Categorical aspects of Hopf algebras

Robert Wisbauer

Abstract. Hopf algebras allow for useful applications, for example in physics. Yet they also are mathematical objects of considerable theoretical interest and it is this aspect which we want to focus on in this survey. Our intention is to present techniques and results from module and category theory which lead to a deeper understanding of these structures. We begin with recalling parts from module theory which do serve our purpose but which may also find other applications. Eventually the notion of Hopf algebras (in module categories) will be extended to Hopf monads on arbitrary categories.

Contents

1. Introduction

The author's interest in coalgebraic structures and Hopf algebras arose from the observation that the categories considered in those situations are similar to those in module theory over associative (and nonassociative) rings. At the beginning in the 1960's, the study of coalgebras was to a far extent motivated by the classical theory of algebras over fields; in particular, the finiteness theorem for comodules brought the investigations close to the theory of finite dimensional algebras. Moreover, comodules for coalgebras C over fields can be essentially handled as modules over the dual algebra C^*.

Bringing in knowledge from module theory, coalgebras over commutative rings could be handled and from this it was a short step to extend the theory to *corings* over *non-commutative* rings (e.g. [**BrWi**]). This allows, for example, to consider for bialgebras B over a commuatative ring R, the tensorproduct $B \otimes_R B$ as coring over B and the Hopf bimodules over B as $B \otimes_R B$-comodules. Clearly this was a conceptual simplification of the related theory and the basic idea could be transferred to other situations. Some of these aspects are outlined in this talk.

Since Lawvere's categorification of general algebra, algebras and coalgebras are used as basic notions in universal algebra, logic, and theoretical computer science, for example (e.g. [**AdPo**], [**Gu**], [**TuPl**]).

The categories of interest there are far from being additive. The transfer of Hopf algebras in module categories to Hopf monads in arbitrary categories provides the chance to understand and study this notion in this wider context.

Generalisations of Hopf theory to *monoidal* categories were also suggested in papers by Moerdijk [**Moer**], Loday [**Lod**] and others. Handling these notions in arbitrary categories may also help to a better understanding of their concepts.

Not surprisingly, there is some overlap with the survey talks [**Wi.H**] and [**Wi.G**]. Here a broader point of view is taken and more recent progress is recorded.

2. Algebras

Let R be an associative and commutative ring with unit. Denote by $\mathbb{M}_R$ the category of (right) R-modules.

2.1. **Algebras.** An R-algebra (A, m, e) is an R-module A with R-linear maps, product and unit,

$$m : A \otimes_R A \to A, \quad e : R \to A,$$

satisfying associativity and unitality conditions expressed by commutativity of the diagrams

$$\begin{tikzcd} A \otimes_R A \otimes_R A \arrow[r, "m\otimes I"] \arrow[d, "I\otimes m"'] & A \otimes_R A \arrow[d, "m"] \\ A \otimes_R A \arrow[r, "m"] & A, \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "I\otimes e"] \arrow[dr, "="'] & A \otimes_R A \arrow[d, "m"] & A \arrow[l, "e\otimes I"'] \arrow[dl, "="] \\ & A. & \end{tikzcd}$$

2.2. **Tensorproduct of algebras.** Given two R-algebras (A, m_A, e_A) and (B, m_B, e_B), the tensor product $A \otimes_R B$ can be made an algebra with product

$$m_{A\otimes B} : \ A \otimes_R B \otimes_R A \otimes_R B \xrightarrow{I\otimes\tau\otimes I} A \otimes_R A \otimes_R B \otimes_R B \xrightarrow{m_A\otimes m_B} A \otimes_R B \ ,$$

and unit $e_A \otimes e_B : R \to A \otimes_R B$, for some R-linear map

$$\tau : B \otimes_R A \to A \otimes_R B.$$

inducing commutative diagrams

$$\begin{tikzcd} B \otimes_R B \otimes_R A \arrow[rr, "m_B\otimes I"] \arrow[d, "I\otimes\tau"'] & & B \otimes_R A \arrow[d, "\tau"] \\ B \otimes_R A \otimes_R B \arrow[r, "\tau\otimes I"] & A \otimes_R B \otimes B \arrow[r, "I\otimes m_B"] & A \otimes_R B, \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "e_B\otimes I"] \arrow[dr, "I\otimes e_B"'] & B \otimes_R A \arrow[d, "\tau"] \\ & A \otimes_R B, \end{tikzcd}$$

and similar diagrams derived from the product m_A and unit e_A of A.

It is easy to see that the canonical twist map

$$\mathrm{tw} : A \otimes_R B \to B \otimes_R, A, \quad a \otimes b \to b \otimes a,$$

satisfies the conditions on τ and this is widely used to define a product on $A \otimes_R B$. However, there are many other such maps of interest.

These kind of conditions can be readily transferred to functors on arbitrary categories and in this context they are known as *distributive laws* (e.g. [**Be**], [**Wi.A**]).

3. Category of A-modules

Let A be an associative R-algebra with unit.

3.1. **A-modules.** A left A-module M is an R-module with an R-linear map $\rho_M : A \otimes_R M \to M$ with commutative diagrams

$$\begin{tikzcd} A \otimes_R A \otimes_R M \arrow[r, "I\otimes\rho_M"] \arrow[d, "m\otimes I"'] & A \otimes_R M \arrow[d, "\rho_M"] \\ A \otimes_R M \arrow[r, "\rho_M"] & A, \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "e\otimes I"] \arrow[dr, "="'] & A \otimes_R M \arrow[d, "\rho_M"] \\ & M. \end{tikzcd}$$

The category ${}_A\mathbb{M}$ of (unital) left A-modules is a Grothendieck category with A a finitely generated projective generator.

Properties of (the ring, module) A are reflected by properties of the category ${}_A\mathbb{M}$. These interdependencies were studied under the title *homological classification of rings.*

To use such techniques for the investigation of the structure of a right A-module M, one may consider the smallest Grothendieck (full) subcategory of ${}_A\mathbb{M}$ which contains M. For this purpose recall that an A-module N is called
M-generated if there is an epimorphism $M^{(\Lambda)} \to N$, Λ an index set, and
M-subgenerated if N is a submodule of an M-generated module.

3.2. **The category $\sigma[M]$.** For any A-module M, denote by $\sigma[M]$ the full subcategory of ${}_A\mathbb{M}$ whose objects are all M-subgenerated modules. This is the smallest Grothendieck category containing M. Thus it shares many properties with ${}_A\mathbb{M}$, however it need not contain neither a projective nor a finitely generated generator. For example, one may think of the category of abelian torsion groups which is just the subcategory $\sigma[\mathbb{Q}/\mathbb{Z}]$ of ${}_\mathbb{Z}\mathbb{M}$ (without non-zero projective objects).

In general, M need not be a generator in $\sigma[M]$. A module $N \in \sigma[M]$ with $\sigma[N] = \sigma[M]$ is said to be a *subgenerator* in $\sigma[M]$. Of course, M is a subgenerator in $\sigma[M]$ (by definition). The notion of a subgenerator also plays a prominent role in the categories considered for coalgebraic structures (e.g. 4.2, 5.3).

An A-module N is a subgenerator in ${}_A\mathbb{M}$ if and only if A embeds in a finite direct sum of copies of N, i.e. $A \hookrightarrow N^k$, for some $k \in \mathbb{N}$. Such modules are also called *cofaithful*.

The notion of singularity in ${}_A\mathbb{M}$ can be transferred to $\sigma[M]$: A module $N \in \sigma[M]$ is called *singular in* $\sigma[M]$ or *M-singular* if $N \simeq L/K$ for $L \in \sigma[A]$ and $K \subset L$ an essential submodule.

3.3. **Trace functor.** The *inclusion functor* $\sigma[P] \to {}_A\mathbb{M}$ has a *right adjoint* $\mathcal{T}^M : {}_A\mathbb{M} \to \sigma[M]$, sending $X \in {}_A\mathbb{M}$ to

$$\mathcal{T}^M(X) := \sum\{f(N) \mid N \in \sigma[M],\ f \in \mathrm{Hom}_A(N, X)\}.$$

3.4. **Functors determined by $P \in {}_A\mathbb{M}$.** Given any A-module P with $S = \mathrm{End}_A(P)$, there is an adjoint pair of functors

$$P \otimes_S - : {}_S\mathbb{M} \to {}_A\mathbb{M}, \quad \mathrm{Hom}_A(P, -) : {}_A\mathbb{M} \to {}_S\mathbb{M},$$

with (co)restriction

$$P \otimes_S - : {}_S\mathbb{M} \to \sigma[P], \quad \mathrm{Hom}_A(P, -) : \sigma[P] \to {}_S\mathbb{M}.$$

and functorial isomorphism

$$\mathrm{Hom}_A(P \otimes_S X, Y) \to \mathrm{Hom}_S(X, \mathrm{Hom}_A(P, Y)),$$

$$\text{unit} \quad \eta_X : X \to \mathrm{Hom}_A(P, P \otimes_S X), \quad x \mapsto [p \mapsto p \otimes x];$$

$$\text{counit} \quad \varepsilon_Y : P \otimes_S \mathrm{Hom}_A(P, Y) \to Y, \quad p \otimes f \mapsto f(p).$$

These functors determine an *equivalence of categories* if and only if η and ε are natural isomorphisms.

In any category $\mathbb{A}$, an object $G \in \mathbb{A}$ is said to be a *generator* provided the functor $\mathrm{Mor}_{\mathbb{A}}(G, -) : \mathbb{A} \to \mathrm{Ens}$ is faithful. It is a property of Grothendieck categories that these functors are even fully faithful ([**Nast**, III, Teoremă 9.1]).

Let $P \in {}_A\mathbb{M}$, $S = \mathrm{End}_A(P)$. Then P is a right S-module and there is a canonical ring morphism

$$\phi : A \to B = \mathrm{End}_S(M), \quad a \mapsto [m \mapsto am].$$

P is called *balanced* provided ϕ is an isomorphism.

3.5. P **as generator in** ${}_A\mathbb{M}$**.** *The following are equivalent:*

(a) *P is a generator in ${}_A\mathbb{M}$;*
(b) $\mathrm{Hom}_A(P, -) : {}_A\mathbb{M} \to {}_S\mathbb{M}$ *is (fully) faithful;*
(c) $\varepsilon : P \otimes_S \mathrm{Hom}_A(P, N) \to N$ *is surjective (bijective), $N \in {}_A\mathbb{M}$;*
(d) *P is balanced and P_S is finitely generated and projective.*

Note that the equivalence of (a) and (d) goes back to Morita [**Mor**]. It need not hold in more general situations. In [**Wi.G**, 2.6] it is shown:

3.6. P **as generator in** $\sigma[P]$**.** *The following are equivalent:*

(a) *P is a generator in $\sigma[P]$;*
(b) $\mathrm{Hom}_A(P, -) : \sigma[P] \to {}_S\mathbb{M}$ *is (fully) faithful;*
(c) $\varepsilon_N : P \otimes_S \mathrm{Hom}_A(P, N) \to N$ *is sur-(bi-)jective, $N \in \sigma[P]$;*
(d) $\phi : A \to B$ *is dense, P_S is flat and*
ε_V *is an isomorphism for all injectives $V \in \sigma[P]$.*

The elementary notions sketched above lead to interesting characterisations of Azumaya R-algebras (R a commutative ring) when applied to A considered as an (A, A)-bimodule, or - equivalently - as a module over $A \otimes_R A^o$.

In this situation we have for any $A \otimes_R A^o$-module M,

$$\mathrm{Hom}_{A \otimes_R A^o}(A, M) = Z(M) = \{m \in M \mid am = ma \text{ for all } a \in A\},$$

and $\mathrm{End}_{A \otimes_R A^o}(A) \simeq Z(A)$, the center of A.

3.7. **Azumaya algebras.** *Let A be a central R-algebra, that is $Z(A) = R$. Then the following are equivalent:*

(a) *A is a (projective) generator in ${}_{A \otimes_R A^o}\mathbb{M}$;*
(b) $A \otimes_R A^o \simeq \mathrm{End}_R(A)$ *and A_R is finitely generated and projective;*
(c) $\mathrm{Hom}_{A \otimes_R A^o}(A, -) : {}_{A \otimes_R A^o}\mathbb{M} \to \mathbb{M}_R$ *is (fully) faithful;*
(d) $A \otimes_R - : \mathbb{M}_R \to {}_{A \otimes_R A^o}\mathbb{M}$ *is an equivalence;*

(e) $\mu : A \otimes_R A^o \to A$ *splits in* ${}_{A\otimes_R A^o}\mathbb{M}$ *(A is R-separable).*

The preceding result can also be formulated for not necessarily associative algebras by referring to the

3.8. **Multiplication algebra.** Let A be a (non-associative) R-algebra with unit. Then any $a \in A$ induces R-linear maps

$$L_a : A \to A,\ x \mapsto ax; \quad R_a : A \to A,\ x \mapsto xa.$$

The *multiplication algebra* of A is the (associative) subalgebra

$$M(A) \subset \mathrm{End}_R(A) \text{ generated by } \{L_a, R_a \mid a \in A\}.$$

Then A is a left module over $M(A)$ generated by 1_A (in general not projective) and $\mathrm{End}_{M(A)}(A)$ is isomorphic to the center of A. By $\sigma[{}_{M(A)}A]$, or $\sigma[A]$ for short, we denote the full subcategory of ${}_{M(A)}\mathbb{M}$ subgenerated by A. (For algebras A without unit these notions are slightly modified, e.g. [**Wi.B**]).

This setting allows to define *Azumaya* also for non-associative algebras (e.g. [**Wi.B**, 24.8]).

3.9. **Azumaya algebras.** *Let A be a central R-algebra with unit. Then the following are equivalent:*

(a) *A is a (projective) generator in ${}_{M(A)}\mathbb{M}$;*

(b) *$M(A) \simeq \mathrm{End}_R(A)$ and A_R is finitely generated and projective;*

(c) *$\mathrm{Hom}_{M(A)}(A, -) : {}_{M(A)}\mathbb{M} \to \mathbb{M}_R$ is (fully) faithful;*

(d) *$A \otimes_R - : \mathbb{M}_R \to {}_{M(A)}\mathbb{M}$ is an equivalence.*

The fact that the generator property of A as $A \otimes_R A^o$-module implies projectivity is a consequence of the commutativity of the corresponding endomorphism ring ($=Z(A)$).

Restricting to the subcategory $\sigma[A]$ we obtain

3.10. **Azumaya rings.** *Let A be a central R-algebra with unit. Then the following are equivalent:*

(a) *A is a (projective) generator in $\sigma[{}_{M(A)}A]$;*

(b) *$M(A)$ is dense in $\mathrm{End}_R(A)$ and A_R is faithfully flat;*

(c) *$\mathrm{Hom}_{M(A)}(A, -) : \sigma[{}_{M(A)}A] \to \mathbb{M}_R$ is (fully) faithful;*

(d) *$A \otimes_R - : \mathbb{M}_R \to \sigma[{}_{M(A)}A]$ is an equivalence.*

For any algebra A, central localisation is possible with respect to the maximal (or prime) ideals of the center $Z(A)$ and also with respect to central idempotents of A.

3.11. **Pierce stalks.** Let A be a (non-associative) algebra and denote by $B(A)$ the set of central idempotents of A which form a Boolean ring. Denote by $\mathcal{X}$ the set of all maximal ideals of $B(A)$. For any $x \in \mathcal{X}$, the set $B(A) \setminus x$ is a multiplicatively closed subset of (the center) of A and we can form the ring of fractions $A_x = AS^{-1}$. These are called the *Pierce stalks* of A (e.g. [**Wi.B**, Section 18]). They may be applied for local-global characterisations of algebraic structures, for example (see [**Wi.B**, 26.8], [**Wi.M**]):

3.12. **Pierce stalks of Azumaya rings.** *Let A be a central (non-associative) R-algebra with unit. Then the following are equivalent:*

(a) *A is an Azumaya algebra;*
(b) *A is finitely presented in $\sigma[A]$ and for every $x \in \mathcal{X}$, A_x is an Azumaya ring;*
(c) *for every $x \in \mathcal{X}$, A_x is an Azumaya ring with center R_x.*

Considering the (A, A)-bimodules for an associative ring A may be regarded as an extension of the module theory over commutative rings to non-commutative rings. Using the multiplication algebra $M(A)$ we can even handle non-associative algebras A. In particular, we can describe a kind of central localisation of semiprime algebras A. This may help to handle notions in non-commutative geometry.

One problem in transferring localisation techniques from semiprime commutative rings to semiprime non-commutative rings is that the latter need not be non-singular as one-sided modules. To guarentee this, additional assumptions on the ring are required (e.g. Goldie's theorem). This is not the case if we consider A in the category $\sigma[A]$.

A module $N \in \sigma[A]$ is called *A-singular* if $N \simeq L/K$ for $L \in \sigma[A]$ and $K \subset L$ an essential $M(A)$-submodule (see 3.2). The following is shown in [**Wi.B**, Section 32].

3.13. **Central closure of semiprime algebras.** *Let A be a semiprime R-algebra and $\widehat{A}$ the injective hull of A in $\sigma[_{M(A)}A]$. Then*

(i) *A is non-singular in $\sigma[_{M(A)}A]$.*
(ii) $\mathrm{End}_{M(A)}(\widehat{A})$ *is a regular, selfinjective, commutative ring, called the* extended centroid.
(iii) $\widehat{A} = A\,\mathrm{Hom}_{M(A)}(A, \widehat{A}) = A\,\mathrm{End}_{M(A)}(\widehat{A})$ *and allows for a ring structure (for $a, b \in A$, $\alpha, \beta \in \mathrm{End}_{M(A)}(\widehat{A})$),*
$$(a\alpha) \cdot (b\beta) := ab\,\alpha\beta.$$
This is the (Martindale) central closure *of A.*
(iv) *$\widehat{A}$ is a simple ring if and only if A is strongly prime (as an $M(A)$-module).*

Not surprisingly - the above results applied to $A = \mathbb{Z}$ yield the rationals $\mathbb{Q}$ as the (self-)injective hull of the integers $\mathbb{Z}$.

A semiprime ring A is said to be *strongly prime (as $M(A)$-module)* if its central closure is a simple ring, and an ideal $I \subset A$ is called *strongly prime* provided the factor ring A/I is strongly prime.

Using this notion, an associative ring A is defined to be a *Hilbert ring* if any strongly prime ideal of A is the intersection of maximal ideals. This is the case if and only if for all $n \in \mathbb{N}$, every maximal ideal $\mathcal{J} \subset A[X_1, \ldots, X_n]$ contracts to a maximal ideal of A or - equivalently - $A[X_1, \ldots, X_n]/\mathcal{J}$ is finitely generated as an $A/\mathcal{J} \cap A$-module (liberal extension). This yields a natural noncommutative version of Hilbert's Nullstellensatz (see [**KaWi**]).

The techniques considered in 3.13 were extended in Lomp [**Lomp**] to study the action of Hopf algebras on algebras.

4. Coalgebras and comodules

The module theory sketched in the preceding section provides useful techniques for the investigation of coalgebras and comodules. In this section R will denote a commutative ring.

4.1. **Coalgebras.** An *R-coalgebra* is a triple (C, Δ, ε) where C is an R-module with R-linear maps

$$\Delta : C \to C \otimes_R C, \quad \varepsilon : C \to R,$$

satisfying coassociativity and counitality conditions.

The tensor product $C \otimes_R D$ of two R-coalgebras C and D can be made to a coalgebra with a similar procedure as for algebras. For this a suitable linear map $\tau' : C \otimes_R D \to D \otimes_R C$ is needed leading to the corresponding commutative diagrams (compare 2.2).

The dual R-module $C^* = \mathrm{Hom}_R(C, R)$ has an associative ring structure given by the *convolution product*

$$f * g = (g \otimes f) \circ \Delta \ \text{ for } f, g \in C^*,$$

with unit ε.

Replacing $g \otimes f$ by $f \otimes g$ (as done in the literature) yields a multiplication opposite to the one given before. This does not do any harm but has some effect on the formalism considered later on.

4.2. **Comodules.** A *left comodule* over a coalgebra C is a pair (M, ϱ^M) where M is an R-module with an R-linear map (coaction)

$$\varrho^M : M \to C \otimes_R M$$

satisfying compatibility and counitality conditions.

A morphism between C-comodules M, N is an R-linear map $f : M \to N$ with $\varrho^N \circ f = (I \otimes f) \circ \varrho^M$. The set (group) of these morphisms is denoted by $\mathrm{Hom}^C(M, N)$.

The category ${}^C\mathbb{M}$ of left C-comodules is additive, with coproduct and cokernels - but not necessarily with kernels.

The functor $C \otimes_R - : \mathbb{M}_R \to {}^C\mathbb{M}$ is right adjoint to the forgetful functor ${}^C\mathbb{M} \to \mathbb{M}_R$, that is, there is an isomorphism

$$\mathrm{Hom}^C(M, C \otimes_R X) \to \mathrm{Hom}_R(M, X), \quad f \mapsto (\varepsilon \otimes I) \circ f,$$

and from this it follows that

$$\mathrm{End}^C(C) \simeq \mathrm{Hom}_R(C, R) = C^*,$$

which is a ring morphism - or antimorphism depending on the choice for the convolution product (see 4.1).

C is a subgenerator in ${}^C\mathbb{M}$, since any C-comodule leads to a diagram

$$\begin{tikzcd} & R^{(\Lambda)} \arrow[d, "h"] & C \otimes_R R^{(\Lambda)} \arrow[r, "\simeq"] \arrow[d, "I\otimes h"] & C^{(\Lambda)} \\ 0 \arrow[r] & M \arrow[r, "\varrho^M"] & C \otimes_R M, & \end{tikzcd}$$

where h is an epimorphism for some index set Λ.

Monomorphisms in ${}^C\mathbb{M}$ need not be injective maps and - as a consequence - generators G in ${}^C\mathbb{M}$ need not be flat modules over their endomorphism rings and the functor $\mathrm{Hom}^C(G, -) : {}^C\mathbb{M} \to \mathrm{Ab}$ need not be full.

All monomorphisms in ${}^C\mathbb{M}$ are injective maps if and only if C is flat as an R-module. In this case ${}^C\mathbb{M}$ has kernels.

There is a close relationship between comodules and modules.

4.3. ***C*-comodules and C^*-modules.** Any C-comodule $\varrho^M : M \to C \otimes_R M$ is a C^*-module by the action

$$\tilde{\varrho}^M : \ C^* \otimes M \xrightarrow{I \otimes \varrho^M} C^* \otimes C \otimes M \xrightarrow{ev \otimes I} M \ .$$

For any $M, N \in {}^C\mathbb{M}$, $\mathrm{Hom}^C(M,N) \subset \mathrm{Hom}_{C^*}(M,N)$ and hence there is a faithful functor

$$\Phi : {}^C\mathbb{M} \to {}_{C^*}\mathbb{M}, \ \ (M, \varrho^M) \mapsto (M, \tilde{\varrho}^M)$$

To make Φ a full functor, the morphism (natural in $Y \in \mathbb{M}_R$)

$$\alpha_Y : C \otimes_R Y \to \mathrm{Hom}_R(C^*, Y), \quad c \otimes y \mapsto [f \mapsto f(c)y],$$

has to be injective for all $Y \in \mathbb{M}_R$ (α-*condition*, see [**BrWi**, 4.3]):

4.4. ${}^C\mathbb{M}$ **a full module subcategory.** *The following are equivalent:*

(a) $\Phi : {}^C\mathbb{M} \to {}_{C^*}\mathbb{M}$ *is a full functor;*

(b) $\Phi : {}^C\mathbb{M} \to \sigma[{}_{C^*}C]$ $(\subset {}_{C^*}\mathbb{M})$ *is an equivalence;*

(c) α_Y *is injective for all* $Y \in \mathbb{M}_R$*;*

(d) C_R *is locally projective.*

This observation shows that under the given conditions the investigation of the category of comodule reduces to the study of C^*-modules, more precisely, the study of the category $\sigma[{}_{C^*}C]$ (see [**BrWi**], [**Wi.F**]).

As a special case we have (see [**BrWi**, 4.7]):

4.5. ${}^C\mathbb{M}$ **a full module category.** *The following are equivalent:*

(a) $\Phi : {}^C\mathbb{M} \to {}_{C^*}\mathbb{M}$ *is an equivalence;*

(b) α *is an isomorphism;*

(c) C_R *is finitely generated and projective.*

4.6. **Natural morphism.** Applying $\mathrm{Hom}_R(X, -)$ to the morphism α_Y leads to the morphism, natural in $X, Y \in \mathbb{M}_R$,

$$\tilde{\alpha}_{X,Y} : \mathrm{Hom}_R(X, C \otimes_R Y) \to \mathrm{Hom}_R(X, \mathrm{Hom}_R(C^*, Y)) \overset{\simeq}{\to} \mathrm{Hom}_R(C^* \otimes_R X, Y).$$

If α_Y is a monomorphism, then $\alpha_{X,Y}$ is a monomorphism,
if α_Y is an isomorphism, then $\alpha_{X,Y}$ is an isomorphism, $X, Y \in \mathbb{M}_R$.

The latter means that the monad $C^* \otimes_R -$ and the comonad $C \otimes_R -$ form an adjoint pair of endofunctors on $\mathbb{M}_R$, while the former condition means a weakened form of adjunction.

It is known (from category theory) that, for the monad $C^* \otimes_R -$, the right adjoint $\mathrm{Hom}_R(C^*, -)$ is a comonad and the category ${}_{C^*}\mathbb{M}$ is equivalent to the category $\mathbb{M}^{\mathrm{Hom}_R(C^*,-)}$ of $\mathrm{Hom}_R(C^*, -)$-comodules (e.g. [**BöBrWi**, 3.5]).

Thus $\alpha : C \otimes - \to \mathrm{Hom}_R(C^*, -)$ may be considered as a comonad morphism yielding a functor

$$\begin{aligned} &\tilde{\Phi} : {}^C\mathbb{M} \longrightarrow \mathbb{M}^{\mathrm{Hom}_R(C^*,-)}, \\ &\qquad M \to C \otimes_R M \ \longmapsto \ M \to C \otimes_R M \xrightarrow{\alpha_M} \mathrm{Hom}_R(C^*, M). \end{aligned}$$

As noticed in 4.4 and 4.5, this functor is fully faithful if and only if α is injective; it is an equivalence provided α is a natural isomorphism.

5. Bialgebras and Hopf algebras

Combining algebras and coalgebras leads to the notion of

5.1. **Bialgebras.** An R-bialgebra is an R-module B carrying an algebra structure (B, m, e) and a coalgebra structure (B, Δ, ε) with compatibility conditions which can be expressed in two (equivalent) ways

(a) $m : B \otimes_R B \to B$ and $e : R \to B$ are coalgebra morphisms;

(b) $\Delta : B \to B \otimes_R B$ and $\varepsilon : B \to R$ are algebra morphisms.

To formulate this, an algebra and a coalgebra structure is needed on the tensor-product $B \otimes_R B$ as defined in 2.2 and 4.1 (with the twist tw map taken for τ). The twist map (or a braiding) can be avoided at this stage by referring to an *entwining map*

$$\psi : B \otimes_R B \to B \otimes_R B,$$

which allows to express compatibility between algebra and coalgebra structure by commutativity of the diagram (e.g. [**BöBrWi**, 8.1])

$$\begin{tikzcd} B \otimes_R B \arrow[r, "m"] \arrow[d, "\Delta\otimes I_B"'] & B \arrow[r, "\Delta"] & B \otimes_R B \\ B \otimes_R B \otimes_R B \arrow[rr, "I_B\otimes\psi"] & & B \otimes_R B \otimes_R B. \arrow[u, "m\otimes I_B"'] \end{tikzcd}$$

In the standard situation this entwining is derived from the twist map as

$$\psi = (m \otimes I) \circ (I \otimes \mathrm{tw}) \circ (\delta \otimes I) : B \otimes_R B \to B \otimes_R B, \; a \otimes b \mapsto a_{\underline{1}} \otimes b a_{\underline{2}}.$$

This is a special case of 6.12 (see also [**BöBrWi**, 8.1]).

5.2. **Hopf modules.** *Hopf modules* for a bialgebra B are R-modules M with a B-module and a B-comodule structure

$$\rho_M : B \otimes_R M \to M, \quad \rho^M : M \to B \otimes_R M,$$

satisfying the compatibility condition

$$\rho^M(bm) = \Delta(b) \cdot \rho^M(m), \;\; \text{for } b \in B, \; m \in M.$$

Here we use that - due to the algebra map Δ - the tensor product $N \otimes_R M$ of two B-modules can be considered as a left B-module via the diagonal action

$$b \cdot (m \otimes n) = \Delta(b)(m \otimes n) = \sum b_{\underline{1}} n \otimes b_{\underline{2}} m.$$

This makes the category ${}_B\mathbb{M}$ monoidal.

If the compatibility between m and Δ is expressed by an entwining map $\psi : B \otimes_R B \to B \otimes_R B$ (see 5.1), then the Hopf modules are characterised by commutativity of the diagram

$$\begin{tikzcd} B \otimes_R M \arrow[r, "\rho_M"] \arrow[d, "I\otimes\rho^M"'] & M \arrow[r, "\rho^M"] & B \otimes_R M \\ B \otimes_R B \otimes_R M \arrow[rr, "\psi\otimes I"] & & B \otimes_R B \otimes_R M. \arrow[u, "I\otimes\rho_M"'] \end{tikzcd}$$

5.3. **Category of Hopf modules.** Morphisms between two B-Hopf modules M and N are R-linear maps $f : M \to N$ which are B-module as well as B-comodule morphisms. With these morphisms, the Hopf modules form an additive category, we denote it by ${}^B_B\mathbb{M}$. Certainly B is an object in ${}^B_B\mathbb{M}$, but in general it is neither a generator nor a subgenerator.

As mentioned above, $B \otimes_R B$ has a (further) left B-module structure induced by Δ, we denote the resulting module by $B \otimes^b B$. It is not difficult to see that $B \otimes^b B$ is an object in ${}^B_B\mathbb{M}$ and is a subgenerator in this category (e.g. [**BrWi**, 14.5]).

Similarly, one may keep the trivial B-module structure on $B \otimes_R B$ but introduce a new comodule structure on it. This is again a Hopf module, denoted by $B \otimes^c B$, and is also a subgenerator in ${}^B_B\mathbb{M}$ (e.g. [**BrWi**, 14.5]).

As for comodules, monomorphisms in ${}^B_B\mathbb{M}$ need not be injective maps unless B is flat as an R-module.

If B is locally projective as an R-module, the comodule structure of the Hopf modules may be considered as B^*-module structure and their module and comodule structures yield a structure as module over the smash product $B\#B^*$. In this case, ${}^B_B\mathbb{M}$ is isomorphic to $\sigma[{}_{B\#B^*}B \otimes^b B]$, the full subcategory of ${}_{B\#B^*}\mathbb{M}$ subgenerated by $B \otimes^b B$ (or $B \otimes^c B$) (e.g. [**BrWi**, 14.15]).

5.4. **Comparison functor.** For any R-bialgebra B, there is a *comparison functor*

$$\phi_B^B : {}_R\mathbb{M} \to {}^B_B\mathbb{M}, \quad X \mapsto (B \otimes_R X, m \otimes I_X, \Delta \otimes I_X),$$

which is full and faithful since, by module and comodule properties, for any $X, Y \in \mathbb{M}_R$,

$$\mathrm{Hom}_B^B(B \otimes_R X, B \otimes_R Y) \simeq \mathrm{Hom}_R^B(X, B \otimes_R Y) \simeq \mathrm{Hom}_R(X, Y),$$

with the trivial B-comodule structure on X. In particular, $\mathrm{End}_B^B(B) \simeq R$.

5.5. **The bimonad** $\mathrm{Hom}_R(B, -)$**.** As mentioned in 4.6, for a monad (comonad) $B \otimes_R -$, the right adjoint functor $\mathrm{Hom}_R(B, -)$, we denote it by $[B, -]$, is a comonad (monad).

An entwining $\psi : B \otimes_R B \to B \otimes_R B$ may be seen as an entwining between the monad $B \otimes_R -$ and the comonad $B \otimes_R -$,

$$\tilde{\psi} : B \otimes_R B \otimes_R - \to B \otimes_R B \otimes_R -$$

and this induces an entwining between the Hom-functors (see [**BöBrWi**, 8.2])

$$\widehat{\psi} : [B, [B, -]] \to [B, [B, -]].$$

This allows to define $[B, -]$-*Hopf modules* (similar to 5.2), the category $\mathbb{M}^{[B,-]}_{[B,-]}$, and a comparison functor (with obvious notation)

$$\phi^{[B,-]}_{[B,-]} : {}_R\mathbb{M} \to \mathbb{M}^{[B,-]}_{[B,-]}, \quad X \mapsto ([B, X], \Delta_X^*, m_X^*).$$

5.6. **Antipode.** For any bialgebra B, a *convolution product* can be defined on the R-module $\mathrm{End}_R(B)$ by putting, for $f, g \in \mathrm{End}_R(B)$, (compare 4.1)

$$f * g = m \circ (f \otimes g) \circ \Delta.$$

This makes $(\mathrm{End}_R(B), *)$ an R-algebra with identity $e \circ \varepsilon$.

An *antipode* is an $S \in \mathrm{End}_R(B)$ which is inverse to the identity map I_B of B with respect to $*$, that is $S * I_B = e \circ \varepsilon = I_B * S$ or - explicitely -

$$m \circ (S \otimes I_B) \circ \Delta = e \circ \varepsilon = m \circ (I_B \otimes S) \circ \Delta.$$

If B has an antipode it is called a *Hopf algebra*.

The existence of an antipode is equivalent to the canonical map

$$\gamma : \; B \otimes_R B \xrightarrow{\delta \otimes I} B \otimes_R B \xrightarrow{I \otimes m} B \otimes_R B$$

being an isomorphism (e.g. [**BrWi**, 15.2]).

The importance of the antipode is clear by the (see [**BöBrWi**, 8.11])

5.7. **Fundamental Theorem.** *For any R-bialgebra B, the following are equivalent:*

(a) *B is a Hopf algebra (i.e. has an antipode);*
(b) *$\phi_B^B : {}_R\mathbb{M} \to {}_B^B\mathbb{M}$ is an equivalence;*
(c) *$\phi_{[B,-]}^{[B,-]} : {}_R\mathbb{M} \to \mathbb{M}_{[B,-]}^{[B,-]}$ is an equivalence;*
(d) *$\mathrm{Hom}_B^B(B,-) : {}_B^B\mathbb{M} \to {}_R\mathbb{M}$ is full and faithful.*

If B_R is flat then (a)-(d) *are equivalent to:*

(e) *B is a generator in ${}_B^B\mathbb{M}$.*

Recall that for B_R locally projective, ${}_B^B\mathbb{M}$ is equivalent to $\sigma[{}_{B\#B^*}B \otimes^b B]$ and thus we have:

5.8. **Corollary.** *Let B be an R-bialgebra with B_R locally projective. Then the following are equivalent:*

(a) *B is a Hopf algebra;*
(b) *B is a subgenerator in ${}_B^B\mathbb{M}$ and $B\#B^*$ is dense in* $\mathrm{End}_R(B)$*;*
(c) *B is a generator in ${}_B^B\mathbb{M}$.*

These characterisations are very similar to those of Azumaya rings (see 3.10). This indicates, for example, that Pierce stalks may also be applied to characterise (properties of) Hopf algebras.

The notion of *bialgebras* addresses one functor with algebra and coalgebra structures. More general, one may consider relationships between distinct algebras and coalgebras:

5.9. **Entwined algebras and coalgebras.** Given an R-algebra (A, m, e) and an R-coalgebra (C, Δ, ε), an *entwining* (between monad $A \otimes_R -$ and comonad $C \otimes_R -$) is an R-linear map

$$\psi : A \otimes_R C \to C \otimes_R A,$$

inducing certain commutative diagrams. This notion was introduced in Brzeziński and Majid [**BrMa**] and is a special case of a mixed *distributive law* (see 6.5). *Entwined modules* are defined as R-modules M which are modules (M, ϱ_M) and comodules (M, ϱ^M), inducing commutativity of the diagram (e.g. [**BrWi**, 32.4])

$$\begin{tikzcd}
A \otimes M \arrow[r, "\varrho_M"] \arrow[d, "I_A \otimes \varrho^M"'] & M \arrow[r, "\varrho^M"] & C \otimes M \\
A \otimes C \otimes M \arrow[rr, "\psi \otimes I"] & & C \otimes A \otimes M. \arrow[u, "I \otimes \varrho_M"']
\end{tikzcd}$$

With morphisms which are A-module as well as C-comodule maps, the entwined modules form a category denoted by ${}_A^C\mathbb{M}$.

$C \otimes_R A$ is naturally a right A-module and ψ can be applied to define a left A-module structure on it,

$$a \cdot (c \otimes b) = \psi(a, c)b, \text{ for } a, b \in A,\ c \in C.$$

Moreover, a coproduct can be defined on $C \otimes_R A$, making $C \otimes_R A$ an A-*coring*, a notion which extends the notion of R-coalgebras to non-commutative base rings A. The category ${}_A^C\mathbb{M}$ of entwining modules can be considered as ${}^{C\otimes A}\mathbb{M}$, the category of left comodules over the coring $C \otimes_R A$ (e.g. [**BrWi**, 32.6]).

To get a comparison functor as in 5.4, we have to require that A is an object in ${}_A^C\mathbb{M}$; this is equivalent to the existence of a *grouplike element* in the A-coring $C \otimes_R A$ (e.g. [**BrWi**, 28.1 and 23.16]).

5.10. **Galois corings.** Let (A, C) be an entwined pair of an algebra A and a coalgebra C. Assume that A is an entwined module by $\varrho^A : A \to C \otimes_A A$. Then there is a comparison functor

$$\phi_A^C : \mathbb{M}_R \to {}_A^C\mathbb{M} : X \mapsto (A \otimes_R X, m \otimes I, \varrho^A \otimes I),$$

which is left adjoint to the (coinvariant) functor $\mathrm{Hom}_A^C(A, -) : {}_A^C\mathbb{M} \to \mathbb{M}_R$.

Moreover, $B = \mathrm{Hom}_A^C(A, A)$ is a subring of A, $\mathrm{Hom}_A^C(A, C \otimes_R A) \simeq A$, and evaluation yields a (canonical) map

$$\gamma : A \otimes_B A \to C \otimes_R A.$$

Now $C \otimes_R A$ is said to be a *Galois A-coring* provided γ is an isomorphism (e.g. [**BrWi**, 28.18]). This describes coalgebra-Galois extensions or non-commutative principal bundles. If - in this case - A_B is a faithfully flat module, then the functor

$$\mathbb{M}_B \to {}_A^C\mathbb{M} : Y \mapsto (A \otimes_B Y, m \otimes I, \varrho^A \otimes I)$$

is an equivalence of categories.

This extends the fundamental theorem for Hopf algebras to entwined structures: If $A = C = H$ is a Hopf algebra, then (H, H) is an entwining, $B = R$, and the resulting γ is an isomorphsism if and only if H has an antipode (see 5.6).

6. General categories

As seen in the preceding sections, the notions of algebras, coalgebras, and Hopf algebras are all buit up on the tensor product. Hence a first step to generalisation is to consider monoidal categories $(\mathbb{V}, \otimes, \mathbb{I})$. For example, entwining structures in such categories are considered in Mesablishvili [**Me**]. Furthermore, opmonoidal monads T on $\mathbb{V}$ were considered by Bruguières and Virelizier (in [**BruVir**, 2.3]) which may be considered as an entwining of the monad T with the comonad $- \otimes T(\mathbb{I})$. The *generalised bialgebras* in Loday [**Lod**], defined as Schur functors (on vector spaces) with a monad structure (operads) and a specified coalgebra structure, may also be seen as a generalisation of entwining structures [**Lod**, 2.2.1].

However, algebras and coalgebras also show up in more general categories as considered in universal algebra, theoretical computer science, logic, etc. (e.g. Gumm [**Gu**], Turi and Plotkin [**TuPl**], Adámek and Porst [**AdPo**]). It is of some interest to understand how the notion of Hopf algebras can be transferred to these settings. In what follows we consider an arbitrary category $\mathbb{A}$.

6.1. **Monads on $\mathbb{A}$.** A *monad on* $\mathbb{A}$ is a triple (F, m, e) with a functor $F : \mathbb{A} \to \mathbb{A}$ and natural transformations

$$m : FF \to F, \quad e : I_{\mathbb{A}} \to F,$$

inducing commutativity of certain diagrams (as for algebras, see 2.1).

F-modules are defined as $X \in \mathrm{Obj}(\mathbb{A})$ with morphisms $\varrho_X : F(X) \to X$ and certain commutative diagrams (as for the usual modules, see 3.1).

The catgegory of F-modules is denoted by $\mathbb{A}_F$. The free functor

$$\phi_F : \mathbb{A} \to \mathbb{A}_F,\ X \mapsto (F(X), m_X)$$

is left adjoint to the forgetful functor $U_F : \mathbb{A}_F \to \mathbb{A}$ by the isomorphism, for $X \in \mathbb{A}$, $Y \in \mathbb{A}_F$,

$$\mathrm{Mor}_{\mathbb{A}_F}(F(X), Y) \to \mathrm{Mor}_{\mathbb{A}}(X, U_F(Y)), \quad f \mapsto f \circ e_X.$$

6.2. **Comonads on $\mathbb{A}$.** A *comonad on* $\mathbb{A}$ is a triple (G, δ, ε) with a functor $G : \mathbb{A} \to \mathbb{A}$ and natural transformations

$$\delta : G \to GG, \quad \varepsilon : G \to I_{\mathbb{A}},$$

satisfying certain commuting diagrams (reversed to the module case).

$\mathbb{G}$-comodules are objects $X \in \mathrm{Obj}(\mathbb{A})$ with morphisms $\varrho^X : X \to G(X)$ in $\mathbb{A}$ and certain commutative diagrams.

The category of G-comodules is denoted by $\mathbb{A}^G$. The free functor

$$\phi^G : \mathbb{A} \to \mathbb{A}^G,\ X \mapsto (G(X), \delta_X)$$

is right adjoint to the forgetful functor $U^G : \mathbb{A}^G \to \mathbb{A}$ by the isomorphism, for $X \in \mathbb{A}^G$, $Y \in \mathbb{A}$,

$$\mathrm{Mor}_{\mathbb{A}^G}(X, G(Y)) \to \mathrm{Mor}_{\mathbb{A}}(U^G(X), Y), \quad f \mapsto \varepsilon_Y \circ f.$$

Monads and comonads are closely related with

6.3. **Adjoint functors.** A pair of functors $L : \mathbb{A} \to \mathbb{B}$, $R : \mathbb{B} \to \mathbb{A}$ is said to be *adjoint* if there is an isomorphism, natural in $X \in \mathbb{A}$, $Y \in \mathbb{B}$,

$$\mathrm{Mor}_{\mathbb{B}}(L(X), Y) \xrightarrow{\simeq} \mathrm{Mor}_{\mathbb{A}}(X, R(Y)),$$

also described by natural transformations $\eta : I_{\mathbb{A}} \to RL$, $\varepsilon : LR \to I_{\mathbb{B}}$. This implies

$$\text{a monad } (RL, R\varepsilon_L, \eta) \text{ on } \mathbb{A}\ , \quad \text{a comonad } (LR, L\eta_R, \varepsilon) \text{ on } \mathbb{B}.$$

L is full and faithful if and only if $\varepsilon : GF \to I_{\mathbb{A}}$ is an isomorphism.

L is an equivalence (with inverse R) if and only if ε and η are natural isomorphisms.

6.4. **Lifting properties.** Compatibility between endofunctors $F, G : \mathbb{A} \to \mathbb{A}$ can be described by *lifting properties*. For this, let $F : \mathbb{A} \to \mathbb{A}$ be a monad and $G : \mathbb{A} \to \mathbb{A}$ any functor on $\mathbb{A}$ and consider the diagram

$$\begin{tikzcd}
\mathbb{A}_F \arrow[r, dotted, "\overline{G}"] \arrow[d, "U_F"'] & \mathbb{A}_F \arrow[d, "U_F"] \\
\mathbb{A} \arrow[r, "G"] & \mathbb{A}.
\end{tikzcd}$$

If a $\overline{G}$ exists making the diagram commutative it is called a *lifting of G*. The questions arising are:

(i) does a lifting $\overline{G}$ exist ?

(ii) if G is a monad - is $\overline{G}$ again a monad (monad lifting)?
(iii) if G is a comonad - is $\overline{G}$ also a comonad (comonad lifting)?

For R-algebras A and B, (i) together with (ii) may be compared with the definition of an algebra structure on $A \otimes_R B$ and leads to diagrams similar to those in 2.1.

For an R-algebras A and an R-coalgebra C, (i) together with (iii) corresponds to the entwinings considered in 5.9.

We formulate this in the general case (e.g. [**Wi.A**, 5.3]).

6.5. **Mixed distributive law (entwining).** *Let (F, m, e) be a monad and (G, δ, ε) a comonad. Then a comonad lifting $\overline{G} : \mathbb{A}_F \to \mathbb{A}_F$ exists if and only if there is a natural transformations*

$$\lambda : FG \to GF$$

inducing commutativity of the diagrams

$$\begin{tikzcd} FFG \arrow[rr, "m_G"] \arrow[d, "F\lambda"'] & & FG \arrow[d, "\lambda"] \\ FGF \arrow[r, "\lambda_F"] & GFF \arrow[r, "Gm"] & GF, \end{tikzcd} \qquad \begin{tikzcd} FG \arrow[r, "F\delta"] \arrow[d, "\lambda"'] & FGG \arrow[r, "\lambda_G"] & GFG \arrow[d, "G\lambda"] \\ GF \arrow[rr, "\delta_F"] & & GGF, \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r, "e_G"] \arrow[dr, "Ge"'] & FG \arrow[d, "\lambda"] \\ & GF, \end{tikzcd} \qquad \begin{tikzcd} FG \arrow[r, "F\varepsilon"] \arrow[d, "\lambda"'] & F \\ GF. \arrow[ur, "\varepsilon_F"'] & \end{tikzcd}$$

Entwining is also used to express compatibility for an endofunctor which is a monad as well as a comonad. Notice that the diagrams in 6.5 either contain the product m *or* the coproduct δ, the unit e *or* the counit ε. Additional conditions are needed for adequate compatibility.

6.6. **(Mixed) bimonad.** An endofunctor $B : \mathbb{A} \to \mathbb{A}$ is said to be a *(mixed) bimonad* if it is

(i) a monad (B, m, e) with $e : I \to B$ a comonad morphism,
(ii) a comonad (B, δ, ε) with $\varepsilon : B \to I$ a monad morphism,
(iii) with an entwining functorial morphism $\psi : BB \to BB$,
(iv) with a commutative diagram

$$\begin{tikzcd} BB \arrow[r, "m"] \arrow[d, "B\delta"'] & B \arrow[r, "\delta"] & BB \\ BBB \arrow[rr, "\psi_B"] & & BBB. \arrow[u, "Bm"'] \end{tikzcd}$$

6.7. **(Mixed) B-bimodules.** For a bimonad B on $\mathbb{A}$, *(mixed) bimodules* are defined as B-modules and B-comodules X satisfying the pentagonal law

$$\begin{tikzcd} B(X) \arrow[r, "\varrho_X"] \arrow[d, "B(\varrho^X)"'] & X \arrow[r, "\varrho^X"] & B(X) \\ BB(X) \arrow[rr, "\psi_X"] & & BB(X). \arrow[u, "B(\varrho_X)"'] \end{tikzcd}$$

B-bimodule morphisms are B-module as well as B-comodule morphisms. We denote the category of B-bimodules by $\mathbb{A}_B^B$.

There is a comparison functor (compare 5.4)

$$\phi_B^B : \mathbb{A} \to \mathbb{A}_B^B, \quad A \longmapsto [BB(A) \stackrel{\mu_A}{\to} B(A) \stackrel{\delta_A}{\to} BB(A)],$$

which is full and faithful by the isomorphisms, functorial in $X, X' \in \mathbb{A}$,

$$\mathrm{Mor}_B^B(B(X), B(X')) \simeq \mathrm{Mor}_B(B(X), X') \simeq \mathrm{Mor}_{\mathbb{A}}(X, X').$$

In particular, this implies $\mathrm{End}_B^B(B(X)) \simeq \mathrm{End}_{\mathbb{A}}(X)$, for any $X \in \mathbb{A}$.

Following the pattern in 5.6 we define an

6.8. **Antipode.** Let B be a bimonad. An *antipode* of B is a natural transformation $S : B \to B$ leading to commutativity of the diagram

$$\begin{tikzcd} B \arrow[r, "\varepsilon"] \arrow[d, "\delta"'] & I \arrow[r, "e"] & B \\ BB \arrow[rr, shift left, "S_B"] \arrow[rr, shift right, "BS"'] & & BB \arrow[u, "m"'] \end{tikzcd}$$

We call B a *Hopf bimonad* provided it has an antipode.

As for Hopf algebras (see 5.6) we observe that the canonical natural transformation

$$\gamma : \; BB \xrightarrow{\delta_B} BBB \xrightarrow{Bm} BB$$

is an isomorphism if and only if B has an antipode (e.g. [**BrWi**, 15.1]).

The Fundamental Theorem for Hopf algebras states that the existence of an antipode is equivalent to the comparison functor being an equivalence (see 5.7). To get a corresponding result in our general setting we have to impose slight conditions on the base category and on the functor (see [**MeWi**, 5.6]):

6.9. **Fundamental Theorem for bimonads.** *Let B be a bimonad on the category $\mathbb{A}$ and assume that $\mathbb{A}$ admits colimits or limits and B preserves them. Then the following are equivalent:*

(a) *B is a Hopf bimonad (see 6.8);*

(b) *$\gamma = Bm \cdot \delta B : BB \to BB$ is a natural isomorphism;*

(c) *$\gamma' = mB \cdot B\delta B : BB \to BB$ is a natural isomorphism;*

(d) *the comparison functor $\phi_B^B : \mathbb{A} \to {}_B^B\mathbb{A}$ is an equivalence.*

Recall that for an R-module B, the tensor functor $B \otimes_R -$ has a right adjoint and we have observed in 5.5 that a bialgebra structure on B can be transferred to the adjoint $\mathrm{Hom}_R(B, -)$.

As shown in [**MeWi**, 7.5], this applies for general bimonads provided they have a right adjoint:

6.10. **Adjoints of bimonads.** *Let B be an endofunctor of $\mathbb{A}$ with right adjoint $R : \mathbb{A} \to \mathbb{A}$. Then B is a bimonad (with antipode) if and only if R is a bimonad (with antipode).*

As a special case we have that for any R-Hopf algebra H, the functor $\mathrm{Hom}_R(H,-)$ is a Hopf monad on $\mathbb{M}_R$. This is not a tensor functor unless H_R is finitely generated and projective.

As pointed out in 5.1, no twist map (or braiding) is needed on the base category to formulate the compatibility conditions for bialgebras (and bimonads). There may exist a kind of braiding relations for bimonads based on distributive laws.

6.11. **Double entwinings.** Let B be an endofunctor on the category $\mathbb{A}$ with a monad structure $\underline{B} = (B, m, e)$ and a comonad structure $\overline{B} = (B, \delta, \varepsilon)$.

A natural transformation $\tau : BB \to BB$ is said to be a *double entwining* provided

(i) τ is a mixed distributive law from the monad $\underline{B}$ to the comonad $\overline{B}$;

(ii) τ is a mixed distributive law from the comonad $\overline{B}$ to the monad $\underline{B}$.

6.12. **Induced bimonad.** Let $\tau : BB \to BB$ be a double entwining with commutative diagrams

$$\begin{tikzcd} BB \arrow[r, "B\varepsilon"] \arrow[d, "m"'] & B \arrow[d, "\varepsilon"] \\ B \arrow[r, "\varepsilon"] & 1, \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "e"] \arrow[d, "e"'] & B \arrow[d, "\delta"] \\ B \arrow[r, "eB"'] & BB, \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "e"] \arrow[dr, "="'] & B \arrow[d, "\varepsilon"] \\ & 1, \end{tikzcd}$$

$$\begin{tikzcd} BB \arrow[r, "m"] \arrow[d, "\delta\delta"'] & B \arrow[r, "\delta"] & BB \\ BBBB \arrow[rr, "B\tau B"'] & & BBBB. \arrow[u, "mm"'] \end{tikzcd}$$

Then the composite

$$\overline{\tau} : BB \xrightarrow{\delta B} BBB \xrightarrow{B\tau} BBB \xrightarrow{mB} BB$$

is a *mixed distributive law* from the monad $\underline{B}$ to the comonad $\overline{B}$ making $(B, m, e, \delta, \varepsilon, \overline{\tau})$ a bimonad (see 6.6).

It is obvious that for any bimonad B, the product BB is again a monad as well as a comonad.

BB is also a *bimonad* provided τ satisfies the *Yang-Baxter equation*, that is, commutativity of the diagram

$$\begin{tikzcd} BBB \arrow[r, "\tau B"] \arrow[d, "B\tau"'] & BBB \arrow[r, "B\tau"] & BBB \arrow[d, "\tau B"] \\ BBB \arrow[r, "\tau B"'] & BBB \arrow[r, "B\tau"'] & BBB\,. \end{tikzcd}$$

If this holds, then BB is a bimonad with

product $\overline{m}:\ BBBB \xrightarrow{B\tau B} BBBB \xrightarrow{mm} BB\,,$

coproduct $\overline{\delta}:\ BB \xrightarrow{\delta\delta} BBBB \xrightarrow{B\tau B} BBBB,$

entwining $\overline{\overline{\tau}}:\ BBBB \xrightarrow{B\tau B} BBBB \xrightarrow{\tau\tau} BBBB \xrightarrow{B\tau B} BBBB\,.$

Finally, if τ is a double entwining satisfying the Yang-Baxter equation and $\tau^2 = 1$, then an *opposite bimonad* B^{op} can be defined for B with

$$\begin{array}{rl} \text{product} & m \cdot \tau : BB \xrightarrow{\tau} BB \xrightarrow{m} B, \\ \text{coproduct} & \tau \cdot \delta : B \xrightarrow{\delta} BB \xrightarrow{\tau} BB. \end{array}$$

If B has an antipode S, then $S : B^{\mathrm{op}} \to B$ is a bimonad morphism provided that

$$\tau \cdot BS = SB \text{ and } \tau \cdot BS = SB.$$

In the classical theory of Hopf algebras, the category $\mathbb{M}_R$ of R-modules over a commutative ring R (or vector spaces) is taken as category $\mathbb{A}$ and tensor functors $B \otimes_R -$ are considered (which have right adjoints $\mathrm{Hom}_R(B, -)$). Here the Fundamental theorem for bimonads 6.9 implies that for Hopf algebras 5.7. The twist map provides a braiding on $\mathbb{M}_R$ and this induces a double entwining on the tensor functor $B \otimes_R -$.

We conclude with a non-additive example of our notions.

6.13. **Endofunctors on Set.** On the category **Set** of sets, any set G induces an endofunctor

$$G \times - : \mathbf{Set} \to \mathbf{Set}, \quad X \mapsto G \times X,$$

which has a right adjoint

$$\mathrm{Map}(G, -) : \mathbf{Set} \to \mathbf{Set}, \quad X \mapsto \mathrm{Map}(G, X).$$

Recall (e.g. from [**Wi.A**, 5.19]) that

(1) $G \times -$ is a monad if and only if G is a monoid;

(2) $G \times -$ is comonad with coproduct $\delta : G \to G \times G$, $g \mapsto (g, g)$;

(3) there is an entwining morphism

$$\psi : G \times G \to G \times G, \quad (g, h) \mapsto (gh, g).$$

Thus for any monoid G, $G \times -$ is a bimonad and

Hopf monads on Set. *For a bimonad $G \times -$, the following are equivalent:*

(a) $G \times -$ *is a Hopf monad;*

(b) $\mathrm{Mor}(G, -)$ *is a Hopf monad;*

(c) G *is a group.*

Here we also have a double entwining given by the twist map

$$\tau : G \times G \times - \mapsto G \times G \times -, \quad (a, b, -) \mapsto (b, a, -).$$

6.14. **Remarks.** After reporting about bialgebras and the compatibilty of their algebra and coalgebra part, we considered the entwining of distinct algebras and coalgebras (see 5.9). Similarly, one may try to extend results for bimonads to the entwining of a monad F and a distinct comonad G on a category $\mathbb{A}$ and to head for a kind of *Fundamental Theorem*, that is, an equivalence between the category $\mathbb{A}_F^G$ and, say, a module category over some coinvariants. For this one has to extend the notion of (co)modules over rings to (co)actions of (co)monads on functors and to introduce the notion of *Galois functors*. Comparing with 5.10, a crucial question is when F allows for a G-coaction. For this a *grouplike natural transformation* $I \to G$ is needed. In cooperation with B. Mesablishvili the work on these problems is still in progress.

References

[AdPo] Adámek, J. and and Porst, H.E., From varieties of algebras to covarieties of coalgebras, Electronic Notes in Theoretical Computer Science 44.1 (2001)

[Be] Beck, J., *Distributive laws*, in Seminar on Triples and Categorical Homology Theory, B. Eckmann (ed.), Springer LNM 80, 119-140 (1969)

[BöBrWi] Böhm, G., Brzeziński, T. and Wisbauer, R., *Monads and comonads in module categories*, arXiv:0804.1460 (2008)

[BruVir] Bruguières, A. and Virelizier, A., *Hopf monads*, Adv. Math. 215(2), 679-733 (2007)

[BrMa] Brzeziński, T. and Majid, Sh., *Comodule bundles*, Commun. Math. Physics 191, 467-492 (1998)

[BrWi] Brzeziński, T. and Wisbauer, R., *Corings and Comodules*, London Math. Soc. Lecture Note Series 309, Cambridge University Press (2003)

[EiMo] Eilenberg, S. and Moore, J.C., *Adjoint functors and triples*, Ill. J. Math. 9, 381-398 (1965)

[Gu] Gumm, H.P., *Universelle Coalgebra*, in: Allgemeine Algebra, Ihringer, Th., Berliner Stud. zur Math., Band 10, 155-207, Heldermann Verlag (2003)

[KaWi] Kaučikas, A. and Wisbauer, R., *Noncommutative Hilbert Rings*, J. Algebra Appl. 3(4), 437-444 (2004)

[Lod] Loday, J.-L., *Generalized bialgebras and triples of operads*, arXiv:math/0611885

[Lomp] Lomp, Ch., *A central closure construction for certain algebra extensions. Applications to Hopf actions*, J. Pure Appl. Algebra 198 (1-3), 297-316 (2005)

[Me] Mesablishvili, B., *Entwining Structures in Monoidal Categories*, J. Algebra 319(6), 2496-2517 (2008)

[MeWi] Mesablishvili, B. and Wisbauer, R., *Bimonads and Hopf monads on categories*, arXiv:math.QA/0710.1163 (2007)

[Moer] Moerdijk, I., Monads on tensor categories, J. Pure Appl. Algebra 168(2-3), 189-208 (2002)

[Mor] Morita,K., *Duality for modules and its applications to the theory of rings with minimum condition*, Sci. Rep. Tokyo Kyoiku Daigaku, Sect. A, 6, 83-142 (1958)

[Nast] Năstăsescu, C., Rings. Modules. Categories. (Inele. Module. Categorii.). (Romanian) Bucuresti: Editura Academiei Republicii Socialiste Romania (1976)

[TuPl] Turi, D. and Plotkin, G., *Towards a mathematical operational Semantics*, Proceedings 12th Ann. IEEE Symp. on Logic in Computer Science, LICS'97, Warsaw, Poland (1997)

[Wi.F] Wisbauer, R., *Foundations of module and ring theory. A handbook for study and research*, Algebra, Logic and Appl., Gordon and Breach, Philadelphia (1991)

[Wi.M] Wisbauer, R., *Modules and structure presheaves for arbitrary rings*, Adv. Math., Beijing 20, No.1, 15-23 (1991).

[Wi.B] Wisbauer, R., *Modules and Algebras: Bimodule Structure and Group Actions on Algebras*, Pitman Mono. PAM 81, Addison Wesley, Longman (1996)

[Wi.A] Wisbauer, R., *Algebras versus coalgebras*, Appl. Categor. Struct. 16(1-2) (2008), 255-295.

[Wi.H] Wisbauer, R., *Hopf monads on categories*, Jain, S.K. (ed.) et al., Noncommutative rings, group rings, diagram algebras and their applications. Intern. conf. University of Madras 2006; American Mathematical Society, Contemporary Mathematics 456, 219-230 (2008).

[Wi.G] Wisbauer, R., *Generators in module and comodule categories*, Proc. of the Faith-Osofsky conference, Zanesville (2007).

Mathematical Institute, Heinrich Heine University, 40225 Düsseldorf, Germany
E-mail address: wisbauer@math.uni-duesseldorf.de