TOROIDAL CROSSINGS AND LOGARITHMIC STRUCTURES

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Abstract. We generalize Friedman's notion of d-semistability, which is a necessary condition for spaces with normal crossings to admit smoothings with regular total space. Our generalization deals with spaces that locally look like the boundary divisor in Gorenstein toroidal embeddings. In this situation, we replace d-semistability by the existence of global log structures for a given gerbe of local log structures. This leads to cohomological descriptions for the obstructions, existence, and automorphisms of log structures. We also apply toroidal crossings to mirror symmetry, by giving a duality construction involving toroidal crossing varieties whose irreducible components are toric varieties. This duality reproduces a version of Batyrev's construction of mirror pairs for hypersurfaces in toric varieties, but it applies to a larger class, including degenerate abelian varieties.

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Introduction

Deligne and Mumford [5] showed that any curve with normal crossing singularities deforms to a smooth curve. This is no longer true for higher dimensional spaces. Friedman [7] discovered that an obstruction for the existence of smoothings with regular total space is an invertible sheaf on the singular locus. He calls spaces with normal crossing singularities *d-semistable* if this sheaf is trivial. So d-semistability

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is a necessary condition for the existence of smoothings with regular total space. This condition, however, is far from being sufficient.

Nowadays, the notion of d-semistability is best understood via *log structures* and *log spaces*. These concepts are due to Fontaine and Illusie, and were largely explored by K. Kato [18]. They now play an important role in crystalline cohomology and deformation theory, and have applications to Hodge theory, étale cohomology, fundamental groups, and mirror symmetry.

Let $\underline{X}$ be an algebraic space. A log space X with underlying algebraic space $\underline{X}$ is a sheaf of monoids $\mathcal{M}_X$ on $\underline{X}$ related to the structure sheaf $\mathcal{O}_{\underline{X}}$ by certain axioms (for details see Section 1). It turns out that a space with normal crossing singularities $\underline{X}$ locally supports local log structures that are unique up to isomorphism, and d-semistability is equivalent with the existence of a global log structure, together with the triviality of the normal sheaf. This has been exploited by Kawamata and Namikawa [21] for the construction of Calabi–Yau manifolds, and by Steenbrink [36] for the construction of mixed Hodge structures. The corresponding theoretical framework is due to Kato [20].

The first goal of this paper is to generalize the notion of d-semistability to spaces that are locally isomorphic to boundary divisors in toric varieties, which one might call *toroidal crossings.* The theory of log structures suggests that such generalization is possible, because spaces with normal crossing singularities are just special instances of log smooth morphisms. Furthermore, it became clear in various areas that spaces with normal crossing singularities do not account for all degenerations that one wants to study. Compare, for example, the work of Kollár and Shepherd–Barron [24] on degenerate surfaces, and of Alexeev [1] on degenerate abelian varieties.

Our main idea is to use the theory of *nonabelian cohomology*, in particular the notion of *gerbe*, to define d-semistability. Roughly speaking, we define a *log atlas* $\mathcal{G}$ on $\underline{X}$ to be a gerbe of local log structures, that is, a collection of locally isomorphic log structures on certain neighborhoods. Now d-semistability is nothing but the triviality of the gerbe class $[\mathcal{G}]$ in a suitable H^2-group, plus the triviality of the normal bundle.

It turns out that the coefficient sheaf of the H^2-cohomology, which is the *band* in the terminology of nonabelian cohomology, becomes abelian if we fix as additional datum the sheaf of monoids $\overline{\mathcal{M}}_X = \mathcal{M}_X/\mathcal{M}_X^\times$. Then the coefficient sheaf becomes the abelian sheaf $\mathcal{A}_X = \mathcal{H}om(\overline{\mathcal{M}}_X, \mathcal{O}_{\underline{X}}^\times)$, and this allows us to calculate $H^2(\underline{X}, \mathcal{A}_X)$ via certain exact sequences. Using such exact sequences, we deduce criteria for the existence of global log structures. The theory works best if we make two additional assumptions: The toric varieties that are local models should be Gorenstein and regular in codimension two. Our main result is: Each log atlas $\mathcal{G}$ determines an invertible sheaf on $\operatorname{Sing}(\underline{X})$ called the *restricted conormal sheaf*, and its extendibility to $\underline{X}$ is equivalent to the existence of a global log structure, and its triviality is equivalent to d-semistability. Olsson independently obtained similar results in the case of normal crossing varieties [29]. He also showed that moduli of log structures yield algebraic stacks [30].

The second goal of this paper is to apply toroidal crossings to mirror symmetry. Our starting point is the observation that from $\overline{\mathcal{M}}_X = \mathcal{M}_X/\mathcal{O}_{\underline{X}}^\times$ it is possible to construct another degenerate variety Y, by gluing together the projective toric varieties $\operatorname{Proj} k[\overline{\mathcal{M}}_{X,\overline{x}}^\vee]$ for $x \in |\underline{X}|$. Furthermore, if X itself consists of projective toric varieties glued to each other along toric subvarieties, then there is a monoid sheaf on Y that at least locally is the sheaf $\overline{\mathcal{M}}_Y$ of a toroidal crossing log structure. This gives an involutive correspondence between certain degenerate varieties endowed with such sheaves of monoids. Applied to hypersurfaces in projective toric varieties it reproduces a degenerate version of Batyrev's mirror construction, but it applies to many more cases, for example degenerate abelian varieties.

Of course, mirror symmetry should do much more than what the naive version presented here does. Our approach indicates that one should try to understand mirror symmetry in terms of limiting data of degenerations of varieties with trivial canonical bundle. By *limiting data* we mean information about the degeneration supported on the central fiber: Most importantly, the log structure induced by the embedding into the total space, and certain cohomology classes on the central fiber obtained by specialization. Mirror symmetry then is a symmetry acting on such limiting data. The explanation of the mirror phenomenon would then be that it relates limiting data of different degenerations. Mark Gross and the second author worked out a correspondence of true log spaces that involves also data encoding the degeneration of a polarization, see [9] and [10].

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1. Algebraic spaces and logarithmic structures

In this section we recall some definitions regarding algebraic spaces and logarithmic structures. For more details on algebraic spaces we refer to the books of Knutson [23] and Laumon and Moret-Bailly [25]. For logarithmic structures the reference is Kato's article [18]. For typographical reasons, we use the following convention throughout: Unadorned symbols $X, U, \ldots$ denote log spaces, whereas underlined symbols $\underline{X}, \underline{U}, \ldots$ denote their underlying algebraic spaces, as in [29].

An *algebraic space* $\underline{X}$ is the quotient of a scheme $\underline{U}$ by an equivalence relation $\underline{R}$, such that $\underline{R}$ is a scheme, the projections $\underline{R} \to \underline{U}$ are étale, and the diagonal $\underline{R} \to \underline{U} \times \underline{U}$ is a quasicompact monomorphism [23]. Here quotient means quotient of sheaves on the site of rings endowed with the étale topology. We prefer to work with algebraic spaces because operations like gluing schemes yield algebraic spaces rather than schemes ([2], Theorem 6.1). Note that, over the complex numbers, proper algebraic spaces correspond to compact Moishezon spaces ([2], Theorem 7.3).

A *point* for $\underline{X}$ is an equivalence class of morphisms $\operatorname{Spec}(K) \to \underline{X}$, where K is a field ([25], Definition 5.2). The collection of all points is a topological space $|\underline{X}|$, whose open sets correspond to open subspaces $\underline{U} \subset \underline{X}$. A morphism of algebraic spaces is called *surjective* if the induced map on the associated topological spaces is surjective.

Let $\mathrm{Et}(\underline{X})$ be the étale site for $\underline{X}$, whose objects are the étale morphisms $\underline{U} \to \underline{X}$, and whose covering families are the surjections. A sheaf on $\underline{X}$ is, by definition, a sheaf on $\mathrm{Et}(\underline{X})$. Given a sheaf $\mathcal{F}$ and a point $x \in |\underline{X}|$, one defines the stalk $\mathcal{F}_{\bar{x}} = \varinjlim \Gamma(\underline{U}, \mathcal{F})$, where the direct limit runs over all affine étale neighborhoods $\underline{U} \to \underline{X}$ endowed with a point $u \in \underline{U}$ such that $\operatorname{Spec} \kappa(u) \to \underline{X}$ represents x. Then $\mathcal{F} \mapsto \mathcal{F}_{\bar{x}}$ defines a fiber functor in the sense of topos theory. According to [16], Exposé VIII, Theorem 3.5, a map between sheaves is bijective if and only if for all points $x \in |\underline{X}|$ the induced map between stalks are bijective. Moreover, by [16], Exposé VIII, Theorem 7.9, the map $x \mapsto \mathcal{F}_{\bar{x}}$ is a homeomorphism between $|\underline{X}|$ and the space of topos-theoretical points for the topos of sheaves on $\mathrm{Et}(\underline{X})$.

Let $\underline{X}$ be an algebraic space. A *log structure* on $\underline{X}$ is a sheaf of monoids $\mathcal{M}_X$ on $\underline{X}$ together with a homomorphism of monoids $\alpha_X : \mathcal{M}_X \to \mathcal{O}_{\underline{X}}$ into the multiplicative monoid $\mathcal{O}_{\underline{X}}$, such that the induced map $\alpha_X^{-1}(\mathcal{O}_{\underline{X}}^\times) \to \mathcal{O}_{\underline{X}}^\times$ is bijective [18]. A *log space* X is an algebraic space endowed with a log structure. In other words, $X = (\underline{X}, \mathcal{M}_X, \alpha_X)$.

A *chart* for a log space X is an étale neighborhood $\underline{U} \to \underline{X}$, together with a monoid P and a homomorphism $P \to \Gamma(\underline{U}, \mathcal{O}_{\underline{X}})$ so that the log space U induced form the log space X is isomorphic to the log space associated to the constant *prelog structure* $P_{\underline{U}} \to \mathcal{O}_{\underline{U}}$ (see [18], Section 1 for details). A log space is called *fine* if it is covered by charts where the monoid P is fine, that is, finitely generated and integral.

Each log space $X = (\underline{X}, \mathcal{M}_X, \alpha_X)$ comes along with a sheaf of monoids

$$\overline{\mathcal{M}}_X = \mathcal{M}_X / \mathcal{M}_X^\times.$$

Using the identifications $\mathcal{M}_X^\times = \alpha_X^{-1}(\mathcal{O}_{\underline{X}}^\times) = \mathcal{O}_{\underline{X}}^\times$, we usually write $\overline{\mathcal{M}}_X = \mathcal{M}_X / \mathcal{O}_{\underline{X}}^\times$. We call it the *ghost sheaf* of the log structure. The stalks of the ghost sheaf are *sharp* monoids, that is, they have no units except the neutral element. Ghost sheaves of fine log structures are not arbitrary. Following [17], Exposé IX, Definition 2.3, we call a monoid sheaf $\mathcal{F}$ *constructible* if its stalks are fine, and any affine étale neighborhood $\underline{U} \to \underline{X}$ admits a decomposition into finitely many constructible locally closed subschemes $\underline{U}_i$ such that the restrictions $\mathcal{F}_{\underline{U}_i}$ are locally constant.

Proposition 1.1. *If X is a fine log space, then its ghost sheaf $\overline{\mathcal{M}}_X$ is a constructible monoid sheaf.*

Proof. This is a local problem by [17], Exposé IX, Proposition 2.8. Hence we easily reduce to the case that $X = \operatorname{Spec}(\mathbb{Z}[P])$ for some fine monoid $P = \sum_{i=1}^r \mathbb{Z} p_i$. Each subset $J \subset \{1, \ldots, r\}$ yields a ring $R_J = S^{-1}\mathbb{Z}[P]/I$, where $S \subset \mathbb{Z}[P]$ is the multiplicative subset generated by all p_i with $i \notin J$, and $I \subset S^{-1}\mathbb{Z}[P]$ is the ideal generated by all p_i with $i \in J$. We obtain locally closed subsets $\underline{X}_J = \operatorname{Spec}(R_J)$. Note that $\underline{X}_J \subset \underline{X}$ is the set of points $x \in \underline{X}$ where the sections p_i are invertible if $i \notin J$, and vanish if $i \in J$. It follows that we have a disjoint decomposition $\underline{X} = \bigcup_J \underline{X}_J$.

To see that $\overline{\mathcal{M}}_X$ is constant along $\underline{X}_J$, fix a point $x \in \underline{X}_J$. Then the germ $\overline{\mathcal{M}}_{X,\bar{x}}$ equals the sharp monoid $(P + \sum_{i \notin J} \mathbb{Z} p_i)/G$, where $G \subset P + \sum_{i \notin J} \mathbb{Z} p_i$ is the subgroup of invertible elements. This does not depend on the point x, hence the assertion. $\square$

Given an algebraic space $\underline{X}$ and two points $x, y \in |\underline{X}|$ with $y \in \overline{\{x\}}$, one has a specialization map $\mathcal{F}_{\bar{y}} \to \mathcal{F}_{\bar{x}}$ (some authors call it a cospecialization map). We say that $\mathcal{F}$ has *surjective specialization maps* if these maps are surjective for all pairs $x, y \in |\underline{X}|$ with $y \in \overline{\{x\}}$. Ghost sheaves are typical examples:

Proposition 1.2. *Let X be a log space. If each point $x \in |\underline{X}|$ admits a chart, then the ghost sheaf $\overline{\mathcal{M}}_X$ has surjective specialization maps.*

Proof. This is a local problem, and we may assume that $\underline{X} = \operatorname{Spec}(A)$ is the spectrum of a henselian local ring with separably closed residue field. Choose a monoid P and a map $f : P \to \Gamma(\underline{X}, \mathcal{O}_{\underline{X}})$ so that X is the associated log space. The cocartesian diagram

$$\begin{array}{ccc} f^{-1}(\mathcal{O}_{\underline{X}}^\times) & \longrightarrow & P_{\underline{X}} \\ \downarrow & & \downarrow \\ \mathcal{O}_{\underline{X}}^\times & \longrightarrow & \mathcal{M}_X \end{array}$$

yields the monoid sheaf $\mathcal{M}_X$. As a consequence, the composite map $P_{\underline{X}} \to \overline{\mathcal{M}}_X$ is surjective, and the ghost sheaf $\overline{\mathcal{M}}_X$ has surjective specialization maps. $\square$

2. Logarithmic atlases

Let $\underline{X}$ be an algebraic space. A natural question to ask is: What is the set of *all* log spaces X with underlying algebraic space $\underline{X}$, up to isomorphism? This is a reasonable moduli problem, as Olsson [30] proved that the fibered category of fine log structures on $\underline{X}$-schemes $\underline{U}$ is an algebraic stack. Here we seek a cohomological approach to classify log structures. This classification problem, however, is non-abelian in nature. To overcome this, we shall fix the ghost sheaf $\overline{\mathcal{M}}_X$ such that the problem becomes abelian. This leads to the desired cohomological descriptions for obstructions, existence, and automorphisms of log structures.

Given a log space X with underlying algebraic space $\underline{X}$, we denote by $\mathcal{A}ut_{X/\underline{X}}$ the sheaf of log space automorphisms $X \to X$ inducing the identity on the underlying algebraic space $\underline{X}$. Such automorphism correspond to bijections $\phi : \mathcal{M}_X \to \mathcal{M}_X$ compatible with $\alpha_X : \mathcal{M}_X \to \mathcal{O}_{\underline{X}}$. They necessarily fix the subsheaf $\mathcal{O}_{\underline{X}}^\times \subset \mathcal{M}_X$ pointwise, and induce a bijection $\overline{\phi} : \overline{\mathcal{M}}_X \to \overline{\mathcal{M}}_X$. Let $\mathcal{A}ut'_{X/\underline{X}} \subset \mathcal{A}ut_{X/\underline{X}}$ be the subsheaf of automorphisms inducing the identity on the ghost sheaf $\overline{\mathcal{M}}_X$. We want to compare $\mathcal{A}ut'_{X/\underline{X}}$ to the abelian sheaf

$$\mathcal{A}_X = \mathcal{H}om(\overline{\mathcal{M}}_X, \mathcal{O}_{\underline{X}}^\times) = \mathcal{H}om(\overline{\mathcal{M}}_X^{\mathrm{gp}}, \mathcal{O}_{\underline{X}}^\times).$$

There is a canonical inclusion $\mathcal{A}_X \subset \mathcal{A}ut(\mathcal{M}_X)$ sending a map $h : \overline{\mathcal{M}}_X \to \mathcal{O}_{\underline{X}}^\times$ to

$$\mathcal{M}_X \longrightarrow \mathcal{M}_X, \quad s \longmapsto s + h(\overline{s}),$$

where $\overline{s} \in \Gamma(\underline{U}, \overline{\mathcal{M}}_X)$ denotes the image of $s \in \Gamma(\underline{U}, \mathcal{M}_X)$, and $\underline{U} \to \underline{X}$ is any affine étale neighborhood.

Proposition 2.1. *Suppose $\underline{X}$ is reduced. Then the inclusion $\mathcal{A}_X \subset \mathcal{A}ut(\mathcal{M}_X)$ factors over the inclusion $\mathcal{A}ut'_{X/\underline{X}} \subset \mathcal{A}ut(\mathcal{M}_X)$.*

Proof. With the preceding notation, we have to check that equality

$$\alpha_X(s) \cdot \alpha_X(h(\overline{s})) = \alpha_X(s)$$

holds inside $\Gamma(\underline{U}, \mathcal{O}_{\underline{X}})$. This is obvious if $\alpha_X(s) = 0$. If $\alpha_X(s)$ is invertible then $\overline{s} = 0$, and equality holds as well. Let $\eta_i \in \underline{U}$, $i \in I$ be the generic points. Since $\underline{X}$

is reduced, there are open neighborhoods $\eta_i \in \underline{U}_i$ so that $\alpha(s)_{\underline{U}_i}$ is either zero or invertible. We infer that the desired equality holds on $\bigcup_{i\in I}\underline{U}_i$. Using again that $\underline{X}$ is reduced, we see that $\alpha_X(s)\cdot\alpha_X(h(\bar{s})) = \alpha_X(s)$ holds on $\underline{U}$. □

Proposition 2.2. *Suppose $\underline{X}$ is reduced and $\overline{\mathcal{M}}_X$ has integral stalks. Then the inclusion $\mathcal{A}_X \subset \mathcal{A}ut'_{X/\underline{X}}$ is bijective.*

Proof. Fix a point $x \in |\underline{X}|$. We have to show that the inclusion $\mathcal{A}_{X,\bar{x}} \subset \mathcal{A}ut'_{X/\underline{X},\bar{x}}$ is bijective. Let $\underline{U} \to \underline{X}$ be an étale neighborhood of x and U the induced log space, and $\phi : \mathcal{M}_U \to \mathcal{M}_U$ a bijection compatible with α_U and inducing the identity on $\overline{\mathcal{M}}_U$. We now construct a homomorphism $h : \overline{\mathcal{M}}_U \to \mathcal{O}_U^\times$ as follows:

Let $\bar{s} \in \Gamma(\underline{V}, \overline{\mathcal{M}}_X)$ be a local section on an étale neighborhood $\underline{V} \to \underline{U}$. Choose a refinement $\underline{W} \to \underline{V}$ so that $\bar{s}_{\underline{W}}$ comes from a section $s \in \Gamma(\underline{W}, \mathcal{M}_X)$. Then the equation $\phi(s) = s + \alpha^{-1}(t)$ defines a section $t \in \Gamma(\underline{W}, \mathcal{O}_{\underline{X}}^\times)$. Since $\overline{\mathcal{M}}_X$ has integral stalks, so has $\mathcal{M}_X$, and we infer from the defining equation that t depends only on $\bar{s}_{\underline{W}}$, and not on the choice of s. Consequently $\mathrm{pr}_0(t) = \mathrm{pr}_1^*(t)$ on $\underline{W}\times_{\underline{V}}\underline{W}$, and t descends to a section $h(\bar{s}) \in \Gamma(\underline{V}, \mathcal{O}_{\underline{X}}^\times)$. Furthermore, this section depends only on $\bar{s}$, and not on the choice of the refinement $\underline{W} \to \underline{V}$.

It follows from the defining equation $\phi(s) = s + \alpha^{-1}(t)$ that $h(\bar{s})$ yields a monoid homomorphism $\Gamma(\underline{V}, \overline{\mathcal{M}}_X) \to \Gamma(\underline{V}, \mathcal{O}_{\underline{X}}^\times)$. Clearly $h(\bar{s})$ is compatible with restrictions. Hence we have defined a sheaf homomorphism $h : \overline{\mathcal{M}}_U \to \mathcal{O}_U^\times$, with $\phi(s) = s + h(\bar{s})$ for any local section $s \in \Gamma(\underline{V}, \mathcal{M}_X)$. In other words, the germ $h_{\bar{x}} \in \mathcal{A}_{X,\bar{x}}$ corresponds to the germ $\phi_{\bar{x}} \in \mathcal{A}ut'_{X/\underline{X},\bar{x}}$ under the canonical inclusion. □

Now let $\underline{X}$ be an algebraic space, and fix as additional datum a sheaf of integral sharp monoids $\overline{\mathcal{M}}_{\underline{X}}$. Let $\mathrm{LS}(\underline{X})$ be the category of pairs (U, φ), where $U = (\underline{U}, \mathcal{M}_U, \alpha_U)$ is a log space, whose underlying algebraic space $\underline{U}$ is an étale neighborhood $\underline{U} \to \underline{X}$, and

$$\varphi : \overline{\mathcal{M}}_U = \mathcal{M}_U/\mathcal{O}_U^\times \longrightarrow \overline{\mathcal{M}}_{\underline{X}}\,|_{\underline{U}} = \overline{\mathcal{M}}_{\underline{U}}$$

is an isomorphism. We call φ a *framing* for the log space U with respect to $\overline{\mathcal{M}}_{\underline{X}}$. The functor

$$\mathrm{LS}(\underline{X}) \longrightarrow \mathrm{Et}(\underline{X}), \quad (U,\varphi) \longmapsto \underline{U}$$

yields a fibered category. The fiber $\mathrm{LS}(\underline{X})_{\underline{U}}$ over an étale neighborhood $\underline{U}$ is equivalent to the category of log structures on $\underline{U}$ whose ghost sheaf is identified with $\overline{\mathcal{M}}_{\underline{U}} = \overline{\mathcal{M}}_{\underline{X}}\,|_{\underline{U}}$. By abuse of notation, we usually write U instead of (U, φ) for the objects in $\mathrm{LS}(\underline{X})$. An inverse image for an $\underline{X}$-morphism of étale neighborhoods $g : \underline{U} \to \underline{V}$ is given by restriction. This also extends from the small étale site $\mathrm{Et}(\underline{X})$ to the big étale site, where the preimage is given by the log structure associated to the prelog structure $g^{-1}(\mathcal{M}_V) \to \mathcal{O}_{\underline{U}}$. Obviously, our fibered category is a *stack* in Giraud's sense [8], Chapter II, Definition 1.2, that is, all descent data are effective.

Now recall that a substack $\mathcal{G} \subset \mathrm{LS}(\underline{X})$ over $\mathrm{Et}(\underline{X})$ is a *subgerbe* if, for each étale neighborhood $\underline{U} \to \underline{X}$, the following axioms hold (see [8], Chapter III, Definition 2.1.3):

(i) The objects in $\mathcal{G}_{\underline{U}}$ are locally isomorphic.
(ii) The morphisms in $\mathcal{G}_{\underline{U}}$ are isomorphisms.
(iii) There is an étale covering $\underline{V} \to \underline{U}$ with $\mathcal{G}_{\underline{V}}$ nonempty.

A gerbe with $\mathcal{G}_{\underline{X}} \neq \emptyset$ is called *neutral*. This means that it is possible to glue the local log structures $V \in \mathcal{G}$, which exists by axiom (iii), in at least one way to obtain a global log structure $X \in \mathcal{G}$. Note that, with respect to inclusion, each subgerbe is contained in a maximal subgerbe, and we may restrict our attention to maximal subgerbes. The following definition is fundamental for the rest of this paper:

Definition 2.3. Let $\underline{X}$ be an algebraic space endowed with a sheaf $\overline{\mathcal{M}}_{\underline{X}}$ of integral sharp monoids. A *log atlas* for $\underline{X}$ with respect to $\overline{\mathcal{M}}_{\underline{X}}$ is a maximal subgerbe $\mathcal{G} \subset \mathrm{LS}(\underline{X})$ over $\mathrm{Et}(\underline{X})$.

The idea is that a log atlas $\mathcal{G}$ tells us how local log structures on $\underline{X}$ should be locally around each point $x \in |\underline{X}|$, up to isomorphism. It does not, however, single out preferred local log structures. Neither does it inform us how to glue these local log structures. Given a log atlas, the problem is to decide whether or not it admits a global log structure. Note that Kawamata and Namikawa [21] used the word log atlas in a very different way, namely to denote global log structures.

Given an object $(U, \varphi) \in \mathcal{G}$, we obtain a homomorphism

$$\mathcal{A}_{\underline{U}} = \mathcal{H}om(\overline{\mathcal{M}}_{\underline{U}}, \mathcal{O}_{\underline{U}}^{\times}) \xrightarrow{\varphi^*} \mathcal{H}om(\overline{\mathcal{M}}_{U}, \mathcal{O}_{\underline{U}}^{\times}) \longrightarrow \mathcal{A}ut'_{U/\underline{U}},$$

which is bijective by Proposition 2.2. In the language of nonabelian cohomology, the abelian sheaf

$$\mathcal{A}_{\underline{X}} = \mathcal{H}om(\overline{\mathcal{M}}_{\underline{X}}, \mathcal{O}_{\underline{X}}^{\times})$$

binds the gerbe $\mathcal{G}$, and $\mathcal{G}$ becomes an $\mathcal{A}_{\underline{X}}$-gerbe ([8], Chapter IV, Definition 2.2.2). In turn, we obtain a gerbe class $[\mathcal{G}] \in H^2(\underline{X}, \mathcal{A}_{\underline{X}})$. The theory of nonabelian cohomology immediately gives the following:

Theorem 2.4. *Let $\mathcal{G}$ be a log atlas on an algebraic space $\underline{X}$ with respect to a sheaf $\overline{\mathcal{M}}_{\underline{X}}$ of integral sharp monoids. Then there is a global log structure $X \in \mathcal{G}$ if and only if the gerbe class $[\mathcal{G}] \in H^2(\underline{X}, \mathcal{A}_{\underline{X}})$ vanishes. In this case, the set of isomorphism classes of $X \in \mathcal{G}$ is a torsor for $H^1(\underline{X}, \mathcal{A}_{\underline{X}})$. Moreover, for each global log structure $X \in \mathcal{G}$, the group of log space automorphisms inducing the identity on the underlying algebraic space $\underline{X}$ and on the sheaf $\overline{\mathcal{M}}_{\underline{X}}$ is $H^0(\underline{X}, \mathcal{A}_{\underline{X}})$.*

Proof. The first statement is [8], Chapter IV, Theorem 3.4.2. The second statement follows from [8], Chapter III, Theorem 2.5.1. The last statement is nothing but Proposition 2.2. □

The preceding result is almost a tautology if we use the *geometric definition* for the universal ∂-functor $H^n(\underline{X}, \mathcal{F})$, where $\mathcal{F}$ is an abelian sheaf and $0 \leq n \leq 2$. In this definition, $H^1(\underline{X}, \mathcal{F})$ is the set of isomorphism classes of $\mathcal{F}$-torsors, and $H^2(\underline{X}, \mathcal{F})$ is the set of equivalence classes of $\mathcal{F}$-gerbes. Given a short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, the coboundary operator maps a section for $\mathcal{F}''$ to the $\mathcal{F}'$-torsor of its preimage in $\mathcal{F}$, and an $\mathcal{F}''$-torsor to the $\mathcal{F}'$-gerbe of its $\mathcal{F}$-liftings.

With these definitions, the cohomology class $[\mathcal{G}] \in H^2(\underline{X}, \mathcal{A}_{\underline{X}})$ of a log atlas $\mathcal{G}$ is the equivalence class of the underlying $\mathcal{A}_{\underline{X}}$-gerbe, and the difference between two isomorphism classes of global log spaces $X, X' \in \mathcal{G}$ is the isomorphism class of the $\mathcal{A}_{\underline{X}}$-torsor $\mathcal{I}som(X', X)$. The situation becomes more illuminating if we use other descriptions for cohomology. We discuss this in the next section.

3. Cohomology and hypercoverings

Let us recall the *combinatorial definition* for cohomology in degrees ≤ 2. Let $\mathcal{F}$ be an abelian sheaf on an algebraic space $\underline{X}$. Then one may describe $H^n(\underline{X}, \mathcal{F})$ for $0 \leq n \leq 2$ as follows.

Suppose we have an étale covering $\underline{U} \to \underline{X}$ and an étale covering $\underline{V} \to \underline{U} \times_{\underline{X}} \underline{U}$. Let $p_0 : \underline{V} \to \underline{U}$ be the composition $v \mapsto (u_0, u_1) \mapsto u_0$, and $p_1 : \underline{V} \to \underline{U}$ be the other composition $v \mapsto (u_0, u_1) \mapsto u_1$. Define

$$(\underline{V}/\underline{U})_2 \subset \underline{V} \times_{\underline{X}} \underline{V} \times_{\underline{X}} \times \underline{V}$$

to be the subspace satisfying the simplicial identities $p_i \circ \mathrm{pr}_j = p_{j-1} \circ \mathrm{pr}_i$, $i < j$, and let $p_j : (\underline{V}/\underline{U})_2 \to V$ be the maps induced by the projections pr_j. Here pr_j denotes the projections $(v_0, v_1, v_2) \mapsto v_j$. Inductively, we define for each $n \geq 2$ subspaces $(\underline{V}/\underline{U})_{n+1} \subset \prod_{i=0}^n (\underline{V}/\underline{U})_n$ and projections $p_j : (\underline{V}/\underline{U})_{n+1} \to (\underline{V}/\underline{U})_n$ as above. This gives a semisimplicial étale covering $(\underline{V}/\underline{U})_\bullet$ of $\underline{X}$, where $(\underline{V}/\underline{U})_1 = \underline{V}$ and $(\underline{V}/\underline{U})_0 = \underline{U}$. In fact, $(\underline{V}/\underline{U})_\bullet$ is the coskeleton for the truncated semisimplicial covering $\underline{V} \rightrightarrows \underline{U}$ (for more on this, see [6], Section 0.7).

Remark 3.1. The maps $p_j : (\underline{V}/\underline{U})_{n+1} \to (\underline{V}/\underline{U})_n$ are indeed étale. To see this, note first that the composite maps $p_i p_j : (\underline{V}/\underline{U})_{n+1} \to (\underline{V}/\underline{U})_{n-1}$ are étale, because $(\underline{V}/\underline{U})_{n+1}$ is defined as a fiber product with respect to étale maps. By induction on n, the maps $p_i : (\underline{V}/\underline{U})_n \to (\underline{V}/\underline{U})_{n-1}$ are étale, and it then follows from [13], Corollary 17.3.5 that $p_j : (\underline{V}/\underline{U})_{n+1} \to (\underline{V}/\underline{U})_n$ are étale as well.

Now let $\mathcal{F}$ be any abelian sheaf on $\underline{X}$. In accordance with the applications we have in mind, we shall write the group law multiplicatively. The sheaf $\mathcal{F}$ yields a cochain complex of abelian groups $C^n(\underline{V}/\underline{U}, \mathcal{F}) = \Gamma((\underline{V}/\underline{U})_n, \mathcal{F})$ with the usual differential $d = \prod p_i^{*(-1)^i}$. Let $H^n(\underline{V}/\underline{U}, \mathcal{F})$ be the corresponding cohomology group. Given other étale coverings $\underline{V}' \rightrightarrows \underline{U}'$ refining the given étale coverings $\underline{V} \rightrightarrows \underline{U}$, we obtain an induced map $H^n(\underline{V}/\underline{U}, \mathcal{F}) \to H^n(\underline{V}'/\underline{U}', \mathcal{F})$. Now let us *define*

$$H^n(\underline{X}, \mathcal{F}) = \varinjlim H^n(\underline{V}/\underline{U}, \mathcal{F}), \quad 0 \leq n \leq 2.$$

This is a ∂-functor: Given a short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ and a 0-cocycle $f \in Z^0(\underline{V}/\underline{U}, \mathcal{F}'')$, we refine $\underline{U}$, choose an $\mathcal{F}$-valued 0-cochain lift $\tilde{f}$ of f, and define $\partial(f) = p_0^*(\tilde{f})/p_1^*(\tilde{f})$. Similarly, given a 1-cocycle $g \in Z^1(\underline{V}/\underline{U}, \mathcal{F}'')$, we pass to a refinement of $\underline{V}$, choose an $\mathcal{F}$-valued 1-cochain lift $\tilde{g}$ of g, and define $\partial(g) = p_0^*(\tilde{g})p_2^*(\tilde{g})/p_1^*(\tilde{g})$. It is not difficult to see that this ∂-functor vanishes on injective sheaves, hence is universal by [11], Proposition 2.2.1. Therefore, the geometric and combinatorial definitions for $H^n(\underline{X}, \mathcal{F})$, $0 \leq n \leq 2$ are canonically isomorphic as ∂-functors.

The canonical isomorphism between geometric and combinatorial definition takes the following explicit form: Suppose we have an $\mathcal{F}$-torsor $\mathcal{T}$. Choose an étale covering $\underline{U} \to \underline{X}$ so that there is a section $s \in \Gamma(\underline{U}, \mathcal{T})$ and set $\underline{V} = \underline{U} \times_{\underline{X}} \underline{U}$. Then $p_0^*(s) = p_1^*(s) \cdot f$ defines a cocycle $f \in Z^1(\underline{V}/\underline{U}, \mathcal{F})$. To see that $\mathcal{T} \mapsto f$ yields the canonical isomorphism, it suffices to check that the induced map is well-defined, additive, and commutes with the coboundary $\partial : H^0 \to H^1$, which is straightforward.

Now suppose $\mathcal{G}$ is an $\mathcal{F}$-gerbe. Choose an étale covering $\underline{U} \to \underline{X}$ admitting an object $T \in \mathcal{G}_{\underline{U}}$, and an étale covering $\underline{V} \to \underline{U} \times_{\underline{X}} \underline{U}$ admitting an isomorphism

$\phi : p_1^*(T) \to p_0^*(T)$. Then the equation

$$g \cdot p_1^*(\phi) = p_0^*(\phi) p_2^*(\phi) \in \operatorname{Isom}(p_1^* p_1^* T, p_0^* p_0^* T)$$

defines a cocycle $g \in Z^2(\underline{V}/\underline{U}, \mathcal{F})$. Note that this equation involves the simplicial identities $p_j^* p_i^*(T) \simeq p_i^* p_{j-1}^*(T)$, $i < j$. To see that $\mathcal{G} \mapsto g$ yields the canonical isomorphism, it suffices to check that the induced map is well-defined, additive, and commutes with $\partial : H^1 \to H^2$, which is again straightforward.

The action of $H^1(\underline{X}, \mathcal{F})$ on the set of isomorphism classes of $\mathcal{G}_{\underline{X}}$ is as follows: Given a $\mathcal{F}$-torsor $\mathcal{T}$, choose a cocycle $f \in Z^1(\underline{V}/\underline{U}, \mathcal{F})$ as above and a global object $T \in \mathcal{G}_{\underline{X}}$. We have a canonical bijection $\phi : p_1^*(T_{\underline{U}}) \to p_0^*(T_{\underline{U}})$ on $\underline{V} = \underline{U} \times_{\underline{X}} \underline{U}$. Then the isomorphism $\phi \circ f : p_1^*(T_{\underline{U}}) \to p_0^*(T_{\underline{U}})$ is another descent datum, that is,

$$p_1^*(\phi) \circ p_1^*(f) = p_0^*(\phi) \circ p_0^*(f) \circ p_2^*(\phi) \circ p_2^*(f) \tag{1}$$

holds as isomorphisms on $(\underline{V}/\underline{U})_2$, with suitable identifications coming from the simplicial identities. Indeed, we have $p_0^*(f) \circ p_2^*(\phi) = p_2^*(\phi) \circ p_0^*(f)$, because $\mathcal{F}$ is abelian, and (1) follows from the cocycle condition for ϕ and f. Summing up, the descend datum $\phi \circ f$ defines another global object $T' \in \mathcal{G}_{\underline{X}}$, together with a bijection $\mathcal{T} \to \mathcal{I}som(T, T')$.

Remark 3.2. Note that we obtain Čech cohomology groups $\check{H}^n(\underline{X}, \mathcal{F})$ if we use $\underline{V} = \underline{U} \times_{\underline{X}} \underline{U}$ instead of étale coverings $\underline{V} \to \underline{U} \times_{\underline{X}} \underline{U}$. In general Čech cohomology groups do not form a ∂-functor on the category of sheaves and differ from true cohomology groups. Note, however, that the canonical map $\check{H}^n(\underline{X}, \mathcal{F}) \to H^n(\underline{X}, \mathcal{F})$ is bijective for all $n \geq 0$ provided that $\underline{X}$ admits an ample invertible sheaf [3]. Furthermore, $\check{H}^2(\underline{X}, \mathcal{F}) \to H^2(\underline{X}, \mathcal{F})$ is bijective if each pair of points admits an affine open neighborhood [34].

4. The sheaf of automorphisms

Let $\underline{X}$ be an algebraic space, endowed with a sheaf of integral sharp monoids $\overline{\mathcal{M}}_{\underline{X}}$. As before, we set $\mathcal{A}_{\underline{X}} = \mathcal{H}om(\overline{\mathcal{M}}_{\underline{X}}, \mathcal{O}_{\underline{X}}^\times)$. The goal now is to compute the cohomology groups $H^1(\underline{X}, \mathcal{A}_{\underline{X}})$ and $H^2(\underline{X}, \mathcal{A}_{\underline{X}})$ in some interesting special cases. To this end we shall relate the sheaf $\mathcal{A}_{\underline{X}}$ to other sheaves via exact sequences. This relies on the following construction.

Suppose that our algebraic space $\underline{X}$ is a noetherian, reduced, and satifies the following condition: For all points $x \in |\underline{X}|$, the integral components of $\operatorname{Spec}(\mathcal{O}_{\underline{X},\bar{x}})$ are normal. This condition holds for the spaces we have in mind for applications, namely boundary divisors in toroidal embeddings. The referees pointed out that such a condition is indeed indispensable. The assumption implies that the normalization $f : \underline{S} \to \underline{X}$ is a finite map. Moreover, f is an isomorphism near each point $x \in |\underline{X}|$ where $\mathcal{O}_{\underline{X},\bar{x}}$ is unibranch, because then $\mathcal{O}_{\underline{X},\bar{x}}$ is integral by [13], Corollary 18.6.13.

Let $\mathcal{I} \subset \mathcal{O}_{\underline{X}}$ be the conductor ideal for f, that is the annihilator ideal of $f_*(\mathcal{O}_{\underline{S}})/\mathcal{O}_{\underline{X}}$, or equivalently the largest coherent $\mathcal{O}_{\underline{X}}$-ideal that is also an $\mathcal{O}_{\underline{S}}$-ideal. The closed subspaces $\underline{D} \subset \underline{X}$ and $f^{-1}(\underline{D}) \subset \underline{S}$ defined by the conductor ideal are the branch space and the ramification space for the finite morphism $f : \underline{S} \to \underline{X}$, respectively. We call $\underline{D} \subset \underline{X}$ the *subspace of nonnormality.* The cartesian diagram

$$\begin{array}{ccc} f^{-1}(\underline{D}) & \longrightarrow & \underline{S} \\ \Big\downarrow & & \Big\downarrow f \\ \underline{D} & \longrightarrow & \underline{X} \end{array}$$

yields sequences of coherent $\mathcal{O}_{\underline{X}}$-modules

$$0 \longrightarrow \mathcal{O}_{\underline{X}} \longrightarrow \mathcal{O}_{\underline{S}} \oplus \mathcal{O}_{\underline{D}} \longrightarrow \mathcal{O}_{f^{-1}(\underline{D})} \longrightarrow 0. \tag{2}$$

Here the map on the left is the diagonal map $t \mapsto (t, \bar{t})$, and the map on the right is the difference map $(t, \bar{s}) \mapsto \bar{t} - \bar{s}$. Similarly, we have a sequence of abelian sheaves on $\underline{X}$

$$1 \longrightarrow \mathcal{O}^\times_{\underline{X}} \longrightarrow \mathcal{O}^\times_{\underline{S}} \times \mathcal{O}^\times_{\underline{D}} \longrightarrow \mathcal{O}^\times_{f^{-1}(\underline{D})} \longrightarrow 1. \tag{3}$$

For the sake of simplicity we have supressed f_* from notation.

Proposition 4.1. *The preceding sequences (2) and (3) are exact.*

Proof. This is a local problem, so we may assume that our algebraic spaces $\underline{X} = \operatorname{Spec}(A)$ and $\underline{S} = \operatorname{Spec}(B)$ are affine. Let $I \subset A$ be the conductor ideal. We treat the additive sequence (2) first. It is easy to see that this sequence is a complex, and exact at $\mathcal{O}_{\underline{X}}$ and $\mathcal{O}_{f^{-1}(\underline{D})}$. To see that the complex is exact in the middle, suppose we have $(t, \bar{s}) \in B \oplus A/I$ with $\bar{t} = \bar{s}$. Subtracting the image of $s \in A$, we may assume that $\bar{s} = 0$. It then follows $t \in I \subset A$, so $(t, 0)$ lies in the image of the diagonal map $A \to B \times A/I$.

It remains to treat the multiplicative sequence (3). Again it is immediate that this sequence is a complex that is exact at the outer terms. To see that the complex is exact in the middle, suppose we have a pair $(t, \bar{s}) \in B^\times \times (A/I)^\times$ with $\bar{t}/\bar{s} = 1$. Then $\bar{s} = \bar{t}$, and we just saw that this implies $t \in A$. Repeating this argument with $(1/t, 1/\bar{s})$, we see that $1/t \in A$, hence $t \in A^\times$. $\square$

Next, consider the constructible sheaf $f_*(\mathbb{Z}_{\underline{S}})$ on $\underline{X}$. Each stalk $f_*(\mathbb{Z}_{\underline{S}})_{\bar{x}}$ is a free $\mathbb{Z}$-modules whose rank is the number of irreducible components in $\operatorname{Spec}(\mathcal{O}_{\underline{X},\bar{x}})$. Let $\rho \in \Gamma(\underline{X}, f_*(\mathbb{Z}_{\underline{S}}))$ be the diagonal section defined by $\rho_{\bar{x}} = (1, \ldots, 1)$, which corresponds to $1 \in \Gamma(\underline{S}, \mathbb{Z})$. We have an evaluation map

$$\rho^* : \mathcal{H}om(f_*(\mathbb{Z}_{\underline{S}}), \mathcal{O}^\times_{\underline{X}}) \longrightarrow \mathcal{O}^\times_{\underline{X}}, \quad s \longmapsto s(\rho)$$

and a sequence of abelian sheaves

$$1 \longrightarrow \mathcal{H}om(f_*(\mathbb{Z}_{\underline{S}}), \mathcal{O}^\times_{\underline{X}}) \xrightarrow{\rho^*} \mathcal{O}^\times_{\underline{X}} \longrightarrow i_*(\mathcal{O}^\times_{\underline{D}}) \longrightarrow 1, \tag{4}$$

where $i : \underline{D} \to \underline{X}$ denotes the closed embedding of the space of nonnormality.

Proposition 4.2. *The preceding sequence (4) is exact.*

Proof. For simplicity we set $\mathcal{B} = \mathcal{H}om(f_*(\mathbb{Z}_{\underline{S}}), \mathcal{O}^\times_{\underline{X}})$. The short exact sequence $0 \to \mathbb{Z}_{\underline{X}} \xrightarrow{\rho} f_*(\mathbb{Z}_{\underline{S}}) \to \mathcal{F} \to 0$ defines an abelian sheaf $\mathcal{F}$ with $i^{-1}(\mathcal{F}) = 0$ for some dense open embedding $i : \underline{U} \to \underline{X}$. Applying $\mathcal{H}om(., \mathcal{O}^\times_{\underline{X}})$, we obtain an exact sequence

$$1 \longrightarrow \mathcal{H}om(\mathcal{F}, \mathcal{O}^\times_{\underline{X}}) \longrightarrow \mathcal{B}_{\underline{X}} \xrightarrow{\rho*} \mathcal{O}^\times_{\underline{X}}.$$

There is an inclusion $\mathcal{H}om(\mathcal{F}, \mathcal{O}^\times_{\underline{X}}) \subset \mathcal{H}om(\mathcal{F}, i_* i^{-1}(\mathcal{O}^\times_{\underline{X}}))$ because $\underline{X}$ has no embedded components. Moreover, $\mathcal{H}om(\mathcal{F}, i_* i^{-1}(\mathcal{O}^\times_{\underline{X}})) = i_* \mathcal{H}om(i^{-1}(\mathcal{F}), i^{-1}(\mathcal{O}^\times_{\underline{X}}))$

by [15], Exposé 1, Corollary 1.5. The latter sheaf vanishes because $i^{-1}(\mathcal{F}) = 0$, and we conclude that $\rho^* : \mathcal{B}_{\underline{X}} \to \mathcal{O}_{\underline{X}}^\times$ is injective.

To see that $\mathcal{O}_{\underline{X}}^\times \to i_*(\mathcal{O}_{\underline{D}}^\times)$ is surjective, fix a point $x \in |\underline{D}|$ and a germ $t \in \mathcal{O}_{\underline{D},\bar{x}}^\times$. Then there is a germ $s \in \mathcal{O}_{\underline{X},\bar{x}}$ mapping to t, and this germ is invertible because $s(\bar{x}) \in \kappa(\bar{x})$ is nonzero.

It remains to see that the sequence (4) is exact in the middle at a given point $x \in |\underline{X}|$. This is obvious on $\underline{X} - \underline{D}$, so we may assume that $x \in |\underline{D}|$, in other words, $\operatorname{Spec}(\mathcal{O}_{\underline{X},\bar{x}})$ is not irreducible. We first check that the sequence (4) is a complex at x. Fix a germ $s_{\bar{x}} \in \mathcal{O}_{\underline{X},\bar{x}}^\times$ coming from a germ $t_{\bar{x}} \in \mathcal{B}_{\underline{X},\bar{x}}$. Choose an affine étale neighborhood $\underline{U} \to \underline{X}$ so that $s_{\bar{x}}, t_{\bar{x}}$ admit representants s, t, and that the canonical map $\operatorname{Spec}(\mathcal{O}_{\underline{X},\bar{x}}) \to \underline{U}$ induces a bijection on the set of irreducible components. Decompose $\underline{U} = \underline{U}_1 \cup \ldots \cup \underline{U}_n$, $n \geq 2$ into irreducible components. Using

$$\mathcal{B}_{\underline{U}} = \mathcal{H}om(\bigoplus_{i=1}^n f_*(\mathbb{Z}_{\underline{U}_i}), \mathcal{O}_{\underline{U}}^\times) = \bigoplus_{i=1}^n \mathcal{H}om(\mathbb{Z}_{\underline{U}_i}, \mathcal{O}_{\underline{U}}^\times),$$

we obtain a decomposition $t = (t_1, \ldots, t_n)$ with $t_i \in \operatorname{Hom}(\mathbb{Z}_{\underline{U}_i}, \mathcal{O}_{\underline{U}}^\times)$, and in turn a factorization $s = s_1 \ldots s_n$ with $s_i = t_i(\rho)$. Let $\eta_i \in \underline{U}_i$ be the generic points. Then $(t_i)_{\bar{\eta}_j} = 1$ for $i \neq j$ because $\mathcal{H}om(\mathbb{Z}_{\underline{U}_i}, \mathcal{O}_{\underline{U}}^\times)$ has support on $\underline{U}_i$. Consequently $(s_i)_{\bar{\eta}_j} = 1$, and therefore $s_i|_{\underline{U}_j} = 1$, since the $\underline{U}_j$ have no embedded components. Making a cyclic permutation, we calculate

$$s_{\underline{D}} = (s_1|_{\underline{U}_2})_{\underline{D}} \cdot (s_2|_{\underline{U}_3})_{\underline{D}} \cdot \ldots \cdot (s_{n-1}|_{\underline{U}_n})_{\underline{D}} \cdot (s_n|_{\underline{U}_1})_{\underline{D}} = 1.$$

Hence s_x maps to $1 \in \mathcal{O}_{\underline{D},\bar{x}}^\times$, and the sequence (4) is a complex.

Finally, suppose a germ $s_{\bar{x}} \in \mathcal{O}_{\underline{X},\bar{x}}^\times$ maps to $1 \in \mathcal{O}_{\underline{D},\bar{x}}^\times$. As above, we choose an affine étale neighborhood $\underline{U} \to \underline{X}$ such that $\operatorname{Spec}(\mathcal{O}_{\underline{X},\bar{x}}) \to \underline{U}$ induces a bijection on the set of irreducible components and that $s_{\bar{x}}$ admits a representant s. Write $\underline{U} = \operatorname{Spec}(A)$, $\underline{U}_i = \operatorname{Spec}(A_i)$, and let $t_i \in A_i$ be the image of $s \in A$. Set $B = A_1 \times \ldots \times A_n$. Then $S_{\underline{U}} = \operatorname{Spec}(B)$, and $t = (t_1, \ldots, t_n) \in B$ is the image of $s \in A$. Since $s_D = 1$, we also have $t_i|_{f^{-1}(\underline{D})} = 1$. Now the exact sequence

$$1 \longrightarrow \mathcal{O}_{\underline{X}}^\times \longrightarrow \mathcal{O}_{\underline{S}}^\times \oplus \mathcal{O}_{\underline{D}}^\times \longrightarrow \mathcal{O}_{f^{-1}(\underline{D})}^\times \longrightarrow 1$$

implies that each pair $(t_i, 1) \in \Gamma(\underline{U}, \mathcal{O}_{\underline{S}}^\times \oplus \mathcal{O}_{\underline{D}}^\times)$ comes from a section $s_i \in \Gamma(\underline{U}, \mathcal{O}_{\underline{X}}^\times)$. Sending the i-th standard generator of $f_*(\mathbb{Z}_{\underline{S}})_{\underline{U}}$ to s_i, we obtain a homomorphism $h : f_*(\mathbb{Z}_{\underline{S}})_{\underline{U}} \to \mathcal{O}_{\underline{U}}^\times$ with $h(\rho) = s$ at the generic points. Since $\underline{X}$ has no embedded points, $h(\rho) = s$ holds globally. In other words the germ s_x lies in the image of $\rho^* : \mathcal{B}_{\underline{X}} \to \mathcal{O}_{\underline{X}}^\times$. □

To apply this calculation to log atlases we first need a comparison result for constructible sheaves.

Proposition 4.3. *Suppose $\underline{X}$ is a noetherian algebraic space satisfying Serre's condition (S_2), and let $i : \underline{U} \to \underline{X}$ be an open embedding containing all points of codimension ≤ 1. Let $\mathcal{F}_1, \mathcal{F}_2$ be two constructible abelian sheaves on $\underline{X}$. If $i_* i^{-1}(\mathcal{F}_1)$ is constructible and $i^{-1}(\mathcal{F}_1) \simeq i^{-1}(\mathcal{F}_2)$, then $\mathcal{H}om(\mathcal{F}_1, \mathcal{O}_{\underline{X}}^\times) \simeq \mathcal{H}om(\mathcal{F}_2, \mathcal{O}_{\underline{X}}^\times)$.*

Proof. Let $\mathcal{K}_j, \mathcal{C}_j$ be kernel and cokernel of the adjunction maps $\mathcal{F}_j \to i_* i^{-1}(\mathcal{F}_j)$, respectively. These are constructible abelian sheaves supported by $\underline{X} - \underline{U}$. Applying the functor $\mathcal{H}om(.,\mathcal{O}_{\underline{X}}^\times)$ to the exact sequences of constructible abelian sheaves

$$0\to\mathcal{K}_j\to\mathcal{F}_j\to\mathcal{F}_j/\mathcal{K}_j\to 0 \quad\text{and}\quad 0\to\mathcal{F}_j/\mathcal{K}_j\to i_*i^{-1}(\mathcal{F}_j)\to\mathcal{C}_j\to 0,$$

we reduce our problem to the following special cases: We have a map $\mathcal{F}_1\to\mathcal{F}_2$ that is either injective or surjective, and furthermore bijective on $\underline{U}$.

First, consider the case that we have a surjective mapping $\mathcal{F}_1\to\mathcal{F}_2$, and let $0\to\mathcal{K}\to\mathcal{F}_1\to\mathcal{F}_2\to 0$ be the corresponding exact sequence. This gives an exact sequence

$$1\longrightarrow\mathcal{H}om(\mathcal{F}_2,\mathcal{O}_{\underline{X}}^\times)\longrightarrow\mathcal{H}om(\mathcal{F}_1,\mathcal{O}_{\underline{X}}^\times)\longrightarrow\mathcal{H}om(\mathcal{K},\mathcal{O}_{\underline{X}}^\times).$$

The adjunction map $\mathcal{O}_{\underline{X}}^\times\to i_*i^{-1}(\mathcal{O}_{\underline{X}}^\times)$ is injective, because $\underline{X}$ has no embedded components, hence there is an injection $\mathcal{H}om(\mathcal{K},\mathcal{O}_{\underline{X}}^\times)\subset\mathcal{H}om(\mathcal{K},i_*i^{-1}(\mathcal{O}_{\underline{X}}^\times))$. We have

$$\mathcal{H}om(\mathcal{K},i_*i^{-1}(\mathcal{O}_{\underline{X}}^\times))=i_*\mathcal{H}om(i^{-1}(\mathcal{K}),i^{-1}(\mathcal{O}_{\underline{X}}^\times))$$

by [15], Exposé I, Corollary 1.5, and conclude that $\mathcal{H}om(\mathcal{F}_2,\mathcal{O}_{\underline{X}}^\times)\to\mathcal{H}om(\mathcal{F}_1,\mathcal{O}_{\underline{X}}^\times)$ is bijective.

Second, suppose we have an injection $\mathcal{F}_1\to\mathcal{F}_2$, and let $0\to\mathcal{F}_1\to\mathcal{F}_2\to\mathcal{C}\to 0$ be the corresponding exact sequence. As above, we have $\mathcal{H}om(\mathcal{C},\mathcal{O}_{\underline{X}}^\times)=1$ and obtain an exact sequence

$$1\longrightarrow\mathcal{H}om(\mathcal{F}_2,\mathcal{O}_{\underline{X}}^\times)\longrightarrow\mathcal{H}om(\mathcal{F}_1,\mathcal{O}_{\underline{X}}^\times)\longrightarrow\mathcal{E}xt^1(\mathcal{C},\mathcal{O}_{\underline{X}}^\times).$$

We finish the proof by checking that $\mathcal{E}xt^1(\mathcal{C},\mathcal{O}_{\underline{X}}^\times)$ vanishes. This is a local problem, so we may assume that $\underline{X}$ is an affine scheme. Let $i:\underline{X}-\underline{U}\to\underline{X}$ be the embedding of the closed subset $\underline{X}-\underline{U}$ of codimension ≥ 2. Then $\mathcal{C}=\mathcal{B}_{\underline{X}}$ for the constructible sheaf $\mathcal{B}=i^{-1}(\mathcal{C})$ on $\underline{X}-\underline{U}$, where $\mathcal{B}_{\underline{X}}=i_!(\mathcal{B})$ denotes extension by zero.

According to [17], Exposé IX, Lemma 2.10, there are finitely many étale neighborhoods $\underline{C}_i\to\underline{X}-\underline{U}$, $1\leq i\leq n$ and local sections $s_i\in\Gamma(\underline{C}_i,\mathcal{B})$ so that the corresponding map $\bigoplus_{i=1}^n\mathbb{Z}_{\underline{C}_i}\to\mathcal{B}$ is surjective. We then say that $\mathcal{B}$ is *generated by n local sections.* Let $\mathcal{B}_1\subset\mathcal{B}$ be the subsheaf generated by $\mathbb{Z}_{\underline{C}_1}$. Using the exact sequence

$$\mathcal{E}xt^1((\mathcal{B}/\mathcal{B}_1)_{\underline{X}},\mathcal{O}_{\underline{X}}^\times)\longrightarrow\mathcal{E}xt^1(\mathcal{B}_{\underline{X}},\mathcal{O}_{\underline{X}}^\times)\longrightarrow\mathcal{E}xt^1((\mathcal{B}_1)_{\underline{X}},\mathcal{O}_{\underline{X}}^\times)$$

and induction on the number n of local sections, it suffices to treat the case that $\mathcal{B}$ is generated by a single local section. In other words, there is an exact sequence $0\to\mathcal{B}'\to\mathbb{Z}_{\underline{C}}\to\mathcal{B}\to 0$, where $\underline{C}\to\underline{X}-\underline{U}$ is an étale neighborhood. Then we have an exact sequence

$$\mathcal{H}om(\mathcal{B}'_{\underline{X}},\mathcal{O}_{\underline{X}}^\times)\longrightarrow\mathcal{E}xt^1(\mathcal{B}_{\underline{X}},\mathcal{O}_{\underline{X}}^\times)\longrightarrow\mathcal{E}xt^1(\mathbb{Z}_{\underline{C},\underline{X}},\mathcal{O}_{\underline{X}}^\times).$$

The term on the left vanishes, and we are reduced to the case $\mathcal{C}=\mathbb{Z}_{\underline{C},\underline{X}}$.

Next, choose an affine open covering $\underline{C}_i\subset\underline{C}$, say $1\leq i\leq m$, so that there are affine étale coverings $\underline{V}_i\to\underline{X}$ with $\underline{C}_i=(\underline{X}-\underline{U})\times_{\underline{X}}\underline{V}_i$ (use [14], Exposé I, Proposition 8.1). Using the surjection $\bigoplus_{i=1}^m\mathbb{Z}_{\underline{C}_i}\to\mathbb{Z}_{\underline{C}}$ and repeating the argument in the preceding paragraph, we reduce to the case $m=1$, and write $\underline{C}=\underline{C}_1$ and $\underline{V}=\underline{V}_1$.

Now $\underline{V}\to\underline{X}$ is étale and $\underline{C}\to\underline{V}$ is a closed embedding. Let $\underline{V}'=\underline{V}-\underline{C}$ be the complementary open subset. Then we have an exact sequence of sheaves $0\to\mathbb{Z}_{\underline{V}',\underline{V}}\to\mathbb{Z}_{\underline{V}}\to\mathbb{Z}_{\underline{C},\underline{V}}\to 0$ on $\underline{V}$. Extending by zero, we obtain an exact sequence $0 \to \mathbb{Z}_{\underline{V}',\underline{X}} \to \mathbb{Z}_{\underline{V},\underline{X}} \to \mathbb{Z}_{\underline{C},\underline{X}} \to 0$ on $\underline{X}$, and in turn a long exact sequence

$$\mathcal{H}^0_{\underline{V}'}(\mathcal{O}^\times_{\underline{X}}) \longrightarrow \mathcal{H}^0_{\underline{V}}(\mathcal{O}^\times_{\underline{X}}) \longrightarrow \mathcal{E}xt^1(\mathbb{Z}_{\underline{C},\underline{X}}, \mathcal{O}^\times_{\underline{X}}) \longrightarrow \mathcal{H}^1_{\underline{V}'}(\mathcal{O}^\times_{\underline{X}}) \longrightarrow \mathcal{H}^1_{\underline{V}}(\mathcal{O}^\times_{\underline{X}}).$$

Here we applied the functor $\mathcal{E}xt^n(\cdot, \mathcal{O}^\times_{\underline{X}})$ and identified $\mathcal{E}xt^n(\mathbb{Z}_{\underline{V},\underline{X}}, \mathcal{O}^\times_{\underline{X}})$ with the sheaf of local cohomology groups $\mathcal{H}^n_{\underline{V}}(\mathcal{O}^\times_{\underline{X}})$ as in [15], Exposé I, Proposition 2.3. The map $\mathcal{H}^0_{\underline{V}'}(\mathcal{O}_{\underline{X}}) \to \mathcal{H}^0_{\underline{V}}(\mathcal{O}_{\underline{X}})$ is surjective, because $\underline{X}$ satisfies Serre's condition (S_2) and $\underline{C} = \underline{V} - \underline{V}'$ has codimension ≥ 2. Hence, by Krull's Principal Ideal Theorem, the map $\mathcal{H}^0_{\underline{V}'}(\mathcal{O}^\times_{\underline{X}}) \to \mathcal{H}^0_{\underline{V}}(\mathcal{O}^\times_{\underline{X}})$ is surjective as well.

The sheaf $\mathcal{H}^1_{\underline{V}}(\mathcal{O}^\times_{\underline{X}})$ is associated to the presheaf $\underline{W} \mapsto \mathrm{Pic}(\underline{V} \times_{\underline{X}} \underline{W})$. The restriction map $\mathrm{Pic}(\underline{V} \times_{\underline{X}} \underline{W}) \to \mathrm{Pic}(\underline{V}' \times_{\underline{X}} \underline{W})$ is injective by [32], Lemma 1.1, so the map on sheaves $\mathcal{H}^1_{\underline{V}'}(\mathcal{O}^\times_{\underline{X}}) \to \mathcal{H}^1_{\underline{V}}(\mathcal{O}^\times_{\underline{X}})$ is injective as well. It follows that $\mathcal{E}xt^1(\mathbb{Z}_{\underline{C},\underline{X}}, \mathcal{O}^\times_{\underline{X}})$ vanishes as desired. □

We now apply this to our sheaf $\mathcal{A}_{\underline{X}} = \mathcal{H}om(\overline{\mathcal{M}}_{\underline{X}}, \mathcal{O}^\times_{\underline{X}})$ of automorphism of log structures. Let $\underline{S}$ be the disjoint union of the irreducible components of $\underline{X}$ and $f : \underline{S} \to \underline{X}$ the corresponding finite birational map, which is the normalization map.

Theorem 4.4. *Let $\underline{X}$ be a noetherian algebraic space satisfying Serre's condition (S_2) and whose integral components of* $\mathrm{Spec}(\mathcal{O}_{\underline{X},\bar{x}})$ *are normal for all $x \in |\underline{X}|$, and let $\overline{\mathcal{M}}_{\underline{X}}$ be a constructible monoid sheaf with integral stalks. Suppose there is an open subset $\underline{U} \subset \underline{X}$ containing all points of codimension ≤ 1 with $\overline{\mathcal{M}}_{\underline{U}} \simeq f_*(\mathbb{N}_{\underline{S}})_{\underline{U}}$. Then $\mathcal{A}_{\underline{X}} = \mathcal{H}om(f_*(\mathbb{Z}_{\underline{S}}), \mathcal{O}^\times_{\underline{X}})$, and we have an exact sequence*

$$1 \longrightarrow \mathcal{A}_{\underline{X}} \longrightarrow \mathcal{O}^\times_{\underline{X}} \longrightarrow \mathcal{O}^\times_{\underline{D}} \longrightarrow 1,$$

where $\underline{D} \subset \underline{X}$ is the branch space for the finite birational morphism $f : \underline{S} \to \underline{X}$.

Proof. To check the first assertion we apply Proposition 4.3 with the constructible abelian sheaves $\mathcal{F}_1 = f_*(\mathbb{Z}_{\underline{S}})$ and $\mathcal{F}_2 = \overline{\mathcal{M}}^{\mathrm{gp}}_{\underline{X}}$. We have to check that $i_*i^{-1}f_*(\mathbb{Z}_{\underline{S}})$ is constructible, where $i : \underline{U} \to \underline{X}$ is the canonical open embedding. We do this by showing that the adjunction map $f_*(\mathbb{Z}_{\underline{S}}) \to i_*i^{-1}f_*(\mathbb{Z}_{\underline{S}})$ is bijective. Fix a point $x \in |\underline{X}|$. Then the stalks of both sides at x are the free group generated by the irreducible components of $\mathrm{Spec}(\mathcal{O}_{\underline{X},\bar{x}})$, and bijectivity follows.

Having $\mathcal{A}_{\underline{X}} = \mathcal{H}om(f_*(\mathbb{Z}_{\underline{S}}), \mathcal{O}^\times_{\underline{X}})$, the second assertion directly follows from Proposition 4.2. □

5. The restricted conormal sheaf

We now use the exact sequence from Theorem 4.4 to compute the cohomology group $H^2(\underline{X}, \mathcal{A}_{\underline{X}})$, which contains the obstruction for the existence of a global log structure. We also compute the cohomology group $H^1(\underline{X}, \mathcal{A}_{\underline{X}})$, which measures how many isomorphism classes of global log structures exists. Throughout, we make the following assumptions: Let $\underline{X}$ be a reduced noetherian algebraic space satisfying Serre's condition (S_2) and such that for all $x \in |\underline{X}|$ the integral components of $\mathrm{Spec}(\mathcal{O}_{\underline{X},\bar{x}})$ are normal. Furthermore, $\overline{\mathcal{M}}_{\underline{X}}$ is a constructible monoid sheaf with integral stalks satisfying the conditions of Theorem 4.4. We set $\mathcal{A}_{\underline{X}} = \mathcal{H}om(\overline{\mathcal{M}}_{\underline{X}}, \mathcal{O}^\times_{\underline{X}})$.

Consider the short exact sequence

$$1 \longrightarrow \mathcal{A}_{\underline{X}} \longrightarrow \mathcal{O}_{\underline{X}}^\times \longrightarrow i_*(\mathcal{O}_{\underline{D}}^\times) \longrightarrow 1,$$

where $i : \underline{D} \to \underline{X}$ is the closed embedding of the space of nonnormality. We have $\mathrm{Pic}(\underline{D}) = H^1(\underline{X}, i_*\mathcal{O}_{\underline{D}}^\times)$, because $R^1 i_*(\mathcal{O}_{\underline{D}}^\times) = 0$ by Hilbert's Theorem 90. The preceding short exact sequence gives a long exact sequence

$$\mathrm{Pic}(\underline{X}) \longrightarrow \mathrm{Pic}(\underline{D}) \longrightarrow H^2(\underline{X}, \mathcal{A}_{\underline{X}}) \longrightarrow \mathrm{Br}'(\underline{X}), \tag{5}$$

where $\mathrm{Br}'(\underline{X}) = H^2(\underline{X}, \mathcal{O}_{\underline{X}}^\times)$ is the cohomological Brauer group. We see that an $\mathcal{A}_{\underline{X}}$-gerbe $\mathcal{G}$ faces two obstructions against neutrality: The first obstruction is the image of the gerbe class $[\mathcal{G}] \in H^2(\underline{X}, \mathcal{A}_{\underline{X}})$ in the cohomological Brauer group $\mathrm{Br}'(\underline{X})$. This obstruction vanishes if and only if there is an invertible $\mathcal{O}_{\underline{D}}$-module $\mathcal{N}_{\underline{D}}$ whose $\mathcal{A}_{\underline{X}}$-gerbe of extensions to invertible $\mathcal{O}_{\underline{X}}$-modules is equivalent to $\mathcal{G}$. Once the first obstruction vanishes, the second obstruction is the extendibility of $\mathcal{N}_{\underline{D}}$ to $\underline{X}$. It turns out that, under suitable assumptions, the Brauer obstruction vanishes automatically:

Proposition 5.1. *Suppose there is a global section $\rho \in \Gamma(\underline{X}, \overline{\mathcal{M}}_{\underline{X}})$ such that the stalks $\rho_{\bar\eta}$ generate $\overline{\mathcal{M}}_{\underline{X},\bar\eta} = \mathbb{N}$ for all generic points $\eta \in |\underline{X}|$. Let $\mathcal{G}$ be a log atlas on $\underline{X}$ with respect to $\overline{\mathcal{M}}_{\underline{X}}$. Then the gerbe class $[\mathcal{G}] \in H^2(\underline{X}, \mathcal{A}_{\underline{X}})$ maps to zero in the cohomological Brauer group* $\mathrm{Br}'(\underline{X})$.

Proof. First note that the map $\mathcal{A}_{\underline{X}} \to \mathcal{O}_{\underline{X}}^\times$ from Theorem 4.4 is nothing but the evaluation map $\rho^*(h) = h(\rho)$.

Let $\mathcal{P}$ be the gerbe of invertible sheaves on étale neighborhoods $\underline{U} \to \underline{X}$. This $\mathcal{O}_{\underline{X}}^\times$-gerbe represents the zero element in $H^2(\underline{X}, \mathcal{O}_{\underline{X}}^\times)$. To check that $[\mathcal{G}]$ maps to zero in $\mathrm{Br}'(\underline{X})$, we have to construct a cartesian functor $\mathcal{G} \to \mathcal{P}$ that is equivariant with respect to the map $\rho^* : \mathcal{A}_{\underline{X}} \to \mathcal{O}_{\underline{X}}^\times$, as explained in [8], Chapter IV, Definition 3.1.4. Let $U \in \mathcal{G}$ be a log space. The exact sequence

$$1 \longrightarrow \mathcal{O}_U^\times \longrightarrow \mathcal{M}_U^{\mathrm{gp}} \longrightarrow \overline{\mathcal{M}}_U^{\mathrm{gp}} \longrightarrow 0$$

yields a coboundary map $H^0(\underline{U}, \overline{\mathcal{M}}_U^{\mathrm{gp}}) \to \mathrm{Pic}(\underline{U})$. Let $\mathcal{N}_U$ be the invertible $\mathcal{O}_{\underline{U}}$-module associated to the $\mathcal{O}_{\underline{U}}^\times$-torsor $\mathcal{M}_U \times_{\overline{\mathcal{M}}_{\underline{U}}} \{\rho_U\}$. Then $U \mapsto \mathcal{N}_U$ is the desired cartesian functor $\mathcal{G} \to \mathcal{P}$. You easily check that the diagram

$$\begin{array}{ccc}
H^0(\underline{U}, \mathcal{A}_{\underline{X}}) & \xrightarrow{\rho^*} & H^0(\underline{U}, \mathcal{O}_{\underline{X}}^\times) \\
\simeq\downarrow & & \downarrow\simeq \\
\mathrm{Aut}(U/\underline{U}) & \longrightarrow & \mathrm{Aut}(\mathcal{N}_U)
\end{array}$$

is commutative and compatible with restrictions. This means $\rho^*([\mathcal{G}]) = [\mathcal{P}] = 0$. $\square$

From now on we assume that a section $\rho \in \Gamma(\underline{X}, \overline{\mathcal{M}}_{\underline{X}})$ as in Proposition 5.1 exists. Then we see that the gerbe class $[\mathcal{G}] \in H^2(\underline{X}, \mathcal{A}_{\underline{X}})$ of a log atlas $\mathcal{G}$ comes from an invertible $\mathcal{O}_{\underline{D}}$-module. It turns out that there is a *canonical choice* as follows: Pick an étale covering $\underline{U} \to \underline{X}$ admitting a log space $U \in \mathcal{G}$. Passing to a finer covering, we also have a section $\tilde\rho \in \Gamma(\underline{U}, \mathcal{M}_U)$ mapping to $\rho \in \Gamma(\underline{U}, \overline{\mathcal{M}}_{\underline{X}})$. Next choose an étale covering $\underline{V} \to \underline{U} \times_{\underline{X}} \underline{U}$ so that there is an isomorphism $\phi : p_1^*(U) \to p_0^*(U)$, which is given by a bijection $\phi : p_1^*(\mathcal{M}_U) \to p_0^*(\mathcal{M}_U)$ fixing the subsheaf $\mathcal{O}_{\underline{V}}^{\times}$ pointwise and inducing the identity on the quotient sheaf $\overline{\mathcal{M}}_{\underline{V}}$. As explained in Section 3, the equation

$$c \cdot p_1^*(\phi) = p_0^*(\phi)p_2^*(\phi) \in \operatorname{Isom}(p_1^*p_1^*U, p_0^*p_0^*U)$$

defines a 2-cocycle $c \in Z^2(\underline{V}/\underline{U}, \mathcal{A}_{\underline{X}})$ representing the gerbe class of the log atlas $\mathcal{G}$. Now the equation

$$\phi(p_0^*(\tilde{\rho})) = e \cdot p_1^*(\tilde{\rho}) \in \Gamma(\underline{V}, p_1^*(\mathcal{M}_U)) \tag{6}$$

defines a cochain $e \in C^1(\underline{V}/\underline{U}, \mathcal{O}_{\underline{X}}^{\times})$. We claim that its restriction $e_{\underline{D}}$ to the ramification locus $\underline{D} \subset \underline{X}$ of $f : \underline{S} \to \underline{X}$ becomes a cocycle. Indeed, using the simplicial identities $p_j^*p_i^* = p_i^*p_{j-1}^*$, $i < j$, we compute

$$\begin{aligned} p_1^*(\phi)(p_0^*p_0^*(\tilde{\rho})) &= p_1^*p_1^*(\tilde{\rho}) \cdot p_1^*(e), \\ p_0^*(\phi)p_2^*(\phi)(p_0^*p_0^*(\tilde{\rho})) &= p_1^*p_1^*(\tilde{\rho}) \cdot p_0^*(e)p_2^*(e). \end{aligned} \tag{7}$$

On the other hand, the two isomorphisms $p_1^*(\phi)$ and $p_2^*(\phi)p_0^*(\phi)$ differ by c, and $c(\rho)_{\underline{D}} = 1$ according to Proposition 4.2, hence $p_1^*(e_{\underline{D}}) = p_0^*(e_{\underline{D}})p_2^*(e_{\underline{D}})$.

The cocycle $e_{\underline{D}} \in Z^1(\underline{V}/\underline{U}, \mathcal{O}_{\underline{D}}^{\times})$ defines an invertible $\mathcal{O}_{\underline{D}}$-module $\mathcal{N}_{\underline{D}}$. In fact, its isomorphism class is an invariant of the log atlas $\mathcal{G}$:

Proposition 5.2. *The isomorphism class of $\mathcal{N}_{\underline{D}}$ depends only on the log atlas $\mathcal{G}$ and the section $\rho \in \Gamma(\underline{X}, \overline{\mathcal{M}}_{\underline{X}})$. It maps to the gerbe class $[\mathcal{G}]$ under the coboundary map* $\operatorname{Pic}(\underline{D}) \to H^2(\underline{X}, \mathcal{A}_{\underline{X}})$.

Proof. Replacing the étale coverings $\underline{V} \rightrightarrows \underline{U}$ by some refinement replaces the cocycle $e_{\underline{D}} \in Z^1(\underline{V}/\underline{U}, \mathcal{O}_{\underline{D}}^{\times})$ by its restriction to some finer covering. Changing the lift $\tilde{\rho} \in \Gamma(U, \mathcal{M}_U)$ by some invertible function changes the cocycle $e_{\underline{D}}$ by a coboundary. Modifying the bijection $\phi : p_1^*(\mathcal{M}_U) \to p_0^*(\mathcal{M}_U)$ with some $h \in C^1(\underline{V}/\underline{U}, \mathcal{A}_{\underline{X}})$ does not affect $e_{\underline{D}}$ at all, because $h(\rho)_{\underline{D}} = 1$ by Proposition 4.2. Summing up, the isomorphism class $\mathcal{N}_{\underline{D}} \in \operatorname{Pic}(\underline{D})$ does not depend on our choices.

To calculate the coboundary $\partial(\mathcal{N}_{\underline{D}})$, we use the cochain $e \in C^1(\underline{V}/\underline{U}, \mathcal{O}_{\underline{X}}^{\times})$ from Equation (6) as a lift for the cocycle $e_{\underline{D}}$. Then the cocycle $h \in Z^2(\underline{V}/\underline{U}, \mathcal{A}_{\underline{X}})$ defined by $h(\rho) = p_0^*(e)p_2^*(e)/p_1^*(e)$ represents $\partial(\mathcal{N}_{\underline{D}})$. On the other hand, the bijections $p_1^*(\phi)$ and $p_0^*(\phi)p_2^*(\phi)$ differ by $p_0^*(e)p_2^*(e)/p_1^*(e)$ on $p_1^*p_1^*(\tilde{\rho}) = p_2^*p_1^*(\tilde{\rho})$, according to Equation (7). By Proposition 4.2, this means that these bijections differ by h, and we conclude $\partial(\mathcal{N}_{\underline{D}}) = [\mathcal{G}]$. □

By abuse of notation, we call the invertible $\mathcal{O}_{\underline{D}}$-module $\mathcal{N}_{\underline{D}}$ in Proposition 5.2 the *restricted conormal sheaf* of the log atlas $\mathcal{G}$. The main result of this section is the following classification result:

Theorem 5.3. *There is a global log structure $X \in \mathcal{G}$ if and only if the restricted conormal sheaf $\mathcal{N}_{\underline{D}}$ extends to an invertible $\mathcal{O}_{\underline{X}}$-module.*

Proof. Proposition 4.2 gives an exact sequence

$$\operatorname{Pic}(\underline{X}) \longrightarrow \operatorname{Pic}(\underline{D}) \longrightarrow H^2(\underline{X}, \mathcal{A}_{\underline{X}}).$$

According to Proposition 5.2, the restricted conormal sheaf $\mathcal{N}_{\underline{D}}$ maps to the gerbe class of $\mathcal{G}$, and the assertion follows. □

For the rest of this section we study the action of $H^1(\underline{X}, \mathcal{A}_{\underline{X}})$ on the isomorphism class of global log structures $X \in \mathcal{G}$. First note that each $U \in \mathcal{G}$ comes along with an exact sequence of abelian groups

$$1 \longrightarrow \mathcal{O}_{\underline{U}}^{\times} \longrightarrow \mathcal{M}_U^{\mathrm{gp}} \longrightarrow \overline{\mathcal{M}}_U^{\mathrm{gp}} \longrightarrow 0,$$

and defines a $\mathcal{O}_{\underline{U}}^{\times}$-torsor $\mathcal{M}_U^{\mathrm{gp}} \times_{\overline{\mathcal{M}}_U^{\mathrm{gp}}} \{\rho_U\}$, hence an invertible $\mathcal{O}_{\underline{U}}$-module $\mathcal{N}_U$. We call $\mathcal{N}_U$ the *conormal sheaf* of the log structure. Its restriction to $\underline{D}$ is isomorphic to $\mathcal{N}_{\underline{D}}$, by the very definition of the restricted conormal sheaf below Equation (6).

Proposition 5.4. *Let $X \in \mathcal{G}$ be a global log structure, $\mathcal{N}_X$ its conormal sheaf, $\alpha \in H^1(\underline{X}, \mathcal{A}_{\underline{X}})$ a cohomology class, and $\mathcal{L} = \rho^*(\alpha)$ its image in* $\mathrm{Pic}(\underline{X})$. *Then the conormal sheaf of the global log structure $X + \alpha \in \mathcal{G}$ is isomorphic to $\mathcal{N}_X \otimes \mathcal{L}$.*

Proof. Choose an étale covering $\underline{U} \to \underline{X}$ and a cocycle $h \in Z^1(\underline{V}/\underline{U}, \mathcal{A}_{\underline{X}})$ representing the cohomology class α. Here $\underline{V} = \underline{U} \times_{\underline{X}} \underline{U}$. Let $\phi : p_1^*(U) \to p_0^*(U)$ be the canonical isomorphism such that (U, ϕ) is a descent datum for X. Consequently $(U, \phi h)$ is a descent datum for $X + \alpha$. Refining $\underline{U}$, we may also choose a lift $\tilde{\rho} \in \Gamma(\underline{U}, \mathcal{M}_U)$ for ρ. Then the cocycle $e \in Z^1(\underline{V}/\underline{U}, \mathcal{O}_{\underline{X}}^{\times})$ defined by $e \cdot p_1^*(\tilde{\rho}) = p_0^*(\tilde{\rho})$ represents the conormal sheaf $\mathcal{N}_X$. It follows that $e \cdot h(\rho)$ is both a cocycle for the conormal sheaf of $X + \alpha$ and the tensor product $\mathcal{N}_X \otimes \mathcal{L}$. □

Corollary 5.5. *Let $\mathcal{N}$ be an invertible $\mathcal{O}_{\underline{X}}$-module extending the restricted conormal sheaf $\mathcal{N}_{\underline{D}}$. Then the set of isomorphism classes of global log spaces $X \in \mathcal{G}$ whose conormal sheaf is isomorphic to $\mathcal{N}$ is a torsor for the cokernel $\Gamma(\mathcal{O}_{\underline{D}}^{\times})/\Gamma(\mathcal{O}_{\underline{X}}^{\times})$ of the restriction map $\Gamma(\mathcal{O}_{\underline{X}}^{\times}) \to \Gamma(\mathcal{O}_{\underline{D}}^{\times})$.*

Proof. There is a global log space $X \in \mathcal{G}$ by Theorem 5.3, and its conormal sheaf $\mathcal{N}_X$ extends the restricted conormal sheaf $\mathcal{N}_{\underline{D}}$. Proposition 4.2 gives an exact sequence

$$\Gamma(\mathcal{O}_{\underline{X}}^{\times}) \longrightarrow \Gamma(\mathcal{O}_{\underline{D}}^{\times}) \longrightarrow H^1(\underline{X}, \mathcal{A}_{\underline{X}}) \longrightarrow \mathrm{Pic}(\underline{X}) \longrightarrow \mathrm{Pic}(\underline{D}).$$

Using Proposition 5.4, we we may change the global log structure X by some element in $H^1(\underline{X}, \mathcal{A}_{\underline{X}})$ so that its conormal sheaf becomes isomorphic to $\mathcal{N}$. Moreover, all such log structures differ by elements in the subgroup $\Gamma(\mathcal{O}_{\underline{D}}^{\times})/\Gamma(\mathcal{O}_{\underline{X}}^{\times})$. □

We can say more about the action of the subgroup $\Gamma(\mathcal{O}_{\underline{D}}^{\times})/\Gamma(\mathcal{O}_{\underline{X}}^{\times}) \subset H^1(\underline{X}, \mathcal{A}_{\underline{X}})$ on the global log structures:

Proposition 5.6. *The action of $\Gamma(\mathcal{O}_{\underline{D}}^{\times})/\Gamma(\mathcal{O}_{\underline{X}}^{\times})$ on the set of isomorphism classes of global log structures $X \in \mathcal{G}$ does not change the sheaf of sets $\mathcal{M}_X$ and the surjective map $\mathcal{M}_X \to \overline{\mathcal{M}}_{\underline{X}}$.*

Proof. Given an invertible function $s_{\underline{D}} \in H^0(\underline{D}, \mathcal{O}_{\underline{D}}^{\times})$, choose an étale covering $\underline{U} \to \underline{X}$ so that $s_{\underline{D}}$ extends to a cochain $s \in \Gamma(\underline{U}, \mathcal{O}_{\underline{X}}^{\times})$. Then there is a 1-cocycle $h \in Z^1(\underline{V}/\underline{U}, \mathcal{A}_{\underline{X}})$ with $h(\rho) = p_0^*(s)/p_1^*(s)$, where $\underline{V} = \underline{U} \times_{\underline{X}} \underline{U}$.

Given a log space $X \in \mathcal{G}$, the canonical isomorphism $\phi : p_0^*(U) \to p_1^*(U)$ yields a descent datum (U, ϕ) defining the log space X. As discussed before Remark 3.2, $(U, \phi h)$ is another descent datum defining another log space $X' = (\underline{X}, \mathcal{M}_{X'}, \alpha_{X'})$, and the torsor $\mathrm{Isom}(X, X')$ corresponds to the cohomology class of the coboundary $\partial(s_{\underline{D}}) \in H^1(\underline{X}, \mathcal{A}_{\underline{X}})$. We now exploit that the 1-cocycle h is defined in terms of s, which lives inside $\mathcal{O}_{\underline{X}}^{\times} \subset \mathcal{M}_X$: Indeed, the commutative diagram

$$\begin{tikzcd} p_1^*(\mathcal{M}_U) \arrow[r, "\phi h"] \arrow[d, "p_1^*(s)"'] & p_0^*(\mathcal{M}_U) \arrow[d, "p_0^*(s)"] \\ p_1^*(\mathcal{M}_U) \arrow[r, "\phi"'] & p_0^*(\mathcal{M}_U) \end{tikzcd}$$

constitutes a bijection of descent data, hence a bijection of set-valued sheaves $\mathcal{M}_{X'} \to \mathcal{M}_X$. This map is compatible with the surjections to $\overline{\mathcal{M}}_{\underline{X}}$, because the images of $p_i^*(s)$ in $\overline{\mathcal{M}}_{\underline{V}}$ vanish. □

6. Gorenstein toric varieties

Our next goal is to study log atlases whose log spaces $U \in \mathcal{G}$ are locally isomorphic to a boundary divisors in toroidal embeddings. We come to this in the next section. Here we collect some facts on boundary divisors in toric varieties, which we shall use later.

Fix a ground field k of arbitrary characteristic $p \geq 0$. Recall that affine *toric varieties* are of the form $\underline{Z} = \operatorname{Spec} k[\sigma^\vee \cap M]$. Here M is a finitely generated free abelian group, σ is a convex rational polyhedral cone in $N \otimes_{\mathbb{Z}} \mathbb{R}$ not containing nontrivial linear subspaces, and $N = \operatorname{Hom}(M, \mathbb{Z})$. Note that monoids of the form $P = \sigma^\vee \cap M$ are precisely the fine saturated torsionfree monoids, and we have $M = P^{\text{gp}}$. Here *saturated* means that that each $p \in P^{\text{gp}}$ with $np \in P$ for some integer $n > 0$ lies in P.

From now on we usually write $P = \sigma^\vee \cap M$. To avoid confusion of the additive composition law for the monoid P and the multiplicative composition law for the ring $k[P]$, we use exponential notation $\chi^p \in k[P]$ for elements $p \in P$. We refer to the books of Kempf et al. [22] and Oda [28] for the theory or toric varieties and toroidal embeddings.

The inclusion of monoids $P \subset k[P]$ defines a log space Z with underlying space $\underline{Z}$. Its ghost sheaf $\overline{\mathcal{M}}_Z = \mathcal{M}_Z/\mathcal{O}_Z^\times$ is nothing but the sheaf of effective Cartier divisors that are invariant under the canonical action of the torus $\underline{T} = \operatorname{Spec} k[M]$. Consider the complement $\underline{Z}_0 = \underline{Z} - \underline{T}$ endowed with its reduced structure. We call $\underline{Z}_0$ the *boundary divisor* of the affine toric variety $\underline{Z}$. It inherits the structure of a log space Z_0 from the ambient log space Z. From now on we denote by Z, Z_0 the log spaces whose underlying schemes are toric varieties and their boundary divisors, respectively.

The reflexive rank one sheaf $\mathcal{O}_{\underline{Z}}(\underline{Z}_0)$ corresponding to the boundary divisor $\underline{Z}_0 \subset \underline{Z}$ is a dualizing sheaf for $\underline{Z}$, according to [28], Corollary 3.3 and the Remark thereafter. Consequently, the Weil divisor $\underline{Z}_0 \subset \underline{Z}$ is Cartier if and only if the toric variety $\underline{Z}$ is Gorenstein. In terms of the cone σ, this means that there is an element $\rho_\sigma \in \sigma^\vee \cap M$ such that the linear form $\rho_\sigma \in N^\vee$ takes value 1 on the integral generator of each 1-dimensional face $\sigma_i \subset \sigma$. In terms of the monoid $P = \sigma^\vee \cap M$, this translates into the following condition: There is a unique element $\rho_\sigma \in P$ with $\rho_\sigma + P = \operatorname{int}(P)$, as Stanley explained in [35], Theorem 6.7. Here $\operatorname{int}(P) = (\operatorname{int} \sigma^\vee) \cap M$ is the set of lattice points inside the topological interiour $\operatorname{int} \sigma^\vee$ of the real cone $\sigma^\vee$.

We are mainly interested in this situation. Then the Cartier divisor $\underline{Z}_0 \subset \underline{Z}$ corresponds to the section $\rho_\sigma \in \Gamma(\underline{Z}, \overline{\mathcal{M}}_{\underline{Z}}) = P$, and we shall also denote by $\rho_\sigma \in \Gamma(\underline{Z}_0, \overline{\mathcal{M}}_{\underline{Z}_0}) = P$ the induced section. To summarize the situation:

Proposition 6.1. *Let $\underline{Z} = \operatorname{Spec}(k[\sigma^\vee \cap M])$ be a Gorenstein toric variety. Then the boundary divisor $\underline{Z}_0$ is Cohen-Macaulay, Gorenstein, generically smooth, and has normal crossing singularities in codimension one.*

Proof. Without any hypothesis, the schemes $\underline{Z}$ and $\underline{Z}_0$ are Cohen-Macaulay by Ishida's Criterion (see [28], page 126). By assumption, $\underline{Z}$ is Gorenstein and $\underline{Z}_0$ is Cartier, so $\underline{Z}_0$ is Gorenstein as well. The toric variety $\underline{Z}$ is smooth in codimension ≤ 1, and has A_n-singularities in codimension two. Saying that a point $z \in \underline{Z}$ of codimension two has an A_n-singularity means that the complete local ring $\mathcal{O}^\wedge_{\underline{Z},z}$ is isomorphic to $\kappa(z)[[x^{n+1}, y^{n+1}, xy]]$. A local computation shows that $\underline{Z}_0$ is generically smooth, and is a Cartier divisor inside an A_n-singularity in codimension one. Therefore $\underline{Z}_0$ has normal crossing singularities in codimension one. $\square$

Let us now consider the ghost sheaf $\overline{\mathcal{M}}_{Z_0}$ of the log space Z_0. Later, we have to glue isomorphic copies of such sheaves. The following result tells us that the cocycle condition then holds automatically:

Proposition 6.2. *Let $Z = \operatorname{Spec}(k[\sigma^\vee \cap M])$ be a toric variety with its canonical log structure. Then the sheaf of groups $\mathcal{A}ut(\overline{\mathcal{M}}_{Z_0})$ is trivial.*

Proof. We have to check that $\mathcal{A}ut(\overline{\mathcal{M}}_{Z_0})_{\bar{x}} = 0$ for a given point $x \in Z_0$. Clearly we may assume that x lies in the closed orbit. The generic points $\eta_i \in Z_0$ correspond to the invariant Weil divisors on $\underline{Z}$, which correspond to the extremal rays $\sigma_i \subset \sigma$. We have $\overline{\mathcal{M}}_{Z_0,\bar{x}} = \sigma^\vee \cap M/\sigma^\perp \cap M$, and the localization map $\overline{\mathcal{M}}_{Z_0,\bar{x}} \to \overline{\mathcal{M}}_{Z_0,\bar{\eta}_i}$ is nothing but the canonical map to $\sigma_i^\vee \cap M/\sigma_i^\perp \cap M$. The direct sum of these maps

$$\sigma^\vee \cap M/\sigma^\perp \cap M \longrightarrow \bigoplus_i \sigma_i^\vee \cap M/\sigma_i^\perp \cap M$$

is injective. Since any automorphism of $\overline{\mathcal{M}}_{Z_0}$ obviously induces the identity on $\overline{\mathcal{M}}_{Z_0,\bar{\eta}_i} = \mathbb{N}$, it has to induce the identity on $\overline{\mathcal{M}}_{Z_0,\bar{x}}$ as well. $\square$

We now turn to a problem that occurs if $\underline{Z}$ is singular in codimension two: Although $\underline{Z}_0$ is normal crossing in codimension one, the ghost sheaf $\overline{\mathcal{M}}_{Z_0}$ does not look like the ghost sheaf of a normal crossing singularity. But we definitely need this property to apply Theorem 4.4. To overcome this problem we make another assumption, namely that the toric variety $\underline{Z}$ satisfies the regularity condition (R_2), in other words, $\underline{Z}$ is regular in codimension ≤ 2. In terms of the cone $\sigma \subset N \otimes \mathbb{R}$, this means that for each 2-dimensional face $\sigma' \subset \sigma$, the two integral vectors generating σ' form a basis for $(\sigma' - \sigma') \cap N$, which is a free abelian group of rank two.

Let $\underline{S}$ be the disjoint union of the irreducible components of $\underline{Z}_0$, and $f : \underline{S} \to \underline{Z}_0$ the canonical map. Note that this is in fact the normalization of $\underline{Z}_0$.

Proposition 6.3. *Let $Z = \operatorname{Spec}(k[\sigma^\vee \cap M])$ be a toric variety satisfying regularity condition (R_2), endowed with its canonical log structure. Then there is an open subset $U \subset Z_0$ containing all points of codimension ≤ 1 such that $\overline{\mathcal{M}}_U \simeq f_*(\mathbb{N}_S)|_U$.*

Proof. This is a local problem because $\mathcal{A}ut(\overline{\mathcal{M}}_{Z_0}) = 0$ by Proposition 6.2. Replacing Z by some affine invariant open subsets, we may assume that Z is regular, and then the assertion is trivial. $\square$

Summing up, we can say that for boundary divisors $\underline{Z}_0$ in Gorenstein toric varieties $\underline{Z} = \mathrm{Spec}(k[\sigma^\vee \cap M])$ satisfying regularity condition (R_2), our results from Section 4 and Section 5 do apply.

7. Gorenstein toroidal crossings

In this section we explore log atlases whose log spaces $U \in \mathcal{G}$ are locally boundary divisors in Gorenstein toroidal embeddings that are regular in codimension ≤ 2. Throughout, we fix a ground field k of characteristic $p \geq 0$, and let $\underline{X}$ be an algebraic k-space of finite type. We also fix a constructible monoid sheaf $\overline{\mathcal{M}}_{\underline{X}}$ with surjective specialization maps and a global section $\rho \in \Gamma(\underline{X}, \overline{\mathcal{M}}_{\underline{X}})$.

Suppose we have a log atlas $\mathcal{G}$ on $\underline{X}$ with respect to $\overline{\mathcal{M}}_{\underline{X}}$. A *gtc-chart* consists of the following: A Gorenstein toric variety $Z = \mathrm{Spec}\, k[\sigma^\vee \cap M]$ viewed as a log space and satisfying the regularity condition (R_2), an affine scheme $\underline{U}$ endowed with étale maps $\underline{X} \leftarrow \underline{U} \rightarrow \underline{Z}_0$, and a bijection $\varphi : \overline{\mathcal{M}}_{Z_0}|_{\underline{U}} \rightarrow \overline{\mathcal{M}}_{\underline{X}}|_{\underline{U}}$, such that the following two conditions hold: First, the bijection φ maps the section $\rho_\sigma|_{\underline{U}} \in \Gamma(\underline{U}, \overline{\mathcal{M}}_{Z_0})$ corresponding to the Cartier divisor $\underline{Z}_0 \subset \underline{Z}$ to our given section $\rho|_{\underline{U}}$. Second, we have $(U, \varphi) \in \mathcal{G}$, where U is the log structure induced from the log space Z_0.

By abuse of notation, we usually omit the toric variety Z and the identification φ from the notation and speak about gtc-charts $\underline{X} \leftarrow \underline{U} \rightarrow \underline{Z}_0$. Moreover, we say that a given point $x \in |\underline{X}|$ lies in a gtc-chart $\underline{X} \leftarrow \underline{U} \rightarrow \underline{Z}_0$ if it is in the image of $|\underline{U}| \rightarrow |\underline{X}|$.

Definition 7.1. A log atlas $\mathcal{G}$ on $\underline{X}$ with respect to $\overline{\mathcal{M}}_{\underline{X}}$ is called a *gtc-atlas* if each point $x \in |\underline{X}|$ lies in at least one gtc-chart $\underline{X} \leftarrow \underline{U} \rightarrow \underline{Z}_0$.

The symbol *gtc* abbreviates *Gorenstein toroidal crossings.* This terminology is justified as follows: According to [14], Exposé I, Proposition 8.1, there is an étale covering $\underline{U}' \rightarrow \underline{U}$ and an étale map $\underline{Z}' \rightarrow \underline{Z}$ fitting into a cartesian diagram

$$\begin{array}{ccc} \underline{U}' & \longrightarrow & \underline{Z}' \\ \downarrow & & \downarrow \\ \underline{Z}_0 & \longrightarrow & \underline{Z}. \end{array}$$

Note that $\underline{Z}' - \underline{U}' \subset \underline{Z}'$ is a *toroidal embedding* (see [22], Definition 1 on page 54), so gtc-charts locally identify $\underline{X}$ with the boundary divisor of a Gorenstein toroidal embedding. If $\underline{Z}$ is a regular toric variety, then $\underline{X}$ has normal crossing singularities. The notion of gtc-charts generalize normal crossing singularities to a broader class of singularities, which one might call *Gorenstein toroidal crossing* singularities.

The existence of a gtc-atlas poses certain local conditions on the algebraic space $\underline{X}$. Let $\underline{S}$ be the disjoint union of the irreducible components of $\underline{X}$ and $f : \underline{S} \rightarrow \underline{X}$ the corresponding birational finite map.

Proposition 7.2. *Suppose $\underline{X}$ admits a gtc-atlas $\mathcal{G}$ with respect to $\overline{\mathcal{M}}_{\underline{X}}$. Then $\underline{X}$ is Cohen-Macaulay, Gorenstein, reduced, and has normal crossing singularities in codimension ≤ 1. There is an open subset $\underline{U} \subset \underline{X}$ containing all points of codimension ≤ 1 such that $\overline{\mathcal{M}}_{\underline{U}} = f_*(\mathbb{N}_{\underline{S}})|_{\underline{U}}$. Furthermore, $\rho_{\bar{\eta}}$ generates $\mathcal{M}_{\underline{X},\bar{\eta}} = \mathbb{N}$ for each generic point $\eta \in \underline{X}$.*

Proof. The first assertion follows from the corresponding properties for boundary divisors in Gorenstein toric varieties satisfying regularity condition (R_2), as in

Proposition 6.1. The second assertion is local by Proposition 6.2, and therefore follows from Proposition 6.3. The last assertion is obvious. □

This tells us that the results from Section 4 and Section 5 do apply. In particular, a gtc-atlas $\mathcal{G}$ comes along with its restricted conormal sheaf $\mathcal{N}_{\underline{D}}$ on the subspace of nonnormality $\underline{D} \subset \underline{S}$, and $\mathcal{G}$ admits a global log space $X \in \mathcal{G}$ if and only if the restricted conormal sheaf extends to an invertible sheaf on $\underline{X}$.

Our next goal is to relate gtc-atlases to local infinitesimal deformations. Suppose $\mathcal{G}$ is a gtc-atlas on $\underline{X}$ with respect to $\overline{\mathcal{M}}_{\underline{X}}$ and ρ. Fix a point $x \in |\underline{X}|$ and choose a gtc-chart $\underline{X} \leftarrow \underline{U} \rightarrow \underline{Z}_0$ containing x, with $\underline{U}$ affine. Let $Z = \operatorname{Spec} k[\sigma^\vee \cap M]$ be the corresponding Gorenstein toric variety viewed as a log space, $\rho_\sigma \in \sigma^\vee \cap M$ the monomial defining the Cartier divisor $\underline{Z}_0 \subset \underline{Z}$, and $\chi^{\rho_\sigma} \in k[\sigma^\vee \cap M]$ the corresponding equation. Then $\chi^{2\rho_\sigma}$ defines another Cartier divisor $\underline{Z}_1 \subset \underline{Z}$, and $\underline{Z}_0 \subset \underline{Z}_1$ is an infinitesimal extension with ideal $(\chi^{\rho_\sigma})/(\chi^{2\rho_\sigma}) \simeq \mathcal{O}_{\underline{Z}_0}$. According to [14], Exposé I, Theorem 8.3, there is an étale map $\underline{U}_1 \rightarrow \underline{Z}_1$ fitting into a cartesian diagram

$$\begin{array}{ccc} \underline{U} & \longrightarrow & \underline{U}_1 \\ \downarrow & & \downarrow \\ \underline{Z}_0 & \longrightarrow & \underline{Z}_1, \end{array}$$

and $\underline{U} \subset \underline{U}_1$ is a first order extension with ideal $\mathcal{O}_{\underline{U}}$. The isomorphism class of such extensions correspond to classes in

$$\operatorname{Ext}^1(\Omega^1_{\underline{U}/k}, \mathcal{O}_{\underline{U}}) = H^0(\underline{U}, \mathcal{E}xt^1(\Omega^1_{\underline{X}/k}, \mathcal{O}_{\underline{X}})).$$

The latter groups are isomorphic because we assumed that $\underline{U}$ is affine. Of course, the class of $\underline{U} \subset \underline{U}_1$ depends on the choice of the gtc-chart $\underline{X} \leftarrow \underline{U} \rightarrow \underline{Z}_0$ and the étale map $\underline{U}_1 \rightarrow \underline{Z}_1$. However, we get rid of this dependence if we pass to the limit and allow rescaling:

Proposition 7.3. *The $\mathcal{O}_{\underline{X},\bar{x}}$-submodule in $\mathcal{E}xt^1(\Omega^1_{\underline{X}/k}, \mathcal{O}_{\underline{X}})_{\bar{x}}$ generated by the extension class of $\underline{U} \subset \underline{U}_1$ depends only on the gtc-atlas $\mathcal{G}$.*

Proof. Suppose we have two gtc-charts $\underline{X} \leftarrow \underline{U} \rightarrow \underline{Z}_0$ and $\underline{X} \leftarrow \underline{U}' \rightarrow \underline{Z}'_0$ containing x, with certain affine Gorenstein toric varieties $Z = \operatorname{Spec} k[P]$ and $Z' = \operatorname{Spec} k[P']$. Replacing $\underline{U}$ and $\underline{U}'$ by some common affine étale neighborhood, we may assume $\underline{U} = \underline{U}'$. Choose a point $u \in \underline{U}$ representing x, let $f : \underline{U} \rightarrow \underline{Z}_0$ and $f' : \underline{U} \rightarrow \underline{Z}'_0$ be the canonical maps, and set $z = f(u)$ and $z' = f'(u)$.

Recall that among the toric orbits in the toric variety Z there is a minimal toric orbit, which is the unique closed toric orbit. Replacing P by a suitable localization $P + f\mathbb{Z}$, $f \in P$, and $\underline{U}$ by an open subset, we may assume that the points $f(u) \in Z$ and $f'(u) \in Z'$ are contained in the closed toric orbit. We then have $P/P^\times = \overline{\mathcal{M}}_{X,\bar{x}} = P'/P'^\times$. This identification of monoids extends to an identification of groups $(P/P^\times)^{\mathrm{gp}} = (P'/P'^\times)^{\mathrm{gp}}$, because the monoids in question are saturated. Moreover, the free abelian groups $P^\times$ and $P'^\times$ have the same rank, because both $\dim(\underline{Z}_0)$ and $\dim(\underline{Z}'_0)$ equal the dimension of $\underline{X}$ in a neighborhood of x. We infer that there is an (uncanonical) bijection $b : P \rightarrow P'$ covering the canonical identification $P/P^\times = P'/P'^\times$. The morphism $f : \underline{U} \rightarrow \underline{Z}$ is defined via the composition

$$P_{\underline{U}} \longrightarrow \mathcal{M}_{Z_0}|_U \longrightarrow f^*(\mathcal{M}_{Z_0}) \xrightarrow{\alpha} \mathcal{O}_{\underline{U}}, \tag{8}$$

and the analogous statement holds for $f' : \underline{U} \to \underline{Z}'$. The commutative diagram

$$\begin{array}{ccccc}
\operatorname{Spec}(\mathcal{O}_{\underline{X},\bar{x}}) & \xrightarrow{f'_{\bar{x}}} & \operatorname{Spec}(\mathcal{O}_{\underline{Z}'_0,\bar{z}'}) & \longrightarrow & \operatorname{Spec}(\mathcal{O}_{\underline{Z}',\bar{z}'}) \\
{\scriptstyle\mathrm{id}}\downarrow & & \downarrow{\scriptstyle g_0} & & \\
\operatorname{Spec}(\mathcal{O}_{\underline{X},\bar{x}}) & \xrightarrow[f_{\bar{x}}]{} & \operatorname{Spec}(\mathcal{O}_{\underline{Z}_0,\bar{z}}) & \longrightarrow & \operatorname{Spec}(\mathcal{O}_{\underline{Z}_0,\bar{z}})
\end{array}$$

defines a bijection $g_0 = f_{\bar{x}} {f'}_{\bar{x}}^{-1}$. Note that $f_{\bar{x}}$ and $f'_{\bar{x}}$ are isomorphisms, because f and f' are étale. We now seek to construct a bijection

$$g : \operatorname{Spec}(\mathcal{O}_{\underline{Z}'_0,\bar{z}'}) \longrightarrow \operatorname{Spec}(\mathcal{O}_{\underline{Z}_0,\bar{z}})$$

extending g_0. Replacing $\underline{U}$ by some smaller affine étale neighborhood, we may assume that there is an isomorphism of log structures $\phi : f^*(\mathcal{M}_{Z_0}) \to {f'}^*(\mathcal{M}_{Z'_0})$. We have inclusions of sheaves $P_{\underline{U}} \subset f^*(\mathcal{M}_{Z_0})$ and $P'_{\underline{U}} \subset {f'}^*(\mathcal{M}_{Z_0})$, and these constant submonoid sheaves both surject onto $\overline{\mathcal{M}}_U$. Consequently, the equation

$$\phi(p) = h(p) \cdot b(p), \quad p \in P$$

inside the stalk ${f'}^*(\mathcal{M}_{Z_0})_{\bar{x}}$ defines a map $h : P \to \mathcal{O}^\times_{\underline{X},\bar{x}}$. As in the proof of Proposition 2.2, we infer that h is a homomorphism of monoids. Since P^{gp} is free, we may lift h to a monoid homomorphism $h : P \to \mathcal{O}^\times_{\underline{Z}',\bar{z}'}$.

To proceed, let $k[P']^{\mathrm{sh}} = \mathcal{O}_{Z',\bar{z}'}$ be the strict henselization of $k[P']$ at the prime ideal corresponding to $z' \in Z'$. The map $P \to k[P']^{\mathrm{sh}}$, $p \mapsto h(p)\chi^{b(p)}$ defines a homomorphism $k[P] \to k[P']^{\mathrm{sh}}$, which by (8) makes the diagram

$$\begin{array}{ccc}
\operatorname{Spec}(\mathcal{O}_{\underline{X},\bar{x}}) & \xrightarrow{f'} & \operatorname{Spec}(\mathcal{O}_{\underline{Z}',\bar{z}'}) \\
{\scriptstyle\mathrm{id}}\downarrow & & \downarrow \\
\operatorname{Spec}(\mathcal{O}_{\underline{X},\bar{x}}) & \xrightarrow[f]{} & \operatorname{Spec}(k[P])
\end{array}$$

commutative. Therefore the preimage of the maximal ideal in $k[P']^{\mathrm{sh}}$ under the map $k[P] \to k[P']^{\mathrm{sh}}$ is the prime ideal in $k[P]$ corresponding to $z \in \underline{Z}$. In turn, we obtain a homomorphism $k[P]^{\mathrm{sh}} \to k[P']^{\mathrm{sh}}$, where $k[P]^{\mathrm{sh}} = \mathcal{O}_{\underline{Z},\bar{z}}$ is the strict henselization of $k[P]$ at the prime ideal for $z \in Z$. This homomorphism defines the desired morphism g making the diagram

$$\begin{array}{ccccc}
\operatorname{Spec}(\mathcal{O}_{\underline{X},\bar{x}}) & \longrightarrow & \operatorname{Spec}(\mathcal{O}_{\underline{Z}_0,\bar{z}}) & \longrightarrow & \operatorname{Spec}(\mathcal{O}_{\underline{Z},\bar{z}}) \\
{\scriptstyle\mathrm{id}}\downarrow & & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g} \\
\operatorname{Spec}(\mathcal{O}_{\underline{X},\bar{x}}) & \longrightarrow & \operatorname{Spec}(\mathcal{O}_{\underline{Z}'_0,\bar{z}'}) & \longrightarrow & \operatorname{Spec}(\mathcal{O}_{\underline{Z}',\bar{z}'})
\end{array}$$

commutative. The rest is easy: Choose affine étale neighborhoods $\underline{W} \to \underline{Z}$ and $\underline{W}' \to \underline{Z}'$ so that there is an isomorphism $g : \underline{W} \to \underline{W}'$ representing the germ $g : \operatorname{Spec}(\mathcal{O}_{\underline{Z}',\bar{z}'}) \to \operatorname{Spec}(\mathcal{O}_{\underline{Z},\bar{z}})$, and replace $\underline{U}$ by some smaller étale neighborhood so that there is a commutative diagram

$$\begin{array}{ccccc}
\underline{U} & \longrightarrow & \underline{W}_0 & \longrightarrow & W \\
{\scriptstyle\mathrm{id}}\downarrow & & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g} \\
\underline{U} & \longrightarrow & \underline{W}'_0 & \longrightarrow & W'.
\end{array}$$

Now let $\underline{U} \subset \underline{U}_1$ and $\underline{U} \subset \underline{U}_1{}'$ be the corresponding first order extensions defined by $\underline{W}$ and $\underline{W}'$, respectively. According to [13], Theorem 18.1.2, we have an isomorphism $\underline{U}_1 \simeq \underline{W}_1 \times_{\underline{W}_1{}'} \underline{U}_1{}'$, and conclude that the first order extensions $\underline{U}_1, \underline{U}_1{}'$ generate the same cyclic $\mathcal{O}_{\underline{X},\bar{x}}$-submodule in $\mathcal{E}xt^1(\Omega^1_{\underline{X}/k}, \mathcal{O}_{\underline{X}})_{\bar{x}}$. □

Next we ask whether the collection of cyclic submodules in $\mathcal{E}xt^1(\Omega^1_{\underline{X}/k}, \mathcal{O}_{\underline{X}})_{\bar{x}}$, $x \in |\underline{X}|$ generated by gtc-charts are the stalks of a coherent subsheaf. This is indeed true at least over the subspace of nonnormality:

Theorem 7.4. *Suppose $\underline{X}$ admits a gtc-atlas $\mathcal{G}$ with respect to $\overline{\mathcal{M}}_{\underline{X}}$ and ρ. Let $\underline{D} \subset \underline{X}$ be the subspace of nonnormality, and $\mathcal{N}_{\underline{D}} \in \operatorname{Pic}(\underline{D})$ the restricted conormal sheaf of $\mathcal{G}$. Then there is an injection $\mathcal{N}_{\underline{D}}^\vee \subset \mathcal{E}xt^1(\Omega^1_{\underline{X}/k}, \mathcal{O}_{\underline{X}})_{\underline{D}}$ whose stalks are the cyclic $\mathcal{O}_{\underline{X}}$-submodules generated by gtc-charts.*

Proof. Choose gtc-charts $\underline{X} \leftarrow \underline{U}_i \to \underline{Z}_{i0}$ so that the disjoint union $\underline{U} = \bigcup \underline{U}_i$ is an étale covering of $\underline{X}$. Let $\underline{Z}_i = \operatorname{Spec}(k[P_i])$ with $P_i = \sigma_i^\vee \cap M_i$ be the corresponding Gorenstein toric variety, and $\rho_i \in P_i$ the element defined by the Cartier divisor $\underline{Z}_{i0} \subset \underline{Z}_i$. Then $\chi^{2\rho_i} \in k[P_i]$ defines a first order extension $\underline{Z}_{i0} \subset \underline{Z}_{i1}$, and by [13], Theorem 18.1.2 there is a cartesian diagram

$$\begin{array}{ccc} \underline{U}_i & \longrightarrow & \underline{U}_{i1} \\ \big\downarrow & & \big\downarrow \\ \underline{Z}_{i0} & \longrightarrow & \underline{Z}_{i1}, \end{array}$$

whose vertical arrows are étale. We have to understand how the first order extensions $\underline{U}_i \subset \underline{U}_{i1}$ differ on the overlaps $\underline{U}_{ij} = \underline{U}_i \times_{\underline{X}} \underline{U}_j$. Fix a point $u \in \underline{U}_{ij}$, and choose an affine étale neighborhood $\underline{V}' \to \underline{U}_{ij}$ of u so that there is an isomorphism $\phi: U_j|_{\underline{V}'} \to U_i|_{\underline{V}'}$ of log spaces. Such isomorphism is given by a bijection $\phi: p_1^*(\mathcal{M}_{U_j}) \to p_0^*(\mathcal{M}_{U_j})$. Here p_0 and p_1 are the projections from V' onto the second and first factor of $\underline{U}_{ij}$, respectively (compare Section 3). Note that $\underline{V}' = \underline{V}'_{ijx}$ depends on i, j, x, but we suppress this dependence to keep notations simple.

The sections $\rho_i \in \Gamma(\underline{U}_i, \mathcal{M}_{U_i})$ are lifts for $\rho|_{\underline{U}_i} \in \Gamma(\underline{U}_i, \overline{\mathcal{M}}_X)$, hence $\phi(\rho_j|_{\underline{V}'}) = e' \cdot \rho_i|_{\underline{V}'}$ for some $e' \in \Gamma(\underline{V}', \mathcal{O}_{\underline{X}}^\times)$. Recall from Section 5 that the restricted conormal sheaf $\mathcal{N}_{\underline{D}}$ is defined in terms of such e'.

Let $x \in |\underline{X}|$, $z_i \in \underline{Z}_i$, and $z_j \in \underline{Z}_j$ be the images of $u \in \underline{U}_{ij}$. In the proof of Proposition 7.3, we constructed a bijection $g: \operatorname{Spec}(\mathcal{O}_{\underline{Z}_j,\bar{z}_j}) \to \operatorname{Spec}(\mathcal{O}_{\underline{Z}_i,\bar{z}_i})$ inducing a commutative diagram

$$\begin{array}{ccccc} \operatorname{Spec}(\mathcal{O}_{\underline{X},\bar{x}}) & \longrightarrow & \operatorname{Spec}(\mathcal{O}_{\underline{Z}_{j0},\bar{z}_j}) & \longrightarrow & \operatorname{Spec}(\mathcal{O}_{\underline{Z}_j,\bar{z}_j}) \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle g_0} & & \big\downarrow{\scriptstyle g} \\ \operatorname{Spec}(\mathcal{O}_{\underline{X},\bar{x}}) & \longrightarrow & \operatorname{Spec}(\mathcal{O}_{\underline{Z}_{i0},\bar{z}_i}) & \longrightarrow & \operatorname{Spec}(\mathcal{O}_{\underline{Z}_i,\bar{z}_i}). \end{array}$$

By its very definition, the map g sends $\chi^{\rho_j} \in \mathcal{O}_{\underline{Z}_j,\bar{z}_j}$ to $e' \cdot \chi^{\rho_i} \in \mathcal{O}_{\underline{Z}_i,\bar{z}_i}$. If follows that the extension class $\lambda_j \in \Gamma(\underline{U}_j, \mathcal{E}xt^1(\Omega^1_{\underline{X}/k}, \mathcal{O}_{\underline{X}}))$ of $\underline{U}_j \subset \underline{U}_{j1}$ and the extension class $\lambda_i \in \Gamma(\underline{U}_j, \mathcal{E}xt^1(\Omega^1_{\underline{X}/k}, \mathcal{O}_{\underline{X}}))$ of $\underline{U}_i \subset U_{i1}$ are related by $\lambda_j|_{\underline{V}'} = e' \cdot \lambda_i|_{\underline{V}'}$, at least after refining $\underline{V}'$. This explains why the local extension classes λ_i do not necessarily satisfy the cocycle condition. However, we showed in Section 5 below Equation (7) that the cocycle condition for e' holds after restricting to the space of nonnormality $\underline{D} \subset \underline{X}$.

To be precise, set $\underline{V} = \bigcup \underline{V}'$, where the disjoint union runs over all étale neighborhoods $\underline{V}' = \underline{V}'_{i,j,x}$. Then the canonical map $\underline{V} \to \underline{U} \times_{\underline{X}} \underline{U}$ is an étale covering. In this set-up, $\lambda \in C^1(\underline{V}/\underline{U}, \mathcal{E}xt^1(\Omega^1_{\underline{X}/k}, \mathcal{O}_{\underline{X}}))$ is a cochain. Restricting to $\underline{D}$ we obtain another cochain $\lambda_{\underline{D}} \in C^1(\underline{V}/\underline{U}, \mathcal{E}xt^1(\Omega^1_{\underline{X}/k}, \mathcal{O}_{\underline{X}}) \otimes \mathcal{O}_{\underline{D}})$. On each $\underline{U}_i$, the section $\rho_i \in \Gamma(\underline{U}_i, \mathcal{M}_{U_i})$ defines a trivialization of $\mathcal{N}_{\underline{D}}$, so we get an identification

$$C^1(\underline{V}/\underline{U}, \mathcal{E}xt^1(\Omega^1_{\underline{X}/k}, \mathcal{O}_{\underline{X}}) \otimes \mathcal{O}_{\underline{D}}) = C^1(\underline{V}/\underline{U}, \mathcal{E}xt^1(\Omega^1_{\underline{X}/k}, \mathcal{O}_{\underline{X}}) \otimes \mathcal{N}_{\underline{D}}).$$

Now $\lambda_{\underline{D}}$, viewed as a cochain with values in $\mathcal{E}xt^1(\Omega^1_{\underline{X}/k}, \mathcal{O}_{\underline{X}}) \otimes \mathcal{N}_{\underline{D}})$, satisfies the cocycle condition, according to the arguments below Equation (7). Consequently, $\lambda_{\underline{D}}$ defines a global section of $\mathcal{E}xt^1(\Omega^1_{\underline{X}/k}, \mathcal{O}_{\underline{X}}) \otimes \mathcal{N}_{\underline{D}}$, and in turn the desired homomorphism $\mathcal{N}^\vee_{\underline{D}} \to \mathcal{E}xt^1(\Omega^1_{\underline{X}/k}, \mathcal{O}_{\underline{X}})_{\underline{D}}$. A local computation shows that this map is bijective in codimension ≤ 1. Here we use the assumption that our toric varieties Z_i are regular in codimension ≤ 2. Since $\underline{D}$ has no embedded component by Ishida's Criterion ([28], page 126), the map $\mathcal{N}^\vee_{\underline{D}} \to \mathcal{E}xt^1(\Omega^1_{\underline{X}/k}, \mathcal{O}_{\underline{X}})$ is injective everywhere. ☐

For gtc-atlases, the restricted conormal sheaf $\mathcal{N}_{\underline{D}}$ thus has two interpretations. First in terms of cocycles obtained from $\rho \in \Gamma(\underline{X}, \overline{\mathcal{M}}_{\underline{X}})$ as in Section 5, and second in term of first order extensions $\underline{U} \subset \underline{U}_1$ as in Theorem 7.4. We now state a generalization of Kato's result, who considered spaces with normal crossing singularities ([20], Theorem 11.7):

Theorem 7.5. *Let $\mathcal{G}$ be a gtc-atlas on $\underline{X}$ with respect to $\overline{\mathcal{M}}_{\underline{X}}$ and $\rho \in \Gamma(\underline{X}, \overline{\mathcal{M}}_{\underline{X}})$, and $\underline{D} \subset \underline{X}$ the space of nonnormality. Then there is a global log space $X \in \mathcal{G}$ if and only if the restricted normal sheaf $\mathcal{N}^\vee_{\underline{D}} \subset \mathcal{E}xt^1(\Omega^1_{\underline{X}/k}, \mathcal{O}_{\underline{X}})$ extends to an invertible $\mathcal{O}_{\underline{X}}$-module.*

Proof. This is Theorem 5.3 in the special case of gtc-atlases. ☐

Remark 7.6. In the normal crossings case the gerbe $\mathcal{G}$ on $\underline{X}$ is uniquely determined by the requirement that $\overline{\mathcal{M}}_{\underline{X}} = f_* \mathbb{N}_{\underline{S}}$ for $f : \underline{S} \to \underline{X}$ the normalization. This is due to the fact that such log structures are locally unique as shown in [20], see also [21]. Indeed, if $\underline{X} \to \operatorname{Spec} k[z_1, \ldots, z_n]/(z_1 \ldots z_r)$ is étale in $x \in |\underline{X}|$ then there exist $m_1, \ldots, m_r \in \mathcal{M}_{\underline{X},\bar{x}}$ generating $\overline{\mathcal{M}}_{\underline{X},\bar{x}} = \mathbb{N}^r$ and with $\alpha_X(m_i) = z_i$, $i = 1, \ldots, r$. For any other choices $m'_1, \ldots, m'_r$ the map $m_i \mapsto m'_i$ defines uniquely an automorphism of $\mathcal{M}_{\underline{X},\bar{x}}$ fixing $\mathcal{O}^\times_{\underline{X},\bar{x}}$.

This argument does not work if $\overline{\mathcal{M}}_{\underline{X},\bar{x}}$ has relations. For example, consider the quadruple point $\underline{X} = \operatorname{Spec} k[z_1, z_2, z_3, z_4]/(z_1 z_3, z_2 z_4)$ with $\overline{\mathcal{M}}_{\underline{X}}$ the ghost sheaf induced by the embedding into the toric variety $\operatorname{Spec} k[z_1, z_2, z_3, z_4]/(z_1 z_3 - z_2 z_4)$, with k separably closed. Then the set of isomorphism classes of log structures on $\underline{X}$ is canonically $(\mathbb{N} \setminus \{0\})^4 \times k^\times$, as explained in [10], Example 3.13. Note this example is normal crossings away from the distinguished closed point of multiplicity 4 and hence the non-uniqueness is concentrated at this point.

For the general case $\underline{X} = \operatorname{Spec} k[P]/(\chi^\rho)$ with a Gorenstein toric monoid P, $\rho \in P$ the distinguished element, and $\overline{\mathcal{M}}_{\underline{X}}$ the ghost sheaf induced by the embedding into $\operatorname{Spec} k[P]$, Proposition 3.14 in the same paper says the following. Let $\underline{x} \in \underline{X}$ be the distinguished closed point. Then the set of isomorphism classes of germs at $\underline{x}$ of gtc-structures on $\underline{X}$ with ghost sheaf $\overline{\mathcal{M}}_{\underline{X}}$ injects into $\mathcal{E}xt^1(\overline{\mathcal{M}}^{\mathrm{gp}}_{\underline{X}}, \mathcal{O}^\times_{\underline{X}})_{\underline{x}}$ by associating the extension class. Moreover, there is an explicit description of both

$\mathcal{E}xt^1(\overline{\mathcal{M}}^{\mathrm{gp}}_{\underline{X}}, \mathcal{O}^\times_{\underline{X}})_{\underline{x}}$ and the image of the germs of log structures in terms of functions h_p, $p \in P$, on open subsets of $\underline{X}$. The function h_p is defined on the complement of $V(\chi^p) \subset \underline{X}$. Conversely, given $(h_p)_{p\in P}$ such that h_p extends to X by 0 then $p \mapsto h_p$ defines a chart for the corresponding log structure.

This description also suggests a notion of *type* for germs of log structures on $\underline{X}$, namely if their representatives (h_p), (h'_p) differ only by invertible functions [10], Definition 3.15. Globally two log structures are of the same type if they are of the same type at each point. Log structures of the same type have charts with image in the same toric variety and inducing the same combinatorial identification of prime components with toric prime divisors. In the example of the quadruple point fixing the type means choosing an element in $(\mathbb{N} \setminus \{0\})^4$.

Note that in any case $\overline{\mathcal{M}}_{\underline{X}}$ is naturally a subsheaf of $f_*\mathbb{N}_{\underline{S}}$ for $f : \underline{S} \to \underline{X}$ the normalization, and this subsheaf determines the type of log structure. Indeed, it suffices to check this for $\underline{X}$ the boundary divisor in a toric variety. Let $g_1 : \underline{X} \to \operatorname{Spec} k[P]$, $g_2 : \underline{X} \to \operatorname{Spec} k[P]$ be isomorphisms of $\underline{X}$ with the boundary divisor of the toric variety $\operatorname{Spec} k[P]$ inducing the same embedding of P into $\mathbb{N}^r$, r the number of irreducible components of $\underline{X}$. Then for any $p \in P$ the orders of vanishing of $g_1^*(\chi^p)$ and $g_2^*(\chi^p)$ along the toric prime divisors agree and hence there exists $h_p \in \Gamma(\mathcal{O}_{\underline{X}})$ with $g_1^*(\chi^p) = h_p g_2^*(\chi^p)$. This shows that the two log structures induced by g_1 and g_2 are of the same type. In particular, comparing the type of log structures for a given set of charts is a finite problem that in practice can often be done by hand.

Taken together this gives a three-step solution to the problem of constructing gtc structures on a given algebraic space $\underline{X}$: First determine the type of gtc structure by covering $\underline{X}$ with finitely many charts of the same type on overlaps as discussed. In the next step one needs to compare the selected sections of $\mathcal{E}xt^1(\overline{\mathcal{M}}^{\mathrm{gp}}_{\underline{X}}, \mathcal{O}^\times_{\underline{X}})$ and adjust if necessary. Although this step is still abelian in nature, it is probably the most difficult one in practice. On the other hand, on the (semi-stable) normal crossings locus where $\overline{\mathcal{M}}_{\underline{X}} = f_*\mathbb{N}_{\underline{S}}$ the subsheaf of $\mathcal{E}xt^1(\overline{\mathcal{M}}^{\mathrm{gp}}_{\underline{X}}, \mathcal{O}^\times_{\underline{X}})$ parametrizing log structures of semi-stable type is trivial and hence has a unique section. This follows from the mentioned explicit description of this sheaf in [10], and it reflects the uniqueness of the gerbe $\mathcal{G}$ on such spaces discussed above. Thus this second step is simple on the normal crossings locus. The third and last step is an application of the theorem above.

That this is indeed a viable approach has been shown in [10]. In this paper $\underline{X}$ is a union of toric varieties and the result is a classification of gtc structures in terms of a certain, computable sheaf cohomology group on a real integral affine manifold B built on the dual intersection complex of $\underline{X}$. In this case the given cell decomposition of B already determines the ghost sheaf.

8. Triple points and quadruple points

It is now time to illustrate the general theory with some concrete examples. The examples are normal crossing except at finitely many points. According to Remark 7.6 to define the gerbe $\mathcal{G}$ it suffices to specify charts at these points.

Example 8.1. We start by looking at 3-dimensional affine toric varieties $Z = \operatorname{Spec} k[\sigma^\vee \cap \mathbb{Z}^3]$ that are Gorenstein and (R_2), such that the boundary divisor Z_0 has three irreducible components. Let $\rho \in \sigma^\vee \cap \mathbb{Z}^3$ be the unique element with

$\rho + \sigma^\vee \cap \mathbb{Z}^3 = (\operatorname{int}\sigma^\vee) \cap \mathbb{Z}^3$. After changing coordinates, we may assume that $\rho = (0,0,1)$. Let $H \subset \mathbb{R}^3$ be the affine hyperplane defined by the affine equation $\rho^\vee = 1$. Then the cone σ is generated by a *lattice triangle* in H generated by $v_1, v_2, v_3 \in H$ such that the vertices are the only boundary lattice points.

Applying an integral linear coordinate change fixing $\rho \in \mathbb{Z}^3$, we may assume $v_1 = (0,0,1)$, $v_2 = (1,0,1)$ and $v_3 = (a,b,1)$ for some $a, b \in \mathbb{Z}$. Making further coordinate changes using the matrices

$$\begin{pmatrix} 1 & & \\ & \pm 1 & \\ & & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & \pm 1 & \\ & 1 & \\ & & 1 \end{pmatrix}$$

we end up with $0 \leq b$ and $0 \leq a < b$. The condition that the segments $\overline{v_1 v_3}$ and $\overline{v_2 v_3}$ contain no additional lattice point means that both $a, a-1$ are prime to b. Note that b is necessarily odd, because either a or $a-1$ is even. Moreover, $b \geq 3$ implies $a \geq 2$. The case $v_3 = (0,1,1)$ yields the regular toric variety. The simplest nontrivial case is therefore $v_3 = (2,3,1)$, which defines the unique isomorphism class of lattice triangle with one interior lattice point and three boundary lattice points.

The boundary divisor $Z_0 \subset Z$ decomposes into three irreducible components $Z_0 = Z_{01} \cup Z_{02} \cup Z_{03}$ corresponding to the vectors v_1, v_2, v_3. Each Z_{0i} is a 2-dimensional affine toric variety. Its cone is the image of σ under the canonical projection $\mathbb{Z}^3 \to \mathbb{Z}^3/\mathbb{Z}v_i$. Since $\det(v_1, v_2, v_3) = b$, the Z_{0i} are affine toric surfaces containing the rational Gorenstein singularity of type A_{b-1}. Note that the underlying scheme $\underline{Z}_0$ is determined up to isomorphism by the integer $b \geq 1$. This is because the normalization map $\amalg \underline{Z}_{0i} \to \underline{Z}_0$ is determined in codimension ≤ 1, compare the discussion in [31], Section 2.

On the other hand, the log space Z_0 depends on the integer a. How many such a are possible? Suppose for a moment that $b = p^n$ is an odd prime power. Then both $a, a-1$ are prime to p if and only if a is neither in $p\mathbb{Z}/(p^n)$ nor in $1 + p\mathbb{Z}/(p^n)$. Hence there are $p^n - 2p^{n-1} = p^{n-1}(p-2)$ choices for a. In general, decompose $b = \prod p_i^{n_i}$ into prime factors. Then there are $\prod p_i^{n_i - 1}(p_i - 2)$ possibilities for a.

Now suppose we have a 2-dimensional algebraic k-scheme $\underline{X}$ that is normal crossing in codimension ≤ 1 and whose irreducible components $\underline{X}_i$ are normal. Let $x_j \in \underline{X}$ be the closed points where at least three irreducible components meet. Away from the x_j our gtc-atlas is uniquely determined by the requirement that $\overline{\mathcal{M}}_{\underline{X}}$ agree with $f_* \mathbb{N}_{\underline{X}}$, $f : \underline{S} \to \underline{X}$ the normalization. We assume that each closed point $\underline{x}_j \in \underline{X}$ that is not normal crossing is étale locally isomorphic to $\underline{Z}_0 = \underline{Z}_{j0}$ at the origin for certain odd integers $b_j \geq 1$. The choice of integers $0 \leq a_j < b_j$ such that both $a_j, a_j - 1$ are prime to b_j now specifies a gtc-chart $\mathcal{G}$ on $\underline{X}$ that is naturally compatible with the already chosen gtc-atlas on the complement of the $\underline{x}_j$.

Example 8.2. Let us now consider another example. Let $Z = \operatorname{Spec} k[\sigma^\vee \cap \mathbb{Z}^3]$ be a 3-dimensional Gorenstein toric variety satisfying (R_2), such that the boundary divisor Z_0 has four irreducible components. Now the cone $\sigma \subset N \otimes \mathbb{R}$ is generated by a *lattice tetragon* in the affine hyperplane $H \subset N \otimes \mathbb{R}$ whose vertices are the only boundary lattice points. Let $v_1, \ldots, v_4 \in H$ be the vertices of such a lattice tetragon. After an integral coordinate change, we may assume $v_1 = (0,0,1)$, $v_2 = (1,0,1)$,

$v_3 = (a, b, 1)$ with $0 \leq a < b$ and $\gcd(a-1, b) = 1$, and $v_4 = (c, d, 1)$ with $\gcd(c, d) = \gcd(c-a, d-b) = 1$. The convexity condition is $ad - bc > 0$ and $d > 0$.

Let $Z_{01}, \ldots, Z_{04} \subset Z_0$ be the irreducible components corresponding to the vectors $v_1, \ldots, v_4 \in \sigma$, respectively. Each Z_{i0} is a Gorenstein toric variety. We have $\det(v_4, v_1, v_2) = d$, so the invariant closed point on Z_{01} is the rational Gorenstein singularity of type A_{d-1}. Similarly, Z_{02} has type A_{b-1}, and Z_{03} has type $A_{b-d+ad-bc-1}$, and Z_{04} has type $A_{ad-bc-1}$.

Let us now concentrate on the special case $a = b = d = 1$ and $c = 0$, that is $v_3 = (1, 1, 1)$ and $v_4 = (0, 1, 1)$. This corresponds to the unique lattice tetragon containing precisely four lattice points. Then every irreducible component Z_{i0} is smooth. The boundary divisor Z_0 is a complete intersection isomorphic to the spectrum of $A = k[x, y, u, v]/(xy, uv)$. Note that we may view Z_0 as the product of two 1-dimensional normal crossings. The space of nonnormality $D \subset Z_0$ is the union of the four coordinate axis in $\mathbb{A}^4_k$, given by the subring in $k[x] \times k[y] \times k[u] \times k[v]$ of polynomials with identical constant term. Using the coordinates x, y, u, v, we calculate

$$\begin{aligned}\operatorname{Ext}^1(\Omega^1_{Z_0/k}, \mathcal{O}_{Z_0}) &= A/(\frac{\partial}{\partial x}xy, \frac{\partial}{\partial y}xy) \oplus A/(\frac{\partial}{\partial u}uv, \frac{\partial}{\partial v}uv) \\ &= k[u, v]/(uv) \oplus k[x, y]/(xy).\end{aligned}$$

Under this identification, the restricted conormal sheaf $\mathcal{N}_D \subset \mathcal{E}xt^1(\Omega^1_{Z_0/k}, \mathcal{O}_{Z_0})$ corresponds to the diagonal submodule $(f(u, v), f(x, y))$.

Here is an example for a proper algebraic surface having such a quadruple point: Let $S \to \mathbb{P}^1_k$ be a Hirzebruch surface of degree $e \geq 0$. We denote by C_1 the unique section with $C_1^2 = -e$, and choose another section $C_3 \subset S$ with $C_2^2 = e$. Let $C_2, C_3 \subset S$ be the fibers over $0, \infty \in \mathbb{P}^1_k$, respectively. Then $C = C_1 \cup C_2 \cup C_3 \cup C_4$ forms a 4-cycle of smooth rational curves. Now choose an isomorphism $C_2 \to C_4$ sending $C_1 \cap C_2, C_2 \cap C_4$ to $C_1 \cap C_4, C_4 \cap C_3$, respectively, and let $C_1 \to C_3$ be a similar isomorphism. Then define X to be the proper algebraic space obtained from S by identifying C_1, C_3 and C_2, C_4 with respect to these maps. Then X has normal crossing singularities except for a single closed point $x \in |\underline{X}|$, whose preimage on S are the nodal points of C. Étale locally near x, the space X is isomorphic to the boundary divisor Z_0. Hence, as in the prevous example, X is endowed with a gtc-atlas $\mathcal{G}$ with the property that the ghost sheaf agrees with $f_*\mathbb{N}_{\underline{S}}$ away from x, $f : \underline{S} \to \underline{X}$ the normalization. We examined similar surfaces in connection with degenerations of primary Kodaira surfaces [33].

9. Smooth log atlases

In this short section we propose a tentative generalization of gtc-atlases using the concept of smoothness in the category of log spaces. Recall that a morphism $f : X \to Y$ of fine log spaces is called *smooth* if, étale locally, there are charts $P_X \to \mathcal{M}_X$, $Q_Y \to \mathcal{M}_Y$, and $Q \to P$ for f such that the induced morphism $\underline{X} \to \underline{Y} \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P]$ of algebraic spaces is étale, and that kernel and the torsion part of the cokernel for $Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ are groups of order prime to the characteristic of the ground field. Equivalently, the morphism $f : X \to Y$ satisfies the lifting criterion for log Artin rings similar to the classical lifting criterion for smoothness of schemes. It turns out that smooth log spaces behave very much like smooth spaces, and can be treated with similar methods. For more details on smooth morphism of log spaces we refer to [18], Section 3.

We now consider the following situation. Fix a ground field k and a fine monoid Q. Let $(\mathrm{Spec}(k), Q)$ be the log structure associated to the prelog structure

$$Q \longrightarrow k, \quad q \mapsto \begin{cases} 1 & \text{if } q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The geometric stalk of $\mathcal{M}_{(\mathrm{Spec}(k),Q)}$ is $(k^{\mathrm{sep}})^\times \oplus Q$. Now let $\underline{X}$ be an algebraic k-space of finite type endowed with a constructible monoid sheaf $\overline{\mathcal{M}}_{\underline{X}}$ with fine stalks. We also assume that we have a fixed monoid homomorphism $\rho : Q \to \overline{\mathcal{M}}_{\underline{X}}$. We propose the following definition:

Definition 9.1. A log atlas $\mathcal{G}$ on $\underline{X}$ with respect to $\overline{\mathcal{M}}_{\underline{X}}$ is called *smooth* if there is an étale covering $\underline{U} \to \underline{X}$, a log space $U \in \mathcal{G}$, and a smooth morphism of log spaces $U \to (\mathrm{Spec}(k), Q)$ compatible with $\rho : Q \to \overline{\mathcal{M}}_{\underline{X}}$.

Note that a morphism $U \to (\mathrm{Spec}(k), Q)$ compatible with ρ is nothing but a lifting $\tilde{\rho} : Q \to \mathcal{M}_U$ of $\rho : Q \to \overline{\mathcal{M}}_U$, thanks to the splitting of $\mathcal{M}_{(\mathrm{Spec}(k),Q)}$. Observe that gtc-atlases are smooth log atlases: In this special case we have $Q = \mathbb{N}$, and the fixed morphism $\rho : Q \to \overline{\mathcal{M}}_{\underline{X}}$ corresponds to the fixed section $\rho \in \Gamma(\underline{X}, \overline{\mathcal{M}}_{\underline{X}})$.

We expect that the notion of smooth log atlases will be crucial in studying degenerations and deformations over higher dimensional base schemes.

10. Kato fans

In this section we recall a combinatorial object introduced by Kato [19] under the name *fan*. To avoid confusion with toric geometry, we shall use the term *Kato fan*. This concept will be a convenient framework for our mirror construction in the next two sections. To keep the discussion within limits we work in the category of schemes locally of finite type over a ground field k rather than algebraic spaces. Also the log structures are now defined on the Zariski site. See [27] for a detailed comparison between log structures on the Zariski and étale sites. Essentially this only rules out self-intersecting components in our construction, confer Kato's comment in [19], Remark 1.8. We may avoid this restriction with a little more effort, confer [10], Section 2.2.

Recall from [19], Definitions 9.1 and 9.3 that a monoidal space is a topological space T endowed with a sheaf of sharp monoids M_T, and that a *Kato fan* is a monoidal space (T, M_T) that is locally of the form

$$(\mathrm{Spec}(P), M_{\mathrm{Spec}(P)}),$$

where $\mathrm{Spec}(P)$ is the set of prime ideals in some monoid P. Here the notation is adopted from commutative algebra. In multiplicative notation, $I \subset P$ is an ideal if $PI \subset I$, and it is a prime ideal if $P \setminus I$ is a submonoid of P ([19], Definition 5.1). The spectrum $\mathrm{Spec}(P)$ is the set of prime ideals in P with the topology generated by $D(f) = \{\mathfrak{p} \in \mathrm{Spec}(P) \mid f \notin \mathfrak{p}\}$ for $f \in P$. The sections of $M_{\mathrm{Spec}(P)}$ over $D(f)$ are

$$S^{-1}P/(S^{-1}P)^\times \quad \text{with } S = \{f^n \mid n \geq 0\}.$$

Similarly, for a prime ideal $\mathfrak{p} \subset P$ we write $P_{\mathfrak{p}} = S^{-1}P/(S^{-1}P)^\times$, with $S = P \setminus \mathfrak{p}$. This is the stalk of $M_{\mathrm{Spec}(P)}$ at $\mathfrak{p}$.

The affine Kato fan $\mathrm{Spec}(P)$ is finite if P is finitely generated. A Kato fan T is *locally of finite type* if the monoids P can be chosen to be finitely generated. In contrast to the situation in [19] we will not be able to restrict to integral monoids as we will see shortly. A Kato fan that is locally of finite type is locally finite. A convenient way to think about locally finite topological spaces is as partially ordered sets via

$$x \leq y \quad \Longleftrightarrow \quad x \in \overline{\{y\}}.$$

Reversing this ordering leads to the *dual space* F^*. In other words, $F^* = F$ as sets, but $U \subset F^*$ is open iff $U \subset F$ is closed. A sheaf P on F is equivalent to a set of monoids P_x indexed by $x \in F$, together with a compatible system of generization maps $\varphi_{yx} : P_x \to P_y$ for any $x \leq y$.

Kato fans arise in log geometry as follows. For a scheme $\underline{X}$ with fine log atlas $\mathcal{G}$ and $x \in \underline{X}$ denote by $I(\mathcal{G}, x) \subset \mathfrak{m}_x$ the ideal generated by the image of $P \backslash \alpha_x^{-1}(\mathcal{O}_{\underline{X},x}^\times)$ for any chart $\alpha_x : P \to \mathcal{O}_{\underline{X},x}$ at x. Note that $I(\mathcal{G}, x)$ depends only on $\mathcal{G}$ and not on the particular chart. We are interested in equivalence classes of log structures with the same ghost sheaf and the same set of ideals $\mathcal{I}(\mathcal{G}, x)$.

Definition 10.1. Let $\underline{X}$ be a scheme endowed with a sheaf of fine sharp monoids $\overline{\mathcal{M}}_{\underline{X}}$. Suppose we have an étale covering $\underline{U}_i \to \underline{X}$ and log spaces U_i together with an identification $\overline{\mathcal{M}}_{U_i} \simeq \overline{\mathcal{M}}_{\underline{U}_i}$. Let $q_i : \mathcal{M}_{U_i} \to \overline{\mathcal{M}}_{\underline{U}_i}$ be the quotient map. We call (U_i, q_i) a *pre-gtc atlas* if:

(i) For each i there exists an étale map $\underline{U}_i \to \underline{Z} = \mathrm{Spec}\, k[\sigma^\vee \cap M]/(\chi^{\rho_i})$ to the reduced boundary divisor of a Gorenstein toric variety inducing the log space U_i.

(ii) For any $x \in \underline{X}$ and $p \in \overline{\mathcal{M}}_{\underline{X},x}$ the ideal $\mathcal{I}(U_i, x) \subset \mathcal{O}_{\underline{X},x}$ generated by $\alpha_i(q_i^{-1}(p))$ is independent of the choice of i with $x \in \underline{U}_i$.

In the situation of the definition the pull-backs of $\rho_i \in \Gamma(\underline{Z}, \overline{\mathcal{M}}_{\underline{Z}})$ glue to a distinguished section $\rho \in \Gamma(\underline{X}, \overline{\mathcal{M}}_{\underline{X}})$. This is true because for a Gorenstein sharp toric monoid P there is a unique element $\rho \in P$ with the property $P \setminus (\rho + P) = \partial P$. Moreover, for any $x \in \underline{X}$ there is a well-defined ideal $\mathcal{I}(\overline{\mathcal{G}}, x) \subset \mathfrak{m}_x$ by taking $\mathcal{I}(U_i, x)$ for any i with $x \in \underline{U}_i$. A scheme with a pre-gtc atlas induces a Kato fan (cf. [19], Proposition 10.1 for an analogue for toroidal varieties):

Proposition 10.2. *Let $\underline{X}$ be a scheme endowed with a sheaf of fine sharp monoids $\overline{\mathcal{M}}_{\underline{X}}$, together with a pre-gtc atlas $\overline{\mathcal{G}}$. Let $\rho \in \Gamma(\underline{X}, \overline{\mathcal{M}}_{\underline{X}})$ be the distinguished section. Then*

(i) *The ideal $I(\overline{\mathcal{G}}, x) \subset \mathcal{O}_{\underline{X},x}$ is a prime ideal for every $x \in \underline{X}$.*

(ii) *The set $F(\underline{X}) = \{x \in \underline{X} \mid I(\overline{\mathcal{G}}, x) = \mathfrak{m}_x\}$ endowed with the subspace topology from $\underline{X}$ and the monoid sheaf $M_{F(\underline{X})} = \overline{\mathcal{M}}_{\underline{X}}/(\rho)|_{F(\underline{X})}$ is a Kato fan locally of finite type.*

(iii) *There is a morphism $\pi : (\underline{X}, \overline{\mathcal{M}}_{\underline{X}}/\!\!/(\rho)) \to (F(\underline{X}), M_{F(\underline{X})})$ mapping $x \in \underline{X}$ to the point of $F(\underline{X}) \subset \underline{X}$ corresponding to the prime ideal $I(\overline{\mathcal{G}}, x) \subset \mathcal{O}_{\underline{X},x}$, and the canonical map $\pi^{-1} M_{F(\underline{X})} \to \overline{\mathcal{M}}_{\underline{X}}/\!\!/(\rho)$ is bijective.*

Remark 10.3. In the statement of the proposition we are taking a certain quotient of a monoid M by an ideal $J = (\rho) = \rho + M$. This quotient is defined as the set consisting of $M \setminus J$ together with one more point ∞. For $m, m' \neq 0$ set $m + m' = \infty$ if one of m, m' equals ∞, or if $m + m' \in J$ as sum in M. Otherwise the sum $m + m'$ is taken in M.

This construction has the following categorical meaning. Consider the category of monoid homomorphisms $M \to M'$ mapping J to an attractive point $\infty \in M'$, that is, with $\infty + m = \infty$ for all $m \in M' \setminus \{0\}$. Then

$$\varphi : M \longrightarrow M/\!\!/J, \quad m \longmapsto \begin{cases} m, & m \in M \setminus J \\ \infty, & m \in J, \end{cases}$$

is an initial object in this category. Note that unless $J = 0$ our quotients are never integral and that any ideal in $M/\!\!/J$ contains ∞. Such ideal quotients in the category of monoids are compatible with ideal quotients in the category of rings in the following sense. Let $(\chi^J) \subset k[M]$ be the ideal generated by monomials χ^m with $m \in J$. Then there is a canonical isomorphism $k[M/\!\!/J]/(\chi^\infty) = k[M]/(\chi^J)$.

The referee pointed out that this is indeed not a proper quotient in the category of monoids, which is why we use the double slash notation.

Proof of Proposition 10.2. Because the problem is local we may restrict ourselves to the case that $\underline{X}$ has an étale morphism to $\operatorname{Spec} k[P]/(\chi^\rho) = \operatorname{Spec} k[P/\!\!/(\rho)]/(\chi^\infty)$ inducing the log structure, where $P = \overline{\mathcal{M}}_{\underline{X},x}$. For $\operatorname{Spec} k[P]$, Kato proved statements (1)–(3) in [19], Section 10. In particular, for each prime ideal $\mathfrak{p} \subset P$ there is exactly one point $x \in \operatorname{Spec} k[P]$ such that $P \setminus \mathfrak{p}$ generates the maximal ideal at x, and conversely. Therefore the points $x \in F(\underline{X})$ are in one-to-one correspondence with prime ideals in P contained in $P \setminus (\rho + P)$. Hence $F(\underline{X}) = \operatorname{Spec}(P/\!\!/(\rho))$. Statement (3) follows from the corresponding statement for $\operatorname{Spec} k[P]$ by dividing out the ideal (ρ). $\square$

The Kato fan $(F(\underline{X}), M_{F(\underline{X})})$ in Proposition 10.2 is a hull for $(\underline{X}, \overline{\mathcal{M}}_{\underline{X}}/\!\!/(\rho))$ rather than for $(\underline{X}, \overline{\mathcal{M}}_{\underline{X}})$. For our construction in the next section we need an additional structure on $F(\underline{X})$ coming from the sheaf $\overline{\mathcal{M}}_{\underline{X}}$ on $\underline{X}$.

Definition 10.4. A *gtc-structure* on a monoidal space (F, M_F) is a sheaf P of Gorenstein sharp toric monoids, together with an isomorphism $P/\!\!/(\rho + P) \simeq M_F$ for $\rho \in \Gamma(F, P)$ the distinguished section. A *gtc-fan* is a Kato fan with a gtc-structure. The notation will be (F, P, ρ).

If P is a Gorenstein sharp toric monoid with distinguished element ρ then the restriction of $M_{\operatorname{Spec} P}$ to $\operatorname{Spec}(P/\!\!/(\rho))$ is a gtc-structure on $(\operatorname{Spec}(P/\!\!/(\rho)), M_{\operatorname{Spec}(P/\!\!/(\rho))})$. Hence the following is a direct consequence from the proof of Proposition 10.2.

Proposition 10.5. *The Kato fan $(F(\underline{X}), M_{F(\underline{X})})$ from Proposition 10.2 has a gtc-structure $(P_{F(\underline{X})}, \rho)$.*

We call $(F(\underline{X}), P_{F(\underline{X})}, \rho)$ the *gtc-fan associated to* $(\underline{X}, \overline{\mathcal{M}}_{\underline{X}}, \rho)$. Next we show how to construct a space with toric components from a gtc Kato fan. For a toric monoid $P = \sigma \cap \mathbb{Z}^d$, $P^{\text{gp}} = \mathbb{Z}^d$, there is a one-to-one correspondence between faces τ of σ and those submonoids $Q \subset P$ whose complement is a prime ideal, by taking the integral points of τ. Such submonoids are commonly called *faces* of P. Its (co-) dimension is the (co-) dimension of τ in σ. Faces of codimension 1 are *facets*. We write $P^\vee$ for the dual monoid $\operatorname{Hom}(P, \mathbb{N})$.

Let (F, P, ρ) be a gtc Kato fan. For any $x \in F$ we have the ring $k[P_x^\vee]$. Evaluation at $\rho \in \Gamma(P)$ defines a grading $P_x^\vee \to \mathbb{N}$. We thus obtain a projective scheme

$$\underline{Y}_x = \operatorname{Proj}(k[P_x^\vee]).$$

The generization maps for the stalks of P tell how to glue these spaces according to the following lemma.

Lemma 10.6. *For any toric monoid P and $\mathfrak{p} \in \operatorname{Spec}(P)$ there exists a canonical surjective morphism $k[P^\vee] \to k[P_\mathfrak{p}^\vee]$. These morphisms are natural with respect to inclusion of prime ideals.*

Proof. Let $S = P \setminus \mathfrak{p}$ be the face associated to $\mathfrak{p}$. As the elements of S are invertible in $S^{-1}P$ the homomorphism $P \to P_\mathfrak{p}$ is surjective. Dualizing gives an injection $P_\mathfrak{p}^\vee \to P^\vee$. The image comprises those $\varphi : P \to \mathbb{N}$ with $\varphi(S) = 0$, because S^{gp} is the kernel of $P^{\mathrm{gp}} \to P_\mathfrak{p}^{\mathrm{gp}}$. Therefore $P^\vee \setminus P_\mathfrak{p}^\vee$ is an ideal. Letting $I \subset k[P^\vee]$ be the ring-theoretic ideal generated by χ^m with $m \in P^\vee \setminus P_\mathfrak{p}^\vee$, we obtain the desired surjection

$$k[P^\vee] \longrightarrow k[P^\vee]/I = k[P_\mathfrak{p}^\vee].$$

If $\mathfrak{p} \subset \mathfrak{q}$ there is a factorization $\varphi_\mathfrak{q} : k[P^\vee] \to k[P_\mathfrak{p}^\vee] \to k[P_\mathfrak{q}^\vee]$, and this gives naturality. □

For $x \leq y$ there exists a prime ideal $\mathfrak{p}$ of $P = P_x$ and an isomorphism $P_y \simeq P_\mathfrak{p}$ such that $\varphi_{yx} : P_x \to P_y$ is the localization map $P \to P_\mathfrak{p}$. This follows because locally around y the monoidal space (F, P) is isomorphic to

$$(\operatorname{Spec}(P_y /\!\!/ (\rho)), M_{\operatorname{Spec}(P_y)}|_{\operatorname{Spec}(P_y /\!\!/ (\rho))}).$$

So we can apply Lemma 10.6. The epimorphism $q_{yx} : k[P_x^\vee] \to k[P_y^\vee]$ thus obtained respects the grading. For any $x \leq y$ we therefore get a closed embedding $\varphi_{xy} : \underline{Y}_y \to \underline{Y}_x$. By compatibility with localization the $\underline{Y}_x$, $x \in F$, together with the closed embeddings φ_{xy} form a directed system of projective toric schemes.

Lemma 10.7. *The direct limit $\varinjlim \underline{Y}_x$ exists as a reduced k-scheme locally of finite type, and the maps $\underline{Y}_x \to \varinjlim \underline{Y}_x$ are closed embeddings. If F is finite then $\varinjlim \underline{Y}_x$ is projective.*

Proof. We may assume that F is finite. If there is only one closed point $z \in F$, the direct limit is $\underline{Y}_z$, because the φ_{xy} are closed embeddings. In the general case, fix a closed point $z \in F$, let $F_1 \subset F$ be the set of points that are generizations of z, and let $F_2 \subset F$ be the set of points that are generizations of a closed point different from z. Let $\underline{Y}_1, \underline{Y}_2, \underline{Y}_{12}$ be the direct limits corresponding to $F_1, F_2, F_1 \cap F_2$, respectively. These are projective schemes by induction on the cardinality of F. We now view $\underline{Y} = \varinjlim \underline{Y}_x$ as a coproduct $\underline{Y}_1 \amalg_{\underline{Y}_{12}} \underline{Y}_2$. According to [2], Theorem 6.1, the coproduct exists as a reduced algebraic space over k, with $\underline{Y}_i \to \underline{Y}_1 \amalg_{\underline{Y}_{12}} \underline{Y}_2$ closed embeddings with images covering $\underline{Y}_1 \amalg_{\underline{Y}_{12}} \underline{Y}_2$ set-theoretically. Repeating this construction with the compatible system of ample line bundles $L_x \to \underline{Y}_x$ corresponding to the ample invertible sheaves $\mathcal{O}_{\underline{Y}_x}(1)$, we infer that the algebraic space $\underline{Y}$ carries a line bundle whose restriction to each irreducible component is ample. Hence $\underline{Y}$ is a projective scheme. □

We write $\underline{Y}_{(F,P,\rho)} = \varinjlim \underline{Y}_x$.

11. A naive mirror construction

Let (F, P, ρ) be a gtc Kato fan, and $\underline{Y}_{(F,P,\rho)} = \varinjlim \underline{Y}_x$ the corresponding projective scheme from Lemma 10.7. Our next goal is to define a pre-gtc atlas on $\underline{Y}_{(F,P,\rho)}$.

This requires some additional data leading to a selfdual structure, which in turn gives a baby version of mirror symmetry.

First note that we have a canonical identification $\mathrm{Spec}(P)^* \simeq \mathrm{Spec}(P^\vee)$ for any toric monoid $P = \sigma \cap \mathbb{Z}^d$, by sending $\tau \cap P$ to $(\mathbb{R}\tau)^\perp \cap P^\vee$. We exploit this as follows: For any closed point $x \in F$ there is a continuous map

$$\underline{Y}_x = \mathrm{Spec}(k[P_x^\vee]) \longrightarrow \mathrm{Spec}(P_x^\vee) \simeq (\mathrm{Spec}(P_x))^* \subset F^*.$$

The collection of these maps descends to a continuous map $\underline{Y}_{(F,P,\rho)} \to F^*$. This map should come from a pre-gtc atlas on F^*. Thus one ingredient to define the desired pre-gtc atlas on $Y_{(F,P,\rho)}$ will be a monoid sheaf Q over F^* with section $\rho^* \in \Gamma(F^*, Q)$ making (F^*, Q, ρ) into a gtc Kato fan. Of course, we also need a compatibility condition relating (F, P, ρ) to (F^*, Q, ρ^*). We call a map $\lambda : Q \to A$ from a monoid into an abelian group *affine* if $\lambda - \lambda(0)$ is a homomorphism of monoids.

Definition 11.1. A *gtc duality datum* consists of the following:

(i) A gtc Kato fan (F, P, ρ).
(ii) A gtc Kato fan (F^*, Q, ρ^*).
(iii) A *compatibility datum* between (F, P, ρ) and (F^*, Q, ρ^*) as follows: For any generic point $y \in F$ and any closed point $x \le y$ we have an affine injection

$$\lambda_{xy} : Q_y^\vee \longrightarrow P_x^{\mathrm{gp}}$$

identifying $Q_y^\vee$ with the cone with vertex $\rho(x)$ over a subset of the face of P_x^{gp} corresponding to $y \in \mathrm{Spec}(P_x) \subset F$, such that the one-dimensional face of $Q_y^\vee$ containing $\lambda_{xy}^{-1}(0)$ equals $Q_x^\vee \subset Q_y^\vee$.

The notation will be $(F, P, Q, \lambda = \{\lambda_{xy}\})$.

The following picture illustrates compatibility data. The four long dash-dotted lines are the rays of P_x, so we are looking from inside P_x. Let the facet of P_x containing the polygon Δ correspond to $y \in F$. Then the indicated cone over Δ represents $Q_y^\vee$.

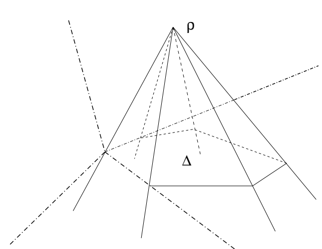

Fig. 1

Note that a compatibility datum is uniquely defined by the system of polytopes Δ, one for each facet of any P_x with $x \in F$ a closed point. The compatibility between the polytopes is best expressed by saying that they give rise to a sheaf Q on the dual space F^*.

We now explain how a gtc-duality datum gives rise to a pre-gtc atlas on $\underline{Y}_{(F,P,\rho)}$.

Construction 11.2. (Construction of pre-gtc structure.) We shall cover $\underline{Y}_{(F,P,\rho)}$ by divisors in affine toric schemes, one for each generic point $y \in F$. Let $x \le y$ be a closed point. To $y \in \operatorname{Spec}(P_x) \subset F$ belongs a facet $S \subset P_x$. Let $w \in P_x^\vee$ be the generator of the one-dimensional face dual to S. Denote by $(P_x^\vee)_{(w)}$ the submonoid of $(P_x^\vee)^{\mathrm{gp}}$ of terms of the form $p - a \cdot w$, $p \in P_x^\vee$, $a \in \mathbb{Z}$ with $p(\rho) = a \cdot w(\rho)$. The notation comes from interpreting $(P_x^\vee)_{(w)}$ as homogeneous localization of $P_x^\vee$ with respect to the grading defined by ρ. The injection

$$\lambda_{xy} - \rho : Q_y^\vee \longrightarrow P_x^{\mathrm{gp}}$$

induces a bijection of groups $(Q_y)^{\mathrm{gp}} \simeq (P_x^\vee)^{\mathrm{gp}}$. We view Q_y as submonoid of $(P_x^\vee)^{\mathrm{gp}}$ via this bijection. With this understood we have

$$(P_x^\vee)_{(w)} = Q_y \cap \rho^\perp. \tag{9}$$

Indeed, if $p - aw \in (P_x^\vee)_{(w)}$ then $(p - aw)(\rho) = 0$ by definition. To check that the image is in Q_y it suffices to evaluate its $\mathbb{R}$-linear extension on $v - \rho$, for all vertices v of the polygon $\Delta \subset S^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{R}$ spanning $Q_y^\vee$:

$$(p - aw)(v - \rho) = (p - aw)(v) = p(v) \geq 0.$$

Conversely, let $q \in Q_y \cap \rho^\perp$. Then for any vertex $v \in \Delta$ as before we have $q(v) = q(v - \rho) \geq 0$. On the other hand, for any $v \in P_x \setminus S$ it holds $w(v) > 0$. Therefore for $a \gg 0$ it holds $(q + aw)(v) \geq 0$ for all $v \in P_x$. Hence $q + aw \in P_x^\vee$ and $a = (q + aw) - aw \in (P_x^\vee)_{(w)}$.

Since $P_x^\vee$ is a sharp monoid, $0 \in \Delta$ is a vertex. When we view $Q_y^\vee$ as submonoid of P_x^{gp} as before, this vertex corresponds to $-\rho \in Q_y^\vee$. It follows that $\mathbb{R}_{\geq 0} \cdot (-\rho) \subset Q_y^\vee$ is a one-dimensional face. Hence $Q_y \cap \rho^\perp$ is a facet of Q_y, and (9) identifies $\operatorname{Spec}\big(k[(P_x^\vee)_{(w)}]\big)$ with an irreducible component of the divisor $\chi^{\rho^*} = 0$ in $\operatorname{Spec}(k[Q_y])$:

$$\operatorname{Spec}(k[(P_x^\vee)_{(w)}]) \subset \operatorname{Spec}(k[Q_y]/(\chi^{\rho^*}).$$

The left-hand side is a standard affine open subset of $\underline{Y}_x = \operatorname{Proj}(k[P_x^\vee])$, which we denote by $U_{x,y}$. Hence

$$\underline{Y}_x = \bigcup_y \underline{U}_{x,y},$$

where y runs over all generic points $y \in F$ with $y \geq x$. Moreover, by what we have just said, $\underline{U}_{x,y}$ embeds as an irreducible component into $\operatorname{Spec} k[Q_y]/(\chi^{\rho^*})$. Restriction also yields closed embeddings $\underline{U}_{x,y} \cap \underline{Y}_z \to \operatorname{Spec}(k[Q_y])$ for any $z \in F$ with $x \leq z \leq y$. These form a directed system of closed embeddings, parametrized by all $z \leq y$. It is compatible with the directed system defined by the Y_x via a system of open embeddings. Thus

$$\underline{U}_y := \bigcup_{x \leq y} \underline{U}_{x,y} \simeq \operatorname{Spec}(k[Q_y]/(\chi^{\rho^*}) \tag{10}$$

is an open subscheme of $\underline{Y}_{(F,P,\rho)}$. The collection of the closed embeddings $\underline{U}_y \to \operatorname{Spec} k[Q_y]$, which is uniquely defined by the duality datum, defines our pre-gtc atlas.

It remains to check the compatibility condition in the definition of pre-gtc atlas (Definition 10.1 (ii)). Let $x \in F^*$ and $q_x \in Q_x$. For generic points $y, y' \in F^*$ with $x \in \overline{y} \cap \overline{y}'$ we have to show equality of the ideals $\mathcal{I}$, $\mathcal{I}'$ on $\underline{U}_y \cap \underline{U}_{y'}$ generated by q via the two gtc-charts indexed by y and y'. Denote $Q = Q_y$ and choose a lift $q \in Q$ of q_x under the generization map $Q \to Q_x$. It suffices to compare the ideals on one of the open sets

$$\underline{U}_h = \operatorname{Spec}(k[Q]_{(h)}), \quad h \in k[Q]$$

generating the topology.

Let $v_1, \ldots, v_n$ be generators of the one-dimensional faces of $Q^\vee$ and $\underline{U}_i \subset \underline{U}_h$ the irreducibe component corresponding to v_i. By definition $I = (\chi^q)$. Precisely for those i with $q(v_i) = 0$ the function χ^q is non-zero at the generic point of $\underline{U}_i$. Therefore χ^q defines a Cartier divisor on the subspace $\underline{Z} \subset \underline{U}_h$ corresponding to the ideal generated by

$$\{p \in Q \,|\, q(v_i) = 0 \Rightarrow p(v_i) \neq 0 \text{ for all } i\}.$$

The associated Weil divisor is $\sum_{q(v_j)\neq 0} q(v_j) \cdot [\underline{Z} \cap \underline{U}_j]$. The essential observation is that both $\underline{Z}$ and this divisor depend only on q_x. Hence, denoting by f a generator of I', there exists $e \in k[Q]_{(h)}$, invertible on $\underline{Z}$, with $f|_Z = (e \cdot \chi^q)|_{\underline{Z}}$. But f and χ^q vanish at the generic points of the closure of $\underline{U}_h \setminus \underline{Z}$, and hence $f = e \cdot \chi^q$ everywhere. This shows $I' = I$.

Next we describe the canonical involution on the set of all duality data:

Construction 11.3. (Mirror duality data.) Let (F, P, Q, λ) be a duality datum. The mirror duality datum will be (F^*, Q, P, λ^*), and we have to define the dual compatibility datum λ^*. Let $\rho^* \in \Gamma(F^*, Q)$ be the distinguished section, and let $x \leq y \in F$ be a closed and a generic point, respectively. Recall that the given compatibility datum gives an affine injection $\lambda_{xy} : Q_y^\vee \to P_x^{\mathrm{gp}}$ with $\lambda_{xy}(0) = \rho$. The dual compatibility datum λ^* is defined by the formula

$$\lambda^*_{yx} = ((\lambda_{xy} - \rho)^{\mathrm{gp}})^\vee + \rho^* : P_x^\vee \longrightarrow Q_y^{\mathrm{gp}}.$$

This indeed works:

Lemma 11.4. *The collection (F^*, Q, P, λ^*) is a duality datum.*

Proof. We have to verify the compatibility condition Definition 11.1,(iii). Since $(\rho^*)^\perp \cap P_x$ is the facet belonging to y we see that $-\rho^*$ spans the one-dimensional face of $P_x^\vee$ corresponding to $y \in F^*$. Since $\rho^\perp \cap Q_y$ is a facet of Q_y and P_x it remains to show that

$$\lambda^*_{yx}(P_x^\vee) \cap \rho^\perp \subset Q_y.$$

For the following computation we view $P_x^\vee$ and $Q_y^\vee$ as subsets of Q_y^{gp} and P_x^{gp} respectively. Let $m \in P_x^\vee$ with $(\rho^* + m)(\rho) = 0$. We have to show that $(\rho^* + m)(p) \geq 0$ for any $p \in Q_y^\vee$. Since $Q_y^\vee$ is generated by elements of the form $q - \rho$ with $q \in P_x \cap (\rho^*)^\perp$, we may restrict to such elements. Now compute

$$(\rho^* + m)(q - \rho) = \rho^*(q) - (\rho^* + m)(\rho) + m(q).$$

The first two terms vanish, while $m(q) \geq 0$ since $m \in P_x^\vee$, $q \in P_x$. □

It is clear from the definition of λ^* that the mirror of the mirror (F^*, Q, P, λ^*) is the original duality datum (F, P, Q, λ). In other words, passing to the mirror duality datum defines an involution on the set of duality data.

12. Batyrev's mirror construction, degenerate abelian varieties

In this section we illustrate our naive mirror construction with two examples.

Example 12.1. (Batyrev's mirror construction.) Let $\Delta \subset \mathbb{R}^n$ be a polytope with integral vertices $v_i \in \mathbb{Z}^n$. We assume that Δ is *reflexive*, which means (1) the origin is the only interior lattice point of Δ, and (2) the polar polytope $\Delta^\circ = \{m \in (\mathbb{R}^n)^\vee \,|\, \langle m, v\rangle \geq -1\}$ has integral vertices. Then also the polar polytope is reflexive.

From Δ we obtain a duality datum as follows. Let F be the set of proper faces $\sigma \subsetneq \Delta$, where the relation $\leq$ of points corresponds to inclusion $\subset$ of faces. For each face σ define a monoid P_σ as the quotient of the "wedge monoid"

$$\mathbb{Z}^n \cap \mathbb{R}_{\geq 0} \cdot \{p_2 - p_1 \,|\, p_1 \in \sigma, p_2 \in \Delta\}$$

by its invertible elements. If $\sigma \subset \tau$ there is a canonical surjection $P_\sigma \to P_\tau$ making these monoids into a sheaf P on F. Similarly, the polar polytope induces a sheaf Q on the dual topological space F^*. For a face $\sigma \subset \Delta$ the monoid of integral points of the cone over σ is canonically dual to Q_σ. For every vertex $v \in \sigma$ we therefore obtain an affine embedding $Q_\sigma^\vee \hookrightarrow P_v$, and these provide the compatibility datum Definition 11.1 (iii). The Gorenstein property of both P and Q follow from reflexivity of Δ.

By going through the construction we see that $\underline{Y}_{(F,P,\rho)}$ is the boundary divisor (the complement of the big cell) in the toric variety $\mathbb{P}(\Delta)$. The pre-gtc-atlas comes from the embedding into $\mathbb{P}(\Delta)$, so in this case actually glues to a logarithmic structure. The conormal sheaf $\mathcal{N}_{\underline{D}}$ is the conormal sheaf of this embedding. As it is never trivial, none of the global logarithmic structures in the specified gtc-atlas is log-smooth over the standard log point.

The space $\underline{Y}_{(F^*,Q,\rho^*)}$ for the mirror duality datum gives the boundary divisor in $\mathbb{P}(\Delta^\circ)$. So here we retrieve part of the Batyrev construction of mirror pairs of hypersurfaces in toric varieties defined by reflexive polyhedra [4]. To go further one would need to control the desingularization procedure involved in Batyrev's construction under this process. This is beyond the scope of this paper and will be further discussed in [10].

Example 12.2. (Degenerate abelian varieties.) Let $f : \mathbb{Z}^n \to \mathbb{Z}$ be a convex mapping, and $C_f \subset \mathbb{R}^{n+1}$ the boundary of the convex hull of the graph $\Gamma_f = \{(v, f(v)) \in \mathbb{Z}^{n+1} \,\big|\, v \in \mathbb{Z}^n\}$. Then C_f is a multi-faceted paraboloid with integral vertices. We assume all faces to be bounded. Let F be the locally finite topological space with points the faces of C_f and the ordering "$\leq$" defined by inclusion of faces. Denote by $\pi : \mathbb{Z}^{n+1} \to \mathbb{Z}^n$ the projection onto the n first coordinates. For each face $\sigma \subset C_f$ define

$$Q_\sigma^\vee = \{(t \cdot v, t) \in \mathbb{Z}^{n+1} \,|\, v \in \pi(\sigma), t \in \mathbb{R}_{\geq 0}\}.$$

As this ist the set of integral points of the cone over $\pi(\sigma)$, embedded into the affine hyperplane $\{1\} \times \mathbb{R}^n$, there are compatible inclusions $Q_\sigma^\vee \longrightarrow Q_\tau^\vee, \quad \sigma \subset \tau$. Therefore the duals Q_σ form the stalks of a sheaf Q on F^*. The projections $Q_\sigma^\vee \to \mathbb{N}$ onto the first coordinate define a section ρ^* of Q, and (F^*, Q, ρ^*) is a gtc Kato fan.

Next we define the sheaf P on F. By abuse of notation, for $\sigma \in F$ let $\langle\sigma\rangle$ denote the saturated subgroup of $\mathbb{Z}^{n+1}$ generated by $v - v'$ with $v, v' \in \sigma \cap \mathbb{Z}^{n+1}$. Define

$$P_\sigma \subset \mathbb{Z}^{n+1}/\langle\sigma\rangle$$

to be the saturated submonoid generated by $w - v$, where $v \in \sigma \cap \mathbb{Z}^{n+1}$ and $w \in C_f$, that is, $w = (\pi(w), t)$ with $t \geq f(\pi(w))$. For $\sigma \subset \tau$ we have canonical surjections $P_\sigma \to P_\tau$, and this defines the sheaf P on F. For $\sigma \in F$ and $w \in \pi(\sigma)$ the equivalence class of $(w, f(w) + 1)$ in P_σ defines the germ of the section ρ at σ. One can show that (F^*, Q, ρ^*) is a gtc Kato fan.

For the compatibility datum let $v = (v_0, t_0) \in C_f$ be a vertex and $\sigma \subset C_f$ a facet with $v \in \sigma$. Then

$$\lambda_{v\sigma} : Q_\sigma^\vee \longrightarrow P_v^{\mathrm{gp}}, \quad (P, t) \longmapsto (P, t_0 + 1 - t)$$

is an affine embedding identifying $Q_\sigma^\vee$ with the integral points of the cone over σ with vertex $\rho_\sigma = (v_0, t_0 + 1)$. (F, P, Q, Λ) is a gtc-duality datum, with $\underline{Y}_{(F^*,Q,\rho^*)}$ only *locally* of finite type.

One can show that the mirror space $\underline{Y}_{(F,P,\rho)}$ is of the same form, with defining function obtained by discrete Legendre transform from f [9], [10].

To obtain a degenerate abelian variety one assumes that $f = q + r$ with $q(x) = x^t A x + b^t x + c$ a strictly convex quadratic function with integral coefficients, and $r : \mathbb{Z}^n \to \mathbb{Z}$ a Λ'-periodic function for a sublattice $\Lambda' \subset \Lambda := \mathbb{Z}^n$ of finite index. The Λ'-action on Λ lifts to an affine action on $\mathbb{Z}^{n+1}$ leaving Γ_f invariant by

$$T_w(v, \lambda) = (v + w, \lambda + 2w^t A v + q(w) - c).$$

The induced Λ'-action on the duality datum defines an étale, quasicompact equivalence relation on $\underline{Y}_{(F^*,Q,\rho^*)}$. The quotient $\underline{Y}_{(F^*,Q,\rho^*)}/\Lambda'$ is the central fiber of the degeneration of polarized abelian varieties associated to $q + r$ by Mumford's construction [26]. The quotient of the gtc-atlas gives the log structure associated to the degeneration. So here there actually is a log-smooth morphism to the standard log point. Up to changing the gluing of the irreducible components any maximally degenerate polarized abelian variety is of this form [1], Section 5.7. In the mirror picture Λ^* is the sublattice of $\Lambda^\vee$ generated by the slopes of f, while $(\Lambda^*)' = \Lambda'$ with action induced from the action on Γ_f.

For an explicit two-dimensional example take $\Lambda' = 2\mathbb{Z}^2$, $q(x, y) = x^2 - xy + y^2$ and $r(v) = 1$ for $v \in \Lambda'$ and $r(v) = 0$ otherwise. Then $\underline{Y}_{(F^*,Q,\rho^*)}/\Lambda'$ is a union of 3 copies of $\mathbb{P}^1 \times \mathbb{P}^1$ and in each copy, the pull-back of the singular locus is a 4-gon of lines. The mirror $\underline{Y}_{(F,P,\rho)}/(\Lambda^*)'$ is a union of 2 copies of $\mathbb{P}^2$ and a $\mathbb{P}^2$ blown up in 3 points. The pull-back of the singular locus is a union of 3 lines for $\mathbb{P}^2$, and a 6-gon of rational curves containing the exceptional curves for the other component.

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