PATHOLOGIES IN COHOMOLOGY OF NON-PARACOMPACT HAUSDORFF SPACES

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Abstract. We construct a non-paracompact Hausdorff space for which Čech cohomology does not coincide with sheaf cohomology. Moreover, the sheaf of continuous real-valued functions is neither soft nor acyclic, and our space admits non-numerable principal bundles.

Introduction

Recall that a topological space X is called *paracompact* if it is Hausdorff, and each open covering admits a refinement that is locally finite. This notion was introduced by Dieudonné [4] as early as 1944 and has turned out to be extremely useful in general topology and sheaf theory. For example, Godement showed that *Čech cohomology* coincides with *sheaf cohomology* on paracompact spaces ([6], Theorem 5.10.1). For general spaces, all that can be said is that there is a spectral sequence

$$\check{H}^p(X, \mathcal{H}^q(\mathcal{F})) \Longrightarrow H^{p+q}(X, \mathcal{F})$$

computing the "true" sheaf cohomology from the Čech cohomology of the presheaves of sheaf cohomology (loc. cit., Theorem 5.9.1). Grothendieck observed that for many irreducible spaces, for example $X = \mathbb{C}^2$ with the Zariski topology, this spectral sequence does not degenerate for suitable $\mathcal{F}$, such that Čech cohomology does not coincide with sheaf cohomology ([7], page 178). On the other hand, Artin [1] established that for "most" separated schemes, Čech cohomology agrees with sheaf cohomology when computed in the étale topology.

Although the known counterexamples are very common in the realm of algebraic geometry, they are perhaps not so natural from the standpoint of algebraic or general topology, since the spaces are not Hausdorff. In my opinion, it would be desirable to have further counterexamples satisfying the Hausdorff axiom, the more so in light of Artin's result.

The goal of this note is to provide such a space. The construction roughly goes as follows: We start with an infinite wedge sum $X = \bigvee_{i=1}^{\infty} D^2$ of closed 2-disks, and replace the CW-topology at the intersection of the 2-disks by some coarser topology. This topology is choose fine enough to keep the space Hausdorff, yet coarse enough so that a variant of Grothendieck's argument holds true.

It turns out that our space has other pathological features as well: The sheaf of continuous real-valued functions is neither soft nor acyclic. Although X_{crs} is contractible, it carries nontrivial principal S^1-bundles. These are necessarily non-numerable, whence do not come from the universal bundle.

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1. The construction

We start by constructing an infinite 2-dimensional CW-complex X. Its 0-skeleton is a sequence e_n^0, $n \geq 0$ of 0-cells. The first 0-cell $x = e_0^0$ will play a special role throughout, and we shall call it the *origin*. To form the 1-skeleton X^1, we connect the origin to each e_n^0, $n \geq 1$ with two 1-cells called $e_{\pm n}^1$. To complete the construction, we choose homeomorphisms

$$\varphi_n : S^1 \longrightarrow e_0^0 \cup e_n^0 \cup e_n^1 \cup e_{-n}^1 \subset X^1,$$

and use these as attaching maps for the 2-cells e_n^2, $n \geq 1$. This gives an infinite 2-dimensional CW-complex X, which one may visualize as follows.

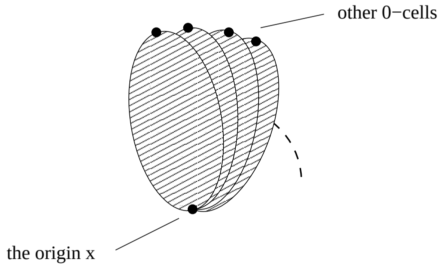

Figure 1: The CW-complex X

Being a CW-complex, the space X is paracompact [10]. With our goal in mind we now replace the CW-topology by some coarser topology: Let τ be the collection of all subsets $U \subset X$ that are open in the CW-topology, and either do not contain the origin x, or contain almost all pointed closed 2-cells $\overline{e_n^2} \smallsetminus e_n^0$. This collection of subsets obviously satisfies the axioms of a topology, and we call this topology τ the *coarser topology.* Here and throughout, almost all means all but finitely many. The set X, endowed with the coarser topology, is denoted X_{crs}.

Proposition 1.1. *The space X_{crs} is Hausdorff but not paracompact.*

Proof. Clearly, the identity map $X \to X_{\text{crs}}$ is continuous, and becomes a homeomorphism outside the origin. Thus X_{crs} is Hausdorff outside the origin x. Given $y \neq x$, we choose two disjoint open neighborhoods $x \in U$, $y \in V$ on the CW-complex X. By shrinking V, we may assume that V intersects only one closed 2-cell. By enlarging U, we may assume that U contains all remaining closed 2-cells, while staying disjoint from V. Then U, V are open in the coarse topology, thus X_{crs} is Hausdorff.

To see that the space is not paracompact, let $U_0 \subset X_{\text{crs}}$ be the complement of $\bigcup_{n\geq 1} e_n^0$, and $U_n \subset X$ by the pointed closed 2-cell $\overline{e}_n^2 \smallsetminus \{x\}$. This gives an open covering $X_{\text{crs}} = U_0 \cup U_1 \cup \ldots$. Every refinement of this covering fails to be locally finite: for any neighborhood of the origin intersects for almost all $n \geq 1$ any neighborhood of e_n^0 contained in U_n, which are pairwise different. Thus our space is not paracompact. $\square$

Remark 1.2. The space X_{crs} is not *regular*: Consider the origin x and the closed sets $A = \bigcup_{n\geq 1} e_n^0$. Then every open neighborhood of x intersects every open neighborhood of A. On the other hand, the space X_{crs} is *pointwise paracompact*, a property also called *metacompactness*: Every open covering admits a refinement that is pointwise finite. Clearly, each closed 2-cell $e_n^2 \subset X_{\mathrm{crs}}$ is compact, hence X_{crs} is a countable union of compacta, in other words, our space is *σ-compact.* In particular, it is *Lindelöf*, which means that every open covering has a countable subcovering. The reader may consult Steen and Seebach [11] for other counterexamples in this direction.

Remark 1.3. The *Kelley topology* (also called the *compactly generated topology*) on a space Y consists of those subsets $V \subset Y$ such that $V \cap K \subset K$ is open for each compact subset $K \subset X$. This topology plays a role for infinite CW-complexes, for example, to define products. One easily checks that each compact subset $K \subset X_{\mathrm{crs}}$ is also compact with respect to the CW-topology. From this it follows that the Kelley topology of X_{crs} coincides with the CW-topology.

2. Čech and sheaf cohomology

Recall that for every abelian sheaf $\mathcal{F}$ on any topological space Y, the canonical map $\check{H}^1(Y, \mathcal{F}) \to H^1(Y, \mathcal{F})$ is bijective, and we have a short exact sequence

$$0 \longrightarrow \check{H}^2(Y, \mathcal{F}) \longrightarrow H^2(Y, \mathcal{F}) \longrightarrow \check{H}^1(Y, \mathcal{H}^1(\mathcal{F})) \longrightarrow 0,$$

as explained in [7], page 177. Thus Čech cohomology does not coincide with sheaf cohomology provided $\check{H}^1(Y, \mathcal{H}^1(\mathcal{F})) \neq 0$.

Our task is therefore to find such a situation. Consider the CW-complex X and the space X_{crs} constructed in the preceding section. The following fact will be useful:

Lemma 2.1. *For each open subset $V \subset X_{\mathrm{crs}}$, the sheaf cohomology groups $H^p(V, \mathbb{Z})$, $p \geq 0$ are the same, whether computed in the CW-topology or in the coarser topology.*

Proof. Let $i : X \to X_{\mathrm{crs}}$ be the identity map, which is continuous. We have a canonical map $\mathbb{Z}_{X_{\mathrm{crs}}} \to i_*(\mathbb{Z}_X)$ of abelian sheaves, where the left hand side is the sheaf of locally constant integer-valued functions on X_{crs}, and the right hand side is the direct image sheaf of the corresponding sheaf on X. We first check that this map is bijective. The question is local on X_{crs}, and bijectivity is obvious outside the origin. Injectivity holds because the mapping i is surjective. Since there are arbitrarily small open neighborhoods $x \subset V \subset X_{\mathrm{crs}}$ that are pathconnected and hence connected in the CW-topology, the canonical map $\mathbb{Z}_{X_{\mathrm{crs}},x} \to i_*(\mathbb{Z}_X)_x$ is bijective as well.

In light of the Leray–Serre spectral sequence

$$H^p(X_{\mathrm{crs}}, R^q i_*(\mathbb{Z}_X)) \Longrightarrow H^{p+q}(X, \mathbb{Z}_X),$$

it suffices to check that $R^p i_*(\mathbb{Z}_X) = 0$ for all $p > 0$. This is again local, and holds for trivial reasons outside the origin. Since there are arbitrarily small open neighborhoods $x \in V \subset X_{\mathrm{crs}}$ that are contractible in the CW-topology, and singular cohomology coincides with sheaf cohomology for CW-complexes ([2], Chapter III, Section 1), vanishing holds at the origin as well. ☐

Let $U \subset X$ be the complement of the 1-skeleton $X^1 \subset X$, that is, the union of all 2-cells, and $\mathbb{Z}_U$ be the abelian sheaf of locally constant integer-valued functions on U. Clearly, U is open in the coarser topology. Thus the inclusion map $i : U \to X_{\text{crs}}$ is continuous. From this we obtain an abelian sheaf $\mathcal{F} = i_!(\mathbb{Z}_U)$ on X_{crs}, called *extension by zero*. It is defined by the rule

$$\Gamma(V, \mathcal{F}) = \begin{cases} \Gamma(V, \mathbb{Z}_U) & \text{if } V \subset U; \\ 0 & \text{else,} \end{cases}$$

compare [8], Expose I. Its first cohomology is easily computed:

Proposition 2.2. *Let $V \subset X_{\text{crs}}$ be an open subset with $H^1(V, \mathbb{Z}) = 0$. Then we have a canonical identification*

$$H^1(V, \mathcal{F}) = H^0(V \cap X^1, \mathbb{Z})/H^0(V, \mathbb{Z}).$$

Proof. The short exact sequence $0 \to \mathcal{F} \to \mathbb{Z}_{X_{\text{crs}}} \to \mathbb{Z}_{X^1} \to 0$ induces a long exact sequence

$$H^0(V, \mathbb{Z}) \longrightarrow H^0(V \cap X^1, \mathbb{Z}) \longrightarrow H^1(V, \mathcal{F}) \longrightarrow H^1(V, \mathbb{Z}),$$

and the result follows. □

For this sheaf, Čech cohomology does not coincide with sheaf cohomology, in a rather drastic way:

Theorem 2.3. *For the abelian sheaf $\mathcal{F} = i_!(\mathbb{Z}_U)$ on the topological space X_{crs} the group $\check{H}^1(X_{\text{crs}}, \mathcal{H}^1(\mathcal{F}))$ is uncountable. In particular, the inclusion $\check{H}^2(X, \mathcal{F}) \subset H^2(X, \mathcal{F})$ is not bijective.*

Proof. Let $\mathfrak{U} = (U_\alpha)_{\alpha \in I}$ be an open covering of X_{crs}. By definition, the corresponding group $\check{H}^1(\mathfrak{U}, \mathcal{H}^1(\mathcal{F}))$ is the first cohomology of the complex

$$\prod_{\alpha} H^1(U_\alpha, \mathcal{F}) \longrightarrow \prod_{\alpha<\beta} H^1(U_{\alpha\beta}, \mathcal{F}) \longrightarrow \prod_{\alpha<\beta<\gamma} H^1(U_{\alpha\beta\gamma}, \mathcal{F}). \tag{1}$$

Here we employ the usual abbreviation $U_{\alpha\beta} = U_\alpha \cap U_\beta$ et cetera. The coboundary maps are the usual one, for example $(s_\alpha) \mapsto (s_\beta|U_{\alpha\beta} - s_\alpha|U_{\alpha\beta})$, and we have chosen a total order on the index set I. By definition, Čech cohomology equals

$$\check{H}^1(X, \mathcal{H}^1(\mathcal{F})) = \varinjlim_{\mathfrak{U}} \check{H}^1(\mathfrak{U}, \mathcal{H}^1(\mathcal{F})),$$

where the direct limit runs over all open coverings ordered by the refinement relation. For a precise definition of the maps in the direct system, and their well-definedness, we refer to [6], Chapter II, Section 5.7.

In general, it can be difficult to control such direct limits. However, one may restrict to open coverings forming a cofinal subsystem. Therefor, we may assume that our open covering satisfies the following five additional assumptions: (i) Each U_α and the intersection $U_\alpha \cap X^1$ are, if nonempty, contractible in the CW-topology. (ii) Each 0-cell is contained in precisely one U_α. (iii) If some U_α contains a 0-cell e^0_n, $n \geq 1$, then it is contained in the corresponding closed 2-cell $\overline{e^2_n}$. (iv) We suppose that the index set I is well-ordered. This allows us to regard the natural numbers $0, 1, \ldots \in I$ as indices. After reindexing, we stipulate that $x \in U_0$ and $e^0_n \subset U_n$. (v) Finally, if a closed 2-cell $\overline{e^2_n}$ is contained in $U_0 \cup U_n$, then it is disjoint from all other U_α.

From now on, we only consider open coverings $\mathfrak{U}$ satisfying these five condition. Choose $m \geq 1$ so that U_0 contains all pointed closed 2-cells $\overline{e_n^2} \smallsetminus e_n^0$, $n \geq m$. Condition (i) implies that for $V = U_0 \cap U_n = U_n \smallsetminus e_n^0$, $n \geq m$ we have $H^1(V, \mathbb{Z}) = 0$, and furthermore

$$H^0(V \cap X^1) = \mathbb{Z}^{\oplus 2} \quad \text{and} \quad H^0(V, \mathbb{Z}) = \mathbb{Z},$$

the latter sitting diagonally in the former. Note that this is the key step in Grothendieck's argument [7], page 178. Now Proposition 2.2 gives us an identification

$$\prod_{n=m}^{\infty} H^1(U_0 \cap U_n) = \prod_{n=m}^{\infty} \mathbb{Z}.$$

In light of Proposition 2.2, Condition (i) ensures that the term on the left in the complex (1) vanishes. Condition (ii) and (v) tell us that the triple intersections $U_0 \cap U_n \cap U_\alpha$ are empty for $n \geq m$ and all indices $\alpha \neq 0, n$. The upshot is that we have a canonical inclusion

$$\prod_{n=m}^{\infty} \mathbb{Z} = \prod_{n=m}^{\infty} H^1(U_0 \cap U_n, \mathcal{F}) \subset \check{H}^1(\mathfrak{U}, \mathcal{H}^1(\mathcal{F})).$$

If $\mathfrak{U}'$ is a refinement of $\mathfrak{U}$ satisfying the same five conditions, the induced map $\check{H}^1(\mathfrak{U}, \mathcal{H}^1(\mathcal{F})) \to \check{H}^1(\mathfrak{U}', \mathcal{H}^1(\mathcal{F}))$ restricts to the canonical projection

$$\prod_{n=m}^{\infty} \mathbb{Z} \longrightarrow \prod_{n=m'}^{\infty} \mathbb{Z}$$

on the subgroups considered above, where we tacitly choose $m' \geq m$. Since forming direct limits is exact, we obtain an inclusion

$$\varinjlim_{m} \prod_{n=m}^{\infty} \mathbb{Z} \subset \varinjlim_{\mathfrak{U}} \check{H}^1(\mathfrak{U}, \mathcal{H}^1(\mathcal{F})) = \check{H}^1(X, \mathcal{H}^1(\mathcal{F})).$$

Again using that forming direct limits is exact, we may rewrite the left hand side as

$$\varinjlim_{m} \left(\prod_{n=1}^{\infty} \mathbb{Z} \Big/ \prod_{n=1}^{m-1} \mathbb{Z} \right) = \left(\prod_{n=1}^{\infty} \mathbb{Z} \right) \Big/ \left(\varinjlim_{m} \prod_{n=1}^{m-1} \mathbb{Z} \right) = \left(\prod_{n=1}^{\infty} \mathbb{Z} \right) \Big/ \left(\bigoplus_{n=1}^{\infty} \mathbb{Z} \right),$$

which is uncountable. □

3. Continuous functions and principal bundles

We finally examine pathological properties of continuous functions and principal bundles on X_{crs}. Let us write $\mathcal{C}_{X_{\text{crs}}}$ for the sheaf of continuous real-valued functions on X_{crs}.

Proposition 3.1. *We have $H^1(X_{\text{crs}}, \mathcal{C}_{X_{\text{crs}}}) \neq 0$.*

Proof. Recall that Čech cohomology agrees with sheaf cohomology in degree one. Thus our task is to construct a nontrivial Čech cohomology class. Consider the open covering $\mathfrak{U}$ given by $U_0 = X_{\text{crs}} \smallsetminus \bigcup_{n \geq 1} e_n^0$ and $U_n = \overline{e_n^2} \smallsetminus \{x\}$, $n \geq 1$. Choose germs of continuous functions $f_n : (U_n, e_n^0) \to \mathbb{R}$ having an isolated zero at e_n^0. Then its reciprocal $1/f_n$ is defined on some open punctured neighborhood of $e_n^0 \subset U_n$, where it is necessarily unbounded. On the other hand, for any continuous function $g : U_0 \to \mathbb{R}$ there is some $m \geq 0$ so that g is bounded on $\bigcup_{n=m}^{\infty} \overline{e_n^2} \cap U_0$. Whence

$1/f_n$ cannot be written as the difference of continuous functions coming from U_0 and U_n, for $n \geq m$. The same applies for any refinement $\mathfrak{U}'$ satisfying the five conditions formulated in the proof for Theorem 2.3. The upshot is that for all refinements $\mathfrak{U}'$ with U'_n sufficiently small, we obtain a well-defined tuple

$$(1/f_n)_{n\geq m} \in \prod_{n=m}^{\infty} H^0(U'_0 \cap U'_n, \mathcal{C}_{X_{\mathrm{crs}}})$$

that is a cocycle whose class in $\check{H}^1(\mathfrak{U}', \mathcal{C}_{X_{\mathrm{crs}}})$ is nonzero. Recall that $m \geq 1$ is any integer so that U'_0 contains $\overline{e_n^2} \smallsetminus e_n^0$ for all $n \geq m$. Since this holds for all such refinements $\mathfrak{U}'$, it follows that the class in the direct limit $\check{H}^1(X_{\mathrm{crs}}, \mathcal{C}_{X_{\mathrm{crs}}})$ is nonzero as well. $\square$

Remark 3.2. On normal spaces Y, the Uryson Lemma ensures that the sheaf $\mathcal{C}_Y$ is *soft*, that is, the canonical map $H^0(Y, \mathcal{C}_Y) \to H^0(A, i^{-1}(\mathcal{C}_Y))$ is surjective for all closed subsets A, where $i : A \to Y$ denotes the inclusion map. According to [6], Chapter II, Theorem 4.4.3, soft sheaves on paracompact spaces Y are acyclic.

For our space X_{crs}, it is easy to check that the canonical map for the discrete closed subset $A = \bigcup_{n\geq 1} e_n^0$ is not surjective. Summing up, the sheaf of continuous real-valued functions on X_{crs} is neither soft nor acyclic.

Next consider the sheaf $\mathcal{S}^1_{X_{\mathrm{crs}}}$ of continuous functions taking values in the circle group $S^1 = \mathbb{R}/\mathbb{Z}$. It sits in the exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{C}_{X_{\mathrm{crs}}} \longrightarrow \mathcal{S}^1_{X_{\mathrm{crs}}} \longrightarrow 0,$$

where the map on the right is $t \mapsto e^{2\pi i t}$. From this we get an exact sequence

$$H^p(X_{\mathrm{crs}}, \mathbb{Z}) \longrightarrow H^p(X_{\mathrm{crs}}, \mathcal{C}_{X_{\mathrm{crs}}}) \longrightarrow H^p(X_{\mathrm{crs}}, \mathcal{S}^1_{X_{\mathrm{crs}}}) \longrightarrow H^{p+1}(X_{\mathrm{crs}}, \mathbb{Z}).$$

The outer terms vanish by Proposition 2.1, and we conclude that

$$H^p(X_{\mathrm{crs}}, \mathcal{C}_{X_{\mathrm{crs}}}) = H^p(X_{\mathrm{crs}}, \mathcal{S}^1_{X_{\mathrm{crs}}})$$

for all $p > 0$. From the preceding Proposition we get $H^1(X_{\mathrm{crs}}, \mathcal{S}^1_{X_{\mathrm{crs}}}) \neq 0$. In other words, there are nontrivial principal S^1-bundles over X_{crs}. This is in stark contrast to the following fact:

Proposition 3.3. *The space X_{crs} is contractible.*

Proof. The CW-topology and the coarse topology induce the same topology on the compact subsets $\overline{e_n^2} \subset X_{\mathrm{crs}}$, which are thus homeomorphic to the 2-disk. Choose homotopies $h_n : \overline{e_n^2} \times I \to \overline{e_n^2}$ between the identity and the constant map to the origin so that $h_n(x,t) = x$ for all $t \in I$, and $h_n(y,t) \notin e_n^0$ for all $t > 0$ and all y. The first condition ensures that the homotopies glue to a map $h : X \times I \to X$, which is continuous with respect to the CW-topology. From the second condition one easily infers that it remains continuous when regarded as a map $h : X_{\mathrm{crs}} \times I \to X_{\mathrm{crs}}$. Thus h is a homotopy from the identity on X_{crs} to the constant map $X_{\mathrm{crs}} \to \{x\}$. $\square$

Let G be a topological group. Recall that a G-principal bundle $P \to Y$ is called *numerable* if it can be trivialized on some numerable covering $Y = \bigcup_{\alpha\in I} V_\alpha$. The latter means that there is a partition of unity $f_\beta : X \to [0,1]$, $\beta \in J$ so that the open covering $f^{-1}(]0,1])$, $\beta \in J$ is locally finite and refines the given covering V_α, $\alpha \in I$. According to Milnor's construction of the classifying space

$$BG = G \star G \star G \star \ldots$$

as a countable join [9], together with Dold's analysis ([5], Section 7 and 8), the isomorphism classes of numerable bundles correspond to the homotopy classes of continuous maps $Y \to BG$. We conclude:

Corollary 3.4. *The only principal G-bundles over X_{crs} that are numerable are the trivial ones.*

Remark 3.5. Non-numerable $\mathbb{Z}$-bundles based on a construction with the long line appear in [3]. A non-numerable principal $\mathbb{R}$-bundle over a non-Hausdorff space is sketched in [12], page 350.

Remark 3.6. The results in this section hold true if one uses a simpler space, obtained by attaching only 1-cells e^1_n and no 2-cells, rather than pairs of 1-cells $e^1_{\pm n}$ and 2-cells e^2_n. Of course, the coarser topology is defined in the same way.

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