ENRIQUES MANIFOLDS

KEIJI OGUISO AND STEFAN SCHRÖER

Final version, February 4, 2011

ABSTRACT. Using the theory of hyperkähler manifolds, we generalize the notion of Enriques surfaces to higher dimensions, explore their properties, and construct several examples using group actions on Hilbert schemes of points or moduli spaces of stable sheaves.

CONTENTS

Introduction 1
1. Bogomolov decomposition and hyperkähler manifolds 3
2. Notion of Enriques manifolds 5
3. Calabi-Yau manifolds via Hilbert schemes of points 8
4. First examples of Enriques manifolds 11
5. Stable sheaves on K3 surfaces 12
6. Bielliptic surfaces 14
References 17

INTRODUCTION

Naturally, Enriques surfaces play a prominent role in the Enriques classification of algebraic surfaces. They are, by definition, minimal surfaces of Kodaira dimension $\kappa = 0$ with $b_2 = 10$. An equivalent condition is that they are not simply connected and have a K3 surface as universal covering. There are numerous deep results concerning Enriques surfaces, for example about their geometry, automorphism groups, or periods. In light of this richness it is natural to ask whether there is a natural generalization of Enriques surfaces to higher dimensions. The goal of this paper is to introduce the notion of *Enriques manifolds*, explore their basic properties, and construct several interesting examples.

Recall that a *hyperkähler manifold* is a smooth compact simply-connected Kähler manifold X with the property that $H^0(X, \Omega^2_X)$ is generated by a symplectic form. Beauville [3] showed that such manifolds are, together with complex tori and Calabi–Yau manifolds, the basic building blocks for Kähler manifolds with $c_1 = 0$. There is a profound theory for hyperkähler manifolds (see Huybrechts [17]), which largely runs parallel to the theory of K3 surfaces. Indeed, one should view hyperkähler manifolds as the correct generalization of K3 surfaces to higher dimensions. Therefore, we define an *Enriques manifold* as a connected complex space Y

2000 *Mathematics Subject Classification.* 14J28, 14J32.

that is not simply connected and whose universal covering $X = \widetilde{Y}$ is a hyperkähler manifold.

It turns out that the fundamental group $\pi_1(Y)$ is a finite cyclic group. We call its order $d \geq 2$ the *index* of the Enriques manifold Y. This number d is a divisor of $n+1$, where $\dim(Y) = 2n$, and moreover meets the condition $\varphi(d) < b_2(X)$, where φ is Euler's phi function. A natural question arises: Which integers d appear as indices for Enriques manifolds? In other words, which cyclic groups can act freely on some hyperkähler manifold?

In some sense, there are not too many known examples of hyperkähler manifolds. Beauville [3] constructed two infinite series, namely the Hilbert scheme of points for K3 surfaces, and his *generalized Kummer variety* $\mathrm{Km}^n(A)$, defined as a Bogomolov factor in $\mathrm{Hilb}^{n+1}(A)$ for abelian surfaces A. Furthermore, there are two sporadic examples of O'Grady (see [27], [28]). The first idea to construct Enriques manifolds is to look at Hilbert schemes for Enriques surfaces or bielliptic surfaces, but this does not work out. Rather, it leads to an interesting new construction of *Calabi–Yau manifolds*:

Theorem. *Let S be an Enriques surface or a bielliptic surface, and $n \geq 2$. Then $\mathrm{Hilb}^n(S)$ has a finite étale covering that is a Calabi–Yau manifold or is the product of a Calabi–Yau manifold with an elliptic curve, respectively.*

However, if one starts with an Enriques surface S', say with universal covering S, and an *odd* number $n \geq 1$, then the induced action of $G = \pi_1(S')$ on $X = \mathrm{Hilb}^n(S)$ is free, and the corresponding quotient is an Enriques manifold Y of dimension $\dim(Y) = 2n$ and index $d = 2$. There is a variant with generalized Kummer varieties, and the preceding construction can be extended from Hilbert schemes of points to moduli spaces of sheaves:

Theorem. *Suppose S' is an Enriques surface whose corresponding K3 surface has Picard number $\rho(S) = 10$. Let $v = (r, l, \chi - r) \in H^{\mathrm{ev}}(S, \mathbb{Z})$ be a primitive Mukai vector with $v^2 \geq 0$ and $\chi \in \mathbb{Z}$ odd. Then for very general polarizations $H \in \mathrm{NS}(S)_{\mathbb{R}}$, the moduli space $X = M_H(v)$ is a hyperkähler manifold endowed with a free action of $G = \pi_1(S')$, and $Y = X/G$ is an Enriques manifold of dimension $v^2 + 2$ and index $d = 2$.*

Recall that a *bielliptic surface* S has, by definition, a finite étale covering A that is an abelian surface. To construct examples of Enriques of higher index, we use the classification of bielliptic surface due to Bagnera and de Franchis and study the induced action of $G = \pi_1(S)$ on $\mathrm{Hilb}^n(A)$. This yields:

Theorem. *There are Enriques manifolds with index $d = 2, 3, 4$.*

The paper is organized as follows: In the first section we recall several results about the Bogomolov decomposition of manifolds with trivial first Chern class and the theory of hyperkähler manifolds. In the second section, we introduce the notion of Enriques manifolds and collect their basic properties. In the third section, we examine Hilbert schemes of points for Enriques surfaces and bielliptic surfaces. The first examples of Enriques manifolds appear in Section 4 as quotients of Hilbert schemes of points for the K3 covering of an Enriques surface. We extend this construction to moduli spaces of stable sheaves in Section 5. In Section 6 we use the classification of bielliptic surfaces to construct Enriques manifolds whose universal covering are Beauville's generalized Kummer varieties.

Remark. Our paper overlaps with the paper *Higher dimensional Enriques varieties and automorphisms of generalized Kummer varieties*, arXiv:1001.4728v1 by Samuel Boissière, Marc Nieper-Wisskirchen, Alessandra Sarti, which appeared on the arXiv after our paper was completed.

Acknowledgement. The authors started this research at the Centro di Ricerca Matematica Ennio De Giorgi during the workshop on *Groups in Geometry* in September 2008. We thank the Centro for providing a stimulating research environment. The second author visited the first author in September 2009 at the Osaka University and thanks the Department of Mathematics for its hospitality. The second author was supported by a DFG grant within the Forschergruppe FOR 790 *Classification of Algebraic Surfaces and Compact Complex Manifolds*. Finally, we thank the referee, Alessandra Sarti and Samuel Boissiére for comments.

1. Bogomolov decomposition and hyperkähler manifolds

Throughout this paper, we shall work over the complex numbers. Given a complex manifold Y, we regard its first Chern class $c_1(Y)$ as an element in the rational vector space $H^2(Y,\mathbb{Q})$. In this section we recall some results on compact Kähler manifolds Y with $c_1(Y)=0$, which are due to Beauville, Bogomolov, Fujiki, Huybrechts, Mukai, O'Grady, Yoshioka, and others.

The fundamental result is that such manifolds Y admit a finite étale covering $X\to Y$ of the form $X=\prod_{i=1}^r X_i$ where the factors X_i are complex tori, Calabi–Yau manifolds, or hyperkähler manifolds ([3] and [7]). Such a factorization on a finite étale cover is called a *Bogomolov decomposition*. The fundamental group of Y is an extension of a finite group by a free abelian group. Obviously, $\pi_1(Y)$ is finite if and only if no Bogomolov factor is a complex torus. In this case, a Bogomolov factorization exists only on the universal covering $X=\widetilde{Y}$.

Throughout the paper, the term *Calabi–Yau manifold* denotes a compact connected Kähler manifold X of dimension ≥ 3 that is simply connected, has $\omega_X=\mathcal{O}_X$, and $h^{p,0}(X)=0$ for $0<p<\dim(X)$. With this definition, Calabi–Yau manifolds are automatically projective, by Kodaira's embedding Theorem. There is no common agreement about the term "Calabi–Yau manifold", and some authors use it to denote manifolds with $c_1=0$. Also, it is sometimes useful to replace the assumption that X is Kähler by the weaker assumption that X is is bimeromorphic to some Kähler manifold, which then is equivalent to being Moishezon.

Recall that a *hyperkähler manifold* is a compact connected Kähler manifold X that is simply-connected with $H^0(X,\Omega_X^2)=\mathbb{C}\sigma$, where σ is a symplectic form. In other words, it induces nondegenerate alternating pairing on all tangent spaces $\Theta_X(x)$, $x\in X$. Let us recall some facts on such manifolds. The existence of a symplectic form $\sigma\in H^0(X,\Omega_X^2)$ ensures that $\dim(X)=2n$ is even and that the dualizing sheaf $\omega_X=\mathcal{O}_X$ is trivial. Moreover, one knows that the algebra of holomorphic forms $\bigoplus_p H^0(X,\Omega_X^p)$ is generated by the symplectic form, such that

$$h^{p,0}(X)=\begin{cases}1 & \text{if } p \text{ is even,}\\ 0 & \text{if } p \text{ is odd,}\end{cases}$$

thus $\chi(\mathcal{O}_X)=n+1$. There is an elaborate general theory about hyperkähler manifolds parallel to the theory of K3 surfaces, see Huybrechts [17].

Up to deformation equivalence, the only known examples of hyperkähler manifolds are the Hilbert scheme of points $\mathrm{Hilb}^n(S)$ for K3 surfaces S and Beauville's generalized Kummer varieties $\mathrm{Km}^n(A)$ for abelian surfaces A, both introduced by Beauville [3], and two sporadic examples M_6, M_{10} constructed by O'Grady as desingularizations of certain moduli spaces of sheaves on K3 or abelian surfaces ([27] and [28]). Here are some numerical invariants for these hyperkähler manifolds:

X	$\mathrm{Hilb}^n(S)$	$\mathrm{Km}^n(A)$	M_6	M_{10}
$\dim(X)$	$2n$	$2n$	6	10
$\chi(\mathcal{O}_X)$	$n+1$	$n+1$	4	6
$b_2(X)$	23	7	8	24

Next, let us recall some facts about symmetric products and Hilbert schemes of points. Given an arbitrary compact complex space V, we denote by $\mathrm{Sym}^n(V)$ the symmetric product. Let $\pi : V^n \to \mathrm{Sym}^n(V)$ be the quotient map. If V is normal, so is $\mathrm{Sym}^n(V)$, and the norm map $\pi_*(\mathcal{O}_{V^n}^\times) \to \mathcal{O}_{\mathrm{Sym}^n(V)}^\times$ induces a homomorphism

$$\mathrm{Pic}(V) \longrightarrow \mathrm{Pic}(\mathrm{Sym}^n(V)), \quad \mathcal{L} \longmapsto \mathcal{L}^{(n)}.$$

Using the interpretation $\mathcal{L}^{(n)} = \pi_*(\otimes_{i=1}^n \mathrm{pr}_i^*(\mathcal{L}))^{S_n}$, we easily infer $H^0(V,\mathcal{L}) = H^0(\mathrm{Sym}^n(V), \mathcal{L}^{(n)})$, such that the preceding homomorphism is injective. If V is Gorenstein, so is $\mathrm{Sym}^n(V)$, and we have $\omega_{\mathrm{Sym}^n(V)} = (\omega_V)^{(n)}$.

According to Grothendieck's observation ([13], Expose IX, Remark 5.8), we have an identification

$$\pi_1(\mathrm{Sym}^n(V)) = H_1(V,\mathbb{Z}), \quad n \geq 2.$$

If V is normal with only quotient singularities, then $\mathrm{Sym}^n(V)$ is normal with only quotient singularities. Under this assumption, the canonical map

$$\pi_1(Z) \longrightarrow \pi_1(\mathrm{Sym}^n(V)) = H_1(V,\mathbb{Z})$$

is bijective for any resolution of singularities $Z \to \mathrm{Sym}^n(V)$, by [20], Theorem 7.8. Now consider the Hilbert scheme, or rather Douady space, of points $\mathrm{Hilb}^n(V)$, and let

$$\gamma : \mathrm{Hilb}^n(V) \longrightarrow \mathrm{Sym}^n(V), \quad A \longmapsto \sum_{x\in V} \mathrm{length}(\mathcal{O}_{A,x})x$$

be the *Hilbert–Chow morphism*, which sends a subscheme to the corresponding zero-cycle (see [19] for more details). In general, the Hilbert scheme of points is much more complicated than the symmetric product. However, if V is a smooth surface, then the Hilbert–Chow morphism is a crepant resolution of singularities. We refer to Beauville's paper [3] or the monograph of Brion and Kumar [10], Chapter 7 for detailed discussions.

Now let us recall Beauville's generalized Kummer surface. Let A be an abelian surface, and consider the composite map

$$\mathrm{Hilb}^{n+1}(A) \longrightarrow \mathrm{Sym}^{n+1}(A) \longrightarrow A,$$

where the first arrow is the Hilbert–Chow morphism, and the second arrow is the addition map. This actually is the Albanese map. The fiber over the origin $\mathrm{Km}^n(A) \subset \mathrm{Hilb}^{n+1}(A)$ is called the *generalized Kummer variety*, and is a hyperkähler manifold $X = \mathrm{Km}^n(A)$ of dimension $2n$. In other words, $\mathrm{Km}^n(A)$ is defined as a Bogomolov factor for $\mathrm{Hilb}^{n+1}(A)$.

Moduli spaces of coherent sheaves provide further examples. Let S be a K3 surface. $v \in H^{\mathrm{ev}}(S, \mathbb{Z})$ a Mukai vector, and $H \in \mathrm{NS}(S)_{\mathbb{R}}$ a polarization. Mukai [22] showed that the moduli space $M_H(v)$ of H-stable sheaves $\mathcal{F}$ on S with Mukai vector $v(\mathcal{F}) = v$ is smooth of dimension $v^2 + 2$, where $v^2 = (v, v)$ comes from the Mukai pairing (for details, see [23] or [16], Chapter 6). It turns out that for H generic and v primitive, $M_H(v)$ is actually a hyperkähler manifold (see [26] and [30]), which is deformation equivalent to $\mathrm{Hilb}^n(S)$, $n = (v^2 + 2)/2$. Using moduli spaces of stable sheaves on abelian surfaces and the Fourier–Mukai transform, Yoshioka [30] constructed hyperkähler manifolds $K_H(v)$ that are deformation equivalent to $\mathrm{Km}^n(A)$.

There are more examples of hyperkähler 4-folds, all of them deformation equivalent to $\mathrm{Hilb}^2(S)$: Beauville and Donagi [4] showed that the variety of lines on a smooth cubic hyperplanes in $\mathbb{P}^5$ is a hyperkähler 4-fold. Iliev and Ranestad [18] proved that the variety of sums of powers for a general cubic hyperplane as above is an another such examples. O'Grady [29] constructed hyperkähler 4-folds as double covers of certain sextic hyperplane in $\mathbb{P}^5$. Debarre and Voisin [11] showed that for $V = \mathbb{C}^{\oplus 10}$ and $\sigma \in \Lambda^3(V^\vee)$ general, the scheme of 6-dimensional subvector spaces of V on which σ vanishes is a hyperkähler 4-fold.

2. Notion of Enriques manifolds

In the classification of surfaces, *Enriques surfaces* are defined as minimal surfaces S with Kodaira dimension $\kappa = 0$ and second Betti number $b_2 = 10$. A different but equivalent definition is that S is not simply connected, and its universal cover is a K3 surface. Viewing hyperkähler manifolds as the correct generalization of K3 surfaces, we propose the following generalization of Enriques surfaces to higher dimensions.

Definition 2.1. An *Enriques manifold* is a connected complex manifold Y that is not simply connected and whose universal cover X is a hyperkähler manifold.

Obviously, Enriques manifolds Y are compact, of even dimension $\dim(Y) = 2n$, and with finite fundamental group. Averaging a Kähler metric on X over its G-translates, one sees that Y is a Kähler manifold. The 2-dimensional Enriques manifolds are precisely the Enriques surfaces, whose fundamental group is cyclic of order $d = 2$. In higher dimensions, this order is a basic numerical invariant:

Definition 2.2. The *index* of an Enriques manifold Y is the order $d \geq 2$ of its fundamental group $\pi_1(Y)$.

Let Y be an Enriques manifold of dimension $\dim(Y) = 2n$, and $X \to Y$ be its universal covering. Fix a base point $y \in Y$. The natural action of $\pi_1(Y, y)$ on X induces a representation on the 1-dimensional vector space $H^0(X, \Omega^2_X)$, which corresponds via the trace to a homomorphism $\rho : \pi_1(Y, y) \to \mathbb{C}^\times$.

Lemma 2.3. *The homomorphism of groups $\rho : \pi_1(Y, y) \to \mathbb{C}^\times$ is injective.*

Proof. Let $G \subset \ker(\rho)$ be a cyclic subgroup, say of order $m = |G|$, and consider the complex manifold $Z = X/G$. Let $f : X \to Z$ be the canonical projection. Then the $\mathcal{O}_Z$-algebra $f_*(\mathcal{O}_X)$ takes the form $\mathcal{O}_Z \oplus \mathcal{L} \oplus \mathcal{L}^{\otimes 2} \oplus \ldots \oplus \mathcal{L}^{\otimes(m-1)}$ for some $\mathcal{L} \in \mathrm{Pic}(Z)$ of order m, where the multiplication is given by some trivialization $\mathcal{L}^{\otimes m} \to \mathcal{O}_Z$. We have $\chi(\mathcal{L}^{\otimes i}) = \chi(\mathcal{O}_Z)$ because $\mathcal{L}$ is numerically trivial, whence $\chi(\mathcal{O}_X) = |G|\chi(\mathcal{O}_Z)$.

On the other hand, we have $H^0(Z,\mathcal{F}) = H^0(X, f^*(\mathcal{F}))^G$ for every coherent sheaf $\mathcal{F}$ on Z. The canonical map $f^*(\Omega^1_Z) \to \Omega^1_X$ is bijective, because $f : X \to Z$ is étale, and consequently $H^0(Z,\Omega^p_Z) = H^0(X,\Omega^p_X)^G$ for all $p \geq 0$. The group $H^0(X,\Omega^p_X)$ vanishes for p odd, and is generated by the p-form $\sigma \wedge \ldots \wedge \sigma$ for p even. Using that σ is G-invariant, we conclude that the canonical maps $H^0(Z,\Omega^p_Z) \to H^0(X,\Omega^p_X)$ are bijective. Hodge symmetry yields

$$\chi(\mathcal{O}_X) = \sum_p (-1)^p h^{p,0}(X) = \sum_p (-1)^p h^{p,0}(Z) = \chi(\mathcal{O}_Z),$$

and this number equals $n + 1 \neq 0$. Whence $|G| = 1$, and the representation ρ is faithful. □

Proposition 2.4. *Let Y be an Enriques manifold of dimension $\dim(Y) = 2n$. Then $\pi_1(Y)$ is a cyclic group whose order $d \geq 2$ is a divisor of $n + 1$.*

Proof. Being a finite subgroup of the multiplicative group of a field, the fundamental group must be cyclic. The universal covering X has $\chi(\mathcal{O}_X) = n + 1$, and we have $\chi(\mathcal{O}_X) = d\chi(\mathcal{O}_Y)$. □

Remark 2.5. The representation on $H^0(X,\Omega^2_X)$ yields a canonical identification $\pi_1(Y) = \mu_d(\mathbb{C})$ with the group of d-th roots of unity. Consequently we have a canonical generator $e^{2\pi\sqrt{-1}/d}$ of the fundamental group.

In the following, Y denotes an Enriques manifold of dimension $\dim(Y) = 2n$ and index $d \geq 2$, with universal cover X. Index and dimension control the Hodge numbers $h^{p,q} = \dim H^q(Y,\Omega^p_Y)$ for $q = 0$ and $p = 0$:

Proposition 2.6. *We have*

$$h^{0,p}(Y) = h^{p,0}(Y) = \begin{cases} 1 & \text{if } 2d \mid p \text{ and } p \leq 2n, \\ 0 & \text{else.} \end{cases}$$

In particular, $\chi(\mathcal{O}_Y) = (\dim(Y) + 2)/2d$.

Proof. Using $H^0(Y,\Omega^p_Y) = H^0(X,\Omega^p_X)^G$, we have $h^{p,0}(X) = 0$ for p odd or $p > 2n$. Consider the case $p \leq 2n$ even. Then the canonical map

$$\bigotimes_{i=1}^{p/2} H^0(X,\Omega^2_X) \longrightarrow H^0(X,\Omega^p_X), \quad \sigma_1 \otimes \ldots \otimes \sigma_{p/2} \longmapsto \sigma_1 \wedge \ldots \wedge \sigma_{p/2} \tag{1}$$

is bijective, and $H^0(X,\Omega^2_X)$ is 1-dimensional. The statement now follows from Lemma 2.3, together with Hodge symmetry. □

Corollary 2.7. *Every Enriques manifold Y is projective, and the same holds for its universal cover X.*

Proof. The inclusion $H^{1,1}(Y)_\mathbb{R} \subset H^2(Y,\mathbb{R})$ is bijective, since $h^{2,0} = h^{0,2} = 0$. Using that the Kähler cone inside $H^{1,1}(Y)_\mathbb{R}$ is nonempty and open, we conclude that there must be an integral Kähler class on Y. According to Kodaira's Embedding Theorem (compare [12], p. 191), the Kähler manifold Y is projective. Pulling back this integral Kähler class, we obtain an integral Kähler class on X, so X is projective as well. □

Proposition 2.8. *The group $\operatorname{Pic}(Y)$ is finitely generated. Its torsion subgroup is a cyclic group of order d, which is generated by the canonical class $\omega_Y \in \operatorname{Pic}(Y)$.*

Proof. The map $f^{-1}(\mathcal{O}_Y^\times) \to \mathcal{O}_X^\times$ is bijective because f is étale. Whence we have a spectral sequence $H^r(G, H^s(X, \mathbb{G}_m)) \Rightarrow H^{r+s}(Y, \mathbb{G}_m)$, which yields an exact sequence

$$0 \longrightarrow H^1(G, \mathbb{C}^\times) \longrightarrow \operatorname{Pic}(Y) \longrightarrow \operatorname{Pic}(X)^G,$$

where G acts trivially on $\mathbb{C}^\times$. In turn, the group cohomology $H^1(G, \mathbb{C}^\times)$ is the kernel of the map $\mathbb{C}^\times \to \mathbb{C}^\times$, $z \mapsto z^d$, which is cyclic of order d. We have $\operatorname{Pic}^0(X) = 0$ because its tangent space $H^1(X, \mathcal{O}_X)$ vanishes. It follows that $\operatorname{Pic}(X)$ is finitely generated. Its torsion part vanishes because X is simply connected. Hence $\operatorname{Pic}(Y)$ is finitely generated, and its torsion part is cyclic of order d.

The canonical class $\omega_Y \in \operatorname{Pic}(Y)$ has finite order because $f^*(\omega_Y) = \omega_X$ is trivial. Let $r \mid d$ be its order. Then the induced G-action on $\omega_X^{\otimes r} = f^*(\omega_Y^{\otimes r})$ is trivial. On the other hand, the representation on $H^0(X, \omega_X^{\otimes r})$ has trace $\tau^{rn} : G \to \mathbb{C}^\times$, which follows from (1). Here τ denotes the trace of the representation on $H^0(X, \Omega_X^2)$. Therefore $d \mid rn$. Proposition 2.4 implies that $d \mid r$, whence $d = r$. □

There is a strong relation between the index of Y and the second Betti number of X. Let $\varphi(d)$ be the order of the multiplicative group $(\mathbb{Z}/d\mathbb{Z})^\times$. Recall that $\varphi(d) = \prod p_i^{\nu_i - 1}(p_i - 1)$ if $d = \prod_i p_i^{\nu_i}$ is the prime factorization.

Proposition 2.9. *We have $\varphi(d) < b_2(X)$.*

Proof. This result is due to Nikulin for K3-surfaces ([25], Theorem 3.1); his argument works in our situation as follows: First note that the restriction of the Beauville–Bogomolov form q_X to $H^{1,1}(X)_\mathbb{R} \subset H^2(X, \mathbb{R})$ has index $(1, b_2(X) - 3)$, and that $q_X(\alpha) > 0$ for all Kähler classes $\alpha \in H^2(X, \mathbb{R})$, as explained in [17], Section 1.9. Using that X is projective, we infer that q_X is nondegenerate on $\operatorname{NS}(X) \subset H^2(X, \mathbb{R})$. Let $T \subset H^2(X, \mathbb{Z})$ be the orthogonal complement of $\operatorname{NS}(X) \subset H^2(X, \mathbb{Z})$ with respect to the Beauville–Bogomolov bilinear form q_X, which clearly is a lattice of rank $< b_2(X)$. Choose a generator $g_0 \in G$. By the Cayley–Hamilton Theorem, it suffices to check that the minimal polynomial for the endomorphism $g_0 \in \operatorname{End}(T_\mathbb{Q})$ is the d-th cyclotomic polynomial Φ_d, which has degree $\varphi(d)$. Suppose this is not the case. Then there is a nontrivial $g \in G$ admitting an eigenvector $x \in T_\mathbb{Q}$ with eigenvalue 1. Choose a generator $\sigma \in H^{2,0}(X)$ and write $g^*(\sigma) = \xi\sigma$ for some nontrivial $\xi \in \mathbb{C}^\times$. The G-invariance of the Beauville–Bogomolov form yields

$$q_X(x, \sigma) = q_X(g^*(x), g^*(\sigma)) = q_X(x, \xi\sigma) = \xi q_X(x, \sigma).$$

Since $\operatorname{NS}(X) \subset H^2(X, \mathbb{Z})$ is the orthogonal complement of σ, we have $q_X(x, \sigma) \neq 0$, whence $\xi = 1$, contradiction. □

Let us tabulate the possible indices for the Betti numbers for Beauville's families of hyperkähler manifolds:

b_2	X	possible indices d
7	$\operatorname{Km}^n(A)$	$2 - 10, 12, 14, 18, 24$
23	$\operatorname{Hilb}^n(S)$	$2 - 28, 30, 32, 33, 34, 36, 38, 40, 42, 44, 46, 50, 54, 66$

Of course, Proposition 2.4 gives stronger restrictions if n has few divisors. For example, with $n = 2$ or $n = 3$ we only have the possibilities $d = 3$ or $d = 2, 4$, respectively. With O'Grady's 6-dimensional example M_6 only $d = 2, 4$ are possible, and with his 10-dimensional example M_{10} only $d = 2, 3, 6$ may occur.

We finally touch upon the subject of birationally equivalent Enriques manifolds:

Proposition 2.10. *Let Y and Y' be two Enriques manifolds that are birationally equivalent. Then they have the same index.*

Proof. The fundamental group is a birational invariant for smooth complex manifolds (compare [12], Section 4.2), whence $d(Y) = d(Y')$. $\square$

3. Calabi-Yau manifolds via Hilbert schemes of points

The prime goal of this paper is to construct examples of Enriques manifolds. The first idea that comes to mind is to look at Hilbert schemes of Enriques surfaces. This, however, does not lead to Enriques manifolds:

Theorem 3.1. *Let S be an Enriques surface and $Y = \operatorname{Hilb}^n(S)$ for some $n \geq 2$. Then $\pi_1(Y)$ is cyclic of order two, and the universal covering X of Y is a Calabi–Yau manifold.*

Proof. We have $h^p(\mathcal{O}_S) = 0$ for all $p > 0$. Let $\mathrm{pr} : S^n \to S^{n-1}$ be a projection. Then $R^p\,\mathrm{pr}_*(\mathcal{O}_{S^n}) = 0$ for all $p > 0$. Induction on n and the Leray–Serre spectral sequence yields $h^p(\mathcal{O}_{S^n}) = 0$ for all $p > 0$. Now consider the canonical projection $g : S^n \to \operatorname{Sym}^n(S)$. The resulting inclusion $\mathcal{O}_{\operatorname{Sym}^n(S)} \subset g_*(\mathcal{O}_{S^n})$ is the inclusion of invariants with respect to the permutation action of the symmetric group, whence is a direct summand. It follows that $h^p(\mathcal{O}_{\operatorname{Sym}^n(S)}) = 0$ for all $p > 0$. In turn, we have $h^p(\mathcal{O}_Y) = 0$ for all $p > 0$ because the singularities on the symmetric product are rational. In particular, $\chi(\mathcal{O}_Y) = 1$.

As discussed in Section 1, the dualizing sheaf ω_Y has order two, and the fundamental group $\pi_1(Y)$ is cyclic of order two. Taking $\dim(Y) = 2n > 2$ into account, we deduce the assertion from the following Lemma. $\square$

Lemma 3.2. *Let Y be a compact Kähler manifold with $c_1(Y) = 0$. Suppose $\chi(\mathcal{O}_Y) = 1$ and $\pi_1(Y)$ is cyclic of order two. Then the universal covering of Y is either a K3 surface or a Calabi–Yau manifold of even dimension.*

Proof. Let $X \to Y$ be the universal covering. The manifold X is compact because $\pi_1(Y)$ is finite. Consider the Bogomolov decomposition $X = \prod_{i=1}^r X_i$, where the factors are Calabi–Yau manifolds or hyperkähler manifolds. Complex tori do not appear, because Y has finite fundamental group. We have

$$2 = |\pi_1(Y)|\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_X) = \prod_{i=1}^r \chi(\mathcal{O}_{X_i}).$$

In particular, $\chi(\mathcal{O}_{X_i}) \neq 0$, so no Bogomolov factor is an odd-dimensional Calabi-Yau manifold. Consequently $\chi(\mathcal{O}_{X_i}) \geq 2$, which gives the estimate $2 \geq 2^r$, whence $r = 1$. Thus $X = X_1$ is either an even-dimensional Calabi-Yau manifold, or hyperkähler. In the latter case, we have $\dim(X) = 2m$ and $\chi(\mathcal{O}_X) = m+1$, so $m = 1$ and X is a K3 surface. $\square$

What are the Bogomolov factors for the Hilbert scheme of points for bielliptic surfaces? In this situation, Calabi–Yau manifolds of odd dimension show up. In contrast to the even-dimensional case, the numbers $\chi(\mathcal{O}_V)$ are then not so helpful. Rather than working with Euler characteristics, we shall work with graded algebras. Suppose V is a compact Kähler manifold. Then we have the cohomology algebra

$$H^\bullet(V,\mathbb{C}) = \bigoplus_i H^i(V,\mathbb{C}) = \bigoplus_{p,q} H^q(X,\Omega^p),$$

which contains the algebra of holomorphic forms

$$R^\bullet(V, \mathbb{C}) = \bigoplus_p H^0(V, \Omega^p) \subset H^\bullet(V, \mathbb{C})$$

as a subalgebra. Note that these algebras are graded-commutative, and that $\chi(\mathscr{O}_V) = \sum_p (-1)^p h^{p,0}(V)$. We shall use the following well-known properties of the algebra of holomorphic forms:

Lemma 3.3. *Let $f : V' \to V$ be a proper dominant morphism of compact Kähler manifolds. Then the pullback map $R^\bullet(V) \to R^\bullet(V')$ is injective. It is even bijective provided f is bimeromorphic.*

The following observation will allow us to obtain some information on the Bogomolov factors in the Hilbert scheme of points for bielliptic surfaces:

Proposition 3.4. *Let X and Y be compact connected Kähler manifolds of dimension $2n+1$ with $c_1 = 0$. Suppose that there is an finite étale covering $f : \tilde{X} \to X$ so that $\tilde{X}$ is the product of an elliptic curve and a hyperkähler manifold. Assume there is a rational dominant map $r : X \dashrightarrow Y$ and that $\pi_1(Y)$ is finite. Then the universal covering $\tilde{Y}$ is a Calabi–Yau manifold.*

Proof. We first reduce to the case $\pi_1(Y) = 0$. Consider the following commutative diagram:

$$\begin{tikzcd}
\tilde{X} \arrow[d] \arrow[dd, bend right, "f"'] & & \\
\hat{X} \arrow[d] & \tilde{X}' \arrow[l] \arrow[r] \arrow[d] & \tilde{Y} \arrow[d] \\
X \arrow[rr, dashed, bend right, "r"'] & X' \arrow[l] \arrow[r] & Y
\end{tikzcd}$$

Here $\tilde{Y} \to Y$ is the universal covering, X' is a smooth compact Kähler manifold, $X' \to X$ is a bimeromorphic proper morphism, and $X' \to Y$ is a dominant proper morphism. The square to the right is cartesian, such that $\tilde{X}' \to X'$ is finite étale. By bimeromorphic invariance of the fundamental group, the induced map $\pi_1(X') \to \pi_1(X)$ is bijective, hence there is a finite étale covering $\hat{X} \to X$ making the square to the left cartesian. We may also assume that our given finite étale covering $\tilde{X} \to X$ factors over $\hat{X}$, by passing to a larger covering. Replacing Y, X by $\widetilde{Y}, \widetilde{X}$, we may assume that Y is simply connected and that $X = M \times E$ is the product of a hyperkähler manifold M and an elliptic curve E.

The idea now is to use the algebra of holomorphic forms

$$R^\bullet(V) = \bigoplus_{p \geq 0} H^0(V, \Omega^p_V)$$

for various schemes V occurring in our situation. Let $Y = Y_1 \times \ldots \times Y_d$ be the Bogomolov decomposition. There are no abelian factors because $\pi_1(Y) = 0$. Consider the chain of injective pull back maps

$$R^\bullet(X) \longrightarrow R^\bullet(X') \longleftarrow R^\bullet(Y)$$

induced from the preceding diagram. The map on the left is bijective since the proper morphism $X' \to X$ is bimeromorphic, by Lemma 3.3. Therefore, we may regard $R^\bullet(Y) \subset R^\bullet(X)$ as a subalgebra. Moreover, $R^\bullet(X) = R^\bullet(M) \otimes R^\bullet(E)$. Note that the product of elements of odd degree is zero, because $\dim(E) = 1$.

Seeking a contradiction, we now suppose that one Bogomolov factor Y_i is hyperkähler, say of dimension $2n_i \leq 2n-2$. We infer that there is a nonzero element $\sigma \in R^\bullet(Y)$ of degree two with $\sigma^{n_i+1} = 0$. On the other hand, $R^\bullet(M)$ is necessarily generated by σ, and therefore $\sigma^{n_i+1} \neq 0$, contradiction. Whence all Bogomolov factors Y_i are Calabi–Yau manifolds, say of dimension $\dim(Y_i) = n_i$, such that $2n+1 = n_1 + \ldots + n_d$. Let $\sigma_i \in R^\bullet(Y_i)$ be a nonzero element of degree n_i. Note that $\sigma_i^2 = 0$, and $\sigma_i\sigma_j \neq 0$ for $i \neq j$.

Now suppose there are two odd-dimensional Bogomolov factors, say Y_1, Y_2. Then $\sigma_1\sigma_2 \neq 0$ inside $R^\bullet(Y)$. But since the degrees of the pullbacks $r^*(\sigma_i) \in R^\bullet(X)$ are odd, we have $r^*(\sigma_1)r^*(\sigma_2) = 0$, contradiction. Whence there is at most one odd-dimensional Bogomolov factor. Since $\dim(Y)$ is odd, there must be precisely one such factor.

It remains to show that there are no even-dimensional Bogomolov factors. Suppose there is such a factor, say Y_1. Set $m = n_1/2$. Clearly, $m \leq n-1$. In order to work with functions fields, let us now assume that X is a algebraic (In the general case, one has to work with general points rather then function fields). Choose transcendence basis $t_1, \ldots, t_{2m} \in K(Y_1)$ in the function field of Y_1. Using the canonical projection, we may regard $K(Y_1) \subset K(Y)$ as a subfield of the function field of Y. Then the differentials $dt_1, \ldots, dt_{2m} \in \Omega^1_{K(Y)/\mathbb{C}}$ are linearly independent. Choose a transcendence basis $s_1, \ldots, s_{2n} \in K(M)$ and $s_{2n+1} \in K(E)$, such that $ds_1, \ldots, ds_{2n+1} \in \Omega^1_{K(X)/\mathbb{C}}$ form a basis. Since $2m < 2n$, at least one of the original $ds_1, \ldots, ds_{2n}$ does not lie in the image of the injection $\Omega^1_{K(Y_1)/\mathbb{C}} \otimes K(X) \to \Omega^1_{K(X)/\mathbb{C}}$. Say this is ds_{2n}.

Now consider the nonzero element $\sigma_1 \in R^{2m}(Y_1)$. Up to some factor from $\mathbb{C}^\times$, we have $r^*(\sigma_1) = \sigma^m \otimes 1$ inside $R^{2m}(X) = R^{2m}(M) \otimes R^0(E)$, where $\sigma \in H^0(M, \Omega^2_M)$ is a symplectic form on M. Since σ is nondegenerate, the contraction $\langle \sigma^m, \partial/\partial s_{2m} \rangle \in \Omega^{2m-1}_{K(X)/\mathbb{C}}$ with the derivation $\partial/\partial s_{2m}$ is nonzero. On the other hand, we have $\langle r^*(\sigma_1), \partial/\partial s_{2m} \rangle = 0$, contradiction. $\square$

Recall that a *bielliptic* surface is a minimal surface S with $\kappa = 0$ and $b_2 = 2$. Equivalently, S has an étale covering that is an abelian surface, but is not an abelian surface itself. Note that the term *hyperelliptic surface* is also frequently used. We shall discuss such surfaces more thoroughly in Section 6.

Theorem 3.5. *Let S be a bielliptic surface and $Y = \mathrm{Hilb}^n(S)$ for some $n \geq 2$. Then there is an étale covering $\widetilde{Y} \to Y$ so that $\widetilde{Y}$ is the product of an elliptic curve and a Calabi–Yau manifold of dimension $2n-1$.*

Proof. Since $h^{1,0}(S) = 1$, the Albanese variety of S is an elliptic curve E. Let $a : S \to E$ be the Albanese map. Then a is surjective, and $\mathcal{O}_E = a_*(\mathcal{O}_S)$, by the universal property. Consider the composite map

$$f : Y = \mathrm{Hilb}^n(S) \longrightarrow \mathrm{Sym}^n(S) \longrightarrow \mathrm{Sym}^n(E) \longrightarrow E, \tag{2}$$

where the first arrow is the Hilbert–Chow morphism, the second arrow is induced from the Albanese map, and the last arrow is given by addition. Clearly, $f : Y \to E$ is surjective with $\mathcal{O}_E = f_*(\mathcal{O}_Y)$. As discussed in Section 1, we have

$\pi_1(Y) = H_1(S,\mathbb{Z})$, thus $b_1(Y) = 2$ and $h^{1,0}(Y) = 1$. It follows that $f : Y \to E$ is the Albanese map.

Let $Y_0 = f^{-1}(0)$ be the fiber over the origin. The idea now is to apply Proposition 3.4 to Y_0. As discussed in Section 1 Y has $c_1 = 0$, thus the same holds for Y_0. Next, we have to verify that $\pi_1(Y_0)$ is finite. Clearly, the Albanese map is a Serre fibration, such that we have an exact sequence

$$\pi_2(E) \longrightarrow \pi_1(Y_0) \longrightarrow \pi_1(Y) \longrightarrow \pi_1(E) \longrightarrow 0.$$

The term on the left vanishes, and the fundamental groups of Y and E are finitely generated abelian groups of rank two. It follows that $\pi_1(Y_0)$ is finite.

It remains to construct a birational map $X_0 \dashrightarrow Y_0$ as in Proposition 3.4. To do so, choose a finite abelian covering $g : A \to S$ for some abelian surface A and let $X = \mathrm{Hilb}^n(A)$. Then X has a finite covering that is the product of an abelian surface and the hyperkähler manifold $\mathrm{Km}^n(A)$. Consider the composite map $A \to S \to E$. The kernel of this map is a subgroup scheme, whose connected component of the origin $A_0 \subset A$ is an elliptic curve. Consider the composite map

$$X = \mathrm{Hilb}^n(A) \longrightarrow \mathrm{Sym}^n(A) \longrightarrow A.$$

The fiber $X_0 \subset X$ over the elliptic curve $A_0 \subset A$ has $c_1 = 0$, and admits a finite étale covering that is a product of an elliptic curve and $\mathrm{Km}^n(A)$. The finite surjection $\mathrm{Sym}^n(A) \to \mathrm{Sym}^n(S)$ defines a dominant rational map $\mathrm{Hilb}^n(A) \dashrightarrow \mathrm{Hilb}^n(S)$, and it is easy to see that the latter restricts to a dominant rational map $X_0 \to Y_0$. Thus we may apply Proposition 3.4 to finish the proof. □

4. First examples of Enriques manifolds

In this section we shall construct the first examples of higher-dimensional Enriques manifolds. Suppose that S' is an Enriques surface, and let $S \to S'$ be its universal covering. Then S is a K3 surface endowed with a free action of $G = \mathbb{Z}/2\mathbb{Z}$ corresponding to a fixed point free involution $\iota : S \to S$. This induces a G-action on $\mathrm{Hilb}^n(S)$. In light of Proposition 2.4, such an action cannot be free for n even.

Proposition 4.1. *Suppose $n \geq 1$ is odd. Then the induced G-action on $X = \mathrm{Hilb}^n(S)$ is free, such that $Y = X/G$ is an Enriques manifold of dimension* $\dim(Y) = 2n$ *with index $d = 2$.*

Proof. There is no ι-invariant zero-cycle $\sum_{i=1}^n x_i$ of odd length on S, because the involution $\iota : S \to S$ is fixed point free. Thus the induced G-action on the symmetric product $\mathrm{Sym}^n(S)$ is free. Since the Hilbert-Chow morphism $\mathrm{Hilb}^n(S) \to \mathrm{Sym}^n(S)$ is equivariant, the G-action on $\mathrm{Hilb}^n(S)$ must be free as well. □

We now turn to Beauville's generalized Kummer varieties. Fix two elliptic curves E and E', and consider the abelian surface $A = E \times E'$. Choose a point $a' \in E'$ of order two and an arbitrary point $a \in E$, and consider the involution

$$\iota : A \longrightarrow A, \quad (b, b') \longmapsto (-b + a, b' + a').$$

Such maps were studied in connections with cohomologically trivial automorphisms and are attributed to Lieberman, compare [24]. The induced action of $G = \mathbb{Z}/2\mathbb{Z}$ is free, because it is free on the second factor. Now consider the induced action on the Hilbert scheme $\mathrm{Hilb}^{n+1}(A)$ and the symmetric product $\mathrm{Sym}^{n+1}(A)$. Note that the addition map $s : \mathrm{Sym}^{n+1}(A) \to A$ is not equivariant.

Proposition 4.2. *Suppose $n \geq 1$ is odd and that $a \in E$ satisfies $(n+1)a = 0$ and $(n+1)/2 \cdot a \neq 0$. Then the subset $\mathrm{Km}^n(A) \subset \mathrm{Hilb}^{n+1}(A)$ is G-invariant, and the induced G-action on $X = \mathrm{Km}^n(A)$ is free. Whence $Y = X/G$ is an Enriques manifold of dimension $\dim(Y) = 2n$ and index $d = 2$.*

Proof. Let $Z \subset \mathrm{Sym}^n(A)$ be the subscheme of zero-cycles of length $n+1$ on A whose sum in A is the origin $0 \in A$. Then we have a commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & Z & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hilb}^{n+1}(A) & \longrightarrow & \mathrm{Sym}^{n+1}(A) & \longrightarrow & A, \end{array}$$

whose squares are cartesian. The Hilbert–Chow morphism is equivariant, thus it suffices to check that the subset $Z \subset \mathrm{Sym}^{n+1}(A)$ is G-invariant and disjoint from the fixed locus.

Let $\sum_{i=1}^{n+1} x_i$ be a zero-cycle of length $n+1$ on A, and write $x_i = (b_i, b_i')$ with respect to the decomposition $A = E \times E'$. Suppose that $\sum_{i=1}^{n+1} x_i = 0$, where now summation is the actual sum in A. Then $\sum_i b_i = 0$ in E and $\sum_i b_i' = 0$ in E'. Applying the involution ι yields a zero-cycle on A summing up to

$$\sum_i \iota(x_i) = \sum_i (-b_i + a, b_i' + a') = \sum_i (-b_i, b_i') + (n+1)(a, a') = (0,0).$$

It follows that $Z \subset \mathrm{Sym}^{n+1}(A)$ is G-invariant.

Now let $p \in \mathrm{Sym}^{n+1}(A)$ be a G-fixed zero-cycle. We check that $p \notin Z$: Since G acts freely on A, the zero-cycle has the form $p = \sum_{i=1}^m (x_i + \iota(x_i))$ for some closed points $x_1, \ldots, x_m \in A$ where $m = (n+1)/2$. As above, write $x_i = (b_i, b_i')$ with respect to the decomposition $A = E \times E'$. Computing the sum $\sum_{i=1}^m (x_i + \iota(x_i))$ in A and projecting onto E, we obtain $\sum_{i=1}^m (b_i - b_i + a) = ma \neq 0$. It follows $p \notin Z$. $\square$

We now introduce the following slightly vague but useful shorthand notation $Y = Q_d X$ in order to refer to a construction of Enriques manifolds Y of index $d \geq 2$ as a quotient of a class of hyperkähler manifolds X by some free action of $G = \mathbb{Z}/d\mathbb{Z}$. For example, we write $Q_2 \mathrm{Hilb}^n(S)$ for the quotients of Hilbert schemes for K3 surfaces, and $Q_2 \mathrm{Km}^n(E \times E')$ to denote the quotients of Beauville's generalized Kummer varieties attached to the product of elliptic curve. We shall generalize these constructions in the next two sections.

5. Stable sheaves on K3 surfaces

We now generalize the construction $Q_2 \mathrm{Hilb}^n(S)$ from the preceding section using moduli spaces of stable sheaves. Throughout this section, S' is an Enriques surface, with universal covering $S \to S'$, such that S is a $K3$-surface endowed with a free action of $G = \pi_1(S')$, corresponding to a free involution $\iota : S \to S$.

Recall that if $\mathcal{F}$ is a coherent sheaf on S of rank $r = \mathrm{rank}(\mathcal{F})$, first Chern class $l = c_1(\mathcal{F})$, and Euler characteristic $\chi = \chi(\mathcal{F})$, then its Mukai vector is

$$v(\mathcal{F}) = \mathrm{ch}(\mathcal{F})\sqrt{\mathrm{Todd}(S)} = (r, l, \chi - r) \in H^{\mathrm{ev}}(S, \mathbb{Z}).$$

Let $v \in H^{\mathrm{ev}}(S, \mathbb{Z})$ be a Mukai vector with $v^2 \geq 4$, and $H \in \mathrm{NS}(S)_{\mathbb{R}}$ be a polarization. If Mukai vector and polarization are G-invariant, then the G-action on S induces a G-action on the moduli space $M_H(v)$ of H-stable sheaves on S with Mukai vector $v(\mathcal{F}) = v$, which is a smooth scheme of dimension $v^2 + 2$ with a symplectic structure but not necessarily proper. On the other hand, if v is primitive and H is very general, then $M_H(v)$ is proper, and indeed a hyperkähler manifold. We have to ensure that the preceding conditions hold simultaneously.

Proposition 5.1. *The following three conditions are equivalent:*

(i) *The K3 surface S has Picard number $\rho(S) = 10$.*
(ii) *The canonical map $\mathrm{Pic}(S') \to \mathrm{Pic}(S)$ is surjective.*
(iii) *The G-action on $\mathrm{Pic}(S)$ is trivial.*

Proof. The spectral sequence $H^p(G, H^q(S, \mathcal{O}_S^\times)) \Rightarrow H^{p+q}(S', \mathcal{O}_{S'}^\times)$ yields an exact sequence

$$0 \longrightarrow H^1(G, \mathbb{C}^\times) \longrightarrow \mathrm{Pic}(S') \longrightarrow \mathrm{Pic}(S)^G \longrightarrow H^2(G, \mathbb{C}^\times).$$

The term on the right vanishes because the group $\mathbb{C}^\times$ is divisible and G acts trivially. Furthermore, Enriques surfaces have Picard number $\rho = 10$, and the statement easily follows. $\square$

Proposition 5.2. *Suppose S' is an Enriques surface whose corresponding K3 surface S has Picard number $\rho(S) = 10$. Let $v = (r, l, \chi - r) \in H^{\mathrm{ev}}(S, \mathbb{Z})$ be a Mukai vector and $H \in \mathrm{NS}(S)_{\mathbb{R}}$ be a polarization. Then v, H are G-invariant, such that the G-action on S induces a G-action on $M_H(v)$. If $\chi \in \mathbb{Z}$ is odd and $M_H(v) \neq \emptyset$, then this G-action on $M_H(v)$ is free.*

Proof. According to Proposition 5.1, the G-action on $\mathrm{Pic}(S)$ is trivial. Consequently the classes $l, H \in \mathrm{NS}(S)_{\mathbb{R}}$ are invariant, hence also the Mukai vector $v \in H^{\mathrm{ev}}(S, \mathbb{Z})$. It follows that $\mathcal{F} \mapsto \iota^*(\mathcal{F})$ defines a G-action on the moduli space $M_H(v)$ of H-stable sheaves $\mathcal{F}$ with Mukai vector $v(\mathcal{F}) = v$.

Now suppose $\chi \in \mathbb{Z}$ is odd. Seeking a contradiction, we assume that the G-action has a fixed point $x \in M_H(v)$. The corresponding coherent sheaf $\mathcal{F}$ is then isomorphic to $\iota_*(\mathcal{F})$. Choose an isomorphism $h : \mathcal{F} \to \iota_*(\mathcal{F})$. Using $\iota_*(\iota_*(\mathcal{F})) = \mathcal{F}$, we may regard $\iota_*(h) \circ h$ as an endomorphism of $\mathcal{F}$. Being stable, the sheaf $\mathcal{F}$ is simple, whence $\iota_*(h) \circ h = \lambda\,\mathrm{id}$ for some scalar $\lambda \in \mathbb{C}^\times$. Multiplying h with a square root of $1/\lambda$, we may assume $\lambda = 1$, and this means that $h : \mathcal{F} \to \iota_*(\mathcal{F})$ defines a G-linearization.

Let $p : S \to S'$ be the canonical projection. By descend we have $\mathcal{F} = p^*(\mathcal{F}')$ for some coherent sheaf $\mathcal{F}'$ on S'. Consequently $\chi = \chi(\mathcal{F}) = |G|\chi(\mathcal{F}')$ is even, contradiction. $\square$

Theorem 5.3. *Suppose S' is an Enriques surface whose corresponding K3 surface has Picard number $\rho(S) = 10$. Let $v = (r, l, \chi - r) \in H^{\mathrm{ev}}(S, \mathbb{Z})$ be a primitive Mukai vector with $v^2 \geq 0$ and $\chi \in \mathbb{Z}$ odd. Then for very general polarizations $H \in \mathrm{NS}(S)_{\mathbb{R}}$, the moduli space $X = M_H(v)$ is a hyperkähler manifold endowed with a free G-action, and $Y = X/G$ is an Enriques manifold of dimension $v^2 + 2$ and index $d = 2$.*

Proof. The moduli space $M_H(v)$ is a hyperkähler manifold because H is very general and χ is primitive ([26], together with [30], Proposition 4.12. Compare also [16], Chapter 6.2). The G-action is free by Proposition 5.2. $\square$

Remark 5.4. Under the assumption of the preceding theorem, we have

$$2n = \dim M_H(v) = v^2 + 2 = l^2 - 2r(\chi - r) + 2$$

with n odd,in accordance with Proposition 2.4. Indeed, $l^2 \equiv 0$ modulo 4, because any invertible sheaf on S comes from S', and $S \to S'$ has degree two, and the intersection form on $\mathrm{Pic}(S')$ is even. Moreover $-2r(\chi - r) \equiv 0$ modulo 4 regardless of r, because χ is odd.

We now check that the conditions of Proposition 5.1 indeed hold for very general Enriques surfaces. For this we need the Global Torelli Theorem for Enriques surfaces, which is due to Horikawa ([14] and [14]; see also [2]) Let $L = E_8(-1)^{\oplus 2} \oplus H^{\oplus 3}$ be the K3 lattice, endowed with the involution

$$\iota : L \longrightarrow L, \quad (x, y, z_1, z_2, z_3) \longmapsto (y, x, -z_1, z_3, z_2).$$

Then each Enriques surface S' admits a marking, that is, an equivariant isometry $L \to H^2(S, \mathbb{Z})$. Obviously, the antiinvariant sublattice $L' \subset L$ is isomorphic to $E_8(-2) \oplus H \oplus H(2)$. Inside the period domain for K3 surfaces

$$D = \{[\sigma] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (\sigma \cdot \sigma) = 0, (\sigma \cdot \bar{\sigma}) > 0\},$$

the period domain for Enriques surfaces is defined as the intersection

$$D' = D \cap \mathbb{P}(L' \otimes \mathbb{C}).$$

The discrete group $\Gamma' = \{g' \in \mathrm{Aut}(L') \mid \exists g \in \mathrm{Aut}(L) \text{ with } g\iota = \iota g \text{ and } g' = g|L' \}$ acts on D' properly discontinuously, such that the quotient D'/Γ' is a normal complex space. In fact, it acquires the structure of a quasiprojective scheme. To obtain the coarse moduli space of Enriques surfaces, one has to remove certain divisors. For each $d \in L'$, consider the divisor $H'_d = \{[\sigma] \in D' \mid (\sigma \cdot d) = 0\} \subset D'$, and let $H' = \bigcup H_d$ be the union over all $d \in L'$ with $(d \cdot d) = -2$. It is known that $H'/\Gamma' \subset D'/\Gamma'$ is an irreducible divisor [1]. The Global Torelli Theorem asserts that

$$D'/\Gamma' \smallsetminus H'/\Gamma' = (D' \smallsetminus H')/\Gamma'$$

is a coarse moduli space for Enriques surfaces, such that its closed points bijectively correspond to isomorphism classes of Enriques surfaces. We note in passing that the coarse moduli space is quasiaffine [9].

Proposition 5.5. *The set of points in the coarse moduli space $(D' \smallsetminus H')/\Gamma'$ corresponding to isomorphism classes of Enriques surfaces S' whose universal covering S has Picard number $\rho(S) = 10$ is the complement of the union of countable many prime divisors.*

Proof. This condition on marked Enriques surface S' means that no $d \in L'$ is contained in $\mathrm{NS}(S) \subset H^2(S, \mathbb{Z})$. In other words, each $d \in L'$ must be orthogonal to $\sigma \in H^{2,0}(S) \subset H^2(S, \mathbb{C})$. Consequently, the period $[\sigma] \in D'$ lies in the complement of the union $\bigcup_{d \in L'} H'_d$. The latter is union is locally finite, because no point in D' is contained in an intersection $H'_{d_1} \cap \ldots \cap H'_{d_s}$ where $d_1, \ldots, d_s \in L'$ generate a subgroup of finite index. Therefore, the union $\bigcup_{d \in L'} H'_d \subset D$ is a divisor, which is clearly Γ'-invariant. Hence the subset $(\bigcup_{d \in L'} H'_d)/\Gamma' \subset D'/\Gamma'$ is locally a Weil divisor, whence itself the countable union of prime divisors. $\square$

6. Bielliptic surfaces

We now generalize the construction method $Q_2 \,\mathrm{Km}^n(A)$ using the theory of bielliptic surfaces. Recall that a minimal surfaces S of Kodaira dimension $\kappa = 0$ and second Betti number $b_2 = 2$ is called *bielliptic*, or *hyperelliptic*. An equivalent condition is that S is not isomorphic to an abelian surface but it admits a finite étale covering by an abelian surface. It turns out that the canonical class $\omega_S \in \mathrm{Pic}(S)$ has finite order $d \in \{2,3,4,6\}$, and that the corresponding finite étale covering $A \to S$ is indeed an abelian surface. Note that this is an abelian Galois covering whose Galois group G is cyclic of order d, and that

$$\mathrm{pr}_*(\mathcal{O}_A) = \mathcal{O}_S \oplus \omega_S \oplus \ldots \oplus \omega_S^{\otimes d-1}.$$

We call $A \to S$ the *canonical covering* of S. It turns out that A is isogeneous to a product of elliptic curves. More precisely, there is a finite étale Galois covering $\tilde{A} \to S$ factoring over A, where $\tilde{A} = E \times F$ is a product of elliptic curves with Galois group of the form $G \times \tilde{T}$, where $\tilde{T} = \ker(\tilde{A} \to A)$ is a finite subgroup and the Galois action of G and $\tilde{T}$ on $\tilde{A} = E \times F$ split into direct product actions.

There are only seven possibilities, and the whole situation was classified by Bagnera and de Franchis. We now recall this result in a form we shall use: Consider the complex roots of unity

$$i = e^{2\pi i/4}, \quad \omega = e^{2\pi i/3}, \quad \zeta = e^{2\pi i/6},$$

and write $E = \mathbb{C}/(\mathbb{Z} + \tau_1\mathbb{Z})$ and $F = \mathbb{C}/(\mathbb{Z} + \tau_2\mathbb{Z})$, where $\tau_1, \tau_2 \in \mathbb{H}$ are periods. Also, let $z \in F$ be an arbitrary element. The classification is as follows (compare [8] and [5]):

τ_2	d	$G \times \tilde{T}$	action of generators on $\tilde{A} = E \times F$	
arbitrary	2	$\mathbb{Z}/2\mathbb{Z}$	$(e,f) \mapsto (e+1/d, -f+z)$	
ζ	3	$\mathbb{Z}/3\mathbb{Z}$	$(e,f) \mapsto (e+1/d, \omega f+z)$	
i	4	$\mathbb{Z}/4\mathbb{Z}$	$(e,f) \mapsto (e+1/d, if+z)$	
ζ	6	$\mathbb{Z}/6\mathbb{Z}$	$(e,f) \mapsto (e+1/d, \zeta f+z)$	
arbitrary	2	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$(e,f) \mapsto (e+1/d, -f+z)$,	$(e,f) \mapsto (e+\tau_1/2, f+1/2)$
ζ	3	$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	$(e,f) \mapsto (e+1/d, \omega f+z)$,	$(e,f) \mapsto (e+\tau_1/3, f+(1+\zeta)/3)$
i	4	$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$(e,f) \mapsto (e+1/d, if+z)$,	$(e,f) \mapsto (e+\tau_1/2, f+(1+i)/2)$

So far, the element $z \in F$ is irrelevant, because each action is conjugate via a suitable translation to the "standard action" with $z = 0$. Now fix an integer $n \geq 1$, and let S be a hyperelliptic surface. The G-action on the canonical covering A induces an action on $\mathrm{Hilb}^{n+1}(A)$. Note that the addition map $\mathrm{Hilb}^{n+1}(A) \to A$ is not equivariant. Nevertheless, we seek conditions under which the zero fiber is invariant, and here our $z \in F$ comes into play:

Proposition 6.1. *Suppose that $d \mid n+1$ and that $z \in F[n+1]$. Then the subset $\mathrm{Km}^n(A) \subset \mathrm{Hilb}^{n+1}(A)$ is G-invariant.*

Proof. Let $\mathrm{Sym}^n_0(A) \subset \mathrm{Sym}^{n+1}(A)$ be the subscheme of zero-cycles summing up to the origin $0 \in A$. Then we have a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Km}^n(A) & \longrightarrow & \mathrm{Sym}^{n+1}_0(A) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Hilb}^{n+1}(A) & \longrightarrow & \mathrm{Sym}^{n+1}(A) & \longrightarrow & A,
\end{array}$$

whose squares are cartesian. The Hilbert–Chow morphism is equivariant, thus it suffices to check that the subset $\mathrm{Sym}^{n+1}_0(A) \subset \mathrm{Sym}^{n+1}(A)$ is G-invariant.

Let $\sum_{i=1}^{n+1} x_i$ be a zero-cycle of length $n+1$ on A summing up to zero. Choose lifts $\tilde{x}_i \in \tilde{A}$ and write $\tilde{x}_i = (e_i, f_i)$ with respect to the decomposition $\tilde{A} = E \times F$. Let $g \in G$ be the canonical generator and write $g \cdot (e, f) = (e + 1/d, \xi f + z)$ with $\xi \in \{-1, \omega, i, \zeta\}$, as in the Table. Application of the this automorphism yields a zero-cycle on $\tilde{A}$ summing up to

$$\sum_{i=1}^{n+1} g\tilde{x}_i = \sum_i (e_i + 1/d, \xi f_i + z) = (1 \times \xi)(\sum_i \tilde{x}_i) + (n+1)(1/d, z).$$

The second summand vanishes by our assumptions, and $\sum_i \tilde{x}_i$ lies in $\tilde{T}$. It remains to check that $\tilde{T} \subset \tilde{A}$ is invariant under the automorphism given by $(1 \times \xi)$, which is an easy direct computation. □

Next, we study fixed points on Hilbert schemes and symmetric products:

Proposition 6.2. *If there is a G-fixed point $p \in \operatorname{Sym}^{n+1}(A)$, then $d \mid n+1$.*

Proof. By induction on $n \geq 1$. Write $p = \sum_{i=1}^{n+1} x_i$. Since G acts freely on A, the G-orbit $G \cdot x_1 \subset A$ consists of d pairwise different points, and is contained in the support of p, such that $p - Gx_1$ is a G-fixed zero cycle on A of length $n+1-d$. The latter is divisible by d by induction, whence the same holds for $n+1$. □

We now make an auxiliary computation: Let S be a bielliptic surface whose canonical class $\omega_S \in \operatorname{Pic}(S)$ has order d, and consider the action of $G = \mathbb{Z}/d\mathbb{Z}$ on $\tilde{A} = E \times F$. Let $g \in G$ be the canonical generator, and write the action as $g(e, f) = (e + 1/d, \xi f + z)$ as in the table, with $\xi = -1, \omega, i, \zeta$ for $d = 2, 3, 4, 6$, respectively. Suppose that there is a G-fixed point $p \in \operatorname{Sym}^{n+1}(A)$. As in the proof for the preceding Proposition, we have $p = \sum_{i=1}^m \sum_{j=0}^{d-1} g^j(x_i)$, where $m = (n+1)/d$ and $x_1, \ldots, x_m \in A$ are suitable closed points. Choose lifts $\tilde{x}_i \in \tilde{A}$.

Lemma 6.3. *Assumptions as in the preceding paragraph. Then the F-component of the sum $\sum_{i=1}^m \sum_{j=0}^{d-1} g^j(\tilde{x}_i) \in \tilde{A} = E \times F$ equals $\sum_{k=1}^{d-1} (d-k)\xi^{k-1} mz \in F$.*

Proof. Write $\tilde{x}_i = (e_i, f_i)$ with respect to the decomposition $\tilde{A} = E \times F$. Computing the sum $\sum_{i=1}^m \sum_{j=0}^{d-1} g^j(x_i)$ in $\tilde{A}$ and projecting onto F, we obtain

$$\begin{aligned} &\textstyle\sum_{i=1}^m \sum_{j=0}^{d-1} (\xi^j f_i + (\xi^{j-1} + \xi^{j-2} + \ldots + \xi^0) z) \\ = \quad &\textstyle\sum_{i=1}^m (1 + \xi + \ldots + \xi^{d-1}) f_i + \sum_{k=1}^{d-1} (d-k)\xi^{k-1} mz. \end{aligned}$$

Obviously, the d-th root of unity $\xi \neq 1$ is a root of the polynomial $1 + T + \ldots + T^{d-1}$, and the result follows. □

We come to the main result of this section. Recall that $\widetilde{T} \subset \widetilde{A}$ is the kernel of $\widetilde{A} \to A$. Its image under the projection $\widetilde{A} = E \times F \to F$ is called $T \subset F$.

Theorem 6.4. *Suppose S is a hyperelliptic surface whose canonical class has order d, and $A \to S$ be its canonical covering. Let $n \geq 1$ be an integer with $d \mid n+1$, and $z \in F[n+1]$. Write $m = (n+1)/d$ and assume:*

(i) *If $d = 2$ then $mz \notin T$.*
(ii) *If $d = 3$ then $T = 0$ and $mz \notin \mathbb{Z}(1+\zeta)/3$.*
(iii) *If $d = 4$ then $T = 0$ and $2mz \notin \mathbb{Z}(1+i)/2$.*

Then the subset $\mathrm{Km}^n(A) \subset \mathrm{Hilb}^{n+1}(A)$ *is* G*-invariant, and the induced* G*-action on* $X = \mathrm{Km}^n(A)$ *is free, such that* $Y = X/G$ *is an Enriques manifold of dimension* $\dim(Y) = 2n$ *and index* d*.*

Proof. We already saw in Proposition 6.1 that $\mathrm{Km}^n(A) \subset \mathrm{Hilb}^{n+1}(A)$ is invariant. Seeking a contradiction, we suppose that the induced action on $X = \mathrm{Km}^n(A)$ is not free.

Let us first consider the cases $d = 2$ and $d = 3$. Then there is a fixed point on X, and its image under the Hilbert–Chow morphism is a fixed point $p \in \mathrm{Sym}_0^{n+1}(A)$. Since G acts freely on A, we may write $p = \sum_{i=1}^m \sum_{j=0}^{d-1} g^j(x_i)$, where $x_1, \ldots, x_m \in A$ are closed points, and $g \in G$ is the canonical generator. Choose lifts $\tilde{x}_i \in \tilde{A}$, and write $\tilde{x_i} = (e_i, f_i)$ with respect to the decomposition $\tilde{A} = E \times F$. We now use Lemma 6.3 to compute the F-component of the sum $\sum_{i=1}^m \sum_{j=0}^{d-1} g^j(\tilde{x}_i) \in \tilde{A}$:

In case $d = 2$, the F-component is given by $mz \in F$, which is not contained in $T \subset F$ by assumption. Whence $\sum_{i=1}^m \sum_{j=0}^{d-1} g^j(\tilde{x}_i)$ is not contained in the kernel $\tilde{T}$ for the homomorphism $\tilde{A} \to A$. On the other hand, we have $\sum_{i=1}^m \sum_{j=0}^{d-1} g^j(x_i) = 0$ in A, contradiction.

Now suppose $d = 3$. Then the F-component is given by $(2 + \omega)mz$. One easily computes that $(1+\zeta)/3$ generates the kernel of $(2+\omega)$ viewed as an endomorphism of $F[3]$. So our assumption ensures that $(2+\omega)mz \neq 0$, and we obtain a contradiction as above.

It remains to treat the cases $d = 4$. Choose a point $p \in \mathrm{Sym}_0^{n+1}(A)$ whose stabilizer is nonzero. In case $d = 4$, this point is fixed by the unique subgroup $G' \subset G$ of order two. Applying the preceding paragraph to the hyperelliptic surface $S' = A/G'$, we obtain a contradiction. □

In all cases, the element $mz \in F[d]$ or suitable multiples have to avoid a 1-dimensional vector subspace in a 2-dimensional vector space over certain finite fields. This can always be done, so the cases are indeed nonvacuous. Thus:

Theorem 6.5. *There are Enriques manifolds of index* $d = 2, 3, 4$*.*

Remark 6.6. As pointed out by Sarti and Boissiére, the case $d = 6$ seems impossible here.

References

[1] D. Allcock: The period lattice for Enriques surfaces. Math. Ann. 317 (2000), 483–488.

[2] W. Barth, C. Peters, A. Van de Ven: Compact complex surfaces. Ergeb. Math. Grenzgebiete (3) 4, Springer, Berlin, 1984.

[3] A. Beauville: Variétés Kähleriennes dont la première classe de Chern est nulle. J. Differential Geom. 18 (1983) 755–782.

[4] A. Beauville, R Donagi: La variété des droites d'une hypersurface cubique de dimension 4. C. R. Acad. Sci. Paris Sér. I Math. 301 (1985), 703–706.

[5] C. Bennett, R. Miranda: The automorphism groups of the hyperelliptic surfaces. Rocky Mountain J. Math. 20 (1990), 31–37.

[6] F. Bogomolov: Kähler manifolds with trivial canonical class. Izv. Akad. Nauk SSSR Ser. Mat. 38 (1974), 11–21.

[7] F. Bogomolov: The decomposition of Kähler manifolds with a trivial canonical class. Mat. Sb. (N.S.) 93 (135) (1974), 573–575, 630.

[8] E. Bombieri, D. Mumford: Enriques' classification of surfaces in char. p. II. In: W. Baily, T. Shioda (eds.), Complex analysis and algebraic geometry, pp. 23–42. Cambridge University Press, London, 1977.

[9] R. Borcherds: The moduli space of Enriques surfaces and the fake Monster Lie superalgebra. Topology 35 (1996), 699–710.

[10] M. Brion, S. Kumar: Frobenius splitting methods in geometry and representation theory. Progress in Mathematics 231. Birkhäuser, Boston, 2005.

[11] O. Debarre, C. Voisin: Hyper-Kähler fourfolds and Grassmann geometry. Preprint (2009), arXiv:math.AG/0904.3974.

[12] P. Griffiths, J. Harris: Principles of algebraic geometry. Wiley-Interscience, New York, 1978.

[13] A. Grothendieck et al.: Revêtements étales et groupe fondamental. Lect. Notes Math. 224, Springer, Berlin, 1971.

[14] E. Horikawa: On the periods of Enriques surfaces. I. Math. Ann. 234 (1978), 73–88.

[15] E. Horikawa: On the periods of Enriques surfaces. II. Math. Ann. 235 (1978), 217–246.

[16] D. Huybrechts, M. Lehn: The geometry of moduli spaces of sheaves. Aspects of Mathematics E31. Vieweg, Braunschweig, 1997.

[17] D. Huybrechts: Compact hyper-Kähler manifolds: basic results. Invent. Math. 135 (1999), 63–113.

[18] A. Iliev, K. Ranestad: $K3$ surfaces of genus 8 and varieties of sums of powers of cubic fourfolds. Trans. Amer. Math. Soc. 353 (2001), 1455–1468.

[19] B. Iversen: Linear determinants with applications to the Picard scheme of a family of algebraic curves. Lecture Notes in Mathematics 174. Springer, Berlin, 1970.

[20] J. Kollár: Shafarevich maps and plurigenera of algebraic varieties. Invent. Math. 113 (1993), 177–215.

[21] H Lange: Hyperelliptic varieties. Tohoku Math. J. 53 (2001), 491–510.

[22] S. Mukai: Symplectic structure of the moduli space of sheaves on an abelian or K3 surface. Invent. Math. 77 (1984), 101-116.

[23] S. Mukai: On the moduli space of bundles on K3 surfaces. I. In: M. Atiyah et al. (eds.), Vector bundles on algebraic varieties. Tata Inst. Fundam. Res. 11 (1987), 341–413.

[24] S. Mukai, Y. Namikawa: Automorphisms of Enriques surfaces which act trivially on the cohomology groups. Invent. Math. 77 (1984), 383–397.

[25] V. Nikulin: Finite automorphism groups of Kähler K3 surfaces. Trans. Mosc. Math. Soc. 2 (1980), 71–135.

[26] K. O'Grady: The weight-two Hodge structure of moduli spaces of sheaves on a $K3$ surface. J. Algebraic Geom. 6 (1997), 599–644.

[27] K. O'Grady: Desingularized moduli spaces of sheaves on a $K3$. J. Reine Angew. Math. 512 (1999), 49–117.

[28] K. O'Grady: A new six-dimensional irreducible symplectic variety. J. Algebraic Geom. 12 (2003), 435–505.

[29] K. O'Grady: Irreducible symplectic 4-folds and Eisenbud-Popescu-Walter sextics. Duke Math. J. 134 (2006), 99–137.

[30] K. Yoshioka: Moduli spaces of stable sheaves on abelian surfaces. Math. Ann. 321 (2001), 817–884.

Department of Mathematics, Osaka University, Toyonaka 560-0043 Osaka, Japan
E-mail address: `oguiso@math.sci.osaka-u.ac.jp`

Mathematisches Institut, Heinrich-Heine-Universität, 40225 Düsseldorf, Germany
E-mail address: `schroeer@math.uni-duesseldorf.de`