HILBERT'S THEOREM 90 AND ALGEBRAIC SPACES

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Abstract. In modern form, Hilbert's Theorem 90 tells us that $R^1\epsilon_*(\mathbb{G}_m) = 0$, where $\epsilon : X_{\text{ét}} \to X_{\text{zar}}$ is the canonical map between the étale site and the Zariski site of a scheme X. I construct examples showing that the corresponding statement for algebraic spaces does not hold.

Introduction

Originally, Hilbert's Theorem 90 is the following number theoretical result [5]: Given a cyclic Galois extension $K \subset L$ of number fields, each $y \in L^\times$ of norm $N(y) = 1$ is of the form $y = x/x^\sigma$ for some $x \in K^\times$ and a given generator $\sigma \in G$ of the Galois group. More generally, Speiser [12] proved that $H^1(G, L^\times) = 1$ for arbitrary Galois extensions (compare the discussion in [8]).

The latter statement has a geometric interpretation: Each line bundle on the étale site of $\operatorname{Spec}(k)$ is trivial. In this form, it admits a far-reaching generalization: If $\epsilon : X_{\text{ét}} \to X_{\text{zar}}$ is the canonical map from the étale site to the Zariski site of a scheme X, then $R^1\epsilon_*(\mathbb{G}_m) = 0$ (see [9], page 124). The result entails, among other things, that the map of Picard groups $\operatorname{Pic}(X_{\text{zar}}) \to \operatorname{Pic}(X_{\text{ét}})$ is bijective, and that the map of Brauer groups $\operatorname{Br}(X_{\text{zar}}) \to \operatorname{Br}(X_{\text{ét}})$ is injective.

It is natural to ask whether a similar statement holds for algebraic spaces instead of schemes. Recall that an *algebraic space* is the quotient $X = U/R$ of a scheme X by an étale equivalence relation $R \rightrightarrows X$. Here the quotient takes place in the topos $(\mathrm{Sch})^\sim_{\text{ét}}$, that is, as a sheaf on the étale site.

Unfortunately, such a generalization does not hold. The goal of this paper is to construct counterexamples, that is, algebraic spaces X and invertible $\mathcal{O}_X$-modules $\mathcal{L}$ such that the open subspaces $V \subset X$ trivializing $\mathcal{L}$ do not cover X. The first example is a nonseparated smooth 1-dimensional *bug-eyed cover* in Kollár's sense [7]. The second example is a nonnormal proper algebraic space obtained by identifying points on suitable nonprojective smooth proper schemes.

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1. Line bundles on algebraic spaces

In this section we recall some basic facts on algebraic spaces and their line bundles. Let $(\mathrm{Sch})_{\text{ét}}$ be the site of schemes endowed with the Grothendieck topology

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generated by the étale surjective morphisms, and $(\mathrm{Sch})^{\sim}_{\text{ét}}$ be the corresponding topos of sheaves. By definition, a sheaf $X \in (\mathrm{Sch})^{\sim}_{\text{ét}}$ is an *algebraic space* if $X = U/R$ for some scheme U and some étale equivalence relation $R \rightrightarrows U$ such that the induced morphism $R \to U \times U$ is quasicompact [6].

Given an algebraic space X, let $\text{Ét}(X)$ be the category of algebraic X-spaces whose structure map $Y \to X$ is étale. The étale surjections $Y_1 \to Y_2$ define a topology on $\text{Ét}(X)$, and we write $X_{\text{ét}}$ for the corresponding site. Let me give a down-to-earth description of sheaves $\mathcal{F}$ on this site. For each scheme U endowed with an étale map $U \to X$, we obtain via restriction a sheaf $\mathcal{F}_U$ on the étale site of étale U-schemes. If $f : U \to V$ is an X-morphisms, we have a map $\theta_f : \mathcal{F}_V \to f_*\mathcal{F}_U$. Such systems $(\mathcal{F}_U, \theta_f)$ are not arbitrary. Consider the following two conditions: (1) If $f : U \to V$ and $g : V \to W$ are X-maps, then the diagram

$$\begin{array}{ccc} \mathcal{F}_W & \xrightarrow{\theta_{gf}} & (gf)_*\mathcal{F}_U \\ {\scriptstyle \theta_g}\downarrow & & \downarrow{\scriptstyle \simeq} \\ g_*(\mathcal{F}_V) & \xrightarrow[g_*(\theta_f)]{} & g_*(f_*\mathcal{F}_U) \end{array}$$

is commutative. (2) If $f : U \to V$ is étale, then the map $\theta_f^\sharp : f^{-1}\mathcal{F}_V \to \mathcal{F}_U$ is bijective. Here the mapping $\theta_f^\sharp$ corresponds to θ_f with respect to the canonical adjunction $\mathrm{Hom}(f^{-1}\mathcal{F}_V, \mathcal{F}_U) \simeq \mathrm{Hom}(\mathcal{F}_V, f_*\mathcal{F}_U)$.

Proposition 1.1. *The assignment $\mathcal{F} \mapsto (\mathcal{F}_U, \theta_f)$ yields an equivalence between the category of sheaves on $X_{\text{ét}}$ and the category of systems $(\mathcal{F}_U, \theta_f)$ satisfying conditions* (1) *and* (2)*.*

Proof. Let $\mathcal{C}$ be the site of étale X-schemes with the induced étale topology. By the Comparison Lemma ([3], Exposé III, Théorèm 4.1), the inclusion $\mathcal{C} \subset X_{\text{ét}}$ induces an equivalence on the corresponding categories of sheaves. Now suppose $\mathcal{F}$ is a sheaf on $\mathcal{C}$. Then the system $(\mathcal{F}_U, \theta_f)$ satisfies condition (1) because $\mathcal{F}$ is a presheaf. If $f : U \to V$ is étale, then $\theta_f^\sharp$ is bijective because $\mathcal{F}$ is a sheaf in the étale topology, and condition (2) holds as well.

Conversely, given such a system, we define $\Gamma(U, \mathcal{F}) = \Gamma(U, \mathcal{F}_U)$. Indeed, this is a presheaf by condition (1), and a sheaf by condition (2). One easily checks that the functors $\mathcal{F} \mapsto (\mathcal{F}_U, \theta_f)$ and $(\mathcal{F}_U, \theta_f) \mapsto \mathcal{F}$ are inverse equivalences of categories. □

For example, the sheaves $\mathcal{O}_U$, together with the maps $\theta_f : \mathcal{O}_V \to f_*(\mathcal{O}_U)$, correspond to the structure sheaf $\mathcal{O}_X$ of an algebraic space X. Similar, we have the sheaf of units $\mathcal{O}_X^\times$. The cohomology group $\mathrm{Pic}(X_{\text{ét}}) = H^1(X_{\text{ét}}, \mathcal{O}_X^\times)$ is the group of isomorphism classes of invertible $\mathcal{O}_X$-modules.

Besides the étale topology, the category $\text{Ét}(X)$ carries the coarser *Zariski topology* as well. Here the covering families are the surjections of the form $\coprod X_i \to X$, where the $X_i \subset X$ are open subspaces, and we demand that $X_i \times_X X' \to X'$ remains an open embedding for any base change $X' \to X$. Write X_{zar} for the corresponding site. The sheaves on X_{zar} admit a similar description in terms of families $(\mathcal{F}_U, \theta_f)$ satisfying condition (1), and condition (2'), where we demand that $\theta_f^\sharp : f^{-1}\mathcal{F}_V \to \mathcal{F}_U$ is bijective whenever $f : U \to V$ is of the form $U = \coprod V_i$ with open subschemes $V_i \subset V$. In particular, we have a structure sheaf $\mathcal{O}_{X_{\text{zar}}}$ and a unit sheaf $\mathcal{O}_{X_{\mathrm{zar}}}^\times$. Let $\operatorname{Pic}(X_{\mathrm{zar}}) = H^1(X_{\mathrm{zar}}, \mathcal{O}_{X_{\mathrm{zar}}}^\times)$ be the corresponding group of line bundles.

The identity functor on $\text{Ét}(X)$ is a continuous functor $\epsilon : X_{\text{ét}} \to X_{\mathrm{zar}}$ of sites, and we have $\epsilon_*(\mathcal{O}_{X_{\text{ét}}}) = \mathcal{O}_{X_{\mathrm{zar}}}$ by descent theory. So for each invertible $\mathcal{O}_{X_{\mathrm{zar}}}$-module $\mathcal{L}$, the canonical map $\mathcal{L} \to \epsilon_*\epsilon^*\mathcal{L}$ is bijective, and we obtain an injection $\operatorname{Pic}(X_{\mathrm{zar}}) \subset \operatorname{Pic}(X_{\text{ét}})$.

Proposition 1.2. *Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Its isomorphism class lies in the subgroup* $\operatorname{Pic}(X_{\mathrm{zar}}) \subset \operatorname{Pic}(X_{\text{ét}})$ *if and only if there is a covering with open subspaces $Y_i \subset Y$ with $\mathcal{L}_{Y_i} \simeq \mathcal{O}_{Y_i}$.*

Proof. The spectral sequence for the composition $\Gamma(X_{\text{ét}}, \mathcal{O}_{X_{\text{ét}}}^\times) = \Gamma(X_{\mathrm{zar}}, \epsilon_*\mathcal{O}_{X_{\text{ét}}}^\times)$ yields an exact sequence

$$0 \longrightarrow \operatorname{Pic}(X_{\mathrm{zar}}) \longrightarrow \operatorname{Pic}(X_{\text{ét}}) \longrightarrow H^0(X_{\mathrm{zar}}, R^1\epsilon_*\mathcal{O}_{X_{\text{ét}}}^\times).$$

The condition precisely means that the image of the invertible sheaf $\mathcal{L}$ under the canonical map $\operatorname{Pic}(X_{\text{ét}}) \to H^0(X_{\mathrm{zar}}, R^1\epsilon_*\mathcal{O}_{X_{\text{ét}}}^\times)$ vanishes. The statement now follows from the exact sequence. $\square$

2. Bug-eyed covers

In this section, we use Kollár's bug-eyed covers to construct a smooth 1-dimensional *nonseparated* algebraic space X and an invertible sheaf $\mathcal{L}$ such that the open subspaces $W \subset X$ trivializing $\mathcal{L}$ do not form a covering.

Fix a ground field k of characteristic $\neq 2$. Set $A = k[[T]]$ and $A' = k[[T^2]]$, and let $Y = \operatorname{Spec}(A)$ and $Y' = \operatorname{Spec}(A')$ be the corresponding affine schemes. The inclusion $A' \subset A$ defines a flat double covering $p : Y \to Y'$. The open subset $U \subset Y$ given by the generic point is the locus where f is étale. The generator $\sigma \in G$ of the group $G = \mathbb{Z}/2\mathbb{Z}$ acts on A via $T^\sigma = -T$, which defines a free G-action on U. Consider the étale equivalence relation

$$R = \Delta_Y \amalg U \longrightarrow Y \times Y,$$

where the embedding of U is given by $U \xrightarrow{\mathrm{id} \times \sigma} U \times U \subset Y \times Y$. Let $X = Y/R$ be the corresponding quotient sheaf in $(\mathrm{Sch}/k)^\sim_{\text{ét}}$. By definition, X is a smooth algebraic space. It is nonseparated because the injection $R \to Y \times Y$ is not closed.

The map $p : Y \to Y'$ factors over X, and the induced projection $X \to Y'$ induces a bijection of points. The algebraic space X is a *bug-eyed cover* in Kollár's sense [7]. It is not a scheme. Otherwise, the morphism $X \to Y'$ would be an isomorphism by Zariski's Main Theorem, and $Y \to X$ would be both étale and ramified.

Proposition 2.1. *We have* $\operatorname{Pic}(X_{\text{ét}}) = \mathbb{Z}/2\mathbb{Z}$.

Proof. The scheme Y is local, hence every invertible $\mathcal{O}_X$-module $\mathcal{L}$ has $\mathcal{L}_Y \simeq \mathcal{O}_Y$. Thus, $\operatorname{Pic}(X_{\text{ét}})$ is the cohomology of the complex

$$\Gamma(Y, \mathcal{O}_X^\times) \xrightarrow{d_0} \Gamma(Y^2, \mathcal{O}_X^\times) \xrightarrow{d_1} \Gamma(Y^3, \mathcal{O}_X^\times).$$

Here Y^n are the n-fold fiber products over X. If $p_i : Y^{n+1} \to Y^n$ denotes the projection omitting the i-th factor, the differentials are $d_0(s) = p_0^*(s)/p_1^*(s)$ and $d_1(s) = p_0^*(s)p_2^*(s)/p_1^*(s)$.

Clearly, we have $Y^n = U^n \cup \Delta_Y$, where $U^n \cap \Delta_Y = \Delta_U$. Since the G-action is free on the open subset $U \subset Y$, we have a bijection

$$U \times G^n \longrightarrow U^{n+1}, \quad (u, g_1, \ldots, g_n) \longmapsto (u, ug_1, \ldots, ug_1g_2 \ldots g_n).$$

In turn, we may identify the n-cochains $\Gamma(Y^{n+1}, \mathcal{O}_X^\times)$ with the the group of functions $c : G^n \to P^\times$ satisfying $c(0,\ldots,0) \in A^\times$. Here $P = k[[T]][T^{-1}]$ is the fraction field of $A = k[[T]]$. The differentials take the form

$$d_0(c)(g) = c(0)/c(0)^g \quad \text{and} \quad d_1(c)(g,h) = c(h)^g c(g)/c(gh),$$

conforming with the usual definition of group cohomology ([2], page 59). We have $d_0(c)(0) = 1$, and $d_0(c)(\sigma)$ is a power series of the form $\lambda_0 + \lambda_1 T + \lambda_2 T^2 + \ldots$ with $\lambda_0 = 1$. One easily checks that a 1-cochain $c : G \to P^\times$ is a 1-cocycle if and only if $c(0) = 1$, and $p = c(\sigma)$ satisfies $p \cdot p^\sigma = 1$. Clearly, the 1-cocycle $c : G \to P^\times$ with $c(0) = 1$ and $c(\sigma) = -1$ is not a coboundary, so $\mathrm{Pic}(X_{\text{ét}})$ is nonzero. On the other hand, by Hilbert's Theorem 90, each $p \in P^\times$ with $p \cdot p^\sigma = 1$ is of the form $p = r/r^\sigma$ for some $r \in P^\times$. Writing $r = T^n s$ with $s \in A^\times$, we have $p = (-1)^n s/s^g$, and infer $\mathrm{Pic}(X_{\text{ét}}) = \mathbb{Z}/2\mathbb{Z}$. □

The smooth 1-dimensional nonseparated algebraic space X is our first counterexample to Hilbert's Theorem 90 for algebraic spaces:

Theorem 2.2. *The canonical inclusion* $\mathrm{Pic}(X_{\text{zar}}) \subset \mathrm{Pic}(X_{\text{ét}})$ *is not surjective.*

Proof. The scheme Y is local, so the space of points for X has a unique closed point. Consequently, any Zariski covering of X contains a copy of X. So any line bundle on X_{zar} is trivial, that is, $\mathrm{Pic}(X_{\text{zar}}) = 0$. On the other hand, $\mathrm{Pic}(X_{\text{ét}}) \neq 0$ by Proposition 2.1. □

3. Nonnormal proper algebraic spaces

Fix an algebraically closed ground field k. In this section, we shall construct a *proper* algebraic space X and an invertible sheaf $\mathcal{L}$ such that the open subspaces $W \subset X$ trivializing $\mathcal{L}$ do not form a covering.

The starting point is a proper smooth k-scheme Y containing two irreducible closed curves $C_1, C_2 \subset Y$ such that $C_1 + C_2$ is numerically trivial. This implies that the generic points $\eta_i \in C_i$ do not admit any common affine neighborhood in Y. Examples of such schemes appear in [11], page 75. Obviously, they are nonprojective. Even worse, they do not admit embeddings into toric varieties ([13], Theorem A). Recall that the support $\mathrm{Supp}(D) \subset Y$ of a Cartier divisor $D \in \mathrm{Div}(Y)$ is the union of its positive and negative part. We have the following useful property:

Proposition 3.1. *Each* $D \in \mathrm{Div}(Y)$ *with* $D \cdot C_1 > 0$ *and* $C_1 \not\subset \mathrm{Supp}(D)$ *has* $C_2 \subset \mathrm{Supp}(D)$.

Proof. Decompose $D = \sum n_i D_i$ into prime divisors with $n_i \neq 0$. Since $C_1 \not\subset D_i$, the intersection number $D_i \cdot D_1$ is the length of the scheme $D_i \cap C_1$, hence nonnegative. So there is at least one prime divisor with $D_i \cdot C_1 > 0$. It follows $D_i \cdot C_2 < 0$, hence $C_2 \subset D_i$. In other words, $C_2 \subset \mathrm{Supp}(D)$. □

Now fix two closed points $y_1 \in C_1$ and $y_2 \in C_2$. Let $Y' \subset Y$ be the reduced closed subscheme corresponding to $\{y_1, y_2\}$, and define an étale sheaf $X \in (\mathrm{Sch}/k)^\sim_{\text{ét}}$ by the cocartesian square

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ \mathrm{Spec}(k) \arrow[r] & X \end{tikzcd}$$

Note that $(\mathrm{Sch}/k)^{\sim}_{\text{ét}}$, being a topos, admits all colimits ([3], Exposé II, Theorem 4.1). Intuitively, X is obtained from Y by identifying the points $y_1, y_2 \in Y$. The sheaf X is not a scheme. Otherwise, an affine open neighborhood for the point $p(y_1) = p(y_2) \in X$ would give a common affine open neighborhood for the pair $y_1, y_2 \in Y$.

Proposition 3.2. *The étale sheaf X is a proper algebraic space.*

Proof. That X is an algebraic space follows immediately from [1], Theorem 6.1. Let me give a more direct argument as follows. Fix two copies $v_1', v_2' \in V'$ and $v_1'', v_2'' \in V''$ of $y_1, y_2 \in Y$, and set $V = V' \amalg V''$. Identifying $v_1' \in V$ with $v_2'' \in V$ and $v_2' \in V$ with $v_1'' \in V$, we obtain a scheme U. The group $G = \mathbb{Z}/2\mathbb{Z}$ acts freely on U by interchanging V' and V''. Clearly, $X = U/G$ is the quotient of this action in the topos of étale sheaves. So $R = U \times_X U$ is nothing but $U \times G$, which is a scheme. Consequently, $X = U/R$ is an algebraic space.

The algebraic space X is separated because the embedding $Y \times G \to Y \times Y$, $(y, g) \mapsto (y, yg)$ is closed. As $Y \to \mathrm{Spec}(k)$ is universally closed and $p : Y \to X$ is surjective, $X \to \mathrm{Spec}(k)$ is universally closed as well. Therefore, X is proper. □

Proposition 3.3. *There is an exact sequence* $1 \to k^\times \to \mathrm{Pic}(X_{\text{ét}}) \to \mathrm{Pic}(Y) \to 0$.

Proof. Let $p : Y \to X$ be the canonical projection. Then the sequence

$$1 \longrightarrow \mathcal{O}_X^\times \longrightarrow p_*(\mathcal{O}_Y^\times) \oplus k^\times \longrightarrow p_*(\mathcal{O}_{Y'}^\times) \longrightarrow 1$$

is exact. Indeed, one easily checks this, as in [4], Lemma 5.1, after base change with an affine étale cover $U \to X$. In turn, we obtain an exact sequence

$$\Gamma(\mathcal{O}_Y^\times) \oplus k^\times \longrightarrow \Gamma(\mathcal{O}_{Y'}^\times) \longrightarrow \mathrm{Pic}(X_{\text{ét}}) \longrightarrow \mathrm{Pic}(Y) \oplus \mathrm{Pic}(k) \longrightarrow \mathrm{Pic}(Y').$$

Being semilocal, the schemes $\mathrm{Spec}(k)$ and Y' have no Picard groups. The cokernel for the map on the left is isomorphic to $k^\times$, and the result follows. □

The proper algebraic space X is another counterexample to Hilbert's Theorem 90 for algebraic spaces:

Theorem 3.4. *The canonical inclusion* $\mathrm{Pic}(X_{\text{zar}}) \subset \mathrm{Pic}(X_{\text{ét}})$ *is not surjective.*

Proof. Choose an invertible $\mathcal{O}_Y$-module $\mathcal{M}$ with $\mathcal{M} \cdot C_1 > 0$. For example, $\mathcal{M}$ could by the invertible sheaf corresponding to the reduced complement of any affine open neighborhood for $y_1 \in Y$.

Let $p : Y \to X$ be the canonical map. According to Proposition 3.3, there is an invertible $\mathcal{O}_X$-module $\mathcal{L}$ with $\mathcal{M} = p^*(\mathcal{L})$. Suppose there is an open subset $W \subset X$ containing the point $p(y_1) = p(y_2)$ and trivializing $\mathcal{L}$. Then $\mathcal{M}$ is trivial on the open subscheme $p^{-1}(W) \subset Y$. By [10], Theorem 3.3, there is a Cartier divisor $D \in \mathrm{Div}(X)$ representing $\mathcal{M}$ with support disjoint from $y_1, y_2 \in Y$. In particular, C_1 and C_2 are not contained in $\mathrm{Supp}(D)$, contradicting Proposition 3.1. □

Question 3.5. Does $\mathrm{Pic}(X_{\text{zar}}) = \mathrm{Pic}(X_{\text{ét}})$ at least hold for smooth proper algebraic spaces? What about the case that X is normal and proper?

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