HOMOGENEOUS COORDINATES AND QUOTIENT PRESENTATIONS FOR TORIC VARIETIES

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ABSTRACT. Generalizing cones over projective toric varieties, we present arbitrary toric varieties as quotients of quasiaffine toric varieties. Such quotient presentations correspond to groups of Weil divisors generating the topology. Groups comprising Cartier divisors define free quotients, whereas $\mathbb{Q}$-Cartier divisors define geometric quotients. Each quotient presentation yields homogeneous coordinates. Using homogeneous coordinates, we express quasicoherent sheaves in terms of multigraded modules and describe the set of morphisms into a toric variety.

1. INTRODUCTION

The projective space $\mathbb{P}^n$ is the quotient of the pointed affine space $\mathbb{A}^{n+1} \setminus 0$ by the diagonal $\mathbb{G}_m$-action. A natural question to ask is whether this generalizes to other toric varieties. Indeed: Cox [4] and others showed that each toric variety X is the quotient of a smooth quasiaffine toric variety $\hat{X}$.

This quasiaffine toric variety $\hat{X}$ and the corresponding homogeneous coordinate ring $\Gamma(\hat{X}, \mathcal{O}_{\hat{X}})$, however, are very large and entail redundant information. For toric varieties with enough invariant Cartier divisors, Kajiwara [7] found smaller homogeneous coordinate rings.

The goal of this paper is to generalize homogeneous coordinates and to study them from a geometric viewpoint. In our language, homogeneous coordinates correspond to quotient presentations. Both the constructions of Cox and Kajiwara are quotient presentations; other examples are cones over quasiprojective toric varieties. Given any particular toric variety, our approach provides flexibility in the choice of homogeneous coordinate rings.

Roughly speaking, a *quotient presentation* for a toric variety X is a quasiaffine toric variety $\hat{X}$, together with an affine surjective toric morphism $q \colon \hat{X} \to X$ such that groups of invariant Weil divisors on X and $\hat{X}$ coincide. The global sections $S = \Gamma(\hat{X}, \mathcal{O}_{\hat{X}})$ are the corresponding *homogeneous coordinates* for X.

Homogeneous coordinates are useful for various purposes. For example, Cox [3] described the set of morphism $r \colon Y \to X$ from a scheme Y into a smooth toric variety X in terms of homogeneous coordinates. Subsequently, Kajiwara [7] generalized this to toric varieties with enough effective Cartier divisors. Using homogeneous coordinates, Brion and Vergne [2] determined Todd classes on simplicial toric varieties. Eisenbud, Mustata and Stillman [5] recently applied homogeneous coordinates to calculate cohomology groups of coherent sheaves.

This article is divided into five sections. In the first section, we define the concept of quotient presentations and give a characterization in terms of fans. Section 2

contains a description of quotient presentations in terms of groups of Weil divisors. Such groups of Weil divisors are not arbitrary. Rather, they generalize the concept of an ample invertible sheaf or an ample family of sheaves.

In Section 3, we relate quotient presentations to geometric invariant theory. Quotient presentations defined by Cartier or $\mathbb{Q}$-Cartier divisors are free or geometric quotients, respectively. Because quotients for group actions tend to be nonseparated, it is natural (and requires no extra effort) to consider nonseparated toric prevarieties as well.

In Section 4, we shall express quasicoherent sheaves on toric varieties in terms of multigraded modules over homogeneous coordinate rings. In the last section, we describe the functor $h_X(Y) = \mathrm{Hom}(Y, X)$ represented by a toric variety X in terms of sheaf data on Y related to homogeneous coordinates.

2. Quotient presentations

Throughout we shall work over an arbitrary ground field k. A *toric variety* is an equivariant torus embedding $T \subset X$, where X is a separated normal algebraic k-variety. As usual, N denotes the lattice of 1-parameter subgroups of the torus T, and M is the dual lattice of characters. Recall that toric varieties correspond to finite fans Δ in the lattice N. We shall encounter *toric prevarieties* as well: These are equivariant torus embeddings as above, but with X possibly nonseparated.

Let $q \colon \hat{X} \to X$ be a surjective toric morphism of toric prevarieties. Then we have a pullback homomorphism $q^* \colon \mathrm{CDiv}^T(X) \to \mathrm{CDiv}^{\hat{T}}(\hat{X})$ for invariant Cartier divisors. There is also a strict transform for invariant Weil divisors defined as follows. Let $U \subset X$ be the union of all T-orbits of codimension ≤ 1, and $\hat{U} \subset \hat{X}$ its preimage. Each invariant Weil divisor on X becomes Cartier on U, and the composition

$$\mathrm{WDiv}^T(X) = \mathrm{CDiv}^T(U) \xrightarrow{q^*} \mathrm{CDiv}^{\hat{T}}(\hat{U}) \subset \mathrm{WDiv}^{\hat{T}}(\hat{U}) \subset \mathrm{WDiv}^{\hat{T}}(\hat{X})$$

defines the *strict transform* $q^\sharp \colon \mathrm{WDiv}^T(X) \to \mathrm{WDiv}^{\hat{T}}(\hat{X})$ on the groups of invariant Weil divisors. Note that $q^\sharp$ is injective.

Definition 2.1. A *quotient presentation* for a toric prevariety X is a quasiaffine toric variety $\hat{X}$, together with a surjective affine toric morphism $q \colon \hat{X} \to X$ such that the strict transform $q^\sharp \colon \mathrm{WDiv}^T(X) \to \mathrm{WDiv}^{\hat{T}}(\hat{X})$ is bijective.

This notion is local: Given that $\hat{X}$ is quasiaffine, a toric morphism $q \colon \hat{X} \to X$ is a quotient presentation if and only if for each invariant affine open subset $U \subset X$ the induced toric morphism $q^{-1}(U) \to U$ is a quotient presentation.

Example 2.2. The cones $\mathbb{R}_+(1, 0)$, and $\mathbb{R}_+(0, 1)$ in the lattice $\hat{N} = \mathbb{Z}^2$ define the quasiaffine toric variety $\hat{X} = \mathbb{A}^2 \setminus 0$. The projection $\mathbb{Z}^2 \to \mathbb{Z}^2/\mathbb{Z}(1, 1)$ yields a quotient presentation $q \colon \mathbb{A}^2 \setminus 0 \to X$ for the projective line $X = \mathbb{P}^1$. We could use the projection onto $\mathbb{Z}^2/\mathbb{Z}(1, -1)$ as well. This defines a quotient presentation $q \colon \mathbb{A}^2 \setminus 0 \to X$ for the affine line $X = \mathbb{A}^1 \cup \mathbb{A}^1$ with origin doubled, which is a nonseparated toric prevariety.

Here comes a characterization of quotient presentations in terms of fans. For simplicity, we are content with the separated case. Suppose that $q \colon \hat{X} \to X$ is a toric morphism of toric varieties given by a map of fans $Q \colon (\hat{N}, \hat{\Delta}) \to (N, \Delta)$.

Theorem 2.3. *The toric morphism $q\colon \hat{X} \to X$ is a quotient presentation if and only if the following conditions hold:*

   (i) *The lattice homomorphism $Q\colon \hat{N} \to N$ has finite cokernel.*

   (ii) *The fan $\hat{\Delta}$ is a subfan of the fan of faces of a strongly convex cone $\bar{\sigma} \subset \hat{N}_{\mathbb{R}}$.*

   (iii) *The assignment $\sigma \mapsto Q_{\mathbb{R}}(\sigma)$ defines bijections $\hat{\Delta}^{\max} \to \Delta^{\max}$ and $\hat{\Delta}^{(1)} \to \Delta^{(1)}$.*

   (iv) *For each primitive lattice vector $\hat{v} \in \hat{N}$ generating a ray $\hat{\rho} \in \hat{\Delta}$, the image $Q(\hat{v}) \in N$ is a primitive lattice vector.*

Proof. Suppose the conditions hold. The cone $\bar{\sigma} \subset \hat{N}_{\mathbb{R}}$ yields a toric open embedding $\hat{X} \subset X_{\bar{\sigma}}$, hence $\hat{X}$ is quasiaffine.

To see that the map $q\colon \hat{X} \to X$ is surjective, consider an affine chart $X_{\sigma} \subset X$, where $\sigma \in \Delta$ is a maximal cone. Since Q induces a bijection of maximal cones, there is a $\hat{\sigma} \in \hat{\Delta}^{\max}$ such that $Q_{\mathbb{R}}(\hat{\sigma}) = \sigma$. Moreover, Q was assumed to have a finite cokernel, so $q\colon \hat{T} \to T$ is surjective. Since q is equivariant, this implies $X_{\sigma} = q(X_{\hat{\sigma}})$.

To check that the map $q\colon \hat{X} \to X$ is affine, keep on considering X_{σ}. It is easy to see that the inverse image of X_{σ} is

$$(2.3.1) \qquad q^{-1}(X_{\sigma}) = \bigcup_{\hat{\tau} \in \hat{\Delta};\ Q_{\mathbb{R}}(\hat{\tau}) \subset \sigma} X_{\hat{\tau}}.$$

Using the bijection $\hat{\Delta}^{(1)} \to \Delta^{(1)}$ we see that $Q_{\mathbb{R}}^{-1}(\sigma)$ contains no element of $\hat{\Delta}^{(1)} \setminus \hat{\sigma}^{(1)}$. Consequently, the only cones of $\hat{\Delta}$ mapped by $Q_{\mathbb{R}}$ into σ are the faces of $\hat{\sigma}$. By the above formula, this means $q^{-1}(X_{\sigma}) = X_{\hat{\sigma}}$. So we see that $q\colon \hat{X} \to X$ is affine.

It remains to show that the strict transform is bijective. As to this, recall first that the invariant prime divisors of X are precisely the closures of the T-orbits $\operatorname{Spec} k[\rho^{\perp} \cap M] \subset X$ where $\rho \in \Delta^{(1)}$.

We calculate the strict transform of a T-stable prime divisor $D \subset X$ corresponding to a ray $\rho \in \Delta^{(1)}$. Since $\hat{\Delta}^{(1)} \to \Delta^{(1)}$ is bijective, there is a unique ray $\hat{\rho} \in \hat{\Delta}^{(1)}$ with $Q_{\mathbb{R}}(\hat{\rho}) = \rho$. It follows from 2.3.1 that $q^{\sharp}(D)$ is a multiple of the $\hat{T}$-invariant prime divisor $\hat{D}$ corresponding to $\hat{\rho}$. Note that $q^{-1}(X_{\rho}) = X_{\hat{\rho}}$. To calculate the multiplicity of $\hat{D}$ in $q^{\sharp}(D)$, it suffices to determine the pullback of $D \in \operatorname{CDiv}^{T}(X_{\rho})$ via $q\colon X_{\hat{\rho}} \to X_{\rho}$.

On the affine chart X_{ρ}, every invariant Cartier divisor is principal, and if v is the primitive lattice vector in ρ then the assignment $m \mapsto \langle m, v \rangle D$ induces a natural isomorphism $M/\rho^{\perp} \simeq \operatorname{CDiv}^{T}(X_{\rho})$. Since we have

$$q^{*}(\operatorname{div}(\chi^{m})) = \operatorname{div}(\chi^{m} \circ q) = \operatorname{div}(\chi^{m \circ Q}),$$

the pullback $q^{*}\colon \operatorname{CDiv}^{T}(X_{\rho}) \to \operatorname{CDiv}^{\hat{T}}(X_{\hat{\rho}})$ corresponds to the map $Q^{*}\colon M/\rho^{\perp} \to \hat{M}/\hat{\rho}^{\perp}$. By condition (iv), this map is an isomorphism and hence $q^{*}(D) = \hat{D}$.

Again using bijectivity of $\hat{\Delta}^{(1)} \to \Delta^{1}$, you conclude that the strict transform is bijective. Thus the conditions are sufficient. Using similar arguments, you see that the conditions are also necessary. $\qquad\square$

Example 2.4. Suppose $\sigma \subset \mathbb{R}^{3}$ be a strongly convex cone generated by four extremal rays $\mathbb{R}_{+}v_{1}, \ldots, \mathbb{R}_{+}v_{4}$, defining a fan Δ in $N = \mathbb{Z}^{3}$. Let $\hat{\Delta}$ be the fan of all faces of the first quadrant in $\hat{N} = \mathbb{Z}v_{1} \oplus \ldots \oplus \mathbb{Z}v_{4}$. Then the canonical surjection $Q\colon \hat{N} \to N$ gives a quotient presentation.
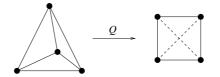
FIGURE 1. A quotient presentation of a non simplicial affine toric variety

The induced map on nonzero cones looks like Figure 1. The fan $\hat{\Delta}$ comprises 16 cones, whereas Δ contains only 10 cones. You see that two 2-dimensional cones and all 3-dimensional cones in $\hat{\Delta}$ map to the maximal cone in Δ.

Example 2.5. Let Δ be a polytopal fan in the lattice N, and for each ray $\rho \in \Delta^{(1)}$ let $v_\rho \in \rho$ be the primitive lattice vector. Consider polytopes $P \subset N_{\mathbb{R}}$ having edges $w_\rho = n_\rho^{-1} v_\rho$ with $n_\rho \in N$, where all $\rho \in \Delta^{(1)}$ occur. Each such polytope defines a quotient presentation of the projective toric variety X associated to Δ:

Set $\hat{N} = N \oplus \mathbb{Z}$. Let $\bar{\sigma} \subset \hat{N}_{\mathbb{R}}$ be the cone generated by $P \times (0,1)$, and $\hat{\Delta}$ the fan of all strict faces $\hat{\sigma} \subsetneq \bar{\sigma}$. Then the canonical projection $Q \colon \hat{N} \to N$ defines a quotient presentation $q \colon \hat{X} \to X$. In fact, these quotient presentations are precisely those obtained from affine cones over X. A typical picture is Figure 2.
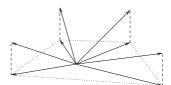

FIGURE 2. A quotient presentation of a projective toric surface

3. Enough effective Weil divisors

The goal of this section is to describe, up to isomorphism, the set of all quotient presentations of a fixed toric prevariety X. Recall that we have a canonical map

$$\text{div} \colon M \longrightarrow \text{WDiv}^T(X), \quad m \mapsto \text{div}(\chi^m),$$

where $\chi^m \in \Gamma(T, \mathcal{O}_X)$ is the character function corresponding to $m \in M$. Suppose $q \colon \hat{X} \to X$ is a quotient presentation. The inverse $q_* \colon \text{WDiv}^{\hat{T}}(\hat{X}) \to \text{WDiv}^T(X)$ of the strict transform yields a factorization

$$M \longrightarrow \hat{M} \longrightarrow \text{WDiv}^T(X)$$

of the canonical map $\text{div} \colon M \to \text{WDiv}^T(X)$. We seek to reconstruct the quotient presentation from such sequences.

Definition 3.1. A *triangle* is an abstract lattice $\hat{M}$, together with a sequence $M \to \hat{M} \to \text{WDiv}^T(X)$, such that the following holds: The composition is the canonical map $\text{div} \colon M \to \text{WDiv}^T(X)$, the map $M \to \hat{M}$ is injective, and for each invariant affine open subset $U \subset X$ there is an $\hat{m} \in \hat{M}$ whose image $D \in \text{WDiv}^T(X)$ is effective with support $X \setminus U$.

Roughly speaking, the image of $\hat{M} \to \mathrm{WDiv}^T(X)$ contains enough Weil divisors, such that it generates the topology of X. Recall that a scheme Y is separated if the diagonal morphism $Y \to Y \times Y$ is a closed embedding. We say that Y is of *affine intersection* if the diagonal is an affine morphism. In other words, there is an affine open covering $U_i \subset Y$ such that the $U_i \cap U_j$ are affine.

Theorem 3.2. *Let X be a toric prevariety of affine intersection. For each quotient presentation $q \colon \hat{X} \to X$, the corresponding sequence $M \to \hat{M} \to \mathrm{WDiv}^T(X)$ is a triangle. Up to isomorphism, this assignment yields a bijection between quotient presentations and triangles.*

Proof. Suppose $q \colon \hat{X} \to X$ is a quotient presentation. Given an invariant affine open subset $U \subset X$, the preimage $\hat{U} \subset \hat{X}$ is affine as well. There is an effective invariant principal divisor $\hat{D} \subset \hat{X}$ with support $\hat{X} \setminus \hat{U}$, because $\hat{X}$ is quasiaffine. So $D = q_*(\hat{D})$ is an effective Weil divisor with support $X \setminus U$. By construction, $D \in \mathrm{WDiv}^T(X)$ lies in the image of $\hat{M}$.

Conversely, suppose that $M \longrightarrow \hat{M} \xrightarrow{\phi} \mathrm{WDiv}^T(X)$ is a triangle. Set for short $\tilde{M} := \mathrm{WDiv}^T(X)$. Let $\tilde{N}$ and $\hat{N}$ denote the dual lattices of $\tilde{M}$ and $\hat{M}$ respectively. Dualizing the triangle, we obtain a sequence

$$\tilde{N} \xrightarrow{\psi} \hat{N} \xrightarrow{Q} N.$$

For each prime divisor $E \in \tilde{M}$, let $E^* \in \tilde{N}$ denote the dual base vector. For every invariant open set $U \subset X$ we have the submonoid $\tilde{N}_+(U)$ generated by the E^*, where $E \in \mathrm{WDiv}^T(U)$ is a prime divisor. Let $\hat{\sigma}_U \subset \hat{N}_{\mathbb{R}}$ be the cone generated by $\psi(\tilde{N}_+(U))$. For example, $\hat{\sigma}_T = \{0\}$.

We claim that $\hat{\sigma}_U \subset \hat{\sigma}_X$ is a face provided that $U \subset X$ is an *affine* invariant open subset. Indeed by assumption, there is an $\hat{m} \in \hat{M}$ such that $D = \phi(\hat{m})$ is an effective Weil divisor with support $X \setminus U$. So for each prime divisor $E \in \tilde{M}$, we have

$$\langle \psi(E^*), \hat{m} \rangle = \langle E^*, \phi(\hat{m}) \rangle = \langle E^*, D \rangle \geq 0,$$

with equality if and only if $E^* \in \tilde{N}_+(U)$. So $\hat{m}$ is a supporting hyperplane for $\hat{\sigma}_X$ cutting out $\hat{\sigma}_U$ and the claim is verified. In particular, since $\hat{\sigma}_T = \{0\}$ is a face of $\hat{\sigma}_X$, this cone is strictly convex.

For later use, let us also calculate $Q(\psi(E^*))$. If v_ρ denotes the primitive lattice vector in the ray ρ corresponding to the divisor $E \in \tilde{M}$, we have

$$\langle Q(\psi(E^*)), m \rangle = \langle E^*, \mathrm{div}\chi^m \rangle = \langle v_\rho, m \rangle.$$

That implies $Q(\psi(E^*)) = v_\rho$. So in particular, $\psi(E^*)$ is a primitive lattice vector, and ψ induces a bijection between the rays of $\tilde{N}_+(X)$ and $\hat{\sigma}_X$.

Let $\hat{\Delta}$ be the fan in $\hat{N}$ generated by the faces $\hat{\sigma}_U$, where $U \subset X$ ranges over all invariant affine open subsets. By construction, this defines a quasiaffine toric variety $\hat{X}$.

It remains to construct the quotient presentation $q \colon \hat{X} \to X$. First, we do this locally over an invariant affine open subset $U \subset X$. Let $\sigma_U \subset N_{\mathbb{R}}$ be the corresponding cone, and let $\hat{U} \subset \hat{X}$ be the affine open subset defined by $\hat{\sigma}_U$.

Clearly, the map $Q \colon \hat{N} \to N$ has a finite cokernel, since $M \to \hat{M}$ was is assumed to be injective. We have $Q(\hat{\sigma}_U) = \sigma_U$. Moreover, it follows from we saw above that the map $Q \circ \psi$ induces a bijection between the sets of primitive lattice vectors

generating the rays of $\tilde{N}_+(U)$ and σ_U. Therefore the induced map $v_{\hat{\rho}} \to Q(v_{\hat{\rho}})$ gives a bijection between the primitive lattice vectors generating the rays in $\hat{\sigma}_U$ and σ_U.

By Proposition 2.3, the associated toric morphism $\hat{U} \to U$ is a quotient presentation. To obtain the desired quotient presentation $q\colon \hat{X} \to X$, we glue the local patches. Let $U_1, U_2 \subset X$ be two affine charts. The intersection $U := U_1 \cap U_2$ is affine, and the rays of σ_U are in bijection with the invariant prime divisors in $U_1 \cap U_2$. On the other hand, the rays of $\hat{\sigma}_1 \cap \hat{\sigma}_2$ are the images of the duals to the prime divisors in $\mathrm{WDiv}^T(U_1 \cap U_2)$. This implies that $Q(\hat{\sigma}_1 \cap \hat{\sigma}_2) = \sigma_U$. $\qquad\square$

For the following examples, assume that X is a toric variety without nontrivial torus factor. Equivalently, the map $M \to \mathrm{WDiv}^T(X)$ is injective. Such toric varieties are called *nondegenerate*.

Example 3.3. Obviously, the factorization $M \to \mathrm{WDiv}^T(X) \overset{\mathrm{id}}{\to} \mathrm{WDiv}^T(X)$ is a triangle. The corresponding quotient presentation was introduced by Cox [4]. It is the largest quotient presentation in the sense that it dominates all other nondegenerate quotient presentations of X.

Example 3.4. Suppose that for each invariant affine open subset $U \subset X$, the complement $X \setminus U$ is the support of an effective Cartier divisor. Then the factorization $M \to \mathrm{CDiv}^T(X) \to \mathrm{WDiv}^T(X)$ is a triangle. The corresponding quotient presentation $q\colon \hat{X} \to X$ was studied by Kajiwara [7]. He says that X has *enough Cartier divisors*. Note that such toric varieties are *divisorial* schemes in the sense of Borelli [1].

Example 3.5. Suppose X is a quasiprojective toric variety. Choose an ample Cartier divisor $D \in \mathrm{WDiv}^T(X)$. Then $M \to M \oplus \mathbb{Z}D \to \mathrm{WDiv}^T(X)$ is a triangle. The corresponding quotient presentation $q\colon \hat{X} \to X$ is nothing but the $\mathbb{G}_m$-bundle obtained from the vector bundle $L \to X$ associated to the ample sheaf $\mathcal{O}_X(D)$.

Next, we come to existence of quotient presentations:

Proposition 3.6. *A toric prevariety admits a quotient presentation if and only if it is of affine intersection.*

Proof. Suppose $q\colon \hat{X} \to X$ is a quotient presentation and consider two invariant affine charts X_1, X_2 of X. Since q is an affine toric morphism, the preimages $\hat{X}_i := q^{-1}(X_i)$ are invariant affine charts of $\hat{X}$.

The restriction of q defines a quotient presentation $\hat{X}_1 \cap \hat{X}_2 \to X_1 \cap X_2$. Since $\hat{X}$ is separated, the intersection $\hat{X}_1 \cap \hat{X}_2$ is even affine. Property 2.3 (iii) implies that the image $X_1 \cap X_2 = q(\hat{X}_1 \cap \hat{X}_2)$ is again an affine toric variety.

Conversely, let X be of affine intersection. Choose a splitting $M = M' \oplus M''$, where $M' \subset M$ is the kernel of the canonical map $M \to \mathrm{WDiv}^T(X)$. It suffices to show that the canonical factorization

$$M \longrightarrow M' \oplus \mathrm{WDiv}^T(X) \longrightarrow \mathrm{WDiv}^T(X)$$

is a triangle. Let $U \subset X$ be an invariant affine open subset. We have to check that the complement $D = X \setminus U$ is a Weil divisor. For each invariant affine open subset $V \subset X$, the intersection $U \cap V$ is affine, so $V \cap D$ is a Weil divisor. Hence D is a Weil divisor. $\qquad\square$

4. Free and geometric quotient presentations

In this section we shall relate quotient presentations to geometric invariant theory. Fix a toric prevariety X, together with a quotient presentation $q\colon \hat{X} \to X$ defined by a triangle $M \to \hat{M} \to \mathrm{WDiv}^T(X)$. Let $G \subset \hat{T}$ be the kernel of the induced homomorphism $\hat{T} \to T$ of tori. The question is: In what sense is X a quotient of the G-action on $\hat{X}$?

Note that $G = \mathrm{Spec}\, k[W]$, such that $W = \hat{M}/M$ is the character group of the group scheme G. Such group schemes are called *diagonalizable*. The G-action on $\hat{X}$ corresponds to a W-grading on

$$q_*(\mathcal{O}_{\hat{X}}) = \mathcal{R} = \bigoplus_{w \in W} \mathcal{R}_w$$

for certain coherent $\mathcal{O}_X$-modules $\mathcal{R}_w$. We call them the *weight modules* of the quotient presentation. To describe the weight modules, consider the commutative diagram

$$
\begin{array}{ccccccc}
M & \longrightarrow & \mathrm{WDiv}^T(X) & \longrightarrow & \mathrm{Cl}(X) & \longrightarrow & 0 \\
{\scriptstyle Q^*}\downarrow & & {\scriptstyle q^\sharp}\downarrow{\scriptstyle \simeq} & & {\scriptstyle q^\sharp}\downarrow & & \\
\hat{M} & \longrightarrow & \mathrm{WDiv}^{\hat{T}}(\hat{X}) & \longrightarrow & \mathrm{Cl}(\hat{X}) & \longrightarrow & 0.
\end{array}
$$

The snake lemma yields a map $W \to \mathrm{Cl}(X)$. Hence each character $w \in W$ gives an isomorphism class of invariant reflexive fractional ideals:

Lemma 4.1. *Each weight module $\mathcal{R}_w$ is an invariant reflexive fractional ideal. The isomorphism class $[\mathcal{R}_w] \in \mathrm{Cl}(X)$ is the image of $-w$.*

Proof. First, suppose that the quotient presentation $q\colon \hat{X} \to X$ is defined by an inclusion of rings $k[\sigma^\vee \cap M] \subset k[\hat{\sigma}^\vee \cap \hat{M}]$. The weight module $\mathcal{R}_w \subset \mathcal{R}$ is given by the homogeneous component $R_w \subset k[\hat{\sigma}^\vee \cap \hat{M}]$ of degree $w \in W$.

Let $v_\rho \in N$ and $v_{\hat{\rho}} \in \hat{N}$ be the primitive lattice vectors generating the rays in $\sigma^{(1)}$ and $\hat{\sigma}^{(1)}$, respectively. Choose $\hat{m} \in \hat{M}$ representing $w \in W$. Note that the $\hat{T}$-invariant Weil divisor $q_*(\mathrm{div}(\chi^{\hat{m}}))$ on $\hat{X}$ is given by the function

$$\hat{m}\colon \sigma^{(1)} \longrightarrow \mathbb{Z}, \quad \rho \mapsto \langle \hat{m}, v_{\hat{\rho}} \rangle.$$

The reflexive fractional ideal $R \subset k(X)$ over the ring $k[\sigma^\vee \cap M]$ corresponding to $-[w] \in \mathrm{Cl}(X)$ is generated by the monomials $\chi^m \in k[M]$ with $m \geq -\hat{m}$ as functions on $\sigma^{(1)}$. Obviously, the map $\chi^m \mapsto \chi^{Q^*(m)+\hat{m}}$ induces the desired bijection $R \to R_w$. This is compatible with localization, hence globalizes. $\square$

Suppose a diagonalizable group scheme G acts on a scheme Y. An invariant affine morphism $f\colon Y \to Z$ with $\mathcal{O}_Z = f_*(\mathcal{O}_Y)^G$ is called a *good quotient*. Note that this implies $f(\bigcap W_i) = \bigcap f(W_i)$ for each family of invariant closed subsets $W_i \subset Y$. Moreover, $f\colon Y \to Z$ is a categorial quotient.

Proposition 4.2. *Each quotient presentation $q\colon \hat{X} \to X$ is a good quotient for the G-action on $\hat{X}$.*

Proof. The problem is local, so we can assume that $q\colon \hat{X} \to X$ is given by an inclusion of rings $k[\sigma^\vee \cap M] \subset k[\hat{\sigma}^\vee \cap \hat{M}]$. By Lemma 4.1, the ring of invariants $k[\hat{\sigma}^\vee \cap \hat{M}]^G$ is nothing but $k[\sigma^\vee \cap M]$. $\square$

Sometimes we can do even better. Suppose a diagonalizable group scheme G acts on a scheme Y. An invariant morphism $f\colon Y \to Z$ such that the corresponding morphism $G \times_Z Y \to Y \times_Z Y$, $(g, y) \mapsto (gy, y)$ is an isomorphism is called a *principal homogeneous G-space*. Equivalently, the projection $Y \to Z$ is a principal G-bundle in the flat topology ([8] III Prop. 4.1).

Proposition 4.3. *The quotient presentation $q\colon \hat{X} \to X$ is a principal homogeneous G-space if and only if $\hat{M} \to \mathrm{WDiv}^T(X)$ factors through the group of invariant Cartier divisors.*

Proof. Suppose that $\hat{M}$ maps to $\mathrm{CDiv}^T(X)$. According to Lemma 4.1, the homogeneous components in $q_*(\mathcal{O}_{\hat{X}}) = \bigoplus_{w \in W} \mathcal{R}_w$ are invertible. You easily check that the multiplication maps $\mathcal{R}_w \otimes \mathcal{R}_{w'} \to \mathcal{R}_{w+w'}$ are bijective. So by [6] Proposition 4.1, the quotient presentation $\hat{X} \to X$ is a principal homogeneous G-space. Hence the condition is sufficient. Reversing the arguments, you see that the condition is necessary as well. $\square$

Example 4.4. Regular toric prevarieties are factorial, hence their quotient presentations are principal homogeneous spaces. Consequently, an arbitrary quotient presentation is a principal homogeneous space in codimension 1.

For the next result, let us recall another concept from geometric invariant theory. Suppose a diagonalizable group scheme G acts on a scheme Y. A good quotient $Y \to Z$ is called a *geometric quotient* if it separates the G-orbits.

Proposition 4.5. *Suppose $q\colon \hat{X} \to X$ is a quotient presentation. Then X is a geometric quotient for the G-action on $\hat{X}$ if and only if $\hat{M} \to \mathrm{WDiv}^T(X)$ factors through the group of invariant $\mathbb{Q}$-Cartier divisors.*

Proof. First, we check sufficiency. Let $\hat{M}' \subset \hat{M}$ be the preimage of the subgroup $\mathrm{CDiv}^T(X) \subset \mathrm{WDiv}^T(X)$. The group scheme $H = \mathrm{Spec}\, k[\hat{M}/\hat{M}']$ is finite, so its action on $\hat{X}$ is automatically closed. Consequently, the quotient $\hat{X}' = \hat{X}/H$ is a geometric quotient. You directly see that $\hat{X}'$ is quasiaffine. Consider the induced toric morphism $q'\colon \hat{X}' \to X$. The strict transforms in

$$\mathrm{WDiv}^T(X) \longrightarrow \mathrm{WDiv}^{\hat{T}'}(\hat{X}') \xrightarrow{(q')^\sharp} \mathrm{WDiv}^{\hat{T}}(\hat{X})$$

are injective, and their composition is bijective. So the map on the right is bijective, hence $q'\colon \hat{X}' \to X$ is another quotient presentation. By construction, its triangle $M \to \hat{M}' \to \mathrm{WDiv}^T(X)$ factors through $\mathrm{CDiv}^T(X)$. According to Proposition 4.3, it is a geometric quotient. So $q\colon \hat{X} \to X$ is the composition of two geometric quotients, hence a geometric quotient.

The condition is also necessary. Suppose X is a geometric quotient, that means the fibers $q^{-1}(x)$ are precisely the G-orbits. By definition, G acts freely on $\hat{T}$. By semicontinuity of the fiber dimension, the stabilizers $G_{\hat{x}} \subset G$ for $\hat{x} \in \hat{X}$ must be finite. Note that the stabilizers are constant along the $\hat{T}$-orbits. Hence the stabilizers generate a finite subgroup $H \subset G$.

Set $\hat{X}' = \hat{X}/H$. As above, we obtain a quotient presentation $q'\colon \hat{X}' \to X$. By construction, X is a free geometric quotient for the action of $G' = G/H$. Now [9], Proposition 0.9, ensures that $q'\colon \hat{X}' \to X$ is a principal homogeneous G'-space. By Proposition 4.3, the triangle $M \to \hat{M}' \to \mathrm{WDiv}^T(X)$ factors through $\mathrm{CDiv}^T(X)$.

This implies that $\hat{M} \to \mathrm{WDiv}^T(X)$ factors through the group of invariant $\mathbb{Q}$-Cartier divisors. $\square$

Example 4.6. Simplicial toric varieties are $\mathbb{Q}$-factorial, hence their quotient presentations are geometric quotients. It follows that arbitrary quotient presentations are geometric quotients in codimension 2.

5. HOMOGENEOUS COORDINATES AND MULTIGRADED MODULES

Throughout this section, fix a toric prevariety X of affine intersection and choose a quotient presentation $q\colon \hat{X} \to X$. The goal of this section is to relate quasicoherent $\mathcal{O}_X$-modules to multigraded modules over homogeneous coordinate rings. This generalizes the classical approach for $X = \mathbb{P}^n$, and results of Cox [4] and Kajiwara [7] as well.

We propose the following definition of homogeneous coordinates. By assumption, the toric variety $\hat{X}$ is quasiaffine, so the affine hull $\bar{X} = \mathrm{Spec}\,\Gamma(\hat{X}, \mathcal{O}_{\hat{X}})$ is an affine toric variety. We call the ring $S = \Gamma(\hat{X}, \mathcal{O}_{\hat{X}})$ the *homogeneous coordinate ring* with respect to the quotient presentation $q\colon \hat{X} \to X$. Let $M \subset \hat{M} \to \mathrm{WDiv}^T(X)$ be its triangle and set $W = \hat{M}/M$. The action of the diagonalizable group scheme $G = \mathrm{Spec}\,k[W]$ on $\hat{X}$ induces a G-action on the affine hull $\bar{X}$, which corresponds to a W-grading $S = \bigoplus S_w$.

Suppose F is a W-graded S-module. Then F corresponds to a quasicoherent G-linearized $\mathcal{O}_{\bar{X}}$-module $\mathcal{M}$. Let $i\colon \hat{X} \to \bar{X}$ be the open inclusion. The restriction $i^*(\mathcal{M})$ is a G-linearized quasicoherent $\mathcal{O}_{\hat{X}}$-module. Because $q\colon \hat{X} \to X$ is affine, this corresponds to a W-grading on

$$q_*(i^*(\mathcal{M})) = \bigoplus_{w \in W} q_*(i^*(\mathcal{M}))_w.$$

Definition 5.1. The sheaf $\tilde{F} = q_*(i^*(\mathcal{M}))_0$ is called the *associated $\mathcal{O}_X$-module* for the W-graded S-module F.

For example, the $\mathcal{O}_X$-module associated to S is nothing but $\tilde{S} = \mathcal{O}_X$. Clearly, $F \mapsto \tilde{F}$ is an exact functor from the category of W-graded S-modules to the category of quasicoherent $\mathcal{O}_X$-modules. You easily check that the functor commutes with direct limits and sends finitely generated modules to coherent sheaves.

We can pass from quasicoherent sheaves to graded modules as well. Suppose $\mathcal{F}$ is a quasicoherent $\mathcal{O}_X$-module. Decompose $q_*(\mathcal{O}_{\hat{X}}) = \bigoplus_{w \in W} \mathcal{R}_w$ into weight modules. Then $\Gamma_*(\mathcal{F}) = \bigoplus_{w \in W} \Gamma(\hat{X}, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{R}_w)$ is a W-graded S-module.

Definition 5.2. We call $\Gamma_*(\mathcal{F})$ the W-graded S-module *associated to the quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$*.

For example, $\Gamma_*(\mathcal{O}_X) = S$. Obviously, $\mathcal{F} \mapsto \Gamma_*(\mathcal{F})$ is a functor from the category of quasicoherent $\mathcal{O}_X$-modules to the category of W-graded S-modules.

Proposition 5.3. *There is a canonical isomorphism $\mathcal{F} \simeq (\Gamma_*(\mathcal{F}))^\sim$ for each quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$.*

Proof. By definition, we have $(\Gamma_*(\mathcal{F}))^\sim = (q_* i^* i_* q^*(\mathcal{F}))_0$. Since $i\colon \hat{X} \to \bar{X}$ is an open embedding, $i^* i_*(\mathcal{M}) \simeq \mathcal{M}$ holds for each quasicoherent $\mathcal{O}_{\hat{X}}$-module $\mathcal{M}$. This gives $(q_* i^* i_* q^*(\mathcal{F}))_0 \simeq (q_* q^*(\mathcal{F}))_0$. Because $\mathcal{R}_0 = \mathcal{O}_X$, we have $(q_* q^*(\mathcal{F}))_0 \simeq \mathcal{F}$. Consequently, $\mathcal{F} \simeq (\Gamma_*(\mathcal{F}))^\sim$. $\square$

We see that the functor $F \mapsto \tilde{F}$ from graded modules to quasicoherent sheaves is surjective on isomorphism classes. It might happen, however, that $\tilde{F} = 0$ although $F \neq 0$. The next task is to understand the condition $\tilde{F} = 0$. To do so, we first have to generalize the classical notions of irrelevant ideals and Veronese subrings.

The reduced closed subset $\bar{X} \setminus \hat{X}$ is an invariant closed subset inside the affine toric variety $\bar{X}$. We call the corresponding $\hat{M}$-homogeneous ideal $S_+ \subset S$ the *irrelevant ideal*. Note that $S_+ = S$ holds if and only if X is affine.

Suppose $U \subset X$ is an invariant open subset. Let $W_U \subset W$ be the subgroup of all $w \in W$ such that the corresponding invariant reflexive fractional ideal $\mathcal{R}_w$ is invertible over $U \subset X$. Following a standard notation, we call the subring

$$S^{(W_U)} = \bigoplus_{w \in W_U} S_w \subset \bigoplus_{w \in W} S_w = S$$

the *Veronese subring* with respect to $W_U \subset W$. Given a W-graded S-module F, we have the Veronese submodule $F^{(W_U)} = \bigoplus_{w \in W_U} F_w$ as well. This is a W_U-graded $S^{(W_U)}$-module.

Theorem 5.4. *Suppose F is a finitely generated W-graded S-module. Then the condition $\tilde{F} = 0$ holds if and only if there is an invariant affine open covering $X = U_1 \cup \ldots \cup U_n$ such that some power $S_+^k \subset S$ of the irrelevant ideal annihilates the Veronese submodules $F^{(W_{U_1})}, \ldots, F^{(W_{U_n})}$.*

Proof. Choose $\hat{m}_i \in \hat{M}$ such that the homogeneous elements $s_i = \chi^{\hat{m}_i} \in S$ define effective Weil divisors with support $X \setminus U_i$. Then $I = (s_1, \ldots, s_n) \subset S$ has the same radical as the irrelevant ideal $S_+ \subset S$.

Suppose that $\tilde{F} = 0$. Then the restrictions $\mathcal{F}_i = \tilde{F} \mid U_i$ are zero as well. Note that the preimage $\hat{U}_i = q^{-1}(U_i)$ is affine, with global section ring S_{s_i}. The Veronese subring $S_i = (S_{s_i})^{(W_{U_i})}$ defines a factorization

$$\hat{U}_i \longrightarrow \mathrm{Spec}(S_i) \longrightarrow U_i.$$

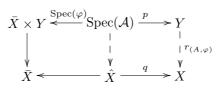According to Proposition 4.1, the map on the right is a principal bundle for the action of the diagonalizable group scheme $G_i = \mathrm{Spec} k[W_{U_i}]$.

Setting $F_i = (F_{s_i})^{(W_{U_i})}$, we conclude that the S_i-module F_i is zero as well. Hence $s_i^{k_i} \cdot F_i = 0$ for some integer $k_i > 0$, because F_i is of finite type. Consequently $I^k \cdot F_i = 0$ with $k = \max\{k_i\}$. This shows that the condition is necessary. The converse is similiar. $\square$

6. Morphisms into toric varieties

Throughout this section, fix a toric prevariety X of affine intersection. We seek to describe the functor $h_X(Y) = \mathrm{Hom}(Y, X)$ represented by X in terms of sheaf data on Y. Here Y ranges over the category of k-schemes. To do so, choose a quotient presentation $q \colon \hat{X} \to X$. Let $S = \Gamma(\hat{X}, \mathcal{O}_{\hat{X}})$ be the homogeneous coordinate ring and set $\bar{X} = \mathrm{Spec}(S)$.

Given a k-scheme Y, we shall deal with pairs $(\mathcal{A}, \varphi)$ such that $\mathcal{A}$ is a W-graded quasicoherent $\mathcal{O}_Y$-algebra with $\mathcal{A}_0 = \mathcal{O}_Y$, and $\varphi \colon S \otimes \mathcal{O}_Y \to \mathcal{A}$ is a W-graded homomorphism of $\mathcal{O}_Y$-algebras. For simplicity, we refer to such pairs as *S-algebras*. An S-algebra $(\mathcal{A}, \varphi)$ yields a diagram

$$\begin{array}{ccccc}
\bar{X} \times Y & \xleftarrow{\mathrm{Spec}(\varphi)} & \mathrm{Spec}(\mathcal{A}) & \xrightarrow{p} & Y \\
\downarrow & & \downarrow & & \downarrow r_{(\mathcal{A},\varphi)} \\
\bar{X} & \longleftarrow & \hat{X} & \xrightarrow{q} & X
\end{array}$$

The problem is to construct the dashed arrows. For this, we need a base-point-freeness condition. Recall that the irrelevant ideal $S_+ \subset S$ is the ideal of the closed subscheme $\bar{X} \setminus \hat{X}$.

Definition 6.1. An S-algebra $(\mathcal{A}, \varphi)$ is called *base-point-free* if for each $y \in Y$ there is an $\hat{M}$-homogeneous $s \in S_+$ such that the germ $\varphi(s) := \varphi(s \otimes 1) \in \mathcal{A}_y$ is a unit.

This is precisely what we need:

Proposition 6.2. *Each base-point-free S-algebra $(\mathcal{A}, \varphi)$ defines, in a canonical way, a morphism $r_{(\mathcal{A},\varphi)} \colon Y \to X$.*

Proof. First, we claim that $\mathrm{Spec}(\mathcal{A}) \to \bar{X}$ factors through the open subset $\hat{X} \subset \bar{X}$. For $y \in Y$ choose $s \in S_+$ such that $\varphi(s)$ is a unit in $\mathcal{A}_y$. Then $\varphi(s)$ is invertible on a p-saturated neighbourhood of $p^{-1}(y) \subset \mathrm{Spec}(\mathcal{A})$. Clearly, this neighbourhood is mapped into $\bar{X}_s \subset \hat{X}$.

According to [9] Theorem 1.1, the projection $\mathrm{Spec}(\mathcal{A}) \to Y$ is a categorical quotient for the G-action defined by the W-grading on $\mathcal{A}$ (here we use the assumption $\mathcal{O}_Y = \mathcal{A}_0$). The composition $\mathrm{Spec}(\mathcal{A}) \to \hat{X} \to X$ is G-invariant. So the universal property of categorical quotients gives a commutative diagram

$$(6.2.1) \qquad\qquad \begin{array}{ccc}
\mathrm{Spec}(\mathcal{A}) & \longrightarrow & \hat{X} \\
\downarrow & & \downarrow \\
Y & \longrightarrow & X,
\end{array}$$

which defines the desired morphism $r_{(\mathcal{A},\varphi)} \colon Y \to X$. $\qquad\square$

Remark 6.3. The assignment $(\mathcal{A}, \varphi) \mapsto r_{(\mathcal{A},\varphi)}$ is functorial in the following sense: Given a base-point-free S-algebra $(\mathcal{A}, \varphi)$ on Y and a morphism $f \colon Y' \to Y$. Then the preimage $(\mathcal{A}', \varphi') = (f^*\mathcal{A}, f^*\varphi)$ is a base-point-free S-algebra on Y', and the corresponding morphisms satisfy $r_{(\mathcal{A}',\varphi')} = r_{(\mathcal{A},\varphi)} \circ f$.

We call an $\hat{M}$-homogeneous element $s \in S_+$ *saturated*, if $\hat{X}_s = q^{-1}(q(\hat{X}_s))$ holds. In that case, $X_s := q(\hat{X}_s)$ is an affine invariant open subset with $\Gamma(X_s, \mathcal{O}_X) = S_{(s)}$. Recall that X is covered by the sets X_s with $s \in S_+$ saturated. We define $Y_{\varphi(s)} \subset Y$ to be the (open) subset of all $y \in Y$ where the germ $\varphi(s) \in \mathcal{A}_y$ is a unit.

Lemma 6.4. *With the preceding notation, we have $Y_{\varphi(s)} = r_{(\mathcal{A},\varphi)}^{-1}(X_s)$ for each saturated $s \in S_+$.*

Proof. Let $y \in Y_{\varphi(s)}$. Then $\varphi(s)$ is invertible on a neighbourhood of the fibre of $\mathrm{Spec}(\mathcal{A}) \to Y$ over y. Looking at the commutative diagram 6.2.1, we see that s is invertible at some point of the fibre of $q \colon \hat{X} \to X$ over $x := r_{(\mathcal{A},\varphi)}(y)$. Since s is saturated, this means $x \in X_s$. The reverse inclusion is clear by definition. $\qquad\square$

Different base-point-free S-algebras may define the same morphism. To overcome this, we need an equivalence relation. Suppose $(\mathcal{A}_1, \varphi_1)$ and $(\mathcal{A}_2, \varphi_2)$ are two base-point-free S-algebras. Call them *equivalent* if for each saturated $s \in S_+$, say of degree $w \in W$, the following holds:

(i) The open subsets $Y_{\varphi_i(s)} \subset Y$ coincide for $i = 1, 2$.

(ii) Over $Y_{\varphi_1(s)} = Y_{\varphi_2(s)}$, the $S_s^{(w)}$-algebras $\mathcal{A}_1^{(w)}$ and $\mathcal{A}_2^{(w)}$ are isomorphic.

Here $S_s^{(w)} \subset S_s$ is the Veronese subring with degrees in $\mathbb{Z}w \subset W$.

Proposition 6.5. *Two base-point-free S-algebras on Y define the same morphism $Y \to X$ if and only if they are equivalent.*

Proof. Suppose that $(\mathcal{A}_i, \varphi_i)$ are two base-point-free S-algebras, which define two morphisms $r_i : Y \to X$, with $i = 1, 2$. First, assume that $r_1 = r_2$. Let $s \in S_+$ be saturated. Using Lemma 6.4, we infer $Y_{\varphi_1(s)} = Y_{\varphi_2(s)}$. To check the second condition for equivalence, note that

$$\mathcal{A}_i^{(w)}|_{Y_{\varphi_i(s)}} = \mathcal{O}_{Y_{\varphi_i(s)}}[\varphi_i(s), \varphi_i(s)^{-1}] \quad \text{and} \quad S_s^{(w)} = \Gamma(X_s, \mathcal{O}_X)[s, s^{-1}]$$

are Laurent polynomial algebras. So the map $\varphi_1(s) \mapsto \varphi_2(s)$ induces the desired isomorphism.

Conversely, assume that the base-point-free S-algebras are equivalent. Let $s \in S_+$ be saturated, and let $w \in W$ be its degree. Consider the partial quotients

$$\operatorname{Spec}(\mathcal{A}_i) \to \operatorname{Spec}(\mathcal{A}_i^{(w)}) \to Y_{\varphi_i(s)} \quad \text{and} \quad \hat{X}_s \to \operatorname{Spec}(S_s^{(w)}) \to X_s$$

Then the isomorphism $\mathcal{A}_2^{(w)} \to \mathcal{A}_1^{(w)}$ induces the identity on $Y_{\varphi_1(s)} = Y_{\varphi_2(s)}$. Thus the morphism $\operatorname{Spec}(\mathcal{A}_1^{(w)}) \to \operatorname{Spec}(S_s^{(w)})$ induces both, $r_i : Y_{\varphi_i(s)} \to X_s$. $\square$

We come to the main result of this section:

Theorem 6.6. *The assignment $(\mathcal{A}, \varphi) \mapsto r_{(\mathcal{A}, \varphi)}$ yields a functorial bijection between the set of equivalence classes of base-point-free S-algebras on Y and the set of morphisms $Y \to X$.*

Proof. In Remark 6.3, we already saw that the assignment is functorial in Y. By Proposition 6.5, it is well-defined on equivalence classes and gives an injection from the set of equivalence classes to the set of morphisms. It remains to check that the identity morphism $\operatorname{id} : X \to X$ arises from a base-point-free S-algebra. Indeed: you easily check that $\mathcal{R} = q_*(\mathcal{O}_{\hat{X}})$, together with the adjunction map $S \otimes \mathcal{O}_X \to \mathcal{R}$ is a base-point-free S-algebra defining the identity on X. $\square$

As an application, we generalize the result of Kajiwara in [7]:

Proposition 6.7. *Suppose the characteristic sequence $M \subset \hat{M} \to \operatorname{WDiv}^T(X)$ of the quotient presentation $q : \hat{X} \to X$ factors through the group of Cartier divisors. Then two base-point-free S-algebras define the same morphism into X if and only if they are isomorphic.*

Proof. Let $(\mathcal{A}, \varphi)$ be a base-point-free S-algebra on the scheme Y defining a morphism $r : Y \to X$. Set $\mathcal{R} = q_*(\mathcal{O}_{\hat{X}})$. The map $\operatorname{Spec}(\mathcal{A}) \to \hat{X}$ defines a homomorphism $\mathcal{R} \otimes_{\mathcal{O}_X} \mathcal{O}_Y \to \mathcal{A}$. Clearly, it suffices to show that this map is bijective. The problem is local, so we may assume that X is affine, hence each weight module $\mathcal{R}_w \subset \mathcal{R}$ is trivial and $S_+ = S$ holds. According to Lemma 6.4, for each

$\hat{M}$-homogeneous unit $s \in S$, the image $\varphi(s) \in \Gamma(Y, \mathcal{A})$ is a global unit. Since each weight module $\mathcal{R}_w$ is generated by such a homogeneous unit, we infer that $\mathcal{R} \otimes \mathcal{O}_Y \to \mathcal{A}$ is bijective. $\square$

In general, the homogeneous components of a base-point-free S-algebra might be noninvertible. However, this does not happen for quotient presentations that are principal bundles:

Corollary 6.8. *Assumptions as in Proposition 6.7. Then each base-point-free S-algebra $(\mathcal{A}, \varphi)$ has invertible homogeneous components $\mathcal{A}_w \subset \mathcal{A}$.*

Proof. By assumption, $\mathcal{R} = q_*(\mathcal{O}_{\hat{X}})$ has invertible homogeneous components. By the preceding Proposition, each base-point-free S-algebra $(\mathcal{A}, \varphi)$ is isomorphic to the preimage $r^*_{(\mathcal{A}, \varphi)}(\mathcal{R})$. $\square$

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