii) *The boundary at infinity $\overline{P} \setminus P$ contains at most one point.*

Proof. Choose some integer $n > 0$ such that the stable base locus $\operatorname{SBs}(-K_X)$ is the base locus of $-nK_X$. Let $F \subset X$ be the fixed curve of $-nK_X$, and let $M = -nK_X - F$ be the movable part. Since P is 2-dimensional, we may furthermore assume that $M \neq 0$, such that $h^0(-nK_X) > 1$. Let $F' \subset F$ be the union of all connected components $C \subset F$ with $M \cdot C > 0$, and set $F'' = F - F'$.

The idea is to construct the compactification $\overline{P}$ as a suitable contraction of X. To do so, we have to define the corresponding negative definite curves. Let $R \subset X$ be the union of all curves $C \subset X$ with $C \cap F' = \emptyset$ and $C \cdot M = 0$. Note that $F'' \subset X$. By the Hodge index theorem, the curve R is negative definite.

Claim. *The curve $F' \subset X$ is also negative definite.*

To prove this, let $C \subset F'$ be a connected component and $F_1 + \ldots + F_r$ be its integral components. We have to show that the intersection matrix $(F_i \cdot F_j)$ is negative definite. Seeking a contradiction, we first assume that some Weil divisor supported by C has positive selfintersection. As in the proof of [23] Proposition 3.2, we find a curve $A \subset X$ with $\operatorname{Supp}(A) = \operatorname{Supp}(C)$, such that $A \subset X$ is linear equivalent to a curve $B \subset X$ not containing any F_i. Decompose $A = \sum \lambda_i F_i$ and $C = \sum \mu_i F_i$. We may assume that $\lambda_1/\mu_1 \geq \lambda_i/\mu_i$ for all indices $1 \leq i \leq r$. Then

$$\lambda_1 C = \sum_{i=1}^{r} \lambda_1 \mu_i F_i = \sum_{i=1}^{r} \mu_1 \lambda_i F_i + \sum_{i=2}^{r} (\lambda_1 \mu_i - \lambda_i \mu_1) F_i,$$

which is linearly equivalent to the effective divisor $\mu_1 B + \sum_{i=2}^{r} (\lambda_1 \mu_i - \lambda_i \mu_1) F_i$ not containing F_1. Consequently, $F_1 \not\subset \mathrm{SBs}(-K_X)$, contradiction.

Next, assume that $(F_i \cdot F_j)$ is negative, but not definite. By [3] Lemma I.2.10, the intersection form is negative semidefinite, and we find a curve $A \subset X$ with $\mathrm{Supp}(A) = \mathrm{Supp}(C)$ and $A \cdot F_i = 0$ for $1 \leq i \leq r$. Riemann–Roch for normal surfaces gives

$$h^0(tA) + h^2(tA) \geq \frac{A^2}{2} t^2 - \frac{A \cdot K_X}{2} t + \chi(\mathcal{O}_X) + \psi(t)$$

for some bounded error function $\psi(t)$. Since $K_X \cdot A < 0$ and $K_X - tA$ is not effective, we conclude that some tA is linearly equivalent to a curve $B \subset X$ disjoint from C. As above, this contradicts $C \subset \mathrm{SBs}(-K_X)$. QED for the claim.

We proceed with the proof of the Theorem. By Proposition 1.5, the negative definite curve $F' \cup R \subset X$ is contractible. Let $f\colon X \to Z$ be its contraction. We show that $Z = \overline{P}$ is the desired compactification of the anticanonical model $P = P(-K_X)$. We have to construct an open embedding $P \subset Z$. Given a section $s \in H^0(-nK_X)$, let $t \in H^0(M + F')$ be the corresponding section. You easily check that $D_+(s) = X_s^{\mathrm{aff}}$ is isomorphic to the open subset $X_t^{\mathrm{aff}} \subset Z$. Patching these isomorphisms gives the desired open embedding $P \subset Z$. By construction, the boundary at infinity $Z \setminus P$ is the image of $\mathrm{SBs}(-K_X) \setminus F''$, which is also $\mathrm{SBs}(-K_Z)$.

Note that $-nK_Z = f_*(-nK_X) = f_*(M)$ is movable, so $\mathrm{SBs}(-K_Z)$ is discrete. First, suppose that Z is $\mathbb{Q}$-Gorenstein. By the Fujita–Zariski theorem ([7] Thm. 1.10), the stable base locus $\mathrm{SBs}(-K_Z)$ is empty, so $Z = P$. Second, suppose that Z contains a non-$\mathbb{Q}$-Gorenstein point $z \in Z$. Since rational surface singularities are $\mathbb{Q}$-factorial, the point $z \in Z$ is a nonrational singularity. By Theorem 2.2, all other singularities are rational. Applying the Fujita–Zariski Theorem on the resolution of $z \in Z$, we deduce that $\mathrm{SBs}(-K_X) = \{z\}$. Hence we have a disjoint union $Z = P \cup \{z\}$. In either case, the open subset $P \subset Z$ is the $\mathbb{Q}$-Gorenstein locus. □

Here is a sufficient condition for 2-dimensional anticanonical models:

Proposition 6.2. *If $K_X^2 > 0$ and K_X is not pseudoeffective, then the anticanonical model $P(-K_X)$ is a surface.*

Proof. Riemann-Roch for normal surfaces [8] and Serre duality yield

$$h^0(-nK_X) + h^0((1+n)K_X) \geq K_X^2 \cdot n^2 + \chi(\mathcal{O}_X) + \psi(n)$$

for certain bounded error function $\psi(n)$. Hence there is an integer $n > 0$ and a curve $C \subset X$ representing $-nK_X$. By [23] Proposition 3.2, the complement $U = X - C$ has 2-dimensional global section ring $\Gamma(U, \mathcal{O}_X)$, since $C^2 > 0$. Because the affine hull $U^{\mathrm{aff}} = \operatorname{Spec} \Gamma(U, \mathcal{O}_X)$ is an open subset of $P(-K_X)$, the assertion follows. □

Example 6.3. Here are normal del Pezzo surfaces with non-finitely generated canonical rings, which also occur in [2]. As in Section 3, let B be a curve of genus $g \geq 2$, and $D \in \mathrm{Div}(B)$ a divisor of degree $0 < d < 2g-2$, and $Y = \mathbb{P}(\mathcal{O}_B \oplus \mathcal{O}_B(D))$

the corresponding ruled surface. According to Proposition 3.2, the negative section $S \subset Y$ is contractible, and the corresponding contraction $g\colon Y \to X$ gives a del Pezzo surface. Let $x \in X$ be the resulting singular point. You easily check that $\mathcal{O}_{X,x}$ is $\mathbb{Q}$-Gorenstein if and only if the divisors K_B and D are linearly dependent in $\operatorname{Pic}(B) \otimes \mathbb{Q}$. In this case, the anticanonical model is $P(-K_X) = X$, and the anticanonical ring is finitely generated. Otherwise, we have $P(-K_X) = X - \{x\}$, such that the anticanonical ring is not finitely generated.

References

[1] M. Artin: Some numerical criteria for contractibility of curves on algebraic surfaces. Am. J. Math. 84, 485–496 (1962).
[2] L. Bădescu: Anticanonical models of ruled surfaces. Ann. Univ. Ferrara, Nuova Ser. 29, 165–177 (1983).
[3] W. Barth, C. Peters, A. Van de Ven: Compact complex surfaces. Ergeb. Math. Grenzgebiete (3) 4, Springer, Berlin, 1984.
[4] S. Bosch, W. Lütkebohmert, M. Raynaud: Néron models. Ergeb. Math. Grenzgebiete (3) 21. Springer, Berlin, 1990.
[5] I. Cheltsov: Del Pezzo surfaces with nonrational singularities. Math. Notes 62, No.3, 377–389 (1997).
[6] T. Fujisawa: On non-rational numerical Del Pezzo surfaces. Osaka J. Math. 32, 613–636 (1995).
[7] T. Fujita: Semipositive line bundles. J. Fac. Sci. Univ. Tokyo 30, 353–378 (1983).
[8] J. Giraud: Improvement of Grauert-Riemenschneider's theorem for a normal surface. Ann. Inst. Fourier 32, 13–23 (1982).
[9] A. Grothendieck et al.: Théorie des intersections et théorème de Riemann-Roch. Lect. Notes Math. 225. Springer, Berlin, 1971.
[10] R. Hartshorne: Ample subvarieties of algebraic varieties. Lect. Notes Math. 156. Springer, Berlin, 1970.
[11] F. Hidaka, K. Watanabe: Normal Gorenstein surfaces with ample anti-canonical divisor. Tokyo J. Math. 4, 319–330 (1981).
[12] F. Hidaka, M. Tomari: On singularities arising from the contraction of the minimal section of a ruled surface. Manuscr. Math. 65, 329–347 (1989).
[13] S. Kleiman: Toward a numerical theory of ampleness. Ann. Math. (2) 84, 293–344 (1966).
[14] J. Kollár: Rational curves on algebraic varieties. Ergeb. Math. Grenzgebiete (3) 32. Springer, Berlin, 1995.
[15] S. Mori: Threefolds whose canonical bundles are not numerically effective. Ann. Math. 116, 133–176 (1982).
[16] D. Mumford: The topology of an normal surface singularity of an algebraic variety and criterion for simplicity. Publ. Math., Inst. Hautes Étud. Sci. 9, 5–22 (1961).
[17] D. Mumford: Lectures on curves on an algebraic surface. Princeton University Press, Princeton, 1966.
[18] D. Mumford: Enriques' classification of surfaces in char p I. In: D. Spencer, S. Iyanaga (eds.), Global Analysis, pp. 325–339. Princeton University Press, Princeton, 1969.
[19] D. Mumford: Abelian varieties. Tata Institute of Fundamental Research Studies in Mathematics 5. Oxford University Press, London, 1970.
[20] F. Sakai: Anticanonical models of rational surfaces. Math. Ann. 269, 389–410 (1984).
[21] F. Sakai: Ruled fibrations on normal surfaces. J. Math. Soc. Japan 40, 249–269 (1988).
[22] S. Schröer: On non-projective normal surfaces. Manuscr. Math. 100, 317–321 (1999).
[23] S. Schröer: On contractible curves on normal surfaces. J. Reine Angew. Math. 524, 1–15 (2000).
[24] H. Tango: On the behavior of extensions of vector bundles under the Frobenius map. Nagoya Math. J. 48, 73–89 (1972).
[25] J. Tate: Genus change in inseparable extensions of function fields. Proc. Am. Math. Soc. 3, 400–406 (1952).
[26] O. Zariski: The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface. Ann. Math. (2) 76, 560–615 (1962).

Mathematische Fakultät, Ruhr-Universität, 44780 Bochum, Germany
Current address: M.I.T. Mathematical Department, 77 Massachusetts Avenue, Cambridge, MA 02139-4307, USA
E-mail address: s.schroeer@ruhr-uni-bochum.de