WILDLY RAMIFIED ACTIONS AND SURFACES OF GENERAL TYPE ARISING FROM ARTIN–SCHREIER CURVES

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Abstract. We analyse the diagonal quotient for the product of certain Artin–Schreier curves. The smooth models are almost always surfaces of general type, with Chern slopes tending asymptotically to 1. The calculation of numerical invariants relies on a close examination of the relevant wild quotient singularity in characteristic p. It turns out that the canonical model has $q-1$ rational double points of type A_{q-1}, and embeds as a divisor of degree q in $\mathbb{P}^3$, which is in some sense reminiscent of the classical Kummer quartic.

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Introduction

It is a classical fact in complex geometry that the singular Kummer surface $A/\{\pm 1\}$ attached to an abelian surface A has sixteen rational double points, and an irreducible principal polarization embeds it as a quartic surface in $\mathbb{P}^3$, compare Hudson's classical monography [18], or for a modern account [12]. Our starting point was an analogous computation in characteristic $p=3$ for the diagonal action of the additive group $G=\mathbb{F}_3$ on the selfproduct $A=E\times E$, where $E: y^2=x^3+x$ is the supersingular elliptic curve, viewed as an Artin–Schreier covering. It turns out that this is a special case of a rather general construction, which works for all primes p, in fact for all prime powers $q=p^s$. It starts with certain *Artin–Schreier curves* and leads, with a few exceptions for small prime powers, to *surfaces of general type*.

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The goal of this paper is to describe the geometry of these surfaces, and we obtain a fairly complete description. The construction goes as follows: Fix an algebraically closed ground field k of characteristic $p > 0$ and consider Artin–Schreier curves of the form

$$C: \quad f(y) = x^q - x,$$

where f is a monic polynomial of degree $\deg(f) = q - 1$. These curves carry a translation action of the additive group $G = \mathbb{F}_q \simeq (\mathbb{Z}/p\mathbb{Z})^{\oplus s}$, and we consider the diagonal action on the product $C \times C'$ of two such Artin–Schreier curves. The quotient $(C \times C')/G$ is a normal surface containing a unique singularity.

Such singularities are examples of *wild quotient singularities*, i.e., the characteristic of the ground field divides the order of the group G. Few examples of wild quotient singularities occur in the literature, and little is known in general. Artin [4] gave a complete classification for wild $\mathbb{Z}/2\mathbb{Z}$-quotient singularities in dimension two, and general $\mathbb{Z}/p\mathbb{Z}$-quotient singularities were studied further by Peskin [25]. Peculiar properties of wild S_n-quotient singularities in relation to punctual Hilbert schemes appear in [28]. In light of the scarcity of examples, it is useful to have more classes of wild quotient singularities in which computations are feasible.

Lorenzini initiated a general investigation of wild quotient singularities on surfaces [21], which play an important role in understanding the reduction behaviour of curves over discrete valuation fields. He compiled a list of open questions [22]. In a recent paper, Lorenzini studied wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularities resulting from diagonal actions on products of $C \times C'$, where one or both factors are ordinary curves [23]. In some sense, we treat the opposite situation, as Artin-Schreier curves have vanishing p-rank, and our main concern is the interplay between the local structure coming from the the wild quotient singuarity and the global geometry of the algebraic surface. We have choosen Artin–Schreier coverings that are concrete enough so that explicit computations are possible. Note that our set-up includes wild quotient singularities with respect to elementary abelian groups, and not ony cyclic groups.

Consider the minimal resolution of singularities

$$X \longrightarrow (C \times C')/G$$

of our normal surface. According to Lorenzini's general observation, the exceptional divisor of a wild quotient singularity in dimension two consists of projective lines, and its dual graph is a tree [21]. Using an explicit formal equation for the singularitiy, we show that the dual graph is even star-shaped, with $q + 1$ terminal chains attached to the central node, each of length q, as depicted in Figure 1. The two basic numerical invariants of surface singularities are the genus $p_f = h^1(\mathcal{O}_Z)$ of the fundamental cycle $Z \subset X$ and the *geometric genus* $p_g = h^1(\mathcal{O}_{nZ})$, $n \gg 0$. The latter is usually very difficult to compute. We obtain

$$p_f = (q-1)(q-2)/2 \quad \text{and} \quad p_g = q(q-1)(q-2)/6,$$

the latter under the assumption that our prime power q is prime. This relies on a computation of the global l-adic Euler characteristic of the surface, combined with a determination of the G-invariant part in $H^0(C \times C', \Omega^1_{C \times C'})$, which in turn depends on a problem in modular representation theory related to tensor products of Jordan matrices. In contrast, the singularities occuring in [23], where at least one factor in $C \times C'$ is an ordinary curve, are all rational.

Having a good hold on the structure of the resolution of singularities, we determine the global invariants of the smooth surface X. It turns out that $H^1(X, \mathcal{O}_X) = 0$, and that $\mathrm{Pic}(X)$ is a free abelian group of finite rank. Moreover, the algebraic fundamental group $\pi_1(X)$ vanishes. Passing to the minimal model

$$X \longrightarrow S,$$

we show that the minimal surface S is of general type for $q \geq 5$, a K3 surface for $q = 4$, and a weak del Pezzo surface for $q = 2, 3$. Their Chern invariants are given by the formula

$$c_1^2 = q^3 - 8q^2 + 16q \qquad \text{and} \qquad c_2 = q^3 - 4q^2 + 6q \qquad\qquad (q \geq 4).$$

The resulting *Chern slopes* asymptotically tend to $\lim_{q\to\infty} c_1^2/c_2 = 1$, and one may say that our surfaces show no pathological behaviour with respect to surface geography. The determination of the Euler characteristic $c_2 = e$ depends on Dolgachev's formula

$$e(X) = e(X_{\bar{\eta}})e(B) + \sum (e(X_a) - e(X_{\bar{\eta}}) + \delta_a)$$

for l-adic Euler characteristic for schemes fibered over curves [9], where δ_a is *Serre's measure of wild ramification*. Its computation is quite easy in our situation, given the explicit nature of the Artin–Schreier curves.

The surface S has a surprisingly simple *projective description*, which is reminiscent to Kummer's quartic surfaces $A/\{\pm 1\} \subset \mathbb{P}^3$. Passing to the normal surface $\bar{S}$ with $q-1$ rational double points of type A_{q-1} obtained by contracting all terminal chains in the fundamental cycle, the image $\bar{Z} \subset \bar{S}$ of the fundamental cycle remains Cartier, and defines the embedding:

Theorem. *The invertible sheaf $\bar{\mathcal{L}} = \mathcal{O}_{\bar{S}}(\bar{Z})$ is very ample, has $h^0(\bar{\mathcal{L}}) = 4$ and embeds the normal surface $\bar{S}$ as a divisor of degree q in $\mathbb{P}^3$, sending the rational double points into a line.*

In fact, a canonical divisor is $K_{\bar{S}} = (q-4)\bar{Z}$. So for $q \geq 5$, the closed embedding $\Phi_{\bar{Z}} : \bar{S} \to \mathbb{P}^3$ can be viewed as an *m-canonical map*, for the fractional value $m = 1/(q-4)$. In the simplest case $q = 5$, this is in line with classification results of Horikawa on minimal surfaces of general type with $K^2 = 5$ and vanishing irregularity [17]. The projective description also allows us to deduce that our surfaces S admit a lifting into characteristic zero, at least in the category of algebraic spaces. It would be interesting to determine the homogeneous polynomial describing the image $\bar{S} \subset \mathbb{P}^3$, but we have made no attempt to do so.

The paper is organized as follows: In Section 1 we review some relevant facts on Artin–Schreier curve, all of them well-known. In Section 2 we study the normal surface $(C \times C')/G$, obtained as the quotient of the product of two Artin–Schreier curves with respect to the diagonal action. We find an explicit equation for the singularitiy, and a dimension formula for the global sections of the dualizing sheaf. Section 3 contains an analysis of the minimal resolution of singularities $X \to (C \times C')/G$. Notable results are formulas for the fundamental cycle and its genus, as well as some bounds on the arithmetic genus. In Section 4 we prove that $H^1(X, \mathcal{O}_X) = 0$, such that the Picard scheme is reduced and 0-dimensional. This relies on a general fact about group actions with fixed points, which seems to be of independent interest, and is verified with Grothendieck's theory of G-equivariant cohomology. In Section 5 we determine the place of the smooth surface X in the Enriques classification. Among other things, this depends on the geometry of the fibration $X \to \mathbb{P}^1$ induced from the projections on $C \times C'$. Section 6 contains our analysis of projective models for the surfaces. Finally, in Section 7 we take up questions from surface geography and compute Chern invariants. This mainly relies on Dolgachev's formula for l-adic Euler characteristics for fibered schemes.

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1. Artin–Schreier curves

Let $p > 0$ be a prime number and k be an algebraically closed ground field of characteristic p. Consider *Artin–Schreier curves* of the form

$$C: \quad f(y) = x^q - x,$$

where the left side of the defining equation is a monic polynomial $f(y) = y^{q-1} + \mu_2 y^{q-2} + \ldots + \mu_q$ of degree $q-1$ with coefficients from the ground field k. In other words, $C \subset \mathbb{P}^2$ is defined by the homogeneous equation

$$Y^{q-1}Z + \mu_2 Y^{q-2} Z^2 + \ldots + \mu_q Z^q = X^q - XZ^{q-1} \tag{1}$$

of degree $\deg(C) = q$ inside the projective plane $\mathbb{P}^2 = \operatorname{Proj}(k[X,Y,Z])$. Homogeneous and inhomogeneous coordinates are related by $x = X/Z$ and $y = Y/Z$. The curve C is smooth, with numerical invariants

$$h^0(\mathcal{O}_C) = 1 \quad \text{and} \quad h^1(\mathcal{O}_C) = (q-1)(q-2)/2 \quad \text{and} \quad \deg(K_C) = q(q-3). \tag{2}$$

Now let $G = \mathbb{F}_q \subset k$ be the additive group of all scalars λ satisfying $\lambda^q = \lambda$, viewed as an elementary abelian p-group. We may also regard it as a subgroup

$$G = \left\{ \begin{pmatrix} 1 & \\ \lambda & 1 \end{pmatrix} \mid \lambda \in \mathbb{F}_q \right\} \subset \mathrm{GL}(2,k).$$

Thus the elements $\lambda \in G$ act on $\mathbb{P}^2$ via $X \mapsto X + \lambda Z, \quad Y \mapsto Y, \quad Z \mapsto Z$. This action leaves the homogeneous equation (1) invariant, whence induces an action on C. This action is free, except for a single fixed point $a = (0:1:0) \in C$. Dehomogenizing in another way by setting $u = X/Y$, $w = Z/Y$, we see that an open neighborhood of the fixed point is the spectrum of the coordinate ring

$$k[u,w]/(u^q - uw^{q-1} - P(w)),$$

where $P(w) = f(1/w)w^q = w + \mu_2 w^2 + \ldots + \mu_q w^q$, and the group elements $\lambda \in G$ act via $u \mapsto u + \lambda w$, $w \mapsto w$. Since

$$\frac{\partial}{\partial w}(u^q - uw^{q-1} - P(w)) = uw^{q-2} - (1 + 2\mu_2 w + \ldots + (-\mu_{q-1} w^{q-2}))$$

becomes a unit in the local ring $\mathcal{O}_{C,a}$, there is a unique way to write the indeterminate w as a formal power series $w(u) = \sum \alpha_i u^i$ in the variable u so that $u^q - uw(u)^{q-1} - P(w(u)) = 0$ ([8], §4, No. 7, Corollary to Proposition 10). Using the latter condition, one easily infers that the initial coefficients are

$$\alpha_0 = \ldots = \alpha_{q-1} = 0 \quad \text{and} \quad \alpha_q = 1. \tag{3}$$

The upshot is that the inclusion

$$k[[u]] \subset k[[u,w]]/(u^q - uw^{q-1} - P(w)) = \mathcal{O}^\wedge_{C,a}$$

is bijective, and the group elements $\lambda \in G$ act on the formal completion $k[[u]] = \mathcal{O}^\wedge_{C,a}$ via

$$u \longmapsto u + \lambda u^q + \text{higher order terms.} \tag{4}$$

From this we infer that the filtration given by the *higher ramification subgroups* $G = G_0 \supset G_1 \supset G_2 \supset \ldots$ takes the simple form

$$G_i = \begin{cases} G & \text{if } i \leq q-1; \\ 0 & \text{if } i \geq q. \end{cases} \tag{5}$$

Recall that $G_i \subset G$ is defined as the *decomposition group* of the i-th infinitesimal neighborhood of the closed point, that is, the subgroup of those $\sigma \in G$ with the property $\sigma(u) - u \in \mathfrak{m}_a^{i+1}$. The corresponding function i_G on G is given by

$$i_G(\sigma) = \begin{cases} q & \text{if } \sigma \neq 0; \\ \infty & \text{if } \sigma = 0. \end{cases}$$

We refer to Serre's monograph [30] for the theory of higher ramification groups. These groups will play a crucial role in Section 7 in the determination of Euler characteristics. The Hurwitz Formula for the quotient map $C \to C/G$ of degree q, in the form of [30], Chapter VI, Proposition 7, gives

$$2 - (q-1)(q-2) = 2 - 2g_C = q(2 - 2g_{C/G}) - a_G(0) = q(2 - 2g_{C/G}) - (q-1)q,$$

where g_C, $g_{C/G}$ denotes genus, and a_G is the character of the Artin representation attached to the fixed point, which by definition has $a_G(0) = \sum_{\sigma \neq 0} i_G(\sigma)$. It follows that $g_{C/G} = 0$, whence $C/G = \mathbb{P}^1$. In light of the defining equations, this was of course clear from the very beginning: the quotient map $C \to \mathbb{P}^1$ is a classical Artin–Schreier covering of the projective line. Using that the quotient map is étale away from and totally ramified at the fixed point, one deduces:

Proposition 1.1. *A canonical divisor for the curve C is given by $K_C = q(q-3)a$, where $a \in C$ is the fixed point for the G-action.*

It is not difficult to give an explicit basis for the vector space of global 1-forms on C:

Proposition 1.2. *The rational differentials $x^i y^j \, dy$, $0 \leq i+j \leq q-3$ are everywhere defined and constitute a basis for $H^0(C, \Omega^1_C)$.*

Proof. Consider the two coordinate rings

$$R = k[x,y]/(x^q - x - y^q P(1/y)) \quad \text{and} \quad R' = k[u,w]/(u^q - uw^{q-1} - P(w))$$

for our curve C. The relation $dx = (...)dy$ reveals that Ω^1_R is freely generated by dy. Similarly, the relation $0 = (w^{q-2}u - P'(w))dw - w^{q-1}du$ and $P'(0) = 1$ shows that $\Omega^1_{R'}$ is freely generated by du, locally at the point $u = w = 0$. Given a polynomial $f(x,y)$, we thus express the differential $f(x,y)dy$ in terms of u, w, using $y = Y/Z = 1/w$ and $x = X/Z = u/w$:

$$f(x,y)dy = -f(u/w, 1/w)w^{-2}dw = \frac{1}{P'(w) - w^{q-2}u} w^{q-3} f(u/w, 1/w)du$$

Hence the rational differential $f(x,y)dy$ is everywhere defined if $w^{q-3}f(u/w,1/w)$, which lies in the field of fractions for R', actually lies in R'. This indeed holds for the monomials $f(x,y)=x^iy^j$, provided $0\leq i+j\leq q-3$, by (3). The resulting $(q-1)(q-2)/2$ elements $x^iy^jdy\in H^0(C,\Omega^1_C)$ are clearly linearly independent over k. They must constitute a basis, because the vector space in question is of dimension $(q-1)(q-2)/2$. □

The induced action of the $\lambda\in G$ on $H^0(C,\Omega^1_C)$ is $x^iy^jdy\mapsto(x+\lambda)^iy^jdy$. Whence $H^0(C,\Omega^1_C)$ is the direct sum of the G-invariant subspaces

$$V_j\subset H^0(C,\Omega^1_C),\quad 0\leq j\leq q-3$$

that are generated by x^iy^jdy, $0\leq i\leq q-3-j$. Obviously, $\dim(V_j)=q-2-j$.

Proposition 1.3. *The G-invariant subspaces $V_j\subset H^0(C,\Omega^1_C)$ are indecomposable as G-modules, and the fixed spaces $V_j^G\subset V_j$ are 1-dimensional.*

Proof. We first check that the fixed space $V_j^G\subset V_j$ are 1-dimensional. Clearly, y^jdy is invariant. Seeking a contradiction, we suppose there is a monic polynomial $f(x)$ with $q-3-j\geq\deg(f)>1$ so that the differential $f(x)y^jdy$ is invariant. Factoring $f(x)=\prod(x-\omega_i)$, we see that the set of roots $\{\omega_1,\ldots,\omega_d\}$ is invariant under the substitution $\omega\mapsto\omega+\lambda$, $\lambda\in G$. Hence $\deg(f)\geq q$, contradiction.

Suppose V_j is decomposable, such that we have decomposition $V_j=V_j'\oplus V_j''$ into nonzero G-invariant subspaces. Since G is commutative, there is a basis of V_j' in which all $\lambda\in G$ act via lower triangular matrices (see [8], Chapter VII, §5, No. 9 Proposition 19). The last member x of such a basis is then a common eigenvector for all $\lambda\in G$. Since each $\lambda\in G$ has order p, all eigenvalues are $\epsilon=1$, whence x is G-fixed. The same applies to V_j'', giving a contradiction to $\dim(V_j^G)=1$. □

Note this implies that the fixed space $H^0(C,\Omega^1_C)^G$ is of dimension $q-2$.

2. Products of Artin–Schreier curves

Now choose a second Artin–Schreier curve C' of the form discussed in the previous section, and consider the product $C\times C'$, endowed with the diagonal G-action. In this section we start to study the quotient $(C\times C')/G$, which is a normal surface whose singular locus consists of one point $s\in(C\times C')/G$, the image of the fixed point (a,a'). The projections $\mathrm{pr}_1: C\times C'\to C$ and $\mathrm{pr}_2:C\times C'\to C'$ induce fibrations

$$\varphi_1:(C\times C')/G\to C/G=\mathbb{P}^1\quad\text{and}\quad\varphi_2:(C\times C')/G\to C'/G=\mathbb{P}^1,$$

respectively. Choose coordinates on the copies of projective lines so that the fixed points $a\in C$ and $a'\in C'$ map to the origin $0\in\mathbb{P}^1$, and consider the fibers $\varphi_1^{-1}(0),\varphi_2^{-1}(0)\subset(C\times C')/G$. Recall that the *multiplicity of a fiber* is the greatest common divisor for the multiplicities of its integral components.

Proposition 2.1. *The fibers $\varphi_i^{-1}(0)\subset(C\times C')/G$ have multiplicity q, and a canonical divisor is given by $K_{(C\times C')/G}=(q-3)(\varphi_1^{-1}(0)+\varphi_2^{-1}(0))$. Its selfintersection number is $K^2_{(C\times C')/G}=2q(q-3)^2$.*

Proof. The fiber $\varphi_i^{-1}(0)$ is clearly irreducible. According to (4), the G-action on the preimage of $0\in\mathbb{P}^1=C/G$ in C is trivial. Whence outside the singularity, the fiber $\varphi_i^{-1}(0)$ is the quotient of $C\otimes_k\mathcal{O}_{C,a}/\mathfrak{m}^q$, where the action on the right factor is trivial, which is an Artin ring of length q. It follows that the fiber has multiplicity q.

As to the canonical divisor, we have $K_{C\times C'} = q(q-3)(\mathrm{pr}_1^{-1}(a) + \mathrm{pr}_2^{-1}(a'))$ by Proposition 1.1. This is the preimage of $(q-3)(\varphi_1^{-1}(0) + \varphi_2^{-1}(0))$. Using that the projection $C \times C' \to (C\times C')/G$ is étale in codimension one, together with [26], Theorem 2.7, we infer that $(q-3)(\varphi_1^{-1}(0) + \varphi_2^{-1}(0))$ is indeed a canonical divisor. Its selfintersection number is

$$K^2_{(C\times C')/G} = \frac{1}{q} K^2_{C\times C'} = 2q(q-3)^2,$$

by the Projection Formula. □

The canonical divisor $K_{(C\times C')/G} = (q-3)(\varphi_1^{-1}(0) + \varphi_2^{-1}(0))$ is clearly Cartier, such that the normal surface $(C\times C')/G$ is Gorenstein. Let us now examine its singularity. Throughout the paper, it will be crucial to understand the local invariants of this singularity, in order to determine global invariants for smooth models of $(C\times C')/G$. Let u, u', w, w' be four variables and set

$$A = k[[u,u']] = k[[u,w,u',w']]/(u^q - uw^{q-1} - P(w), u'^q - u'w'^{q-1} - Q(w')),$$

which is the complete local ring at the fixed point $(a,a') \in C \times C'$. Here

$$P(w) = w + \mu_2 w^2 + \ldots + \mu_q w^q \quad \text{and} \quad Q(w') = w' + \mu_2' w'^2 + \ldots + \mu_q' w'^q$$

are polynomials stemming from the left hand side of the Artin-Schreier equations, as discussed in Section 1. The group elements $\lambda \in G$ act via

$$u \longmapsto u + \lambda w, \quad u' \longmapsto u' + \lambda w', \quad w \longmapsto w, \quad w' \longmapsto w'.$$

Clearly, the elements $w, w', wu' - w'u \in A$ are invariant and satisfy the relation

$$(wu' - w'u)^q = (ww')^{q-1}(wu' - w'u) + w^q Q(w') - w'^q P(w).$$

Therefore, we obtain a homomorphism of k-algebras

$$k[[a,b,c]]/(c^q - (ab)^{q-1}c - a^q Q(b) + b^q P(a)) \longrightarrow A^G, \tag{6}$$

where a, b, c are indeterminates and $a \mapsto w$, $b \mapsto w'$, $c \mapsto wu' - w'u$.

Proposition 2.2. *The preceding homomorphism (6) is bijective.*

Proof. Clearly, the local ring $R = k[[a,b,c]]/(c^q - (ab)^{q-1}c - a^q Q(b) + b^q P(a))$ is 2-dimensional and Cohen–Macaulay. Computing the jacobian ideal, one sees that the singular locus consists of the closed point, whence R is normal. According to Galois Theory, the finite extension $A^G \subset A$ has generically rank $q = \mathrm{ord}(G)$. By the Main Theorem of Zariski ([14], Corollary 4.4.9), it therefore suffices to check that $R \subset A$ has generically rank q. Obviously, the extensions $k[[a,b]] \subset R$ and $k[[a,b]] \subset A$ have rank q and q^2, respectively, and the statement follows by transitivity of ranks. □

This local description of the singularity enables us to determine the scheme structure for the canonical divisor $K_{(C\times C')/G} = (q-3)(\varphi_1^{-1}(0) + \varphi_2^{-1}(0))$, viewed as a reduced Weil divisor:

Corollary 2.3. *The reduced Weil divisors $\varphi_i^{-1}(0)_{\mathrm{red}}$ are isomorphic to the projective line, and the schematic intersection $\varphi_1^{-1}(0)_{\mathrm{red}} \cap \varphi_2^{-1}(0)_{\mathrm{red}}$ has length one.*

Proof. We saw in the proof of Proposition 2.1 that $\varphi_i^{-1}(0)_{\mathrm{red}}$ is isomorphic to C/G, at least outside the singular point on the ambient surface. However, at this singularity, $\varphi_i^{-1}(0)$ is formally isomorphic to the spectrum of $k[[a,b,c]]/(c^q,a)$, which follows from Proposition 2.2. Hence its reduction is regular. Similarly, the union $\varphi_1^{-1}(0)_{\mathrm{red}}\cup\varphi_2^{-1}(0)_{\mathrm{red}}$ is formally isomorphic to the spectrum of $k[[a,b,c]]/(c,ab)$, and the result follows. ☐

We would like to compute the dimension of the vector space of global sections for the dualizing sheaf $\omega_{(C\times C')/G}$, but are only able to do so directly in the special case $q=p$:

Proposition 2.4. *For $q=p$, we have $h^0(\omega_{(C\times C')/G})=(2p^3-9p^2+13p-6)/6=(2p-3)(p-2)(p-1)/6$.*

Proof. We have a commutative diagram

$$\begin{array}{ccc} H^0(U,\omega_{C\times C'})^G & \longleftarrow & H^0(\omega_{C\times C'})^G \\ \uparrow & & \uparrow \\ H^0(V,\omega_{(C\times C')/G}) & \longleftarrow & H^0(\omega_{(C\times C')/G}) \end{array}$$

where $U\subset C\times C'$ is the locus where G acts freely, $V\subset (C\times C')/G$ is the smooth locus, and $U\to V$ is the induced finite étale Galois covering with Galois group G. The latter ensures that the vertical map on the left is bijective. Since our dualizing sheaves are invertible, the horizontal maps are bijective (compare, for example, [16], Proposition 1.11). It follows that that the canonical map $H^0(\omega_{(C\times C')/G})\to H^0(\omega_{C\times C'})^G$ is bijective.

Recall that $H^0(C,\omega_C)=\bigoplus_{j=0}^{p-3}V_j$, where V_j are indecomposable G-submodule of dimension $d=p-2-j$. Since $G=\mathbb{F}_p$, the action of the generator $1\in\mathbb{F}_p$ is given, in a suitable basis, by the Jordan matrix

$$J_d(1)=\begin{pmatrix}1 & & & \\ 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1\end{pmatrix}\in\mathrm{Mat}(d,k),$$

which determines the G-module up to isomorphism. Computing the dimension of the G-fixed part for $H^0(\omega_{C\times C'})=H^0(C,\omega_C)\otimes H^0(C',\omega_{C'})$ thus reduces to extracting the number of blocks in the Jordan normal form of tensor products of certain Jordan matrices. Write

$$J_d(1)\otimes J_{d'}(1)=\bigoplus_r J_r(1)^{\oplus\lambda_r}$$

for certain multiplicities $\lambda_r\geq 0$. Note that it is a notorious unsolved problem in linear algebra to find the Jordan decomposition of tensor products of nilpotent Jordan matrices in positive characteristics. However, it is well-known that $\sum_r\lambda_r=\min(d,d')$ (see, for example, [29], Proposition 3.2). Hence the G-fixed part of such a tensor product representation is of dimension $\min(d,d')$, and we get

$$h^0(\omega_{(C\times C')/G})=\sum_{d,d'=1}^{p-2}\min(d,d').$$

Summing over $l = \min(d, d')$ rather than (d, d'), we rewrite the latter sum as

$$\sum_{l=1}^{p-2}(2(p-2-l)+1)l = \sum_{l=1}^{p-2}(-2l^2 + (2p-3)l).$$

The statement follows by applying the formulas $1+2+\ldots+n = n(n+1)/2$ and $1^2+2^2+\ldots+n^2 = n(n+1)(2n+1)/6$. ☐

Remark 2.5. In order to make similar computations in the general case $q = p^s$, one would need to understand the modular representations of the elementary abelian group $\mathbb{F}_q = (\mathbb{Z}/p\mathbb{Z})^s$. Such representations can be expressed in terms of s commuting nilpotent matrices, or equivalently via modules of finite length over the polynomial ring in s indeterminates over $\mathbb{F}_p$. For this situation, little seems to be known about multiplicities in tensor products.

3. Invariants of the singularity

In this section we study in more detail the 2-dimensional ring

$$R = k[a,b,c]/(c^q - (ab)^{q-1}c - a^qQ(b) + b^qP(a)),$$

whose formal completion gives the singularity of the surface $(C \times C')/G$. As in the proof of Proposition 2.2, one sees that R is normal, and the maximal ideal $\mathfrak{m} = (a,b,c)$ corresponds the unique singularity $s \in \operatorname{Spec}(R)$. According to [21], Theorem 2.5, the exceptional divisor on the minimal resolution of singularities consists of projective lines, and has as dual graph a tree. To understand its structure, we first consider a partial resolution of singularities: Let

$$f : Y \longrightarrow \operatorname{Spec}(R)$$

be the blowing-up of the maximal ideal $\mathfrak{m} \subset R$, and denote by $E = f^{-1}(s)$ the exceptional divisor. This is the Cartier divisor $E \subset Y$ with ideal $\mathcal{O}_Y(1) \subset \mathcal{O}_Y$.

Proposition 3.1. *The surface Y is normal, and* $\operatorname{Sing}(Y) = \{s_0, \ldots, s_q\}$ *consists of $q+1$ closed points, whose local rings are rational double points of type A_{q-1}. We have $E_{\text{red}} = \mathbb{P}^1$ and $qE_{\text{red}} = E$. Furthermore, the inclusions $iE_{\text{red}} \subset (i+1)E_{\text{red}}$ are infinitesimal extensions by the sheaf $\mathcal{O}_{\mathbb{P}^1}(-i)$, $0 \leq i \leq q-1$.*

Proof. The blowing-up Y is covered by three affine charts, the a-chart, the b-chart, and the c-chart. The coordinate ring of the a-chart is generated by $a, b/a, c/a$, subject to the relation

$$(c/a)^q - a^{q-1}(b/a)^{q-1}c/a - Q(a \cdot b/a) + (b/a)^qP(a) = 0.$$

Computing partial derivatives, one sees that the singular locus is given by $(c/a) = a = 0$ and $(b/a)^q = (b/a)$, and the latter means $b/a = i$, $i \in \mathbb{F}_q$. Writing the left hand side of the preceding equation in the form

$$(c/a)^q - a\Psi(c/a, b/a, a),$$

one easily computes that $c/a, a, \Psi$ form a regular system of parameters in the formal completion $k[[c/a, b/a - i, a]]$. The upshot is that singularity is a rational double point of type A_{q-1}. The situation on the b-chart is symmetric, whereas the c-chart turns out to be disjoint from the exceptional divisor.

On the a-chart, the exceptional divisor is given by $a = 0$, whence its coordinate ring is $k[b/a, c/a]/(c/a)^q$. Its reduction is defined by $c/a = 0$. We infer that E_{red} is isomorphic to $\mathbb{P}^1 = \operatorname{Proj} k[a/b, b/a]$. The ideal of $iE_{\text{red}} \subset (i+1)E_{\text{red}}$ is generated by $(c/a)^i$ and $(c/b)^i$ on the a- and b-chart, respectively. The statement about the infinitesimal extensions follows. $\square$

Now let $g : X \to Y$ be the minimal resolution of singularities of Y, the latter being a partial resolution of $\operatorname{Spec}(R)$. For each singular point $s_i \in Y$, $0 \leq i \leq q$, write $g^{-1}(s_i) = A_{i,1} \cup \ldots \cup A_{i,q-1}$ as a chain of projective lines $A_{i,j}$, such that $A_{i,j} \cdot A_{i,j+1} = 1$. Let $A_0 \subset X$ be the strict transform of $E_{\text{red}} \subset Y$, which is another projective line.

According to Lorenzini [21], Theorem 2.5, the dual graph for the exceptional divisor on the resolution of a wild quotient singularity is necessarily a tree. To our knowledge, no examples are known where the tree is not star-shaped. In [22], Question 1.1, Lorenzini askes whether trees with more that one node are possible. In our situation, Equation (3) immediately shows that A_0 intersects each chain of rational curves $g^{-1}(s_i)$ in a terminal component of the chain, say $A_{i,1}$. Thus the reduced exceptional divisor

$$A_0 + \sum_{i=0}^{q} \sum_{j=1}^{q-1} A_{i,j}$$

is a strictly normal crossing made out of projective lines, and its dual graph looks like this:

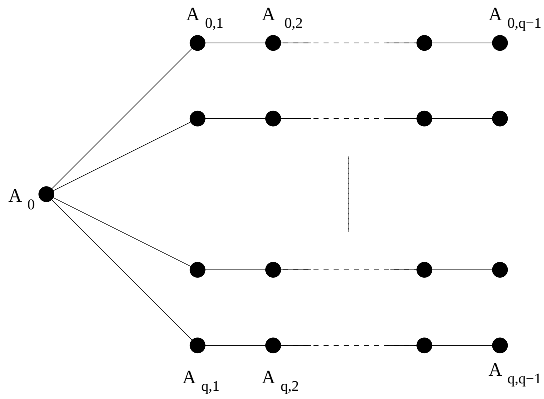

Figure 1: Dual graph for exceptional divisor on X.

Proposition 3.2. *We have $E^2 = -q$, and $E \cdot E_{\text{red}} = -1$, and $A_0^2 = -q$. In particular, the composite map $X \to \operatorname{Spec}(R)$ is the minimal resolution of singularities.*

Proof. As in the proof of Proposition 3.1, we have

$$E_{\text{red}} = \operatorname{Spec} k[a/b] \cup \operatorname{Spec} k[b/a],$$

and the invertible sheaf $\mathcal{O}_Y(1)|_{E_{\text{red}}}$ is given by $a = a/b \cdot b$ on the two charts. Viewing $a/b \in k[a/b, b/a]^\times$ as a cocycle for $\mathcal{O}_Y(1)|_{E_{\text{red}}}$, one immediately verifies that this invertible sheaf has degree 1. Thus

$$E \cdot E_{\text{red}} = \deg \mathcal{O}_{E_{\text{red}}}(E) = \deg \mathcal{O}_{E_{\text{red}}}(-1) = -1.$$

Since $E = qE_{\text{red}}$, the selfintersection number $E^2 = -q$ also follows.

Now consider the first infinitesimal neighborhood $E_{\mathrm{red}} \subset 2E_{\mathrm{red}} \subset E$. In light of Proposition 3.1, this is an infinitesimal extension of $E_{\mathrm{red}} = \mathbb{P}^1$ by the invertible sheaf $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(-1)$. Let $Y' \to Y$ be the blowing-up of the singular points $s_0, \ldots, s_q \in Y$. Then the exceptional divisors are disjoint unions of pairs of rational curves, each pair intersecting transversely at one point whose local ring on Y' is a rational double point of type A_{q-3}. Moreover, the strict transform of E_{red} lies in the smooth locus of Y'. According to the theory of *ribbons* developed by Bayer and Eisenbud, the strict transform of $2E_{\mathrm{red}}$ on Y', that is, the blowing-up of the scheme $2E_{\mathrm{red}}$ with respect to the centers $s_0, \ldots, s_q \in 2E_{\mathrm{red}}$, is an infinitesimal extension of $\mathbb{P}^1$ by the invertible sheaf $\mathcal{L}(\sum a_i) = \mathcal{O}_{\mathbb{P}^1}(q)$, compare [6], Theorem 1.9. It follows that $A_0^2 = \deg(\mathcal{O}_{A_0}(A_0)) = -q$. In particular, the exceptional divisor for the resolution of singularities $X \to \operatorname{Spec}(R)$ contains no (-1)-curve, hence is minimal. $\square$

We next compute the *fundamental cycle* $Z \subset X$ for the resolution of singularities $h : X \to \operatorname{Spec}(R)$, a notion introduced by M. Artin [2]. By definition, this is the smallest effective cycle Z whose support equals the exceptional divisor and has intersection number ≤ 0 on each irreducible component of the exceptional divisor. One way to compute the fundamental cycle is with a *computation sequence* $Z_{\mathrm{red}} = Z_0 \subset Z_1 \subset \ldots \subset Z_r = Z$, where in each step $Z_{s+1} - Z_s$ is an integral component of the exceptional divisor with $Z_s \cdot (Z_{s+1} - Z_s) > 0$. Using the exact sequence

$$0 \longrightarrow \mathcal{O}_{Z_{s+1}-Z_s}(-Z_s) \longrightarrow \mathcal{O}_{Z_{s+1}} \longrightarrow \mathcal{O}_{Z_s} \longrightarrow 0, \tag{7}$$

one inductively infers that $h^0(\mathcal{O}_{Z_{s+1}}) = h^0(\mathcal{O}_{Z_s})$, in particular $h^0(\mathcal{O}_Z) = 1$. Consequently the schematic image $h(Z) \subset \operatorname{Spec}(R)$ is nothing but the reduced singular point $s \in \operatorname{Spec}(R)$. Indeed, one should view the fundamental cycle as an approximation to the schematic fiber $h^{-1}(s) \subset X$, the latter usually containing embedded components.

Proposition 3.3. *The fundamental cycle is given by the formula*

$$Z = qA_0 + \sum_{i=0}^{q} \sum_{j=1}^{q-1} (q-j) A_{ij},$$

and its selfintersection number is $Z^2 = -q$.

Proof. Let Z' be the cycle on the right hand side. One easily computes the intersection numbers

$$\begin{aligned} Z' \cdot A_0 &= -q^2 + (q+1)(q-1) = -1, \\ Z' \cdot A_{ij} &= (q-j+1) - 2(q-j) + (q-j-1) = 0, \end{aligned} \tag{8}$$

whence the fundamental cycle Z is contained in Z', by the minimality property of fundamental cycles. Seeking a contradiction, we suppose $Z \subsetneqq Z'$. The effective cycle $Z' - Z$ has intersection numbers

$$\begin{aligned} (Z' - Z) \cdot A_0 &= -1 - (Z \cdot A_0), \\ (Z' - Z) \cdot A_{ij} &= -Z \cdot A_{ij} \geq 0. \end{aligned}$$

Since nonzero effective exceptional cycles are not nef on all exceptional curves (see [2], proof of Proposition 2), there is no possibility but $Z \cdot A_0 = 0$. Now write $Z = \lambda_0 A_0 + \sum_{i=0}^{q} \sum_{j=1}^{q-1} \lambda_j A_{ij}$ with coefficients $1 \leq \lambda_0 \leq q$ and $1 \leq \lambda_j \leq q - j$.

Note that the coefficients λ_j do not depend on i, due to the obvious symmetry of the dual graph in Figure 1. We have

$$0 = Z \cdot A_0 = -q\lambda_0 + (q+1)\lambda_1,$$

whence $q \mid \lambda_1$, contradicting $1 \leq \lambda_1 \leq q-1$. □

Recall that the *canonical cycle* $K_h = K_{X/R}$ is the divisor supported by the exceptional divisor that satisfies the equations $K_{X/R} \cdot C + C^2 = \deg(K_C)$, where C runs through the integral exceptional divisors.

Corollary 3.4. *The canonical cycle is given by $K_h = -(q-2)Z$, with selfintersection number $K_h^2 = -q(q-2)^2$.*

Proof. Using that A_0 and $A_{i,j}$ are copies of the projective lines with selfintersection numbers $A_0^2 = -q$, $A_{i,j}^2 = -2$, one deduces the result from the intersection numbers (8). The selfintersection then follows from Proposition 3.3. □

We are now in position to compute the arithmetic genus of the fundamental cycle

$$p_f = h^1(\mathcal{O}_Z) = 1 - \chi(\mathcal{O}_Z),$$

which is also called the *fundamental genus* of the singularity, confer [33].

Corollary 3.5. *The fundamental genus of the singularity R is given by the formula $p_f = (q-1)(q-2)/2$.*

Proof. Riemann–Roch yields $-2\chi(\mathcal{O}_Z) = \deg(K_Z) = Z^2 + K_h \cdot Z = (3-q)Z^2$. We have $Z^2 = -q$, and finally obtain

$$p_f = 1 - \chi(\mathcal{O}_Z) = 1 - (3-q)q/2 = (q-1)(q-2)/2,$$

as claimed. □

From this we deduce:

Corollary 3.6. *The singularity R is a rational double point if and only if $q = 2$, and minimally elliptic if and only if $q = 3$.*

Proof. According to [2], Theorem 3, the singularity is rational if and only if $h^1(\mathcal{O}_Z) = 0$. In light of Corollary 3.5, this happens precisely when $q = 2$. By Laufer [20], Theorem 3.4, the condition $K_h = -Z$ is one of several equivalent defining property of *minimally elliptic singularities*. By Corollary 3.4, this happens if and only if $q = 3$. □

Remark 3.7. For $q = 2$, this singularity is actually a rational double point of type D_4^1, according to Artin's list [5]. Indeed, the equation $c^q - (ab)^{q-1}c - a^q b + b^q a = 0$ is a special case of Artin's normal form for wild $\mathbb{Z}/2\mathbb{Z}$-quotient singularities in dimension two [4], compare also [28]. For $q = 3$, the minimally elliptic singularity appears in Laufer's classification ([20], Table 3 on page 1294) under the designation $A_{1,\star,0} + A_{1,\star,0} + A_{1,\star,0} + A_{1,\star,0}$.

Remark 3.8. Shioda [31] and Katsura [19] obtained rather similar results for the action of the sign involution on abelian surfaces in characteristic $p = 2$.

We next want to compute the *geometric genus*

$$p_g = \operatorname{length} R^1 h_*(\mathcal{O}_X) = h^1(\mathcal{O}_{nZ}), \quad n \gg 0$$

of the singularity. Except for rational double points and minimally elliptic singularities, this invariant is difficult to compute. We have at least some bounds. It will turn out later that in the special case $q = p$, these bounds are actually equalities (Corollary 7.4).

Proposition 3.9. *The geometric genus of the singularity R satisfies the inequalities $p_g \leq q(q-1)(q-2)/6$. For $q \geq 5$, we moreover have $h^1(\mathcal{O}_{2Z}) > h^1(\mathcal{O}_Z)$, in particular $p_g > p_f$.*

Proof. It is convenient to work on the partial resolution Y rather than on the full resolution X. The Leray–Serre spectral sequence gives an exact sequence

$$0 \longrightarrow R^1 f_*(\mathcal{O}_Y) \longrightarrow R^1 h_*(\mathcal{O}_X) \longrightarrow f_*(R^1 g_*(\mathcal{O}_X)).$$

The term on the right vanishes, since Y has only rational singularities, whence $p_g = \operatorname{length} R^1 f_*(\mathcal{O}_Y)$. Let $D = -K_f = (q-2)E$ be the anticanonical cycle, where $E = f^{-1}(s)$. It follows with the Grauert–Riemenschneider Vanishing Theorem (see [11], Theorem 1.5) that $R^1 f_*(\mathcal{O}_Y(-D)) = R^1 h_*(g^*\mathcal{O}_Y(-D)) = 0$, whence the canonical surjection

$$H^0(R^1 f_*(\mathcal{O}_Y)) \longrightarrow H^1(D, \mathcal{O}_D)$$

is bijective. Moreover, Riemann–Roch gives $\chi(\mathcal{O}_D) = (D + K_f) \cdot D = 0$, such that $p_g = h^1(\mathcal{O}_D) = h^0(\mathcal{O}_D)$.

According to Proposition 3.1 and Proposition 3.2, we have

$$E_{\text{red}} = \mathbb{P}^1 \quad \text{and} \quad E = qE_{\text{red}} \quad \text{and} \quad E \cdot E_{\text{red}} = -1.$$

Consider the integral Weil divisors $(i/q)E = iE_{\text{red}}$. We deduce from Proposition 3.1 that the kernel $\mathcal{K}_i$ in the exact sequence

$$0 \longrightarrow \mathcal{K}_i \longrightarrow \mathcal{O}_{((i+1)/q)E} \longrightarrow \mathcal{O}_{(i/q)E} \longrightarrow 0$$

is an invertible sheaf on $E_{\text{red}} = \mathbb{P}^1$ of degree

$$\deg(\mathcal{K}_i) = \lfloor i/q \rfloor - q\,\{i/q\}$$

where $\lfloor i/q \rfloor$ and $\{i/q\}$ denotes integral and fractional parts, respectively. Let us tabulate the kernels $\mathcal{K}_i$ for $0 \leq i < q(q-2)$ in a matrix of size $(q-2) \times q$:

$$\begin{matrix} \mathcal{O}_{\mathbb{P}^1}(0) & \mathcal{O}_{\mathbb{P}^1}(-1) & \mathcal{O}_{\mathbb{P}^1}(-2) & \ldots & & \mathcal{O}_{\mathbb{P}^1}(1-q) \\ \mathcal{O}_{\mathbb{P}^1}(1) & \mathcal{O}_{\mathbb{P}^1}(0) & \mathcal{O}_{\mathbb{P}^1}(-1) & \ldots & & \mathcal{O}_{\mathbb{P}^1}(2-q) \\ \vdots & \vdots & & \ddots & & \vdots \\ \mathcal{O}_{\mathbb{P}^1}(q-3) & \mathcal{O}_{\mathbb{P}^1}(q-4) & \ldots & & \mathcal{O}_{\mathbb{P}^1}(-1) & \mathcal{O}_{\mathbb{P}^1}(-2) \end{matrix} \tag{9}$$

Only kernels of degree ≥ 0 may contribute to $h^0(\mathcal{O}_D)$, and the total possible contribution is

$$\begin{aligned} &(q-2) \cdot h^0(\mathcal{O}_{\mathbb{P}^1}) + (q-3) \cdot h^0(\mathcal{O}_{\mathbb{P}^1}(1)) + \ldots + 1 \cdot h^0(\mathcal{O}_{\mathbb{P}^1}(q-3)) \\ &= \sum_{i=1}^{q-2} i(q-1-i) \\ &= (q-1)\sum_{i=1}^{q-2} i - \sum_{i=1}^{q-2} i^2 \\ &= (q-1)(q-1)(q-2)/2 - (q-2)(q-1)(2q-3)/6 \\ &= q(q-1)(q-2)/6. \end{aligned}$$

But some coboundary maps in the exact sequence

$$H^0(\mathcal{O}_{((i+1)/q)E}) \longrightarrow H^0(\mathcal{O}_{(i/q)E}) \xrightarrow{\partial} H^1(\mathcal{K}_i) \longrightarrow H^1(\mathcal{O}_{((i+1)/q)E}) \tag{10}$$

might be nonzero, so we only get an upper bound $p_g \leq q(q-1)(q-2)/6$, rather than an equality.

To obtain a lower bound, we consider the cycle $2E \subset Y$. The second row in (9) reveals that some cycle $E \subset F \subset 2E$ has

$$h^0(\mathcal{O}_F) = h^0(\mathcal{O}_{\mathbb{P}^1}(0)) + h^0(\mathcal{O}_{\mathbb{P}^1}(1)) + h^0(\mathcal{O}_{\mathbb{P}^1}(0)) = 4 \quad \text{and} \quad h^1(\mathcal{O}_F) = h^1(\mathcal{O}_E).$$

Now suppose that we would have $h^1(\mathcal{O}_F) = h^1(\mathcal{O}_{2E})$. Then in each step leading from F to $2E$ the coboundary map in (10) must surjects onto the respective cohomology groups

$$H^1(\mathcal{O}_{\mathbb{P}^1}(-2)), \quad H^1(\mathcal{O}_{\mathbb{P}^1}(-3)), \quad \ldots \quad H^1(\mathcal{O}_{\mathbb{P}^1}(2-q)).$$

It follows that

$$4 = h^0(\mathcal{O}_F) > h^1(\mathcal{O}_{\mathbb{P}^1}(2-q)) = q-3,$$

whence $q \leq 5$. This already implies the inequality $p_f < p_g$ for $q > 5$.

It remains to rule out the case $h^1(\mathcal{O}_E) = h^1(\mathcal{O}_{2E})$ and $q = 5$. The preceding paragraph reveals that than $h^0(\mathcal{O}_{2E}) = 1$, and the short exact sequence

$$0 \longrightarrow \mathcal{O}_E(-E) \longrightarrow \mathcal{O}_{2E} \longrightarrow \mathcal{O}_E \longrightarrow 0$$

yields $\chi(\mathcal{O}_E(-E)) = 0$. On the other hand, Riemann–Roch gives

$$\chi(\mathcal{O}_E(-E)) = -Z^2 + \chi(\mathcal{O}_E) = -q + 1 - (q-1)(q-2)/2 = -q(q+1)/2 < 0,$$

contradiction. □

4. Vanishing of Irregularity

We continue to study the normal surface $(C \times C')/G$. Let $f : Y \to (C \times C')/G$ be the blowing-up of the unique singularity. We saw in the preceding section that the singular locus $\mathrm{Sing}(Y)$ consists of $q+1$ rational double points of type A_{q-1}. Let $g : X \to Y$ be the minimal resolution of these double points. We now dispose off the *irregularity* $h^1(\mathcal{O}_X)$.

Proposition 4.1. *The irregularity $h^1(\mathcal{O}_X)$ vanishes.*

Proof. Since Y contains only rational singularities, it suffices to check $h^1(\mathcal{O}_Y) = 0$. Let $\tilde{Y} \to C \times C'$ be the blowing-up of the reduced fixed point (a, a'). We claim that the schematic preimage on $\tilde{Y}$ of the singular point $s \in (C \times C')/G$ is a Cartier divisor, such that the universal property of blowing-ups gives a commutative diagram

$$\begin{array}{ccc} C \times C' & \longleftarrow & \tilde{Y} \\ \downarrow & & \downarrow \\ (C \times C')/G & \longleftarrow & Y. \end{array}$$

This is a local problem. Using the notation from Section 2, we have to understand what happens with the ideal

$$(w, w') = (w, w', wu' - w'u) \subset k[[u, u']]$$

on $\tilde{Y}$. But $(w, w') = (u^q, u'^q)$ according to (3), and it obvious that the latter ideal becomes invertible upon blowing-up of (u, u').

The G-action on $C \times C'$ induces a G-action on $\tilde{Y}$, and the canonical morphism $\tilde{Y}/G \to Y$ is an isomorphism by the Main Theorem of Zariski. It follows from (3) that the G-action on the cotangent space $\mathfrak{m}/\mathfrak{m}^2$ for the fixed point $(a, a') \in C \times C'$ is trivial, whence the G-action on the exceptional curve $\tilde{E} \subset \tilde{Y}$ is trivial as well. Now Lemma 4.2 below ensures that the induced map $H^1(Y, \mathcal{O}_Y) \to H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}})$ is injective.

To finish the argument, consider the reduced fiber union $F = \varphi_1^{-1}(0)_{\text{red}} \cup \varphi_2^{-1}(0)_{\text{red}}$ for the two projections $\varphi_i : (C \times C')/G \to \mathbb{P}^1$. Its preimage on $\tilde{Y}$ contains the strict transform $\tilde{F} \subset \tilde{Y}$ of $C \times \{a'\} \cup \{a\} \times C' \subset C \times C'$, which is isomorphic to a disjoint union $C \amalg C'$, and we have a commutative diagram

$$\begin{array}{ccc}
H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) & \longrightarrow & H^1(\tilde{F}, \mathcal{O}_{\tilde{F}}) \\
\uparrow & & \uparrow \\
H^1(Y, \mathcal{O}_Y) & \longrightarrow & H^1(F, \mathcal{O}_F).
\end{array}$$

The term $H^1(F, \mathcal{O}_F)$ vanishes by Corollary 2.3. Since the map on the left is injective, it suffices to check that the the restriction map $H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) \to H^1(\tilde{F}, \mathcal{O}_{\tilde{F}})$ is injective. Indeed, the maps

$$H^1(C, \mathcal{O}_C) \oplus H^1(C', \mathcal{O}_{C'}) = H^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) \longrightarrow H^1(\tilde{F}, \mathcal{O}_{\tilde{F}}) = H^1(C, \mathcal{O}_C) \oplus H^1(C', \mathcal{O}_{C'})$$

are all bijective. $\square$

In the course of the preceding proof we have used a fact that appears to be of independent interest. Let us formulate it in a rather general way: Suppose X is a scheme over a field k, and G be a finite group acting on X so that the quotient $Y = X/G$ exists as a scheme.

Lemma 4.2. *Assumptions as above. Suppose additionally that $k = H^0(X, \mathcal{O}_X)$ and that there is a rational fixed point $x \in X$. Then the canonical map $H^1(Y, \mathcal{O}_Y) \to H^1(X, \mathcal{O}_X)$ is injective.*

Proof. The idea is to use G-equivariant cohomology $H^r(X, G, \mathcal{O}_X)$, which was introduced in [13]. Consider the two spectral sequences with E_2-terms

$$E_2^{r,s} = H^r(Y, \mathcal{H}^s(G, \mathcal{O}_X)) \quad \text{and} \quad E_2^{r,s} = H^r(G, H^s(X, \mathcal{O}_X))$$

abutting to $H^{r+s}(X, G, \mathcal{O}_X)$, where $\mathcal{H}^s(G, \mathcal{O}_X)$ denotes the sheaf of cohomology groups. They give rise to a commutative diagram

(11)

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
 & & H^1(Y, \mathcal{H}^0(G, \mathcal{O}_X)) \arrow[d] \arrow[dr] & \\
0 \arrow[r] & H^1(G, H^0(X, \mathcal{O}_X)) \arrow[r] \arrow[dr] & H^1(X, G, \mathcal{O}_X) \arrow[r] \arrow[d] & H^0(G, H^1(X, \mathcal{O}_X)) \\
 & & H^0(G, \mathcal{H}^1(G, \mathcal{O}_X)) &
\end{tikzcd}$$

with exact row and column, and the composition given by the upper diagonal arrow

$$H^1(Y, \mathcal{O}_Y) = H^1(Y, \mathcal{H}^0(G, \mathcal{O}_X)) \longrightarrow H^0(G, H^1(X, \mathcal{O}_X)) \subset H^1(X, \mathcal{O}_X)$$

is our map in question. By a diagram chase, it therefore suffices to check that the other composition $H^1(G, H^0(X, \mathcal{O}_X)) \to H^0(G, \mathcal{H}^1(G, \mathcal{O}_X))$ is injective. Now comes in our rational fixed point $x \in X$: Composing further with the restriction map induced by $\{x\} \subset X$, we obtain

$$H^1(G, H^0(X, \mathcal{O}_X)) \longrightarrow H^0(G, \mathcal{H}^1(G, \kappa(x))) = H^1(G, \kappa(x)).$$

By assumption, the map $H^0(X, \mathcal{O}_X) \to \kappa(x)$ is bijective, whence the assertion. □

Remark 4.3. Let $\nu : X \to Y$ be the quotient map. If X is normal, and the order of G is prime to the characteristic of k, then the existence of a trace map shows that $\mathcal{O}_Y \subset \nu_*(\mathcal{O}_X)$ is a direct summand, such that $H^r(Y, \mathcal{O}_Y) \to H^r(X, \mathcal{O}_X)$ is injective for all $r \geq 0$.

Remark 4.4. On the other hand, if Y is an Enriques surface in characteristic $p = 2$ with $\mathrm{Pic}^\tau_Y = \mu_2$, and $X \to Y$ is the K3-covering, such that $G = \pi_1(Y)$ is cyclic of order two and $Y = X/G$, then $H^1(Y, \mathcal{O}_Y)$ is 1-dimensional, whereas $H^1(X, \mathcal{O}_X)$ vanishes. We note in passing that this situation is somewhat typical:

Lemma 4.5. *Suppose G is cyclic, acts freely on X, and $k = H^0(X, \mathcal{O}_X)$. Then the kernel of the canonical map $H^1(Y, \mathcal{O}_Y) \to H^1(X, \mathcal{O}_X)$ is at most 1-dimensional.*

Proof. Making a diagram chase in (11), the dimension of the kernel is bounded by the dimension of $H^1(G, H^0(X, \mathcal{O}_X))$. Clearly, $H^0(X, \mathcal{O}_X) = k$ is the trivial G-module. Let $n = \mathrm{ord}(G)$. Then $H^1(G, k)$ is isomorphic to the kernel of the multiplication map $n : k \to k$, whence a k-vector space of dimension at most one. □

5. Place in the Enriques classification

We now study the global geometry of the normal surface $(C \times C')/G$ in more detail. Recall that $f : Y \to (C \times C')/G$ is the blowing-up of the singularity. We saw in Section 3 that Y is normal and contains $(q-1)$ rational double points of type A_{q-1}. Let $g : X \to Y$ be the minimal resolution of these singularities. Then the composite map $h : X \to (C \times C')/G$ is the minimal resolution of singularities. Let $X \to S$ be the contraction to a minimal model S. We display our surfaces and maps in a commutative diagram:

$$\begin{tikzcd}
C \times C' \arrow[dr] & & & X \arrow[r] \arrow[dll, "h"'] \arrow[dl, "g"] \arrow[dd, "\psi_1,\psi_2"] & S \\
& (C \times C')/G \arrow[drr, "\varphi_1,\varphi_2"'] & Y \arrow[l, "f"] & & \\
& & & \mathbb{P}^1 &
\end{tikzcd}$$

The top row contains the smooth surfaces, the middle row the normal surfaces, and the arrows φ_i, ψ_i are the maps induced from the two projections $\mathrm{pr}_1 : C \times C' \to C$ and $\mathrm{pr}_2 : C \times C' \to C'$.

The goal of this section is to determine the place of X, or rather its minimal model S, in the Enriques classification of surfaces, in dependence on the prime power $q = p^s$. An elementary argument involving only intersection numbers already gives:

Proposition 5.1. *If $q \geq 7$, then the surface X is of general type.*

Proof. We first compute the number K_X^2 on the smooth surface X. Obviously $K_X = K_h + h^* K_{(C\times C')/G}$, such that $K_X^2 = K_h^2 + K_{(C\times C')/G}^2$. But $K_h^2 = -q(q-2)^2$ by Proposition 3.4, whereas $K^2_{(C\times C')/G} = 2q(q-3)^2$ according to Proposition 2.1. The upshot is

$$K_X^2 = q(q^2 - 8q + 14) = q((q-4)^2 - 2). \tag{12}$$

Now suppose that $q \geq 7$. Then $K_X^2 > 0$, and the Theorem of Riemann–Roch gives

$$\chi(\omega_X^{\otimes t}) = (t^2 - t)K_X^2/2 + \chi(\mathcal{O}_X).$$

So for $t \gg 0$ either tK_X or $(1-t)K_X$ is effective. The latter is impossible, because the canonical divisor on X maps to the canonical divisor on $(C \times C')/G$, which is effective. Thus tK_X, and in turn tK_S is effective. Since $K_S^2 \geq K_X^2 > 0$, it follows from the Enriques classification of surfaces that the minimal surface S is of general type. $\square$

To understand the remaining cases, and the geometry of the contraction $X \to S$ as well, we have to analyze the two fibrations $\psi_i : X \to \mathbb{P}^1$, $i = 1, 2$ induced by the projections $\mathrm{pr}_1 : C \times C' \to C$ and $\mathrm{pr}_2 : C \times C' \to C'$. Let us first record:

Lemma 5.2. *We have $\psi_{i*}(\mathcal{O}_X) = \mathcal{O}_{\mathbb{P}^1}$.*

Proof. By the Main Theorem of Zariski, it suffices to check this at the generic point. Let $E = k(C)$ be the function field of C, such that $L = E^G = k(\mathbb{P}^1)$ is the function field of the projective line. By construction, the generic fiber X_η of $f_1 : X \to \mathbb{P}^1$ is isomorphic to $(C \otimes_k E)/G = (C_L \otimes_L E)/G$. In other words, the generic fiber is a twisted form of C_L with respect to the étale topology. This description ensures that $H^0(X_\eta, \mathcal{O}_{X_\eta}) = L$, and the result follows. $\square$

Now let $F_1 \subset X$ and $F_2 \subset X$ be the respective schematic fibers for the projections $\psi_1 : X \to \mathbb{P}^1$ and $\psi_2 : X \to \mathbb{P}^1$ containing the exceptional divisor $A_0 + \sum_{i=0}^{q} \sum_{j=1}^{q-1} A_{i,j}$ for the resolution of singularities $X \to (C \times C')/G$. Up to multiplicities, the fiber F_i, $i = 1, 2$ is the union of this exceptional divisor $A_0 + \sum_{i=0}^{q} \sum_{j=1}^{q-1} A_{i,j}$ with another integral curve $B_i \subset X$, which is birational to $\mathbb{P}^1 = C/G = C'/G$. One can say more about the curves $B_1, B_2 \subset X$:

Proposition 5.3. *The curves $B_1, B_2 \subset X$ are (-1)-curves, and the reduction of the fiber union $F_1 \cup F_2 \subset X$ has only simple normal crossings.*

Proof. By symmetry, it suffices to treat B_1. Being an integral component of a reducible fiber, it has selfintersection $B_1^2 < 0$. Thus it suffices to check that $K_X \cdot B_1 < 0$. In light of Corollary 3.4 and Proposition 3.3, we have $K_h \cdot B_1 \leq -(q-2)$. The image of qB_1 on the normal surface $(C \times C')/G$ is a schematic fiber. Using the projection formula and computing intersections on $C \times C'$, we deduce the value $h_*(B_1) \cdot K_{(C\times C')/G} = q - 3$. The upshot is that

$$K_X \cdot B_1 = K_h \cdot B_1 + K_{(C\times C')/G} \cdot h_*(B_1) \leq -1,$$

whence B_1 is a (-1)-curve. Consequently, $K_h \cdot B_1 = -(q-2)$. In light of Corollary 3.4, it follows that B_i intersects the exceptional divisor $A_0 + \sum_{i=0}^{q} \sum_{j=1}^{q-1} A_{i,j}$ in precisely one component where the fundamental cycle attains its minimal multiplicity, that is, $j = q-1$. We conclude that the reduced fiber $F_{1,\mathrm{red}}$ is simple normal crossing. □

Let us choose the indices i for the $A_{i,j}$ so that $B_1 \cdot A_{0,q-1} = B_2 \cdot A_{q,q-1} = 1$. Thus the dual graph of the fiber union $F_1 \cup F_2$ looks like this:

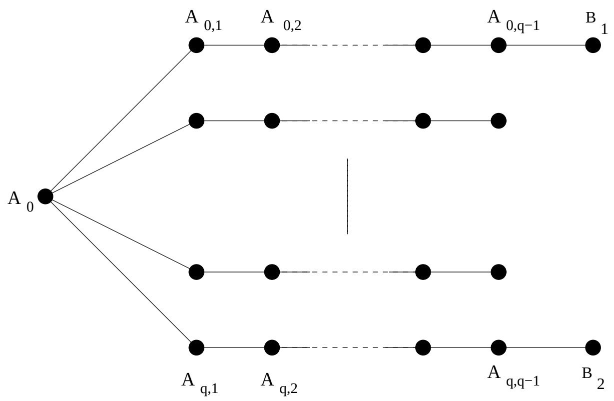

Figure 2: Dual graph for fiber union $F_1 \cup F_2$ on X.

We are now in position to compute the multiplicities occurring in the schematic fibers $F_i \subset X$:

Proposition 5.4. *The schematic fibers are given by* $F_1 = Z + \sum_{j=1}^{q-1} jA_{0,j} + qB_1$ *and* $F_2 = Z + \sum_{j=1}^{q-1} jA_{q,j} + qB_2$.

Proof. It suffices to treat F_1. By definition of the fundamental cycle Z, we have

$$Z + \sum_{j=1}^{q-1} jA_{0,j} + qB_1 = qA_0 + \sum_{i=1}^{q} \sum_{j=1}^{q-1} (q-j)A_{i,j} + \sum_{j=1}^{q-1} qA_{0,j} + qB_1,$$

and it is a straightforward computation that this cycle is numerically trivial on all irreducible components of F_1. It is therefore a rational multiple of F_1. Our cycle contains B_1 with multiplicity q. But qB_1 is the strict transform of the fiber for $\varphi_1 : (C \times C')/G \to \mathbb{P}^1$, by Proposition 2.1. Thus our cycle coincides with F_1. □

Note that the component $A_{0,q-1}$ has multiplicity one in the fiber $F_2 \subset X$, and similarly for $A_{q,q-1} \subset F_1 \subset X$. In particular, the fibers are neither multiple nor wild. Moreover:

Corollary 5.5. *The sheaf* $R^1\psi_{i*}(\mathcal{O}_X)$ *is locally free of rank* $(q-1)(q-2)/2$, *and the formation of the direct image* $\mathcal{O}_{\mathbb{P}^1} = \psi_{i*}(\mathcal{O}_X)$ *commutes with base change.*

Proof. This follows from [27], Theorem 7.2.1, because each geometric fiber of ψ_i contains a reduced irreducible component. □

This leads to a very useful consequence concerning fundamental group:

Corollary 5.6. *The fundamental group* $\pi_1(X)$ *vanishes.*

Proof. Let $X' \to X$ be a finite étale covering with X' nonempty. We have to check that it has a section. Consider the projection $\psi = \psi_1 : X \to \mathbb{P}^1$, and let

$$X' \longrightarrow T' \longrightarrow \mathbb{P}^1$$

be the Stein factorization for the composition $\psi' : X' \to \mathbb{P}^1$, which is given by $T' = \operatorname{Spec}(\psi'_* \mathcal{O}_{X'})$. As explained in [14], Remark 7.8.10, the finite morphism $T' \to \mathbb{P}^1$ is étale, since the equality $\mathcal{O}_{\mathbb{P}^1} = \psi_*(\mathcal{O}_X)$ commutes with base change. Moreover, the fiber $\psi^{-1}(0) \subset X$ is simply connected, by the description in Proposition 5.4. Using this as in the proof for [15], Exposé X, Theorem 1.3, we infer that the canonical map $X' \to X \times_{\mathbb{P}^1} T'$, which is finite étale, is actually an isomorphism. But the projective line is simply connected, so $T' \to \mathbb{P}^1$, whence also $X' \to X$ has a section. □

For later use, we record:

Proposition 5.7. *The Picard group* $\operatorname{Pic}(X)$ *is a finitely generated free abelian group.*

Proof. It already follows from Proposition 4.1 that $\operatorname{Pic}^0(X) = 0$, whence $\operatorname{Pic}(X) = \operatorname{NS}(X)$ is finitely generated. If $l \neq p$ is a prime different from the characteristic, then the elements $\mathcal{L} \in \operatorname{Pic}(X)$ of order l yield nontrivial μ_l-torsors $X' \to X$, which are finite étale coverings of degree l. Since the fundamental group vanishes, there are no such elements.

Finally, suppose there is an element $\mathcal{L}$ of order p. To handle this case, first note that the Artin–Schreier curve C has p-rank $\sigma = 0$. This follows from [32], Theorem 3.5, applied to $C \to \mathbb{P}^1$. In other words $H^1(C, \mathbb{Z}/p\mathbb{Z}) = 0$, or equivalently, $\operatorname{Pic}(C)$ contains no element of order p. Then the generic fiber X_η for the second projection $\psi_2 : X \to \mathbb{P}^1$, which is a twisted form of C over the function field of $\mathbb{P}^1$, admits no invertible sheaf of order p. Thus $\mathcal{L}|X_\eta$ is trivial. It follows that $\mathcal{L} \simeq \mathcal{O}_X(D)$, where D is a divisor supported by fibers of $\psi_2 : X \to \mathbb{P}^1$. Now recall that the intersection form on the integral components of a given fiber is negative semidefinite. Moreover, the radical is generated by the fiber, since the fibers have multiplicity one by Proposition 5.4. Using that $\mathcal{L}$ is numerically trivial, we first deduce that each D_a, $a \in \mathbb{P}^1$ is a multiple of the fiber, and then that D is linearly equivalent to the zero divisor, contradiction. □

The geometry of our surface X simplifies further if we contract in a different way. Indeed, the curves $B_1 + \sum_{i=1}^{q-1} A_{0,j}$ and $B_2 + \sum_{i=1}^{q-1} A_{q,j}$ are two disjoint *exceptional curves of the first kind*, comprising altogether $2q$ irreducible components. Let

$$X \longrightarrow \tilde{S}$$

be their contraction, such that $\tilde{S}$ is a smooth surface with

$$K_{\tilde{S}}^2 = K_X^2 + 2q. \tag{13}$$

By choosing another minimal model S if necessary, we may tacitly assume that the contraction $X \to S$ factors over $\tilde{S}$. Let $\tilde{A}_0, \tilde{A}_{i,j} \subset \tilde{S}$ be the images of the curves $A_0, A_{i,j} \subset X$ for $i \neq 0, q$. Then

$$\tilde{Z} = \sum_{i=1}^{q-1} \sum_{j=1}^{q-1} (q-j) \tilde{A}_{i,j}$$

is the image of the fundamental cycle $Z \subset X$ for the resolution of singularities $X \to (C \times C')/G$.

Proposition 5.8. *We have $K_{\tilde{S}} = (q-4)\tilde{Z}$, and $K_{\tilde{S}}^2 = q(q-4)^2$.*

Proof. Recall that $K_X^2 = q(q^2-8q+14)$ by (12). The map $X \to \tilde{S}$ contracts successively $2q$ (-1)-curves, such that

$$K_{\tilde{S}}^2 = K_X^2 + 2q = q(q^2-8q+16) = q(q-4)^2.$$

The canonical class $K_{\tilde{S}}$ is the image of the canonical class K_X. Recall that we have

$$K_h = -(q-2)Z \quad \text{and} \quad h^*K_{(C\times C')/G} = (q-3)(F_1+F_2).$$

The latter coincides with the cycle $2(q-3)Z$, up to components that are contracted by $X \to \tilde{S}$. Consequently $K_{\tilde{S}} = (2(q-3)-(q-2))\tilde{Z} = (q-4)\tilde{Z}$. □

We now have an explicit description of the minimal model:

Theorem 5.9. *For $q \geq 4$, the surface $\tilde{S}$ is minimal, such that $S = \tilde{S}$.*

Proof. We have $\tilde{A}_0^2 = 2-q \neq -1$. Consequently, there is no (-1)-curve supported by $\tilde{Z} \subset \tilde{S}$. Since $K_{\tilde{S}}$ is effective, there is no other (-1)-curve on $\tilde{S}$, and the result follows. □

From this we easily determine the place in the Enriques classification of surfaces. Recall that a *weak del Pezzo surface* is a surface whose anticanonical divisor is nef and big.

Corollary 5.10. *The mimimal surface S is of general type if $q \geq 5$, a K3-surface for $q = 4$, and a weak del Pezzo surface for $q = 2, 3$.*

Proof. First suppose $q \geq 5$. Then $S = \tilde{S}$ is minimal, and K_S is effective with $K_S^2 > 0$. By the Enriques classification, S is of general type. For $q = 4$, we have $K_S = 0$. Whence S is either abelian, bielliptic, quasibielliptic, K3 or Enriques. But $H^1(S, \mathcal{O}_S) = 0$ by Proposition 4.1, whence S is either K3 or a classical Enriques surface. In the latter case, $\pi_1(S)$ is cyclic of order two. In light of Proposition 5.6, our S must be a K3 surface. Finally, suppose $q \leq 3$. Then $-K_{\tilde{S}} = (4-q)\tilde{Z}$ is effective, and one easily checks that $\tilde{Z} \subset \tilde{S}$ is not an exceptional curve of the first kind. Thus $-K_{\tilde{S}}$ is nef. It is also big, according to Proposition 5.8. The same necessarily holds for $-K_S$, hence S is a weak del Pezzo surface. □

6. Canonical models and canonical maps

In this section we introduce projective models for our surfaces. If $q \geq 5$ then $S = \tilde{S}$ is a minimal surface of general type, and the homogeneous spectrum

$$\bar{S} = \operatorname{Proj} H^0(S, \bigoplus_{t\geq 0} \omega_S^{\otimes t})$$

is called the *canonical model* of S. The canonical morphism $S \to \bar{S}$ is the contraction of all (-2)-curves, and the singularities on the normal surface $\bar{S}$ are at most rational double points. Clearly, the integral curves $\tilde{A}_{i,j} \subset \tilde{S} = S$ are (-2)-curves, which get contracted. It turns out that there are no more:

Proposition 6.1. *For $q \geq 5$, the canonical model $\bar{S}$ is obtained by contracting the (-2)-curves $\tilde{A}_{i,j}$, $1 \leq i, j \leq q-1$, such that the singular locus of $\bar{S}$ comprises exactly $q-1$ rational double points of type A_{q-1}.*

Proof. The exceptional curve for the contraction $\tilde{S} \to \bar{S}$ is the union of all (-2)-curves, so all $\tilde{A}_{i,j}$, $1 \leq i, j \leq q-1$ get contracted. Suppose there would be another (-2)-curve $\tilde{E} \subset \tilde{S}$, such that $\tilde{E}$ is disjoint from the canonical divisor $K_{\tilde{S}} = (q-4)\tilde{Z}$. This implies that its strict transform $E \subset (C \times C')/G$ is contained in a smooth fiber of one of the projections $\psi_i : (C \times C')/G \to \mathbb{P}^1$, contradiction. $\square$

Note that the contraction $\tilde{S} \to \bar{S}$ of the (-2)-curves $\tilde{A}_{i,j}$ makes sense for all prime powers $q = p^s$, and it turns out that the normal surface $\bar{S}$ has a very satisfactory projective description. To see this, let $\bar{Z} \subset \bar{S}$ be the Weil divisor defined as the image of $Z \subset X$. Recall that the latter is nothing but the fundamental cycle for the resolution of singularities $X \to (C \times C)/G$. Then clearly

$$\bar{Z}^2 = q \quad \text{and} \quad \bar{Z}_{\text{red}} = \mathbb{P}^1 \quad \text{and} \quad \bar{Z} = q\bar{Z}_{\text{red}} \quad \text{and} \quad K_{\bar{S}} = (q-4)\bar{Z}.$$

Since the local Picard group of a rational double point of type A_{q-1} is cyclic of order q, the Weil divisor $\bar{Z} \subset \bar{S}$ is actually Cartier. The invertible sheaf $\bar{\mathcal{L}} = \mathcal{O}_{\bar{S}}(\bar{Z})$ defines a rational map $\Phi_{\bar{Z}} : S \dashrightarrow \mathbb{P}^n$, with $n + 1 = h^0(\bar{\mathcal{L}})$.

Theorem 6.2. *The invertible sheaf $\bar{\mathcal{L}} = \mathcal{O}_{\bar{S}}(\bar{Z})$ is very ample, has $h^0(\bar{\mathcal{L}}) = 4$, and the image of the closed embedding $\Phi_{\bar{Z}} : \bar{S} \to \mathbb{P}^3$ is a normal surface of degree q whose singular locus consists of $q-1$ rational double points of type A_{q-1}, all lying on a line in $\mathbb{P}^3$.*

Proof. We first check that $\bar{\mathcal{L}}$ is globally generated. Let $\bar{F}_i \subset \bar{S}$, $i = 1, 2$ be the image of the schematic fibers $F_i = \psi_i^{-1}(0)$ for the two projections $\psi_i : X \to \mathbb{P}^1$. Using the multiplicities computed in Proposition 5.4, we deduce $\bar{F}_1 = \bar{Z} = \bar{F}_2$. Now let $\bar{F}_i' \subset \bar{S}$ be the image of the schematic fibers $F_i' = \psi_i^{-1}(\infty)$. Then $\bar{F}_i, \bar{F}_i'$ are linearly equivalent. Using that F_1' intersects F_2 only in the component B_2, we deduce that and $\bar{Z} \cap \bar{F}_1'$ is the image of the contracted curve B_2 under the canonical map $X \to \bar{S}$. By symmetry, $\bar{Z} \cap \bar{F}_2'$ is the image of B_1. The upshot is that $\bar{Z} \cap \bar{F}_1' \cap \bar{F}_2'$ is empty, hence $\bar{\mathcal{L}}$ is globally generated. We also showed that $h^0(\bar{\mathcal{L}}) \geq 3$.

By the same ideas we verify that the resulting morphism $\Phi_{\bar{Z}} : \bar{S} \to \mathbb{P}^n$ is generically injective, where $n + 1 = h^0(\bar{\mathcal{L}})$. Let $x, x' \in X$ be two points not lying on the cycle $Z \cup F_1 \cup F_2$ with $\psi_1(x) \neq \psi_1(x')$. The image on $\bar{S}$ of the schematic fiber $\psi_1^{-1}(\psi_1(x)) \subset X$ is a divisor linearly equivalent to $\bar{Z}$, which contains the image of x, but not the image of x'. Consequently, our map $\Phi_{\bar{Z}}$ is generically injective.

Next, we verify $h^0(\bar{\mathcal{L}}) = 4$. To this end, consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\bar{S}} \longrightarrow \bar{\mathcal{L}} \longrightarrow \bar{\mathcal{L}}|_{\bar{Z}} \longrightarrow 0.$$

Since $h^1(\mathcal{O}_{\bar{S}}) = 0$, we have $h^0(\bar{\mathcal{L}}) = h^0(\bar{\mathcal{L}}|_{\bar{Z}}) + 1$. The Weil divisors $i\bar{Z}_{\text{red}} \subset \bar{S}$ yield short exact sequences

$$0 \longrightarrow \mathcal{K}_i \longrightarrow \mathcal{O}_{(i+1)\bar{Z}_{\text{red}}} \longrightarrow \mathcal{O}_{i\bar{Z}_{\text{red}}} \longrightarrow 0.$$

For $0 \leq i < q$, the kernels are $\mathcal{K}_i = \mathcal{O}_{\mathbb{P}^1}(1-i)$, which follows from the intersection number $\bar{Z} \cdot \bar{Z}_{\text{red}} = 1$ and Proposition 3.1, together with an application of [6], Theorem 1.9. This gives the estimate $h^0(\bar{\mathcal{L}}_{\bar{Z}}) \leq h^0\mathcal{O}_{\mathbb{P}^1}(1) + h^0\mathcal{O}_{\mathbb{P}^1} = 3$. If $h^0(\bar{\mathcal{L}}) = 3$, then $\Phi_{\bar{Z}} : \bar{S} \to \mathbb{P}^2$ would be a finite surjective map of degree q, contradicting generic injectivity. Thus $h^0(\bar{\mathcal{L}}) = 4$.

Summing up, we have a morphism $\Phi_{\bar{Z}} : \bar{S} \to \mathbb{P}^3$ whose image $\hat{S} \subset \mathbb{P}^3$ is a divisor of degree q, and the induced map $\nu : \bar{S} \to \hat{S}$ is the normalization map. Clearly, $\hat{S}$ is Cohen–Macaulay and Gorenstein. Using $\bar{\mathcal{L}} = \Phi_{\bar{Z}}^*(\mathcal{O}_{\mathbb{P}^3}(1))$ and $\omega_{\hat{S}} = \mathcal{O}_{\hat{S}}(q-4)$, we deduce $\omega_{\bar{S}} = \nu^*(\omega_{\hat{S}})$, such that relative dualizing sheaf $\omega_{\bar{S}/\hat{S}}$ is trivial. From this it follows that the conductor locus for the finite birational morphism $\nu : \bar{S} \to \hat{S}$ is empty, such that ν is an isomorphism. Consequently $\bar{\mathcal{L}}$ is very ample. Finally observe that the image of $\bar{Z}_{\mathrm{red}}$, which contains the singular locus of $\bar{S}$, is a line, because $\bar{Z} \cdot \bar{Z}_{\mathrm{red}} = 1$. □

Recall that a proper k-scheme V_0 is called *liftable in the category of schemes*, if there exists a local ring $(R, \mathfrak{m}_R)$ of characteristic zero with $k = R/\mathfrak{m}_R$, together with a proper flat R-scheme V with $V \otimes_W k = V_0$. If such V exists at least as an algebraic space, we say that V_0 is *liftable in the category of algebraic spaces.*

Corollary 6.3. *The surface S is liftable in the category of algebraic spaces.*

Proof. We first consider the projective normal surface $V_0 = \bar{S}$. Let $f_0 \in k[X_0, \ldots, X_4]$ be a homogeneous polynomial of degree $\deg(f_0) = q$ so that $\bar{S} = V_+(f)$ as subscheme of $\mathbb{P}^3 = \operatorname{Proj} k[X_0, \ldots, X_4]$. Choose a homogeneous polynomial f of degree $\deg(f) = q$ with coefficients in the ring of Witt vectors $R = W(k)$ reducing to f_0 modulo p, and consider the proper R-scheme $V = V_+(f) \subset \mathbb{P}^3_R$. Then $V \to \operatorname{Spec}(R)$ is flat and projective, such that $\bar{S}$ is liftable in the category of schemes.

According to a general result of Artin and Brieskorn [3], there exists a finite extension $R \subset R'$ and a simultaneous minimal resolution of singularities $W \to V \otimes_R R'$. Here, however, one knows only that the total space W is an algebraic space, although the individual fibers are projective. Thus $S = W_0$ is liftable in the category of algebraic spaces. □

7. Numerical invariants and geography

The *Chern invariants* of a smooth proper surface are the numbers

$$c_1^2 = K^2 \quad \text{and} \quad c_2 = e.$$

They are paramount for minimal surfaces of general type, and the study of occurrence and distribution of Chern invariants for minimal surfaces of general type is referred to as *surface geography.*

Theorem 7.1. *The Chern invariants for the smooth surface $\tilde{S}$ are given by the formulas*

$$K^2 = q^3 - 8q^2 + 16q \quad \textit{and} \quad e = q^3 - 4q^2 + 6q,$$

and $\chi(\mathcal{O}_{\tilde{S}}) = (q^3 - 6q^2 + 11q)/6$.

Proof. We already computed the value of K^2 in Proposition 5.8. Recall that $X \to \tilde{S}$ is a sequence of $2q$ blowing-ups of $\tilde{S}$, such that $e(X) = e(\tilde{S}) + 2q$. To determine the Euler characteristic, we first examine the surface X and its fibration $\psi_1 : X \to \mathbb{P}^1$. According to Dolgachev's formula [9] we have

$$e(X) = e(X_{\bar{\eta}})e(\mathbb{P}^1) + \sum (e(X_a) - e(X_{\bar{\eta}}) + \delta_a),$$

where $X_{\bar{\eta}}$ is the geometric generic fiber, the sum runs over all closed point $a \in \mathbb{P}^1$, and δ_a is *Serre's measure of wild ramification* attached to the Galois module $M_a = H^1(X_{\bar{\eta}}, \mathbb{Z}/l\mathbb{Z})$ at the point $a \in \mathbb{P}^1$. Here l is any prime number different from p. This invariant is given by

$$\delta_a = \sum_{i \geq 1} \frac{1}{[G : G_i]} \dim_{\mathbb{F}_l}(M_a/M_a^{G_i}),$$

Here G is the Galois group of a finite Galois extension of the function field $\kappa(\eta)$ trivializing the Galois module M_a, and $G_i \subset G$ are the ramification subgroups for the induced extension of discrete valuation rings. By the very construction, we may choose this extension induced from $C \times C' \to (C \times C')/G$, such that $G = \mathbb{F}_q$.

In light of (5), we have $\delta_a = (q-1)\dim_{\mathbb{F}_l}(M_a/M_a^G)$. According to Proposition 5.7, the Picard group $\mathrm{Pic}(X)$ is finitely generated. Since the map $\mathrm{Pic}(X) \to \mathrm{Pic}(X_\eta)$ is surjective, the group $\mathrm{Pic}(X_\eta)$ is finitely generated as well. Thus we may choose our prime l so that $\mathrm{Pic}(X_\eta)$ contains no nontrivial l-torsion. Since the Brauer group of the function field of $\mathbb{P}^1$ vanishes by Tsen's Theorem, it then follows that $M_a^G = 0$, whence $\dim_{\mathbb{F}_l}(M_a/M_a^G) = (q-1)(q-2)$. In turn, we obtain the value for $e(\tilde{S}) = e(X) - 2q$. Finally, Riemann–Roch for surfaces $\chi = (K^2 + e)/12$ yields the formula for $\chi(\mathcal{O}_{\tilde{S}})$. $\square$

Remark 7.2. There are two unfortunate misprints in [9]: The correction terms in Theorem 1.1 should appear with the sign lost in the proof while passing from equation (3.2) to (3.3). In the definition of Serre's measure of wild ramification δ on top of page 305, the sum should run only for $i \geq 1$, which comes from the fact that the Swan character is the difference of the Artin character and the augmentation character. Compare also [24].

We now can read off the *geometric genus* $p_g = h^2(\mathcal{O}_X) = h^0(\omega_S)$:

Corollary 7.3. *The surface S has geometric genus $p_g = (q^3 - 6q^2 + 11q - 6)/6 = (q-2)(q-3)(q-1)/6$.*

Proof. According to Proposition 4.1, we have $h^1(\mathcal{O}_S) = 0$. The statement thus follows from $\chi(\mathcal{O}_S) = (q^3 - 6q^2 + 11q)/6$. $\square$

Using our global invariants, we now can show that our bound on the geometric genus $p_g = \mathrm{length}\, R^1h_*(\mathcal{O}_X)$ of the singularity on $(C \times C')/G$ in Proposition 3.9 is actually an equality, at least in a special case:

Corollary 7.4. *Suppose $q = p$ is prime. Then the singularity on $(C \times C')/G$ has geometric genus $p_g = p(p-1)(p-2)/6$.*

Proof. The Leray–Serre spectral sequence for $h : X \to (C \times C')/G$ gives an exact sequences

$$H^1(X, \mathcal{O}_X) \longrightarrow H^0(R^1h_*(\mathcal{O}_X)) \longrightarrow H^2(\mathcal{O}_{(C\times C')/G}) \longrightarrow H^2(\mathcal{O}_X) \longrightarrow 0.$$

The term on the left vanishes by Proposition 4.1. We thus have

$$p_g = h^0(\omega_{(C\times C')/G}) - h^2(\mathcal{O}_X).$$

The first summand was computed in Proposition 2.4, the second in Corollary 7.3, and the result follows. $\square$

Corollary 7.5. *Suppose $q \geq 5$. Then the surface of general type S has plurigenera*

$$P_m = \frac{q^3 - 8q^2 + 16q}{2}(m^2 - m) + \frac{q^3 - 6q^2 + 11q}{6}$$

for all $m \geq 2$.

Proof. We have $H^1(X, \omega_X^{\otimes m}) = H^1(X, \omega^{\otimes(1-m)}) = 0$ for all $m \geq 2$ according to general results of Ekedahl ([10], Theorem 1.7), and the value for

$$P_m = h^0(\omega_S^{\otimes m}) = \chi(\omega_S^{\otimes m}) = \frac{m^2 - m}{2} K_S^2 + \chi(\mathcal{O}_S), \quad m \geq 2$$

follows from Riemann–Roch and the preceding Theorem. □

Remark 7.6. The *Chern quotients* of our surfaces S asymptotically tend to

$$\lim_{q \to \infty} c_1^2/c_2 = 1.$$

Note also that we always have $3c_2 - c_1^2 = 2q(q-1)^2 > 0$, such that the *Bogomolov–Miyaoka–Yau inequality* $c_1^2 < 3c_2$ holds. Therefore our surfaces S show no exotic behavior with respect to geography.

Remark 7.7. In the case $q = 4$, we have $K_S = 0$, and the preceding Theorem gives $e(S) = 24$. This yields another proof that S is a K3 surface rather than an Enriques surface.

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