WEAK DEL PEZZO SURFACES WITH IRREGULARITY

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Abstract. I construct normal del Pezzo surfaces, and regular weak del Pezzo surfaces as well, with positive irregularity $q > 0$. This can happen only over nonperfect fields. The surfaces in question are twisted forms of nonnormal del Pezzo surfaces, which were classified by Reid. The twisting is with respect to the flat topology and infinitesimal group scheme actions. The twisted surfaces appear as generic fibers for Fano-Mori contractions on certain threefolds with only canonical singularities.

Introduction

Suppose that X is a smooth and projective scheme over the complex numbers. The Kawamata-Viehweg Vanishing Theorem asserts that $H^i(X, \omega_X \otimes \mathcal{L}) = 0$ for all integers $i > 0$ and all invertible sheaves $\mathcal{L}$ that are nef and big (see [18] and [39]). In the special case that X is a *weak Fano variety*, in other words, the dual of the dualizing sheaf is nef and big, we may apply the Kawamata-Viehweg Vanishing Theorem with $\mathcal{L} = \omega_X^\vee$ and conclude that $H^i(X, \mathcal{O}_X) = 0$ for all integers $i > 0$. *It is unknown whether or to what extend this particular vanishing holds true for Fano or weak Fano varieties in positive characteristics.*

The Kawamata-Viehweg Vanishing Theorem is a generalization of the Kodaira Vanishing Theorem [20], which deals with ample rather than nef and big invertible sheaves. It is well-known that the Kodaira Vanishing Theorem does not hold true in positive characteristics. Raynaud [29] constructed the first counterexamples, which are fibered surfaces whose generic fiber is regular but not smooth. The surfaces are mostly of general type. A rather different set of counterexamples is due to Lauritzen [22] relying on representation theory: He used homogeneous schemes of the form G/B, where $B \subset G$ is a nonreduced Borel subgroup scheme in some linear algebraic group. Using more elementary methods, Lauritzen and Rao [23] further constructed smooth Fano varieties of dimension $d \geq 6$ so that Kodaira vanishing fails for some ample invertible sheaves $\mathcal{L} \neq \omega_X^\vee$.

Esnault [8] gives a completely different aspect involving crystalline cohomology: Her results, which apply to a much wider class than just Fano varieties, tell us that for smooth Fano varieties over perfect fields k, with the ring of Witt vectors W and field of fractions $W \subset K$, the following holds: The part with slopes $\lambda \in [0, 1[$ inside the crystalline cohomology groups $H^i_{\text{cris}}(X/W) \otimes_W K$ vanishes for $i > 0$. On the other hand, this part

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$(H^i_{\text{cris}}(X/W)\otimes_W K)_{[0,1[}$ is isomorphic to Serre's Witt vector cohomology $H^i(X, W\mathcal{O}_X)\otimes_W K$. In turn, the group $H^i(X, W\mathcal{O}_X)$ is related to ordinary cohomology groups $H^i(X, \mathcal{O}_X)$ by exact sequences, but it seems difficult to gain control over torsion phenomena. Note that Berthelot, Bloch and Esnault [3] extended the bijection between the slope $[0,1[$-part and Witt vector cohomology to singular schemes, with rigid cohomology instead of crystalline cohomology.

There are some positive results in low dimensions. If follows from the classification of smooth del Pezzo surfaces, which are the 2-dimensional Fano varieties, that $H^1(S, \mathcal{O}_S) = 0$ holds regardless to the characteristic. The same holds for weak del Pezzo surfaces. For a nice account, see [7]. Shepherd-Barron [38] established the vanishing $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$ for smooth Fano threefolds. On the other hand, Reid [30] constructed nonnormal del Pezzo surfaces with $H^1(S, \mathcal{O}_S) \neq 0$.

My original motivation for this work was to construct *regular* Fano varieties over nonperfect fields that have $H^1(X, \mathcal{O}_X) \neq 0$. The point here is that regularity does not imply geometric regularity (= formal smoothness) over nonperfect fields. I did not quite succeed in my goals, but I came close to it. The main result of this paper is as follows:

THEOREM. *Over every nonperfect field of characteristic $p = 2$, there are weak del Pezzo surfaces that are regular, and normal del Pezzo surfaces S with only factorial rational double points of type A_1 as formal singularities, both with $h^1(\mathcal{O}_S) \neq 0$.*

These del Pezzo surfaces are twisted forms of Reid's nonnormal del Pezzo surfaces, and the weak del Pezzo surface is obtained by resolving the singularity. Such del Pezzo surfaces necessarily become nonnormal after passing to the perfect closure of the ground field. Indeed, it follows from the work of Hidaka and Watanabe [17] and myself [32] that $h^1(\mathcal{O}_S) = 0$ for geometrically normal del Pezzo surfaces.

The existence of such wild del Pezzo surfaces S over nonperfect fields has consequences for the structure theory of algebraic varieties X over algebraically closed fields. Namely, such del Pezzo surfaces might arise as generic fibers in some Fano-Mori contractions of fiber type, obtained by contracting an extremal ray. To my knowledge, the geometry of fibrations $f : X \to B$ whose generic fiber X_η is not geometrically regular or geometrically normal has not been studied systematically, except for the quasielliptic surfaces of Bombieri and Mumford, see [4] and [5].

The existence of Fano-Mori contractions of extremal rays on smooth threefolds in arbitrary characteristics was established by Kollár in [21]. In Remark 1.2, he raised the question whether there are contractions of fiber type whose generic geometric fibers are nonnormal del Pezzo surfaces. We shall see that our exotic del Pezzo surfaces appear as generic fibers $S = Z_\eta$ for some Fano-Mori contraction $f : Z \to E$ of fiber type, where Z is a threefold, and E is the supersingular elliptic curve in characteristic two. Unfortunately, my results are not strong enough to make the total space smooth. However, the threefold

Z will be locally of complete intersection, locally factorial, with only canonical singularities. The anticanonical divisor is nef and has Kodaira dimension two, and the first higher direct image $R^1 f_*(\mathcal{O}_Z)$ is nonzero.

There are several papers dealing with Fano threefolds in positive characteristics. For example, Shepherd-Barron [38] obtained a classification for Picard number $\rho = 1$, and Megyesi [25] treated the case of Fano varieties of index ≥ 2. Saito showed that on Fano threefolds with Picard number $\rho = 2$ there are no fibrations whose geometric generic fiber is a nonnormal del Pezzo surface [31]. Mori and Saito [26] have further results on wild hypersurface bundles.

Here is a plan for the paper: In Section 1, we collect some general facts on twisting and twisted forms, and give a criterion for regularity of twisted forms. In Section 2, we recall Reid's construction of nonnormal del Pezzo surfaces Y in terms of glueing along a double line to a rational cuspidal curve, and discuss the glueing process in detail. In Section 3, we shall see that the resulting del Pezzo surface Y is locally of complete intersection. In Section 4, I analyse the Picard group and the dualizing sheaf on these del Pezzo surfaces. Section 5 contains a discussion of curves of degree one. In particular, we shall explain how and why some of these curves are Cartier divisors, and others are only Weil divisors. In Section 6, we shall prove that the cotangent sheaf modulo torsion is locally free of rank two. This seems to be a rather special situation. An immediate consequence is that the tangent sheaf is locally free. Under suitable assumptions, we moreover identify a global vector field $\delta \in H^0(Y, \Theta_{Y/k})$ that defines an α_2-action. Unfortunately, this vector field has a unique zero at the so-called point at infinity $y_\infty \in Y$. However, we check in Section 7 that zeros of vector fields are inevitable, by computing the splitting type of the tangent sheaf. It turns out that our choice of δ is in some sense the best possible. In Section 8, we use the α_2-action to construct twisted forms Y' of our del Pezzo surface Y. It turns out that the twisted forms are normal, with only one singularity. A formal analysis reveals that this singularity is a rational double point of type A_1, which is moreover factorial. In Section 9 we use our results to construct some interesting Fano-Mori contractions. In Section 10, we finally discuss ampleness and semiampleness for line bundles on Y. Here we see that it is impossible to realize Y as hypersurface or double covering of some $\mathbb{P}^n$. The smallest embeddings have codimension three, and the smallest coverings have degree four.

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1. Twisted forms

In this section I discuss some useful general aspects of twisting and twisted form that we shall apply later to nonnormal del Pezzo surfaces. Suppose S is a scheme of finite type over a base field k. Another k-scheme S' is called a *twisted form* of S if there is a nonzero

k-algebra R with $S_R \simeq S'_R$. Such schemes S' are automatically of finite type by descent theory (see [15], Exposé VIII, Proposition 3.3).

LEMMA 1.1. *If S' is a twisted form of S, then there is a finite field extension $k \subset E$ with $S_E \simeq S'_E$.*

PROOF. Choose an isomorphism $f : S_R \to S'_R$. As explained in [13], Theorem 8.8.2, there is a k-subalgebra of finite type $R_\alpha \subset R$ and an isomorphism $f_\alpha : S_{R_\alpha} \to S'_{R_\alpha}$ inducing f. Choose a maximal ideal $\mathfrak{m} \subset R_\alpha$. Then $E = R_\alpha/\mathfrak{m}$ is a finite field extension of k, and we may restrict f_α to E. $\square$

If follows that the set of isomorphism classes of twisted forms S' is a subset of the nonabelian cohomology set $H^1(k, \mathrm{Aut}_{S/k})$, where we may use the finite flat topology. A nice account of this correspondence in the context of Galois cohomology appears in Serre's book [36]. The full theory is exposed at length in Giraud's treatise [9]. The basic construction goes as follows: Let T be a torsor under $\mathrm{Aut}_{S/k}$ with action from the left. Then we may form the product $S \times T$ and obtain the quotient by the diagonal action

$$S \wedge T = \mathrm{Aut}_{S/k} \backslash (S \times T), \quad (s,t) \sim (gs, gt).$$

Note that $\mathrm{Aut}_{S/k}$ also acts from the right on S via $sg = g^{-1}s$, so we may rewrite the equivalence relation in the particularly attractive form $(s,t) \sim (sg^{-1}, gt)$.

The result $S \wedge T$, which is a sheaf in the finite flat topology, is a sheaf twisted form of S. This sheaf, however, is not necessarily representable by a scheme. We shall discuss this below. Conversely, if S' is a twisted form of S, then $T = \underline{\mathrm{Isom}}(S', S)$ is an $\mathrm{Aut}_{S/k}$-torsor with action from the left, and the canonical map $S \wedge T \to S'$, $(s,t) \mapsto t^{-1}(s)$ is an isomorphism.

Now let $G \subset \mathrm{Aut}_{S/k}$ be a subgroup scheme. Then we have an induced map on nonabelian cohomology $H^1(k, G) \to H^1(k, \mathrm{Aut}_{S/k})$. Given any G-torsor T, we may form $S \wedge T = G \backslash (S \times T)$ to produce twisted forms. Note that the twisted form might be trivial, although the torsor is nontrivial. More precisely:

LEMMA 1.2. *The twisted form $S \wedge T$ is isomorphic to S if and only if there is a G-equivariant morphism $T \to \mathrm{Aut}_{S/k}$.*

PROOF. As explained in [9] Chapter III, Proposition 3.2.2 , we have a sequence

$$H^0(k, \mathrm{Aut}_{S/k}) \longrightarrow H^0(k, G\backslash \mathrm{Aut}_{S/k}) \longrightarrow H^1(k, G) \longrightarrow H^1(k, \mathrm{Aut}_{S/k}),$$

which is exact in the following sense: The G-torsors T inducing trivial $\mathrm{Aut}_{S/k}$-torsors come from the sections $x \in G\backslash \mathrm{Aut}_{S/k}$. The G-torsor coming from such a section x is the fiber over x under the projection $\mathrm{Aut}_{S/k} \to G\backslash \mathrm{Aut}_{S/k}$, whence the assertion. $\square$

We are mainly interested in the case that the group scheme G is finite. Then there are almost no problems with representability:

LEMMA 1.3. *Suppose G is a finite group scheme. Then $S' = S \wedge T$ is an algebraic space. It is a scheme if either G is infinitesimal, or if S is quasiprojective.*

PROOF. It follows from [15], Exposé VIII, Corollary 7.7, that the torsor T is representable by a scheme. The quotient $S' = G\backslash(S \times T)$ exists as an algebraic space, according to very general results of Keel and Mori [19]. If G is infinitesimal or if S is quasiprojective then the G-invariant affine open subset $U_\alpha \subset S \times T$ form a covering. By [16], Exposé V, Theorem 4.1 the quotient S' of $S \times T$ by the free G-action exists as a scheme. □

From now on we assume for convenience that the group scheme G is finite, and that $S' = S \wedge T$ is a scheme, where T is a G-torsor.

LEMMA 1.4. *The scheme S is locally of complete intersection or smooth if and only if S' is locally of complete interesection or smooth, respectively.*

PROOF. This follows from [14], Corollary 19.3.4 and Proposition 17.7.1. □

In contrast, regularity or nonregularity does not transfer to twisted forms. It is possible to remove singularities by passing to twisted forms, which is indeed the leitmotiv of this paper. Of course, such things may happen only in positive characteristics. I found the following basic fact very useful.

THEOREM 1.5. *Let $A \subset S$ be a G-invariant subscheme whose ideal is locally generated by regular sequences. If the twisted form $A' = A \wedge T$ is a regular scheme, then the twisted form $S' = S \wedge T$ is a regular scheme at all points on the subset $A' \subset S'$.*

PROOF. Consider the commutative diagram

$$\begin{array}{ccc} A \times T & \longrightarrow & S \times T \\ \downarrow & & \downarrow \\ A' & \longrightarrow & S'. \end{array}$$

The vertical maps are surjective and flat, because G acts freely on $A \times T$ and $S \times T$. Moreover, the diagram is cartesian. By assumption, the embedding $A \subset S$ is regular. Hence the induced embedding $A \times T \subset S \times T$ is regular as well. According to [14], Proposition 19.1.5, this implies that the embedding $A' \subset S'$ is regular. By assumption, the scheme A' is regular. If follows that the scheme S' is regular at all points $s \in A'$, by [14], Proposition 19.1.1. □

We shall mainly apply this in the cases that $A \subset S$ is either a Cartier divisor or an Artin subscheme. Let me record the latter:

COROLLARY 1.6. *Let $A = Gs$ be the orbit of a rational point $s \in S$, and let $s' \in S'$ be a closed point in $A' \subset S'$. If the scheme T is reduced and $Gs \subset S$ is a regular embedding, then the local ring $\mathcal{O}_{S',s'}$ is regular.*

Proof. The orbit $A = Gs$ of our finite group scheme G is isomorphic to the homogeneous space G/H, where $H = G_s$ is the isotropy group scheme. The projection $G/H\times T \to H\backslash T$, $(gH, t) \mapsto Hg^{-1}t$ is well-defined, and induces a bijection $G\backslash(G/H \times T) \to H\backslash T$. If follows that the twisted form $A' = A \wedge T$ is isomorphic to $H\backslash T$. By assumption, the Artin scheme T is reduced, whence the qotient scheme $H\backslash T$ is reduced as well. In other words, the twisted form A' is regular, and the Theorem applies. □

Example 1.7. Consider the global field $k = \mathbb{F}_2(t)$ in characteristic two and the 1-dimensional scheme $S = \operatorname{Spec} k[u^2, u^3]$, which contains a cuspidal singularity at the origin. The finite infinitesimal group scheme $G = \alpha_2$ acts on S via the derivation $u^3 \mapsto 1$. It also acts on the $T = \operatorname{Spec} k[\sqrt{t}]$ via the derivation $\sqrt{t} \mapsto 1$. The twisted form $S' = S \wedge T$ then must be a regular curve. Indeed, it is the spectrum of the subalgebra $k[u^2, u^3 + \sqrt{t}] \subset k[u^2, u^3, \sqrt{t}]$.

Now suppose that $x \in S$ is a rational point that is fixed under the G-action. Setting $A = \{x\}$, we see that the twisted form $A' = A \wedge T$ is given by another rational point $x' \in S'$. The following tells us that it is impossible to remove singularities by twisting at points that are both fixed and singular.

Proposition 1.8. *Assumptions as above. If the local ring $\mathcal{O}_{S,x}$ is not regular, then the local ring $\mathcal{O}_{S',x'}$ is not regular as well.*

Proof. Let $B \subset S \times T$ be the preimage of $x \in S$ under the projection map, which coincides with the preimage of $x' \in S'$ under the quotient map. Suppose $s' \in S'$ is a regular point. Then the residue field $\kappa(s')$ has finite projective dimension. Hence $\mathcal{O}_B$ has finite projective dimension as well, because the quotient map $S \times T \to S'$ is flat. Choose a resolution $\ldots \to F_1 \longrightarrow F_0 \to \kappa(s) \to 0$ with finitely generated free $\mathcal{O}_{S,x}$-modules. Pulling back under the flat projection map, we obtain a free resolution $\ldots \to F_1' \longrightarrow F_0' \to \mathcal{O}_B \to 0$. Whence the kernel of some $F_{i+1}' \to F_i'$ is free. By descent theory, the kernel of $F_{i+1} \to F_i$ must also be free. In other words, $x \in S$ is regular. □

2. Glueing along ribbons

Throughout the following sections, we shall study the geometry of certain nonnormal del Pezzo surfaces Y with irregularity $h^1(\mathcal{O}_Y) > 0$. Such surfaces were first constructed by Reid [30]. A key point in his construction is the use of certain infinitesimal neighborhoods called ribbons. Reid's construction works roughly as follows:

We fix a base field k of characteristic $p = 2$. Let $X = \mathbb{P}^2$ be the projective plane, and $A = \mathbb{P}^1$ be the projective line. Choose an embedding $A \subset X$ of degree one, and let $B = A^{(1)}$ be the first order infinitesimal neighborhood, which is a nonreduced quadric. Let C be the rational cuspidal curve with arithmetic genus $h^1(\mathcal{O}_C) = 1$, whose normalization is $A \to C$. The idea now is to extend the nonflat normalization map $A \to C$ to a flat morphism $\varphi : B \to C$ of degree two, and obtain the desired del Pezzo surface Y via the cocartesian diagram

$$\begin{array}{ccc} B & \longrightarrow & X \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\nu} \\ C & \longrightarrow & Y. \end{array}$$

Note that the normalization map $\nu : X \to Y$ is a homeomorphism. One way to think about this is that we thinned out the structure sheaf $\mathcal{O}_X$ by artificially removing sections satisfying certain conditions on B to obtain the structure sheaf $\mathcal{O}_Y$, as explained in Serre's book [37], Chapter IV, §1.1. In some sense, we introduced a curve of cusps $C \subset Y$, which itself contains a cuspidal singularity. Naturally, the singularity on this curve of singularities plays a crucial role in the whole affair.

To make this construction explicit and to explore its properties, it seems inevitable to introduce coordinates. Choose indeterminates u, v and cover the projective plane $X = \mathbb{P}^2$ in the usual way by three affine open subschemes

$$X = \operatorname{Spec} k[u,v] \,\cup\, \operatorname{Spec} k[u^{-1}, vu^{-1}] \,\cup\, \operatorname{Spec} k[uv^{-1}, v^{-1}]. \tag{1}$$

We sometimes denote this open affine covering by $X = U \cup U' \cup U''$. We shall see that our constructions do not work well in homogeneous coordinates, and it seems necessary to introduce inhomogeneous coordinates. The projective line $A = \mathbb{P}^1$ shall be embedded into the projective plane $X = \mathbb{P}^2$ by setting

$$A = \operatorname{Spec} k[u,v]/(v) \,\cup\, \operatorname{Spec} k[u^{-1}, vu^{-1}]/(vu^{-1}). \tag{2}$$

We write the rational cuspidal curve C with arithmetic genus $p_a = 1$ as the union of two affine open subschemes

$$C = \operatorname{Spec} k[u^2, u^3] \,\cup\, \operatorname{Spec} k[u^{-1}]. \tag{3}$$

Then we have a canonical morphism $A \to C$, which is the normalization map. Finally, consider the first order infinitesimal neighborhood $B = A^{(1)}$ inside the projective space $X = \mathbb{P}^2$. This nonreduced quadric is given by

$$B = \operatorname{Spec} k[u, \epsilon] \,\cup\, \operatorname{Spec} k[u^{-1}, \epsilon u^{-1}], \tag{4}$$

where ϵ denotes the residue class of v modulo v^2. The inclusion $A \subset B$ is a *ribbon* in the sense of Bayer and Eisenbud [6]. This means that the ideal $\mathcal{I} \subset \mathcal{O}_B$ of the closed embedding $A \subset B$ satisfies $\mathcal{I}^2 = 0$ and that $\mathcal{I}$ is an invertible $\mathcal{O}_A$-module. Note that the first condition implies that the $\mathcal{O}_B$-module structure on $\mathcal{I}$ indeed comes from an $\mathcal{O}_A$-module structure. We have an exact cotangent sequence of $\mathcal{O}_A$-modules

$$0 \longrightarrow \mathcal{I} \longrightarrow \Omega^1_{B/k} \otimes \mathcal{O}_A \longrightarrow \Omega^1_{A/k} \longrightarrow 0, \tag{5}$$

where we use $\mathcal{I} = \mathcal{I}/\mathcal{I}^2$. Pulling back the extension along the universal derivation $d : \mathcal{O}_A \to \Omega^1_{A/k}$, we obtain an extension of sheaves of k-vector spaces

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_B \longrightarrow \mathcal{O}_A \longrightarrow 0.$$

One may recover the multiplication in $\mathcal{O}_B$ by exploiting the fact that $d : \mathcal{O}_A \to \Omega^1_{A/k}$ is a derivation. Note that in particular there is a cartesian diagram of $\mathcal{O}_B$-modules

$$\begin{tikzcd}
\Omega^1_{B/k} \otimes \mathcal{O}_A \arrow[r] & \Omega^1_{A/k} \\
\mathcal{O}_B \arrow[u, "d\otimes 1"] \arrow[r] & \mathcal{O}_A. \arrow[u, "d"']
\end{tikzcd}$$

The normalization map $A \to C$ induces an $\mathcal{O}_A$-linear map $\Omega^1_{C/k} \otimes \mathcal{O}_A \to \Omega^1_{A/k}$. We may use the latter map to pull back the extension (5). As explained in [6], Theorem 1.6 the splittings of this induced extension of $\mathcal{O}_A$-modules correspond bijectively to the desired extension $\varphi : B \to C$ of the normalization map $A \to C$ along the inclusion $A \subset B$. In other words, we are looking for commutative diagrams

$$\begin{tikzcd}
\Omega^1_{B/k} \otimes \mathcal{O}_A \arrow[r] & \Omega^1_{A/k} \\
\Omega^1_{C/k} \otimes \mathcal{O}_A \arrow[u, "d\varphi\otimes 1"] \arrow[r, "\mathrm{id}"'] & \Omega^1_{C/k} \otimes \mathcal{O}_A \arrow[u]
\end{tikzcd}$$

of $\mathcal{O}_A$-modules. We thus happily arrived at a linearization of the problem.

To proceed, we merely have to compute the sheaf of differentials $\Omega^1_{B/k}$ and $\Omega^1_{C/k}$, together with their restrictions to A. We start with the nonreduced quadric B. The $\mathcal{O}_B$-module $\Omega^1_{B/k}$ is freely generated by the differentials

$$du, d\epsilon \quad \text{and} \quad d(u^{-1}), d(\epsilon u^{-1})$$

over the two open subsets $B \cap U$ and $B \cap U'$, respectively. The corresponding 1-cocycle for the locally free $\mathcal{O}_B$-module $\Omega^1_{B/k}$ is the 2×2-matrix

$$\begin{pmatrix} u^{-2} & \epsilon u^{-2} \\ 0 & u^{-1} \end{pmatrix},$$

because $d(u^{-1}) = u^{-2}du$ on the overlap, and similarly for $d(\epsilon u^{-1})$.

We next turn to the rational cuspidal curve C. The $\mathcal{O}_C$-module $\Omega^1_{C/k}$ is generated by the differentials

$$d(u^2), d(u^3) \quad \text{and} \quad d(u^{-1})$$

over the two open subsets, respectively. On the first open subset, we have a single relation $u^4 d(u^2) = 0$, because we are in characteristic $p = 2$. It follows that $\Omega^1_{C/k}$ modulo torsion is invertible, with generators $d(u^3)$ and $d(u^{-1})$, and corresponding 1-cocycle u^{-4}. We refer to [34], Section 3 for further results.

Now to the desired $\mathcal{O}_A$-linear map $d\varphi\otimes 1:\Omega^1_{C/k}\otimes\mathcal{O}_A\to\Omega^1_{B/k}\otimes\mathcal{O}_A$. Any such map is of the form

$$d(u^3)\longmapsto u^2du+P(u)d\epsilon\quad\text{and}\quad d(u^{-1})\longmapsto d(u^{-1})+Q(u^{-1})d(\epsilon u^{-1})$$

for some polynomials $P(u)$ and $Q(u^{-1})$ with coefficients from k. The 1-cocycles computed in the preceding paragraph impose the condition $P(u)u^{-3}=Q(u^{-1})$. The upshot is that the polynomials $P=P(u)$ of degree ≤ 3 correspond to such $\mathcal{O}_A$-linear maps. The corresponding morphism $\varphi:B\to C$ is given in coordinates by

$$u^2\longmapsto u^2,\quad u^3\longmapsto u^3+\epsilon P\quad\text{and}\quad u^{-1}\longmapsto u^{-1}+\epsilon u^{-4}P.$$

Throughout, we call P the *glueing polynomial* and write it as

$$P=\alpha_3u^3+\alpha_2u^2+\alpha_1u+\alpha_0,$$

with scalars $\alpha_3,\ldots,\alpha_0\in k$. For the constructions we have in mind it is important that the morphism $\varphi:B\to C$ of degree two is flat. This condition depends on the constant term of the glueing polynomial:

PROPOSITION 2.1. *The morphism $\varphi:B\to C$ is flat if and only if the constant term α_0 in the glueing polynomial $P\in k[u]$ is nonzero.*

PROOF. Clearly, φ is flat outside the singular point $c\in C$. So φ is flat if and only if the Artin scheme $\varphi^{-1}(c)\subset B$ has length two. Clearly, the fiber in question is the spectrum of the Artin ring

$$k[u,\epsilon]/(u^2,u^3+\epsilon P)=k\oplus ku\oplus k\epsilon\oplus ku\epsilon/(\epsilon P).$$

If $\alpha_0=0$, this is a k-vector space of dimension $d=3$ or $d=4$. If $\alpha_0\neq 0$, the residue class of P is a unit, and the k-vector space has dimension $d=2$. ☐

From now on we assume that the glueing polynomial $P=\alpha_3u^3+\ldots+\alpha_0$ has a nonzero constant term, such that our morphism $\varphi:B\to C$ is flat. Moreover, we regard our rational cuspidal curve as

$$C=\operatorname{Spec}k[u^2,u^3+\epsilon P]\;\cup\;\operatorname{Spec}k[u^{-1}+\epsilon u^{-4}P].$$

In other words, we view $\mathcal{O}_C$ as a subsheaf of $\mathcal{O}_B$ with respect to our morphism $\varphi:B\to C$, and not merely as a subsheaf of $\mathcal{O}_A$. We call the rational point $y_\infty\in C$ that constitutes the complement of the affine open subset $\operatorname{Spec}k[u^2,u^3+\epsilon P]\subset C$ the *point at infinity.* We shall see that it plays a special role. This is already apparent in the following fact.

PROPOSITION 2.2. *The $\mathcal{O}_C$-module $\mathcal{T}=\varphi_*\mathcal{O}_B/\mathcal{O}_C$ is invertible, and isomorphic to $\mathcal{O}_C(y_\infty)$, where $y_\infty\in C$ is the point at infinity.*

Proof. Since $\varphi : B \to C$ is flat of degree two, $\mathcal{T}$ must be invertible. We compute

$$\deg(\mathcal{T}) = \chi(\mathcal{T}) - \chi(\mathcal{O}_C) = \chi(\mathcal{O}_B) - 2\chi(\mathcal{O}_C) = 1.$$

Hence $H^0(C, \mathcal{T})$ is 1-dimensional. To compute a nonzero section, we use the affine fpqc-covering $V \amalg V' \to C$ given by the formal completion $V = \operatorname{Spec} k[[u^2, u^3 + \epsilon P]]$ and the affine open subset $V' = \operatorname{Spec} k[u^{-1} + \epsilon u^{-4} P]$. One easily sees that the residue classes of the unit $u \in \Gamma(V, \mathcal{O}_B)$ and the nilpotent $\epsilon u^{-1} P \in \Gamma(V', \mathcal{O}_B)$ generate the quotient sheaf $\mathcal{T}$ as an $\mathcal{O}_C$-module. On the overlap $V \times_C V'$, we have $(u^3 + \epsilon P)u^{-2} \cdot u \equiv \epsilon u^{-1} P$ modulo $\mathcal{O}_C$. Using

$$\frac{1}{(u^3 + \epsilon P)u^{-2}} = u^{-1} + \epsilon u^{-4} P,$$

we see that the local sections $1 \cdot u \in \Gamma(V, \mathcal{T})$ and $(u^{-1} + \epsilon u^{-4} P) \cdot \epsilon u^{-1} P \in \Gamma(V', \mathcal{T})$ glue together and define a global section, which vanishes precisely at the point at infinity $y_\infty \in C$. $\square$

3. The geometric construction

We keep the notation from the preceding section, and use the flat morphism $\varphi : B \to C$ to form the cocartesian square

$$\begin{array}{ccc} B & \longrightarrow & X \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\nu} \\ C & \longrightarrow & Y. \end{array} \tag{6}$$

The surface Y is our desired del Pezzo surface, as we shall see in due course. Pushouts like the above exists as algebraic spaces according to a very general criterion of Artin [1], Theorem 6.1. The morphism $\nu : X \to Y$ is the normalization, and the cartesian square is also cocartesian. In particular, we have $\nu^{-1}(C) = B$. Fortunately, we may immediately forget about the category of algebraic spaces.

Proposition 3.1. *The algebraic space Y is a projective scheme.*

Proof. It suffices to find an ample invertible $\mathcal{O}_Y$-module. As explained in [33], Proposition 4.1, the pushout diagram yields an exact sequence of abelian sheaves

$$1 \longrightarrow \mathcal{O}_Y^\times \longrightarrow \mathcal{O}_X^\times \times \mathcal{O}_C^\times \longrightarrow \mathcal{O}_B^\times \longrightarrow 1,$$

which results in an exact sequence of abelian groups

$$0 \longrightarrow \operatorname{Pic}(Y) \longrightarrow \operatorname{Pic}(X) \oplus \operatorname{Pic}(C) \longrightarrow \operatorname{Pic}(B). \tag{7}$$

Clearly, the preimage map $\operatorname{Pic}(C) \to \operatorname{Pic}(B)$ is surjective. Hence there is an invertible $\mathcal{O}_C$-module $\mathcal{L}$ with $\mathcal{L}_B = \mathcal{O}_B(1)$. Consequently, there is an invertible $\mathcal{O}_Y$-module $\mathcal{O}_Y(1)$ whose preimage on X is isomorphic to $\mathcal{O}_X(1)$. According to [11], Proposition 2.6.2, the invertible sheaf $\mathcal{O}_Y(1)$ must be ample, so the algebraic space Y is projective. $\square$

REMARK 3.2. The proof works under fairly general assumptions: X might be any projective scheme, B a one-dimensional subscheme, and $\varphi : B \to C$ a morphism of curves that is generically an isomorphism.

It is not difficult to write down the coordinate rings for the affine open covering $Y = V \cup V' \cup V''$ corresponding to the affine open covering $X = U \cup U' \cup U''$ defined in (1). Indeed, the diagram

$$\begin{array}{ccc} \mathcal{O}_B & \longleftarrow & \mathcal{O}_X \\ {\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\nu} \\ \mathcal{O}_C & \longleftarrow & \mathcal{O}_Y \end{array} \tag{8}$$

is cartesian, and this implies that

$$\begin{aligned} V &= \operatorname{Spec} k[u^2, u^3 + vP, v^2, v^2u, v^3, v^3u], \\ V' &= \operatorname{Spec} k[u^{-2}, u^{-1} + vu^{-4}P, v^2u^{-2}, v^2u^{-3}, v^3u^{-3}, v^3u^{-4}], \\ V'' &= \operatorname{Spec} k[uv^{-1}, v^{-1}]. \end{aligned} \tag{9}$$

Let me explain this for the first open subset V: Clearly, the six given elements $u^2, u^3 + vP, \ldots, v^3u$ lie in $\mathcal{O}_V$. Moreover, any monomial of the form u^mv^n with $m, n \geq 2$ is a monomial in the elements $u^2, v^2, v^2u, v^3, v^3u$. Finally, the residue classes of $u^2, u^3 + vP$ generate the quotient sheaf $\mathcal{O}_C|_V$. Whence the given elements generate $\mathcal{O}_V$.

The two open subsets $V, V' \subset Y$ need uncomfortably many generators. It is possible to compute, with computer algebra, a Gröbner basis for the ideal of relations, but this sheds little light on the situation. However, we shall see that things clear up under passing to suitable localizations or completions.

The scheme Y has as reduced singular locus the rational cuspidal curve $C \subset Y$. Our ultimate goal is to construct twisted forms of Y that are regular, or at least normal. This can only happen if the singularities on Y are not too bad. Recall that a locally noetherian scheme S is called *locally of complete intersection* if for all points $s \in S$, the formal completion of the stalk $\mathcal{O}_{S,s}$ is of the form $\mathcal{O}_{S,s}^\wedge = R/I$, where R is a regular local noetherian ring and $I \subset R$ is an ideal generated by a regular sequence ([14], Definition 19.3.1). The whole paper hinges on the following observation.

THEOREM 3.3. *The scheme Y is locally of complete intersection.*

PROOF. We shall determine the local generators and relations explicitly. This will be useful later, when we compute the cotangent sheaf of the singular scheme Y. To start with, consider the affine open subset $V \subset Y$ occurring in (9), with coordinate ring $A = k[u^2, u^3 + vP, v^2, v^2u, v^3, v^3u]$. To simplify notation, we give names to the generators:

$$a = u^2, \quad b = u^3 + vP, \quad c = v^2, \quad e = v^2u, \quad c' = v^3, \quad f = v^3u. \tag{10}$$

The idea now is to localize so that fewer than six generators suffice. First, let us take the affine open subset $V_{P^2} \subset V$ obtained by inverting the element

$$P^2 = \alpha_3^2 a^3 + \alpha_2^2 a^2 + \alpha_1^2 a + \alpha_0^2 \in A.$$

Recall that $P = \alpha_3 u^3 + \ldots + \alpha_0$ is the glueing polynomial in the indeterminate u defining our glueing map $\varphi : B \to C$. I contend that $A_{P^2} = k[a,b,e]_{P^2}$ as subrings inside the function field $k(u,v)$. Indeed, we have $c = (b^2 + a^3)/P^2$ and compute

$$\begin{aligned} cb &= ea + c'(\alpha_0 + \alpha_2 a) + f(\alpha_1 + \alpha_3 a), \\ eb &= ca^2 + c'(\alpha_1 a + \alpha_3 a^2) + f(\alpha_0 + \alpha_2 a). \end{aligned}$$

The matrix of coefficients at c', f has determinant

$$\det \begin{pmatrix} \alpha_0 + \alpha_2 a & \alpha_1 + \alpha_3 a \\ \alpha_1 a + \alpha_3 a^2 & \alpha_0 + \alpha_2 a \end{pmatrix} = P^2,$$

and hence we may express c', f in terms of $a, b, e, 1/P^2$. The upshot is that the canonical inclusion $k[a,b,e]_{P^2} \subset A_{P^2}$ is bijective.

According to the work of Avramov [2], the property of being locally of complete intersection is stable under localization. Therefore, it remains to check that $k[a,b,e]$ is of complete intersection. But this is trivial: We write the 2-dimensional ring $k[a,b,e]$ as a quotient of a polynomial ring in three indeterminates, and since the latter is factorial, the ideal of relations is generated by a single element. For later use, I write down the such a relation; it is

$$P^4 e^2 + b^4 a + a^7. \tag{11}$$

To continue, let us look at the affine open subset $V_Q \subset V$ given by localizing the element $Q = a^2 + c(\alpha_1 + \alpha_3 a)^2 \in A$. With this choice, I claim that $A_Q = k[a,b,c,c']_Q$. The argument is very similar to the one in the preceding paragraph, and reveals how to come up with a denominator like Q: We compute

$$\begin{aligned} cb &= ea + f(\alpha_1 + \alpha_3 a) + c'(\alpha_0 + \alpha_2 a), \\ c'b &= ec(\alpha_1 + \alpha_3 a) + fa + c^2(\alpha_0 + \alpha_2 a). \end{aligned}$$

The coefficients at e, f comprise a matrix, whose determinant is

$$\det \begin{pmatrix} a & \alpha_1 + \alpha_3 a \\ c(\alpha_1 + \alpha_3 a) & a \end{pmatrix} = a^2 + c(\alpha_1 + \alpha_3 a)^2 = Q.$$

Hence we may express the generators e, f in terms of $a, b, c, c', 1/Q$, and therefore $A_Q = k[a,b,c,c']_Q$. It remains to see that $R = k[a,b,c,c']$ is a complete intersection. The generators $a, c \in R$ are algebraically independent, and we have relations

$$b^2 + a^3 + cP^2 \quad \text{and} \quad c'^2 = c^3. \tag{12}$$

I claim that the canonical surjection

$$R' = k[a,c][x_1,x_2]/(x_1^2+a^3+cP^2, x_2^2+c^3) \longrightarrow R, \quad x_1 \longmapsto b, \quad x_2 \longmapsto c'$$

is bijective, where x_1, x_2 are indeterminates. Indeed, both rings in question are 2-dimensional, and R is integral, so it suffices to check that the ring R' on the left is integral. The inclusion $k[a,c] \subset R'$ is purely inseparable, so $\operatorname{Spec}(R')$ is irreducible. Being a complete intersection, the affine scheme $\operatorname{Spec}(R')$ has no embedded components. Hence, to check that the 2-dimensional ring R' is reduced we may replace it by the local Artin ring $k(a,c)[x_1,x_2]/(x_1^2+a^3+cP^2, x_2^2+c^3)$. By Lemma 3.4 below, it suffices to check that the differentials

$$d(a^3+cP^2) = (a^2+\alpha_1^2 c+\alpha_3^2 ca^2)da + P^2 dc \quad \text{and} \quad d(c^3) = c^2 dc$$

from $\Omega^1_{k(a,c)/k}$ are linearly independent, and the latter is obvious. If follows that the affine open subset $V \subset Y$ is locally of complete intersection, because the open subsets $V_{P^2}, V_Q \subset V$ cover the singular locus $\operatorname{Sing}(V) = V \cap C$.

To finish the proof, it suffices to see that the formal completion

$$R = k[[u^{-2}, u^{-1}+vu^{-4}P, v^2u^{-2}, v^2u^{-3}, v^3u^{-3}, v^3u^{-4}]]$$

of the second affine open subset $V' \subset Y$ is a complete intersection. We introduce names

$$x = u^{-1}+vu^{-4}P \quad \text{and} \quad y = v^2u^{-2} \quad \text{and} \quad z = v^3u^{-3}. \tag{13}$$

The equation $x^2 = u^{-2}+y(\alpha_3^2+\alpha_2^2u^{-2}+\alpha_1^2xu^{-4}+\alpha_0^2u^{-6})$, viewed as a recursion relation for u^{-2}, reveals that the generator u^{-2} is a formal power series in x^2, y. Next, we decompose the glueing polynomial $P = \alpha_3u^3+\ldots+\alpha_0$ into even and odd part $P = P_{\mathrm{ev}}+P_{\mathrm{odd}}$, such that $P_{\mathrm{ev}}u^{-2}$ and $P_{\mathrm{odd}}u^{-3}$ are polynomials in u^{-2}. Computing

$$xy = v^2u^{-3}+v^3u^{-4}P_{\mathrm{ev}}u^{-2}+zP_{\mathrm{odd}}u^{-3},$$
$$xz = v^2u^{-3}yP_{\mathrm{ev}}u^{-2}+v^3u^{-4}+y^2P_{\mathrm{odd}}u^{-3},$$

we see that the matrix of coefficients at v^2u^{-3}, v^3, u^{-4} has determinant

$$\det\begin{pmatrix} 1 & P_{\mathrm{ev}}u^{-2} \\ yP_{\mathrm{ev}}u^{-2} & 1 \end{pmatrix} = 1-yP_{\mathrm{ev}}^2u^{-4},$$

which is a unit. Whence it is possible to express the generators v^2u^{-3}, v^3u^{-4} as formal power series in x,y,z, so the inclusion $k[[x,y,z]] \subset R$ is bijective. But the ring $k[[x,y,z]]$ is obviously a complete intersection, with relation $y^3+z^2=0$. $\square$

In the preceding proof, we needed the following fact.

Lemma 3.4. *Let K be a field of characteristic $p>0$, and $f_1,\ldots,f_n \in K$ elements so that the differentials $df_1,\ldots,df_n \in \Omega^1_{K/K^p}$ are linearly independent. Then the local Artin ring $K[x_1,\ldots,x_n]/(x_1^p-f_1,\ldots,x_n^p-f_n)$ is a field.*

PROOF. Let $A = K[x_1, \ldots, x_n]$ be the polynomial algebra and set $y_i = x_i^p - f_i$. Let $\mathfrak{m} \subset A$ be the maximal ideal containing the ideal $(y_1, \ldots, y_n)$. It suffices to check that the residue classes $y_i \in \mathfrak{m}/\mathfrak{m}^2$ are linearly independent, according to [12], Proposition 17.1.7.

By assumption, we find K^p-derivations $D_i : K \to K$ with $D_i(f_j) = \delta_{ij}$ (Kronecker delta). The cotangent sequence

$$0 \longrightarrow \Omega^1_{K/K^p} \otimes A \longrightarrow \Omega^1_{A/K^p} \longrightarrow \Omega^1_{A/K} \longrightarrow 0$$

is exact and splits, because the ring extension $K \subset A$ is smooth. Hence we may extend our D_i to K^p-derivations $D_i : A \to A$, which have $D_i(y_j) = \delta_{ij}$. These derivations induce linear maps $D_i : \mathfrak{m}/\mathfrak{m}^2 \to A/\mathfrak{m}$. If follows that the residue classes of y_i are linearly independent. $\square$

REMARK 3.5. The condition that differentials $df_1, \ldots, df_n \in \Omega^1_{K/K^p}$ are linearly independent exactly means that the elements $f_1, \ldots, f_n \in K$ are *p-linearly independent.*

4. Picard scheme and dualizing sheaf

We keep the notation from the preceding section, such that Y is an integral projective surface locally of complete intersection, which is defined by the cocartesian square (6). It is nonnormal, with reduced nonsmooth locus $C \subset Y$ and normalization $X = \mathbb{P}^2$. In this section we study invertible sheaves on Y. In some sense, everything reduces to the curve $C \subset Y$:

PROPOSITION 4.1. *The restriction map* $\mathrm{Pic}_{Y/k} \to \mathrm{Pic}_{C/k}$ *of Picard schemes is an isomorphism.*

PROOF. The exact sequence of abelian sheaves $1 \to \mathcal{O}_Y^\times \to \mathcal{O}_X^\times \times \mathcal{O}_C^\times \to \mathcal{O}_B^\times \to 1$ induces an exact sequence of Picard schemes

$$0 \longrightarrow \mathrm{Pic}_{Y/k} \longrightarrow \mathrm{Pic}_{X/k} \times \mathrm{Pic}_{C/k} \longrightarrow \mathrm{Pic}_{B/k}\,.$$

Hence it suffices to check that the restriction map $\mathrm{Pic}_{X/k} \to \mathrm{Pic}_{B/k}$ is an isomorphism. Recall that B is the first order infinitesimal neighborhood of a line A inside $X = \mathbb{P}^2$. Clearly, the restriction map $\mathrm{Pic}_{X/k} \to \mathrm{Pic}_{A/k}$ is an isomorphism, so it suffices to check that the restriction map $\mathrm{Pic}_{B/k} \to \mathrm{Pic}_{A/k}$ is an isomorphism. Let $\mathcal{I} \subset \mathcal{O}_B$ be the ideal for the closed embedding $A \subset B$. Then $\mathcal{I}^2 = 0$ and we have an exact sequence of abelian sheaves $0 \to \mathcal{I} \to \mathcal{O}_B^\times \to \mathcal{O}_A^\times \to 1$, which gives an exact sequence

$$H^1(B, \mathcal{I}) \longrightarrow \mathrm{Pic}_{B/k} \longrightarrow \mathrm{Pic}_{A/k} \longrightarrow H^2(B, \mathcal{I}).$$

The outer terms vanish, because the abelian sheaf $\mathcal{I}$ is isomorphic to the $\mathcal{O}_A$-module $\mathcal{O}_A(-1)$, whence the assertion holds. $\square$

We conclude that the Picard scheme $\operatorname{Pic}_{Y/k} = \operatorname{Pic}_{C/k}$ is reduced and therefore smooth, and 1-dimensional. In particular, its tangent space $H^1(Y, \mathscr{O}_Y)$ is 1-dimensional. The Picard scheme sits inside a split extension

$$0 \longrightarrow \mathbb{G}_a \longrightarrow \operatorname{Pic}_{Y/k} \longrightarrow \mathbb{Z} \longrightarrow 0. \tag{14}$$

The map on the right is given by sending an invertible $\mathscr{O}_Y$-module $\mathscr{L}$ to the degree of the restriction $\mathscr{L}_C$. To simplify notation, we set $\deg(\mathscr{L}) = \deg(\mathscr{L}_C)$ and call this integer the *degree* of $\mathscr{L}$.

Proposition 4.2. *An invertible $\mathscr{O}_Y$-module $\mathscr{L}$ is ample if and only if $\deg(\mathscr{L}) > 0$.*

Proof. According to [11], Proposition 2.6.2, the invertible sheaf $\mathscr{L}$ is ample if and only if its preimage $\nu^*(\mathscr{L})$ is ample. If follows easily from the definition of $d = \deg(\mathscr{L})$ that $\nu^*(\mathscr{L}) = \mathscr{O}_X(d)$. Hence $\mathscr{L}$ is ample if and only if $d > 0$. □

Being locally of complete intersection, the proper scheme Y also has an invertible dualizing sheaf ω_Y. It is straightforward to compute its degree:

Proposition 4.3. *The degree of the dualizing sheaf is $\deg(\omega_Y) = -1$.*

Proof. Consider the Tschirnhausen module $\mathscr{T} = \nu_*\mathscr{O}_X/\mathscr{O}_Y$. Its annulator ideal $\mathfrak{c} \subset \mathscr{O}_Y$ is called the *conductor ideal* for the normalization map $\nu : X \to Y$. According to Proposition 2.2, the $\mathscr{O}_Y$-module $\mathscr{T}$ is an invertible $\mathscr{O}_C$-module, and hence $\mathfrak{c} = \mathscr{O}_Y(-C)$. Since the square (6) defining Y is cocartesian, the induced ideal on X satisfies $\mathfrak{c} = \mathfrak{c}\mathscr{O}_X$. As the square is also cartesian, we have $\mathfrak{c}\mathscr{O}_X = \mathscr{O}_X(-B)$.

The conductor is closely related to duality: The equality $\mathfrak{c} = \mathscr{H}om(\nu_*\mathscr{O}_X, \mathscr{O}_Y)$ shows that the conductor ideal has a natural $\mathscr{O}_X$-module structure, and coincides with the relative dualizing sheaf $\omega_{X/Y}$. The latter satisfies $\omega_X = \omega_{X/Y} \otimes \nu^*\omega_Y$. Clearly, the projective plane $X = \mathbb{P}^2$ has dualizing sheaf $\omega_X = \mathscr{O}_X(-3)$. Together with $\omega_{X/Y} = \mathscr{O}_X(-2)$, it follows $\nu^*(\omega_Y) = \mathscr{O}_X(-1)$. □

In particular, the dualizing sheaf ω_Y is antiample. We thus call our Y a *nonnormal del Pezzo surface*. Let me point out that its irregularity is $h^1(\mathscr{O}_Y) = 1$, which is highly unusual for del Pezzo surfaces, even for singular ones (compare [17], Corollary 2.5 and [32], Theorem 2.2 and [30]).

To determine the isomorphism class of ω_Y in the Picard group, it suffices to compute its restriction to C. Recall that the Tschirnhausen module $\mathscr{T} = \nu_*\mathscr{O}_X/\mathscr{O}_Y$ is the invertible $\mathscr{O}_C$-module $\mathscr{O}_C(y_\infty)$, where $y_\infty \in C$ is the point at infinity.

Proposition 4.4. *With the preceding notation, we have $\omega_Y|_C = \mathscr{O}_C(-y_\infty)$.*

Proof. We consider the relative dualizing sheaf $\omega_{C/Y} = \mathscr{E}xt^1_{\mathscr{O}_Y}(\mathscr{O}_C, \mathscr{O}_Y)$, which satisfies $\omega_C = \omega_{C/Y} \otimes \omega_Y|_C$. The exact sequence $0 \to \mathfrak{c} \to \mathscr{O}_C \to \mathscr{O}_C \to 0$ yields an exact sequence

$$0 \longrightarrow \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{O}_Y) \longrightarrow \mathcal{H}om_{\mathcal{O}_Y}(\mathfrak{c}, \mathcal{O}_Y) \longrightarrow \mathcal{E}xt^1_{\mathcal{O}_Y}(\mathcal{O}_C, \mathcal{O}_Y) \longrightarrow 0.$$

We have an obvious inclusion $\mathcal{O}_X \subset \mathcal{H}om(\mathfrak{c}, \mathcal{O}_Y)$, and we now check that it is bijective. The composition map

$$\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_Y) \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(\mathfrak{c}, \mathcal{O}_X) \longrightarrow \mathcal{H}om_{\mathcal{O}_Y}(\mathfrak{c}, \mathcal{O}_Y)$$

is bijective, because the conductor ideal is invertible as $\mathcal{O}_X$-module. For the same reason, the evaluation map

$$\mathcal{H}om_{\mathcal{O}_X}(\mathfrak{c}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathfrak{c} \to \mathcal{O}_X$$

is bijective. Composing the previous maps, we obtain a chain of inclusion $\mathcal{O}_X \subset \mathcal{H}om(\mathfrak{c}, \mathcal{O}_Y) \subset \mathcal{O}_X$, which clearly is bijective. The upshot is that the relative dualizing sheaf $\omega_{C/Y}$ coincides with the Tschirnhausen module $\mathcal{T}$. Using that $\omega_C = \mathcal{O}_C$ and Proposition 2.2, we deduce the assertion. □

5. Cartier divisors and Weil divisors

Our del Pezzo surface Y has a natural polarization furnished by the ample invertible sheaf $\omega_Y^\vee$. Given any Weil divisor D on Y, we define its *degree* by the intersection number $\deg(D) = \omega_Y^\vee \cdot D$. In this section we have a closer look at curves $D \subset Y$ of degree one. Much of the geometry of Y is captured by these curves. Any such curve is the schematic image of a line L on the normalization $X = \mathbb{P}^2$. To begin with, we compute some cohomology groups.

PROPOSITION 5.1. *Let $\mathcal{L}$ be an invertible $\mathcal{O}_Y$-module of degree d. Then we have $\chi(\mathcal{L}) = d(d+1)/2$. Moreover, $H^2(Y, \mathcal{L}) = 0$ for $d \geq 0$, and $H^1(Y, \mathcal{L}) = 0$ for $d \geq 1$.*

PROOF. The exact sequence $0 \to \mathcal{L} \to \mathcal{L}_X \to \mathcal{L} \otimes \mathcal{T} \to 0$ of coherent sheaves gives $\chi(\mathcal{L}) = \chi(\mathcal{L}_X) - \chi(\mathcal{L} \otimes \mathcal{T})$. Here $\mathcal{T} = \mathcal{O}_X/\mathcal{O}_Y$ is the Tschirnhausen module. We clearly have $\chi(\mathcal{L}_X) = (d+2)(d+1)/2$. According to Proposition 2.2, $\mathcal{T}$ is an invertible $\mathcal{O}_C$-module of degree one. It follows that $\chi(\mathcal{L} \otimes \mathcal{T}) = d+1$, and the assertion on the Euler characteristic follows.

Now suppose $d \geq 0$. Then the term on the left in the exact sequence

$$H^1(C, \mathcal{L} \otimes \mathcal{T}) \longrightarrow H^2(Y, \mathcal{L}) \longrightarrow H^2(X, \mathcal{L}_X)$$

vanishes. The term on the right is Serre dual to $H^0(X, \mathcal{L}_X^\vee(-3))$, and vanishes as well. Hence $H^2(Y, \mathcal{L}) = 0$.

Finally, suppose that the degree is $d \geq 1$. We now use the short exact sequence $0 \to \mathcal{L} \to \mathcal{L}_X \oplus \mathcal{L}_C \to \mathcal{L}_B \to 0$, which gives an exact sequence

$$H^0(X, \mathcal{L}_X) \oplus H^0(C, \mathcal{L}_C) \longrightarrow H^0(B, \mathcal{L}_B) \longrightarrow H^1(Y, \mathcal{L}) \longrightarrow H^1(X, \mathcal{L}_X) \oplus H^1(C, \mathcal{L}_C).$$

The sum on the right vanishes, and the map on the left is surjective. The latter follows from the exact sequence

$$H^0(X,\mathscr{L}) \longrightarrow H^0(B,\mathscr{L}_B) \longrightarrow H^1(X,\mathscr{L}(-2)) = 0.$$

The upshot is that $H^1(Y,\mathscr{L}) = 0$. $\square$

For $d = 1$, this means that $H^0(Y,\mathscr{L})$ is 1-dimensional. In other words, each invertible sheaf $\mathscr{L}$ of degree one, there is precisely one effective Cartier divisor $D \subset Y$ with $\mathscr{L} = \mathscr{O}_Y(D)$. Note that this applies in particular to the antidualizing sheaf $\mathscr{L} = \omega_Y^\vee$. It turns out that the position of these Cartier divisors is determined by restricting to the reduced singular locus $C \subset Y$:

Proposition 5.2. *Let $\mathscr{L}$ be an invertible sheaf of degree $d = 1$. Then the restriction map $H^0(Y,\mathscr{L}) \to H^0(C,\mathscr{L}_C)$ is bijective.*

Proof. The exact sequence $0 \to \mathscr{L}_X(-B) \to \mathscr{L}_X \to \mathscr{L}_B \to 0$ gives an exact sequence

$$H^0(X,\mathscr{L}_X(-B)) \longrightarrow H^0(X,\mathscr{L}_X) \longrightarrow H^0(B,\mathscr{L}_B) \longrightarrow H^1(X,\mathscr{L}_X(-B)).$$

Both outer terms vanish, and hence the restriction map $H^0(X,\mathscr{L}_X) \to H^0(B,\mathscr{L}_B)$ is bijective.

Using the exact sequence $0 \to \mathscr{L} \to \mathscr{L}_X \oplus \mathscr{L}_C \to \mathscr{L}_B \to 0$, we obtain a short exact sequence

$$0 \longrightarrow H^0(Y,\mathscr{L}) \longrightarrow H^0(X,\mathscr{L}_X) \oplus H^0(C,\mathscr{L}_C) \longrightarrow H^0(B,\mathscr{L}_B) \longrightarrow 0.$$

In light of the preceding paragraph, the restriction map $H^0(Y,\mathscr{L}) \to H^0(C,\mathscr{L}_C)$ must be bijective. $\square$

We conclude that given a rational point $y \in C$ in the smooth locus of C, there is precisely one Cartier divisor $D \subset Y$ of degree one passing through y. Of course, there is a continuous family of Weil divisors of degree one passing through that point. Any such Weil divisor is the image of a unique line on $X = \mathbb{P}^2$. How to distinguish between such Cartier divisors and Weil divisors?

Proposition 5.3. *Let $L \subset X$ be a line not contained in the conductor locus $B \subset X$, and $D \subset Y$ be its image, and $y \in C \cap D$ be the unique intersection point with the singular locus $C \subset Y$. Then the Weil divisor D is Cartier if and only if the schematic preimage $\nu^{-1}(y) \subset X$ is contained in L.*

Proof. Suppose that $D \subset Y$ is Cartier. Then the preimage $\nu^{-1}(D) \subset X$ is Cartier as well, and clearly contains $\nu^{-1}(y)$. The inclusion $L \subset \nu^{-1}(D)$ is an equality outside the conductor locus $B \subset X$. Since both subschemes are Cartier, they must be equal. If follows that $\nu^{-1}(y) \subset L$.

Now suppose $\nu^{-1}(y) \subset L$. Let $D' \subset Y$ be the unique Cartier divisor of degree one passing through y. Its preimage $L' = \nu^{-1}(D') \subset X$ is a line containing $\nu^{-1}(y)$, which is an Artin subscheme of length two. However, through any Artin subscheme of length two on $X = \mathbb{P}^2$, there passes precisely one line. We conclude $L = L'$, whence $D = D'$ is Cartier. □

We finally determine what kind of scheme a Weil divisor of degree one is.

PROPOSITION 5.4. *Let $D \subset Y$ be a Weil divisor of degree one.*

(i) *If $D \subset Y$ is Cartier or if $D = C$, then the curve D is isomorphic to the rational cuspidal curve with arithmetic genus $p_a = 1$.*
(ii) *If $D \subset Y$ is not Cartier and $D \neq C$, then the curve D is isomorphic to the projective line $\mathbb{P}^1$.*

PROOF. The case $D = C$ is clear, because C is by definition the rational cuspidal curve with $p_a = 1$. Now suppose that D is Cartier. Then $\omega_D = \omega_Y(D) \mid_D$ has degree zero. It follows that $-2\chi(\mathcal{O}_C) = \deg(\omega_D) = 0$, whence $h^1(\mathcal{O}_C) = 1$. Clearly, D is birational and homeomorphic to the projective line, and the assertion follows.

Finally, suppose that $D \neq C$ is not Cartier. Let $L \subset X$ be the unique line with $D = \nu(L)$, and consider the birational morphism $f : L \to D$. According to Proposition 5.3, the fiber $f^{-1}(y) = L \cap \nu^{-1}(y)$ is an Artin scheme of length one. It follows that f is an isomorphism. □

To close this section, we look again at the antidualizing sheaf $\omega_Y^\vee$. We know from Proposition 5.1 that there is *only one* effective anticanonical divisor $D \subset Y$. We note in passing an interesting consequence: The anticanonical divisor $D \subset Y$ is invariant under any automorphism of Y.

6. The tangent sheaf

We keep the notation from the preceding sections, such that Y is a nonnormal del Pezzo surface. To construct twisted forms of Y, we have to understand the group scheme $\mathrm{Aut}_{Y/k}$ and its Lie algebra $H^0(Y, \Theta_{Y/k})$. In this section we shall see that the tangent sheaf $\Theta_{Y/k}$ is locally free of rank two, and that it is not difficult to determine its global sections. This is somewhat surprising, because the cotangent sheaf $\Omega^1_{Y/k}$ is not locally free along the singular curve $C \subset Y$. However, we shall see that the trouble only comes from the torsion subsheaf $\tau \subset \Omega^1_{Y/k}$. This effect seems to be special to positive characteristics.

THEOREM 6.1. *The coherent $\mathcal{O}_Y$-module $\Omega^1_{Y/k}/\tau$ is locally free of rank two.*

PROOF. This is a local problem in Y. First, consider the affine open subset $V \subset Y$ that is the spectrum of

$$k[u^2, u^3 + vP, v^2, v^2u, v^3, v^3u] = k[a, b, c, e, c', f],$$

as in Equation (10). As explained in the proof for Proposition 3.3, it is advisable to localize further, using $P^2 = \alpha_3^2 a^3 + \alpha_2^2 a^2 + \alpha_1^2 a + \alpha_0^2$ and $Q = a^2 + c(\alpha_1 + \alpha_3 a)^2$ as denominators. We saw that V_{P^2} is an open subset inside the spectrum of

$$k[a,b,e]/(P^4e^2 + b^4a + a^7).$$

The module of differentials is generated by da, db, de modulo the relation $(b^4 + a^6)da$. Since Y is generically smooth, the coefficient $b^4 + a^6$ must be a regular element, hence the differential da is torsion. We conclude that the differentials db, de form a basis of $\Omega^1_{Y/k}$ modulo torsion over $V_{P^2} \subset Y$. A similar argument using Equation (12) gives that the differentials db, dc' form a basis for Ω^1_Y modulo torsion over $V_Q \subset Y$.

It remains to treat the affine open subset $V' \subset Y$. Since the point at infinity $y_\infty \in V'$ is the only singularity not contained in V, it suffices to consider the formal completion $R = k[[x,y,z]]/(y^3 - z^2)$ of $\mathcal{O}_{Y,y_\infty}$ as in Equation (13). The separated completion $\widehat{\Omega}^1_{R/k}$ modulo torsion is free, with basis dx, dz. Since the two functions $x = u^{-1} + vu^{-4}P$ and $z = u^3v^{-3}$ are already contained in $\mathcal{O}_{Y,y_\infty}$, it follows that dx, dz are a basis modulo torsion in some affine neighborhood of $y_\infty \in Y$. □

Corollary 6.2. *The tangent sheaf $\Theta_{Y/k}$ is locally free of rank two.*

Proof. Dualizing the exact sequence $0 \to \tau \to \Omega^1_{Y/k} \to \Omega^1_{Y/k}/\tau \to 0$, we see that the canonical map $\mathcal{H}om(\Omega^1_{Y/k}/\tau, \mathcal{O}_Y) \to \mathcal{H}om(\Omega^1_{Y/k}, \mathcal{O}_Y)$ is bijective. □

Our next task is to compute the Lie algebra of global sections for the tangent sheaf. We are mostly interested in the behavior of derivations near the singular locus $C \subset Y$, whence we shall describe $H^0(Y, \Theta_{Y/k})$ as a subalgebra of $H^0(V_{P^2}, \Theta_{Y/k})$. We just saw that $\Omega^1_{Y/k}$ modulo torsion has basis a on $V_{P^2} \subset Y$ given by db, de. We denote by D_b, D_e the dual basis of $\Theta_{Y/k}$ on V_{P^2}. The following result gives an implicit description of $H^0(Y, \Theta_{Y/k})$, which will give enough information for our purposes.

Proposition 6.3. *The Lie algebra $H^0(Y, \Theta_{Y/k})$ consists of all derivations of the form $fD_b + gD_e$, where $f, g \in k[u^2, u^3 + vP, v^2, v^2u, v^3, v^3u]$ are polynomials so that the rational functions*

$$f\frac{u}{v^2P} + g\frac{Pv + P'uv + u^3}{v^4P} \quad \textit{and} \quad f\frac{1}{v^2P} + g\frac{u^2 + P'v}{v^4P} \tag{15}$$

are contained in $k[u/v, 1/v]$.

Proof. Since the cotangent sheaf satisfies Serre's condition (S_2) and the complement of $V_{P^2} \cup V'' \subset Y$ is finite, the restriction map $H^0(Y, \Theta_{Y/k}) \to H^0(V_{P^2} \cup V'', \Theta_{Y/k})$ must be bijective. The latter group is the kernel of the difference map

$$H^0(V_{P^2}, \Theta_{Y/k}) \oplus H^0(V'', \Theta_{Y/k}) \longrightarrow H^0(V_{P^2} \cap V'', \Theta_{Y/k}) \tag{16}$$

coming from Čech cohomology. To determine the kernel, we first compute with differentials rather than derivations:

$$db = (u^2 + P'v)du + Pdv, \quad de = v^2 du$$
$$d(u/v) = 1/vdu + u/v^2 dv, \quad d(1/v) = 1/v^2 dv,$$

as follows from (10). Consequently,

$$A = \begin{pmatrix} u^2 + P'v & v^2 \\ P & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1/v & 0 \\ u/v^2 & 1/v^2 \end{pmatrix} \tag{17}$$

are the base change matrices for base changes from db, de to du, dv, and from $d(u/v), d(1/v)$ to du, dv, respectively. It follows that $B^{-1}A$ is the base change matrix from db, de to $d(u/v), d(1/v)$, and whence

$${}^t(B^{-1}A)^{-1} = \begin{pmatrix} u/v^2P & (Pv + P'uv + u^3)/v^4P \\ 1/v^2P & (u^2 + P'v)/v^4P \end{pmatrix}$$

is the base change matrix from the dual basis D_b, D_e to $D_{u/v}, D_{1/v}$. Using this base change matrix, we compute the kernel in the exact sequence (16), and the assertion follows. $\square$

Recall that the normalization of Y is the projective plane $X = \mathbb{P}^2$. Pulling back to X, we see that $H^0(X, \Theta_{Y/k} \otimes \mathcal{O}_X)$ is given by derivations of the form $fD_b + gD_e$, where $f, g \in k[u, v]$ are polynomials so that the two rational functions in (15) lie in $k[u/v, 1/v]$. In particular, the rational derivation $\delta = PD_e$ defines a global section of $\Theta_{Y/k} \otimes \mathcal{O}_X$. It does not, however, always come from a global section of $\Theta_{Y/k}$:

Corollary 6.4. *Suppose the glueing polynomial P is even, that is, $P = \alpha_2 u^2 + \alpha_0$. Then the rational vector field $\delta = PD_e$ lies in $H^0(Y, \Theta_{Y/k})$. Moreover, we have $\delta \circ \delta = 0$.*

Proof. The first statement follows from the preceding proposition. We have $\delta(e) = \alpha_2 a + \alpha_0$ and $\delta(b) = 0$, and therefore $\delta \circ \delta = 0$. $\square$

From now on we assume that the glueing polynomial P is even. Using the base change matrices in (17), we easily express the global vector field $\delta = PD_e$ in terms of other rational derivations, and obtain

$$\delta = PD_e = (Pv + u^3)v^{-4}D_{u/v} + u^2v^{-4}D_{1/v} = Pv^{-2}D_u + u^2v^{-2}D_v.$$

In the next section, we shall interpret δ as a group scheme action of α_2 on Y. The fixed points for the group scheme action correspond to the zeros of the vector field. By definition, $\delta = PD_e$ has no zeros on the open subset V_{P^2}. Clearly, it vanishes on V'' to first order along the closed subscheme given by $1/v = 0$. It remains to determine the precise behavior of the vector field near the curve of singularities $C \subset Y$.

Proposition 6.5. *The only zero of the global vector field $\delta = PD_e$ lying on the singular curve $C \subset Y$ is the point at infinity $y_\infty \in C$. With respect to the formal coordinates $\mathcal{O}^\wedge_{Y,y_\infty} = k[[x, y, z]]/(y^3 - z^2)$, we have $\delta = xD_z$.*

Proof. Using $\delta = Pv^{-2}D_u + u^2v^{-2}D_v$ and the definitions of x, y, z in Equation (13), one computes $\delta(x) = \delta(y) = 0$ and $\delta(z) = x$, whence $\delta = xD_z$. It remains to write δ on $V_Q \subset V$ in terms of the basis $D_b, D_{c'}$, where $c' = v^3$. Note that $Q = a + c(\alpha_3 u^3 + \alpha_1 u) = a$, because we assume the glueing polynomial to be even. We have $\delta(b) = 0$ and compute $\delta(v^3) = u^2 = a$, whence $\delta = aD_{c'}$ has no zero on the affine open subset $V_Q \subset Y$. □

7. Splitting type of tangent sheaf

In this section we determine the restriction $\Theta_{Y/k}|_D$ of the tangent sheaf to Weil divisors $D \subset Y$ of degree one. This will show that our choice of global vector field $\delta = PD_e$ is, in some sense, the best possible choice.

The computation with the tangent sheaf is rather easy, because we may dually work with the cotangent sheaf modulo torsion. The latter sits in an exact sequence

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega^1_{Y/k} \otimes \mathcal{O}_D \longrightarrow \Omega^1_{D/k} \longrightarrow 0, \tag{18}$$

where $\mathcal{I} = \mathcal{O}_Y(-D)$ is the ideal of the Weil divisor. For the following arguments, note that the torsion subsheaf $\tau \subset \Omega^1_{Y/k}$ is locally a direct summand, because $\Omega^1_{Y/k}/\tau$ is locally free, whence this subsheaf commutes with base change.

Proposition 7.1. *Let $D \subset Y$ be a Cartier divisor of degree one. Then the tangent sheaf splits as $\Theta_{Y/k}|_D \simeq \Theta_{D/k} \oplus \omega_Y^\vee|_D$. Both summands are invertible $\mathcal{O}_D$-modules, of degree four and one, respectively.*

Proof. The $\mathcal{O}_D$-module $\mathcal{I}/\mathcal{I}^2$ is invertible of degree $-D^2 = -1$, and the canonical map $\mathcal{I}/\mathcal{I}^2 \to \Omega^1_{Y/k} \otimes \mathcal{O}_D$ on the left in the cotangent sequence (18) is injective, because $D \subset Y$ is Cartier. According to Proposition 5.4, the scheme D is the rational cuspidal curve with arithmetic genus $p_a = 1$. It follows $\mathcal{I}/\mathcal{I}^2 \simeq \omega_Y|_D$, and that $\Omega^1_{D/k}$ modulo torsion is an invertible sheaf of degree -4. Since Y is smooth at the generic point $\eta \in D$, the torsion in $\Omega^1_{Y/k} \otimes \mathcal{O}_D$ maps to the torsion of $\Omega^1_{D/k}$, and this map must be bijective because $\Omega^1_{Y/k}/\tau$ is locally free of rank two, which contains $\mathcal{I}/\mathcal{I}^2$ locally as a direct summand. The result now follows by taking duals, and the fact that there are no nontrivial extension of invertible sheaves of degree one by invertible sheaves of degree four on D. □

In particular, we see that the invertible sheaf $\det(\Theta_{Y/k})$ has degree five.

Proposition 7.2. *Let $D \subset Y$ be a Weil divisor of degree one that is not Cartier and not the singular curve C. Then $\Theta_{Y/k}|_D \simeq \mathcal{O}_D(3) \oplus \mathcal{O}_D(2)$ is a direct sum of invertible $\mathcal{O}_D$-modules of degree two and three.*

Proof. In this case, Proposition 5.4 tells us that $D \simeq \mathbb{P}^1$. Hence $\Omega^1_{D/k}$ is invertible of degree -2. It follows that the torsion of $\Omega^1_{Y/k} \otimes \mathcal{O}_D$ maps to zero in $\Omega^1_{Y/k}$. The induced map $(\Omega^1_{Y/k}/\tau) \otimes \mathcal{O}_D \to \Omega^1_{D/k}$ has invertible kernel, which must have degree -3. The result follows after dualizing, and the fact that there are non nontrivial extensions of $\mathcal{O}_{\mathbb{P}^1}(3)$ by $\mathcal{O}_{\mathbb{P}^1}(2)$. $\square$

PROPOSITION 7.3. *Let $C \subset Y$ be the reduced singular locus. Then the tangent sheaf splits as $\Theta_{Y/k}|_C \simeq \Theta_{C/k} \oplus \mathcal{O}_C(y_\infty)$. Both summands are invertible, of degree four and one, respectively.*

PROOF. I claim that the torsion in $\Omega^1_{Y/k}$ maps to the torsion in $\Omega^1_{C/k}$. It suffices to check this on the formal completion $R' = k[[x,y,z]]/(y^3+z^2)$ of the affine open subset $V' \subset Y$ at the point at infinity. The curve $C \subset Y$ has ideal (y,z), and the torsion is generated by dy, whence the claim follows. Whence we have an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \Omega^1_{Y/k} \otimes \mathcal{O}_C/(\text{torsion}) \longrightarrow \Omega^1_{C/k}/(\text{torsion}) \longrightarrow 0$$

for some coherent $\mathcal{O}_C$-module $\mathcal{K}$. Since both terms on the right are locally free, the $\mathcal{O}_C$-module $\mathcal{K}$ must be invertible. It must have degree -1, because $\Omega^1_{C/k}$ modulo torsion is invertible of degree -4. Such an extension of $\mathcal{O}_C$-modules must split. Dualizing it, we obtain $\Theta_{Y/k}|_C = \Theta_{C/k} \oplus \mathcal{K}^\vee$.

It remains to see that $\mathcal{K}^\vee \simeq \mathcal{O}_C(y_\infty)$. We use our global vector field $\delta = PD_e$. By Proposition 6.5, the restriction $\delta \otimes 1$ vanishes only at $y_\infty \in C$, and has vanishing order one there. Decompose $\delta\otimes 1 = \delta' + \delta''$, where δ' is a global vector field on C, and $\delta'' \in H^0(C, \mathcal{K}^\vee)$. If $\delta'' = 0$, then $\Theta_{C/k}$ would have degree one, contradiction. Hence $\delta'' \neq 0$, and it follows $\mathcal{K}^\vee \simeq \mathcal{O}_C(y_\infty)$. $\square$

REMARK 7.4. Let $\mathfrak{m} \subset \mathcal{O}_Y$ be the maximal ideal for the point at infinity $y_\infty \in Y$. The preceding result tells us that any global vector field on Y, the corresponding tangent vector in $\Theta_{Y/k}(y_\infty) \subset \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$ is tangent to the curve of singularities $C \subset Y$. We conclude that our $\delta = PD_e$ is in some sense the best possible choice when it comes to twisting in Section 8.

REMARK 7.5. Suppose $\mathcal{F}$ is a locally free sheaf of rank n on the projective plane $X = \mathbb{P}^2$. The restriction to any line $L \subset X$ splits into a direct sum of invertible sheafs $\mathcal{F}_L \simeq \mathcal{O}_L(d_1) \oplus \ldots \oplus \mathcal{O}_L(d_n)$, say with $d_1 \leq \ldots \leq d_n$. This sequence of integers is called the *splitting type* of $\mathcal{F}$ along the line L. The preceding results tell us: The generic splitting type of $\mathcal{F} = \nu^*(\Theta_{Y/k})$ is given by the sequence $(2,3)$. The generic splitting type degenerates to the special splitting type $(1,4)$ on those lines $L \subset \mathbb{P}^2$ whose image in $D \subset Y$ is Cartier or equals C.

8. Twisted del Pezzo surfaces

We keep the assumptions as in the preceding section, such that Y is a nonnormal del Pezzo surfaces, defined by an even glueing polynomial $P = \alpha_2 u^2 + \alpha_0$. Then we have a global vector field $\delta \in H^0(Y, \Theta_{Y/k})$ with $\delta \circ \delta = 0$ given by the formula $\delta = PD_e$. Such vector fields correspond to actions of the group scheme α_2, which is finite and infinitesimal.

Recall that we have $\alpha_2 = \operatorname{Spec} k[\epsilon]$ as a scheme. Its values on k-algebras R is the group $\alpha_2(R) = \{f \in R \mid f^2 = 0\}$, with addition as group law. The action $\alpha_2 \times Y \to Y$ is given by the formula

$$\mathcal{O}_Y \longrightarrow k[\epsilon] \otimes_k \mathcal{O}_Y, \quad s \longmapsto \delta(s)\epsilon \otimes s.$$

A rational point $y \in Y$ is a fixed point for the α_2-action if and only if $\delta(y) = 0$ as a section of $\Theta_{Y/k}$, or equivalently $\delta(\mathfrak{m}_y) \subset \mathfrak{m}_y$ as derivation $\delta : \mathcal{O}_Y \to \mathcal{O}_Y$.

As explained in Section 1, any α_2-torsor T yields a twisted form $Y' = Y \wedge T$ of our nonnormal del Pezzo surface Y. Note that the projections $Y \leftarrow Y \times T \to Y'$ are universal homeomorphisms, and we may identify points on Y with points on Y'. Any such twisted form Y' is locally of complete intersection. Moreover, $\omega_{Y'}$ is antiample, and we have $h^1(\mathcal{O}_{Y'}) = h^1(\mathcal{O}_Y) = 1$. Whence the twisted form Y' is another del Pezzo surface, possibly with less severe singularities than Y.

Any α_2-torsor is of the form $T = \operatorname{Spec} k(\sqrt{\lambda})$ for some scalar $\lambda \in k$, with action given by the derivation $\sqrt{\lambda} \mapsto 1$. The torsor is nontrivial if and only if $\lambda \in k$ is not a square. We now can formulate the main result of this paper:

Theorem 8.1. *Let k be a nonperfect field of characteristic two, $\lambda \in k$ be a nonsquare, and $T = \operatorname{Spec} k(\sqrt{\lambda})$ the corresponding α_2-torsor. Then the twisted form $Y' = Y \wedge T$ is a normal del Pezzo surface with $h^1(\mathcal{O}_{Y'}) = 1$. It has a unique singularity $y'_\infty \in Y'$, which corresponds to the point at infinity $y_\infty \in Y$.*

Proof. First observe that Y' is smooth outside the curve corresponding to the reduced singular locus $C \subset Y$. According to Proposition 6.5, the point at infinity $y_\infty \in Y$ is a fixed point on the singular locus. The corresponding point on the twisted form $y'_\infty \in Y'$ then must be a singularity, by Proposition 1.8.

It remains to see what the effect of twisting is on the affine open subset $V \subset Y$ at the singular locus. Recall that the open subset $V_{P^2} \subset V$ is given by the algebra $k[a, b, e, P^{-1}]/(P^4e^2 + b^4a + a^7)$, and that $\delta = PD_e$, compare the proof for Theorem 3.3. We first analyse the rational point $y \in Y$ corresponding to the origin $a = b = e = 0$. We have $\delta(a) = \delta(b) = 0$ and $\delta(e) = P$, whence the orbit $Gy \subset Y$ is given by the ideal (a, b, e^2). Clearly, this ideal is generated by a, b, which is a regular sequence. Now Theorem 1.5 tells us that Y' is regular near the point corresponding to $y \in Y$.

Next we treat the singular locus of V_{P^2} outside the origin. Consider the Cartier divisor $A \subset V_{P^2}$ supported by the singular locus given by the ring element $c = v^2 = (b^2 + a^3)/P^2$. This element is invariant, and we have

$$k[a, b, e, P^{-1}]/(P^4e^2 + b^4a + a^7, b^2 + a^3) = k[a, b, e, P^{-1}]/(b^2 + a^3, e^2).$$

Twisting this algebra, we obtain as twisted algebra $k(\sqrt{\lambda})[a, b, P^{-1}]/(b^2 + a^3)$, which defines a rational cuspidal curve over the quadratic extension field $k(\sqrt{\lambda})$. The latter is regular outside the origin. Using Theorem 1.5 again, we conclude that the twisted form Y' is regular on the open subset corresponding to $V_{P^2} \subset Y$.

Finally, we treat the other open subset $V_Q \subset V$, which is given by

$$k[a,b,c,c',a^{-1}]/(b^2+a^3+cP^2, c'^2=c^3).$$

Here our derivation takes the form $\delta = aD_{c'}$. Again, consider the Cartier divisor $A \subset V_Q$ supported by the singular locus given by c. We have

$$k[a,b,c,c']/(b^2+a^3+cP^2, c'^2=c^3, c) = k[a,b,c']/(b^2+a^3, c'^2),$$

and we may argue as above. The upshot is that the twisted form Y' is regular outside the point at infinity $y'_\infty \in Y'$. □

It is not difficult to analyse the singularity:

THEOREM 8.2. *The singularity $y'_\infty \in Y'$ is a rational double point of type A_1. The minimal resolution of singularities $r: \tilde{Y}' \to Y'$ is obtained by blowing up the reduced singular point. The exceptional divisor $E = r^{-1}(y'_\infty)$ is isomorphic to a regular quadric in $\mathbb{P}^2_k$ that is a twisted form of the double line. The regular surface $\tilde{Y}'$ is a weak del Pezzo surface with $h^1(\mathcal{O}_{\tilde{Y}'}) = 1$.*

PROOF. As explained in the proof for Proposition 3.3, the completion $\mathcal{O}^\wedge_{Y,y_\infty}$ is the algebra $R = k[[x,y,z]]/(y^3+z^2)$, and furthermore $\delta = xD_z$. It follows that the completion $\mathcal{O}^\wedge_{Y',y'_\infty}$ is the subalgebra $R \subset k[\sqrt{\lambda}, x, y, z]/(y^3+z^2)$ generated by the invariants x, y, z', where $z' = z + \sqrt{\lambda}x$, which has defining relation $z'^2 = y^3 + \lambda x^2$.

Consider the blowing up $Z \to \operatorname{Spec}(R)$ of the maximal ideal (x,y,z'). It is covered by two charts: The x-chart

$$x, y/x, z'/x \quad \text{modulo} \quad (z'/x)^2 = (y/x)^3 x + \lambda, \tag{19}$$

and the z'-chart

$$x/z', y/z', z' \quad \text{modulo} \quad z'^2 = (y/z')^3 z' + \lambda(x/z')^2. \tag{20}$$

The exceptional divisor $E \subset Z$ is given by setting x and z' to zero, respectively. Whence E is covered by $y/x, z'/x$ modulo $(z'/x)^2 = \lambda$ and $x/z, y/z'$ modulo $(x/z')^2 = 1/\lambda$. The exceptional divisor is evidently regular, and hence the blowing up $\tilde{Y}'$ is regular as well. Furthermore, we easily infer that $E = r^{-1}(y'_\infty)$ is isomorphic to a regular quadric in $\mathbb{P}^2_k$ that becomes a double line after adjoining $\sqrt{\lambda}$.

We infer that $R^1r_*(\mathcal{O}_{\tilde{Y}'}) = 0$, so the singularity is rational. It also follows that the map $H^1(Y', \mathcal{O}_{Y'}) \to H^1(\tilde{Y}', \mathcal{O}_{\tilde{Y}'})$ is bijective. Finally, write the relative dualizing sheaf in the form $\omega_{\tilde{Y}'/Y'} = \mathcal{O}_{\tilde{Y}'}(nE)$ for some integer n. Using

$$-2 = \deg(\omega_E) = \omega_{\tilde{Y}'}(E) \cdot E = (n+1)E^2, \tag{21}$$

we conclude $n = 0$ and $E^2 = -2$. In other words, the singularity is a rational double point of type A_1. Moreover, we have $\omega_{\tilde{Y}'} = r^*(\omega_{Y'})$, and hence the antidualizing sheaf for $\tilde{Y}'$ is nef and big. In other words, the regular surface $\tilde{Y}'$ is a weak del Pezzo surface. □

According to Mumford's result [27], the only normal surface singularities over the complex numbers whose formal completion are factorial are the rational double points of type E_8 (for arbitrary algebraically closed ground fields, see [24], §25). The situation is more complicated over nonclosed ground fields.

Corollary 8.3. *The complete local rings $\mathcal{O}^\wedge_{Y',y'}$ of our twisted del Pezzo surface Y' are factorial.*

Proof. Let $D \subset Y'$ be a Weil divisor, and $\tilde{D} \subset \tilde{Y}'$ be its strict transform. The exceptional divisor $E \subset \tilde{Y}'$ carries no invertible sheaf of degree one. Rather, it is a cyclic group generated by the invertible sheaf $\mathcal{O}_E(E)$, which has degree two. Write $\tilde{D} \cdot E = 2n$ for some integer n. Then $(\tilde{D} + nE) \cdot E = 0$. This implies that the invertible sheaf $\mathcal{L} = \mathcal{O}_{\tilde{Y}'}(\tilde{D} + nE)$ is trivial on the formal completion along E, because $H^1(\tilde{Y}', \mathcal{O}_{mE}) = 0$ for all integers $m \geq 0$. It follows that the coherent $\mathcal{O}_{Y'}$-module $r_*(\mathcal{L})$ is invertible. Therefore, the Weil divisor $D \subset Y'$ must be Cartier. The same argument applies for formal Weil divisors on $\operatorname{Spec}(\mathcal{O}^\wedge_{Y',y'})$. □

9. Fano-Mori contractions

We now use the results of the preceding section on del Pezzo surfaces over nonperfect ground fields to construct some interesting Fano-Mori contractions of fiber type over algebraically closed fields. Let us now work, for simplicity, over an algebraically closed ground field k of characteristic two.

Choose an abelian variety A' with a-number $a(A) \geq 1$. This mean that there exists at least one embedding $\alpha_2 \subset A'$, and in turn an α_2-action on the abelian variety via translations. In dimension one, for example, we could choose a supersingular elliptic curve with Weierstrass equation of the form $y^2 + y = x^3 + a_4x + a_6$, with action given by the derivation $x \mapsto 1$, $y \mapsto x^2 + a_4$. Note that in characteristic two, all supersingular elliptic curves are isomorphic. The quotient $A = A'/\alpha_2$ is again an abelian variety, and the quotient map $A' \to A$ is a purely inseparable isogeny of degree two.

We now fix once and for all an embedding $\alpha_2 \subset A'$ and consider the corresponding α_2-action on A' via translations. Let Y be the nonnormal del Pezzo surface constructed in the preceding sections. We assume that the glueing polynomial P is even, such that we have the global vector field $\delta = PD_e$ corresponding to an α_2-action on Y. The product $Z' = Y \times A'$ carries the diagonal action, and we may take the quotient $Z = \alpha_2 \backslash Z'$. The projection $f' : Z' \to A'$ induces a projection $f : Z \to A$. To understand its fibers, consider the function fields $K' = k(A')$ and $K = k(A)$. Then $K \subset K'$ is a purely inseparable quadratic field extension, and hence of the form $K' = K(\sqrt{\lambda})$ for some nonsquare $\lambda \in K$. We may view $T = \operatorname{Spec} K'$ as an α_2-torsor over K.

PROPOSITION 9.1. *The generic fiber Z_η of the projection $f : Z \to A$ is the twisted form $Y_K \wedge T$, which is a normal del Pezzo surface. For all closed points $\sigma \in A$, the fiber Z_σ is isomorphic to the nonnormal del Pezzo surface Y.*

PROOF. Taking quotients by free group actions commutes with arbitrary base change. Given a point $\sigma \in A$ with residue field $\kappa = \kappa(\sigma)$, and $T \subset A'$ be its preimage. Making base change with respect to $T \to A$, we see that the fiber Z_σ is the quotient of $Y_\kappa \times_{\operatorname{Spec}(\kappa)} T$ by the diagonal action, so that $Z_\sigma = Y_\kappa \wedge T$. If σ is a closed point, the torsor T is trivial, and hence $Z_\sigma = Y$. If σ is the generic point, then the torsor T is nontrivial. According to Theorem 8.1, the twisted form is then normal. □

The point at infinity $y_\infty \in Y$ is invariant under the α_2-action. Whence it defines a section $s : A \to Z$, whose image is the quotient of $\{y_\infty\} \times A'$ by the diagonal action.

PROPOSITION 9.2. *The scheme Z is normal and locally of complete intersection. The reduced singular locus of Z equals the image of the section $s(A) \subset Z$.*

PROOF. The morphism $f : Z \to A$ is flat, because the composition $Z' \to A$ is flat and the quotient $Z' \to Z$ is faithfully flat. The base A and all fibers Z_a are locally of complete intersection, whence Z is locally of complete intersection.

According to Theorem 8.1, the singular locus of the generic fiber ist $s(A)_\eta$. It follows that $s(A) \subset \operatorname{Sing}(Z)$. The translation action of A' on $Z' = Y \times A'$ via the second factor commutes with the diagonal α_2-action, whence induces an action of A on the quotient Z. For any closed point $z \in Z$, the induced map $f : Az \to A$ is surjective. Using that the singular locus is invariant under this action, we infer that $\operatorname{Sing}(Z) \subset s(A)$. In particular, Z is regular in codimension one. It follows that Z is normal. □

PROPOSITION 9.3. *The blowing up $r : \tilde{Z} \to Z$ with center the reduced subscheme $s(A) \subset Z$ is a resolution of singularities. The singularities of Z are canonical.*

PROOF. According to 8.2, the generic fiber $\tilde{Z}_\eta$ is regular. Moreover, the A-action on Z leaves the center of the blowing up $s(A) \subset Z$ invariant, and hence the action extends to the relative homogeneous spectrum $\tilde{Z} = \mathcal{P}roj\,(\oplus(\mathcal{I}^n/\mathcal{I}^{n+1}))$, where $\mathcal{I} \subset \mathcal{O}_Z$ denotes the ideal of the center. We now may argue as in the preceding proof and infer that $\tilde{Z}$ must be regular.

Let $E \subset \tilde{Z} = r^{-1}(s(A))$ be the exceptional divisor. It must be flat over the base of the projection $\tilde{Z} \to A$ because it carries an A-action. The relative dualizing sheaf $\omega_{\tilde{Z}/Z}$ is of the form $\mathcal{O}_{\tilde{Z}}(nE)$ for some integer n, which is called the *discrepancy* for the resolution of singularities. The singularities on the threefold Z are called *canonical* if $n \geq 0$. According to (21), we have $n = 0$, and hence Z is canonical. □

Recall that a morphism of proper normal scheme $f : V \to W$ is called a *Fano-Mori contraction* if $\mathcal{O}_W \to f_*(\mathcal{O}_V)$ is bijective, the total space V is $\mathbb{Q}$-Gorenstein, and $\omega_V^\vee$ is f-ample.

PROPOSITION 9.4. *The morphism $f : Z \to A$ is a Fano-Mori contraction. The $\mathcal{O}_A$-module $R^1f_*(\mathcal{O}_Z)$ is invertible and commutes with base change.*

PROOF. For all closed points $a \in A$, we have $Z_a = Y$, and hence $h^0(\mathcal{O}_{Z_a}) = 1$ and $h^2(\mathcal{O}_{Z_a}) = 0$. If follows that the canonical map $\mathcal{O}_A \to f_*(\mathcal{O}_Z)$ is bijective, and that the coherent $\mathcal{O}_A$-module $R^1f_*(\mathcal{O}_Z)$ is locally free, of rank $h^1(\mathcal{O}_{Z_a}) = 1$.

By Proposition 9.2, the scheme Z is Gorenstein. Let $C \subset Z$ be an integral curve, and $C' \subset Z'$ be its preimage. We have $\omega_Z \otimes \mathcal{O}_{Z'} = \mathrm{pr}_1^*(\omega_Y)$. If follows that $C \cdot \omega_Z \leq 0$, with equality if and only if the induced map $f : C \to A$ is finite. Summing up, $f : Z \to A$ is a Fano-Mori contraction. □

10. Maps to projective spaces

We now return to our nonnormal del Pezzo surface Y. In this final section we study maps to projective spaces, which are defined in terms of semiample invertible sheaves. The upshot will be that it is not possible to define Y in a simple way as a hypersurface in projective space, or a finite covering of projective space. Obviously, the same then holds for twisted forms $Y' = Y \wedge T$.

The various geometric properties of invertible sheaves on Y can be nicely expressed in terms of degrees.

THEOREM 10.1. *An invertible $\mathcal{O}_Y$-module $\mathcal{L} \neq \mathcal{O}_Y$ of degree $d = \deg(\mathcal{L})$ is:*

(i) *semiample if and only if $d \geq 0$;*
(ii) *ample if and only if $d \geq 1$;*
(iii) *globally generated if and only if $d \geq 2$;*
(iv) *very ample if and only if $d \geq 3$;*

PROOF. For this we may assume that the ground field k is algebraically closed. We already proved assertion (ii) in Proposition 4.2. Concerning (i), recall that semiampleness means that some tensor power is globally generated. Suppose $\mathcal{L}$ is semiample. Then the restriction $\mathcal{L}_C$ is semiample as well, and hence $d \geq 0$. Conversely, suppose $d \geq 0$. If $d > 1$, then $\mathcal{L}$ is ample, and if $d = 0$, then the sheaf $\mathcal{L}^{\otimes 2}$ is trivial by the exact sequence (14). In both cases $\mathcal{L}$ is semiample.

Next, we prove (iii). Suppose that $\mathcal{L}$ is globally generated. Then the restriction $\mathcal{L}_C$ to the cuspidal curve of arithmetic genus $p_a = 1$ is globally generated as well, and this implies that $d \geq 2$ by Lemma 10.3 below. Conversely, suppose the degree is $d \geq 2$. Decompose $\mathcal{L} = \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_d$ into a tensor product of invertible sheaves of degree one. According to Proposition 5.1, we have $h^0(\mathcal{L}_i) = 1$, whence there are unique Cartier divisors $C_i \subset Y$

with $\mathcal{L}_i = \mathcal{O}_Y(C_i)$. It follows that the base locus of $\mathcal{L}$ is contained in $\bigcup_{i=1}^d C_i$. The exact sequence $0 \to \mathcal{L}(-C_i) \to \mathcal{L} \to \mathcal{L}_{C_i} \to 0$ yields an exact sequence

$$H^0(Y, \mathcal{L}) \longrightarrow H^0(C_i, \mathcal{L}_{C_i}) \longrightarrow H^1(Y, \mathcal{L}(-C_i)).$$

The term on the right vanishes by Proposition 5.1, because $\mathcal{L}(-C_i)$ has degree $d-1 \geq 1$. To finish the argument, it suffices to check that $\mathcal{L}_{C_i}$ is globally generated. According to Proposition 5.4, the C_i are isomorphic to the rational cuspidal curve with $p_a = 1$. By Lemma 10.3 below, $\mathcal{L}_{C_i}$ is globally generated.

It remains to prove (iv), which is the most interesting part. Suppose first that $\mathcal{L}$ is very ample. Then the restriction $\mathcal{L}_C$ is very ample as well, and this implies $d \geq 3$ by Lemma 10.3 below. Conversely, suppose $d \geq 3$. Let $A \subset Y$ be an Artin subscheme of length two. We have to show that $H^0(Y, \mathcal{L}) \to H^0(A, \mathcal{L}_A)$ is surjective. The idea is to use Cartier divisors of degree two.

Let $\mathcal{N}$ be an invertible $\mathcal{O}_Y$-module of degree two. We already saw that $\mathcal{N}$ is globally generated, whence defines a morphism $r_{\mathcal{N}} : Y \to \mathbb{P}^2$. The image $r_{\mathcal{N}}(A) \subset \mathbb{P}^2$ is an Artin scheme of length ≤ 2, and hence $\mathcal{N}$ has a nonzero global section whose zero scheme $D \subset Y$ contains A. The exact sequence $0 \to \mathcal{L}(-D) \to \mathcal{L} \to \mathcal{L}_D \to 0$ yields an exact sequence

$$H^0(Y, \mathcal{L}) \longrightarrow H^0(D, \mathcal{L}_D) \longrightarrow H^1(Y, \mathcal{L}(-D)).$$

The term on the right vanishes by Proposition 5.1, because $\mathcal{L}(-D)$ has degree ≥ 1. Hence it suffices to show that $H^0(D, \mathcal{L}_D) \to H^0(A, \mathcal{L}_A)$ is surjective.

Now suppose for a moment that $\omega_Y \otimes \mathcal{N}^{\otimes 2} \neq \mathcal{L}$ and that $r(A) \subset \mathbb{P}^2$ has length one. Using the latter, we see that $\mathcal{N}$ has another nonzero section whose zero scheme $D' \subset Y$ having no irreducible component in common with D and containing A. Set $A' = D \cap D'$. Then $A \subset A'$, and the inclusion $A' \subset D$ is Cartier. The exact sequence $0 \to \mathcal{L}_D(-A') \to \mathcal{L}_D \to \mathcal{L}_{A'} \to 0$ yields an exact sequence

$$H^0(D, \mathcal{L}_D) \longrightarrow H^0(A', \mathcal{L}_{A'}) \longrightarrow H^1(D, \mathcal{L}_D(-A')).$$

The term on the right sits inside the exact sequence

$$H^1(Y, \mathcal{L} \otimes \mathcal{N}^\vee) \longrightarrow H^1(D, \mathcal{L}_D(-A')) \longrightarrow H^2(Y, \mathcal{L} \otimes \mathcal{N}^{\otimes -2}).$$

In this sequence, the term on the left vanishes by Proposition 5.1, since we have $\deg(\mathcal{L} \otimes \mathcal{N}^\vee) < 0$. The term on the right is Serre dual to $H^0(Y, \mathcal{M})$, where $\mathcal{M} = \omega_Y \otimes \mathcal{L}^{\otimes -1} \otimes \mathcal{N}^{\otimes 2}$. This sheaf has degree $3-d$, and hence $H^0(Y, \mathcal{M})$ vanishes for $d > 3$. In the boundary case $d = 3$ we also have $H^0(Y, \mathcal{M}) = 0$, because we are presently assuming that $\omega_Y \otimes \mathcal{N}^{\otimes 2} \neq \mathcal{L}$. Combining these observations, we see that the restriction map $H^0(Y, \mathcal{L}) \to H^0(A, \mathcal{L}_A)$ is surjective.

To complete the proof, we now may assume that for all invertible sheaves $\mathcal{N}$ of degree two with $\omega_Y \otimes \mathcal{N}^{\otimes 2} \neq \mathcal{L}$ the image $r_{\mathcal{N}}(A) \subset \mathbb{P}^2$ has length two. Our goal now is to find a global section of $\mathcal{L}$ whose zero scheme intersects A but does not contain A. This implies that $H^0(Y,\mathcal{L}) \to H^0(A,\mathcal{L}_A)$ is surjective, because we already know that $\mathcal{L}$ is globally generated. By our assumption, for any $\mathcal{N}$ of degree two with $\omega_Y \otimes \mathcal{N}^{\otimes 2} \neq \mathcal{L}$, we find a global section of $\mathcal{N}$ whose zero scheme $D \subset Y$ intersects A but does not contain A.

We now have to distinguish the cases that $n = \deg(\mathcal{L})$ is even or odd. I only go through the case that $n = 2m+1$ is odd, the even case being similar. Choose a Cartier divisor $D' \subset Y$ of degree two disjoint from A. The equation of invertible sheaves $\mathcal{L} \simeq \mathcal{N}((m-1)D'+E)$ defines an invertible sheaf $\mathcal{O}_Y(E)$ of degree one. Since $H^0(Y,\mathcal{O}_Y(E)) = 1$, the effective Cartier divisor $E \subset Y$ is also unique. Now note that $E \subset Y$ is the image of a line $L \subset \mathbb{P}^2$, and that the restriction map $H^0(Y,\mathcal{O}_Y(E)) \to H^0(C,\mathcal{O}_C(E))$ is bijective, according to Proposition 5.2. From this we infer that there is at most one invertible sheaf of degree one $\mathcal{O}_Y(E)$ with $A \subset E$. So tensoring $\mathcal{N}$ with some general numerically trivial invertible sheaf, we may assume that $A \not\subset E$. If A is disjoint from E, then $D+(m-1)D'+E$ is the desired Cartier divisor representing $\mathcal{L}$ that intersect but does not contain A. If $A \cap E$ is nonempty, we simply replace D by a linearly equivalent Cartier divisor disjoint form A, and conclude as above. □

Suppose that the invertible $\mathcal{O}_Y$-module $\mathcal{L}$ is globally generated. In other words, its degree is $d \geq 2$. Set $n = d(d+1)/2 - 1$, and let $r_{\mathcal{L}} : Y \to \mathbb{P}^n$ be the resulting morphism defined by $\mathcal{L}$.

COROLLARY 10.2. *If $d = 2$, then the morphism $r_{\mathcal{L}} : Y \to \mathbb{P}^2$ is flat, surjective, of degree four, and all fibers are Artin schemes of complete intersection. There is no surjection to the projective plane of smaller degree. If $d = 3$, then $r_{\mathcal{L}}$ is a closed embedding $Y \subset \mathbb{P}^5$. There is no closed embedding into any projective space of smaller dimension.*

PROOF. Suppose $d = 2$. The morphism $r_{\mathcal{L}} : Y \to \mathbb{P}^2$ is flat because Y is Cohen-Macaulay and $\mathbb{P}^2$ is regular ([35], page IV-37, Proposition 22). The other statements follow immediately from Theorem 10.1. □

In the course of the proof for Theorem 10.1, we used the following facts.

LEMMA 10.3. *Let C be the rational cuspidal curve of arithmetic genus $p_a = 1$, and $\mathcal{L} \neq \mathcal{O}_C$ be an invertible $\mathcal{O}_C$-module of degree d. Then $\mathcal{L}$ is globally generated if and only if $d \geq 2$, and very ample if and only if $d \geq 3$.*

PROOF. Of course, we may assume that the ground field k is algebraically closed. The arguments are similar to the case of elliptic curves. The problem, however, is that some Weil divisors on C are not Cartier.

Let us first prove that the numerical conditions are necessary. Suppose that $\mathcal{L}$ is globally generated, so $d \geq 0$. The case $d = 0$ is impossible, because $\mathcal{L}$ is nontrivial by assumption. Hence $d \geq 1$, and the usual argument gives $h^0(C,\mathcal{L}) = d$. The invertible sheaf $\mathcal{L}$ is ample, whence the morphism $r_{\mathcal{L}} : C \to \mathbb{P}^{d-1}$ is finite, and therefore $d \geq 2$. If, furthermore, $\mathcal{L}$ is very ample, we must have $d \geq 3$.

The converse is more interesting. Suppose $d \geq 2$, and let $y \in C$ be any closed point, which is a Weil divisor of length one. I claim that there is a Cartier divisor $D \subset C$ of degree at most two that contains y and has $\mathcal{L} \not\simeq \mathcal{O}_C(D)$. Suppose this for the moment. The short exact sequence $0 \to \mathcal{L}(-D) \to \mathcal{L} \to \mathcal{L}_D \to 0$ yields an exact sequence

$$H^0(C, \mathcal{L}) \longrightarrow H^0(D, \mathcal{L}_D) \longrightarrow H^1(C, \mathcal{L}(-D)). \tag{22}$$

The term on the right is Serre dual to $H^0(C, \mathcal{L}^\vee(D))$. The invertible sheaf $\mathcal{L}^\vee(D)$ has degree $n \leq 2-d \leq 0$, and in the case $n = 0$ is nontrivial. Whence it has no global section, and it follows that $\mathcal{L}$ has a section that does not vanish at y.

Let us now verify the claim. There is nothing to prove if $y \in C$ is contained in the regular locus. So let us assume that it is the singular point, and write $\mathcal{O}^\wedge_{C,y} = k[[u^2, u^3]]$. For any scalar $\lambda \in k$, the element $u^2 + \lambda u^3$ defines a Cartier divisor $D_\lambda \subset C$ of length two with support y.

Finally, suppose that $d \geq 3$. Let $A \subset C$ be a closed subset of length two. We have to see that the restriction map $H^0(C, \mathcal{L}) \to H^0(A, \mathcal{L}_A)$ is surjective. It follows from the above that there is a Cartier divisor $D \subset A$ of length four containing A, and with $\mathcal{L} \not\simeq \mathcal{O}(D)$. In the case $d \geq 4$, one proceeds easily as above to see that $H^0(C, \mathcal{L}) \to H^0(A, \mathcal{L}_A)$ is surjective. For $d = 3$ we argue as follows: We have $h^0(C, \mathcal{L}) = 3$, and we already know that $\mathcal{L}$ is globally generated, hence there is a finite morphism $r_\mathcal{L} : C \to \mathbb{P}^2$, which does not factor over a line $\mathbb{P}^1 \subset \mathbb{P}^2$. Whence $r_\mathcal{L}$ is birational onto its image $r_\mathcal{L}(C)$, which must be a cubic. Any cubic has arithmetic genus $p_a = 1$. Since C also has arithmetic genus $p_a = 1$, the birational morphism $r_\mathcal{L}$ must be an isomorphism. □

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