

THE STRONG FRANCHETTA CONJECTURE IN ARBITRARY CHARACTERISTICS

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ABSTRACT. Using Moriwaki's calculation of the \mathbb{Q} -Picard group for the moduli space of curves, I prove the strong Franchetta Conjecture in all characteristics. That is, the canonical class generates the group of rational points on the Picard scheme for the generic curve of genus $g \geq 3$. Similar results hold for generic pointed curves. Moreover, I show that Hilbert's Irreducibility Theorem implies that there are many other nonclosed points in the moduli space of curves with such properties.

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INTRODUCTION

Let M_g be the coarse moduli space of smooth curves of genus $g \geq 3$ over an arbitrary ground field k . Deligne and Mumford [8] showed that M_g is an irreducible algebraic scheme. Let $\eta \in M_g$ its the generic point and $C = M_{g,1} \rightarrow M_g$ the tautological curve. The generic fiber C_η is a smooth curve of genus g over the function field $\kappa(\eta)$ of the moduli space M_g . We call it the *generic curve*.

Franchetta [10] conjectured $\text{Pic}(C_\eta) = \mathbb{Z}K_{C_\eta}$. Arbarello and Cornalba [2] proved it over the complex numbers using Harer's calculation [26] of the second homology for the mapping class group of Riemann surfaces. The latter is a purely topological result. Later, Arbarello and Cornalba [3] gave an algebro-geometric proof over the

complex numbers. Mestrano [37] and Kouvidakis [33] deduced the *strong Franchetta Conjecture* over \mathbb{C} , which states that the rational points in the Picard scheme $\text{Pic}_{C_\eta/\eta}$ are precisely the multiples of the canonical class.

The first goal of this paper is to give an algebraic proof for the strong Franchetta Conjecture in all characteristics $p \geq 0$. The idea is to construct special stable curves showing that any divisor class violating Franchetta's Conjecture must be nontorsion. Having this, we use Moriwaki's calculation [39] of $\text{Pic}(\overline{M}_{g,n+1}) \otimes \mathbb{Q}$ in characteristic $p > 0$ to infer the strong Franchetta Conjecture. I also prove the Franchetta Conjecture for *generic pointed curves*: Their Picard groups are freely generated by the canonical class and the marked points. Actually, I use the pointed case as an essential step in the proof for the unpointed case.

The second goal of this paper is to show that there are many other nonclosed points $x \in M_{g,n}$ such that the marked points and the canonical class generates $\text{Pic}(C_x)$, at least up to torsion. This seems to be new even in characteristic zero. We shall see that over uncountable ground fields, there is an uncountable dense set of such points with $\dim \overline{\{x\}} \leq 2$. This relies on *Hilbert's Irreducibility Theorem* for function fields. The idea is to view C_η as the generic fiber of some fibered surface Y , extend this to a family of fibered surfaces $\mathfrak{Y} \rightarrow S$, and apply Hilbert's Irreducibility Theorem and the *Tate–Shioda Formula* to the resulting family of Néron–Severi scheme $s \mapsto \text{NS}(\mathfrak{Y}_s)$. Such specialization arguments are problematic in characteristic $p > 0$, because ungeometric properties like regularity behave badly in families. However, we overcome these difficulties by using the theory of *geometric unibranch singularities*.

Here is a plan for the paper. The first section contains some general facts on curves, Picard schemes, and moduli spaces. In Section 2 we examine curves of compact type and ordinary abelian varieties. In Section 3 we prove that the Mordell–Weil group of the generic Jacobian is torsion free. This result is further improved in Section 4. Section 5 contains the proof for the strong Franchetta Conjecture. As an application I deduce in Section 6 that the generic curve in characteristic $p = 2$ does not admit a tamely ramified morphism to the projective line. In Section 7 we construct an explicit stable curve X of genus g over the rational function field F so that the $\text{Pic}_{X/F}^0$ -torsor $\text{Pic}_{X/F}^1$ has order $2g - 2$ in the Weil–Châtelet group.

The next two sections contain some general result on schemes: In Section 8, we show that geometrically unibranch schemes are good substitutes for normal schemes in characteristic $p > 0$. In Section 9 we use Hilbert's Irreducibility Theorem, which comes from Galois theory, to study Picard numbers in families of proper schemes. We shall apply these results in Section 10: There we first construct fibered surfaces with small Picard numbers over transcendental extension fields. Specializing them, we show that there are many nonclosed points $x \in M_{g,n}$ such that $\text{Pic}(C_x)$ has the same rank as $\text{Pic}(C_\eta)$. The last section contains a list of open problems.

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1. PRELIMINARIES

Let us collect some well-known facts on algebraic curves and their Picard schemes. Throughout this paper we fix an arbitrary ground field k of characteristic $p \geq 0$. We will, however, also deal with algebraic schemes over extension fields $k \subset F$. A *curve* over F is a proper 1-dimensional F -scheme X with $F = H^0(X, \mathcal{O}_X)$. Each curve comes along with its *Picard scheme* $\text{Pic}_{X/F}$, which is a smooth group scheme of finite type. We denote by $\text{Pic}(X/F) = \text{Pic}_{X/F}(F)$ its group of rational points. If X contains a rational point, this is nothing but the Picard group of X . In general we have an exact sequence

$$0 \longrightarrow \text{Pic}(X) \longrightarrow \text{Pic}(X/F) \longrightarrow \text{Br}(F) \longrightarrow \text{Br}(X),$$

where $\text{Br}(F)$ is the Brauer group ([22], Corollary 5.3). Tsen's Theorem implies the following (compare [22], Theorem 1.1):

Lemma 1.1. *Suppose that k is algebraically closed and that F is the function field of a smooth algebraic curve over k . Then $\text{Pic}(X) = \text{Pic}(X/F)$.*

Note that we do not lose rational points by passing to field extensions:

Lemma 1.2. *Let $F \subset F'$ be a field extension, and $X' = X \otimes_F F'$ the induced curve. Then the canonical map $\text{Pic}(X/F) \rightarrow \text{Pic}(X'/F')$ is injective.*

Proof. We have $\text{Pic}_{X'/F'} = \text{Pic}_{X/F} \otimes F'$, hence the fiber over the rational point $0 \in \text{Pic}_{X/F}$ is nothing but the rational point $0 \in \text{Pic}_{X'/F'}$. \square

Let $\text{Pic}_{X/F}^0 \subset \text{Pic}_{X/F}$ be the connected component of the origin. Equivalently, $\text{Pic}_{X/F}^0$ is the subgroup scheme given by numerically trivial line bundles. We sometimes write $J = \text{Pic}_{X/F}^0$ and call it the *Jacobian* of X . If X is geometrically irreducible, the invertible sheaves of degree d comprise the other connected component $\text{Pic}_{X/F}^d \subset \text{Pic}_{X/F}$. These are torsors under the group scheme $\text{Pic}_{X/F}^0$, in other words, elements of the *Weil-Châtelet* group $H^1(k, \text{Pic}_{X/F}^0)$. Here cohomology is with respect to the fppf topology. However, étale cohomology gives the same result because $\text{Pic}_{X/F}^0$ is smooth ([22], Theorem 11.7). The torsor $\text{Pic}_{X/F}^d$ is trivial if and only if it contains a rational point. If $\text{Pic}_{X/F}^0$ is an abelian variety, $\text{Pic}^0(X/F)$ is also called the *Mordell-Weil group*.

Given an integer $g \geq 3$, let M_g be the *coarse moduli space* of smooth curves of genus g over k . Mumford [41] showed that this is an algebraic k -scheme. Moreover, Deligne and Mumford [8] proved that M_g is irreducible. If k is algebraically closed, its closed points correspond to isomorphism classes of smooth curves over k . In any case, the coarse moduli space $C = M_{g,1}$ of pointed smooth curves defines a tautological family of curves $C \rightarrow M_g$.

We call C_η the *generic curve of genus g* , where $\eta \in M_g$ is the generic point. This is justified as follows: Let \bar{F} be an algebraic closure of the function field $F = \kappa(\eta)$, and \bar{X} a smooth curve of genus g over \bar{F} corresponding to the geometric point $\text{Spec}(\bar{K}) \rightarrow M_g$. Then $C_\eta \otimes_F \bar{F} \simeq \bar{X} / \text{Aut}(\bar{X})$. However, the generic curve of genus $g \geq 3$ has trivial automorphism group (see [43] for an algebraic proof). So the generic curve C_η is indeed a smooth curve of genus g over the function field $F = k(\eta)$. In contrast, the generic curve in genus two is the projective line.

As often, it will be important to consider *stable pointed curves* of genus $g \geq 3$ as well. Over an algebraically closed field, stability means that X has only ordinary

double points, and each smooth rational component contains either two double points, or two marked points, or a double point and a marked point. Such curves form a coarse moduli space $\overline{M}_{g,n}$, which is irreducible and projective. Here g is the (arithmetic) genus, and n is the number of marked points $x_i \in X$. We also have a tautological family $\pi : \overline{M}_{g,n+1} \rightarrow \overline{M}_{g,n}$ sending an $(n+1)$ -pointed stable curve (X, x_1, \dots, x_{n+1}) to the n -pointed stable curve (X', x'_1, \dots, x'_n) obtained by forgetting the last marked point and contracting the possibly occurring rational component in (X, x_1, \dots, x_n) violating stability.

Let $\eta_n \in \overline{M}_{g,n}$ be the generic point and $C = \overline{M}_{g,n+1}$ the tautological family of curves. The generic fiber C_{η_n} is a smooth curve of genus g , which is endowed with marked rational points $c_1, \dots, c_n \in C_{\eta_n}$. We call C_{η_n} the *generic n -pointed curve of genus g* .

2. CURVES OF COMPACT TYPE

Let $k \subset F$ be a field extension. A stable curve X over F is called of *compact type* if $\text{Pic}_{X/F}^0$ is an abelian variety. Equivalently, the map $\text{Pic}_{X/F}^0 \rightarrow \text{Pic}_{\tilde{X}/F}^0$ induced by the normalization $\tilde{X} \rightarrow X$ is injective. If F is algebraically closed, the condition means that the irreducible components $X_i \subset X$ are smooth, and the configuration of the X_i is tree-like ([6], Chapter 9, Proposition 10). Such curves have nice properties:

Lemma 2.1. *Let $F \subset F'$ be a purely transcendental field extension, X a stable curve over F , and $X' = X \otimes F'$ the induced curve. If X is of compact type, then the preimage map $\text{Pic}(X) \rightarrow \text{Pic}(X')$ is bijective.*

Proof. By the usual limit argument ([18], Theorem 8.5.2), we may assume that F' is finitely generated. Applying induction, we reduce to the case $\text{trdeg}(F') = 1$. Now we may view F' as the function field of \mathbb{P}_F^1 .

The task is to check that $\text{Pic}(X) \rightarrow \text{Pic}(X')$ is surjective. First we do the special case that X is smooth. Then $X \times \mathbb{P}_F^1$ is factorial, so the restriction map $\text{Pic}(X \times \mathbb{P}_F^1) \rightarrow \text{Pic}(X')$ is surjective. Using $\text{Pic}(X \times \mathbb{P}_F^1) = \text{Pic}(X) \oplus \mathbb{Z}$ we deduce that $\text{Pic}(X) \rightarrow \text{Pic}(X')$ is surjective as well.

Now suppose X is arbitrary, and let $\tilde{X} \rightarrow X$ be the normalization. The maps $\text{Pic}(X) \rightarrow \text{Pic}(\tilde{X})$ and $\text{Pic}(X') \rightarrow \text{Pic}(\tilde{X}')$ are bijective, because X is of compact type. Applying the preceding special case to \tilde{X} , we infer that $\text{Pic}(X) \rightarrow \text{Pic}(X')$ is surjective. \square

Recall that, in characteristic $p > 0$, an abelian variety A over F has a p -rank f defined by $\text{Hom}(\mu_p, A \otimes \overline{F}) = (\mathbb{Z}/p\mathbb{Z})^f$. The abelian variety is called *ordinary* if $f = \dim(A)$. This condition means that the group of geometric p -torsion points in $A(\overline{F})$ is as large as possible, namely $(\mathbb{Z}/p\mathbb{Z})^{\dim(A)}$. In characteristic zero, abelian varieties are ordinary by definition. The following specialization result will play a role in the sequel:

Lemma 2.2. *Suppose B is a discrete valuation ring with residue field F and field of fractions Q . Let $Z \rightarrow \text{Spec}(B)$ be a relative stable curve such that Z_0 is of compact type and $\text{Pic}_{Z_0/F}^0$ is ordinary. If $\text{Pic}^0(Z_0/F)$ is torsion free, then $\text{Pic}^0(Z_\eta/Q)$ is torsion free as well.*

Proof. Consider the relative generalized Jacobian $J = \text{Pic}_{Z/B}^0$. This is a separated group scheme of finite type over B by [9], Proposition 3.4. It is an abelian scheme

because Z_0 is of compact type. Let $J[m]$ be the relative kernel of the multiplication map $[m] : J \rightarrow J$. Then $J[m]$ is a finite flat group scheme over B whose fibers have length m^{2g} , where g is the genus of Z_0 .

We claim that $J[m](Q) = 0$. Indeed, suppose we have a point $x \in J[m](Q)$. We now assume that the characteristic $p > 0$ is positive, the case $p = 0$ being similar. Decompose $m = lq$, where l is prime to p , and q is a power of p . By construction, the geometric closed fiber of $J[m]$ is isomorphic to $(\mathbb{Z}/l\mathbb{Z})^{2g} \oplus (\mathbb{Z}/q\mathbb{Z})^g \oplus \mu_q^g$, where μ_q is the local group scheme of q -th roots of unity (here we use that J_0 is ordinary). The same holds for all geometric fibers, because the number of geometric connected components is lower semicontinuous in proper flat families ([17], Proposition 15.5.9). By [18], Proposition 15.5.1, each of the $l^{2g}q^g$ geometric points in the closed fiber extend to disjoint sections over the strict henselization $B \subset B^{\text{sh}}$, which defines $l^{2g}q^g$ different geometric points over the generic fiber. It now follows from [23], Exposé VIII, Theorem 4.1 that the closure $\overline{\{x\}} \subset J[m]$ is disjoint from the zero section, hence the specialization map $J[m](Q) \rightarrow J[m](F)$ is injective. Since $J(F)$ is torsion free we have $J[m](Q) = 0$. \square

3. TORSION POINTS

Fix an integer $n \geq 0$ and a genus $g \geq 3$. Let C_{η_n} be the generic n -pointed curve of genus g . In this section we take care of torsion points:

Proposition 3.1. *The group $\text{Pic}^0(C_{\eta_n}/\eta_n)$ is torsion free.*

The proof requires some preparation. First note that by Lemma 1.2, we may replace the ground field k by any extension field. For the rest of the section, we assume that k is algebraically closed, and write $F = k(T)$ for the rational function field in one indeterminate.

Proposition 3.2. *There is an ordinary elliptic curve E over the rational function field F satisfying $\text{Pic}^0(E) = 0$.*

Proof. The idea is to use special Halphen pencils. Fix an ordinary elliptic curve E_0 over k . Such a curve exists because there are only finitely many supersingular elliptic curves ([27], Chapter IV, Corollary 4.23). Let $x \in E_0$ be the origin and consider the closed embedding $E_0 \subset \mathbb{P}_k^2$ defined via $\mathcal{O}_{E_0}(3x) \simeq \mathcal{O}_{E_0}(1)$.

Lines and quadrics in \mathbb{P}_k^2 are uniquely determined by their intersection with the cubic E_0 , because the restriction maps $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(n)) \rightarrow H^0(E_0, \mathcal{O}_{E_0}(3nx))$ are bijective for $n = 1, 2$. Let $L \subset \mathbb{P}_k^2$ be the unique line with $L \cap E_0 = 3x$. Setting $E_\infty = 3L$, we have $E_0 \cap E_\infty = 9x$. Consider the pencil of cubics E_t , $t \in \mathbb{P}_k^1$ generated by E_0 and E_∞ . Then E_t is integral for $t \neq \infty$. Indeed, any quadric $Q \subset E_t$ satisfies $Q \cap E_0 = 6x$, hence $Q = 2L$. Moreover, any line $H \subset E_t$ has $H \cap E_0 = 3x$, hence $H = L$.

Now let $g' : Y' \rightarrow \mathbb{P}_k^2$ be the blowing-up of the nonreduced center $E_0 \cap E_\infty$, and $f' : Y' \rightarrow \mathbb{P}_k^1$ the induced fibration. The Jacobian of the generic fiber Y'_η is the desired elliptic curve E . To see this, note that $R' = g'^{-1}(E_0 \cap E_\infty)$ is isomorphic to the projective line over the Artin algebra $k[v]/(v^9)$. A local computation as in [40], page 417 reveals that Y' is smooth except for a rational double point $y' \in Y'$ of type A_8 lying on R' . Let $Y \rightarrow Y'$ be the minimal resolution of this singularity. Its exceptional locus is a string of eight smooth rational (-2) -curves. It follows that the strict transform $R \subset Y$ of R'_{red} is a smooth rational (-1) -curve.

Let $f : Y \rightarrow \mathbb{P}_k^1$ be the induced fibration and consider its generic fiber Y_η . We now check that R generates $\text{Pic}(Y_\eta)$. Indeed, the restriction map $\text{Pic}(Y) \rightarrow \text{Pic}(Y_\eta)$ is surjective, and $\text{Pic}(Y)$ is generated by the exceptional curves for the sequence of blowing ups $g : Y \rightarrow \mathbb{P}_k^2$ and $g^*(L)$. The strict transform of L is disjoint from Y_η , because the strict transform of $3L$ is disjoint from Y_η , and we conclude that the restriction of $f^*(L)$ to Y_η is a multiple of R .

Summing up, the Jacobian $E = \text{Pic}_{Y_\eta/\eta}^0$ contains no rational point but the origin. Then $\text{Pic}^0(E) = 0$, because E is isomorphic to its own Jacobian. To see that the elliptic curve E is ordinary, look at the relative Jacobian of $f : Y \rightarrow \mathbb{P}_k^1$ near $0 \in \mathbb{P}_k^1$. Its closed fiber E_0 is ordinary. Since this is an open condition, the generic fiber E is ordinary as well. \square

Remark 3.3. (i) The rational double point $y' \in Y'$ maps to $\infty \in \mathbb{P}_k^1$. To see this, write $\mathcal{O}_{\mathbb{P}^2, x}^\wedge = k[[u, v]]$ so that $E_0, L \subset \mathbb{P}_k^2$ correspond to $u = 0$ and $v^3 = u$, respectively. Then $\mathcal{O}_{Y', y'}^\wedge = k[[u, v, w]]/(uw = v^9)$. You easily check that the equation $v^3 = u$ remains indecomposable inside $k[[u, v, w]]/(uw = v^9)$. This equation defines the preimage $g^*(L) \subset Y'$, which decomposes as a Weil divisor into two irreducible components. So the strict transform of L is not Cartier, hence passes through the singular point $y' \in Y'$.

(ii) The degenerate fiber $X_\infty \subset X$ is of type II^* in Kodaira's notation, that is, it corresponds to the root lattice \tilde{E}_8 . To see this, note that the string of eight (-2) -curves in X_∞ hits the strict transform of L in precisely one point. Moreover, the strict transform is not a (-1) -curve, because the intersection form on X_∞ is negative semidefinite. Glancing at Kodaira's classification ([31], Theorem 6.2), we deduce that X_∞ must be of type II^* .

Next, we construct a stable curve X of genus g over the rational function field $F = k(T)$ as follows: Let E_1, \dots, E_g be copies of the elliptic curve E from Proposition 3.2, and choose rational points $p_1, \dots, p_g \in \mathbb{P}_F^1$. Let $X = E_1 \cup \dots \cup E_g \cup \mathbb{P}_F^1$ be the curve obtained by identifying the rational point $p_i \in \mathbb{P}_F^1$ with the origins $0 \in E_i$ for $i = 1, \dots, g$. This curve is stable because $g \geq 3$. Note that we may view it as an n -pointed curve, simply by choosing rational points in the rational component. The curve X has the following properties:

Proposition 3.4. *The curve X is of compact type, the abelian variety $\text{Pic}_{X/F}^0$ is ordinary, and its Mordell–Weil group is $\text{Pic}^0(X) = 0$.*

Proof. The normalization $\tilde{X} \rightarrow X$ is the disjoint union of g copies of E and a projective line. The canonical map $\text{Pic}_{X/F} \rightarrow \text{Pic}_{\tilde{X}/F}$ is an isomorphism, so $\text{Pic}_{X/F}^0 = E \times \dots \times E$, and the result follows. \square

Proof of Proposition 3.1: Let $X = E_1 \cup \dots \cup E_g \cup \mathbb{P}_F^1$ be the stable curve of genus g over the rational function field $F = k(T)$ constructed above. Pick n rational points $x_i \in X$ contained in the rational component \mathbb{P}_F^1 and disjoint from the elliptic components E_i . Let $\mathfrak{Y} \rightarrow \text{Spf}(A)$ be the formal versal deformation of the n -pointed stable curve X . Then A is a formal power series ring in $3g - 3 + n$ variables with coefficients in F . Since $H^2(X, \mathcal{O}_X) = 0$, we may extend any ample invertible \mathcal{O}_X -module to an invertible $\mathcal{O}_{\mathfrak{Y}}$ -module. By Grothendieck's Algebraization Theorem ([15], Theorem 5.4.5), the formal scheme \mathfrak{Y} is the formal completion of a relative curve $Y \rightarrow \text{Spec}(A)$.

Blowing up the closed point in $\text{Spec}(A)$ and localizing at the generic point of the exceptional divisor, we obtain a discrete valuation ring B dominating A . Its field of fractions $B \subset Q$ is also the field of fractions for A . The residue field $B_0 = B/\mathfrak{m}_B$ is a purely transcendental field extension of F . Let $Z \rightarrow \text{Spec}(B)$ be the induced family of stable curves.

The classifying map $\text{Spec}(B) \rightarrow \overline{M}_{g,n}$ is dominant. Hence $Z_Q = C_{\eta_n} \otimes Q$, which induces an injection $\text{Pic}(C_{\eta_n}) \subset \text{Pic}(Z_Q)$. We have $\text{Pic}^0(Z_0) = 0$ by construction, so Lemma 2.2 applies, and we conclude that the group $\text{Pic}^0(Z_Q)$ is torsion free. \square

4. ANOTHER STABLE CURVE

We keep the notation from the previous section, such that C_{η_n} is the generic n -pointed curve of genus g . Let $c_1, \dots, c_n \in C_{\eta_n}$ be the marked points, and consider the free abelian group $P = \mathbb{Z}c_1 \oplus \dots \oplus \mathbb{Z}c_n \oplus \mathbb{Z}K_{C_{\eta_n}}$. The following is a key step in proving Franchetta's Conjecture:

Proposition 4.1. *The cokernel for the map $P \rightarrow \text{Pic}(C_{\eta_n})$ is torsion free.*

This again depends on the existence of certain elliptic curves over function fields. Let $F = k(T)$ be the rational function field, and E the ordinary elliptic curve with $\text{Pic}^0(E) = 0$ from Proposition 3.4.

Proposition 4.2. *For each $m \geq 0$ there is a field extension $F \subset F'$ such that $E' = E \otimes F'$ satisfies $\text{Pic}^0(E') = \mathbb{Z}^{\oplus t}$ for some $t \geq m$.*

Proof. The case $m = 0$ is trivial. We proceed by induction on m . Suppose we already have a field extension F' such that $\text{Pic}^0(E') = \mathbb{Z}^{\oplus t}$ with $t \geq m$. Choose a basis $p_1, \dots, p_t \in \text{Pic}^0(E')$. Let $x \in E'$ be the origin, and $x_i \in E'$ the rational points with $x - x_i \sim p_i$.

Now let A be the henselization of the localized polynomial algebra $F'[U]_{(U)}$, and consider the trivial family $Y = E' \otimes A$. Let $S, S_i \subset Y$ be the sections corresponding to the rational points $x, x_i \in E'$. Let $u, v \in \mathcal{O}_{Y,x}$ be the regular parameter system corresponding to $Y_0, S \subset Y$. Replacing v by a different parameter v' , we obtain, locally around x , a curve $S' \neq S$ with $S' \cap Y_0 = \{x\}$. According to [18], Theorem 18.5.11, this locally defined curve S' defines a section $S_{t+1} \subset Y$ passing through x with $S_{t+1} \neq S$. Then the difference $S - S_{t+1}$ defines a nonzero point $p_{t+1} \in \text{Pic}^0(Y_\eta)$ specializing to zero in $\text{Pic}^0(Y_0)$. It follows that any nonzero multiple of p_{t+1} lies outside the span of the p_1, \dots, p_t .

Next, choose a subalgebra $B \subset A$ that is a localization of a finite $F'[U]_{(U)}$ -algebra at some F' -valued étale point, so that the section S_{t+1} is defined over B . Let F'' be the function field of B and set $E'' = E \otimes F''$. Then $\text{Pic}^0(E'')$ is torsion free according to Proposition 2.2. By construction, it contains a free group of rank $t + 1$.

It remains to check that $\text{Pic}^0(E'')$ is finitely generated. Let C be the normal curve over F' corresponding to the function field F'' , and consider the regular proper surface $X = E' \times C$. Then $\text{Pic}_{X/F'}^0 = \text{Pic}_{E'/F'}^0 \times \text{Pic}_{C/F'}^0$, and the Néron–Severi group $\text{NS}(X) = \text{Pic}_{X/F'}/\text{Pic}_{X/F'}^0$ is finitely generated. Now view E'' as the generic fiber for the projection $X \rightarrow C$. The cokernel $\text{Pic}^0(E'')/\text{Pic}^0(E')$ injects into $\text{NS}(X)$, and we conclude that $\text{Pic}^0(E'')$ is finitely generated. \square

Remark 4.3. The proof shows that we may choose $F \subset F'$ as a finitely generated separable field extension of transcendence degree $\leq m$.

We now use such elliptic curves to construct a geometrically integral n -pointed stable curve of genus g . Choose a field extension $F \subset F'$ so that $\text{Pic}^0(E') = \mathbb{Z}^{\oplus t}$ for some $t \geq 2g - 2 + n$. Pick $2g - 2 + n$ rational points

$$p_1, p'_1, \dots, p_{g-1}, p'_{g-1}, x_1, \dots, x_n \in E'$$

which are part of a basis for $\text{Pic}(E')$. Let X be the stable curve of genus g obtained by identifying the pairs $p_i, p'_i \in E'$ for $i = 1, \dots, g-1$. The rational points $x_i \in E'$ define n rational points $x_i \in X$, which we denote by the same letter. The canonical morphism $\nu : E' \rightarrow X$ is the normalization map for X .

Proposition 4.4. *We have $\nu^*(K_X) = \sum_{j=1}^{g-1} (p_j + p'_j)$.*

Proof. This follows from duality theory for the finite morphism $\nu : E' \rightarrow X$. See, for example, [44], Proposition 2.3. \square

Proof of Proposition 4.1: Let X be the n -pointed stable curve of genus g over the function field F' constructed above. As in the proof of Proposition 3.1, we construct a discrete valuation ring B and a stable curve $Z \rightarrow \text{Spec}(B)$ with regular total space such that the following holds: (i) The residue field $B_0 = B/\mathfrak{m}_B$ is a purely transcendental field extension of F' , and the closed fiber is $Z_0 = X \otimes B_0$. (ii) If $B \subset Q$ denotes the function field, then the classifying map $\text{Spec}(Q) \rightarrow \overline{M}_{g,n}$ is dominant.

Now suppose we have a class $L \in \text{Pic}(C_{\eta_n})$ such that $mL \in P$ for some $m \neq 0$. Write $mL = \lambda K_{C_{\eta_n}} + \sum \lambda_i c_i$ for certain coefficients $\lambda, \lambda_i \in \mathbb{Z}$. The task is to prove $L \in P$. Using the field extension $\kappa(\eta_n) \subset Q$, we obtain a class $L_Q \in \text{Pic}(Z_Q)$. It extends to a divisor $D \in \text{Div}(Z)$, and we have

$$mD = \lambda K_{Z/B} + \sum_{i=1}^n \lambda_i C_i$$

in $\text{Pic}(Z)$, where $C_i \subset Z$ are the marked sections. This is because the divisor Z_0 , being an integral fiber, supports only principal divisors. Pulling back to the normalization $\tilde{Z}_0 = E \otimes B_0$ of the closed fiber Z_0 , we obtain

$$mD|_{\tilde{Z}_0} = \lambda \sum_{j=1}^{g-1} (p_j + p'_j) + \sum_{i=1}^n \lambda_i x_i$$

in $\text{Pic}(E \otimes B_0)$. Since the $p_j + p'_j, x_i$ are part of a basis for $\text{Pic}(E \otimes B_0)$ by Lemma 2.1, all coefficients λ, λ_i are multiples of m . Replacing L by $L - \frac{\lambda}{m} K_{C_{\eta_n}} - \sum \frac{\lambda_i}{m} c_i$, we reduce to the case that $mL = 0$. Then $L = 0$ by Proposition 3.1, and in particular $L \in P$. \square

5. THE STRONG FRANCHETTA CONJECTURE

We come to the first main result of this paper:

Theorem 5.1. *Let k be a field, and $g \geq 3$ and $n \geq 0$ be integers. Let $\eta_n \in \overline{M}_{g,n}$ the generic point in the moduli space of n -pointed stable curves of genus $g \geq 3$, and $C = \overline{M}_{g,n+1}$ the tautological curve. Then the marked points $c_1, \dots, c_n \in C_{\eta_n}$ and the canonical class $K_{C_{\eta_n}}$ freely generate $\text{Pic}(C_{\eta_n}/\eta_n)$.*

Before we prove this, let us recall the definition of certain *tautological classes* in $\text{Pic}(\overline{M}_{g,n}) \otimes \mathbb{Q}$. Let $\pi : \overline{M}_{g,n+1} \rightarrow \overline{M}_{g,n}$ be the projection sending an $(n+1)$ -pointed stable curve (X, x_1, \dots, x_{n+1}) to the n -pointed stable curve (X', x'_1, \dots, x'_n) obtained by contracting any component in (X, x_1, \dots, x_n) violating stability. The *Hodge class* $\lambda \in \text{Pic}(\overline{M}_{g,n}) \otimes \mathbb{Q}$ is defined as the determinant of $\pi_*(\omega_\pi)$, where $\omega_\pi = \omega_{\overline{M}_{g,n+1}/\overline{M}_{g,n}}$ is the relative dualizing sheaf.

We also have canonical sections $s_i : \overline{M}_{g,n} \rightarrow \overline{M}_{g,n+1}$ for $i = 1, \dots, n$ as follows. These morphisms are best described on geometric points: The section s_i sends an n -pointed stable curve (X', x'_1, \dots, x'_n) to the $(n+1)$ -pointed stable curve (X, x_1, \dots, x_{n+1}) defined as follows: We have $X = X' \cup \mathbb{P}^1$, where the point $x'_i \in X'$ is identified with $\infty \in \mathbb{P}^1$. The marked points are $x_i = 0 \in \mathbb{P}^1$, $x_{n+1} = 1 \in \mathbb{P}^1$, and $x_j = x'_j$ for $j \neq i$. The *Witten classes* $\psi_i \in \text{Pic}(\overline{M}_{g,n}) \otimes \mathbb{Q}$ are defined as $\psi_i = s_i^*(\omega_\pi)$.

There are also *boundary classes* $\delta_v \in \text{Pic}(\overline{M}_{g,n}) \otimes \mathbb{Q}$, which are effective Weil divisors supported on $\overline{M}_{g,n} - M_{g,n}$. They correspond to various topological types of degeneration. The idea now is to restrict tautological classes on $\overline{M}_{g,n+1}$ to the generic curve $C_{\eta_n} \subset \overline{M}_{g,n+1}$.

Proposition 5.2. *The subgroup $P \subset \text{Pic}(C_{\eta_n})$ generated by the marked sections and the canonical class contains the restriction of the tautological classes.*

Proof. First note that geometric points c in the generic n -pointed curve C_{η_n} correspond to $(n+1)$ -pointed stable curves (X, x_1, \dots, x_{n+1}) . The curve X is smooth if c is not a marked point. On the other hand, if $c = c_i$ is a marked point, then $X = X' \cup \mathbb{P}^1$ with $x_i, x_{n+1} \in \mathbb{P}^1$. It now follows immediately from their definitions in [39], Section 1 that the restrictions $\delta_v|_{C_{\eta_n}}$ of the boundary classes are supported by the marked points $c_i \in C_{\eta_n}$.

How do Hodge classes and Witten classes restrict to the generic pointed curve? To see this, note that the fiber product $C_{\eta_n} \times_{\overline{M}_{g,n+1}} \overline{M}_{g,n+2}$ is isomorphic to the blowing up of $C_{\eta_n} \times_{\eta_n} C_{\eta_n}$ with respect to the centers (c_i, c_i) for $i = 1, \dots, n$. This also follows from the modular interpretation of geometric points in $\overline{M}_{g,n+1}$. Indeed, the marked points $c_i \in C_{\eta_n}$, $i = 1, \dots, n$ correspond to reducible stable curves of the form $(X' \cup \mathbb{P}^1, x_1, \dots, x_{n+1})$ with $x_i, x_{n+1} \in \mathbb{P}^1$, and the exceptional curve in the blowing up of $(c_i, c_i) \in C_{\eta_n} \times_{\eta_n} C_{\eta_n}$ is given by $(n+2)$ -pointed stable curves $(X' \cup \mathbb{P}^1, x_1, \dots, x_{n+2})$ with $x_i, x_{n+1}, x_{n+2} \in \mathbb{P}^1$. As a consequence we have $\lambda|_{C_{\eta_n}} = 0$, and $\psi_i|_{C_{\eta_n}} = c_i$ for $i = 1, \dots, n$, and $\psi_{n+1}|_{C_{\eta_n}} = K_{C_{\eta_n}}$. \square

Proof of Theorem 5.1: Set $P = \mathbb{Z}c_1 \oplus \dots \oplus \mathbb{Z}c_n \oplus \mathbb{Z}K_{C_{\eta_n}}$. The canonical map $P \rightarrow \text{Pic}(C_{\eta_n})$ is injective. Indeed, this is clear for $n = 0$. In case $n \neq 0$, we have $\text{Pic}(C_{\eta_n}) = \text{Pic}(C_{\eta_n}/\eta_n)$, and it suffices to construct an n -pointed stable curve X of genus g where the marked points and the canonical class are linearly independent. We constructed such a curve in Section 4.

The task is to prove surjectivity. Fix a point $L \in \text{Pic}^d(C_{\eta_n}/\eta_n)$. Having only quotient singularities, the normal scheme $\overline{M}_{g,n+1}$ is \mathbb{Q} -factorial. Hence some multiple mL with $m > 0$ extends to a Cartier divisor class D on $\overline{M}_{g,n+1}$. According to [39], Theorem 5.1, we have $D = a\lambda + \sum_{i=1}^{n+1} b_i\psi_i + \sum_v d_v\delta_v$ for certain integral coefficients, at least after replacing m by a multiple. Recall that λ is the Hodge class, λ_i are the Witten classes, and δ_v are the boundary classes. Restricting to the generic curve and using Proposition 5.2, we infer $mL \in P$.

We now distinguish three cases. First, suppose $n \geq 1$. Then $\text{Pic}(C_{\eta_n}) = \text{Pic}(C_{\eta_n}/\eta_n)$, and Proposition 4.1 implies $L \in P$. Second, suppose $n = 0$ and $d = 0$. Then there are only the two tautological classes λ and ψ_1 besides the boundary classes. The equation

$$mL = (a\lambda + \sum_{i=1}^{n+1} b_i\psi_i + \sum_v d_v\delta_v)|_{C_\eta}$$

boils down to $mL = b_1K_{C_{\eta_n}}$ and therefore $mL = 0$. Now Proposition 3.1 ensures $L = 0$.

So only the case $n = 0$ and $d \neq 0$ remains. Now we argue as follows: The $\text{Pic}_{C_\eta/\eta}^0$ -torsor $\text{Pic}_{C_\eta/\eta}^{d(2g-2)}$ contains both $(2g-2)L$ and dK_{C_η} . These rational points differ by a point in $\text{Pic}^0(C_\eta/\eta) = 0$, so $(2g-2)L = dK_{C_\eta}$. In other words, L is a rational multiple of K_{C_η} . Now consider the affine surjection $C_{\eta_1} \rightarrow C_\eta$ defined on geometric points by $(X, x_1, x_2) \mapsto (X, x_2)$. We already saw that the canonical class $K_{C_{\eta_1}} \in \text{Pic}(C_{\eta_1})$ is a primitive element. Since $\text{Pic}(C_\eta/\eta) \rightarrow \text{Pic}(C_{\eta_1})$ is injective, the canonical class $K_\eta \in \text{Pic}(C_\eta/\eta)$ is primitive as well, and we infer that L is an integral multiple of K_{C_η} . \square

Remark 5.3. By definition, the generic curves C_{η_n} with $n \geq 1$ contain a rational point, hence $\text{Pic}(C_{\eta_n}) = \text{Pic}(C_{\eta_n}/\eta_n)$. So a priori there is no difference between weak and strong Franchetta Conjectures.

Remark 5.4. Moriwaki's calculation [39] of $\text{Pic}(\overline{M}_{g,n}) \otimes \mathbb{Q}$ depends on the existence of certain simply connected coverings of $\overline{M}_{g,n}$ due to Looijenga [36], Pikaart and de Jong [42], and Boggi and Pikaart [5].

To my knowledge, the group $\text{Pic}(\overline{M}_{g,n})$ itself has not been calculated yet, neither in characteristic zero nor in positive characteristics.

6. APPLICATION TO TAME COVERINGS

Let me give an application of the Strong Franchetta Conjecture to tame coverings. Belyi's Theorem [4] states that a complex curve is defined over a number field if and only if it admits a map to $\mathbb{P}_\mathbb{C}^1$ with at most three branch points. Fulton showed that any smooth curve X over a separably closed field of characteristic $p \neq 2$ admits a branched covering $X \rightarrow \mathbb{P}^1$ with tame ramification ([13], Proposition 8.1). Saïdi used this to prove an analog of Belyi's Theorem in odd characteristics: A smooth curve X over an algebraically closed field of characteristic $p > 2$ is defined over a finite field if and only if it is a tamely ramified covering of \mathbb{P}^1 with at most three branch points ([45], Theorem 5.6). It is unknown to what extent these facts hold true in characteristic $p = 2$. We have the following negative result for the generic n -pointed curve C_{η_n} of genus $g \geq 3$:

Theorem 6.1. *Suppose the ground field k has characteristic $p = 2$. Then any finite separable morphism $f : C_{\eta_n} \rightarrow \mathbb{P}_{\eta_n}^1$ has wild ramification.*

Proof. Suppose on the contrary that $f : C_{\eta_n} \rightarrow \mathbb{P}_{\eta_n}^1$ has only tame ramification. This means that, over the algebraic closure, every ramification point has odd ramification index.

Clearly, this implies that for all ramification points $x \in C_{\eta_n}$, say with image $y \in \mathbb{P}_{\eta_n}^1$, the field extension $\kappa(y) \subset \kappa(x)$ is separable. Moreover, the localization of the fiber $f^{-1}(y)$ at x is of the form dx for some odd integer $d \geq 2$. In turn, the

relative canonical class $K_{C_{\eta_n}/\mathbb{P}^1} = \sum (d-1)x$ is 2-divisible. The same holds for the canonical class of $\mathbb{P}^1_{\eta_n}$. Consequently $K_{C_{\eta_n}}$ is 2-divisible, contradicting Theorem 5.1. \square

Remark 6.2. The proof actually shows that $f : C_{\eta_n} \rightarrow X$ has wild ramification if X is any curve whose canonical class is 2-divisible in the Picard scheme. The choice of an $L \in \text{Pic}(X/\eta_n)$ with $2L = K_X$ is sometimes called a *spin structure* or *theta characteristic*, compare [1], Chapter VI, Appendix B.

Remark 6.3. Consider the unique supersingular elliptic curve E in characteristic $p = 2$, which has Weierstrass equation $y^2 + y = x^3$ and invariant $j(E) = 0$. Its automorphism group is a semidirect product of $\mathbb{Z}/3\mathbb{Z}$ and the quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$. Therefore, this curve even admits a Galois covering $f : E \rightarrow \mathbb{P}^1$ of degree $d = 3$, which has three ramification points, all of ramification index $e = 3$. Amusingly, this is the only elliptic curve in characteristic $p = 2$ for which the existence of a tamely ramified function is a priori clear.

It seems to be open whether a given smooth curve X in characteristic $p \geq 0$ has a morphism $f : X \rightarrow \mathbb{P}^1$ whose ramification indices $e \geq 2$ are all odd. For example, we may ask this for the generic elliptic curve E over $\mathbb{Q}(j)$, which has Weierstrass equation

$$y^2 + xy = x^3 - \frac{1728}{j-1728}x - \frac{1}{j-1728}$$

and invariant $j(E) = j$. We can say at least the following:

Proposition 6.4. *There is a finite field extension $\mathbb{Q}(j) \subset F$ and a morphism $f : E \otimes F \rightarrow \mathbb{P}^1_F$ of degree $d = 4$ with four ramification points, all with ramification index $e = 3$.*

Proof. Let $\mathfrak{E} \rightarrow S$ be the universal elliptic curve over the punctured j -line $S = \mathbb{A}_{\mathbb{Q}}^1 - \{0, 1728\}$. Consider the subscheme $U \subset \text{Hom}(\mathfrak{E}, \mathbb{P}^1_S)$ of the relative Hom-scheme that parametrizes morphisms $\mathfrak{E}_s \rightarrow \mathbb{P}^1_s$ of degree $d = 4$ with four ramification points, all with index $e = 3$. It follows from [12], Theorem 1, that the projection $\text{pr} : U \rightarrow S$ is dominant. Then the generic fiber $\text{pr}^{-1}(\eta) \subset U$ contains a closed point u , and the residue field $F = \kappa(u)$ is the desired finite field extension of $\mathbb{Q}(j)$. \square

7. A STABLE CURVE WITH MAXIMAL INDEX

Recall that the *index* $\text{ind}(X) \in \mathbb{Z}$ of a proper curve X over an arbitrary field F is the positive generator for the image of the degree map $\text{Pic}(X/F) \rightarrow \mathbb{Z}$. If X is geometrically integral, $\text{ind}(X)$ is also the order of $\text{Pic}_{X/F}^1$ in the Weil–Châtelet group $H^1(k, \text{Pic}_{X/F}^0)$.

The index of a stable curve of genus g is a divisor of $2g - 2$, because there is always the canonical class. It follows from Theorem 5.1 that the generic curve has maximal index $\text{ind}(C_{\eta}) = 2g - 2$. It would be interesting to construct special curves with maximal index over fields of smaller transcendence degree. I do not know how to achieve this with smooth curves. The goal of this section is to produce an explicit stable curve with maximal index.

Throughout, we assume that our ground field k is algebraically closed, and let $F = k(T)$ be the rational function field. Note that, in our situation, we have $\text{Pic}(X/F) = \text{Pic}(X)$ by Lemma 1.1.

Proposition 7.1. *For each integer $d > 0$ there is a smooth curve Y of genus one over the rational function field F with $\text{ind}(Y) = d$.*

Proof. Let $F \subset F'$ be a cyclic field extension of degree d , say with Galois group $G \simeq \mathbb{Z}/d\mathbb{Z}$. Choose an elliptic curve E over F containing a rational point $x \in E$ of order d . Let $T \subset \text{Aut}(E)$ be the cyclic subgroup of order d generated by the translation $T_x(e) = e + x$.

Set $E' = E \otimes_F F'$. Then $g \in G$ acts on $\text{Aut}(E'/F')$ via $\phi \mapsto g\phi g^{-1}$. This action fixes the subgroup T pointwise, because $x \in E$ is a rational point. We get

$$H^1(G, T) = \text{Hom}(G, T) \simeq \mathbb{Z}/d\mathbb{Z}.$$

The latter identification depends on the choice of a generator $g_0 \in G$. The inclusion $T \subset \text{Aut}(E'/F')$ gives an inclusion $H^1(G, T) \subset H^1(G, \text{Aut}(E'/F'))$. As explained in [46], Chapter II, §1, Proposition 5, the set $H^1(G, \text{Aut}(E'/F'))$ may be viewed as the set of isomorphism classes of twisted forms Y of E whose preimage $Y' = Y \otimes F'$ is isomorphic to E' . Indeed, Y is the quotient of E' by the G -action $g \circ z_g$, where $z_g \in Z^1(G, \text{Aut}(E'/F'))$ is a cocycle representing a given cohomology class.

Now choose the generator $g_0 \mapsto T_x$ of $H^1(G, T)$ and consider the corresponding twisted form Y of E . Then $\text{ind}(Y)$ divides d . Indeed, the reduced divisor $D \subset E$ comprising the multiples of $x \in E$ defines a G -invariant divisor of degree d in E' , hence an element in $\text{Pic}^d(Y/F)$.

For the converse, suppose we have a class in $\text{Pic}^l(Y/F)$. It corresponds to an invertible \mathcal{O}_Y -module \mathcal{L} of degree l by Lemma 1.1. The isomorphism class of the induced invertible $\mathcal{O}_{Y'}$ -module \mathcal{L}' is invariant under both g_0 and $g_0 \circ T_x$, hence also under the translation T_x . But $T_x^*(\mathcal{O}_{E'}(e)) \simeq \mathcal{O}_{E'}(e - x)$ for any rational point $e \in E'$, so $T_x^*(\mathcal{L}') \simeq \mathcal{L}'$ implies $lx = 0$. In turn, d divides l , and we conclude $\text{ind}(Y) = d$. \square

Remark 7.2. The cyclic field extension used above exist. Indeed, if d is prime to the characteristic p , the cyclic extensions of degree d correspond to étale cohomology $H^1(F, \mathbb{Z}/d\mathbb{Z}) = F^*/(F^*)^d$ via the Kummer sequence $0 \rightarrow \mathbb{Z}/d\mathbb{Z} \rightarrow \mathbb{G}_m \rightarrow \mathbb{G}_m \rightarrow 1$. If $d = p^m$, they correspond to étale cohomology $H^1(F, \mathbb{Z}/d\mathbb{Z}) = F/\wp(F)$ via the Artin–Schreier sequence $0 \rightarrow \mathbb{Z}/p^m\mathbb{Z} \rightarrow W_m(F) \rightarrow W_m(F) \rightarrow 0$, where $W_m(F)$ is the sheaf of Witt vectors of length m . In general, decompose $d = lp^m$ with l prime to p , and let F' be the subextension inside the maximal abelian extension $F \subset F^{\text{ab}}$ generated by the linearly disjoint cyclic extensions corresponding to l and p^m ([7], Chapter V, §10, no. 4, Corollary 2 of Theorem 1).

Now we assemble the desired stable curve:

Proposition 7.3. *There is a geometrically integral stable curve X of genus g over the rational function fields F with $\text{ind}(X) = 2g - 2$.*

Proof. Let Y be a smooth curve of genus one with $\text{ind}(Y) = 2g - 2$, as in Proposition 7.1. By Lemma 1.1, there is an invertible \mathcal{O}_Y -module \mathcal{L} of degree $2g - 2$. The Riemann–Roch Theorem gives $h^0(Y, \mathcal{L}) = 2g - 2$, hence \mathcal{L} comes from an effective divisor $D \subset Y$ of length $2g - 2$. By Bertini’s Theorem, the very ample sheaf \mathcal{L} has a global section so that the corresponding divisor $D \subset Y$ is smooth (see [29], Corollary 6.11). Such a subscheme is necessarily of the form $D = \text{Spec}(F')$ for some separable field extension $F \subset F'$ of degree $2g - 2$ because $\text{ind}(Y) = \text{deg}(\mathcal{L})$.

Let $F \subset F'' \subset F'$ be a subextension of degree $g - 1$, such that $F'' \subset F'$ has degree two. The cocartesian diagram

$$\begin{array}{ccc} \mathrm{Spec}(F') & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \mathrm{Spec}(F'') & \longrightarrow & X \end{array}$$

defines an integral curve X . The curve X has a single cuspidal singularity, which breaks up into $g - 1$ nodal singularities over the algebraic closure. Therefore X is stable and geometrically integral. The exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow F'' \oplus \mathcal{O}_Y \longrightarrow F' \longrightarrow 0$$

gives an exact sequence

$$0 \longrightarrow H^0(\mathcal{O}_X) \longrightarrow F'' \oplus H^0(\mathcal{O}_Y) \longrightarrow F' \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(Y, \mathcal{O}_Y) \longrightarrow 0,$$

so $h^1(X, \mathcal{O}_X) = g$. Since $Y \rightarrow X$ is birational, the map $\mathrm{Pic}(X) \rightarrow \mathrm{Pic}(Y)$ is surjective, and its kernel consists of numerically trivial sheaves. We conclude that $\mathrm{ind}(X) = \mathrm{ind}(Y) = 2g - 2$ holds. \square

Remark 7.4. Unfortunately, the curve X is not of compact type and contains no rational point. It is therefore difficult to analyze specialization of points in the Picard scheme of the versal deformation for X . In particular, the indices of curves occurring in the versal deformation are hard to control.

8. GEOMETRICALLY UNIBRANCH SINGULARITIES

Our next goal is to find, beside the generic point $\eta_n \in M_{g,n}$, other nonclosed points $x \in M_{g,n}$ whose curve C_x has the property that the marked section and the canonical class generate $\mathrm{Pic}(C_x)$, at least up to torsion. This will occupy the remaining sections. But first we have to remove a problem in characteristic $p > 0$, which occurs over and over again, namely: Normality is not necessarily preserved under inseparable field extensions. An annoying consequence is, for instance, that the Jacobian $\mathrm{Pic}_{X/F}^0$ of a proper normal scheme X is not necessarily proper.

In this section we sidestep such problems using geometrically unibranch schemes instead of normal schemes. This property is stable under field extensions and, as we shall see, almost as good as normality. Recall that a local ring A is called *geometrically unibranch* if it is irreducible, and the normalization \tilde{A} of the reduction A_{red} induces a bijection $\mathrm{Spec}(\tilde{A}) \rightarrow \mathrm{Spec}(A)$ whose residue field extension is purely inseparable. Equivalently, the strict henselization $A \subset A^{\mathrm{sh}}$ is irreducible ([18], Proposition 18.8.15). A scheme is geometrically unibranch if all its local rings are geometrically unibranch. The nice thing about such schemes is that we may change the structure sheaf \mathcal{O}_X as long as the topological space remains the same:

Lemma 8.1. *Let $f : X \rightarrow Y$ be an integral universal homeomorphism of schemes. Then Y is geometrically unibranch if and only if X is geometrically unibranch.*

Proof. To check this we may assume that $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$ are local. We have $B^{\mathrm{sh}} = B \otimes_A A^{\mathrm{sh}}$ according to [18], Proposition 18.8.10 and Remark 18.8.11. Hence A^{sh} is irreducible if and only if B^{sh} is irreducible, because $A \rightarrow B$ is a universal homeomorphism. In turn, A is geometrically unibranch if and only if B is geometrically unibranch. \square

Let us now consider Jacobians. Surely, Jacobians of geometrically unibranch proper schemes may be nonproper. However, things are not too bad:

Proposition 8.2. *Let X be proper F -scheme in characteristic $p > 0$. If X is geometrically unibranch, then there is an integer $n \geq 0$ such that the schematic image of the multiplication map $[p^n] : \text{Pic}_{X/F}^0 \rightarrow \text{Pic}_{X/F}^0$ is proper.*

Proof. Let $H_n \subset \text{Pic}_{X/F}^0$ be the image of the multiplication map $[p^n]$. Being the image of a homomorphism of algebraic group schemes, the embedding $H_n \subset \text{Pic}_{X/F}^0$ is closed by [24], Exposé VI_B, Proposition 1.2.

Note that we may replace the ground field F by its algebraic closure \bar{F} . Indeed, the scheme $\bar{X} = X \otimes \bar{F}$ is geometrically unibranch by [16], Proposition 6.15.7. Moreover, the image H_n of the multiplication map $[p^n]$ on $\text{Pic}_{X/F}^0$ is proper if and only if the induced image $\bar{H}_n = H_n \otimes \bar{F}$ is proper, according to [23], Exposé VIII, Corollary 4.8.

Next, let $\tilde{X} \rightarrow X$ be the normalization of X_{red} . The \tilde{X} is geometrically normal, hence $\text{Pic}_{\tilde{X}/F}^0$ is proper by [20], Theorem 2.1. Let $\tilde{X} \rightarrow \tilde{X}^{(p^m)}$ be the m -fold relative Frobenius map, which is a finite universal homeomorphism. It admits a factorization

$$\tilde{X} \longrightarrow X \longrightarrow \tilde{X}^{(p^m)}$$

for some $m \geq 0$ by [32], Proposition 6.6, because $\tilde{X} \rightarrow X$ is a finite universal homeomorphism. Applying this result again, this time to the finite universal homeomorphism $X \rightarrow \tilde{X}^{(p^m)}$, we obtain a factorization

$$X \longrightarrow \tilde{X}^{(p^m)} \longrightarrow X^{(p^n)}$$

for some $n \geq 0$. Consequently, the image of $\text{Pic}_{\tilde{X}^{(p^m)}/F}^0$ in $\text{Pic}_{X/F}^0$ contains the image of $\text{Pic}_{X^{(p^n)}/F}^0$, which is nothing but H_n . Using that $\tilde{X}^{(p^m)}$ is normal, we conclude that H_n is proper. \square

The next result tells us that geometrically unibranch schemes behave well in families:

Proposition 8.3. *Let S be an integral noetherian scheme in characteristic $p > 0$, and $f : X \rightarrow S$ a morphism of finite type whose generic fiber X_η is geometrically unibranch. Then there is a nonempty open subset $U \subset S$ such that all fibers X_s , $s \in U$ are geometrically unibranch.*

Proof. Using generic flatness ([17], Theorem 11.1.1), we may replace S by an open subset and assume that f is flat. By the usual limit argument, there is a finite morphism $X' \rightarrow X$ such that $X'_\eta \rightarrow X_\eta$ coincides with the generic fiber of the normalization for X_{red} . By assumption, $X'_\eta \rightarrow X_\eta$ is a universal homeomorphism, hence its geometric fibers are connected. According to [17], Proposition 15.5.1, the whole map $X' \rightarrow X$ has geometrically connected fibers, hence is a universal homeomorphism. We may replace X by X' and assume that X_η is normal.

Next, we define for each integer $n \geq 0$ a scheme X_n as follows. The underlying topological space for X_n is the underlying topological space for X . If $W \subset X$ is an affine open subset, we set

$$\Gamma(W, \mathcal{O}_{X_n}) = \Gamma(W, \mathcal{O}_X) \cap Q(L^{p^n}).$$

Here Q is the function field of S and L is the function field of X . Then each $(X_n)_\eta$ is a normal scheme and the projection morphisms $X_n \rightarrow S$ are proper.

The descending sequence of fields $L \supset L^p \supset \dots$ defines a sequence of universal homeomorphisms $X_0 \rightarrow X_1 \rightarrow \dots$ with $X_0 = X$.

According to [28], Theorem 1, there is an integer $n \geq 0$ such that the geometric fiber Y_η of $Y = X_n$ is geometrically normal over Q . By [17], Theorem 12.2.4, the set $W \subset Y$ of geometrically normal points is open. According to Chevalley's Theorem ([14], Theorem 7.1.4), the image $f(W) \subset S$ is constructible. We have $\eta \in f(W)$ by construction. Being constructible, the set $f(W)$ contains a nonempty open subset $U \subset S$. The fibers Y_s , $s \in U$ are geometrically unibranch. Since $X_s \rightarrow Y_s$ is a finite universal homeomorphism, we may apply Lemma 8.1 and conclude that the X_s , $s \in U$ are geometrically unibranch as well. \square

The other direction holds as well:

Proposition 8.4. *Let S be an integral noetherian scheme, and $f : X \rightarrow S$ a morphism of finite type whose generic fiber X_η is not geometrically unibranch. Then there is a nonempty open subset $U \subset S$ such that all fibers X_s , $s \in U$ are not geometrically unibranch.*

Proof. Clearly, we may assume that S is affine and that X is reduced. Like in the proof for Proposition 8.3, there is a finite morphism $X' \rightarrow X$ such that $X'_\eta \rightarrow X_\eta$ is the normalization. Shrinking S , we may assume that the fibers X'_s have no embedded component by [17], Proposition 9.9.2. According to [17], Corollary 9.7.9, the set $Z \subset X$ over which the geometric fibers of $X' \rightarrow X$ are not connected is constructible. Replacing X by some open subset, we may assume that $Z \subset X$ is closed.

The set $x \in X$ where $\text{codim}_x(Z_{f(x)}, X_{f(x)}) = 0$ is constructible by [17], Proposition 9.9.1, and disjoint from X_η . By Chevalley's Theorem, there is a nonempty open subset $U \subset S$ such that $\text{codim}_x(Z_{f(x)}, X_{f(x)}) > 0$ holds for all $x \in f^{-1}(U)$. In other words, Z_s contains no generic point from X_s for $s \in U$. We infer that the fibers X_s are not geometrically unibranch for $s \in U$. \square

Remark 8.5. In Grothendieck's terminology, the preceding two result tell us that the property of being geometrically unibranch is *constructible* (compare [17], Definition 9.2.1). It then follows from [17], Corollary 9.2.4, that given any morphism $f : X \rightarrow S$ of finite presentation, the set $E \subset S$ of points whose fiber X_t is geometrically unibranch is locally constructible.

Now suppose X be a proper F -scheme. Recall that the *Néron-Severi group scheme* is the quotient $\text{NS}_{X/F} = \text{Pic}_{X/F} / \text{Pic}_{X/F}^0$. This is an étale group scheme over F , whose points are given by finite separable field extensions ([24], Exposé VI_A, 5.5). Its group of rational points is denoted by $\text{NS}(X/F)$, which is a finitely generated abelian group ([25], Exposé XIII, Theorem 5.1). The *Néron-Severi group* is the subgroup $\text{NS}(X)$ generated by $\text{Pic}(X)$.

Lemma 8.6. *The inclusion $\text{NS}(X) \subset \text{NS}(X/F)$ has finite index.*

Proof. Fix a rational point $z \in \text{NS}(X/F)$. The corresponding connected component $T \subset \text{Pic}_{X/F}$ is a torsor under $J = \text{Pic}_{X/F}^0$, and we have to show that its order in $H^1(k, J)$ is finite. Indeed, then T^n contains a rational point x for some $n > 0$, and mx comes from an invertible sheaf for some $m > 0$ because $\text{Br}(F)$ is torsion.

Let $F \subset F'$ be some finite field extension such that $T' = T \otimes F'$ acquires a rational point y . Passing to some multiple, we may also assume that y comes from some invertible sheaf \mathcal{M} .

First, consider the special case that F' is separable. Then we may assume that F' is Galois as well, say with Galois group G . Then $\bigotimes_{g \in G} g^* \mathcal{M}$ is a G -invariant invertible sheaf whose rational point lies on T'^n , hence descends to the desired rational point $x \in T^n$.

Next, consider the special case that F' is purely inseparable. Then the projection $X' = X \otimes F' \rightarrow X$ is a finite universal homeomorphism of F -schemes, so we have a factorization $X' \rightarrow X \rightarrow X'^{(p^n)}$ for some $n \geq 0$. It follows that $\mathcal{M}^{\otimes p^n}$ comes from the invertible \mathcal{O}_X -module $\mathcal{L} = \mathcal{M}^{(p^n)}|_{X'}$.

The general case follows from the special cases, because any finite field extension is given by a purely inseparable extension followed by a separable extension. \square

Proposition 8.7. *Let $f : X \rightarrow Y$ be a universal homeomorphism of proper F -schemes in characteristic $p > 0$. Then kernel and cokernel of the induced map $\mathrm{NS}(Y/F) \rightarrow \mathrm{NS}(X/F)$ are finite p -groups. In particular, we have $\rho(Y) = \rho(X)$.*

Proof. By [32], Proposition 6.6, we have a factorization $X \rightarrow Y \rightarrow X^{(p^n)}$ for some $n \geq 0$. Observe that the cartesian diagram

$$\begin{array}{ccc} X^{(p^n)} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathrm{Spec}(F) & \xrightarrow{\mathrm{Fr}^n} & \mathrm{Spec}(F). \end{array}$$

induces a bijection $\mathrm{NS}(X/F) \rightarrow \mathrm{NS}(X^{(p^n)}/F)$. This is because Fr^n is a universal homeomorphism, and sections for the étale scheme $\mathrm{NS}_{X/F}$ correspond to points $l \in \mathrm{NS}_{X/F}$ such that $\mathrm{Spec} \kappa(l) \rightarrow \mathrm{Spec}(F)$ is a universal homeomorphism ([18], Corollary 17.9.3). It follows that kernel and cokernel of $\mathrm{NS}(X^{(p^n)}/F) \rightarrow \mathrm{NS}(X/F)$ are p -groups. Hence the cokernel of $\mathrm{NS}(Y/F) \rightarrow \mathrm{NS}(X/F)$ is a p -group.

Finally, suppose $L \in \mathrm{NS}(Y/F)$ vanishes on X . As above, there is a factorization $Y \rightarrow X^{(p^n)} \rightarrow Y^{(p^m)}$ for some $m \geq 0$. Then $p^m L$ comes from a point $M \in \mathrm{NS}(X^{(p^n)}/F)$, and $p^n M$ vanishes on X . It follows that some p -power annihilates L . \square

Remark 8.8. The preceding result does not hold in characteristic zero. To see an example, set $X = \mathbb{P}_{\mathbb{C}}^2$, such that $\mathrm{Pic}(X) = \mathbb{Z}$. According to [27], Exercise 5.9 on page 232, there is a first order infinitesimal extension $X \subset Y$ with ideal $\mathcal{O}_X(-3) = \omega_X$ such that $\mathrm{Pic}(Y) = 0$.

9. PICARD NUMBERS IN FAMILIES AND HILBERT SETS

Let S be a scheme and $f : X \rightarrow S$ a proper morphism. How do the Picard numbers $\rho(X_s)$ vary in this family? The function $s \mapsto \rho(X_s)$ can be very nasty. To explore its nature we need the notion of Hilbert sets. Suppose for simplicity that S is integral and noetherian. Let T be an integral scheme and $T \rightarrow S$ an étale dominant morphism of finite type. The generic fiber T_η is an integral scheme given by a finite separable field extension of $\kappa(\eta)$. We define $H_T \subset S$ as the set of all points $s \in S$ where the fiber T_s is integral and nonempty. A subset $H \subset S$ is called a *separable Hilbert set* if there are finitely many $T_i \rightarrow S$, $1 \leq i \leq n$ as above with $H = H_{T_1} \cap \dots \cap H_{T_n}$. Clearly, open sets are separable Hilbert sets. Moreover, separable Hilbert subsets are stable under generization.

Usually, Hilbert sets are defined in terms of irreducible polynomials. They occur in Galois theory in connection with *Hilbert's Irreducibility Theorem*. The latter states, in its original form, that an irreducible polynomial $P(T, X) \in \mathbb{Q}[T, X]$ remains irreducible for infinitely many rational specializations $T = a/b$. Hilbert sets are studied in detail in [11], Chapter 11 and 12. Our definition follows Lang [34] and Serre [47], which is better suited for algebraic geometry. For example, Hilbert sets are closely related to Néron–Severi groups:

Proposition 9.1. *Let S be an integral noetherian scheme, $\eta \in S$ its generic point, and $f : X \rightarrow S$ a proper morphism. Then there are countably many nonempty separable Hilbert subsets $H_i \subset S$ such that $\rho(X_s) = \rho(X_\eta)$ for all $s \in \bigcap H_i$.*

Proof. Replacing S by some nonempty open subset, we may assume that the relative Picard functor $\text{Pic}_{X/S}$ is representable by a scheme, according to [25], Exposé XII, Corollary 1.2. After shrinking S further, we may also assume that there is a group scheme $J \rightarrow S$ of finite type, together with an open embedding $J \subset \text{Pic}_{X/S}$, such that $J_s = \text{Pic}_{X_s/s}^0$ for all $s \in S$. This holds by [20], Lemma 1.2. By generic flatness, we may also assume that $J \rightarrow S$ is flat.

Consider the group scheme $\text{NS}_{X/S}$. By Chow's Lemma, there is a projective surjective morphism $X' \rightarrow X$ such that the composite map $f' : X' \rightarrow S$ is projective. Shrink S so that $\text{NS}_{X'/S}$ is representable by a group scheme. Then $\text{NS}_{X'/S}$ contains only countably many irreducible component. Indeed, this follows from the existence of the Hilbert scheme (see [6], Chapter 8, Theorem 5). Using that $\text{Pic}_{X/S} \rightarrow \text{Pic}_{X'/S}$ is of finite type ([25], Exposé XIII, Theorem 3.5), we conclude that $\text{NS}_{X/S}$ has only countably many irreducible components as well.

Let $A_i \subset \text{NS}_{X/S}$ be such an irreducible component. If $A_i \rightarrow S$ is not dominant, choose a nonempty open subset $H_i \subset S$ disjoint from the image of A_i . If $A_i \rightarrow S$ is dominant, let $U_i \subset S$ be an open subset over which A_i becomes flat, hence étale, and let $H_i \subset U_i$ be the corresponding Hilbert subset.

Then $H = \bigcap H_i$ is the desired countable intersection of Hilbert subsets. Indeed, for each $s \in H$, the points $a \in \text{NS}(X_s/s)$ correspond to the points $b \in \text{NS}(X_\eta/\eta)$ via some dominant irreducible component A_i . Moreover, a is rational if and only if b is rational. In other words, we have constructed a bijective specialization map $\text{NS}(X_\eta/\eta) \rightarrow \text{NS}(X_s/s)$ and conclude $\rho(X_s) = \rho(X_\eta)$. \square

How do countable intersections of Hilbert sets look like? There seems to be no general answer. However, over certain ground fields we can say more;

Lemma 9.2. *Let k be an uncountable field, $k \subset F$ a finitely generated field extension of transcendence degree ≥ 1 , and S an integral F -scheme of finite type of dimension $n \geq 1$. Then any countable intersection $\bigcap H_i$ of nonempty separable Hilbert subsets $H_i \subset S$ contains uncountably many closed points.*

Proof. We may assume that $S = \text{Spec}(R)$ is affine. Let $R \subset Q$ be the field of fractions. Replacing F by its perfect closure in Q and shrinking S , we may assume that S is geometrically reduced. Clearly, we may assume that our separable Hilbert subsets $H_i \subset S$ are given by étale morphisms $T_i \rightarrow S$.

Next, we reduce the problem to a special case. Choose a separating transcendence basis $t_1, \dots, t_n \in Q$ over F . Shrinking S , we obtain a separable morphism $f : S \rightarrow \mathbb{A}_F^n$. Choose nonempty open subsets $T'_i \subset T_i$ so that the composition $T'_i \rightarrow \mathbb{A}_F^n$ is étale, and let $H'_i \subset \mathbb{A}_F^n$ be the corresponding separable Hilbert subsets. Then $f^{-1}(H'_i) \subset H_i$, and we see that it suffices to treat the special case $S = \mathbb{A}_F^n$.

We now proceed by induction on $n \geq 1$. First, suppose $S = \mathbb{A}_F^1$. According to [11], Theorem 12.9, there are nonempty open subsets $U_i \subset \mathbb{A}_k^2$ with

$$\{a + tb \mid a, b \in U_i(k)\} \subset H_i.$$

Since k is uncountable, the countable intersection $\bigcap U_i$ contains uncountably many rational points (choose a line not contained in any $\mathbb{A}_k^2 - U_i$). Consequently $\bigcap H_i$ contains uncountably many rational points as well.

Now suppose the result is already true for $n \geq 1$. Let F' be the function field of \mathbb{A}_F^n , and consider the projection $\mathbb{A}_F^{n+1} \rightarrow \mathbb{A}_F^n$. Its generic fiber is isomorphic to the affine line $\mathbb{A}_{F'}^1$. Set $H = \bigcap H_i$. We just saw that the intersection $H \cap \mathbb{A}_{F'}^1$ contains a rational point $z \in \mathbb{A}_{F'}^1$. Its closure $S' \subset \mathbb{A}_F^{n+1}$ is n -dimensional. Applying our preliminary reduction and the induction hypothesis, we conclude that $H = \bigcap H_i$ contains uncountably many closed points. \square

Remark 9.3. Under the additional hypothesis that S is geometrically reduced, the preceding proof shows that there are uncountably many closed points $s \in \bigcap H_i$ whose residue field extension $F \subset \kappa(s)$ are separable.

Summing up, we obtain the following result about families of Néron–Severi groups:

Theorem 9.4. *Let k be an uncountable field, $k \subset F$ a nonalgebraic finitely generated field extension, S an integral F -scheme of finite type of dimension at least one, and $f : X \rightarrow S$ a proper morphism. Then there are uncountably many closed points $s \in S$ with $\rho(X_s) = \rho(X_{\bar{\eta}})$.*

Proof. Combine Proposition 9.1 and Lemma 9.2. \square

The situation simplifies if we look at Picard numbers of geometric fibers $X_{\bar{s}} = X \otimes \bar{\kappa}(s)$ instead of schematic fibers X_s . More precisely:

Theorem 9.5. *Let S be an integral noetherian scheme and $f : X \rightarrow S$ a proper morphism. Then there are countably many nonempty open subsets $U_i \subset S$ such that $\rho(X_{\bar{s}}) = \rho(X_{\bar{\eta}})$ for all $s \in \bigcap U_i$.*

Proof. As in the proof for Proposition 9.1, we reduce to the case that $\text{Pic}_{X/S}$ and $J \subset \text{Pic}_{X/S}$ are representable by group schemes, and set $\text{NS}_{X/S} = \text{Pic}_{X/S}/J$. For each irreducible component $A_i \subset \text{NS}_{X/S}$, choose a nonempty open subset $U_i \subset S$ over which A_i is finite and flat. Then $\bigcap U_i$ is the desired countable intersection of open subsets.

To see this, fix a point $s \in \bigcap U_i$. Let $R = \mathcal{O}_{S,s}$ be the corresponding local ring, $F = \kappa(s)$ its residue field, $F \subset F^{\text{sep}}$ a separable closure, and $R \subset R^{\text{sh}}$ be the corresponding strict henselization. Then $A_i \otimes R$ is étale over R , and $A_i \otimes R^{\text{sh}}$ decomposes into disjoint sections by [18], Proposition 18.5.19. We conclude that $\text{NS}(X \otimes F^{\text{sep}}/F^{\text{sep}})$ is canonically isomorphic to $\text{NS}(X \otimes Q/Q)$, where Q is the function field of R^{sh} . The canonical mappings $\text{NS}(X \otimes F^{\text{sep}}/F^{\text{sep}}) \rightarrow \text{NS}(X \otimes \bar{F})$ and $\text{NS}(X \otimes Q/Q) \rightarrow \text{NS}(X \otimes \bar{Q})$ are obviously bijective, so $\rho(X_{\bar{s}}) = \rho(X_{\bar{\eta}})$. \square

10. FIBERED SURFACES WITH SMALL PICARD NUMBER

Fix an uncountable algebraically closed ground field k . The task now is to construct fibered surfaces with small Picard number defined over small transcendental extension fields $k \subset F$. The Tate–Shioda formula then implies that their generic

fibers have small Picard group as well. The upshot is that there is a dense set of points $x \in M_{g,n}$ with $\dim \overline{\{x\}} \leq 2$ such that the marked points and the canonical class of the corresponding curve C_x generate $\text{Pic}(C_x) \otimes \mathbb{Q}$.

We start with some notation. A *surface* over F is a proper 2-dimensional F -scheme Y with $\Gamma(Y, \mathcal{O}_Y) = F$. A *fibration* on a surface Y is curve B together with a proper morphism $f : Y \rightarrow B$ satisfying $\mathcal{O}_B = f_*(\mathcal{O}_Y)$. The *Tate–Shioda formula* relates the Picard number $\rho(Y)$ of a fibered surface with the rank of the Mordell–Weil group of the generic fiber (confer [50] and [48]):

Proposition 10.1. *Let Y be a normal surface whose singularities are \mathbb{Q} -factorial, and $f : Y \rightarrow B$ be a fibration. Then we have*

$$\rho(Y) - \text{rank Pic}^0(Y_\eta) / \text{Pic}^0(Y) = 2 + \sum_{b \in B} (\rho(Y_b) - 1).$$

Proof. The intersection form is negative semidefinite on the group of Weil divisors supported by a given fiber $f^{-1}(b) \subset Y$, and the fiber $f^{-1}(b)$ generates the radical over \mathbb{Q} . Consequently, the vertical Weil divisors generate a subgroup of rank $1 + \sum_{b \in B} (\rho(Y_b) - 1)$ inside the Néron–Severi group.

Next, choose a horizontal curve $H \subset Y$. Given any Weil divisor C , we may subtract a suitable rational multiple of H until C_η has degree zero. Since $\text{Pic}(Y) \rightarrow \text{Pic}(Y_\eta)$ has finite cokernel, we conclude that the horizontal Weil divisors generate a subgroup of rank $1 + \text{rank Pic}^0(Y_\eta) / \text{Pic}^0(Y)$ inside the Néron–Severi group. This easily implies the formula. \square

This formula often allows us to control the rank of the Jacobian $\text{Pic}^0(Y_\eta)$ in terms of the Picard number $\rho(Y)$.

Now fix an integer $n \geq 0$ and a genus $g \geq 2$, and let X be an n -pointed stable curve of genus g over k . For simplicity, we also assume $\text{Aut}(X) = 0$. Then the closed point $x \in \overline{M}_{g,n}$ corresponding to X lies in the smooth locus [35]. As in the proof of Proposition 3.1, we blow up the center $x \in \overline{M}_{g,n}$ and localize at the generic point of the resulting exceptional divisor. This produces a discrete valuation ring A whose residue field $F = A/\mathfrak{m}_A$ is a purely transcendental field extension of degree $3g - 4 + n$, and whose field of fractions $A \subset Q$ is the function field of the moduli space $\overline{M}_{g,n}$.

Write $F = k(t_i)$ for some transcendence basis $t_i \in F$, and choose lifts $t_i \in A$. We obtain an inclusion $k[t_i] \subset A$, and in turn a lift $F \subset A$. Then $F \subset Q$ is a finitely generated field extension of transcendence degree one, which corresponds to a proper normal curve B over F . The discrete valuation ring $A \subset Q$ defines a rational point $b_0 \in B$, hence $F = \Gamma(B, \mathcal{O}_B)$, and B is geometrically integral. The tautological curve $C \rightarrow \overline{M}_{g,n}$ defines a fibered regular surface $f : Y \rightarrow B$. By construction, we have $f^{-1}(b_0) = X \otimes F$ and $f^{-1}(\eta) = C_\eta$. To apply the Tate–Shioda formula, we have to ensure the following:

Lemma 10.2. *Suppose the normalization \tilde{X} of X has only rational components. Then the canonical map $\text{Pic}^0(Y) \rightarrow \text{Pic}^0(Y_\eta)$ vanishes.*

Proof. Let $F \subset \bar{F}$ be an algebraic closure. The normalization \tilde{Y} of $\bar{Y} = Y \otimes \bar{F}$ is geometrically normal, so $\text{Pic}_{\tilde{Y}/\bar{F}}^0$ is proper by [20], Theorem 2.1. Moreover, it contains a unique abelian subscheme with the same underlying topological space by [20], Corollary 3.2. Its dual abelian variety is the Albanese variety $\text{Alb}_{\tilde{Y}/\bar{F}}$. The

resulting morphism $\tilde{Y} \rightarrow \text{Alb}_{\tilde{Y}/\bar{F}}$ factors over \tilde{B} , because the normalization of X has genus zero.

According to [25], Exposé XII, Corollary 1.5, the map $\text{Pic}_{\tilde{Y}/\bar{F}}^0 \rightarrow \text{Pic}_{Y/\bar{F}}^0$ is affine. It follows that the cokernel of $\text{Pic}_{B/F}^0 \rightarrow \text{Pic}_{Y/F}^0$ is affine. Using that $\text{Pic}_{Y_\eta/\eta}^0$ is abelian, we infer that the map $\text{Pic}_{Y/F}^0 \rightarrow \text{Pic}_{Y_\eta/\eta}^0$ factors over the origin, hence $\text{Pic}^0(Y) \rightarrow \text{Pic}^0(Y_\eta)$ vanishes. \square

From now on, we shall assume that the normalization of X has only rational components. Then Theorem 5.1, together with the Tate–Shioda formula gives

$$\rho(Y) - (n + 1) = 2 + \sum_{b \in B} (\rho(Y_b) - 1).$$

We now use this formula to specialize Y and maintain control over $\text{Pic}(Y_\eta)$. Recall that $t_i \in F$, $1 \leq i \leq 3g - 4 + n$ is a transcendence basis over k . Consider the rational function field $L = k(t_1)$. Then we may view $L \subset F$ as the function field of \mathbb{P}_L^{3g-5+n} . Our F -schemes X and B extend to proper flat morphisms $\mathfrak{Y} \rightarrow S$ and $\mathfrak{B} \rightarrow S$, respectively, over some open subsets $S \subset \mathbb{P}_L^{3g-5+n}$. Shrinking S , we also have a morphism $f : \mathfrak{Y} \rightarrow \mathfrak{B}$ such that $\mathcal{O}_{\mathfrak{B}} \rightarrow f_*(\mathcal{O}_{\mathfrak{Y}})$ is bijective, and remains bijective after any base change. Let $\mathfrak{Y}_{\eta_s} \subset \mathfrak{Y}_s$ be the generic fiber of the induced fibration $\mathfrak{Y}_s \rightarrow \mathfrak{B}_s$.

Proposition 10.3. *There is an uncountable dense subset of closed points $s \in S$ such that \mathfrak{Y}_{η_s} is a stable n -pointed curve of genus g , and that the marked points and the canonical class generate $\text{Pic}(\mathfrak{Y}_{\eta_s}) \otimes \mathbb{Q}$.*

Proof. Replacing S by some nonempty open subset, we may assume that the following holds: First, the fibers of $\mathfrak{Y} \rightarrow S$ and $\mathfrak{B} \rightarrow S$ are geometrically integral by [17], Theorem 9.7.7, and geometrically unibranch by Proposition 8.3. Second, the maps $\text{Pic}_{\mathfrak{Y}_s/s}^0 \rightarrow \text{Pic}_{\mathfrak{Y}_b/b}^0$ factor over the origin for almost all closed points $b \in \mathfrak{B}_s$. It then follows that $\text{Pic}^0(\mathfrak{Y}_s) \rightarrow \text{Pic}^0(\mathfrak{Y}_{\eta_s})$ vanishes. Third, the generic fibers \mathfrak{Y}_{η_s} are n -pointed smooth curves of genus g .

By Theorem 9.4, there are uncountably many closed points $s \in S$ with $\rho(\mathfrak{Y}_s) = \rho(Y)$. Let $\tilde{\mathfrak{Y}}_s \rightarrow \mathfrak{Y}_s$ be the normalization and $\tilde{\mathfrak{Y}}_s \rightarrow \tilde{\mathfrak{Y}}_s$ a resolution of singularities. Then $\tilde{\mathfrak{Y}}_s$ is a fibered surface with generic fiber $\tilde{\mathfrak{Y}}_{\eta_s} = \mathfrak{Y}_{\eta_s}$. We have $\rho(\tilde{\mathfrak{Y}}_s) = \rho(\mathfrak{Y}_s)$ by Proposition 8.7. The Picard number $\rho(\tilde{\mathfrak{Y}}_s)$ of the desingularization is usually larger. However, any additional classes lie in the fibers of $\tilde{\mathfrak{Y}}_s \rightarrow \mathfrak{B}_s$. The Tate–Shioda formula therefore implies that $\text{Pic}(Y_\eta)$ and $\text{Pic}(\mathfrak{Y}_{\eta_s})$ have the same rank. Recall that Y_η is the generic n -pointed curve of genus g . Using Theorem 5.1, we infer that the canonical class and the marked points generate $\text{Pic}(\mathfrak{Y}_{\eta_s}) \otimes \mathbb{Q}$. \square

We come to the second main result of this paper:

Theorem 10.4. *Let k be an uncountable algebraically closed field. Then there is an uncountable dense set of points $x \in M_{g,n}$ with $\dim \overline{\{x\}} \leq 2$ such that the canonical class and the marked sections generate $\text{Pic}(C_x) \otimes \mathbb{Q}$, where C_x is the curve corresponding to the point $x \in M_{g,n}$.*

Proof. Consider the family $\mathfrak{Y} \rightarrow \mathfrak{B}$ of fibered surfaces over S constructed in Proposition 10.3. We then have a rational map $\mathfrak{B} \dashrightarrow M_{g,n} \otimes L$ whose image is a divisor, because the composition $\mathfrak{B} \dashrightarrow M_{g,n}$ is dominant. After shrinking S , we find an

open subset $\mathfrak{U} \subset \mathfrak{B}$ whose fibers over S are nonempty, such that $\mathfrak{U} \rightarrow M_{g,n} \otimes L$ is everywhere defined and quasifinite. This is because $\dim(\mathfrak{B}) = \dim(M_{g,n}) - 1$.

Consider the uncountable dense set of points $s \in S$ from Proposition 10.3, such that $\text{Pic}(\mathfrak{X}_{\eta_s}) \otimes \mathbb{Q}$ is generated by the canonical class and the marked sections. Let $x \in M_{g,n}$ be the images of the points $\eta_s \in \mathfrak{B}$. By construction, the residue fields $\kappa(x)$ have transcendence degree ≤ 2 over k . Hence the $x \in M_{g,n}$ constitute the desired uncountable dense subset. \square

11. OPEN PROBLEMS

We close the paper by listing some open problems:

(1) Theorem 10.4 states that the marked points and the canonical class generate $\text{Pic}(C_x)$. If $l \in \text{Pic}(C_x/x)$ is a rational point, what denominators are necessary to write l as a linear combination of the canonical class and the marked points? Is it sufficient to allow powers of the characteristic exponent $p \geq 1$ as denominators?

(2) Is it possible to choose a dense set of nonclosed points $x \in M_{g,n}$ as in Theorem 10.4 so that all closures $\overline{\{x\}}$ are 1-dimensional? This seems to rely on improved versions of Hilbert's Irreducibility Theorem for prime fields.

(3) Can we say more about the structure of countable intersections of nonempty Hilbert sets in Theorem 9.4? Obviously, we cannot expect such sets to have a reasonable algebraic structure. Is it possible to write such sets as a countable union of sets with some sort of algebraic structure?

(4) Does the Strong Franchetta Conjecture generalize to other moduli problems, and if so, in what form? What about polarized abelian varieties, or surfaces of general type, or canonically polarized varieties of higher dimensions? For example, Silverberg [49] showed that generic complex abelian varieties have finite Mordell–Weil groups.

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