POINTS IN THE FPPF TOPOLOGY

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Abstract. Using methods from commutative algebra and topos-theory, we construct topos-theoretical points for the fppf topology of a scheme. These points are indexed by both a geometric point and a limit ordinal. The resulting stalks of the structure sheaf are what we call fppf-local rings. We show that for such rings all localizations at primes are henselian with algebraically closed residue field, and relate them to AIC and TIC rings. Furthermore, we give an abstract criterion ensuring that two sites have point spaces with identical sobrification. This applies in particular to some standard Grothendieck topologies considered in algebraic geometry: Zariski, étale, syntomic, and fppf.

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Introduction

One of the major steps in Grothendieck's program to prove the Weil Conjectures was the introduction of topoi [2], thus lying the foundations for étale cohomology. Roughly speaking, a *topos* is a category $\mathscr{E}$ that is equivalent to the category of sheaves $\mathrm{Sh}(\mathcal{C})$ on some *site* $\mathcal{C}$. The latter is a category endowed with a Grothendieck topology, which gives the objects a role similar to the open subsets of a topological space. (Set-theoretical issues will be neglected in the introduction, but treated with care in what follows.)

One may perhaps say that topoi are the true incarnation of our notion of space, keeping exactly what is necessary to pass back and forth between local and global, to apply geometric intuition, and to use cohomology. This comes, of course, at the price of erecting a frightening technical apparatus. Half-way between sites and topological spaces dwell the so-called *locales*, sometimes referred to as *pointless*

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spaces, which are certain ordered sets L having analogous order properties like the collection $\mathscr{T}$ comprising the open subsets from a topological space $(X, \mathscr{T})$.

A considerable part of [2] deals with the notion of points for a topos. Roughly speaking, *topos-theoretical points* are continuous maps of topoi $P : (\mathrm{Set}) \to \mathscr{E}$, where the category of sets is regarded as the topos of sheaves on a singleton space. Such a continuous map consists of a *stalk functor* $P^{-1} : \mathscr{E} \to (\mathrm{Set})$, which must commute with finite inverse limits, and a direct image functor $P_* : (\mathrm{Set}) \to \mathscr{E}$, related by an adjunction. After one chooses an equivalence $\mathscr{E} \simeq \mathrm{Sh}(\mathcal{C})$, the points P can be recovered via certain pro-objects $(U_i)_{i \in I}$ of *neighborhoods* $U_i \in \mathcal{C}$, such that $F_P = P^{-1}(F) = \varinjlim_i \Gamma(U_i, F)$ for all sheaves F on the site $\mathcal{C}$.

The topos-theoretical points P form the *category of points* $\mathrm{Points}(\mathscr{E})$, and their isomorphism classes $[P]$ comprise the *space of points* $|\mathscr{E}|$, where the topology comes from the subobjects of the terminal object $e \in \mathscr{E}$. In the special case that $\mathscr{E} = X_{\mathrm{Zar}} = \mathrm{Sh}(X)$ is the topos of sheaves on a topological space X, there is continuous map $X \to |\mathscr{E}|$, which can be identified with the *sobrification* $X \to X_{\mathrm{sob}}$, in other words, the universal map into a topological space where each irreducible closed subset has a unique generic points. In particular, for schemes X endowed with the Zariski topology we have an identification $X = |\mathscr{E}|$.

Another important result is *Deligne's Theorem*, which asserts that topoi fulfilling certain technical finiteness conditions have *enough points*, that is, stalk functors detect monomorphisms ([3], Appendix to Expose VI). Note, however, that there are examples of topoi having no point at all, and examples of topoi having "large" spaces of points ([2], Expose IV, Section 7).

The goal of this paper is to investigate the space of points $|\mathscr{E}|$ for various topoi occurring in algebraic geometry, in particular for the *fppf topology.* The fppf topos was studied, for example, by Milne ([26], Chapter III, §3 and [27], Chapter III), Shatz ([34], Chapter VI and [31], [32], [33]), Waterhouse [37], and the Stacks Project [35]. Somewhat surprisingly, very little seems to be known about the points. Interesting result on several other Grothendieck topologies were recently obtained by Gabber and Kelly [10].

Note that one ususally defines the fppf topos via sheaves on the "big" site (Sch/X) of all X-schemes. In this paper, however, we will mainly consider the topos obtained by sheaves on the "small" site (fppf/X) comprising flat schemes that are locally of finite presentation.

To gain flexibility and facilitate applications, it seems preferable to work in an axiomatized situation, having nevertheless the fppf topos in mind. One of our main result is a sufficent criterion for adjoint functors between two abstract sites

$$u : \mathcal{C}_f \to \mathcal{C}_z \quad \text{and} \quad v : \mathcal{C}_z \to \mathcal{C}_f \tag{1}$$

to induce homeomorphisms between locales:

Theorem (see Thm. 2.3)**.** *Suppose the adjoint functors above satisfy the conditions (TL 1) – (TL 4) given in Section 2. Then the induced continuous maps of locales* $\epsilon : \mathrm{Loc}(\mathscr{E}_f) \to \mathrm{Loc}(\mathscr{E}_z)$ *is a homeomorphism. Moreover, we get an embedding* $|\mathscr{E}_f|_{\mathrm{sob}} \subset |\mathscr{E}_z|_{\mathrm{sob}}$ *of sober spaces. The latter is an equality if the map* $|\mathscr{E}_f| \to |\mathscr{E}_z|$ *admits a section.*

Note that these conditions are rather technical to formulate, but easy to verify in practice. The result applies for the fppf site $\mathcal{C}_f = (\mathrm{fppf}/X)$ comprising flat X-schemes that are locally of finite presentation, and the Zariski site $\mathcal{C}_z = (\mathrm{Zar}/X)$

of open subschemes, and their corresponding topoi $\mathscr{E}_f = X_{\mathrm{fppf}}$ and $\mathscr{E}_z = X_{\mathrm{Zar}}$. The morphisms in the above sites are the X-morphisms and inclusion maps, respectively. Since we are able to construct sections for the canonical map $|X_{\mathrm{fppf}}| \to |X_{\mathrm{Zar}}|$, we get:

Theorem (see Thm. 6.5)**.** *For every scheme X, the continuous map of topoi $X_{\mathrm{fppf}} \to X_{\mathrm{Zar}}$ induces an identification $|X_{\mathrm{fppf}}|_{\mathrm{sob}} = |X_{\mathrm{Zar}}|_{\mathrm{sob}} = X$ of sober spaces.*

Similar results hold for the Nisnevich topology, the étale topology, and the syntomic topology. Note that for the étale site, the much stronger result $|X_{\mathrm{et}}| = X$ is true by [35], Lemma 44.29.12. This, however, becomes false in the fppf topology, as we shall see below. Unfortunately, our method, as it stands, does not apply to the fpqc site, because fpqc morphisms are not necessarily open maps.

In order to construct explicit points $P : (\mathrm{Set}) \to (X_{\mathrm{fppf}})$ for the fppf topos, we introduce the notion of *fppf-local rings*, which are local rings R for which any fppf algebra A admits a retraction, in other words, the morphisms $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ has a section. Such rings should be regarded as generalizations of algebraically closed fields. However, they have highly unusual properties from the point of view of commuative algebra. For example, their formal completion $\hat{R} = \varprojlim_n R/\mathfrak{m}^n$ coincides with the residue field $\kappa = R/\mathfrak{m}$. Rings with similar properties were studied by Gabber and Romero [11], in the contex of "almost mathematics". Here is another amazing property:

Theorem (see Thm. 4.6)**.** *If R is fppf-local, then the local rings $R_{\mathfrak{p}}$ are strictly local with algebraically closed residue field, for all prime ideals $\mathfrak{p} \subset R$.*

Moreover, we relate fppf-local rings to the so-called *TIC rings* introduced by Enochs [8] and further studied by Hochster [19], and the *AIC rings* considered by Artin [4]. Furthermore, we show that the stalks $\mathscr{O}_{X_{\mathrm{fppf}},P}$ of the structure sheaf at topos-theoretical points are examples of fppf-local rings. Throughout, the term *strictly local* denotes local henselian rings with separably closed residue fields.

We then use ideas of Picavet [29] to construct, for each strictly local ring R and each limit ordinal λ, some faithfully flat, integral ring extension R_λ that is fppf-local. Roughly speaking, the idea is to form the tensor product over "all" finite fppf algebras, and to iterate this via tranfinite recursion, until reching the limit ordinal λ. Note that there is a close analogy to the Steinitz's original construction of algebraic closures for fields ([36], Chapter III).

This is next used to produce, for each geometric point $\bar{a} : \mathrm{Spec}(\Omega) \to X$ on a scheme X and each limit ordinal λ, a topos-theoretical point

$$P = P_{\bar{a},\lambda} : (\mathrm{Set}) \to X_{\mathrm{fppf}}$$

with $\mathscr{O}_{X_{\mathrm{fppf}},P} = (\mathscr{O}_{X,\bar{a}})_\lambda$. Here the main step is to construct a suitable *pro-object* $(U_i)_{i\in I}$ of flat X-schemes locally of finite presentation yielding the stalk functor. The index category I will consists of certain 5-tuples of X-schemes and morphisms between them. This gives the desired section:

Theorem (see Thm. 6.4)**.** *For each limit ordinal λ, the map $a \mapsto P_{\bar{a},\lambda}$ induces a continuous section for the canonical map $|X_{\mathrm{fppf}}| \to X$.*

The paper is organized as follows: In Section 1, we recall some basic definitions and result from topos-theory, also paying special attention to set-theoretical issuses. Following Grothendieck, we avoid the use of the ambiguous notion of "classes", and use universes instead. Section 2 contains the sufficient criterion that two sites have homeomorphic locales and sobrified spaces. This is applied, in Section 3, to the fppf topos and the Zariski topos of a scheme. Here we also discuss relations to the "big" fppf topos, which usually occurs in the literature. In Section 4 we introduce the notion of fppf-local rings and establish their fundamental properties. A construction of fppf-local rings depending on a given strictly local ring and a limit ordinal is described in Section 5. This is used, in the final Section 6, to construct topos-theoretical points for the fppf topos attached to a scheme.

Remark. After this paper was submitted to the arXiv, I was kindly informed by Shane Kelly that related results appear in [10], now a joint paper with Ofer Gabber.

Acknowledgement. I wish to thank the referee for noting some mistakes and giving several suggestions, in particular for pointing out that one has to distinguish between the "big" fppf topos, which is usually used in the literature, and the "small" fppf topos considered here.

1. Recollection: Universes, sites, topoi and locales

In this section, we recall some relevant foundational material about topos-theory from Grothendieck et. al. [2]. Further very useful sources are Artin [1], Johnstone [20],[22],[23], Kashiwara and Shapira [25], and the Stacks Project [35].

Recall that a *universe* is a nonempty set $\mathscr{U}$ of sets satisfying four very natural axioms, which we choose to state in the following form:

(U 1) If $X \in \mathscr{U}$ then $X \subset \mathscr{U}$.
(U 2) If $X \in \mathscr{U}$ then $\{X\} \in \mathscr{U}$.
(U 3) If $X \in \mathscr{U}$ then $\wp(X) \in \mathscr{U}$.
(U 4) If $I \in \mathscr{U}$ and $X_i \in \mathscr{U}$, $i \in I$ then $\bigcup_{i\in I} X_i \in \mathscr{U}$.

In other words, $\mathscr{U}$ is a nonempty transitive set of sets that is stable under forming singletons, power sets, and unions indexed by $I \in \mathscr{U}$. Roughly speaking, this ensures that universes are stable under the set-theoretical operation usually performed in practice.

If $X \in \mathscr{U}$ is an arbitrary element, which a fortiori exists because $\mathscr{U}$ is nonempty, then the power set $\wp(X)$ and hence $\emptyset \in \wp(X)$ and the singleton $S = \{\emptyset\}$ are elements. In turn, the set $I = \wp(S)$ of cardinality two is an element. It follows by induction that all finite ordinals $0 = \emptyset$ and $n + 1 = n \cup \{n\}$ are elements as well. Moreover, (U 2) and (U 4) ensure that for each $X, Y \in \mathscr{U}$, the set $\{X, Y\}$ is an element. The latter statement is the form of the axiom (U 2) given in [2], Expose I, Section 0.

Note that for a pair we have $(X, Y) \in \mathscr{U}$ if and only if $X, Y \in \mathscr{U}$, in light of Kuratowski's definition of pairs $(X, Y) = \{\{X, Y\}, \{Y\}\}$. In particular, it follows that groups or topological spaces are elements of $\mathscr{U}$ if and only if the underlying sets are elements of $\mathscr{U}$. Note also that we adopt von Neumann's definition of *ordinals* ν as sets of sets that are transitive (in the sense $\alpha \in \nu \Rightarrow \alpha \subset \nu$), so that the resulting order relation on ν is a well-ordering (where $\alpha \leq \beta$ means $\alpha = \beta$ or $\alpha \in \beta$). In turn, each well-ordered set is order isomorphic to a unique ordinal.

Following Grothendieck, we assume that any set X is an element of some universe $\mathscr{U}$, which is an additional axiom of set theory. Note that the intersection of universes is a universe, so there is always a unique smallest such universe. The two axioms (U 3) and (U 4) enforce that the cardinality $\aleph_\iota = \mathrm{Card}(\mathscr{U})$ of a universe is *strongly inaccessible*. In fact, the assumption that any set is contained in some universe is equivalent to the assumption that any cardinal is majorized by some strongly inaccessible cardinal. A related notion was already mentioned by Felix Hausdorff under the designation "reguläre Anfangszahlen mit Limesindex", and I cannot resist from quoting the original [18], page 131: "Wenn es also reguläre Anfangszahlen mit Limesindex gibt (und es ist uns bisher nicht gelungen, in dieser Annahme einen Widerspruch zu entdecken), so ist die kleinste unter ihnen von einer so exorbitanten Größe, daß sie für die üblichen Zwecke der Mengenlehre kaum jemals in Betracht kommen wird."

Given a universe $\mathscr{U}$, a set X is called a *$\mathscr{U}$-element* if $X \in \mathscr{U}$. We write $(\mathrm{Set})_{\mathscr{U}}$ for the category of all sets that are $\mathscr{U}$-elements, and likewise denote by $(\mathrm{Grp})_{\mathscr{U}}, (\mathrm{Sch})_{\mathscr{U}}, (\mathrm{Cat})_{\mathscr{U}}$ the categories of all groups, schemes, categories that are $\mathscr{U}$-elements. The same notation is used for any other mathematical structure. By common abuse of notation, we sometimes drop the index, if there is no risk of confuction. Note that a category $\mathcal{C}$ is a $\mathscr{U}$-element if and only if its object set and all its hom sets have this property. Given such a category, we denote by $\mathrm{PSh}(\mathcal{C})$ the category of presheaves, that is, contravariant functors $\mathcal{C} \to (\mathrm{Set})_{\mathscr{U}}$. Given $X \in \mathcal{C}$, one writes the corresponding Yoneda functor as $h_X : \mathcal{C} \to (\mathrm{Set})_{\mathscr{U}}$, $Y \mapsto \mathrm{Hom}_{\mathcal{C}}(Y, X)$.

A set X is called *$\mathscr{U}$-small* if it is isomorphic to some element of $\mathscr{U}$. The same locution is used for any mathematical structure, for example groups, rings, topological spaces, schemes and categories. If there is no risk of confusion, we simple use the term *small* rather than $\mathscr{U}$-small. Note that a category $\mathcal{C}$ is a $\mathscr{U}$-element or $\mathscr{U}$-small if and only if its object set and all its hom sets have the respective property. This has to be carfully distinguished from the following notion: A category $\mathcal{C}$ is called a *$\mathscr{U}$-category* if the sets $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ are $\mathscr{U}$-small for all objects $X, Y \in \mathcal{C}$. Clearly, this property is preserved by equivalences of categories. It is also possible to define a Yoneda functor for $\mathscr{U}$-categories, and not only for categories that are $\mathscr{U}$-elements, by choosing bijections between $\mathrm{Hom}_{\mathcal{C}}(Y, X)$ and elements of $\mathscr{U}$.

Usually, the category $\mathrm{PSh}(\mathcal{C})$ is not $\mathscr{U}$-small, even for $\mathcal{C} \in \mathscr{U}$: Take, for example, the category of presheaves on a singleton space, which is equivalent to the category $(\mathrm{Set})_{\mathscr{U}}$. Suppose its object set $\mathscr{U}$ has the same cardinality κ as an element $X \in \mathscr{U}$. Since the power set $\wp(X) \in \mathscr{U}$ has strictly larger cardinality, and $\wp(X) \subset \mathscr{U}$, we obtain $2^\kappa \leq \kappa$, in contradiction to cardinal arithmetic.

Let $\mathcal{C}$ be a category. A *Grothendieck topology* on $\mathcal{C}$ is a collection $J(X)$ of sieves for each object $X \in \mathcal{C}$, satisfying certain axioms. We do not bother to reproduce the axioms, and refer for details to [2], Expose II, Section 1. Recall that a *sieve* on X is a full subcategory $\mathcal{S} \subset \mathcal{C}/X$ with the property that $A \in \mathcal{S}$ for each morphism $A \to B$ in $\mathcal{C}/X$ with $B \in \mathcal{S}$. Usually, the covering sieves of a Grothendieck topology are specified with the help of *pretoplogies*, which is a collection $\mathrm{Cov}(X)$ of tuples $(X_\alpha \to X)_{\alpha \in I}$ of morphisms $X_\alpha \to X$ for each object $X \in \mathcal{C}$ satisfying similar axioms. These tuples are referred to as *coverings* of X, and the induced Grothendieck topology is the finest one for which the coverings families generate coverings sieves.

A category $\mathcal{C}$ endowed with a Grothendieck topology is called a *site*. To proceed, choose a universe with $\mathcal{C} \in \mathscr{U}$. Then we have the full subcategory $\mathrm{Sh}(\mathcal{C}) \subset \mathrm{PSh}(\mathcal{C})$ of sheaves, that is, contravariant functors $\mathcal{C} \to (\mathrm{Set})_{\mathscr{U}}$ satisfying the sheaf axioms with respect to the covering sieves or covering families. A *$\mathscr{U}$-site* is a $\mathscr{U}$-category $\mathcal{C}$ endowed with a Grothendieck topology, so that there is a full $\mathscr{U}$-small subcategory $\mathcal{D} \subset \mathcal{C}$ so that for each object $X \in \mathcal{C}$, there is a covering family $(X_\alpha \to X)_\alpha$ with $X_\alpha \in \mathcal{D}$. This condition ensures that the category $\mathrm{Sh}(\mathcal{C})$ of $\mathscr{U}$-sheaves remains a $\mathscr{U}$-category.

A *$\mathscr{U}$-topos* is a $\mathscr{U}$-category $\mathscr{E}$ that is equivalent to the category $\mathrm{Sh}(\mathcal{C})$ for some site $\mathcal{C} \in \mathscr{U}$. The central result on topoi is *Giraud's Characterization* ([2], Expose IV, Theorem 1.2). Roughly speaking, it makes the following three assertions: First, it says that one may choose the site $\mathcal{C}$ so that it contains all inverse limits, and that its Grothendieck topology is *subcanonical*, which means that all Yoneda functors h_X, $X \in \mathcal{C}$ satisfy the sheaf axioms. Second, it singles out the topoi among the $\mathscr{U}$-categories in terms of purely categorical properties of $\mathscr{E}$, referring to objects and arrows rather than to coverings. Third, it characterizes the topoi among the $\mathscr{U}$-sites using the *canonical topology* on $\mathscr{E}$, which is the finest topology on $\mathscr{E}$ that turns all Yoneda functors h_F, $F \in \mathscr{E}$ into sheaves.

A *continuous map* $\epsilon : \mathscr{E} \to \mathscr{E}'$ between $\mathscr{U}$-topoi is a triple $\epsilon = (\epsilon_*, \epsilon^{-1}, \varphi)$, where $\epsilon_* : \mathscr{E} \to \mathscr{E}'$ and $\epsilon^{-1} : \mathscr{E}' \to \mathscr{E}$ are adjoint functors, and φ is the adjunction isomorphism. Here ϵ^{-1} is left adjoint and called the *preimage functor*, and ϵ_* is right adjoint and called the *direct image functor*. Moreover, one demands that the preimage functor ϵ^{-1} is left exact, that is, commutes with finite inverse limits. Up to isomorphism, the continuous map ϵ is determined by either of the preimage functor ϵ^{-1} and the direct image functor ϵ_*. The set of all continuous maps $\mathrm{Hom}(\mathscr{E}, \mathscr{E}')$ is itself a category, the morphism being the compatible natural transformations between the direct image and preimage functors.

The $\mathscr{U}$-valued sheaves on a topological space $X \in \mathscr{U}$ form a a topos $\mathscr{E} = \mathrm{Sh}(X)$. In particular, the category $(\mathrm{Set})_{\mathscr{U}}$ can be identified with the category $\mathrm{Sh}(S)$ for the singleton space $S = \{\star\}$. In light of this, a *point in the sense of topos-theory* of a $\mathscr{U}$-topos $\mathscr{E}$ is a continuous map of topoi $P : (\mathrm{Set})_{\mathscr{U}} \to \mathscr{E}$. We denote by $\mathrm{Points}(\mathscr{E})$ the category of points, and by $|\mathscr{E}|$ the set of isomorphism classes $[P]$ of points. This set is endowed with a natural topology: Choose a terminal object $e \in \mathscr{E}$. Given a subobject $U \subset e$, we formally write $U \cap |\mathscr{E}| \subset |\mathscr{E}|$ for the set of isomorphism classes of points P with $P^{-1}(U) \neq \emptyset$, and declare it as open. This indeed constitutes a topology on the set $|\mathscr{E}|$.

Recall that a nonempty ordered set L is called a *locale* if the following axioms hold:

- (LC 1) For all pairs $U, V \in L$, the infimum $U \wedge V \in L$ exists, that is, the largest element that is smaller or equal than both U, V.
- (LC 2) For each family $U_i \in L$, $i \in I$, the supremum $\bigvee_{\alpha \in I} U_i \in L$ exists, that is, the least element that is larger or equal than all U_i.
- (LC 3) The distributive law holds, which means $U \wedge (\bigvee_{i \in I} V_i) = \bigvee_{i \in I}(U \wedge V_i)$.

The ordered set $L = \mathscr{T}$ of open subsets $U \subset X$ of a topological space $(X, \mathscr{T})$ is the paramount example for locales. One should regard locales as abstractions of topological spaces, where one drops the underlying set and merely keeps the topology. In light of this, one defines a *continuous map* $f : L \to L'$ between locales as a monotonous map $f^{-1} : L' \to L$ that respects finite infima and arbitrary suprema. Note the reversal of arrows. Here the notation f^{-1} is purely formal, and does not indicate that f is bijective.

Each locale comes with a Grothendieck topology, and thus can be regarded as site: The covering families $(U_\alpha \to V)_{\alpha\in I}$ are those with $V = \bigvee_\alpha U_\alpha$. In turn, we have the $\mathscr{U}$-topos $\mathrm{Sh}(L)$ of sheaves on the locale $L \in \mathscr{U}$. Conversely, for each $\mathscr{U}$-topos $\mathscr{E}$ we have a locale $\mathrm{Loc}(\mathscr{E})$, which is the ordered set of subojects $U \subset e$ of a fixed terminal object $e \in \mathscr{E}$. Up to canonical isomorphism, it does not depend on the choice of the terminal object.

2. Topoi with same locales

In this section, we establish some facts on continuous maps between certain topoi, which occur in algebraic geometry when various Grothendieck topologies are involved. In order to achieve flexibility and facilitate application, we work in the following axiomatic set-up:

Throughout, fix a universe $\mathscr{U}$. Let $\mathcal{C}_f$ and $\mathcal{C}_z$ be two categories that are $\mathscr{U}$-elements, in which terminal objects exist. Furthermore, suppose these categories are equipped with a pretopology of coverings, such that we regard them as sites. We now suppose that we have adjoint functors

$$u : \mathcal{C}_f \longrightarrow \mathcal{C}_z \quad \text{and} \quad v : \mathcal{C}_z \longrightarrow \mathcal{C}_f,$$

where u is the left adjoint and v is the right adjoint. Let us write the objects of $\mathcal{C}_f$ *formally* as pairs (U, p), and the objects of $\mathcal{C}_z$ by ordinary letters V, and the functors as

$$u(U, p) = p(U) \quad \text{and} \quad v(V) = (V, i).$$

Note that the adjunction, which by abuse of notation is regarded as an identification, takes the form

$$\mathrm{Hom}_{\mathcal{C}_z}(p(U), V) = \mathrm{Hom}_{\mathcal{C}_f}((U, p), (V, i)). \tag{2}$$

Let me emphasize that all *this notation is purely formal*, but based on geometric intuition. The guiding example, which one should have in mind, is that $\mathcal{C}_z$ comprises open subsets of a scheme X, and that $\mathcal{C}_f$ consists of certain flat X-schemes (U, p), where $p : U \to X$ is the structure morphism that is assumed to be universally open. The functor u takes such an X-scheme to its image $p(U) \subset X$, whereas the functor v turns the open subset $V \subset X$ into an X-scheme (V, i), where $i : V \to X$ is the inclusion morphism. Note that, by abuse of notation, we usually write $i : V \to X$ and not the more precise $i_V : V \to X$. Of course, the indices in $\mathcal{C}_f$ and $\mathcal{C}_z$ refer to "flat" and "Zariski", respectively.

We now demand the following four conditions (TL 1) – (TL 4), which conform with geometric intuition:

- (TL 1) The composite functor $u \circ v$ is isomorphic to the identity on $\mathcal{C}_z$.
- (TL 2) For each covering family $(V_\lambda \to V)_\lambda$ in the site $\mathcal{C}_z$, the induced family $((V_\lambda, i_\lambda) \to (V, i))_\lambda$ in the site $\mathcal{C}_f$ is covering.
- (TL 3) For each family $(U_\lambda, p_\lambda)_\lambda$ of objects in $\mathcal{C}_f$, there is a subobject V of the terminal object in $\mathcal{C}_z$ and factorizations $p_\lambda(U_\lambda) \to V$ so that the family $(p_\lambda(U_\lambda) \to V)_\lambda$ is a covering.
- (TL 4) For each object (U, p) in $\mathcal{C}_f$, the representable presheaf $h_{(U,p)}$ is a sheaf.

Let me make the following remarks: The functor $v : \mathcal{C}_z \to \mathcal{C}_f$, being a right adjoint, commutes with inverse limits. Moreover, condition (TL 1) ensures that v is faithful, which allows us to make the identification $i(V) = V$. By our overall assumption on the categories $\mathcal{C}_f$ and $\mathcal{C}_z$, the terminal object appearing in condition (TL 3) does exists. Finally, condition (TL 4) can be rephrased as that the Grothendieck topology on $\mathcal{C}_f$ is finer than the *canonical topology*, which is the finest topology for which every representable presheaf satisfies the sheaf axioms. One also says that the Grothendieck topology on $\mathcal{C}_f$ is *subcanonical*.

Furthermore, I want to point out that we do not assume that our categories $\mathcal{C}_f, \mathcal{C}_z$ all finite inverse limits are representable. However, it is part of the definition for *pretopologies* that for any covering family $(U_\lambda \to U)_\lambda$, the members $U_\lambda \to U$ are *base-changeable* ("quarrable" in [2], Expose II, Definition 1.3), that is, for every other morphism $U' \to U$, the fiber product $U_\lambda \times_U U'$ does exist.

Proposition 2.1. *The functor* $u : \mathcal{C}_f \to \mathcal{C}_z$ *is cocontinuous, and the adjoint functor* $v : \mathcal{C}_z \to \mathcal{C}_f$ *is continuous.*

Proof. For the precise definition of *continuous* and *cocontinuous functors* between sites, we refer to [2], Expose III. The two assertions are equivalent, according to loc. cit. Proposition 2.5, because the functors u and v are adjoint. To check that u is cocontinuous, let $(U, p) \in \mathcal{C}_f$, and $(V_\lambda \to p(U))_\lambda$ be a covering family in $\mathcal{C}_z$. By condition (TL 2), the induced family $((V_\lambda, i_\lambda) \to (p(U), i))_\lambda$ is a covering family in $\mathcal{C}_f$. Form the pull-back

$$\begin{array}{ccc} (U_\lambda, p_\lambda) & \longrightarrow & (V_\lambda, i_\lambda) \\ \downarrow & & \downarrow \\ (U, p) & \longrightarrow & (p(U), i) \end{array}$$

in $\mathcal{C}_f$, which exists because members of covering families are base-changeable. By the axioms for covering families, $((U_\lambda, p_\lambda) \to (U, p))_\lambda$ remains a covering family. The preceding diagram, together with the adjunction, shows that the induced maps $p_\lambda(U_\lambda) \to p(U)$ factor over $V_\lambda \to p(U)$. If follows that u is cocontinuous ([2], Expose III, Definition 2.1). □

Now let $\mathscr{E}_f = \mathrm{Sh}(\mathcal{C}_f)$ and $\mathscr{E}_z = \mathrm{Sh}(\mathcal{C}_z)$ be the $\mathscr{U}$-topoi of sheaves on $\mathcal{C}_f$ and $\mathcal{C}_z$, respectively. We refer to the sheaves on $\mathcal{C}_f$ as *f-sheaves*, and to the sheaves on $\mathcal{C}_z$ as *z-sheaves*. According to [2], Expose IV, Section 4.7, the cocontinuous functor $u : \mathcal{C}_f \to \mathcal{C}_z$ induces a continuous map of topoi

$$\epsilon = (\epsilon_*, \epsilon^{-1}, \varphi) : \mathscr{E}_f \longrightarrow \mathscr{E}_z.$$

Let me make this explicit: The direct image functor ϵ_* sends an f-sheaf F to the z-sheaf $\epsilon_*(F)$ defined by

$$\Gamma(V, \epsilon_*(F)) = \Gamma((V, i), F). \tag{3}$$

Note that, in general, this would be an inverse limit of local sections, indexed by the category of pairs $((U, p), \psi)$, where $\psi : p(U) \to V$ is a morphism in $\mathcal{C}_z$. In our situation, such an inverse limit is not necessary, because the index category has a terminal object $((V, i), \psi)$, where $\psi : i(V) \to V$ is the canonical isomorphism coming from condition (TL 1).

The inverse image functor ϵ^{-1} sends a z-sheaf G to the f-sheaf $\epsilon^{-1}(G)$, defined by

$$\Gamma((U, p), \epsilon^{-1}(G)) = \Gamma(p(U), G). \tag{4}$$

Note that, in general, this would give merely a presheaf, and sheafification is necessary. In our situation, however, the presheaf is already a sheaf, thanks to condition (TL 2).

The adjunction map φ between ϵ^{-1} and ϵ_* is determined by natural transformations

$$G \longrightarrow \epsilon_*\epsilon^{-1}(G) \quad \text{and} \quad \epsilon^{-1}\epsilon_*(F) \longrightarrow F.$$

Here the former comes from identity maps

$$\Gamma(V, G) \xrightarrow{\text{id}} \Gamma(i(V), G) = \Gamma((V, i), \epsilon^{-1}(G)) = \Gamma(V, \epsilon_*\epsilon^{-1}(G)).$$

The latter is the given by restriction maps

$$\Gamma((U, p), \epsilon^{-1}\epsilon_*(F)) = \Gamma(p(U), \epsilon_*(F)) = \Gamma((p(U), i), F) \xrightarrow{\text{res}} \Gamma((U, p), F).$$

For later use, we now establish a technical fact:

Lemma 2.2. *For each object V in $\mathcal{C}_z$, the presheaf h_V is a sheaf. Furthermore, we have $\epsilon^{-1}(h_V) = h_{(V,i)}$.*

Proof. According to condition (TL 4), the presheaf $h_{(V,i)}$ on $\mathcal{C}_z$ is a sheaf. If the follows from condition (TL 2) and the fact that the functor $v : \mathcal{C}_z \to \mathcal{C}_f$ is faithful that the presheaf h_V on $\mathcal{C}_z$ satisfies the sheaf axioms. Finally, for each object (U, p) of $\mathcal{C}_f$, we have

$$\Gamma((U, p), \epsilon^{-1}(h_V)) = \Gamma(p(U), h_V) = \operatorname{Hom}_{\mathcal{C}_z}(p(U), V),$$

where the first equation comes form (4), and the second equation stems from the Yoneda Lemma. Similarly, we have

$$\Gamma((U, p), h_{(V,i)}) = \operatorname{Hom}_{\mathcal{C}_f}((U, p), (V, i)).$$

Using the adjointness of the functors (2), we infer that $\epsilon^{-1}(h_V) = h_{(V,i)}$. □

The continuous map of topoi $\epsilon : \mathscr{E}_f \to \mathscr{E}_z$ induces a functor $\epsilon : \text{Points}(\mathscr{E}_f) \to \text{Points}(\mathscr{E}_z)$ on the category of points. In turn, we get a continuous map of topological spaces $\epsilon : |\mathscr{E}_f| \longrightarrow |\mathscr{E}_z|$. To understand this map, we first look at the induced continuous map of locales

$$\epsilon : \text{Loc}(\mathscr{E}_f) \longrightarrow \text{Loc}(\mathscr{E}_z). \tag{5}$$

Recall that these locales comprise the ordered sets of subobjects of the chosen terminal objects. Let us write $X \in \mathcal{C}_z$ for the terminal object. It follows that $e_z = h_X$ is a terminal object in the topos $\mathscr{E}_z$, and $\text{Loc}(\mathscr{E}_z)$ is the ordered set of subobjects $G \subset e_z$. Being right adjoint, the functor $v : \mathcal{C}_z \to \mathcal{C}_f$ respects inverse limits, whence $(X, i) \in \mathcal{C}_f$ is a terminal object. In turn, $e_f = h_{(X,i)}$ is the terminal object in the topos $\mathscr{E}_f$, and $\text{Loc}(\mathscr{E}_f)$ is the ordered set of subobjects $F \subset e_f$. The continuous map of locales (5) is just the monotonous map

$$\text{Loc}(\mathscr{E}_z) \longrightarrow \text{Loc}(\mathscr{E}_f), \quad G \longmapsto \epsilon^{-1}(G), \tag{6}$$

which, by definition of the hom-sets in the category of locales, goes in the reverse direction.

Theorem 2.3. *The continuous map of locales $\epsilon : \text{Loc}(\mathscr{E}_f) \to \text{Loc}(\mathscr{E}_z)$ is a homeomorphism, that is, the monotonous map (6) is bijective.*

Proof. The argument is analogous to [3], Expose VIII, Proposition 6.1. To see that the monotonous map is injective, suppose we have two subobjects $G, G' \subset e_z$ with $\epsilon^{-1}(G) = \epsilon^{-1}(G')$ as subobjects in $e_f = \epsilon^{-1}(e_z)$. For each $V \in \mathcal{C}_z$, we then have

$$\Gamma(V, G) = \Gamma((V, i), \epsilon^{-1}(G)) = \Gamma((V, i), \epsilon^{-1}(G')) = \Gamma(V, G'),$$

where the outer identifications come from (4). Whence $G = G'$.

For surjectivity, let $F \subset e_f$ be a subobject in $\mathcal{C}_f$. First note that, for each object (U, p) in $\mathcal{C}_f$ over which F has a local section, the set $\Gamma((U, p), F)$ must be a singleton, because F is a subobject of the terminal object. Moreover, if $(U', p') \to (U, p)$ is a morphism, then $\Gamma((U', p'), F)$ stays a singleton.

Now consider the family (U_λ, p_λ) of all objects in $\mathcal{C}_f$ over which F has a local section. Using condition (TL 3), there is a subobject $V \subset X$ of the terminal object and morphisms $p_\lambda(U_\lambda) \to V$ so that the induced family $((U_\lambda, p_\lambda) \to (V, i))_\lambda$ is covering. Consider the fiber products

$$\begin{array}{ccc} (U_{\lambda\mu}, p_{\lambda\mu}) & \longrightarrow & (U_\mu, p_\mu) \\ \downarrow & & \downarrow \\ (U_\lambda, p_\lambda) & \longrightarrow & (V, i), \end{array}$$

which exists because members of covering families are base-changeable. The sheaf axioms give a short exact sequence

$$\Gamma((V, i), F) \longrightarrow \prod_\lambda \Gamma((U_\lambda, p_\lambda), F) \longrightarrow \prod_{\lambda,\mu} \Gamma((U_{\lambda\mu}, p_{\lambda\mu}), F),$$

where the terms in the middle and the right are singletons. If follows that the term on the left is a singleton. The Yoneda Lemma yields a morphism of presheaves $h_{(V,i)} \to F$. Note that the presheaf $h_{(V,i)}$ is actually a sheaf, according to condition (TL 4). Moreover, this morphism is actually an isomorphism: Let (U, p) be an arbitrary object of $\mathcal{C}_f$. Suppose there is a morphism $(U, p) \to (V, i)$. Since V is a subobject of the terminal object, so is (V, i), by the adjunction (2). It follows that the term on the left in

$$\Gamma((U, p), h_{(V,i)}) \longrightarrow \Gamma((U, p), F) \tag{7}$$

is a singleton, whence the map is bijective. Finally, suppose there is no morphism $(U, p) \to (V, i)$, such that there is also no morphism $p(U) \to V$. By the very definition of V, this means $\Gamma((U, p), F) = \emptyset$. Again, the map (7) is bijective. We conclude that $h_{(V,i)} \to F$ is an isomorphism. Using Lemma 2.2, we infer that $F = \epsilon^{-1}(h_V)$. Since V is a subobject of the terminal object in $\mathcal{C}_z$, the sheaf h_V must be a subobject of the terminal object $e_z \in \mathscr{E}_z$, which concludes the proof. □

We finally come to the induced continuous map $\epsilon : |\mathscr{E}_f| \to |\mathscr{E}_z|$ of topological spaces. Recall that the *chaotic topology* on a set has as sole open subsets the whole set and the empty set. A topological space X is called *sober* if each irreducible closed subset has a unique generic point. Each space X comes with the *sobrification* $X \to X_{\text{sob}}$, which is universal with respect to continuous maps into sober spaces, compare [13], Chapter 0, Section 2.9.

Corollary 2.4. *Each fiber of the map $\epsilon : |\mathscr{E}_f| \to |\mathscr{E}_z|$ carries the chaotic topology, and the induced map of sober spaces is an embedding $|\mathscr{E}_f|_{\text{sob}} \subset |\mathscr{E}_z|_{\text{sob}}$. The latter is an equality provided that the map $\epsilon : |\mathscr{E}_f| \to |\mathscr{E}_z|$ admits a section.*

Proof. According to the theorem, each open subset in $|\mathscr{E}_f|$ is the preimage of an open subset in $|\mathscr{E}_z|$. The statement is thus a special case of the following lemma. □

Lemma 2.5. *Let $f : X \to Y$ be a continuous map between topological spaces. Suppose that each open subset in X is the preimages of an open subset in Y. Then all fibers of f carry the chaotic topology, and the induced map of sober space is an embedding $X_{\mathrm{sob}} \subset Y_{\mathrm{sob}}$. The latter is an equality provided that f admits a section.*

Proof. Given $y \in Y$, and let $U = f^{-1}(V)$ be an open subset in X. Then $U \cap f^{-1}(y)$ is either empty or the hole fiber. In turn, the fiber $f^{-1}(y)$ carries the chaotic topology. Likewise, one sees that $f : X \to Y$ is a closed map. Recall that one may view the sobrification Y_{sob} as the space of closed irreducible subsets in Y. Let $Z \subset Y$ be such a subset. Then $f(Z) \subset X$ is a closed irreducible subset as well. Given $x \in f^{-1}f(Z)$ and any open neighborhood $x \in U = f^{-1}(V)$, we see that $f(f^{-1}(V) \cap Z) = V \cap f(Z)$ is nonempty. Whence x is in the closure of Z, which means $x \in Z$. We conclude that $Z \subset X$ equals the preimage of the closed irreducible subset $f(Z)$. In turn, we may regard the induced map on sobrification as an inclusion $X_{\mathrm{sob}} \subset Y_{\mathrm{sob}}$. The space X_{sob} carries the subspace topology, again by our assumption on the open sets.

One easily sees that any set-theoretical section $s : Y \to X$ for f must be continuous. It thus induces a right inverse for the inclusion $X_{\mathrm{sob}} \subset Y_{\mathrm{sob}}$, which then must must be an equality. □

3. Applications to algebraic geometry

We now apply the abstract results of the preceding section to some concrete Grothendieck topologies in algebraic geometry. Let X be a scheme, and fix a universe with $X \in \mathscr{U}$. In what follows, all schemes are tacitly assumed to be $\mathscr{U}$-elements. We denote by (Zar/X) the locale given by the ordered set of open subschemes $V \subset X$, regarded as a site in the usual way, and write X_{Zar} for the ensuing $\mathscr{U}$-topos of sheaves on X.

Let us denote by (fppf/X) the category of X-schemes (U, p), where the structure morphism $p : U \to X$ is flat and locally of finite presentation, following the convention of [35]. The hom sets in this category are formed by arbitrary X-morphisms. Note that any such morphism is automatically locally of finite presentation, by [13], Proposition 6.2.6, but not necessarily flat. If $(U, p), (V, q)$ are two objects and $U \to V$ is a flat X-morphism, than for any other object (V', q') and any X-morphism $V' \to V$, the usual fiber product of schemes $U \times_V V'$ yields an object and whence a fiber product in (fppf/X). It is not clear to me to what extend other fiber products in (fppf/X) exists, which may differ from the ususal fiber products in (Sch/X).

Our category is equipped with the pretopology of *fppf coverings* $((U_\alpha, p_\alpha) \to (U, p))_{\alpha \in I}$, where each $U_\alpha \to U$ is flat, and the induced map $\coprod_\alpha U_\alpha \to U$ is surjective. We regard (fppf/X) as a site, and denote by X_{fppf} the resulting $\mathscr{U}$-topos of sheaves. Note that the category (fppf/X) usually contains hom sets of cardinality ≥ 2, in contrast to the Zariski site.

Clearly, the categories (Zar/X) and (fppf/X) have terminal objects. Consider the functors

$$u : (\mathrm{fppf}/X) \longrightarrow (\mathrm{Zar}/X), \quad (U, p) \longmapsto p(U)$$

and

$$v : (\mathrm{Zar}/X) \longrightarrow (\mathrm{fppf}/X), \quad V \longmapsto (V, i)$$

where $i : V \to X$ denotes the inclusion morphism of an open subscheme. These are well-defined, because any flat morphism locally of finite presentation is universally open ([14], Theorem 2.4.6), and any open embedding is a fortiori flat and locally of finite presentation. Note that, for schemes that fail to be quasiseparated, there are open subschemes whose inclusion morphism is not quasicompact, and in particular not of finite presentation. Nevertheless, they are locally of finite presentation.

Proposition 3.1. *The functor u is left adjoint to v, and this pair of adjoint functors satisfy the conditions (TL 1) – (TL 4) of Section 2.*

Proof. The adjointness follows from the universal property of schematic images of open morphisms. The first two conditions (TL 1) and (TL 2) are trivial. To see (TL 3), let $p_\lambda : U_\lambda \to X$ be flat and locally of finite presentation. The the image is open, and the open subscheme $V = \bigcup_\lambda p_\lambda(U_\lambda)$ of the terminal object $X \in (\mathrm{Zar}/X)$ is covered by the $p_\lambda(U_\lambda)$. Finally, (TL 4) holds by [17], Expose VIII, Theorem 5.2. $\square$

In turn, the functor $u : (\mathrm{fppf}/X) \to (\mathrm{Zar}/X)$ is cocontinuous and induces a morphism of topoi $\epsilon : X_{\mathrm{fppf}} \to X_{\mathrm{Zar}}$. Applying Theorem 2.3 and its Corollary we get:

Proposition 3.2. *The induced continuous map $\epsilon : \mathrm{Loc}(X_{\mathrm{fppf}}) \to \mathrm{Loc}(X_{\mathrm{Zar}})$ of locales is a homeomorphism, and the induced map of sober spaces is an embedding $|X_{\mathrm{fppf}}|_{\mathrm{sob}} \subset |X_{\mathrm{Zar}}|_{\mathrm{sob}} = X$.*

In Section 6, we shall construct a section for the map $|X_{\mathrm{fppf}}| \to |X_{\mathrm{Zar}}|$, whence we actually have an equality $|X_{\mathrm{fppf}}|_{\mathrm{sob}} = |X_{\mathrm{Zar}}|_{\mathrm{sob}} = X$.

This approach carries over to analogous topoi defined with the étale topology, the Nisnevich topology [28], and the syntomic topology [9]. With the obvious notation, we thus get canonical identifications of locales and sober topological spaces. We leave the details to the reader. Note, however, that Theorem 2.3 does not apply to the fpqc topology. This is because flat and quasicompact morphisms are not necessarily open (for example the one induced by the faithfully flat ring extension $\mathbb{Z} \subset \mathbb{Z}\times\mathbb{Q}$, compare [14], Remark 2.4.8). Thus we apparently have no functor v from the fpqc site to the Zariski site. However, one may apply it to the site (fpuo/X) of *flat and universally open morphisms*, which was considered by Romagny [30]. It would be interesting to understand the relation between the fpqc topos and the fpuo topos.

One disadvantage for the site (fppf/X) and the enusing topos X_{fppf} is that it is apparently not functorial with respect to X, by similiar reasons as for the lisse-étale topos. Therfore, in the literature one usually considers the category (Sch/X) of all X-schemes contained in a fixed universe, and endows it with the fppf topology. The covering families are the $(U_\alpha \to Y)_\alpha$, where $U_\alpha \to Y$ are flat X-morphisms that are locally of finite presentation, and $\amalg U_\alpha \to Y$ is surjective. Let me refer to sheaves on this site (Sch/X) as *big fppf sheaves*, whereas we now call sheaves on the site (fppf/X) *small fppf sheaves.* Likewise, we call the resulting topos $X_{\mathrm{fppf}}^{\mathrm{big}}$ the *big fppf topos*, whereas we refer to X_{fppf} as the *small fppf topos.*

Given a big fppf sheaf F over X, we obtain by forgetting superfluous local sections and restriction maps a small fppf sheaf $F|Y$ for each X-scheme Y. The resulting functor

$$X_{\text{fppf}}^{\text{big}} \longrightarrow Y_{\text{fppf}}, \quad F \longmapsto F|Y \tag{8}$$

commutes with direct and inverse limits, because the sheafification functor $F \mapsto F^{++}$ involves over a fixed Y only fppf coverings and their fiber products, which are the same in (Sch/X) and (fppf/Y). In turn, we obtain a functor

$$\mathrm{Points}(Y_{\text{fppf}}) \longrightarrow \mathrm{Points}(X_{\text{fppf}}^{\text{big}}), \quad P \longmapsto P^{\text{big}},$$

where P^{big} is given by the fiber functor $F_{P^{\text{big}}} = (F|Y)_P$. Likewise, we get a continuous map of locales: Fix a terminal object $e \in X_{\text{fppf}}^{\text{big}}$. Then $e|Y$ are terminal objects, and for each subobject $G \subset e$ we get subojects $G|Y \subset e|Y$. Since the forgetful functor (8) commutes with direct and inverse limits, the monotonous map $G \mapsto G|Y$ constitutes a continuous map of locales

$$\mathrm{Loc}(Y_{\text{fppf}}) \longrightarrow \mathrm{Loc}(X_{\text{fppf}}^{\text{big}}). \tag{9}$$

In turn, the canonical map of topological spaces $|Y_{\text{fppf}}| \to |X_{\text{fppf}}^{\text{big}}|$ is continuous as well.

4. fppf-local rings

We now define a class of local rings that generalizes the notion of algebraically closed fields. Such rings will appear as stalks of the structure sheaf $\mathscr{O}_{X_{\text{fppf}}}$ at topos-theoretical points.

Definition 4.1. A ring R is called *fppf-local* if it is local, and every fppf homomorphism $R \to B$ admits a retraction. In other words, the corresponding morphism of schemes $\mathrm{Spec}(B) \to \mathrm{Spec}(R)$ admits a section.

Recall that a ring R is called *totally integrally closed* if for any ring homomorphism $B \to R$ and any integral extension $B \subset B'$, there is homomorphism $B' \to R$ making the diagram

$$\begin{tikzcd} & \mathrm{Spec}(B') \arrow[d] \\ \mathrm{Spec}(R) \arrow[ur] \arrow[r] & \mathrm{Spec}(B) \end{tikzcd}$$

commutative. This was introduced by Enochs [8], and further analyzed by Hochster [19]. One also says that R is a *TIC ring.* Note that such rings are necessarily reduced (see [8], Theorem 1 and also [5]).

Let us call a ring R *absolutely integrally closed* if each monic polynomial $f \in R[T]$ has a root in R. These rings are also called *AIC rings.* Note that an integral domain R is AIC if and only its it is normal and its field of fraction is algebraically closed, and this holds if and only if R is TIC ([8], Proposition 3). Throughout, we call a ring R *integral* if it is an integral domain, that is, a subring of a field.

Proposition 4.2. *Any fppf-local ring is AIC. Moreover, for local rings R, the following three conditions are equivalent:*

(i) *R is TIC.*
(ii) *R is AIC and integral.*
(iii) *R is fppf-local and integral.*

Proof. Suppose that R is fppf-local, and let $f \in R[T]$ be a monic polynomial. The fppf algebra $R[T]/(f)$ contains a root of f, and also admits an R-algebra homomorphism to R. Hence R itself contains a root. Consequently, R is AIC. This also shows that implication (iii)$\Rightarrow$(ii). According to [19], Proposition 7, a local ring that is TIC must be integral. The equivalence of (i) and (ii) follows from [8], Proposition 3. Now suppose that R is TIC. Using the TIC condition with $B = R$ and a finite fppf algebra B', we infer that R is fppf-local. This gives the implication (i)$\Rightarrow$(iii). $\square$

We shall see in Section 5 that there are fppf-local rings that are not integral. I do not know whether there are local AIC rings that are not fppf-local. Neither do I know whether nonzero homomorphic images of fppf-local rings remain fppf-local. We have the following partial results in this direction:

Proposition 4.3. *Let R be an fppf-local ring. For every prime ideal $\mathfrak{p} \subset R$, the domain $R/\mathfrak{p}$ is fppf-local, and the residue field $\kappa(\mathfrak{p})$ is algebraically closed. For every ideal $\mathfrak{a} \subset R$, the residue class ring $R/\mathfrak{a}$ is AIC.*

Proof. Let $\bar{f} \in R/\mathfrak{a}[T]$ be a monic polynomial. Lift it to a monic polynomial $f \in R[T]$. The fppf R-algebra $B = R[T]/(f)$ contains a root of f, and admits an R-algebra homomorphism to R. Whence there is root $a \in R$, whose residue class is a root of $\bar{f}$ in $R/\mathfrak{a}$, such that the latter is AIC. If $\mathfrak{a} = \mathfrak{p}$ is prime, then the AIC domain $R/\mathfrak{p}$ is fppf-local by Proposition 4.2. Any localization of the domain $R/\mathfrak{p}$ stays AIC. In particular, its field of fractions $\kappa(\mathfrak{p})$ is algebraically closed. $\square$

Proposition 4.4. *Let R be an fppf-local ring, and $\mathfrak{p} \subset R$ be a prime ideal. Then $\mathfrak{p}^n = \mathfrak{p}$ for every integer $n \geq 1$.*

Proof. Let $a \in \mathfrak{p}$ be some element, and consider the monic polynomial $f = T^n - a \in R[T]$. The fppf R-algebra $R[T]/(f)$ contains a root for f, and admits an R-algebra homomorphism to R, whence there is an element $b \in R$ with $b^n = a$. Since $\mathfrak{p}$ is prime, we must have $b \in \mathfrak{p}$. $\square$

In particular, all cotangent spaces $\mathfrak{p}/\mathfrak{p}^2$ of an fppf-local ring R vanish, and its formal completion $\hat{R} = \varprojlim_n R/\mathfrak{m}^n$ coincides with the residue field $\kappa = R/\mathfrak{m}$. This yields the following:

Corollary 4.5. *A ring is fppf-local and noetherian if and only if it is an algebraically closed field.*

Proof. The condition is sufficient by Proposition 4.2, or more directly by Hilbert's Nullstellensatz. Conversely, suppose that R is an fppf-local noetherian ring, with residue field $k = R/\mathfrak{m}$. By the Proposition, $\mathfrak{m} \otimes k = \mathfrak{m}/\mathfrak{m}^2 = 0$. The Nakayama Lemma ensures $\mathfrak{m} = 0$, whence $R = k$ is a field. This field is algebraically closed by Proposition 4.2. $\square$

It follows that finite flat algebras over fppf-local rings may fail to be fppf-local: Take $R = k$ an algebraically closed field and $A = k[\epsilon]$ the ring of dual numbers, where $\epsilon^2 = 0$.

We now can state an amazing property of fppf-local rings:

Theorem 4.6. *Let R be an fppf-local ring. For every prime ideal $\mathfrak{p} \subset R$, the local ring $R_{\mathfrak{p}}$ is strictly local, with algebraically closed residue field.*

Proof. The residue field $\kappa(\mathfrak{p})$ is algebraically closed by Proposition 4.3. It remains to check that the local rings $R_{\mathfrak{p}}$ are henselian. Let $\mathfrak{q} \subset R$ be a minimal prime ideal contained in $\mathfrak{p}$. Then the local domain $(R/\mathfrak{q})_{\mathfrak{p}}$ is AIC according to Proposition 4.3, whence henselian, for example by [4], Proposition 1.4. Clearly, the $\operatorname{Spec}(R/\mathfrak{q})_{\mathfrak{p}}$ are the irreducible components of $\operatorname{Spec}(R_{\mathfrak{p}})$. The assertion now follows from the following lemma. □

Lemma 4.7. *A local ring A is henselian if and only if $A/\mathfrak{q}$ is henselian for every minimal prime ideal $\mathfrak{q} \subset A$.*

Proof. The condition is necessary by [16], Proposition 18.5.10. For the converse, write $Y = \operatorname{Spec}(A)$. Let $f : X \to Y$ be a finite morphism, and write $a_1, \ldots, a_n \in X$ for the closed points. Since X is quasicompact, each point $x \in X$ specializes to at least one closed point. Since $Z = \overline{\{x\}}$ is irreducible, the composite morphism $Z \to Y$ is finite, and Y is henselian, it follows that Z is local. Consequently, we have a disjoint union $X = \operatorname{Spec}(\mathscr{O}_{X,x_1}) \cup \ldots \cup \operatorname{Spec}(\mathscr{O}_{X,a_n})$. Writing $X = \operatorname{Spec}(B)$, we conlcude that the maximal ideals $\mathfrak{m}_i \subset B$ corresponding to the closed points $a_i \in X$ are coprime, and thus $B = \prod B_{\mathfrak{m}_i}$. In turn, X is a sum of local schemes. □

We now easily obtain the following useful criterion:

Proposition 4.8. *A ring R is fppf-local if and only if it is local henselian and every finite fppf homomorphism $R \to B$ admits a retraction.*

Proof. According to Theorem 4.6, the condition is necessary. It is sufficient as well: Suppose that R is local henselian, and every finite fppf algebra admits a retraction. Let $R \to C$ be an arbitrary fppf homomorphism. According to [16], Corollary 17.16.2 there is a residue class ring $C/\mathfrak{a}$ that is quasifinite and fppf over R. Since R is henselian, there is a larger ideal $\mathfrak{a} \subset \mathfrak{b}$ so that $B = C/\mathfrak{b}$ is finite and fppf. The latter admits, by assumptions, a retraction $B \to R$, and the composite map $C \to B \to R$ is the desired retraction of C. □

Now let X be a scheme, and $P : (\text{Set}) \to X_{\text{fppf}}$ be a point in the sense of topos-theory. Applying the corresponding fiber functor P^{-1} to the structure sheaf $\mathscr{O}_{X_{\text{fppf}}}$, we get a ring $\mathscr{O}_{X_{\text{fppf}},P} = P^{-1}(\mathscr{O}_{X_{\text{fppf}}})$.

Theorem 4.9. *Under the preceding assumptions, the ring $\mathscr{O}_{X_{\text{fppf}},P}$ is fppf-local.*

Proof. Choose a pro-object $(U_i)_{i\in I}$ in (fppf/X) so that the fiber functor is of the form $P^{-1}(F) = \varinjlim_{i\in I} \Gamma(U_i, F)$. Write $R_i = \Gamma(U_i, \mathscr{O}_{X_{\text{fppf}}})$ and $R = \mathscr{O}_{X_{\text{fppf}},P}$, such that $R = \varinjlim_{i\in I} R_i$. According to Lemma 4.10 below, we may assume that the schemes U_i are affine, in other words, $U_i = \operatorname{Spec}(R_i)$.

We first verify that the ring R is local. In light of [24], Chapter III, Corollary 2.7, it suffices to check that the ringed topos $(X_{\text{fppf}}, \mathscr{O}_{X_{\text{fppf}}})$ is locally ringed. This indeed holds by loc. cit. Criterion 2.4, because for each $U \in (\text{fppf}/X)$ and each $s \in \Gamma(U, \mathscr{O}_U)$, the open subsets $U_s, U_{1-s} \subset U$ where s respectively $1-s$ are invertible form a covering.

We next check that the local ring R is henselian. Let $R \to B$ be étale, and suppose there is a retraction $B/\mathfrak{m} \to k$, where $\mathfrak{m} \subset R$ is the maximal ideal, and $k = R/\mathfrak{m}$. We have to verify that this retraction extends to a retraction $B \to R$, compare [16], Theorem 18.5.11. Localizing B, we may assume that B is local, such that we merely have to check that there is a retraction $B \to R$ at all. According to [16] Proposition 17.7.8, there is an index $i \in I$ and some étale homomorphism

$R_i \to B_i$ with $B = B_i \otimes_{R_i} R$. Set $B_j = R_j \otimes_{R_i} B_i$ for $j \geq i$. Invoking [2], Expose IV, Section 6.8.7 again, we infer that there is an index $j \geq i$ and some R_i-algebra homomorphism $R_j \to B_i$, which gives a retraction $B_i \otimes_{R_i} R_j \to R_j$. This yields a direct system of retractions

$$B_i \otimes_{R_i} R_k = B_i \otimes_{R_i} R_j \otimes_{R_j} R_k \longrightarrow R_j \otimes_{R_j} R_k = R_k.$$

Passing to direct limits with respect to $k \geq j$ yields the desired retraction $B = \varinjlim_{k\geq i}(B_i \otimes_{R_i} R_k) \to R$.

We finally show that R is fppf-local. Let $R \to B$ be a finite fppf homomorphism of rings. It suffices to check that it admits a retraction, by Proposition 4.8. Define B_i as in the preceding paragraph. There is an index $i \in I$ and an homomorphism $R_i \to B_i$ with $B = B_i \otimes_{R_i} R$, according to [15], Theorem 8.8.2. Since the R-module B is free of finite nonzero rank, we may assume that the same holds for the R_i-module B_i, by [15], Corollary 8.5.2.5. Set $V = U_i = \operatorname{Spec}(R_i)$, and $V' = \operatorname{Spec}(B_i)$. Again using [2], Expose IV, Section 6.8.7 and arguing as above, one infers that the desired retraction $B \to R$ exists. ☐

In the course of the preceding proof, we have used the following fact:

Proposition 4.10. *Each topos-theoretical point $P : (\mathrm{Set}) \to (\mathrm{fppf}/X)$ has a fiber functor isomorphic to $F \mapsto \varinjlim_{i\in I} \Gamma((U_i, p_i), F)$ for some pro-object $((U_i, p_i))_{i\in I}$ in (fppf/X) where all the U_i are affine.*

Proof. To simplify notation, write $\mathcal{C}' = (\mathrm{fppf}/X)$ and consider the full subcategory $\mathcal{C} \subset \mathcal{C}'$ of objects (U, p) with U affine, endowed with the induced Grothendieck topology. Let $\mathscr{E}' = X_{\mathrm{fppf}}$ and $\mathscr{E}$ be the ensuing topoi of $\mathscr{U}$-sheaves. Given an object $(U, p) \in \mathcal{C}'$, we denote by $U_\alpha \subset U$ the family of all affine open subschemes, and set $p_\alpha = p|U_\alpha$. Clearly, $(U_\alpha, p_\alpha) \in \mathcal{C}$ and $((U_\alpha, p_\alpha) \to (U, p))_\alpha$ is a covering in $\mathcal{C}'$. By the Comparison Lemma ([2], Expose III, Theorem 4.1), the restriction functor $\mathscr{E}' \to \mathscr{E}$, $F \mapsto F|\mathcal{C}$ is an equivalence of categories. In turn, every fiber functor on $\mathscr{E}'$ is isomorphic to some fiber functor coming from a pro-object in the category $\mathcal{C}$. ☐

5. Construction of fppf-local rings

Let R be a strictly local ring, that is, a henselian local ring with separably closed residue field. Choose a universe $R \in \mathscr{U}$ and some ordinal $\sigma' \notin \mathscr{U}$. Let $\sigma < \sigma'$ be the smallest ordinal that is not an element of $\mathscr{U}$. The goal of this section is to attach, in a functorial way, a direct system $R_\nu \in \mathscr{U}$ of strictly local rings, indexed by the well-ordered set

$$\sigma = \{\nu \mid \nu \text{ ordinal with } \nu < \sigma\}$$

of all smaller ordinals. The transition maps in this direct system will be faithfully flat and integral. For each limit ordinal $\lambda < \sigma$, the local ring R_λ will be fppf-local, that is, every fppf R_λ-algebra admits a retraction. Maybe it goes without saying that all the rings R_ν, $\nu > 0$ are highly non-noetherian.

The construction of the rings is as follows: Consider the category $\mathcal{F} = \mathcal{F}(R)$ of finite fppf R-algebras A with $\operatorname{Spec}(A)$ connected. Note that each such algebra is a fortiori isomorphic to some $R[T_0, \ldots, T_m]/(f_1, \ldots, f_r)$ for some integer $m \geq 0$ and some finite collection of polynomials $f_1, \ldots, f_r$. Whence the set of isomorphism classes of objects in $\mathcal{F}$ does not depend on the chosen universe, up to canonical bijection. Choose a set $I = I(R)$ of such R-algebras, so that each isomorphism class is represented by precisely one element of I. Now consider the set $\Phi = \Phi(R)$ of all finite subsets of the set I, endowed with the order relation coming from the inclusion relation. Clearly, the ordered set Φ is filtered. Each of its elements φ is thus a finite set of certain finite fppf R-algebras. Given an element $\varphi \in \Phi$, we form the finite fppf R-algebra

$$A_\varphi = \bigotimes_{A \in \varphi} A.$$

Here the tensor product denotes the unordered tensor product.

Recall that for an collection of R-modules $(M_j)_{j\in J}$, indexed by some finite set J of cardinality $n \geq 0$, the *unordered tensor product* is the R-module of invariants

$$\bigotimes_{j\in J} M_j = \left(\bigoplus_\eta M_{\eta(1)} \otimes \ldots \otimes M_{\eta(n)} \right)^{S_n}.$$

Here the sum runs over all bijections $\eta : \{1, \ldots, n\} \to J$, and the symmetric group S_n acts from the right on the sum by permuting the summands:

$$(a_{\eta(1)} \otimes \ldots \otimes a_{\eta(n)})_\eta \cdot \sigma = (a_{\eta\sigma(1)} \otimes \ldots \otimes a_{\eta\sigma(n)})_{\eta\sigma}.$$

Note that, for each choice of ordering $J = \{j_1, \ldots, j_n\}$, the obvious inclusion into the sum gives a canonical identification $M_{j_1} \otimes \ldots \otimes M_{j_n} = \bigotimes_{j\in J} M_j$ of the ordinary tensor product with the unordered tensor product. However, the unordered tensor product has the advantage to be functorial, in the strict sense, with respect to indexed R-modules $(M_j)_{j\in J}$. Here a morphism between $(M_j)_{j\in J}$ and $(M_k)_{k\in K}$ is given by a map $m : J \to K$ together with homomorphisms $M_j \to N_{m(j)}$.

This functoriality ensures that $\varphi \mapsto A_\varphi$ is a direct system of R-algebras, and we already remarked that it is filtered. We denote by

$$R_+ = \varinjlim_{\varphi\in\Phi(R)} A_\varphi$$

its direct limit. Clearly, all transition maps in this direct limit are finite fppf, so we may regard each A_φ as an R-subalgebra of R_+.

Lemma 5.1. *The ring R_+ is strictly local, and the homomorphism $R \to R_+$ is flat, integral and local. Moreover, we have $R_+ \in \mathscr{U}$, and* $\mathrm{Card}(R_+) = \mathrm{Card}(R)$.

Proof. Using the notation from the beginning of this section, we start by checking that the tensor products $A_\varphi \simeq A_1 \otimes \ldots \otimes A_n$, where $A_1, \ldots, A_n$ are the elements φ, are local. Obviously, A_φ is a finite fppf R-algebra. By definition, the schemes $\mathrm{Spec}(A_i)$ are connected. Whence $R \to A_i$ are local maps of local rings, because R is henselian. Furthermore, $A_i \otimes_R k$, where $k = R/\mathfrak{m}_R$ is the residue field, is a finite local k-algebra. Their tensor product remains local, because k is separably closed. We conclude that there is a unique prime ideal in A lying over $\mathfrak{m}_R \subset R$. Since $\mathrm{Spec}(A) \to \mathrm{Spec}(R)$ is a closed map, it follows that A is local, and that the map $R \to A$ is local. Passing to the filtered direct limit, the first assertion follows.

Since R, A_φ, $I(R)$ and whence $\Phi(R)$ are elements of the universe $\mathscr{U}$, the same must hold for the direct limit R_+. By faithful flatness, the maps $A_\varphi \to R_+$ are injective, whence $R_+ = \bigcup_{\varphi\in\Phi} A_\varphi$ as a union of subrings that are finite fppf R-algebras. Now recall that R is strictly local, and in particular infinite. Let $\aleph_\iota$ be its cardinality. It easily follows that each subring $A_\varphi \subset R_+$ and the index set Φ

both have the same cardinality. Cardinal arithmetic thus gives $\aleph_\iota \leq \mathrm{Card}(R_+) \leq \aleph_\iota \cdot \aleph_\iota = \aleph_\iota$. Consequently $\mathrm{Card}(R_+) = \mathrm{Card}(R)$. □

The ring R_+, however, is never noetherian: If $n \geq 1$, then $R[T]/(T^n)$ is a finite fppf algebra with connected spectrum, whence isomorphic to some subring of R_+. It follows that there is an element $f \in R_+$ with $f^n = 0$ but $f^{n-1} \neq 0$. In particular, the nilradical $\mathrm{Nil}(R_+)$ is not nilpotent.

Using *transfinite recursion*, we now define a direct system of rings R_ν, $\nu < \sigma$ as follows: To start with, set $R_0 = R$. Suppose the direct system is already defined for all ordinals smaller that some $\nu < \sigma$. We then set

$$R_\nu = \begin{cases} (R_\gamma)_+ & \text{if } \nu = \gamma + 1 \text{ is a successor ordinal;} \\ \varinjlim_{\gamma<\nu} R_\gamma & \text{if } \nu \text{ is a limit ordinal.} \end{cases}$$

Proposition 5.2. *For each ordinal $\nu < \sigma$, the rings R_ν are strictly local, and the transition map $R_\gamma \to R_\nu$, $\gamma \leq \nu$ are local, faithfully flat, and integral.*

Proof. By transfinite induction. The assertion is trivial for $\nu = 0$. Now suppose that $\nu > 0$, and that the assertion is true for all smaller ordinals. If ν is a successor ordinal, the assertion follows from Lemma 5.1. If $\nu = \lambda$ is a limit ordinal, then R_λ is a filtered direct limit of strictly local ring with local transition maps, whence strictly local. Moreover, the transition maps $R_\gamma \to R_\lambda$ for $\gamma < \lambda$ are local, faithfully flat, and integral. □

Proposition 5.3. *For each ordinal $\nu < \sigma$, we have $\dim(R_\nu) = \dim(R)$, and the residue field $k_\nu = R_\nu/\mathfrak{m}_{R_\nu}$ is an algebraic closure of the residue field $k = R/\mathfrak{m}_R$.*

Proof. Since $R \subset R_\nu$ is integral and faithfully flat, the first statement follows from [7], Chapter VIII, §2, No. 3, Theorem 1. As to the second assertion, the field extension $k \subset k_\nu$ is clearly algebraic. Let $P \in R[T]$ be a monic polynomial, consider the finite fppf R-algebra $A = R[T]/(P)$, and let $A_\mathfrak{m}$ be the localization at some maximal ideal $\mathfrak{m} \subset A$. Then P has a root in A, and $A_\mathfrak{m}$ is isomorphic to a subring of R_ν. Whence we have a homomorphism $A \to k_\nu$. It follows that each monic polynomials with coefficients in k has a root in k_ν. □

Proposition 5.4. *For each ordinal $\nu < \sigma$, the ring R_ν is an element of the chosen universe $\mathscr{U}$.*

Proof. By transfinite induction. The assertion is obvious for $\nu = 0$. Now suppose that $\nu > 0$, and that the assertion holds for all smaller ordinals. If $\nu = \gamma + 1$ is a successor ordinal, then $R_\nu = (R_\gamma)_+ \in \mathscr{U}$ by Proposition 5.1. If ν is a limit ordinal, then $R_\nu = \varinjlim_{\gamma<\nu} R_\gamma \in \mathscr{U}$ because the R_γ and the index set, which equals the set ν, are members of the universe $\mathscr{U}$. □

Theorem 5.5. *For each limit ordinal $\lambda < \sigma$, the ring R_λ is fppf-local.*

Proof. Given a fppf homomorphism $R \to B$, we have to show that it admits a retraction. It suffices to treat the case that B is finite fppf, according to Proposition 4.8. Since B is of finite presentation, there is some ordinal $\nu < \lambda$ and some R_ν-algebra B_ν with $B \simeq B_\nu \otimes_{R_\nu} R_\lambda$ (see [14], Lemma 5.13.7.1). Moreover, we may assume that B_ν is finite and fppf. Since tensor products commute with filtered direct limits, the canonical map $\varinjlim_\gamma (B_\nu \otimes_{R_\nu} R_\gamma) \to B$ is bijective, where the direct limit runs over all ordinals $\nu \leq \gamma < \lambda$.

Choose $A_\nu \in I(R_\nu)$ and an isomorphism of R_ν-algebras $h : B_\nu \to A_\nu$. Consider the singleton $\varphi = \{A_\nu\} \in \Phi(R_\nu)$, such that $A_\varphi = A_\nu$ in the notation introduced above. By the very definition of $R_{\nu+1}$, there exists an R_ν-algebra homomorphism $B_\nu \xrightarrow{h} A_\varphi \to R_{\nu+1}$, which gives a retraction $B_\nu \otimes_{R_\nu} R_{\nu+1} \to R_{\nu+1}$. Tensoring with R_γ over $R_{\gamma+1}$, $\gamma \geq \nu + 1$ we get a direct system of retractions

$$B_\nu \otimes_{R_\nu} R_\gamma = B_\nu \otimes_{R_\nu} R_{\nu+1} \otimes_{R_{\nu+1}} R_\gamma \longrightarrow R_{\nu+1} \otimes_{R_{\nu+1}} R_\gamma = R_\gamma.$$

Passing to direct limits yields the desired retraction $B \to R_\lambda$. □

For later use, we record the following fact:

Lemma 5.6. *For each ordinal $\nu < \sigma$ and each finite subset $S \subset R_\nu$, there is an R-subalgebra $B \subset R_\nu$ containing S so that the structure map $R \to B$ is finite and fppf.*

Proof. By transfinite induction. The case $\nu = 0$ is trivial. Now suppose that $\nu > 0$, and that the assertion is true for all smaller ordinals. If ν is a limit ordinal, then $R_\nu = \varinjlim_{\gamma<\lambda} R_\gamma$, there is some ordinal $\gamma < \nu$ with $S \subset R_\gamma$, and the induction hypothesis, together with flatness of $R_\gamma \subset R_\nu$, yields the assertion.

Now suppose that $\nu = \gamma + 1$ is a successor ordinal. Write $R_\nu = \varinjlim A_\varphi$ as a filtered union of finite fppf local R_γ-subalgebras and choose some index φ so that $S \subset A_\varphi$. Since R_γ is local, the underlying R_γ-module of A_φ is free. The same holds for A_φ/R_γ, because the unit element $1 \in A_\varphi$ does not vanish anywhere. Thus we may extend $b_1 = 1$ to an R_γ-basis $b_1, \ldots, b_m \in A_\varphi$, and write

$$s = \sum_k c_{sk} b_k \quad \text{and} \quad b_i \cdot b_j = \sum_k c_{ijk} b_k$$

for some coefficients $c_{sk}, c_{ijk} \in R_\gamma$, where $s \in S$ and $1 \leq i, j, k \leq m$. Form the finite subset $S' = \{c_{sk}, c_{ijk}\} \subset R_\gamma$ comprising all these coefficients. By induction hypothesis, there is a finite fppf R-subalgebra $B' \subset R_\gamma$ containing S'. Now consider the canonical R-linear map

$$\bigoplus_{i=1}^{m} B' b_i \longrightarrow R_\nu.$$

This map factors over $A_\varphi \subset R_\nu$, and it is injective, because the $b_1, \ldots, b_m \in A_\varphi$ are R_γ-linearly independent. Let $B \subset R_\nu$ be its image. By construction, $S \subset B$, and $B' \subset B$ is a direct summand of free B'-modules of finite rank, in particular, an fppf ring extension. It follows that $R \subset B$ is finite fppf. □

Given $\nu < \sigma$, consider the set of all R-subalgebras $B_i \subset R_\nu$, $i \in I_\nu$ so that the structure map $R \to B_i$ is finite fppf. We regard I_ν is an ordered set, where the order relation is the inclusion relation.

Proposition 5.7. *The ordered set I_ν is filtered, each B_i is a local R-algebra such that the structure map $R \to B_i$ is local, finite and fppf, and $R_\nu = \bigcup_{i \in I_\nu} B_i$.*

Proof. If follows from Lemma 5.6 that R_ν is the union of the B_i. To see that the union is filtered, let $B_i, B_j \subset R_\nu$ be two such subrings. Let $S_i \subset B_i$ be an R-basis, and similarly $S_j \subset B_j$. Then $S = S_i \cup S_j$ is a finite subset of R_ν, and Lemma 5.6 gives us the desired subalgebra $B \subset R_\nu$ containing B_i and B_j.

Each B_i is by definition finite fppf over R. Since R is strictly local, it remains to check that $\mathrm{Spec}(B_i)$ is connected. Since $B_i \to R_\nu$ is injective, the image of the continuous map $\mathrm{Spec}(R_\nu) \to \mathrm{Spec}(B_i)$ contains every generic point. Obviously,

$B_i \subset R_\nu$ is integral, whence the continuous map is surjective. Since $\mathrm{Spec}(R_\nu)$ is connected, so must be its continuous image $\mathrm{Spec}(B_i)$. $\square$

The direct system R_ν, $\nu < \sigma$ is functorial: Suppose that $f : R \to R'$ is a local homomorphism between strictly local rings. With the notation introduced at the beginning of this section, we get a functor

$$\mathcal{F}(R) \longrightarrow \mathcal{F}(R'), \quad A \longmapsto A' = A \otimes_R R',$$

and thus induced maps of ordered sets $I(R) \to I(R')$ and $\Phi(A) \to \Phi(A')$, $\phi \mapsto \phi'$. The latter are not necessarily injective, but in any case induce morphisms $A_\phi \to A_{\phi'}$. In turn, we get a natural homomorphism of direct limits $R_+ \to R'_+$. Using transfinite induction, one finally obtains the desired homomorphism $R_\nu \to R'_\nu$ of direct systems. One easily checks that this is functorial.

6. Points in the fppf topos

Let X be a scheme. Choose a universe $X \in \mathscr{U}$, and let σ be the smallest ordinal not contained in this universe. Given a geometric point $\bar{a} : \mathrm{Spec}(\Omega) \to X$ and a limit ordinal $\lambda < \sigma$, we call

$$\mathscr{O}_{X,\bar{a},\lambda} = (\mathscr{O}_{X,\bar{a}})_\lambda$$

the *fppf-local ring attached to the geometric point and the limit ordinal*, as defined in Section 5. The goal now is to construct a point $P = P_{\bar{a},\lambda} : (\mathrm{Set}) \to X_{\mathrm{fppf}}$ in the sense of topos-theory, together with a canonical identification

$$\mathscr{O}_{X_{\mathrm{fppf}},P} = P^{-1}(\mathscr{O}_{X_{\mathrm{fppf}}}) = \mathscr{O}_{X,\bar{a},\lambda}.$$

Actually, the isomorphism class of $P_{\bar{a},\lambda} \in \mathrm{Points}(X_{\mathrm{fppf}})$ depends only on the image point $a \in X$ of the geometric point $\bar{a}$, and will be denoted by $P_{a,\lambda} \in |X_{\mathrm{fppf}}|$. This will give a continuous section $a \mapsto P_{a,\lambda}$ for the canonical map $|X_{\mathrm{fppf}}| \longrightarrow |X_{\mathrm{Zar}}| = X$ of topological spaces.

The main step is the construction of a pro-object in (fppf/X) that will be isomorphic to the pro-object of neighborhood for the topos-theoretical point $P_{\bar{a},\lambda}$. Our first task to to find a suitable index category for such a pro-object:

Fix a geometric point $\bar{a} : \mathrm{Spec}(\Omega) \to X$ and some ordinal $\nu < \sigma$. For the moment, this can be either a limit ordinal or a successor ordinal. We now define the index category $I_{\bar{a},\nu}$ as follows: The objects are 5-tuples

$$(V_0, V_1, \phi, U, \psi),$$

where $a \in V_0 \subset X$ is an affine open neighborhood, V_1 is an affine étale V_0-scheme, $\phi : \mathrm{Spec}(\mathscr{O}_{X,\bar{a}}) \to V_1$ is a morphism, U is a finite fppf V_1-scheme, and $\psi : \mathrm{Spec}(\mathscr{O}_{X,\bar{a},\nu}) \to U$ is a morphism. We demand that the diagram

$$\begin{array}{ccccccc} & & \mathrm{Spec}(\mathscr{O}_{X,a}) & \xleftarrow{\mathrm{can}} & \mathrm{Spec}(\mathscr{O}_{X,\bar{a}}) & \xleftarrow{\mathrm{can}} & \mathrm{Spec}(\mathscr{O}_{X,\bar{a},\nu}) \\ & & {\scriptstyle\mathrm{can}}\downarrow & & {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi} \\ X & \longleftarrow & V_0 & \longleftarrow & V_1 & \longleftarrow & U \end{array} \tag{10}$$

is commutative, and that the resulting morphism of affine schemes

$$\mathrm{Spec}(\mathscr{O}_{X,\bar{a},\nu}) \longrightarrow U \times_{V_1} \mathrm{Spec}(\mathscr{O}_{X,\bar{a}}) \tag{11}$$

is *schematically dominant*, that is, induces an injection on global sections of the structure sheaf. The morphisms

$$(V_0', V_1', \phi', U', \psi') \longrightarrow (V_0, V_1, \phi, U, \psi)$$

in the category $I_{\bar{a},\nu}$ are 3-tuples (h_0, h_1, h), where $h_0 : V_0' \to V_0$ and $h_1 : V_1' \to V_1$ and $h : U' \to U$ are morphisms of schemes. We demand that the diagram

$$\begin{tikzcd}
 & \mathrm{Spec}(\mathscr{O}_{X,a}) \arrow[d] \arrow[ddr] & \mathrm{Spec}(\mathscr{O}_{X,\bar{a}}) \arrow[l] \arrow[d, "\phi'"'] \arrow[ddr, "\phi"] & \mathrm{Spec}(\mathscr{O}_{X,\bar{a},\lambda}) \arrow[l] \arrow[d, "\psi'"'] \arrow[ddr, "\psi"] & \\
X \arrow[dr, "\mathrm{id}"'] & V_0' \arrow[l] \arrow[dr, "h_0"] & V_1' \arrow[l] \arrow[dr, "h_1"] & U' \arrow[l] \arrow[dr, "h"] & \\
 & X & V_0 \arrow[l] & V_1 \arrow[l] & U \arrow[l]
\end{tikzcd} \tag{12}$$

is commutative. Note that h_0 is an inclusion between the two open subschemes $V_0', V_0 \subset X$, and h_1 is a refinement between the étale neighborhoods $V_1', V_1 \to X$ of the geometric point $\bar{a}$.

Given an object $(V_0, V_1, \phi, U, \psi)$, the composite morphism $U \to V_1 \to V_0 \subset X$ is quasifinite and fppf, whence we may regard U as an object in (fppf/X). This yields a covariant functor $I_{\bar{a},\nu} \to (\mathrm{fppf}/X)$, which on morphism is defined as $(h_0, h_1, h) \mapsto h$. The corresponding contravariant functor $I_{\bar{a},\nu}^{\mathrm{op}} \to (\mathrm{fppf}/X)$ is actually a pro-object, which means the following:

Proposition 6.1. *The opposed category $I_{\bar{a},\nu}^{\mathrm{op}}$ is filtered.*

Proof. Working with $I_{\bar{a},\nu}$ rather then the opposed category, we have to check two things: First, for any two given objects, there is some object and morphisms from it to the given objects. Second, any two morphisms with the same domain and range become equal after composing with some morphism from the right.

We start with the former condition: Suppose

$$(V_0', V_1', \phi', U', \psi') \quad \text{and} \quad (V_0'', V_1'', \phi'', U'', \psi''). \tag{13}$$

are two objects. Choose an affine open neighborhood $a \in V_0 \subset X$ contained in $V_0' \cap V_0''$. Base-changing the data to V_0 over X, we easily reduce to the case that $V_0' = V_0 = V_0''$. Similarly, we may assume $V_1' = V_1''$ and $\phi' = \phi''$.

To simplify notation, now write

$$R = \mathscr{O}_{X,\bar{a}} \quad \text{and} \quad R_\nu = (\mathscr{O}_{X,\bar{a}})_\nu$$

Let $B' \subset R_\nu$ be the images of the ring of global sections with respect to the morphism of affine schemes $\mathrm{Spec}(R_\nu) \to U \times_{V_1} \mathrm{Spec}(R)$. The definition of objects in $I_{\bar{a},\nu}$ ensures that $\mathrm{Spec}(B') = U \times_{V_1} \mathrm{Spec}(R)$. Define $B'' \subset R_\nu$ analogously. Then B', B'' are finite fppf as R-algebras, and R-subalgebras inside R_ν. According to Proposition 5.7, they are contained in some larger finite fppf R-subalgebra $B \subset R_\nu$. Since $R = \mathscr{O}_{X,\bar{a}}$ can be regarded as the filtered direct limit of the global section rings of the étale neighborhoods of the geometric point

$$\mathrm{Spec}(\Omega) \subset \mathrm{Spec}(R) \xrightarrow{\phi' = \phi''} V_1' = V_1'',$$

we find some étale neighborhood $V_1 \to V_1' = V_1''$ so that $\mathrm{Spec}(B) \to \mathrm{Spec}(R)$ arises via base-change from some finite fppf scheme $U \to V_1$. Passing to smaller étale neighborhoods, we may assume that the inclusion maps $B', B'' \subset B$ inside R_ν of finite fppf R-algebras are induced by some V_1-morphisms $U \to U'$ and $U \to U''$. Let $\phi : \mathrm{Spec}(R) \to V_1$ and $\psi : \mathrm{Spec}(R_\nu) \to U$ be the canonical morphisms. Then $(V_0, V_1, \phi, U, \psi)$ is an object in $I_{\bar{a},\nu}$ and by construction has morphisms to both of the given objects in (13).

It remains to verify the second condition. Suppose we have two arrows

$$(V_0', V_1', \phi', U', \psi') \rightrightarrows (V_0, V_1, \phi, U, \psi)$$

called (h_0, h_1, h) and (k_0, k_1, k). We have to show that they become equal after composing from the right with some morphism. According to the commutative diagram (12), both $h, k : U' \to U$ are V_1-morphisms. Consider the two morphisms

$$U' \times_{V_1} \operatorname{Spec}(\mathscr{O}_{X,\bar{a}}) \xrightarrow{h,k} U \times_{V_1} \operatorname{Spec}(\mathscr{O}_{X,\bar{a}}) \xrightarrow{\mathrm{pr}} U.$$

coming form base-change. These coincide, because the morphism in (11) is schematically dominant. Since U', U are of finite presentation over V_1, the morphisms

$$U' \times_{V_1} V_1'' \xrightarrow{h,k} U \times_{V_1} V_1'' \xrightarrow{\mathrm{pr}} U$$

coming from base-change of h and k to some étale neighborhood $V_1'' \to V_1$ of the geometric point $\phi : \operatorname{Spec}(\Omega) \to V_1$ become identical ([15], Theorem 8.8.2). Choosing the neighborhood small enough, we may assume that it factors over V_1'. Let $\phi'' : \operatorname{Spec}(\mathscr{O}_{X,\bar{a}}) \to V_1''$ be the canonical map. Define $U'' = U' \times_{V_1} V_1''$ and let $\psi'' : \operatorname{Spec}(\mathscr{O}_{X,\bar{a},\lambda}) \to U''$ be the canonical map. The resulting morphism $(V_0', V_1'', \phi'', U'', \psi'') \to (V_0', V_1', \phi', U', \psi')$ does the job. $\square$

We now have a pro-object

$$I_{\bar{a},\nu} \longrightarrow (\mathrm{fppf}/X), \quad (V_0, V_1, \phi, U, \psi) \longmapsto U$$

and obtain a covariant functor

$$X_{\mathrm{fppf}} \longrightarrow (\mathrm{Set}), \quad F \longmapsto \varinjlim \Gamma(U, F), \tag{14}$$

where the direct limit runs over all objects $(V_0, V_1, \phi, U, \psi) \in I_{\bar{a},\nu}$ This functor respects finite inverse limits, because the opposite of the index category is filtered. In the special case $F = \mathscr{O}_{X_{\mathrm{fppf}}}$, the morphisms $\psi : \operatorname{Spec}(\mathscr{O}_{X,\bar{a},\nu}) \to U$ induce a canonical homomorphism

$$\varinjlim \Gamma(U, \mathscr{O}_{X_{\mathrm{fppf}}}) \longrightarrow \mathscr{O}_{X,\bar{a},\nu} \tag{15}$$

of rings.

Proposition 6.2. *The preceding homomorphism of rings (15) is bijective.*

Proof. The map in question is surjective: Set $R = \mathscr{O}_{X,\bar{a}}$ and $R_\nu = (\mathscr{O}_{X,\bar{a}})_\nu$. According to Proposition 5.7, every element $c \in R_\nu$ is contained in some R-subalgebra $C \subset R_\nu$ so that the homomorphism $R \to C$ is finite fppf. Write $R = \varinjlim_{i \in I} R_i$ as a filtered direct limit with étale neighborhoods $\operatorname{Spec}(R_i) \to X$ of the geometric point $\bar{a} : \operatorname{Spec}(\Omega) \to X$. For some index $j \in I$, there is a finite fppf R_j-algebra C_j with $C_j \otimes_{R_j} R = C$. Then

$$C = C_j \otimes_{R_j} R = C_j \otimes_{R_j} \varinjlim_{i \geq j} R_i = \varinjlim_{i \geq j} (C_j \otimes_{R_j} R_i).$$

Whence there is some index $i \geq j$ so that $c \in C$ lies in the image $C_i = C \otimes_{R_j} R_i$. Replacing j by i and C by C_i, we thus may assume that $c \in C$ is in the image of of C_j.

Set $U = \operatorname{Spec}(C_j)$ and $V_1 = \operatorname{Spec}(R_j)$, and let $\phi : \operatorname{Spec}(R) \to V_1$ and $\psi : \operatorname{Spec}(R_\nu) \to U$ be the canonical morphisms. The image of the structure map $V_1 \to X$, which is étale, is an open neighborhood of $a \in X$, whence contains some affine open neighborhood $a \in V_0$. Base-changing with V_0, we may assume that

$V_1 \to X$ factors over V_0. Replacing V_1 by an affine open neighborhood of the image of ϕ, we may again assume that V_1 is affine. The upshot is that the tuple $(V_0, V_1, \phi, U, \psi)$ is an object of the index category $I_{\bar{a},\nu}$. By construction, the element $c \in R_\nu$ lies in the image of $\Gamma(U, \mathscr{O}_{X_{\mathrm{fppf}}})$.

The map is injective as well: Suppose we have an object $(V_0, V_1, \phi, U, \psi) \in I_{\bar{a},\nu}$ and some local section $s \in \Gamma(U, \mathscr{O}_{X_{\mathrm{fppf}}}) = \Gamma(U, \mathscr{O}_U)$ whose image in R_ν vanishes. By definition of the index category, the map

$$\Gamma(U, \mathscr{O}_U) \otimes_{\Gamma(V_1, \mathscr{O}_{V_1})} R = \Gamma(U \times_{V_1} \operatorname{Spec} R, \mathscr{O}_{U \times_{V_1} \operatorname{Spec} R}) \longrightarrow R_\nu$$

is injective, whence $s \otimes 1$ vanishes in the left hand side. Now regard $R = \varinjlim R_i$ as a filtered direct limit for étale neighborhoods $\operatorname{Spec}(R_i) \to V_1$ of the induced geometric point $\phi : \operatorname{Spec}(\Omega) \to V_1$. Then it vanishes already in $\Gamma(U, \mathscr{O}_U) \otimes_{\Gamma(V_1, \mathscr{O}_{V_1})} R_i$ for some index $i \in I$. Set $V_1' = \operatorname{Spec}(R_i)$ and $U' = U \otimes_{\Gamma(V_1, \mathscr{O}_{V_1})} R_i$, endowed with the canonical morphisms $\phi' : \operatorname{Spec}(R) \to V_1'$ and $\psi' : \operatorname{Spec}(R_\nu) \to U'$. Then $(V_0, V_1', \phi', U', \psi)$ is an object in $I_{\bar{a},\nu}$, endowed with a canonical morphism to $(V_0, V_1, \phi, U, \psi)$, on which the pullback of s vanishes. □

Theorem 6.3. *Suppose our $\nu = \lambda$ is a limit ordinal. Then there is a topos-theoretical point $P_{\bar{a},\lambda} : (\mathrm{Set}) \to X_{\mathrm{fppf}}$ whose inverse image functor $P^{-1}_{\bar{a},\lambda}$ equals the functor in (14).*

Proof. We apply the criterion given in [2], Expose IV, 6.8.7. Let $W \in (\mathrm{fppf}/X)$ be an object, $(W_\alpha \to W)_{\alpha \in \Lambda}$ an fppf covering, $(V_0, V_1, \phi, U, \psi) \in I_{\bar{a},\lambda}$ an index, and $U \to W$ be a X-morphism. We have to find a larger index $(V_0', V_1', \phi', U', \psi')$, some $\alpha \in \Lambda$ and a morphism $U' \to W_\alpha$ making the diagram

$$\begin{array}{ccc} U' & \longrightarrow & W_\alpha \\ \downarrow & & \downarrow \\ U & \longrightarrow & W \end{array}$$

commutative. To simplify notation, set $R = \mathscr{O}_{X,\bar{a}}$ and $R_\lambda = \mathscr{O}_{X,\bar{a},\lambda}$. Consider the base-changes $\operatorname{Spec}(R_\lambda) \times_W W_\alpha$. Choose some $\alpha \in \Lambda$ so that the closed point in the local scheme $\operatorname{Spec}(R_\lambda)$ is in the image of the projection. By flatness, the projection is thus surjective. According to Theorem 4.9, there is a morphism $\operatorname{Spec}(\mathscr{O}_{X,\bar{a},\lambda}) \to W_\alpha$ making the diagram

$$\begin{tikzcd} & & W_\alpha \arrow[d] \\ \operatorname{Spec}(\mathscr{O}_{X,\bar{a},\lambda}) \arrow[r] \arrow[urr] & U \arrow[r] & W \end{tikzcd}$$

commute. Using Proposition 6.2, together with [15], Theorem 8.8.2, we conclude that there is some morphism $(V_0', V_1', \phi', U', \psi') \to (V_0, V_1, \phi, U, \psi)$ and a W-morphism $U' \to W_\alpha$, as desired. □

Let $a \in X$ be the image of the geometric point $\bar{a}$. If $\bar{b}$ is another geometric point on X whose image points b equals a, there is a $\kappa(a)$-isomorphism $\kappa(\bar{a}) \to \kappa(\bar{b})$, which comes from a unique isomorphism of strictly local rings $\mathscr{O}_{X,\bar{a}} \to \mathscr{O}_{X,\bar{b}}$. By functoriality, it extends to an isomorphism $\mathscr{O}_{X,\bar{a},\nu} \to \mathscr{O}_{X,\bar{b},\nu}$, which finally yields an isomorphism of inverse systems

$$(V_0, V_1, \phi, U, \psi) \longmapsto (V_0, V_1, \phi f, U, \psi f).$$

We conclude that the isomorphism class of the topos-theoretical point $P_{\bar{a},\lambda} \in \mathrm{Points}(X_{\mathrm{fppf}})$ only depends on the image point $a \in X$, and we write this isomorphism class as $P_{a,\lambda} \in |X_{\mathrm{fppf}}|$.

Theorem 6.4. *Let $\lambda < \sigma$ be a limit ordinal. Then the map*

$$X = |X_{\mathrm{Zar}}| \longrightarrow |X_{\mathrm{fppf}}|, \quad a \longmapsto P_{a,\lambda}$$

is section for the canonical projection $|X_{\mathrm{fppf}}| \to |X_{\mathrm{Zar}}| = X$.

Proof. Suppose that $V \subset X$ is an open subscheme that is a neighborhood of the topos-theoretical point $P_{\bar{a},\lambda}$. Then there is an index $(V_0, V_1, \psi, U, \phi) \in I_{\bar{a},\lambda}$ having an X-morphism $U \to V$. By the commutative diagram (10), the diagram

$$\begin{array}{ccc} \mathrm{Spec}(\mathscr{O}_{X,\bar{a},\lambda}) & \longrightarrow & \mathrm{Spec}(\mathscr{O}_{X,a}) \\ {\scriptstyle\phi}\downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

is commutative as well. In turn, we have $a \in V$. According to [2], Expose IV, Section 7.1 there is a unique point $a' \in X$ so that the open subschemes of X that are neighborhoods of $P_{\bar{a},\lambda}$ are neighborhoods of a'. It follows that $a' = a$. Hence $a \mapsto P_{\bar{a},\lambda}$ is a section. □

In light of Proposition 3.2 and Lemma 2.5, the existence of a section now yields our main result:

Theorem 6.5. *The continuous map $|X_{\mathrm{fppf}}| \to |X_{\mathrm{Zar}}|$ induces an identification $|X_{\mathrm{fppf}}|_{\mathrm{sob}} = |X_{\mathrm{Zar}}|_{\mathrm{sob}} = X$ of sober spaces.*

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