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ON EMBEDDINGS INTO TORIC PREVARIETIES

J. HAUSEN AND S. SCHRÖER

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Abstract. We give examples of complete normal surfaces that are not embeddable into simplicial toric prevarieties nor toric prevarieties of affine intersection.

Introduction

This note is concerned with embeddability and non-embeddability into toric prevarieties. Włodarczyk [11] has shown that a normal variety Y admits a closed embedding into a toric variety X if and only if every pair $y_1, y_2 \in Y$ is contained in a common affine open neighbourhood. If one drops the latter condition there still exists a closed embedding into some toric *prevariety* X. This means that X may be non-separated.

The above embedding result has the following refinement: Supposing that Y is $\mathbb{Q}$-factorial one can choose X to be simplicial and of affine intersection. In other words, X is $\mathbb{Q}$-factorial and the intersection of any two affine open subsets of X is again affine (see [11] and [7]). Embeddings into toric prevarieties of affine intersection can for example be used to obtain global resolutions of coherent sheaves on Y (see [7]).

The purpose of this note is to find out whether or not every normal variety Y can be embedded into a toric prevariety X which is simplicial or of affine intersection. Unfortunately, the answer is negative: We provide examples of normal surfaces that admit neither embeddings into toric prevarieties of affine intersection nor into simplicial ones.

Together with Włodarczyk's embedding result, these counterexamples imply that there are toric prevarieties of affine intersection that cannot be embedded into simplicial toric prevarieties of affine intersection. An analogous statement is easily seen to hold in the category of toric varieties, since there are complete toric varieties with trivial Picard group (see for example [4]).

1. Non-projective complete surfaces

Here we briefly discuss some properties of complete normal surfaces which are non-projective. First, we have to fix notation. Throughout, we will work over an uncountable algebraically closed ground field k. The word *prevariety* refers to an integral k-scheme of finite type. A *variety* is a separated prevariety, and a *surface* is a 2-dimensional variety. By a *toric* prevariety we mean a prevariety X arising

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from a system of fans in some lattice N in the sense of [1]. For $k = \mathbb{C}$ this notion precisely yields the normal prevarieties X endowed with an effective regular action of a torus T with an open orbit.

The following facts are well known (compare [8], theorem 4.2, and [11]):

Proposition 1.1. *Let S be a normal surface and denote by $s_1, \ldots, s_n \in S$ its non-$\mathbb{Q}$-factorial singularities.*

(1) *The surface S is quasi-projective if and only if the points $s_1, \ldots, s_n$ have a common affine neighbourhood.*
(2) *The surface S admits a closed embedding into a toric variety if and only if every pair s_i, s_j lies in a common affine neighbourhood.*

In view of these statements, the simplest possible candidate for a normal variety without closed embeddings into toric prevarieties of affine intersection is a normal surface having precisely two non-$\mathbb{Q}$-factorial singularities $s_1, s_2 \in S$. In fact we will work with surfaces of this type.

Let S be a complete normal surface. There is a *projective reduction* $r \colon S \to S^{\mathrm{prj}}$ of S, which means that the morphism r is universal with respect to morphisms to projective schemes (see [2]). Since $\mathcal{O}_{S^{\mathrm{prj}}} \to r_*(\mathcal{O}_S)$ is necessarily bijective, the projective scheme S^{prj} is normal and the fibres of r are connected.

If S^{prj} is a curve, any Weil divisor E on S has a decomposition

$$E = E^{\mathrm{vert}} + E^{\mathrm{hor}}$$

into the part E^{vert} consisting of those prime cycles of E that are contained in the fibres of r and the part E^{hor} which consists of the remaining prime cycles. A Weil divisor E on S is called *vertical* if $E = E^{\mathrm{vert}}$, and it is called *horizontal* if $E = E^{\mathrm{hor}}$.

For a Weil divisor D on a variety Y and $y \in Y$, we denote by D_y the Weil divisor obtained from D by omitting all prime cycles not containing the point y. The following statement will be crucial for non-embeddability:

Lemma 1.2. *Let S be a complete normal surface with precisely two non-$\mathbb{Q}$-factorial singularities s_1, $s_2 \in S$. Assume that S^{prj} is a curve and that every vertical Weil divisor on S is $\mathbb{Q}$-Cartier. If E is a $\mathbb{Q}$-Cartier divisor on S with $E_{s_2} \geq 0$ and $E^{\mathrm{hor}} \neq 0$, then $E^{\mathrm{hor}}_{s_1}$ is not effective.*

Proof. Assume that $E^{\mathrm{hor}}_{s_1} \geq 0$ holds. Since E and E^{vert} are $\mathbb{Q}$-Cartier, so is their difference E^{hor}. Consider the decomposition

$$E^{\mathrm{hor}} = E^{\mathrm{hor}}_{+} + E^{\mathrm{hor}}_{-}$$

into positive and negative parts. Since s_1 and s_2 are not contained in E^{hor}_{-}, the surface S is $\mathbb{Q}$-factorial near E^{hor}_{-}. So E^{hor}_{-} is $\mathbb{Q}$-Cartier. Hence E^{hor}_{+} is also $\mathbb{Q}$-Cartier . As $E^{\mathrm{hor}} \neq 0$, there must be at least one horizontal Cartier divisor $C \subset S$.

Consider the invertible sheaf $\mathcal{L} = \mathcal{O}_S(C)$. Obviously, $\mathcal{L}$ is S^{prj}-generated on $S \setminus C$. Since $C \to S^{\mathrm{prj}}$ is finite, we can apply [10], Theorem 1.1 (over affine open subsets Spec(R) $\subset$ S$^{\mathrm{prj}}$), and deduce that $\mathcal{L}^{\otimes n}$ is S^{prj}-generated for some $n > 0$. Consequently, the homogeneous spectrum

$$S' = \mathrm{Proj}(f_*\mathrm{Sym}(\mathcal{L}))$$

is a projective S^{prj}-scheme, hence projective. But $\mathcal{L}$ is ample on the generic fibre of $r \colon S \to S^{\mathrm{prj}}$, contradicting the universal property of the projective reduction. $\square$

Remark 1.3. Surfaces as above really exist, compare [9], 2.5. Here the assumption that the ground field k is uncountable comes in.

We will also make use of the following elementary fact:

Lemma 1.4. *Let $f\colon Y \to X$ be a morphism of integral normal prevarieties. Given a Cartier divisor D on X such that the preimage $E := f^*(D)$ exists as Cartier divisor. Decompose $D = \sum_i \lambda_i D_i$ and $E = \sum_j \mu_j E_j$ into prime cycles. If there is a component E_j with $\mu_j < 0$, then there is a component D_i with $\lambda_i < 0$ and $f(E_j) \subset D_i$.*

Proof. Assume there is no such D_i. Decompose $D = D_+ - D_-$ into positive and negative parts. Then we have $E_j \not\subset f^{-1}(D_-)$. Hence the restriction of E to $Y \setminus f^{-1}(D_-)$ is not effective. On the other hand, the restriction of D to $X \setminus D_-$ is effective, contradiction. □

2. Non-embeddability

Recall that a scheme X is separated if the diagonal morphism $\Delta\colon X \to X \times X$ is a closed embedding. A weaker condition is that Δ is affine. In this situation we call X to be of *affine intersection*. To check this property it suffices to find an open affine covering $X = \bigcup_i X_i$ such that each intersection $X_i \cap X_j$ is affine. Clearly, if X is of affine intersection, every subscheme is so.

Theorem 2.1. *Let S be a complete normal surface with precisely two non-$\mathbb{Q}$-factorial singularities $s_1, s_2 \in S$. Assume that the projective reduction S^{prj} is a curve and that all irreducible components of the fibres of $r\colon S \to S^{\mathrm{prj}}$ are $\mathbb{Q}$-Cartier. Let $f\colon S \to X$ be a morphism to a toric prevariety X. If X is simplicial or of affine intersection, then there is a morphism $\widetilde{f}\colon S^{\mathrm{prj}} \to X$ with $f = \widetilde{f} \circ r$.*

Let us record the following evident

Corollary 2.2. *The surface S is neither embeddable into a toric prevariety of affine intersection nor into a simplicial toric prevariety.*

Proof of theorem 2.1. First we treat the case that X is of affine intersection. Let T denote the acting torus of X. There is a unique T-orbit $B \subset X$ such that $f(S) \subset \overline{B}$ holds. Note that $\overline{B}$ is again a toric prevariety of affine intersection. Replacing X by $\overline{B}$ we may assume that $f(S)$ hits the open T-orbit.

Set $x_1 := f(s_1)$ and $x_2 := f(s_2)$. For $i = 1, 2$ let $X_i \subset X$ be the affine T-stable neighborhoods containing $T{\cdot}x_i$ as closed orbits, respectively. Set $N := \mathrm{Hom}(k^*, T)$ and let $M := \mathrm{Hom}(N, \mathbb{Z})$. Then we have

$$X_i = X_{\sigma_i} = \mathrm{Spec}(k[\sigma_i^\vee \cap M])$$

for certain cones σ_i in the lattice N. Since X is of affine intersection, the set $X_{12} := X_1 \cap X_2$ is of the form $X_{12} = X_{\sigma_{12}}$ for some common face σ_{12} of σ_1 and σ_2. Hence there is a linear form $u \in M$ with

$$u \in \sigma_1^\vee \qquad \text{and} \qquad \sigma_{12} = \sigma_1 \cap u^\perp.$$

Since $f(S)$ hits the open T-orbit, all characters χ^m, $m \in M$, admit pullbacks $f^*(\chi^m)$ as rational functions. In particular χ^u has a pullback ψ with respect to f, and the principal divisor $\mathrm{div}(\chi^u)$ has the principal divisor $\mathrm{div}(\psi)$ as pullback. Since χ^u is defined on X_1, we have

$$\mathrm{div}(\psi)_{s_1} \geq 0.$$

We claim that no component of $\mathrm{div}(\psi)_{s_1}$ contains the point $s_2 \in S$. Otherwise, let $E \subset S$ be such a component. Then there exists a T-stable component $D_1 \subset X_1$ of $\mathrm{div}(\chi^u) \cap X_1$ containing $f(E) \cap X_1$. This component corresponds to an edge $\varrho_1 \subset \sigma_1$. Note that by [5], p. 61 we have $\varrho_1 \not\subset u^\perp$. Take the closure $D := \overline{D_1}$ in X, set $D_2 := D \cap X_2$, and let $\varrho_2 \subset \sigma_2$ be the edge corresponding to $D_2 \subset X_2$. Then $s_2 \in E$ implies

$$x_2 = f(s_2) \in f(E) \cap X_2 \subset D_2.$$

Thus $D \cap X_2$ is non-empty, hence $D_{12} := D \cap X_{12}$ is a T-stable Weil divisor in $X_{12} = X_{\sigma_{12}}$. Let $\varrho_{12} \subset \sigma_{12}$ be the edge corresponding to $D_{12} \subset X_{12}$. Since D_{12} is induced by D_1 and D_2 as well, we conclude $\varrho_1 = \varrho_{12} = \varrho_2$. Thus we have $\varrho_1 \subset \sigma_{12} \subset u^\perp$, contradicting $\varrho_1 \not\subset u^\perp$. As a consequence of our claim, $\mathrm{div}(\psi)_{s_1}$ is an effective $\mathbb{Q}$-Cartier divisor. According to lemma 1.2, the divisor $\mathrm{div}(\psi)_{s_1}$ is vertical.

Next, we show that for every principal divisor $D = \mathrm{div}(\chi^m)$, $m \in M$ its pullback to S is vertical. It suffices to check this for a set of generators of M, for example $M \cap \sigma_2^\vee$. Assume that there is some $m \in M \cap \sigma_2^\vee$ such that $E := f^*(D)$ is not vertical. Decompose $D = \sum_i \lambda_i D_i$ and $E = \sum_j \mu_j E_j$ into prime cycles. Since $D_{x_2} \geq 0$, we also have $E_{s_2} \geq 0$. By lemma 1.2 there must be some horizontal component E_j containing s_1 with $\mu_j < 0$. Lemma 1.4 tells us that there is some component D_i with $\lambda_i < 0$ and $f(E_j) \subset D_i$. In particular we have $x_1 \in D_i$. Let $\varrho_i \subset \sigma_1$ be the edge corresponding to the T-invariant Weil divisor $D_i \cap X_1$. Since $x_2 \notin D_i$ we have $\varrho_i \not\subset \sigma_{12}$. Hence $\varrho_i \not\subset \sigma_1 \cap u^\perp$, and D_i occurs with positive multiplicity in $\mathrm{div}(\chi^u)_{x_1}$. On the other hand, we already have seen that $\mathrm{div}(\psi)_{s_1}$ is vertical. Consequently, $E_j \subset f^{-1}(D_i)$ is also vertical, contradiction. We have shown that $\mathrm{div}(f^*(\chi^m))$ is vertical for all $m \in M$.

As a consequence, the induced morphism $f^{-1}(T) \to T$ is constant along the generic fibre of $r\colon S \to S^{\mathrm{prj}}$. By the rigidity lemma (see for example [3], 1.5), all fibres of r are mapped to points under $f\colon S \to X$. Now [6], 8.11.1 gives a morphism $\widetilde{f}\colon S^{\mathrm{prj}} \to X$ with $f = \widetilde{f} \circ r$. This proves the theorem for toric prevarieties of affine intersection.

It remains to treat the case that X is simplicial. According to [7], Section 1, there is a toric prevariety X' of affine intersection and a local isomorphism $g\colon X \to X'$. As we have seen, $g \circ f$ is constant along the fibres of $r\colon S \to S^{\mathrm{prj}}$. Since the fibres of r are connected, this implies that f is constant along the fibres of r, and we obtain the desired factorization as above.

References

1. A. A'Campo–Neuen, J. Hausen: *Toric prevarieties and subtorus actions.* Geom. Dedicata **87** (2001), 34–65.
2. A. Białynicki–Birula: *Categorical quotients.* J. Algebra **239** (2001), 35–55.
3. H. Clemens, J. Kollar, S. Mori: *Higher dimensional complex geometry.* Astérisque **166** (1988).
4. M. Eikelberg: *Picard groups of compact toric varieties and combinatorial classes of fans.* Result. Math. **23**, No. 3–4, (1993) 251–293.
5. W. Fulton: *Introduction to toric varieties.* Princeton University Press, Princeton, 1993.
6. A. Grothendieck, J. Dieudonné: *Éléments de géometrie algébrique, Chap. II: Étude globale élémentaire de quelques classes de morphismes.* Publications mathématiques de l'IHES **8** (1961).
7. J. Hausen: *On Włodarczyk's embedding theorem.* Int. J. Math. **11**, No. 6 (2000), 811–836.
8. R. Hartshorne: *Ample subvarieties of algebraic varieties.* Lecture Notes in Math. **156**, Springer, Berlin etc., 1970.

9. S. Schröer: *On non-projective surfaces.* Manuscr. Math. **100** (1999), 317–321.
10. S. Schröer: *A characterization of semiampleness and contractions of relative curves.* Kodai Math. J. **24** (2001), 207-213.
11. J. Włodarczyk: *Embeddings in toric varieties.* J. Algebraic Geometry **2** (1993), 705–726.

Fachbereich Mathematik und Statistik, Universität Konstanz, D-78557 Konstanz, Germany.
E-mail address: `Juergen.Hausen@uni-konstanz.de`

Mathematisches Institut, Ruhr-Universität, 44780 Bochum, Germany.
E-mail address: `s.schroeer@ruhr-uni-bochum.de`