KUMMER SURFACES FOR THE SELFPRODUCT OF THE CUSPIDAL RATIONAL CURVE

STEFAN SCHRÖER

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Abstract. The classical Kummer construction attaches to an abelian surface a K3 surface. As Shioda and Katsura showed, this construction breaks down for supersingular abelian surfaces in characteristic two. Replacing supersingular abelian surfaces by the selfproduct of the rational cuspidal curve, and the sign involution by suitable infinitesimal group scheme actions, I give the correct Kummer-type construction for this situation. We encounter rational double points of type D_4 and D_8, instead of type A_1. It turns out that the resulting surfaces are supersingular K3 surfaces with Artin invariant one and two. They lie in a 1-dimensional family obtained by simultaneous resolution, which exists after purely inseparable base change.

Contents

Introduction

Let A be an abelian surface over the complex numbers, and $\iota : A \to A$ the sign involution. The quotient surface $Z = A/\iota$ is a normal surface with 16 rational double points of type A_1, whose minimal resolution X is a K3 surface. One also

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says that X is a Kummer K3 surface; they play a fairly central role in the theory of all complex K3 surfaces.

It is easy to see that the Kummer construction works in positive characteristics $p \neq 2$ as well. In contrast, Shioda [37] and Katsura [22] observed that the Kummer construction breaks down in characteristic $p = 2$ for supersingular abelian surfaces A. In this case, they showed that singularities on the quotient surface Z are elliptic singularities, and that the minimal resolution X is a rational surface.

The goal of this paper is to give a new type of Kummer construction in the supersingular situation at $p = 2$. To explain this construction, let me discuss the case where A is superspecial, that is, isomorphic to the selfproduct $E \times E$ of supersingular elliptic curves. My idea is to replace the supersingular elliptic curve E by a cuspidal rational curve C, and the group action of $\mathbb{Z}/2\mathbb{Z}$ by a suitable group scheme action of the infinitesimal group scheme α_2. In particular, we start with the *nonnormal* surface $Y = C \times C$. Nevertheless, it turns out that the quotient $Z = Y/\alpha_2$ is a normal surface, whose singularities are rational double points of type D_4, B_3, or D_8, at least if the ground field is perfect. The first main result of this paper is the following:

Theorem. *The minimal resolution of singularities $X \to Z$ is a K3 surface.*

There are complications for nonperfect ground fields. It seems that the minimal resolution could be a regular surface with trivial canonical class that is not geometrically regular. This seems to be an interesting subject matter in its own right, but I do not pursue this topic in the present paper.

The Kummer construction also plays an important role in the theory of supersingular K3 surfaces in characteristic $p > 0$. Recall that a K3 surface X is called *supersingular* (in the sense of Shioda) if its Picard number equals the second Betti number, that is, $\rho = 22$. Artin [1] introduced for such K3 surfaces an integer invariant $1 \leq \sigma_0 \leq 10$ called the *Artin invariant*, which can be defined in terms of discriminants of the intersection form. Shioda [36] proved for $p \neq 2$ that the supersingular K3 surfaces with Artin invariant $1 \leq \sigma_0 \leq 2$ are precisely the Kummer K3 surfaces coming from supersingular abelian surfaces. The second main result of this paper is:

Theorem. *Our K3 surfaces X are supersingular with Artin invariant* $1 \leq \sigma_0 \leq 2$.

This depends on an analysis of the degenerate fibers in the quasielliptic fibrations $f : X \to \mathbb{P}^1$ induced from the projections on $Y = C \times C$. Drops in the Artin invariants are due to confluence of a pair of D_4-singularities into a single D_8-singularity.

Oort [28] showed that any supersingular abelian surface is an infinitesimal quotient of a superspecial abelian surface. This implies that supersingular abelian surfaces A form a 1-dimensional family, in which the action of $\mathbb{Z}/2\mathbb{Z}$ is constant. In our new Kummer construction, it is the other way round: The nonnormal surface $Y = C \times C$ does not move, but the action of α_2 lies in a moving family. This is reminiscent of the Moret-Bailly construction [25]. Our construction gives a family $\mathfrak{Z} \to S$ of normal surfaces with rational double points over the punctured affine plane $S = \mathbb{A}^2 - 0$. According to the work of Brieskorn [10] and Artin [4], simultaneous resolutions exist rarely without base change. This is indeed the case in our situation:

Theorem. *The family of normal surface $\mathfrak{Z} \to S$ admits a simultaneous minimal resolution of singularities after certain purely inseparable base change.*

We achieve simultaneous resolution by successively blowing up Weil divisors in multiple fibers on the quasielliptic fibrations $Z \to \mathbb{P}^1$. The purely inseparable base change enters the picture, because the original family contains integral fibers switching to multiple fibers over the perfect closure. It turns out that the resulting family of K3 surfaces $\mathfrak{X} \to S'$, where $S' \to S$ is the purely inseparable base change, is induced from a family of $K3$ surfaces parameterized by the projective line $\mathbb{P}^1$. This is in accordance with results of Rudakov and Shafarevich [30] and Ogus [27] on the moduli of marked supersingular K3 surfaces.

The paper is organized as follows: Section 1 discusses the interplay between restricted Lie algebras and infinitesimal group schemes of height ≤ 1, in particular α_p. Section 2 contains some results on the behavior of quotients with respect to base change and singularities. In Section 3 we examine the rational cuspidal curve C in characteristic two, determine its restricted Lie algebra, and read off all possible α_2-actions. In Section 4, we consider the selfproduct $Y = C \times C$, define α_2-actions on it, and establish some basic properties. This is refined in the following two sections, where we determine the structure of singularities on the quotient $Z = Y/\alpha_2$. I return to global properties of Y in Section 7, where we analyse quasielliptic fibrations. In Section 8 we turn to the the resolution of singularities X, and establish that it is a supersingular K3 surface provided that the parameters do not vanish simultaneously. In Section 9, we determine the Artin invariant. Here we crucially use the structure of the singularities and the quasielliptic fibration. Sections 10 contains some material on blowing up rational surface singularities along curves. We use this in Section 11, where we consider blowing ups of Weil divisors in genus-one fibrations. This is crucial in Section 12, where we view our normal K3 surfaces as lying in a family $\mathfrak{Z} \to S$, and show that this family admits a simultaneous resolution $\mathfrak{X}' \to S'$ after a purely inseparable base change $S' \to S$. In the last section we show that this family is induced from a family $\mathfrak{X} \to \mathbb{P}^1$.

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1. Generalities on α_p-actions

Let k be a ground field of characteristic $p > 0$. Then there is a group scheme α_p whose values on k-algebras R are $\alpha_p(R) = \{r \in R \mid r^p = 0\}$, with addition as group law. As a scheme, we have $\alpha_p = \operatorname{Spec} k[t]/(t^p)$, which is finite and infinitesimal. Such group schemes exist only in characteristic $p > 0$; it is frequently possible to explain characteristic-p-phenomena in terms of α_p-actions. In this section I collect some well-known facts on finite infinitesimal group schemes, and in particular on α_p. Proofs are omitted. The book of Demazure and Gabriel [12] is an exhaustive reference.

The Lie algebra $\mathrm{Lie}(\alpha_p) = \alpha_p(k[\epsilon])$ is the 1-dimensional vector space $k\epsilon \subset k[\epsilon]$. The generator ϵ corresponds to the derivation $\partial : k[t]/(t^p) \to k$ given by the Kronecker delta $\partial(t^i) = \delta_{i1}$. The interpretation $\mathrm{Lie}(\alpha_p) = \mathrm{Der}(k[t]/(t^p), k)$ shows that both the Lie bracket $[D, D'] = D \circ D' - D' \circ D$ and the p-th power operation $D^{[p]} = D \circ \ldots \circ D$ (p-fold composition) vanish. In other words, $\mathrm{Lie}(\alpha_p)$ is a restricted Lie algebra whose Lie bracket and p-map both vanish.

Recall that a *restricted Lie algebra* over k is a Lie algebra $\mathfrak{g}$ endowed with a p-map $\mathfrak{g} \to \mathfrak{g}$, $x \mapsto x^{[p]}$ (sometimes restricted Lie algebras are also called *p-Lie algebras*). By definition, p-maps satisfy three axioms. The first axiom relates them with scalar multiplication:

$$(\lambda x)^{[p]} = \lambda^p x^{[p]} \quad \text{for all } \lambda \in k,\, x \in \mathfrak{g}.$$

The second axiom relates p-maps with Lie brackets:

$$[x^{[p]}, y] = [x, [x, [\ldots, [x, y]]]] \quad \text{for all } x, y \in \mathfrak{g},$$

where the right-hand side is a p-fold iterated Lie bracket. The third axiom relates p-maps to addition:

$$(x + y)^{[p]} = x^{[p]} + \Lambda_p(x, y) + y^{[p]} \quad \text{for all } x, y \in \mathfrak{g},$$

where $\Lambda_p(x, y)$ is a universal expression in terms of iterated Lie brackets due to Jacobson. Rather than explaining this tricky matter, I refer to [12], Chapter II, Section 7.2. Note, however, that in characteristic $p = 2$ things simplify and we have $(x + y)^{[2]} = x^{[2]} - [x, y] + y^{[2]}$.

The group scheme α_p is entirely determined by its restricted Lie algebra $\mathrm{Lie}(\alpha_p)$. This correspondence works, more generally, for finite infinitesimal group schemes G of height ≤ 1. Let me recall the basic steps in this correspondence. If $\mathfrak{g} = \mathrm{Lie}(G)$ is the restricted Lie algebra, one may interpret the restricted universal enveloping algebra $U^{[p]}(\mathfrak{g})$ as the *algebra of distributions* on G at the neutral element $e \in G$, and its k-linear dual $U^{[p]}(\mathfrak{g})^\vee$ as the *algebra of functions* on G. In particular, we have

$$G = \mathrm{Spec}(U^{[p]}(\mathfrak{g})^\vee).$$

Here the diagonal map $\mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$ induces the multiplication in $U^{[p]}(\mathfrak{g})^\vee$. Recall that $U^{[p]}(\mathfrak{g})$ is the quotient of the universal enveloping algebra $U(\mathfrak{g})$ modulo the ideal generated by the elements $x^p - x^{[p]}$ with $x \in \mathfrak{g}$. From this discussion it follows that the canonical map

$$\mathrm{Hom}(G, H) \longrightarrow \mathrm{Hom}(\mathfrak{g}, \mathfrak{h}), \quad \varphi \longmapsto \mathrm{Lie}(\varphi)$$

is bijective for all group schemes H, where $\mathfrak{h} = \mathrm{Lie}(H)$.

Now consider the special case that $H = \mathrm{Aut}_{Y/k}$ for some k-scheme Y, such that $\mathfrak{h} = H^0(Y, \Theta_{Y/k})$. Then the set of G-actions on Y is in correspondence to the set of homomorphisms $\varphi : \mathfrak{g} \to H^0(Y, \Theta_{Y/k})$ of restricted Lie algebras. Indeed, such a homomorphism of restricted Lie algebras induces a homomorphism of k-algebras $U^{[p]}(\mathfrak{g}) \to \mathrm{Diff}(\mathcal{O}_Y, \mathcal{O}_Y)$ into the algebra of differential operators, and the adjoint map $\mathcal{O}_Y \to \mathcal{O}_Y \otimes_k U^{[p]}(\mathfrak{g})^\vee$ then defines the desired action $Y \times G \to Y$.

In the case $G = \alpha_p$, the preceding discussion simplifies as follows. To give a homomorphism of group schemes $\varphi : \alpha_p \to H$ is nothing but to give a vector $\delta \in \mathrm{Lie}(H)$ with $\delta^{[p]} = 0$, by setting $\delta = \mathrm{Lie}(\varphi)(\partial)$. As a special case we have $\mathrm{End}(\alpha_p) = k$ and $\mathrm{Aut}(\alpha_p) = k^\times$. If we change the vector δ by a nonzero factor

$\lambda \in k$, we only compose the homomorphism $\varphi : \alpha_p \to H$ with an automorphism of α_p, but do not change its image. Note that the subset

$$\left\{\delta \in \mathrm{Lie}(H) \mid \delta^{[p]} = 0\right\} \subset \mathrm{Lie}(H)$$

is a cone, but in general not a subgroup. It follows that the space of nonzero homomorphisms $\varphi : \alpha_p \to H$ may have several connected components. For later use we record:

Proposition 1.1. *Let Y be a k-scheme. Then there is a bijection between the set of α_p-actions on Y and the vectors $\delta \in H^0(Y, \Theta_{Y/k})$ with $\delta^{[p]} = 0$.*

It is worthwile to see this correspondence explicitly: Write $G = \operatorname{Spec} k[t]/(t^p)$ and $\mathfrak{g} = k\partial$, where $\partial(t^i) = \delta_{i1}$ is given by the Kronecker delta. The restricted universal enveloping algebra is the truncated polynomial algebra $U^{[p]}(\mathfrak{g}) = k[\partial]/(\partial^p)$. The monomials ∂^i with $0 \leq i < p$ form a basis in $U^{[p]}(\mathfrak{g})$. Its dual basis in $U^{[p]}(\mathfrak{g})^\vee$ are the divided powers $\gamma_i = t^i/i!$. Now suppose we have a vector field $\delta \in H^0(Y, \Theta_{Y/k})$ with $\delta^{[p]} = 0$. Then the corresponding α_p-action on Y is given by the formula

$$\mathcal{O}_Y \longrightarrow \mathcal{O}_Y \otimes U^{[p]}(\mathfrak{g})^\vee, \quad f \longmapsto \sum_{i=0}^{p-1} \frac{\delta^i(f)}{i!} \otimes t^i. \tag{1}$$

Note that the formula involves differential operators δ^i, $1 < i < p$, which are in general not derivations. For $p = 2$, however, the situation simplifies a lot, for we just have $f \mapsto f \otimes 1 + \delta(f) \otimes t$.

Using Formula (1), it is now easy to understand invariant or fixed subschemes: Let $A \subset Y$ be a closed subscheme with ideal $\mathcal{I} \subset \mathcal{O}_Y$. It follows from Formula (1) that A is *invariant* if and only if $\delta(\mathcal{I}) \subset \mathcal{I}$, or equivalently that the composition $\delta : \mathcal{I} \to \mathcal{O}_A$ is zero. The schematic image $\alpha_p \cdot A \subset Y$ of the morphism $G \times A \to Y$ is called the *orbit* of A. Its ideal $\mathcal{J} \subset \mathcal{O}_Y$ is the intersection of the kernels for the k-linear maps $\delta^i : \mathcal{I} \to \mathcal{O}_A$ for $i = 1, \ldots, p-1$, again by Formula (1). For $p = 2$ we simply have $\mathcal{J} = \operatorname{kern}(\delta : \mathcal{I} \to \mathcal{O}_A)$.

The induced action on an invariant closed subscheme $A \subset Y$ is trivial if and only if $\delta(\mathcal{O}_Y) \subset \mathcal{I}$. Equivalently, the composite map $\delta : \mathcal{O}_Y \to \mathcal{O}_A$ vanishes. In particular, a closed point $y \in Y$ with maximal ideal $\mathfrak{m} \subset \mathcal{O}_Y$ is a *fixed point* if and only if $\delta(\mathcal{O}_Y) \subset \mathfrak{m}$. The fixed closed subsets correspond to the quasicoherent ideals containing the abelian subsheaf $\delta(\mathcal{O}_Y) \subset \mathcal{O}_Y$.

2. Quotients, singularities, and base change

We keep the situation as in the preceding section. Let $\delta \in H^0(Y, \Theta_{Y/k})$ be a vector field with $\delta^{[p]} = 0$, which defines an α_p-action on Y. In this section we discuss quotients and their properties. According to [12], Chapter III, Proposition 3.2 quotients by finite infinitesimal group schemes always exist. For α_p this works as follows: The underlying topological space for $Z = Y/\alpha_p$ is homeomorphic to Y, so that the quotient map $Y \to Z$ is a homeomorphism, and the structure sheaf $\mathcal{O}_Z$ is the kernel of the derivation $\delta : \mathcal{O}_Y \to \mathcal{O}_Y$. If Y is of finite type over k, so is Z, and the morphism $Y \to Z$ is finite. In any case, this morphism is integral. From this it follows that there is a unique maximal fixed closed subset $Y^{\alpha_p} \subset Y$, which is called the *fixed scheme*. Its ideal is the quasicoherent ideal that is locally generated by the abelian subsheaf $\delta(\mathcal{O}_Y) \subset \mathcal{O}_Y$. If there are no fixed points, then

$Y \to Z$ is an α_p-torsor in the fppf-topology, by [12], Chapter III, Proposition 3.2. In particular, $Y \to Z$ is flat of degree p, with geometric fibers isomorphic to the spectrum of $\bar{\kappa}(z)[t]/(t^p)$.

We now discuss singular and nonsmooth points on the quotient. Let $y \in Y$ be a point, and $z \in Z$ be its image. We assume that Y and hence Z are of finite type over k. The following two results are very useful in determining the nonsmooth locus on the quotients Z.

Proposition 2.1. *Suppose that Y is of dimension ≥ 2 and satisfies Serre's Condition (S_2). Let $y \in Y$ be an isolated fixed point. Then Z is not smooth near $z \in Z$.*

Proof. The problem is local in Z. Seeking a contradiction, we assume that Z is affine and smooth, and that y is the only fixed point. Set $U = Y - \{y\}$ and $V = Z - \{z\}$. Then the quotient morphism $Y \to Z$ induces an α_p-torsor $U \to V$. According to Ekedahl's purity result [13], Proposition 1.4, this α_p-torsor extends to an α_p-torsor $T \to Z$. By assumption, Y satisfies Serre's condition (S_2), and T is obviously Cohen–Macaulay. It follows that both restriction maps $H^0(T, \mathcal{O}_T) \to H^0(U, \mathcal{O}_T)$ and $H^0(Y, \mathcal{O}_Y) \to H^0(U, \mathcal{O}_Y)$ are bijective, which implies $Y = T$. The maps $\mathcal{O}_Y \to \mathcal{O}_Y \otimes k[t]/(t^p)$ and $\mathcal{O}_T \to \mathcal{O}_T \otimes k[t]/(t^p)$ defining the α_p-actions are uniquely determined by their restriction U. It follows that the action on Y is free, contradiction. $\square$

Over nonperfect ground fields k, one has to distinguish regularity and geometric regularity (= formal smoothness). I do not know if, in the preceding situation, the quotient Z could be regular. The next lemma deals with regularity rather then smoothness.

Proposition 2.2. *Suppose $y \in Y$ is not a fixed point. Let $\mathcal{I} \subset \mathcal{O}_Y$ be the ideal of the orbit $A = \alpha_p \cdot \{y\}$. Then $\mathcal{O}_{Z,z}$ is regular if and only if the ideal $\mathcal{I} \subset \mathcal{O}_{Y,y}$ has finite projective dimension. In this case, $\mathcal{I}$ is generated by $n = \dim(\mathcal{O}_{Y,y})$ elements.*

Proof. The problem is local in Z, so we may assume that Z is affine and that $Y \to Z$ is an α_p-torsor. Suppose first that $\mathcal{O}_{Z,z}$ is regular. Then the maximal ideal $\mathfrak{m} \subset \mathcal{O}_{Z,z}$ has finite projective dimension, and is generated by n elements. The exact sequence $0 \to \mathfrak{m} \to \mathcal{O}_Z \to \kappa(z) \to 0$ induces by flatness an exact sequence

$$0 \longrightarrow \mathfrak{m} \otimes \mathcal{O}_Y \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_A \longrightarrow 0$$

hence $\mathfrak{m} \otimes \mathcal{O}_Y = \mathcal{I}$. Using flatness again, we infer that $\mathcal{I}$ has finite projective dimension and is generated by n elements.

Suppose conversely $\mathcal{I}$ has finite projective dimension, say $\mathrm{pd}(\mathcal{I}) = m$. Choose a resolution

$$0 \longrightarrow M \longrightarrow F_m \longrightarrow \ldots \longrightarrow F_0 \longrightarrow \mathfrak{m} \longrightarrow 0$$

with $F_0, \ldots, F_m$ free and finitely generated. Pulling back, we obtain an exact sequence

$$0 \longrightarrow M \otimes \mathcal{O}_{Y,y} \longrightarrow F_m \otimes \mathcal{O}_{Y,y} \longrightarrow \ldots \longrightarrow F_0 \otimes \mathcal{O}_{Y,y} \longrightarrow \mathcal{I} \longrightarrow 0$$

By Hilbert's Syzygy Theorem, $M \otimes \mathcal{O}_{Y,z}$ is free. It follows from descent theory that already M must be free ([18], Exposé VII, Corollary 1.11). Hence $\mathrm{pd}(\kappa(z)) < \infty$, so the local ring $\mathcal{O}_{Z,z}$ is regular. $\square$

This gives a handy criterion in terms on embedding dimensions on Y for singularities on Z:

Corollary 2.3. *Suppose that $y \in Y$ is a rational point that is not fixed, with* $\operatorname{edim}(\mathcal{O}_{Y,y}) \geq \dim(\mathcal{O}_{Y,y}) + p$. *Then the local ring $\mathcal{O}_{Z,z}$ is not regular.*

Proof. Let $\mathfrak{m} \subset \mathcal{O}_{Y,y}$ be the maximal ideal of $y \in Y$, and $I \subset \mathcal{O}_{Y,y}$ be the ideal for its orbit. Then we have a short exact sequence $0 \to I \to \mathfrak{m} \to \mathfrak{m}/I \to 0$, where the quotient $\mathfrak{m}/I$ is a k-vector space of dimension $p-1$. Tensoring with k gives an exact sequence

$$\operatorname{Tor}_1(\mathfrak{m}, k) \longrightarrow \operatorname{Tor}_1(k,k) \longrightarrow I \otimes k \longrightarrow \mathfrak{m} \otimes k \longrightarrow \mathfrak{m}/I \longrightarrow 0.$$

Whatever the contribution from the Tor terms on the left, the k-vector space $I \otimes k$ has dimension $\geq \operatorname{edim}(\mathcal{O}_{Y,y}) - (p-1) \geq \dim(\mathcal{O}_{Y,y}) + 1$. By the Nakayama Lemma, it is impossible to generate I with $n = \dim(\mathcal{O}_{Y,y})$ elements. According to Proposition 2.2, the local ring $\mathcal{O}_{Z,z}$ must be singular. □

Provided that the group scheme action is free, the scheme Y satisfies Serre's condition (S_n) if and only if the quotient Z satisfies (S_n), by [16], Corollary 6.4.2. For nonfree action, we at least have the following:

Proposition 2.4. *Suppose that Y satisfies Serre's condition (S_2). Then the quotient $Z = Y/\alpha_p$ satisfies Serre's condition (S_2) as well.*

Proof. We may assume that Z is affine. Let $V \subset Z$ be an open subset whose complement has codimension ≥ 2. We have to check that the restriction map $H^0(Z, \mathcal{O}_Z) \to H^0(V, \mathcal{O}_Z)$ is bijective. Injectivity is clear, because $\mathcal{O}_Z \subset \mathcal{O}_Y$ contains no torsion sections. As to surjectivity, suppose $s_V \in H^0(V, \mathcal{O}_Z)$. By assumption, it extends to a section $s \in \Gamma(Y, \mathcal{O}_Y)$. The section $\delta(s) \in H^0(Y, \mathcal{O}_Y)$ vanishes on V, hence vanishes everywhere, whence s is a section for $\mathcal{O}_Z$ extending s_V. □

The following result on dualizing sheaves on quotients is useful in constructing Calabi–Yau quotients; we shall use it in Section 8.

Proposition 2.5. *Suppose that Y is Gorenstein with $\omega_Y = \mathcal{O}_Y$, that the fixed scheme Y^{α_p} is empty, and that $k = \Gamma(Y, \mathcal{O}_Y)$. Then Z is Gorenstein with $\omega_Z = \mathcal{O}_Z$.*

Proof. The morphism $q : Y \to Z$ is flat with Gorenstein fibers, because the group scheme action is free. Hence $\omega_{Y/Z}$ is invertible. By assumption, the dualizing sheaf $\omega_Y = \omega_{Y/Z} \otimes q^*(\omega_Z)$ is invertible. Using descent theory, we infer that ω_Z is invertible, that is, Z is Gorenstein.

We next check that $\omega_{Y/Z}$ is trivial. Let $\mathcal{I} \subset \mathcal{O}_{Y \times_Z Y}$ be the ideal of the diagonal. The group scheme action is free by assumption, hence $Y \times_Z Y = Y \times \alpha_p$. It follows that $\mathcal{I} \simeq \mathcal{O}_Y$ as $\mathcal{O}_Y$-module. According to Kunz [23], page 363 there is a canonical map $\mathcal{I} \to \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{H}om_{\mathcal{O}_Z}(\mathcal{O}_Y, \mathcal{O}_Z), \mathcal{O}_Y)$, which is bijective. The term on the right is the dual of $\omega_{Y/Z}$, which therefore is trivial.

It remains to check that the preimage map $\operatorname{Pic}(Z) \to \operatorname{Pic}(Y)$ is injective. For this we may assume that our ground field k is algebraically closed. Jensen showed in [21], Section 2 that we have an exact sequence

$$0 \longrightarrow D(\alpha_p) \longrightarrow \underline{\operatorname{Pic}}_{Z/k} \longrightarrow \underline{\operatorname{Pic}}_{Y/k}.$$

of group-valued functors. Here $D(\alpha_p)$ is the Cartier dual, which in our case is uncanonically isomorphic to α_p, hence contains no point but the origin. Note that Jensen assumed that Y is proper, but his arguments go through with the weaker assumption $k = \Gamma(Y, \mathcal{O}_Y)$. We infer that $\operatorname{Pic}(Z) \to \operatorname{Pic}(Y)$ is injective, and hence $\omega_Z = \mathcal{O}_Z$. □

We next discuss base-change properties for quotients. Suppose that Y is endowed with a morphism $h : Y \to \operatorname{Spec}(R)$ onto some affine k-scheme, and that our derivation δ lies in $H^0(Y, \Theta_{Y/R}) \subset H^0(Y, \Theta_{Y/k})$. Then the group scheme action $\alpha_p \times Y \to Y$ is an R-morphism, and the quotient Z is an R-scheme. The following base-change property is standard:

Proposition 2.6. *The formation of $Z = Y/\alpha_p$ commutes with flat base change in R. If the fixed scheme Y^{α_p} is empty, it commutes with arbitrary base change in R.*

Proof. Consider the exact sequence of quasicoherent $\mathcal{O}_Z$-modules

$$0 \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_Y \xrightarrow{\delta} \mathcal{O}_Y \tag{2}$$

on Z. Given a flat R-algebra R', we obtain another exact sequence by applying $\otimes_R R'$, so taking the quotient commutes with flat base change.

Now suppose our action is free. The inclusion $\mathcal{O}_Z \subset \mathcal{O}_Y$ remains injective after tensoring with $\kappa(z)$ for all $z \in Z$, because $1 \in \mathcal{O}_{Z,z}$. Since the α_p-action is free, the quasicoherent $\mathcal{O}_Z$-module $\mathcal{O}_Y$ is locally free of rank p. Shrinking Z, we may assume that the inclusion admits a splitting, hence the the exact sequence (2) remains exact after tensoring with arbitrary R-algebras R'. $\square$

For the geometric constructions I have in mind, it is crucial to work with group actions with fixed points. I see no reason why the base-change property should hold in general. To discuss this problem, suppose for simplicity that $Y = \operatorname{Spec}(A)$ is affine, such that $Z = \operatorname{Spec}(B)$, where B is defined by the exact sequence

$$0 \longrightarrow B \longrightarrow A \xrightarrow{\delta} A \longrightarrow \operatorname{coker}(\delta) \longrightarrow 0. \tag{3}$$

The issue is as follows:

Proposition 2.7. *Suppose that Y is R-flat. Then the quotient Z is R-flat if and only if the R-module* $\operatorname{coker}(\delta)$ *has flat dimension ≤ 2. The formation of $Z = Y/\alpha_p$ commutes with arbitrary base change in R if and only if* $\operatorname{coker}(\delta)$ *is R-flat.*

Proof. By assumption, the R-module A is flat. The exact sequence (3) gives $\text{flat-dim}(B) = \text{flat-dim}(\operatorname{coker}(\delta)) - 2$ for flat dimensions, hence the first assertion. For the second assertion, let R' be an R-algebra, and consider the induced sequence

$$0 \longrightarrow B \otimes R' \longrightarrow A \otimes R' \xrightarrow{\delta\otimes 1} A \otimes R'.$$

Suppose that $\operatorname{coker}(\delta)$ is flat. Then the exact sequence (3) is a flat resolution, so the preceding sequence remains exact, which means that the formation of the quotient commutes with base change. Conversely, suppose the preceding sequence stays exact for all R'. Then $\operatorname{Tor}_1^R(\operatorname{coker}(\delta), R') = 0$, whence $\operatorname{coker}(\delta)$ is flat. $\square$

Let me record the following consequence:

Corollary 2.8. *Suppose that $R = k[r, s]$ is a polynomial ring in two variables. Then $Z \to \operatorname{Spec}(R)$ is flat.*

Proof. The ring $R = k[r, s]$ has homological dimension $\text{hom-dim}(R) = 2$, hence also Tor-dimension $\text{Tor-dim}(R) = 2$ (confer [40], Section 4.1). Hence $\operatorname{coker}(\delta)$ has projective dimension ≤ 2, so Proposition 2.7 applies. $\square$

We shall use the following base-change-property in Section 4, which I formulate for a rather special situation: Suppose that $p = 2$ and $A = R[x, y]$ is a polynomial algebra in two variables. Then we have $\delta = fD_x + gD_y$ for some $f, g \in R[x, y]$, where $D_x = \partial/\partial x$ and $D_y = \partial/\partial_y$.

Corollary 2.9. *In the preceding situation, suppose that $f, g \in R[x, y]$ are monic polynomials. Then the formation of $Z = Y/\alpha_2$ commutes with arbitrary base change in R.*

Proof. We check that $\operatorname{coker}(\delta)$ is a free R-module. All terms in the exact sequence (3) are modules over $A' = R[x^2, y^2]$, because the derivation δ is A'-linear. Moreover, A is a free A'-module of rank four, with basis $1, x, y, xy$. The matrix of δ with respect to this basis is

$$\begin{pmatrix} 0 & f & g & 0 \\ 0 & 0 & 0 & g \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

hence $\operatorname{coker}(\delta) = (A'/fA' + gA') \oplus A'^{\oplus 2}/(f, g)A' \oplus A'$. The first summand is a free R-modules, because f, g are monic, and the last summand is obviously free. The middle summand sits in an exact sequence

$$0 \longrightarrow A' \longrightarrow A'^{\oplus 2}/(f, g)A' \xrightarrow{pr_2} A'/gA' \longrightarrow 0,$$

hence is a free R-module as well. □

3. The cuspidal rational curve

Let k be a ground field of characteristic $p = 2$, and C be the cuspidal curve of arithmetic genus $p_a(C) = 1$. To be explicit, we choose an indeterminate u and write

$$C = \operatorname{Spec} k[u^2, u^3] \ \cup \ \operatorname{Spec} k[u^{-1}].$$

The goal of this section is to determine the restricted Lie algebra of $\operatorname{Aut}_{C/k}$ and read off all possible α_2-actions on C. This extends results of Bombieri and Mumford ([7], Proposition 6), who already studied $\operatorname{Aut}_{C/k}$ in connection with quasielliptic fibrations in the Enriques classification of surfaces.

I start by describing the Lie algebra $\mathfrak{g} = H^0(C, \Theta_{C/k})$ in Lie-theoretic terms without referring to geometry: Let $\mathfrak{a}$ be the 3-dimensional restricted Lie algebra over k with trivial Lie brackets $[a, a'] = 0$ and trivial p-map $a^{[p]} = 0$. Let $\mathfrak{b}$ be the 1-dimensional restricted Lie algebra endowed with a generator $b \in \mathfrak{b}$ with $b^{[p]} = b$. Any linear endomorphism of $\mathfrak{a}$ is a derivation, because $\mathfrak{a}$ is commutative. The linear map $\mathfrak{b} \to \mathfrak{gl}(\mathfrak{a})$, $b \mapsto \mathrm{id}_{\mathfrak{a}}$ commutes with Lie brackets. As explained in [8], §1.7 this homomorphism yields a semidirect product Lie algebra $\mathfrak{g} = \mathfrak{a} \rtimes \mathfrak{b}$, whose Lie bracket is

$$[a + \lambda b, a' + \lambda' b] = \lambda a' - \lambda' a.$$

We remark in passing that the ideal $\mathfrak{a} \subset \mathfrak{g}$ is the *derived ideal* $\mathfrak{D}\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, whence $\mathfrak{D}^2\mathfrak{g} = 0$ and $\mathfrak{g}$ is solvable. On the other hand, $\mathfrak{g}$ has trivial center, so $\mathfrak{g}$ it is not nilpotent.

Lemma 3.1. *There is precisely one p-map on the semidirect product Lie algebra $\mathfrak{g} = \mathfrak{a} \rtimes \mathfrak{b}$ such that the canonical inclusions $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$ are restricted inclusions.*

Proof. If it exists, the p-map for $\mathfrak{g}$ is uniquely determined: For $p=2$ the axioms for p-maps give $(x+y)^{[2]} = x^{[2]} + y^{[2]} - [x,y]$. Therefore the p-map in our semidirect product must be

$$(a+\lambda b)^{[2]} = a^{[2]} + (\lambda b)^{[2]} + [a, \lambda b] = \lambda(a+\lambda b). \tag{4}$$

It remains to check that this meets the remaining two axioms for p-maps. Indeed, given any scalar $\mu \in k$, we obviously have

$$(\mu(a+\lambda b))^{[2]} = \mu\lambda(\mu(a+\lambda b)) = \mu^2(a+\lambda b)^{[2]}.$$

Moreover, we easily compute

$$[(a+\lambda b)^{[2]}, a' + \lambda' b] = [\lambda a + \lambda^2 b, a' + \lambda' b] = \lambda^2 a' + \lambda\lambda' a,$$

which equals

$$[a+\lambda b, [a+\lambda b, a' + \lambda' b]] = [a + \lambda b, \lambda a' + \lambda' a] = \lambda(\lambda a' + \lambda' a).$$

Hence Formula (4) indeed defines a p-map. $\square$

The book of Strade and Farnsteiner [38] contains more information on the existence of p-maps in Lie algebras. We now come back to our cuspidal curve C of arithmetic genus one:

Proposition 3.2. *The restricted Lie algebras $H^0(C, \Theta_{C/k})$ and $\mathfrak{g} = \mathfrak{a} \rtimes \mathfrak{b}$ are isomorphic.*

Proof. Consider the two affine open subsets $U = \operatorname{Spec} k[u^2, u^3]$ and $V = \operatorname{Spec} k[u^{-1}]$. The only relation between the generators u^2, u^3 on U is the obvious one, namely $(u^2)^3 = (u^3)^2$. Whence $\Omega^1_{U/k}$ is generated by the differentials $d(u^2), d(u^3)$ modulo the relation $u^4 d(u^2) = 0$. Since $\mathcal{O}_C$ is torsion free, the dual $\Theta_{U/k}$ is a free $\mathcal{O}_U$-module of rank one, generated by the form $d(u^3) \mapsto 1$. On the overlap $U \cap V$, this form becomes $u^{-2}D_u$, where $D_u = \partial/\partial u$ is taking derivative with respect to u. We shall use the same symbol $u^{-2}D_u$ to denote this form on U, although the two individual factors u^{-2} and D_u do not make sense on U.

The affine open subset V is smooth. Here $\Omega^1_{V/k}$ is free of rank one, generated by the differential $d(u^{-1})$. The dual $\Theta_{V/k}$ is free of rank one as well, generated by the form $d(u^{-1}) \mapsto 1$. On the overlap $U \cap V$ we have $d(u^{-1}) = u^{-2}du$, so our form becomes $u^2 D_u$. From this we infer that the Lie algebra $\mathfrak{g}' = H^0(C, \Theta_{C/k})$ is a k-vector space of rank 4, with basis

$$u^{-2}D_u, \quad D_u, \quad uD_u, \quad u^2 D_u.$$

Let $\mathfrak{a} \subset \mathfrak{g}'$ be the subspace generated by the derivations with even coefficients, that is, $u^{-2}D_u, D_u, u^2D_u$. Then one easily computes that both Lie bracket and p-map vanish on $\mathfrak{a}$. Moreover, we have $[uD_u, u^{2i}D_u] = u^{2i}D_u$ for all integers i. Whence $\mathfrak{a} \subset \mathfrak{g}'$ is the derived Lie algebra for $\mathfrak{g}'$. The resulting extension of Lie algebras

$$0 \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{g}' \longrightarrow \mathfrak{b} \longrightarrow 0$$

splits, because $\mathfrak{b}$ is 1-dimensional. The element $uD_u \in \mathfrak{g}'$ defines a splitting, and we have $(uD_u)^{[2]} = uD_u$. We now regard $\mathfrak{g}'$ as a semidirect product. Any such semidirect product is given by a homomorphism $\mathfrak{b} \to \mathfrak{gl}(\mathfrak{a})$. In our case, this map sends the generator uD_u to the identity $\mathrm{id}_{\mathfrak{a}}$. It follows that $\mathfrak{g}' = H^0(C, \Theta_{C/k})$ and $\mathfrak{g} = \mathfrak{a} \rtimes \mathfrak{b}$ are isomorphic as Lie algebras. By Lemma 3.1, they also have the same p-maps. $\square$

Our algebraic computations translate into the following geometric statement:

Corollary 3.3. *The set of α_2-operations on the cuspidal curve C of arithmetic genus one is parameterized by the affine space $\mathbb{A}^3$.*

Proof. According to Proposition 1.1, the set in question is the set of all $a+\lambda b\in\mathfrak{g}$ with $(a+\lambda b)^{[2]}=0$. By formula (4), each vector a from the derived ideal $\mathfrak{a}\subset\mathfrak{g}$ has this property. On the other hand, each vector $a+\lambda b\in\mathfrak{g}$ with $\lambda\neq 0$ has $(a+\lambda b)^{[2]}=\lambda(a+\lambda b)\neq 0$. $\square$

4. The selfproduct for the cuspidal rational curve

Let C be the cuspidal curve of arithmetic genus one over a ground field k of characteristic $p=2$, as in the preceding Section. Throughout we consider the selfproduct $Y=C\times C$, which is a nonnormal integral proper surface with unibranch singularities and normalization $\mathbb{P}^1\times\mathbb{P}^1$. We shall use coordinates u^2,u^3,u^{-1} on the first factor of $C\times C$, and coordinates v^2,v^3,v^{-1} on the second factor:

$$C=\operatorname{Spec} k[u^2,u^3]\cup\operatorname{Spec} k[u^{-1}]\quad\text{and}\quad C=\operatorname{Spec} k[v^2,v^3]\cup\operatorname{Spec} k[v^{-1}].$$

I want to define α_2-actions on $Y=C\times C$ depending on two parameters r,s. In the following, we have to allow parameters from various parameter rings (for example fields, dual numbers, polynomial rings and so forth). To unify notation, let us fix a k-scheme S and two global sections $r,s\in H^0(S,\mathcal{O}_S)$, and consider the scheme

$$Y\times S=(C\times C)\times S,$$

viewed as a proper flat family of surfaces over S. The derivation

$$\delta=(u^{-2}+r)D_u+(v^{-2}+s)D_v$$

defines a global vector field $\delta\in H^0(Y\times S,\Theta_{Y\times S/S})$. A straightforward computation shows that $\delta\circ\delta=0$. Hence δ defines an α_2-action on $Y\times S$ over S. Let us first examine the fixed scheme for this action.

Proposition 4.1. *The fixed scheme $(Y\times S)^{\alpha_2}\subset Y\times S$ is contained in the open subset $\operatorname{Spec}\mathcal{O}_S[u^{-1},v^{-1}]\subset Y\times S$, and its ideal is generated by the two elements $u^{-2}(u^{-2}+r)$ and $v^{-2}(v^{-2}+s)$.*

Proof. The ideal $\mathcal{I}\subset\mathcal{O}_{Y\times S}$ generated by the image of the derivation δ defines the fixed scheme in question. Since $\delta(u^3)=1+ru^2$, we have $\mathcal{I}_y=\mathcal{O}_{Y,y}$ for all points $y\in Y\times S$ outside $\operatorname{Spec}\mathcal{O}_S[u^{-1},v^{-1}]$. Over $\operatorname{Spec}\mathcal{O}_S[u^{-1},v^{-1}]$, we compute

$$\delta=u^{-2}(u^{-2}+r)D_{u^{-1}}+v^{-2}(v^{-2}+s)D_{v^{-1}},\tag{5}$$

and the result follows. $\square$

We see that the geometric fibers of $Y\times S\to S$ contain precisely four fixed points over the open subset $D(rs)\subset S$. These fixed points come together in pairs over the closed subsets $V(r)$ and $V(s)$. On the intersection $V(r,s)=V(r)\cap V(s)$, precisely one fixed point remains.

Next, we examine the quotient scheme $\mathfrak{Z}=Y\times S/\alpha_2$ with respect to the α_2-action on the product family $Y\times S$. Here I use the fracture letter $\mathfrak{Z}$, to emphasize that the resulting morphism $\mathfrak{Z}\to S$ usually is not a product family (this has nothing to do with formal schemes).

Proposition 4.2. *The resulting morphism $\mathfrak{Z}\to S$ is flat and commutes with arbitrary base change in S.*

Proof. We first check the base-change property. By Proposition 2.6, the base-change property holds outside the fixed points. In light of Corollary 2.9 and Formula (5), the base-change property holds near the fixed points as well. To check flatness, it therefore suffice to treat the universal situation $R = k[s,t]$, and then flatness holds by Corollary 2.8. □

Throughout the paper, we shall frequently pass back and forth between the family $\mathfrak{Z} \to S$ and its fibers $\mathfrak{Z}_\sigma$, $\sigma \in S$. We just saw that the fiber $\mathfrak{Z}_\sigma$ is also the quotient of Y_σ by the induced α_2-action. To simplify notation, we usually write $Z = \mathfrak{Z}_\sigma$ to denote fibers; making base change, we then usually assume that S is the spectrum of our ground field k and write $Z = \mathfrak{Z}$.

Proposition 4.3. *The fibers $Z = \mathfrak{Z}_\sigma$ of the flat family $\mathfrak{Z} \to S$ are normal.*

Proof. We may assume that $S = \operatorname{Spec}(k)$ and that k is algebraically closed. By Proposition 2.4, the surface Z is Cohen–Macaulay. It remains to check that it is regular in codimension one. For this we may ignore the fixed points, which are isolated by Proposition 4.1. Let $U \subset Y$ be the regular locus minus the fixed locus; by Proposition 2.2, the quotient Z is regular on the image of U. To finish the argument, let $y \in Y$ be a singular point defined by the maximal ideal $\mathfrak{m} = (u^2, u^3, v^{-1} + \lambda)$ for some $\lambda \in k$ with $\lambda \neq 0, s$. Then the ideal

$$\mathcal{I} = (u^2, (1 + ru^2)(v^{-1} + \lambda) + (v^{-4} + sv^{-2})u^3)$$

is δ-invariant, whence the spectrum of $\mathcal{O}_Y/\mathcal{I} = k[u^3]/(u^6)$ must be the orbit of y. Clearly, the ideal $\mathcal{I}$ has finite projective dimension, whence Z is regular at the image of y. Summing up, Z is regular in codimension one. □

In light of Proposition 2.1, the four fixed points on Y contribute to the singular points $z \in Z$. It turns out that these singularities are *geometrically* isomorphic, because the automorphism group scheme $\operatorname{Aut}_{Z/k}$ acts transitively on them. More precisely:

Proposition 4.4. *Assume that $S = \operatorname{Spec}(k)$, and that $r, s \in k$ are squares. Let $y_1, y_2 \in Y$ be fixed points, and $z_1, z_2 \in Z$ be the corresponding singular points. Then there are automorphisms $\phi : Y \to Y$ with $\phi(y_1) = y_2$ and $\psi : Z \to Z$ with $\psi(z_1) = z_2$ such that the diagram*

$$\begin{tikzcd} Y \arrow[r, "\phi"] \arrow[d] & Y \arrow[d] \\ Z \arrow[r, "\psi"'] & Z \end{tikzcd}$$

commutes.

Proof. In case $r = s = 0$, there is precisely one fixed point $y \in Y$, and we have nothing to prove. Suppose $r \neq 0$. Let $\phi : Y \to Y$ be the involution defined by $u^{-1} \mapsto u^{-1} + \sqrt{r}$. It is easy to see that the corresponding $\mathbb{Z}/2\mathbb{Z}$-action commutes with the α_2-action on Y. It therefore descends to an involution $\psi : Z \to Z$. Now suppose that $s \neq 0$. Then we have similar involutions on Y and Z induced by $v^{-1} \mapsto v^{-1} + \sqrt{s}$. Obviously, the subgroup generated by these involutions acts transitively on the set of singularities on Z coming from fixed points on Y. □

5. The singularity coming from the quadruple point

We keep the notation from the preceding section, but we assume for simplicity that our parameter space $S = \mathrm{Spec}(R)$ is affine. Recall that $Y = (C \times C) \otimes R$, and $\mathfrak{Z} = Y \otimes R/\alpha_2$ is the quotient by the α_2-action defined by the vector field

$$\delta = (u^{-2} + r)D_u + (v^{-2} + s)D_v$$

with parameters $r, s \in R$. Set $A = R[u^2, u^3, v^2, v^3]$, and let $B \subset A$ be the kernel of the derivation $\delta : A \to A$. Then $\mathrm{Spec}(A) \subset Y \otimes R$ is an affine open neighborhood containing fiberwise the point of embedding dimension four, which is defined by the ideal $(u^2, u^3, v^2, v^3) \subset A$, and $\mathrm{Spec}(B) \subset \mathfrak{Z}$ is an affine open neighborhood containing fiberwise the corresponding singularities.

Proposition 5.1. *We have $B = R[u^2, v^2, (1+sv^2)u^3+(1+ru^2)v^3]$ as R-subalgebras inside A.*

Proof. In light of Proposition 4.2, we may assume that $R = k[r, s]$ is a polynomial algebra in two variables. Consider the k-subalgebra $A' = R[u^2, v^2]$ inside A. Obviously, we have $A' \subset B$. Moreover, A is a free A'-module of rank four, with basis $1, u^3, v^3, u^3v^3$. Given $\alpha, \beta, \gamma \in A'$, we compute

$$\delta(\alpha u^3 + \beta v^3 + \gamma u^3 v^3) = \alpha(1 + ru^2) + \beta(1 + sv^2) + \gamma(1 + ru^2)v^3 + \gamma(1 + sv^3)u^3.$$

Suppose this expression vanishes. Comparing coefficients, we see $\gamma(1 + sv^3) = 0$ and hence $\gamma = 0$. Using the factoriality of A', we infer that α and β are multiples of $1+sv^2$ and $1+ru^2$, respectively. Hence $A = k[u^2, v^2, (1+sv^2)u^3+(1+ru^2)v^3]$. □

The invariant subring $B \subset A$ also admits a simple description as a quotient ring:

Proposition 5.2. *We have $B = R[a, b, c]/(c^2 + a^3 + b^3 + s^2a^3b^2 + r^2a^2b^3)$.*

Proof. It suffices to treat the universal situation $R = k[r, s]$. In particular, we may assume that R is noetherian. Set $B' = R[a, b, c]/(c^2 + a^3 + b^3 + s^2a^3b^2 + r^2a^2b^3)$. The mapping

$$a \longmapsto u^2, \quad \text{and} \quad b \longmapsto v^2, \quad \text{and} \quad c \longmapsto (1 + sv^2)u^3 + (1 + ru^2)v^3$$

clearly induces a surjective homomorphism $B' \to B$. To see that it is also injective, it suffices to treat the case $R = k$, by the Nakayama Lemma. Being a complete intersection, the ring B' is Cohen–Macaulay. A straightforward computation of $\Omega^1_{B'/k}$ reveals that the singular locus on $\mathrm{Spec}(B')$ is 0-dimensional. It follows that the 2-dimensional ring B' is integral. Being a surjection between integral 2-dimensional rings $B' \to B$, the map in question is bijective. □

Let me now recall some facts from singularity theory. Suppose $(\mathcal{O}_z, \mathfrak{m}, k)$ is a local noetherian ring that is normal, 2-dimensional, with resolution of singularities $X \to \mathrm{Spec}(\mathcal{O}_z)$. One says that $\mathcal{O}_z$ is a *rational double point* if $H^1(X, \mathcal{O}_X) = 0$ and $\mathrm{mult}(\mathcal{O}_z) = 2$. Each rational singularity comes along with an *irreducible root systems*, which are classified by Dynkin diagrams. This goes as follows: Let $E \subset S$ be the reduced exceptional divisor, $E_i \subset E$ be its integral components, and V the real vector space generated by the E_i endowed with the scalar product $\Phi(E_i, E_j) = -(E_i \cdot E_j)$. Then the vectors $E_i \in V$ form a root basis for an irreducible root system. Note that in characteristic $p = 2, 3, 5$ there are nonisomorphic rational double points with the same root system, as Artin observed in [5].

When the residue field k is algebraically closed, only Dynkin diagrams of type ADE appear, which are precisely Dynkin diagrams with equal root lengths. Such Dynkin diagrams are also called *simply laced*. When k is not algebraically closed, Dynkin diagrams with unequal root lengths show up as well. This means that some field extensions $k \subset H^0(E_i, \mathcal{O}_{E_i})$ are nontrivial. The degree of the field extensions are determined by the Cartan numbers.

Now suppose that the parameter ring R is our ground field k, and let $z \in Z$ be the singularity defined by the maximal ideal $(a, b, c) \subset B$, which corresponds to the closed point $y \in Y$ defined by the maximal ideal $(u^2, u^3, v^2, v^3) \subset A$, which is the unique point of embedding dimension four.

Proposition 5.3. *The local ring $\mathcal{O}_{Z,z}$ is a rational double point. It is of type D_4 if the ground field k contains a third root of unity, and of type B_3 otherwise.*

Figure 1: The Dynkin diagrams D_4 and B_3.

Proof. If the ground field k is algebraically closed, it suffices to check for the defining equation $c^2 = a^3 + b^3 + s^2a^3b^2 + r^2a^2b^3$ in one of the available lists of normal forms for rational double points in characteristic two (see Artin [5] or Greuel and Kröning [15]). To recheck this, and to cover arbitrary ground fields as well, let us do the blowing up $Z' \to \operatorname{Spec}(B)$ of the ideal $(a, b, c) \subset B$. The scheme Z' is covered by two affine open subsets $Z' = D_+(a) \cup D_+(b)$. The first one is the spectrum of $k[a, b', c']$, where we set $b' = b/a$ and $c' = c/a$, modulo the relation

$$c'^2 = a(1 + b'^3) + a^3(s^2 + r^2b')b'^2.$$

A computation with differentials reveals that the reduced locus of nonsmoothness on $D_+(a)$ lying on the exceptional divisor has ideal $I = (a, 1 - b'^3)$.

Now suppose that the ground field k contains a third root of unity $\zeta \in k$. Then $1 - b'^3 = (1 - b')(1 - \zeta b')(1 - \zeta^2 b')$, hence $D_+(a)$ contains three singularities, each with residue field k, and a straightforward computation shows that these are rational double points of type A_1. By symmetry, the other open subset $D_+(b)$ does not contribute any further singularities. From this we infer that $\mathcal{O}_{Z,z}$ is a rational double point of type D_4.

Finally, suppose that k does not contain a third root of unity. Then we have an irreducible decomposition $1 - b'^3 = (1 - b')(1 + b' + b'^2)$, hence $D_+(a)$ contains two singularities. Again we easily see that each one is of type A_1, but one has residue field k, and the other has residue field isomorphic to the splitting field of $1 + b' + b'^2$. If follows that $\mathcal{O}_{Z,z}$ is a rational double point of type B_3. $\square$

Finally, suppose that R is a k-algebra with residue field k, and set $B_0 = B \otimes_R k$ and $R_0 = k$. We now view the R-algebra B as a deformation of the R_0-algebra B_0. To understand this deformation near the singularity $z \in \operatorname{Spec}(B_0)$, we have to pass the completion $\hat{B} = R[[a, b, c]]/(c^2 + a^3 + b^3 + s^2a^3b^2 + r^2a^2b^3)$. It turns out that this deformation is trivial:

Proposition 5.4. *The deformation $\hat{B}$ is isomorphic to the trivial deformation $\hat{B}_0 \hat{\otimes}_{R_0} R$.*

Proof. Set $f = c^2 + a^3 + b^3$, and suppose we have a power series $g \in (a^3b^2, a^2b^3)$. We will show that there is an automorphism φ of $R[[a,b,c]]$ with $\varphi(f+g) = f$, which clearly implies the assertion.

Write $g = \sum \lambda_{ij} a^i b^j$ with $\lambda_{ij} \in R$, and $i,j \geq 2$, and $i+j \geq 5$. Consider the nonzero monomials in g with minimal total degree $i+j$, and pick among them the monomial $\lambda_{mn} a^m b^n$ with minimal a-degree m. Suppose for the moment $m \geq 3$. Then $a \mapsto a + \lambda_{mn} a^{m-2} b^n$ defines an automorphism φ_{mn} of $R[[a,b,c]]$. We have

$$\varphi(a^3) = a^3 + \lambda_{mn} a^m b^n + \lambda_{mn}^2 a^{2m-3} b^{2n} + \lambda_{mn}^3 a^{3m-6} b^{3n},$$

and the last two summands have total degree $> m+n$. A similar computation of $\varphi(a^k)$, $k \geq 4$ gives that $g' = \varphi_{mn}(f+g) - f$ lies in (a^3b^2, a^2b^3), and that g' has only monomials of total degree $> m+n$, or of total degree $m+n$ and a-degree $> m$.

In the case $m = 2$, we have $n \geq 3$, and we may apply the preceding arguments with b instead of a. Proceeding by induction, we see that we obtain the desired automorphism φ in the form $a \mapsto a + \sum \mu_{ij} a^{i-2} b^j$, $b \mapsto b + \sum \eta_{ij} a^i b^{j-2}$ for certain inductively determined coefficients $\mu_{ij}, \eta_{ij} \in R$. □

Remark 5.5. According to [5], there are two isomorphism classes of rational double points of type D_4, which are called D_4^0 and D_4^1. The upper index has to do with the versal deformation: The smaller the upper index, the larger the dimension of the versal deformation. The preceding arguments show that our singularity is of type D_4^0.

6. Singularities coming from fixed points

Our next goal is to analyse the singularities on $\mathfrak{Z} = Y \otimes R/\alpha_2$ coming from the fixed points on $Y \otimes R$ for the α_2-action. In this section, we set $A = R[u^{-1}, v^{-1}]$ and define $B \subset A$ to be the kernel of the derivation $\delta : A \to A$. Then $\operatorname{Spec}(A) \subset Y \otimes R$ is an affine open neighborhood for the fixed points, and $\operatorname{Spec}(B) \subset \mathfrak{Z}$ is an affine open neighborhood for singularities coming from fixed points.

Proposition 6.1. *We have $B = R[u^{-2}, v^{-2}, u^{-1}(v^{-4} + sv^{-2}) + v^{-1}(u^{-4} + ru^{-2})]$ as subalgebras inside A. Fiberwise over R, the B-module A is reflexive of rank two, but not locally free.*

Proof. The arguments for the first statement are as in the proof for Proposition 5.1, and left to the reader. For the second statement, we may assume that $R = k$. According to [34], Section IV, Proposition 11, the B-module A is Cohen–Macaulay. Hence it must be reflexive. Consider the maximal ideal

$$\mathfrak{m} = (u^{-2}, v^{-2}, u^{-1}(v^{-4} + sv^{-2}) + v^{-1}(u^{-4} + ru^{-2}))$$

inside B. We have $A/A\mathfrak{m} = k[u^{-1}, v^{-1}]/(u^{-2}, v^{-2})$, which has length four instead of two. It follows that A is not locally free as B-module. □

Setting $a = u^{-2}$ and $b = v^{-2}$ and $c = u^{-1}(v^{-4} + sv^{-2}) + v^{-1}(u^{-4} + ru^{-2})$, we obtain as in Proposition 5.2 the following description of B as a complete intersection:

Proposition 6.2. *We have $B = R[a,b,c]/(c^2 + a(b^4 + s^2b^2) + b(a^4 + r^2a^2))$.*

In what follows, we assume $R = k$ and write $Z = \mathfrak{Z}$. Let $z \in Z$ be the rational point defined by the ideal $\mathfrak{m} = (a,b,c)$, which is the image of a fixed point on Y. We want to understand the singularity $\mathcal{O}_{Z,z}$. Note that if the ground field k is perfect, the other singularities on Z corresponding to the fixed points on Y are isomorphic to $z \in Z$, by Proposition 4.4. For our purposes it therefore suffices to understand $\mathcal{O}_{Z,z}$. We first treat the generic case:

Proposition 6.3. *Suppose the parameters $r, s \in k$ are both nonzero. Then the local ring $\mathcal{O}_{Z,z}$ is a rational double point of type D_4.*

To see this, one has to make an explicit blowing up $X \to \operatorname{Spec}(B)$ of the ideal (a, b, c) as in the proof for Proposition 5.3. We leave this for the reader and immediately turn to the more interesting case where one of the parameters r, s degenerates:

Proposition 6.4. *Suppose precisely one of the parameters $r, s \in k$ vanishes. Then the local ring $\mathcal{O}_{Z,z}$ is a rational double point of type D_8.*

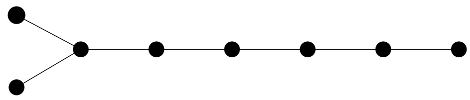

Figure 2: The Dynkin diagram D_8.

Proof. We may assume $s = 0$. Then the defining equation is $c^2 = ab^4 + a^4b + r^2a^2b$ with $r \neq 0$. We view the corresponding affine scheme as a double covering of the affine plane $\operatorname{Spec} k[a, b]$. To produce a resolution for $\mathcal{O}_{Z,z}$ we compute a blowing-up $X \to \operatorname{Spec} k[[a, b]]$ so that the reduced transform of the equation $ab^4 + a^4b + r^2a^2b$ defines a divisor with normal crossings. It turns out that, if t denotes a local equation for this normal crossing divisor, the partial derivatives of t vanish precisely at the singularities of the normal crossing divisor (and luckily nowhere outside the divisor). Hence the equation $c^2 = t$ has at most rational double points of type A_1 (even in characteristic $p = 2$), and one immediately reads off the configuration of the exceptional divisor for a resolution of $\operatorname{Spec}(\mathcal{O}_{Z,z})$. This is indeed the D_8-configuration, and $X \to \operatorname{Spec} k[[a, b]]$ is a sequence of three blowing ups.

I do not want to reproduce these computations in detail; but let me explain the first blowing up of $k[a, b]$: On the $k[a, \frac{b}{a}]$ chart, the transform of $ab^4 + a^4b + r^2a^2b$ is $a^5\frac{b}{a} + a^5(\frac{b}{a})^4 + r^2a^3\frac{b}{a}$. Removing the square factor a^2 we obtain $a\frac{b}{a}(a^2 + a^2\frac{b}{a} + r^2)$, and this is already normal crossing along the exceptional curve $a = 0$.

On the $k[\frac{a}{b}, b]$ chart, the transform divided by square factors is $(\frac{a}{b})^4b^3 + \frac{a}{b}b^3 + r^2(\frac{a}{b})^2b$. There is a unique singularity, which is located at $\frac{a}{b} = b = 0$, but this singularity is not yet normal crossing. Here we have to repeat the process, which I leave to the reader. □

It remains to treat the case of totally degenerate parameters $r = s = 0$. Here we do not get rational singularities. Instead, we get an elliptic singularity. Let me first recall the terminology. Suppose that $\mathcal{O}_z$ is a local ring that is 2-dimensional and normal, with resolution of singularities $X \to \operatorname{Spec}(\mathcal{O}_z)$. The *arithmetic genus* $p_a(\mathcal{O}_z)$ is defined as the maximal arithmetic genus $p_a(D) = 1 - \chi(\mathcal{O}_D)$, where $D \subset X$ ranges over all divisors supported by the exceptional locus. The rational singularities are precisely those with $p_a(\mathcal{O}_z) = 0$, according to [1], Theorem 1.7. Singularities with $p_a(\mathcal{O}_z) = 1$ are called *elliptic*. Wagreich [39] obtained a list of elliptic singularities. For us, the following elliptic singularities are important:

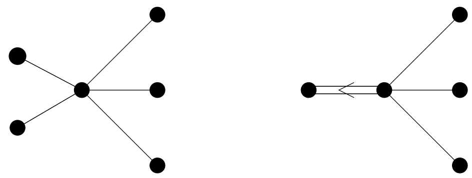

Figure 3: Wagreich's elliptic singularity of type 19_0, and a twisted form.

In the elliptic singularity of type 19_0, the central exceptional curve E_0 has self-intersection number $E_0^2 = -3$, whereas the others have $E_i^2 = -2$. In the twisted form, the exceptional curve on the left is $\mathbb{P}^1_{k'}$ for some quadratic field extension $k \subset k'$.

Proposition 6.5. *Suppose $r = s = 0$. Then $\mathcal{O}_{Z,z}$ is an elliptic singularity. If the ground field k contains a third root of unity, it is the elliptic singularity of type* 19_0 *from Wagreich's list. Otherwise, it is the twisted form in Figure 3.*

Proof. We have $B = k[a,b,c]$ modulo the relation $c^2 = ab(a-b)(a^2+ab+b^2)$. If the ground field k contains a third root of unity, the factor a^2+ab+b^2 splits into linear factors. According to [39], Corollary on page 449, the singularity is then elliptic of type 19_0. If k does not contain a third root of unity, a similar analysis as in the proof for Proposition 5.2 shows that the singularity is the twisted form in Figure 3. $\square$

Remark 6.6. Wagreich's elliptic singularity of type 19_0 also showed up in Katsura's analysis of the classical Kummer construction in characteristic two [22]. I do not know whether there is a structural reason for this.

7. Quasielliptic fibrations

We keep the notation from the preceding section and work over a ground field $R = k$, such that Z is a proper normal surface, defined as the quotient of $Y = C \times C$ by an α_2-action depending on two parameters $r, s \in k$. The goal of this section is to study fibrations on Z.

First note that the composite morphism from the normalization

$$\mathbb{P}^1 \times \mathbb{P}^1 = \tilde{Y} \longrightarrow Y \longrightarrow Z$$

is a finite universal homeomorphism of degree two. It follows that the induced mapping $\operatorname{Pic}(Z) \rightarrow \operatorname{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$ becomes bijective after tensoring with $\mathbb{Z}[1/2]$. Consequently, the proper normal surface Z has Picard number $\rho(Z) = 2$. Moreover, the two projections $Y \rightarrow C$ induce fibrations $Z \rightarrow \mathbb{P}^1$, because $C/\alpha_2 = \mathbb{P}^1$. Throughout, we shall use the projection $\mathrm{pr}_2 : Y = C \times C \rightarrow C$ onto the second factor, and denote by $g : Z \rightarrow \mathbb{P}^1$ the induced fibration, such that the diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\mathrm{pr}_2"'] & Z \arrow[d, "g"] \\ C \arrow[r, "h"'] & \mathbb{P}^1 \end{tikzcd} \tag{6}$$

commutes. In accordance with our notation $C = \operatorname{Spec} k[v^2, v^3] \cup \operatorname{Spec} k[v^{-1}]$, we write the projective line as $\mathbb{P}^1 = \operatorname{Proj}(k[v^2])$. This should cause no confusion.

Proposition 7.1. *The canonical map $\mathcal{O}_{\mathbb{P}^1} \to g_*(\mathcal{O}_Z)$ is bijective, and the fibration $g : Z \to \mathbb{P}^1$ is quasielliptic.*

Proof. Suppose that the inclusion $\mathcal{O}_{\mathbb{P}^1} \subset g_*(\mathcal{O}_Z)$ is not bijective. Using the inclusion $\mathcal{O}_Z \subset \mathcal{O}_Y$ we infer that g factors birationally over C. It also must factor over the normalization of $\mathbb{P}^1 = \tilde{C}$, because Z is normal. In turn, the projection $Y = C \times C \to C$ factors over $\tilde{C}$, which is absurd. Whence $\mathcal{O}_{\mathbb{P}^1} = g_*(\mathcal{O}_Z)$.

Let $\eta \in \mathbb{P}^1$ be the generic point, and $K = \kappa(\eta) = k(v^2)$ be the corresponding function field. The generic fiber $Z_\eta = Z_K$ is a curve over the function field K. Since Z is normal, Z_K is a regular curve. Saying that the fibration $f : Z \to \mathbb{P}^1$ is *quasielliptic* means that Z_K is a twisted form of the cuspidal curve C_K of arithmetic genus one. To check this, let $L = k(v)$ be the function field of C, and consider the commutative diagram

$$\begin{array}{ccc} Y_L & \longrightarrow & Z_K \\ \downarrow & & \downarrow \\ \mathrm{Spec}(L) & \longrightarrow & \mathrm{Spec}(K). \end{array} \tag{7}$$

The morphisms are equivariant with respect to the induced α_2-actions on Y_L and $\mathrm{Spec}(L)$, and the horizontal arrows are α_2-torsors. More precisely, the α_2-action on $Y_L = C_K \times \mathrm{Spec}(L)$ is the diagonal action, coming from actions on C_K and $\mathrm{Spec}(L)$. Passing to the algebraic closure $K \subset \bar{K}$, the induced torsor $\mathrm{Spec}(L \otimes_K \bar{K}) \to \mathrm{Spec}(\bar{K})$ becomes trivial. This implies that the quotient of $Y_L \otimes_K \bar{K}$ by the action is isomorphic to $C_{\bar{K}}$. Summing up, $Z_{\bar{K}}$ and $C_{\bar{K}}$ are isomorphic. $\square$

In what follows, $K \subset L$ denote the function fields for $C \to \mathbb{P}^1$, respectively. Note that, with the notation from the preceding proof, the composite morphism

$$\mathbb{P}^1_L \longrightarrow Y_L \longrightarrow Z_K$$

is a finite universal homeomorphism. Our first task is to determine the set of K-rational points $Z_K(K)$. This depends in a strange way on the parameters:

Proposition 7.2.
(1) *Suppose that $r, s \neq 0$, $\frac{s}{r} \in \mathbb{F}_4$, $\sqrt{r} \in K$. Then the generic fiber Z_K contains precisely four K-rational points.*
(2) *Suppose that either $r = 0, s \neq 0$ or that $\sqrt{r} \notin K, s = 0$ or that $r, s \neq 0$, $\frac{s}{r} \notin \mathbb{F}_4$, $\sqrt{r} \notin K$. Then Z_K contains only one K-rational point.*
(3) *In all other cases, Z_K contains exactly two K-rational points.*

Proof. Let $z \in Z_K$ be a K-rational point. Its preimage under the quotient map $Y_L \to Z_K$ is a closed subscheme of K-length two, which is invariant under the diagonal α_2-action. Hence the preimage must be an α_2-invariant L-rational point on Y_L. The latter might be viewed as an equivariant section for the structure map $Y_L \to \mathrm{Spec}(L)$. Since the diagram

$$\begin{array}{ccc} C_K & \longleftarrow & Y_L \\ \downarrow & & \downarrow \\ \mathrm{Spec}(K) & \longleftarrow & \mathrm{Spec}(L) \end{array}$$

is cartesian, such sections are equivariant K-morphisms $h : \mathrm{Spec}(L) \to C_K$. Note that the α_2-actions on C_K and L are given by the derivations $(u^{-2} + r)D_u$ and $(v^{-2} + s)D_v$, respectively. Summing up, the K-rational points on Z_K correspond to the equivariant K-morphisms $h : \mathrm{Spec}(L) \to C_K$.

Clearly, the image of h must lie in the smooth locus of C_K. So h is given by a ring homomorphism

$$K[u^{-1}] \longrightarrow L, \quad u^{-1} \longmapsto \alpha v^{-1} + \beta$$

for some $\alpha, \beta \in K$. Equivariance means that $(u^{-2}+r)D_u(u^{-1}) = u^{-4}+ru^{-2}$ maps to $(v^{-2}+s)D_v(\alpha v^{-1}+\beta) = \alpha(v^{-4}+sv^{-2})$, which gives the equation

$$(\alpha v^{-1} + \beta)^4 = \alpha(v^{-4} + sv^{-2})$$

Comparing coefficients, we obtain three equations

$$\alpha^4 = \alpha, \quad r\alpha^2 = \alpha s, \quad \beta^2(\beta^2 + r) = 0.$$

The first equation simply means $\alpha \in \mathbb{F}_4$. Unraveling the other two equations, we reach the assertion. For example, consider the case that both $r, s \neq 0$. The second equation then means $\alpha = 0$ or $\alpha = \frac{s}{r}$. This gives two or one possibilities for α, depending on whether $\frac{s}{r}$ is contained in $\mathbb{F}_4$ or not. The third equation means that $\beta = 0$ or $\beta = \sqrt{r}$. Again we have two or one possibility, depending on whether r is a square in K or not. Summing up, we have either one, two, or four rational points in the generic fiber Z_K, depending on the values $\frac{s}{r}$ and $\sqrt{r}$ as claimed in the assertion. The other cases are similar, and left to the reader. □

For example, in the case $r = s = 1$, the homomorphism $K[u^{-1}] \to L$ given by $u^{-1} \mapsto v^{-1}$ corresponds to the rational point on Z_K given by $a = b$, $c = 0$, with maximal ideal $\mathfrak{m} = (a + b)$.

Remark 7.3. For later use we record the outcome if $\sqrt{r} \in K$, for example if K is perfect. Then the number of K-rational points in the generic fiber Z_K depends only on the ratio $(r : s)$ viewed as a point in the projective line $\mathbb{P}^1$. If $r = 0$ we have precisely one rational point. If $s = 0$, or $r, s \neq 0$ and $(r : s) \notin \mathbb{P}^1(\mathbb{F}_4)$ we have precisely two rational points. And in case $r, s \neq 0$, $\frac{s}{r} \in \mathbb{P}^1(\mathbb{F}_4)$ we have four rational points.

Next, we look at the closed fibers $Z_b = g^{-1}(b)$ of the quasielliptic fibration $g : Z \to \mathbb{P}^1$. The situation is particularly simple for fibers over which the α_2-action is free:

Proposition 7.4. *Let $b \in \mathbb{P}^1$ be a rational point so that $Y_b \subset Y$ contains no fixed points. Then $Z_b \subset Z$ is isomorphic to the cuspidal curve of arithmetic genus one.*

Proof. I make the computation for the special case $b = 0$, the other cases being similar. Consider the commutative diagram (7). The fiber $h^{-1}(0) \subset C$ is the spectrum of the Artin ring $A = k[v^2, v^3]/(v^2)$. In other words, we have $A = k[\epsilon]$, where ϵ denotes the residue class of v^3. Our derivation $\delta = (u^{-2}+r)D_u + (v^{-2}+s)D_v$ acts as $\delta(\epsilon) = \epsilon$. The fiber $Y_A \subset Y$ is hence equivariantly isomorphic to $C \times \alpha_2$ with diagonal α_2-action. We have $Y_A/\alpha_2 = Z_0$ because the group scheme action is free on Y_A, by Proposition 2.6. Clearly, the quotient Y_A/α_2 is the rational cuspidal curve of arithmetic genus one, hence the assertion. □

Next, we come to multiple fibers:

Proposition 7.5. *Let $b \in \mathbb{P}^1$ be the rational point defined by the maximal ideal (v^{-2}). Then the curve Z_b is nonreduced, and we have an equality $Z_b = 2(Z_b)_{\mathrm{red}}$ of Weil divisors.*

Proof. Let $h : C \to \mathbb{P}^1$ be the canonical morphism from Diagram (6). The fiber $h^{-1}(b) \subset C$ is the spectrum of the Artin ring $A = k[v^{-1}]/(v^{-2})$. In other words, we have $A = k[\epsilon]$, where ϵ denotes the residue class of v^{-1}. Our derivation $\delta = (u^{-2} + r)D_u + (v^{-2} + s)D_v$ acts as $\delta(\epsilon) = 0$. The fiber $Y_A \subset Y$ is therefore equivariantly isomorphic to $C \otimes k[\epsilon]$, where α_2 acts via the derivation $(u^{-2}+\lambda)D_u$ on the first factor, and trivially on the second factor. Hence the quotient is isomorphic to $\mathbb{P}^1 \otimes k[\epsilon]$. According to Proposition 2.6, this quotient and the fiber Z_b are isomorphic on a dense open set, and the assertion follows. □

In case that the parameter $s \in k$ is a square, the fiber Z_b over the closed point $b \in \mathbb{P}^1$ defined by the maximal ideal $(v^{-2} - s)$ is a double fiber as well. This follows from Proposition 4.4. The situation is different if $s \in k$ is not a square, which can happen only over nonperfect ground fields:

Proposition 7.6. *Suppose that $s \in k$ is not a square. Let $b \in \mathbb{P}^1$ be the rational point defined by the maximal ideal $(v^{-2} - s)$. Then the irreducible curve Z_b is integral, but not geometrically integral.*

Proof. We just discussed that the fiber Z_b has geometric multiplicity two. The fiber $h^{-1}(b) \subset C$ is the spectrum of the Artin ring $k' = k[v^{-1}]/(v^{-2} - s)$, which is a field. As in the preceding proof, we argue that the fiber Z_b is birational to the integral curve $\mathbb{P}^1_{k'}$. □

We shall see in Section 12 that the existence of such integral but geometrically multiple fibers is responsible for the nonexistence of simultaneous resolutions. The following result also gives a hint that some purely inseparable base change is necessary:

Proposition 7.7. *Let $Z \to \mathbb{P}^1$ be the quasielliptic surface defined by parameters $r, s \in k$, and $Z' \to \mathbb{P}^1$ be the quasielliptic surface defined by $r' = t^2 r, s' = t^2 s$ for some nonzero $t \in k$. Then there is a commutative diagram*

$$\begin{array}{ccc} Z_\eta & \longrightarrow & Z'_\eta \\ \downarrow & & \downarrow \\ \operatorname{Spec} \kappa(\eta) & \longrightarrow & \operatorname{Spec} \kappa(\eta), \end{array}$$

such that the horizontal maps are isomorphisms, and $\eta \in \mathbb{P}^1$ is the generic point.

Proof. This is an exercise in Weierstrass equations. The generic fibers Z_η, Z'_η are isomorphic to regular cubics in $\mathbb{P}^2_\eta$ containing a rational point. According to Proposition 5.2, the Weierstrass equation for Z_η is

$$y^2 = (1 + s^2 b^2)x^3 + r^2 b^3 x^2 + b^3,$$

where we set $y = c$ and $x = a$, and $b \in k(\eta)$ is a transcendental generator. Applying the substitutions $b \mapsto t^2 b$ and $y \mapsto t^3 y$ and $x \mapsto t^2 x$, we obtain the corresponding Weierstrass equation for Z'_η up to a factor t^6. □

8. K3 surfaces and rational surfaces

We keep the notation of the preceding two sections, such that Z is a proper normal surface, defined as the quotient $Z = Y/\alpha_2$ and depending on two parameters $r, s \in k$. We also assume that our ground field k is perfect. Let $r : X \to Z$ be the minimal resolution of singularities. Note that we now have three surfaces Y, Z, X, and all three are Cohen–Macaulay, hence each has a dualizing sheaf.

Proposition 8.1. *The proper normal surfaces Y and Z are Gorenstein, and their dualizing sheaves are trivial as invertible sheaves. If at least one parameter $r, s \in k$ is nonzero, the same holds for X.*

Proof. The surface $Y = C \times C$ is Gorenstein and the invertible sheaf ω_Y trivial, because the same holds for the curve C. Let $U \subset Y$ be the complement of the fixed locus, and $V \subset Z$ be the corresponding open subset. The induced morphism $U \to V$ is an α_2-torsor. By Proposition 2.5 the quotient V is Gorenstein and the invertible sheaf ω_V is trivial. Using that Y is Cohen–Macaulay, we infer that the invertible sheaf $\omega_Y = \mathcal{O}_Y$ is trivial as well.

Suppose the parameters $r, s \in k$ do not vanish simultaneously. We saw in Sections 5 and 6 that the singularities on Z are then rational double points, hence the relative dualizing sheaf $\omega_{X/Z}$ is trivial. We conclude that $\omega_X \simeq \mathcal{O}_X$. □

Using the Enriques classification of surfaces, it is now easy to determine the nature of the smooth proper surface X:

Theorem 8.2. *The smooth proper surface X is a K3 surface if at least one parameter $r, s \in k$ is nonzero. Otherwise it is a geometrically rational surface.*

Proof. We may assume that the ground field is algebraically closed. Suppose $r, s \in k$ do not vanish simultaneously. We saw in Proposition 8.1 that ω_X is trivial. It follows that the regular surface X contains no (-1)-curves, in other words, X is minimal. By the Enriques classification of surfaces, X is either an abelian surface, or an Enriques surface, or a K3 surface. Such surfaces have second Betti number $b_2 = 6$, $b_2 = 10$, and $b_2 = 22$, respectively. In Sections 5 and 6 we showed that the normal surface Z contains either five rational double points of type D_4, or one rational double point of type D_4 and two of type D_8. In any case, these singularities contribute twenty exceptional curves on X, and this implies $b_2(X) \geq 20$. It follows that X must be a K3 surface.

Now suppose $r = s = 0$. We shall apply the Castelnuovo Criterion for rationality. According to Proposition 6.5, the normal surface Z contains an elliptic singularity. Let $E \subset X$ be the corresponding exceptional divisor, and $D \subset X$ be a curve supported by E. Using $K_{X/Z} \cdot D + D^2 = K_X \cdot D + D^2 = \deg(\omega_D) = 2p_a(D) - 2$, we infer $K_{X/Z} \cdot D \geq 0$. Since the intersection form on the divisors supported on E is negative definite, the relative canonical class $K_{X/Z}$, which is a divisor supported on E, has negative coefficients. If we choose D with arithmetic genus $p_a(D) = 1$, we have $K_{X/Y} \cdot D = -D^2 > 0$. The upshot is that $K_X = K_{X/Z}$ is negative, and all plurigenera $P_n(X) = h^0(X, \omega_X^{\otimes n})$, $n \geq 0$ vanish. By Serre duality, $H^2(X, \mathcal{O}_X)$ vanishes, hence the Picard scheme $\operatorname{Pic}_{X/k}$ is reduced. Since we have a quasielliptic fibration $X \to \mathbb{P}^1$, the Albanese map for X is trivial, and we conclude that $H^1(X, \mathcal{O}_X) = 0$. Now the Castelnuovo Criterion tells us that X is rational. □

We are mainly interested in K3 surfaces. Let us therefore assume that $r, s \in k$ do not vanish simultaneously, such that X is a K3 surface. Let $f : X \to \mathbb{P}^1$ be the quasielliptic fibration induced by $g : Z \to \mathbb{P}^1$. Suppose for the moment that the ground field k is algebraically closed. Obviously, quasielliptic K3 surfaces are *unirational*, that is, there is a surjective morphism from a rational surface onto X. In our situation, we may choose as rational surface the fiber product $X \times_Z \tilde{Y}$, where

$\tilde{Y} = \mathbb{P}^1 \times \mathbb{P}^1$ is the normalization of Y. Unirational K3 surfaces are *supersingular* (in the sense of Shioda), that is, the Picard number is $\rho(X) = 22$. There is a slight complication over nonclosed ground fields:

Proposition 8.3. *We have $\rho(X) = 22$ if the ground field k contains a third root of unity, and $\rho(X) = 21$ otherwise.*

Proof. We check this with the Tate–Shioda formula. For each $b \in \mathbb{P}^1$, let $\rho(X_b)$ be the number of reducible components in the fiber $X_b \subset X$. Clearly, $\sum(\rho(X_b) - 1)$ is the number of irreducible components in the exceptional divisor for the resolution of singularities $X \to Z$. It follows from the results in Sections 5 and 6 that this sum equals 20 if the ground field k contains a third root of unity, and 19 otherwise. The group $\operatorname{Pic}^0(X_\eta)$ for the generic fiber X_η is 2-torsion. Hence the Tate–Shioda formula for the quasielliptic fibration $f : X \to \mathbb{P}^1$ takes the form

$$\rho(X) = 2 + \sum(\rho(X_b) - 1),$$

and the result follows. $\square$

Our next task is to understand the intersection form on the Néron–Severi group $\mathrm{NS}(X)$, which for K3 surfaces is isomorphic to the Picard group. For this, we have to analyse the reducible fibers of the quasielliptic fibration $f : X \to \mathbb{P}^1$.

Recall that reducible fibers in genus-one fibrations with regular total space X correspond to root systems. The correspondence is as follows: Decompose the fiber $X_b = E_0 + E_1 + \ldots + E_n$ into integral components, and let V be the real vector space generated by the E_i, endowed with the positive semidefinite bilinear form $\Phi(E_i, E_j) = -(E_i \cdot E_j)$. Suppose that E_0 is a component with multiplicity one. Then the remaining $E_1, \ldots, E_n \in V$ form a root basis. Note that $X_b - E_0$ is then the longest root, and its multiplicities can be read off from the Bourbaki tables [9]. One usually uses the symbols for positive semidefinite Coxeter matrix ($\tilde{A}_n$, $\tilde{B}_n, ..., \tilde{G}_2$) to denote the fiber type.

In our situation it is very simple to determine the fiber type of X_b, $b \in \mathbb{P}^1$. We only need to know the types of singularities $z \in Z$ lying in the fiber Z_b, which we determined in Sections 5 and 6, and the multiplicity of the closed fiber $Z_0 \subset Z$, which we did in Section 7. The proofs for the following two assertions are straightforward whence omitted.

Proposition 8.4. *Suppose $Z_b \subset Z$ is the fiber containing the rational double point $z \in Z$ corresponding to the quadruple point $y \in Y$. Then $X_b \subset X$ is a fiber of type $\tilde{D}_4$ if the ground field k contains a third root of unity, and of type $\tilde{B}_3$ otherwise. The strict transform of the closed fiber Z_b corresponds to the white vertices in Figure 4.*

Figure 4: The extended Dynkin diagrams $\tilde{D}_4$ and $\tilde{B}_3$.

We next examine fibers related to the fixed points of the α_2-action on X.

Proposition 8.5. *Suppose $Z_b \subset Z$ is a fiber containing two rational double points $z \in Z$, say of type D_4 or D_8. Then $X_b \subset X$ is a fiber of type $\tilde{D}_8$ or $\tilde{D}_{16}$, respectively.*

The strict transform of the closed fiber Z_b corresponds to the central white vertex, as in Figure 5.

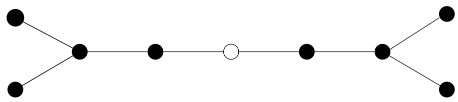

Figure 5: The extended Dynkin diagram $\tilde{D}_8$.

Proposition 8.6. *Suppose $Z_b \subset Z$ is a fiber containing precisely one rational double point $z \in Z$ corresponding to a fixed point on Y. Then $X_b \subset X$ is a fiber of type $\tilde{E}_8$, and the strict transform of the closed fiber Z_b corresponds to the white vertex in Figure 6.*

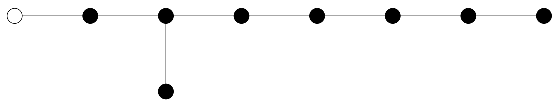

Figure 6: The extended Dynkin diagram $\tilde{E}_8$.

Proof. Clearly, only the fiber types $\tilde{E}_8$ or $\tilde{D}_8$ are possible. In the latter case, the strict transform of the closed fiber Z_b must be one of the four outer vertices. The outer vertices, however, appear with multiplicity one in the fiber, contradicting that Z_b has multiplicity two. □

9. Discriminants and Artin invariants

Let X be a K3 surface over an algebraically closed ground field k of characteristic $p > 0$ with Picard number $\rho(X) = 22$. In other words, X is supersingular. Artin [3] introduced an integer invariant for such surfaces called the *Artin invariant*. One way to define it is in terms of the intersection form on the Néron–Severi group $\mathrm{NS}(X)$. Artin showed that its discriminant is of the form $\operatorname{disc}\mathrm{NS}(X) = -p^{2\sigma_0}$ for some integer $1 \leq \sigma_0 \leq 10$, which is the Artin invariant $\sigma_0(X) = \sigma_0$.

It is not difficult to compute the Artin invariant in the presence of a quasielliptic fibration $f : X \to \mathbb{P}^1$ with a section $A \subset X$. Let $L \subset \mathrm{NS}(X)$ be the subgroup generated by the section $A \subset X$, together with all curves $C \subset X$ inside the closed fibers of the quasielliptic fibration $f : X \to \mathbb{P}^1$. As Ito explained in [20], Section 2, the quotient group $\mathrm{NS}(X)/L$ is a finite group annihilated by p, say of order p^n. This group acts freely and transitively on the set of rational points in the generic fiber X_η. Hence the group order p^n is also the number of rational points on X_η.

For each point $b \in \mathbb{P}^1$, let $C_b \subset X_b$ be the unique integral component with $C_b \cdot A = 1$. The remaining irreducible components $E_i \subset X_b - C_b$ form a root basis, whose type corresponds to the fiber type of X_b. Let $d_b = \det(E_i \cdot E_j)$ be the determinant of the corresponding intersection form. Bourbaki calls this number the *connection index* ([9], chapter IV, §1.9). Clearly, there are only finitely many points $b_1, \ldots, b_m \in \mathbb{P}^1$ whose fibers are reducible. To simplify notation, we set $d_i = d_{b_i}$.

Lemma 9.1. *Under the preceding assumptions, the Artin invariant σ_0 for the quasielliptic K3 surface X is given by the formula $p^{2\sigma_0+2n} = \pm d_1 \ldots d_m$.*

Proof. By definition, the subgroup $L \subset \mathrm{NS}(X)$ has index p^n. According to [35], Chapter III, §2, Proposition 5, we have $\operatorname{disc}\mathrm{NS}(X) = \pm p^{-2n}\operatorname{disc}(L)$, and it remains to compute the discriminant of L.

Let $L' \subset L$ be the subgroup generated by the section $A \subset X$ and a closed fiber $X_b \subset X$. Using $A^2 = -2$ and $A \cdot X_b = 1$ and $X_b \cdot X_b = 0$, we have $\operatorname{disc}(L') = -1$. For each $1 \leq i \leq m$ let $L_i \subset L$ be the subgroup generated by the integral components of $X_{b_i} - C_{b_i}$, where $C_{b_i} \subset X_{b_i}$ is the unique integral component with $C_{b_i} \cdot A = 1$. It is easy to see that we have an orthogonal decomposition $L' \oplus L_1 \oplus \ldots \oplus L_m$, and the result follows. $\square$

We now return to the situation of the preceding section, such that X is our quasielliptic K3 surface in characteristic $p = 2$ depending on two parameters $r, s \in k$, which do not vanish simultaneously. We then obtain a point $(r : s) \in \mathbb{P}^1(k)$. The Artin invariant of X now depends on whether or not this point lies inside the subset $\mathbb{P}^1(\mathbb{F}_4) \subset \mathbb{P}^1(k)$.

Proposition 9.2. *Our supersingular K3 surface X has Artin invariant $\sigma_0 = 1$ if $(r : s) \in \mathbb{P}^1(\mathbb{F}_4)$. Otherwise, it has Artin invariant $\sigma_0 = 2$.*

Proof. Suppose first that $r = 0$. According to Proposition 7.2 and Remark 7.3, the generic fiber X_η contains only $1 = 2^0$ rational point. By Propositions 8.4 and 8.6, there are three reducible fibers, one of type $\tilde{D}_4$, the others of type $\tilde{E}_8$. The connection indices are $d_1 = -4$ and $d_2 = d_3 = 1$. By the formula in Lemma 9.1, we have $\sigma_0 = 1$.

Now suppose that $r \neq 0$, $s = 0$. Then the the generic fiber contains precisely $2 = 2^1$ rational points. By Proposition 9.2, we have one fiber of type $\tilde{D}_4$ and one fiber of type $\tilde{D}_8$, which both have connection index $d_1 = d_2 = -4$. The formula for the Artin invariant again gives $\sigma_0 = 1$.

In case that $r, s \neq 0$ but $\frac{s}{r} \notin \mathbb{F}_4$ we also have only two rational points in the generic fiber, but three fibers of type $\tilde{D}_4$. This implies $\sigma_0 = 2$.

Finally, suppose we have $r, s \neq 0$ and $\frac{s}{r} \in \mathbb{F}_4$. Then we have $4 = 2^2$ rational points in the generic fiber and again three fibers of type $\tilde{D}_4$, and this leads to $\sigma_0 = 1$. $\square$

Remark 9.3. There seems to be a close relation between our family of K3 surfaces and the family studied in [32].

10. Blowing up curves on rational singularities

The next task is to construct simultaneous resolutions of singularities for our flat family of normal K3 surfaces. By the work of Brieskorn [10] and Artin [4], simultaneous resolutions in flat families rarely exist without base change. In our case, it turns out that a purely inseparable base change is necessary. After that, simultaneous resolution is achieved by blowing up Weil divisors inside the quasielliptic fibration, which easily extends to the family. The goal of this section is to collect some useful facts on blowing up curves on rational surface singularities. Throughout, $(\mathcal{O}_z, \mathfrak{m}, k)$ is a 2-dimensional normal local ring, with resolution of singularities $r : X \to \operatorname{Spec}(\mathcal{O}_z)$. We assume that $\mathcal{O}_z$ is a *rational singularity*, that is, $H^1(X, \mathcal{O}_X) = 0$.

Let $I \subset \mathcal{O}_z$ be a reflexive ideal, which defines a curve $C \subset \operatorname{Spec}(\mathcal{O}_z)$ without embedded components. How to compute the schematic fiber $r^{-1}(C) \subset S$ on the resolution of singularities? The following arguments are adapted from Artin's paper [2], where he considered maximal ideals instead of reflexive ideals.

Let $E \subset X$ be the reduced exceptional divisor, and $E = E_1 + \ldots + E_n$ be its decomposition into integral components, and $C' \subset X$ be the strict transform of $C \subset \mathrm{Spec}(\mathcal{O}_z)$. Consider nonzero divisors $Z = \sum r_i E_i$ with coefficients $r_i \geq 0$ satisfying $(Z + C') \cdot E_i \leq 0$ for all $1 \leq i \leq n$. As in [2], page 131 there is a unique minimal cycle Z with these properties. (In Artin's situation, this cycle is called the *fundamental cycle*.)

One may determine this cycle by computing a sequence of cycles $Z_0, Z_1, \ldots$ inductively as follows: Start with $Z_0 = E_i$, where E_i is any component with $E_i \cdot C' > 0$. Suppose we already defined Z_m for some $m \geq 0$. If $(Z_m + C') \cdot E_i \leq 0$ for all $1 \leq i \leq n$, we set $Z = Z_m$ and are done. Otherwise, we have $(Z_m + C') \cdot E_i > 0$ for some integral component E_i. We then define $Z_{m+1} = Z_m + E_i$ and proceed by induction. This algorithm stops after finitely many steps and yields the desired cycle Z.

Lemma 10.1. *We have $r^{-1}(C) = Z \cup C'$ as subschemes of the resolution of singularities X.*

Proof. The arguments are as in the proof for [2], Theorem 4. $\square$

Remark 10.2. This result shows that in the algorithm to compute Z, we may start by letting Z_0 be the fundamental cycle of the singularity $\mathcal{O}_z$. Note that the fundamental cycle corresponds to the longest roots, which can be read off from the Bourbaki tables [9].

Remark 10.3. Mumford [26] defined a linear preimage from the group of 1-cycles on $\mathrm{Spec}(\mathcal{O}_z)$ to the group of 1-cycles with rational coefficients on S. Note that the schematic fiber $r^{-1}(C) \subset X$ usually differs from the linear preimage $r^*(C) \in \mathrm{Div}(X) \otimes \mathbb{Q}$.

Lemma 10.4. *Let I be a reflexive fractional $\mathcal{O}_z$-ideal. Then the k-vector space $I \otimes_{\mathcal{O}_z} k$ is at most 2-dimensional.*

Proof. We may assume that the local ring $\mathcal{O}_z$ is henselian, and that the fractional ideal I is an ideal in $\mathcal{O}_z$, which defines a local curve $C \subset \mathrm{Spec}(\mathcal{O}_z)$. It induces an ideal sheaf $\mathcal{I} = I\mathcal{O}_S$ on the resolution of singularities S. I claim that the canonical map $I \to H^0(S, \mathcal{I})$ is bijective. To see this, consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & \mathcal{O}_z & \longrightarrow & \mathcal{O}_z/I & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(X, \mathcal{I}) & \longrightarrow & H^0(X, \mathcal{O}_X) & \longrightarrow & H^0(X, \mathcal{O}_X/\mathcal{I}). & &
\end{array}$$

The vertical map in the middle is surjective, because $\mathcal{O}_z$ is normal and $X \to \mathrm{Spec}(\mathcal{O}_z)$ is proper and birational. The vertical map on the right is injective, because $\mathcal{O}_z/I$ has no embedded components. Now the Snake Lemma tells us that $I \to H^0(X, \mathcal{I})$ is bijective.

I claim that there are two global sections $f, f' \in H^0(X, \mathcal{I})$ generating the stalk $\mathcal{I}_x$ for all points $x \in X$. To construct f, choose a divisor $D_0 \subset E$ whose support is disjoint from x and $\mathrm{Sing}(E)$, and whose degree on E_i equals the intersection number $-r^{-1}(C) \cdot E_i$ for all $1 \leq i \leq n$. Note that the closed subscheme $r^{-1}(C) \subset X$ indeed has no embedded points, hence is a Cartier divisor, according to Lemma 10.1.

Since $\mathcal{O}_z$ is henselian, we may extend $D_0 \subset E$ to a Cartier divisor $D \subset S$ by [17], Proposition 21.9.11. By construction

$$-D \cdot E_i = r^{-1}(C) \cdot E_i.$$

Using that $\mathcal{O}_z$ is rational, we conclude that the invertible sheaf $\mathcal{O}_X(D + r^{-1}(C))$ is trivial, whence there is an isomorphism $f : \mathcal{O}_X(D) \to \mathcal{I}$. Repeating the same construction, we find another divisor $D' \subset X$ as above, and whose support is disjoint from D. This yields the desired surjection $f + f' : \mathcal{O}_X^{\oplus 2} \to \mathcal{I}$.

We obtain an exact sequence $0 \to \mathcal{L} \to \mathcal{O}_X^{\oplus 2} \to \mathcal{I} \to 0$, which induces a long exact sequence

$$H^0(X, \mathcal{O}_X^{\oplus 2}) \longrightarrow H^0(X, \mathcal{I}) \longrightarrow H^1(X, \mathcal{L}).$$

The sheaf $\mathcal{L}$ is invertible. Taking determinants, one sees that $\mathcal{L}$ is isomorphic to the dual $\mathcal{I}^\vee = \mathcal{O}_X(r^{-1}(C))$. Consequently, $\mathcal{L} \cdot E_i = r^{-1}(C) \cdot E_i \leq 0$. Using [14], Proposition 1.9, we infer $H^1(X, \mathcal{L}) = 0$. Summing up, $H^0(X, \mathcal{I})$ and hence I are generated by two elements. □

Next, consider the class group $\mathrm{Cl}(\mathcal{O}_z)$ of reflexive fractional ideals I. Given a class $[I] \in \mathrm{Cl}(\mathcal{O}_z)$, we form the blowing up

$$Z' = \mathrm{Proj}(\bigoplus_{n \geq 0} I^n).$$

It depends, up to isomorphism, only on the ideal class and not the ideal itself. Note that the induced map $h : Z' \to \mathrm{Spec}(\mathcal{O}_z)$ is projective and birational.

Proposition 10.5. *The resolution of singularities $r : X \to \mathrm{Spec}(\mathcal{O}_z)$ factors over our blowing up $h : Z' \to \mathrm{Spec}(\mathcal{O}_z)$.*

Proof. We may assume that the fractional ideal I is an ideal in $\mathcal{O}_z$, defining a curve $C \subset \mathrm{Spec}(\mathcal{O}_z)$. According to Lemma 10.1, the schematic preimage $r^{-1}(C) \subset X$ is a Cartier divisor. By the universal property of blowing ups, this means that r factors over h. □

The next result tells us that the induced morphism $X \to Z'$ coincides with its Stein factorization, whence is uniquely determined by its exceptional curves.

Proposition 10.6. *The scheme Z' is normal.*

Proof. We may assume that I is an ideal in $\mathcal{O}_z$. The quotient $\mathcal{O}_z/I$ has no embedded primes. Let $Z'' \to \mathrm{Spec}(\mathcal{O}_z)$ be any proper birational morphism, with Z'' reduced. In light of Lipman's results [24], it suffices to check that the canonical map $I \to \Gamma(Z'', I\mathcal{O}_{Z''})$ is bijective. For this, we argue as in Lemma 10.4. □

We infer that the fibers of the blowing up are as small as possible:

Proposition 10.7. *Suppose that $[I] \neq 0$. Then the closed fiber $h^{-1}(z) \subset Z'$ is isomorphic to the projective line $\mathbb{P}^1_k$.*

Proof. Write $I = (f, g)$. This defines a closed embedding $Z' \subset \mathbb{P}^1 \times \mathrm{Spec}(\mathcal{O}_z)$, hence the fiber $h^{-1}(z) \subset Z'$ is a closed subset of $\mathbb{P}^1$. Suppose we have $h^{-1}(z) \neq \mathbb{P}^1$. Then the closed fiber is finite, hence $h : Z' \to \mathrm{Spec}(\mathcal{O}_z)$ is finite and birational. Since $\mathcal{O}_z$ is normal, we conclude that $Z' = \mathrm{Spec}(\mathcal{O}_z)$. In turn, I must be invertible, contradiction. □

In light of the preceding results, it is possible to compute the blowing up $Z' \to \mathrm{Spec}(\mathcal{O}_z)$ of some reflexive fractional ideal I via a resolution of singularities $r : X \to$

$\mathrm{Spec}(\mathcal{O}_z)$ as follows: Let $E \subset X$ be the reduced exceptional divisor, and assume that I is an ideal defining a curve $C \subset \mathrm{Spec}(\mathcal{O}_z)$. The scheme $Z' = \mathrm{Proj}(\bigoplus_{n\geq 0} I^n)$ is obtained from X by contracting all integral components $E_i \subset E$ but one. The next result tells us which ones:

Proposition 10.8. *The integral components $E_i \subset E$ that are contracted by $X \to Z'$ are precisely those with $r^{-1}(C) \cdot E_i = 0$.*

Proof. The Cartier divisor $h^{-1}(C) \subset Z'$ is h-antiample. The projection formula implies that $r^{-1}(C) \cdot E_i < 0$ if E_i is not contracted, and $r^{-1}(C) \cdot E_i = 0$ if E_i is contracted. □

11. Blowing ups in genus-one fibrations

We now apply the results from the previous section to the following situation. Suppose X is a smooth surface endowed with a genus-one fibration $f : X \to \mathbb{P}^1$. We assume that the fibration is relatively minimal, admits a section, and that the generic fiber X_η is a regular curve of arithmetic genus one. Fix a rational point $b \in \mathbb{P}^1$, and decompose the closed fiber $(X_b)_{\mathrm{red}} = E_0 + \ldots + E_n$ into integral components. Let $r : X \to Z$ be the contraction of some integral components in $X_b \subset X$. We seek to recover the minimal resolution of singularities with a sequence

$$X = Z^{(n)} \longrightarrow \ldots \longrightarrow Z^{(1)} \longrightarrow Z^{(0)} = Z$$

of blowing ups so that each step $Z^{(i+1)} \to Z^{(i)}$ has as center a Weil divisor $C^{(i)} \subset Z_b^{(i)}$ inside the fiber. This approach has its merits when it comes to deformations: Such 1-dimensional centers behave much better in families than 0-dimensional centers. A convenient way to describe the centers $C^{(i)}$ is via its strict transforms on X.

Suppose now that the fiber X_b has fiber type $\tilde{D}_8$, and let $r : X \to Z$ be the contraction of all integral components but $E_4 \subset X$, which corresponds to the white vertex as depicted in Figure 7. As usual, the choice of indices is taken from the Bourbaki tables [9]. Throughout, the integral component $E_i \subset X_b$ shall correspond to the vertex with number i. Note that the situation is as in Proposition 8.5.

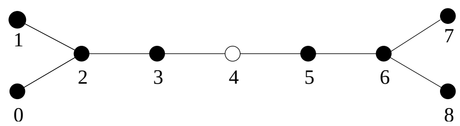

Figure 7: The fiber type $\tilde{D}_8$.

Proposition 11.1. *Under the preceding assumptions, we reach the resolution of singularities with the sequence of blowing ups $X = Z^{(6)} \to \ldots \to Z^{(0)} = Z$, in which the centers $C^{(0)}, \ldots, C^{(5)}$ have strict transforms $E_4, 2E_5, E_3, E_2, E_5, E_6$, respectively. All centers have arithmetic genus $p_a = 0$.*

Proof. By assumption, the normal surface Z contains precisely two singularities, which are rational double points of type D_4. Let $Z^{(1)} \to Z$ be the blowing up whose center $C^{(0)}$ has strict transform E_4. Note that this center is nothing but the half fiber. Using the algorithm in Section 10, we compute its schematic preimage

$F^{(0)} \subset X$ on the minimal resolution of singularities. It turns out that

$$F^{(0)} = E_0 + E_1 + 2E_2 + 2E_3 + E_4 + 2E_5 + 2E_6 + E_7 + E_8,$$

which has $F^{(0)} \cdot E_3 = F^{(0)} \cdot E_5 = -1$. Therefore, the exceptional curve for $Z^{(1)} \to Z$ corresponds to $E_3 + E_5$. We now have to repeat this, quite mechanically. My findings are summarized in the following table:

	RDP	center	schematic preimage of center
$Z^{(0)}$	$2D_4$	E_4	$E_0 + E_1 + 2E_2 + 2\underline{E_3} + E_4 + 2\underline{E_5} + 2E_6 + E_7 + E_8$
$Z^{(1)}$	$2A_3$	$2E_5$	$2E_5 + 2E_6 + E_7 + E_8$
$Z^{(2)}$	$2A_3$	E_3	$E_0 + E_1 + 2\underline{E_2} + E_3$
$Z^{(3)}$	$2A_1 + A_3$	E_2	$\underline{E_0} + \underline{E_1} + E_2$
$Z^{(4)}$	A_3	E_5	$E_5 + 2\underline{E_6} + E_7 + E_8$
$Z^{(5)}$	$2A_1$	E_6	$E_6 + \underline{E_7} + \underline{E_8}$

Table 1: Blowing ups in the $\tilde{D}_8$-fiber.

The second column displays the types of rational double points on $Z^{(i)}$. The third column gives the strict transforms of the centers $C^{(i)} \subset Z^{(i)}$ on X. The last column contains the schematic preimage of the center $C^{(i)}$ on X. The underlined irreducible components are those with nonzero intersection number with the preimage, hence give the exceptional curves for $Z^{(i+1)} \to Z^{(i)}$.

Note that the Weil divisor $C^{(1)}$ is already Cartier, such that $Z^{(2)} \to Z^{(1)}$ is the identity. I have included this seemingly superfluous step here because we need it later when it comes to simultaneous resolutions in families.

It remains to check $p_a(C^{(i)}) = 0$. I do this for $i = 0$ and $i = 1$, the other cases being similar. The strict transform $E_4 \subset X$ for $C^{(0)} \subset Z$ is clearly isomorphic to $\mathbb{P}^1$, so the possibly nonnormal points on $C^{(0)}$ must appear at the singularities $z \in Z$. To see that $C^{(0)}$ is normal, it suffices to check that $E_4 \cap r^{-1}(z) = \operatorname{Spec}(k)$. According to [2], Theorem 4, the fiber $r^{-1}(z)$ is the fundamental cycle for the singularity, which in our case equals $E_5 + 2E_6 + E_7 + E_8$, and we see $E_4 \cdot (E_5 + 2E_6 + E_7 + E_8) = 1$. It follows $p_a(C^{(0)}) = 0$.

As above, one checks that $C^{(1)}_{\text{red}} \simeq \mathbb{P}^1$. The center $C^{(1)}$ and its strict transform $2E_5$ are nonreduced. They are *ribbons* in the terminology of Bayer and Eisenbud, that is, infinitesimal extension of a reduced scheme by an invertible sheaf. The strict transform $2E_5$ is the infinitesimal extension of $\mathbb{P}^1$ by $\mathcal{O}_{\mathbb{P}^1}(2)$. Let $z \in Z^{(1)}$ be the singularity lying on $C^{(1)}$, which is a rational double point of type A_3. As above, one shows that its preimage on E_5 is the intersection $E_5 \cap (2E_6 + E_7 + E_8) = \operatorname{Spec}(k[\epsilon])$. This implies that $C^{(1)}$ is an extension of $\mathbb{P}^1$ by $\mathcal{O}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(2-2)$, as explained in [6], Corollary 1.10. The exact sequence $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \to H^1(C^{(1)}, \mathcal{O}_{C^{(1)}}) \to H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$ gives $p_a(C^{(1)}) = 0$. □

We now turn to the following case: Suppose that X_b has fiber type $\tilde{E}_8$, and that $X \to Z$ is the contraction of all irreducible components $E_i \subset X_b$ except E_1, which corresponds to the white vertex in Figure 8. Note that the situation is precisely as in Proposition 8.6.

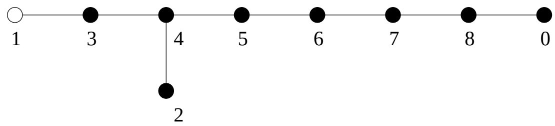

Figure 8: The fiber type $\tilde{E}_8$.

Proposition 11.2. *Under the preceding assumptions, we reach the resolution of singularities with the sequence of blowing ups $X = Z^{(6)} \to \ldots \to Z^{(0)} = Z$, in which the centers $C^{(0)}, \ldots, C^{(5)}$ have strict transforms $E_1, 2E_3, E_3, E_4, 2E_2 + E_3 + 2E_4 + 2E_6, E_8$, respectively. All centers have arithmetic genus $p_a = 0$.*

Proof. By assumption, the normal surface Z contains precisely one singularity, which is a rational double point of type D_8. Let $Z^{(1)} \to Z$ be the blowing up whose center $C^{(0)}$ has strict transform E_1. Note that this is nothing but the half fiber. Using the algorithm in Section 10, we compute its schematic preimage $F^{(0)} \subset X$ on the minimal resolution of singularities. It turns out that

$$F^{(0)} = E_1 + 3E_2 + 4E_3 + 6E_4 + 5E_5 + 4E_6 + 3E_7 + 2E_8 + E_0,$$

which has $F^{(0)} \cdot E_3 = -1$. Therefore, the exceptional curve for $Z^{(1)} \to Z$ corresponds to E_3. Further calculations are summarized as above in the following table:

	RDP	center	schematic preimage of center
$Z^{(0)}$	D_8	E_1	$E_1 + 3E_2 + 4\underline{E_3} + 6E_4 + 5E_5 + 4E_6 + 3E_7 + 2E_8 + E_0$
$Z^{(1)}$	A_7	$2E_3$	$2E_2 + 2E_3 + 4\underline{E_4} + 4E_5 + 4\underline{E_6} + 3E_7 + 2E_8 + E_0$
$Z^{(2)}$	$2A_3$	E_3	$E_2 + E_3 + 2\underline{E_4} + E_5$
$Z^{(3)}$	$2A_1 + A_3$	E_4	$\underline{E_2} + E_4 + \underline{E_5}$
$Z^{(4)}$	A_3	$E_2 + E_3 + 2E_4 + 2E_5 + 2E_6$	$E_2 + E_3 + 2E_4 + 2E_5 + 2E_6 + 2E_7 + 2\underline{E_8} + E_0$
$Z^{(5)}$	$2A_1$	E_8	$\underline{E_7} + E_8 + \underline{E_0}$

Table 2: Blowing ups in the $\tilde{E}_8$-fiber.

The first assertion follows. Checking that $p_a(C^{(i)}) = 0$ is as in the proof for Proposition 11.1. $\square$

12. Simultaneous resolutions and nonseparatedness

Let $S \subset \operatorname{Spec} k[r,s]$ be the complement of the origin, and $\mathfrak{Z} \to S$ be our projective flat family of normal K3 surfaces with rational double points, as constructed in Section 4. In this section we seek a simultaneous resolution of singularities. By Brieskorn's work, such simultaneous resolutions seldomly exist without base change [10]. Artin introduced the *resolution functor* $\operatorname{Res}_{\mathfrak{Z}/S}$ to remedy the situation [4]. Given an S-scheme S', the resolution functor takes as values $\operatorname{Res}_{\mathfrak{Z}/S}(S')$ the set of isomorphism classes of simultaneous resolutions over S', that is, commutative diagrams

$$\begin{array}{ccc} \mathfrak{X}' & \longrightarrow & \mathfrak{Z} \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S, \end{array}$$

where $\mathfrak{X}' \to S'$ is a proper smooth family of K3 surfaces, and for each point $\sigma \in S'$, the canonical morphism $\mathfrak{X}'_\sigma \to \mathfrak{Z}_\sigma$ is the minimal resolution of singularities. Artin's insight was that the resolution functor is representable by an algebraic space that, however, is only locally quasiseparated. The goal of this section is to show that simultaneous resolutions exist after purely inseparable base change:

Theorem 12.1. *There exist a simultaneous resolution $\mathfrak{X}' \to S'$ for the family $\mathfrak{X} \to S$ over the purely inseparable flat base change $S' = S \otimes_{k[r,s]} k[\sqrt{r}, \sqrt{s}]$.*

Proof. Recall that the quadruple point $y \in Y = C \times C$ induces a family of D_4-singularities in $\mathfrak{Z} \to S$. According to Proposition 5.4, this family of singularities is formally trivial. It is then easy to see that a simultaneous resolution already exists over S without base change.

The trouble comes from the other singularities, which indeed necessitate a base change. To proceed we cover the quasiaffine scheme $S' \subset \operatorname{Spec} k[\sqrt{r}, \sqrt{s}]$ by the two affine open subsets $U = D(\sqrt{s})$ and $V = D(\sqrt{r})$. Let us first concentrate on the restriction $\mathfrak{Z}_U \to U$ to the first affine open subset. As discussed in Section 7, the second projection $\mathrm{pr}_2 : Y = C \times C \to C$ induces a family of quasielliptic structures $f : \mathfrak{Z}_U \to \mathbb{P}^1_U$. In accordance with our notation, we write the base as $\mathbb{P}^1_U = \operatorname{Proj} \mathcal{O}_U[v^2]$. For each $\sigma \in U$, the induced fibration $f : \mathfrak{Z}_\sigma \to \mathbb{P}^1_\sigma$ has over $v^{-2} = 0$ and $v^{-2} = s$ degenerate fibers, of type $\tilde{D}_8$ over the open subset $U \cap V$, and of type $\tilde{E}_8$ over the closed subset $U - V$. It is precisely this point where we use base change, in order to apply Proposition 4.4.

Now let $\eta \in U$ be the generic point. We discussed in the preceding section how to obtain the minimal resolution of $\mathfrak{Z}_\eta$ via a sequence of blowing ups

$$\mathfrak{X}_\eta = \mathfrak{Z}^{(6)}_\eta \longrightarrow \ldots \longrightarrow \mathfrak{Z}^{(0)}_\eta = \mathfrak{Z}_\eta$$

whose centers $\mathfrak{C}^{(i)}_\eta \subset \mathfrak{Z}^{(i)}_\eta$ are Weil divisors in the reducible fibers. I claim that this procedure extends to the family $\mathfrak{Z} \to U$. Indeed: The first center $\mathfrak{C}_\eta = \mathfrak{C}^{(0)}_\eta$ is nothing but the half fiber inside the degenerate fibers. Let $\mathfrak{C}_U \subset \mathfrak{Z}_U$ be the schematic closure of $\mathfrak{C}_\eta$. On the smooth part of $\mathfrak{Z}_U \to U$ this is a relative Cartier divisor, and its restrictions $\mathfrak{C}_\sigma$, $\sigma \in U$ gives the half fibers. At the singularities, the schematic fibers $\mathfrak{C}_\sigma$ could pick up embedded components. This, however, is impossible, because $h^1(\mathcal{O}_{\mathfrak{C}_\sigma}) = 0$ by Propositions 11.1 and 11.2, and the Euler characteristic $\chi(\mathcal{O}_{\mathfrak{C}_\sigma})$ is constant by flatness. We deduce that the blowing up of $\mathfrak{C}_U \subset \mathfrak{Z}_U$ yields fiberwise the first step in the resolution of singularities.

Repeating the preceding argument inductively for the other centers, we see that after six steps we reach a simultaneous resolution $\mathfrak{X}_U \to \mathfrak{Z}_U$, at least over the open subset $U \cap V$ where the fiber type is constantly $\tilde{D}_8$. Using the tables in the preceding section we can check that centers in the $\tilde{D}_8$-fibers specialize to the centers in the $\tilde{E}_8$-fibers, hence we obtain the desired resolution of singularities over U. Below is a table describing how the integral components of the $\tilde{D}_8$-fibers specialize to curves on the $\tilde{E}_8$-fiber. Note that this specialization respects intersection numbers. The existence of such a specialization seems to be interesting in its own right.

$\tilde{D}_8$-fiber	E_0	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8
$\tilde{E}_8$-fiber	E_2	E_5	E_4	E_3	E_1	$E_2 + E_3 + 2E_4 + 2E_5 + 2E_6 + E_7$	E_8	E_7	E_0

Table 3: Specialization from $\tilde{D}_8$-fiber to $\tilde{E}_8$-fiber.

The reason for this specialization behavior is as follows: On $\mathfrak{Z}^{(0)}$, the E_4-curve in the $\tilde{D}_8$-fibers specialize to the E_1-curve in the $\tilde{E}_8$-fibers. We then do the first blowing up $\mathfrak{Z}^{(1)} \to \mathfrak{Z}^{(0)}$. According to Tables 1 and Table 2 in Section 11, its center is the family of curves comprising the E_4-curve on the $\tilde{D}_8$-fibers and the E_1-curve on the $\tilde{E}_8$-fibers. The preimage on $\mathfrak{Z}^{(1)}$ of the center is the family of curves comprising the $2E_3 + E_4 + 2E_5$-curve on the $\tilde{D}_8$-fibers and $E_1 + 4E_3$-curve on the $\tilde{E}_8$-fibers. Consequently, both the E_3-component and the E_5-component specialize to the E_3-component. The E_4-components specialize to the E_1-component. Note that the preimage of the center is the Cartier divisor corresponding to the canonical section into $\mathcal{O}_{\mathfrak{Z}^{(1)}}(-1)$.

Next, we do the second blowing up $\mathfrak{Z}^{(2)} \to \mathfrak{Z}^{(1)}$. Now the center is the family comprising the $2E_5$-curve on the $\tilde{D}_8$-fibers and the $2E_3$-curve on the $\tilde{E}_8$-fibers. The preimage on $\mathfrak{Z}^{(2)}$ of the center is the family consisting of the $2E_5$-curve on the $\tilde{D}_8$-fibers and $2E_3 + 4E_6$-curve on the $\tilde{E}_8$-fibers. Hence the E_5-component specializes to the $E_3 + 2E_6$-curve. The E_3-components specializes to the E_3-component, and the E_4-component to the E_1-component.

Now we come to the third blowing up $\mathfrak{Z}^{(3)} \to \mathfrak{Z}^{(2)}$. Here the center is the family comprising the E_3-curve on both the $\tilde{D}_8$-fibers and the $\tilde{E}_8$-fibers. The preimage of the center is the family comprising the $E_3 + 2E_2$-curves on the $\tilde{D}_8$-fibers and $E_3 + 2E_4$-curve on the $\tilde{E}_8$-fibers. So the E_3-component specializes to E_3, and the E_2-component specialize to E_4. The E_4-component keeps on specializing to the E_1-component, because this family is disjoint from the exceptional locus of the blowing-up. The specialization of the E_5-component on the $\tilde{D}_8$-fibers is more interesting: It is necessarily of the form $E_3 + nE_4 + 2E_6$ for some integer $n \geq 0$. Using the fact that the whole fibers specializes to a whole fiber, we infer $n = 2$. The arguments for the remaining two steps in the blowing up process are similar, and left to the reader.

It remains to extend this simultaneous resolution to S'. Using the first projection $\mathrm{pr}_1 : Y = C \times C \to C$ rather than the second projection, we obtain a simultaneous resolution $\mathfrak{X}_V \to V$ for the family $\mathfrak{Z}_V \to V$. The two resolutions $\mathfrak{X}_U \to U$ and $\mathfrak{X}_V \to V$ coincide over the overlap $U \cap V$. Indeed, the degenerate fibers in $\mathfrak{Z} \to \mathbb{P}^1_S$ have constant fiber type $\tilde{D}_8$ over $U \cap V$, and the fiberwise integral components of the exceptional divisors in the simultaneous resolution $\mathfrak{X}_U$ constitute relative Cartier divisors over $U \cap V$. Such simultaneous resolutions are necessarily unique. □

The result may be rephrased by saying that there is a morphism $S' \to \mathrm{Res}_{\mathfrak{Z}/S}$. This does not seem to be an isomorphism, because our simultaneous resolution $\mathfrak{X}' \to S'$ does not appear to be unique. This nonuniqueness leads to nonseparability phenomena, as discussed by Burns and Rapoport in [11], Section 7.

13. Isomorphic fibers in the family

In the preceding section we constructed a smooth family of supersingular K3-surfaces $\mathfrak{X}' \to S'$ with Artin invariants $\sigma_0 \leq 2$, which is defined over the complement $S' \subset \mathbb{A}^2$ of the origin, where $\mathbb{A}^2 = \mathrm{Spec}\, k[\sqrt{r}, \sqrt{s}]$. According to Rudakov and Shafarevich [30], such families should depend only on one effective parameter. This is indeed the case, and can be made explicit as follows. Write $\mathbb{P}^1 = \mathrm{Proj}\, k[\sqrt{r}, \sqrt{s}]$, and consider the canonical projection $S' \to \mathbb{P}^1$.

Theorem 13.1. *Our smooth family $\mathfrak{X}' \to S'$ is isomorphic to the pullback of a smooth family $\mathfrak{X} \to \mathbb{P}^1$ of supersingular K3 surfaces with Artin invariants $\sigma_0 \leq 2$.*

Proof. The assertion would be obvious if the moduli space of polarized K3 surfaces would be fine. This, however, does not seem to be the case. Instead we shall work with moduli space of marked K3 surfaces. For this we have to check that our family admits a marking.

Let $K = k(\sqrt{r}, \sqrt{s})$ be the function field of the pointed affine plane S', and choose a separable closure $K \subset K^{\text{sep}}$. The Galois group $G = \text{Gal}(K^{\text{sep}}/K)$ acts on the the module $N = \text{Pic}(X_{K^{\text{sep}}})$, where $X_K = \mathfrak{X}'_K$ denotes the generic fiber. This action must be trivial: According to the explicit description of fibers and sections in X_K in Sections 5 and 6 and Proposition 7.2, the pullback map $\text{Pic}(X_K) \to \text{Pic}(X_{K^{\text{sep}}})$ is bijective. We conclude that the sheaf $\text{Pic}_{\mathfrak{X}'/S'}$ on S' in the étale topology is constant on some open affine subset $U \subset S'$. Whence there is bijection $N_U \to (\text{Pic}_{\mathfrak{X}'/S'})_U$. This bijection extends to a homomorphism $N \to \text{Pic}_{\mathfrak{X}'/S'}$, thanks to the explicit description of relative Cartier divisors inside the flat family $\mathfrak{X}' \to S'$ discussed in the previous section.

The homomorphism $N \to \text{Pic}_{\mathfrak{X}'/S'}$ is a *marking* in the sense of Ogus [27]. He proved in loc. cit., Theorem 2.7 that the functor of isomorphism classes of N-marked K3 surfaces is representable by a algebraic space S_N. Note that Ogus works in his paper under the general assumption $p \geq 3$, which seems appropriate for several arguments involving quadratic forms. However, the result concerning the existence of S_N holds true in all characteristics. Indeed, it is easy to see that the cofibered groupoid $\mathcal{F}_N$ of N-marked families of K3 surfaces is a stack, and indeed an algebraic stack (= Artin stack). The latter involves Grothendieck's Algebraization Theorem and openness of versality in the usual way. According to Rudakov and Shafarevich [31], Section 8, Proposition 3, the automorphism group of any marked K3 surface is trivial. By [27], Lemma 2.2 this carries over to families of marked K3 surfaces. It follows that the algebraic stack $\mathcal{F}_N$ is equivalent to its coarse moduli space S_N, which is a nonseparated algebraic space. The upshot is that the algebraic space S_N is a fine moduli space.

Our N-marked flat family $\mathfrak{X}' \to S'$ induces a morphism $h : S' \to S_N$. According to Proposition 7.7, this morphism is constant along all pointed lines, that is, fibers of the canonical projection $S' \to \mathbb{P}^1$. The image $h(S') \subset S_N$ is therefore a 1-dimensional algebraic space. Any algebraic space is generically a scheme, whence we obtain a rational morphism $\mathbb{P}^1 \dashrightarrow h(S')$ that extends as a continuous map on the underlying topological spaces. A local computation then shows that the rational morphism extends to a morphism $\mathbb{P}^1 \to h(S')$ of algebraic spaces. Pulling back the universal family from S' to $\mathbb{P}^1$, we obtain the desired family $\mathfrak{X} \to \mathbb{P}^1$ inducing our original family $\mathfrak{X}' \to S'$. □

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Mathematisches Institut, Heinrich-Heine-Universität, 40225 Düsseldorf, Germany
E-mail address: schroeer@math.uni-duesseldorf.de