GENERALISED BIALGEBRAS AND ENTWINED MONADS AND COMONADS

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Abstract. J.-L. Loday has defined generalised bialgebras and proved structure theorems in this setting which can be seen as general forms of the Poincaré-Birkhoff-Witt and the Cartier-Milnor-Moore theorems. It was observed by the present authors that parts of the theory of generalised bialgebras are special cases of results on entwined monads and comonads and the corresponding mixed bimodules. In this article the Rigidity Theorem of J.-L. Loday is extended to this more general categorical framework.

Contents

1. Introduction

The introduction of *entwining structures* between an algebra and a coalgebra by T. Brzeziński and S. Majid in [2] opened new perspectives in the mathematical treatment of quantum principal bundles. It turned out that these structures are special cases of distributive laws treated in J. Beck's paper [1]. The latter were also used by D. Turi and G. Plotkin [16] in the context of operational semantics.

These observations led to a revival of the investigation of various forms of distributive laws. In a series of papers [13, 14, 15] it was shown how they allow for formulating the theory of Hopf algebras and Galois extensions in a general categorical setting.

On the other hand, *generalised bialgebras* as defined in J.-L. Loday [8, Section 2.1], are vector spaces which are algebras over an *operad* $\mathscr{A}$ and

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coalgebras over a cooperad $\mathscr{C}$. Moreover, the operad $\mathscr{A}$ and the cooperad $\mathscr{C}$ are required to be related by a distributive law. Since any operad $\mathscr{A}$ yields a monad $\mathcal{T}_{\mathscr{A}}$ and $\mathscr{A}$-algebras are nothing but $\mathcal{T}_{\mathscr{A}}$-modules, and similarly any cooperad $\mathscr{C}$ yields a comonad $\mathcal{G}_{\mathscr{C}}$ and $\mathscr{C}$-coalgebras are nothing but $\mathcal{G}_{\mathscr{C}}$-comodules, generalised bialgebras have interpretations in terms of bimodules over a bimonad in the sense of [14].

The purpose of the present paper is to make this relationship more precise (as proposed in [14, 2.3]). On the one hand, given a monad $\mathcal{T}$ and a comonad $\mathcal{G}$ on a category $\mathbb{A}$ together with a mixed distributive law λ between them, we provide in Theorem 4.1 conditions for which a functor from $\mathbb{A}$ to the category of TG-bimodules in $\mathbb{A}$ is an equivalence of categories. In Theorem 5.8 we concentrate on the special case when the monad and the comonad share the same underlying functor. These are general theorems, which do not depend on the shape of the functors underlying the monads and comonads. On the other hand, when considering operads and cooperads the functors involved are analytic in the sense of A. Joyal [7] and, in particular, are graded and connected. In Section 6, we show that the general theory together with the grading lead to the Rigidity Theorem [8, 2.5.1] as a special case (without using idempotents as in [8]). In Corollary 6.9 we focus on the special case when the operad and cooperad share the same underlying analytic functor and prove (under mild conditions) that the category of generalised bialgebras is equivalent to the category of vector spaces.

2. Comodules and adjoint functors

In this section we provide basic notions and properties of comodule functors and adjoint pairs of functors. Throughout the paper $\mathbb{A}$ and $\mathbb{B}$ will denote any categories.

2.1. **Monads and comonads.** Recall that a *monad* $\mathcal{T}$ on $\mathbb{A}$ is a triple (T, m, e) where $T : \mathbb{A} \to \mathbb{A}$ is a functor with natural transformations $m : TT \to T$, $e : 1 \to T$ satisfying associativity and unitality conditions. A $\mathcal{T}$-*module* is an object $a \in \mathbb{A}$ with a morphism $h : T(a) \to a$ subject to associativity and unitality conditions. The (Eilenberg-Moore) category of $\mathcal{T}$-modules is denoted by $\mathbb{A}_{\mathcal{T}}$ and there is a free functor

$$\phi_{\mathcal{T}} : \mathbb{A} \to \mathbb{A}_{\mathcal{T}},\ a \mapsto (T(a), m_a),$$

which is left adjoint to the forgetful functor

$$U_{\mathcal{T}} : \mathbb{A}_{\mathcal{T}} \to \mathbb{A},\ (a, h) \mapsto a.$$

Dually, a *comonad* $\mathcal{G}$ on $\mathbb{A}$ is a triple (G, δ, ε) where $G : \mathbb{A} \to \mathbb{A}$ is a functor with natural transformations $\delta : G \to GG$, $\varepsilon : G \to 1$, and $\mathcal{G}$-*comodules* are objects $a \in \mathbb{A}$ with morphisms $\theta : a \to G(a)$. Both notions are subject to coassociativity and counitality conditions. The (Eilenberg-Moore) category of $\mathcal{G}$-comodules is denoted by $\mathbb{A}^{\mathcal{G}}$ and there is a cofree functor

$$\phi^{\mathcal{G}} : \mathbb{A} \to \mathbb{A}^{\mathcal{G}},\ a \mapsto (G(a), \delta_a),$$

which is right adjoint to the forgetful functor

$$U^{\mathcal{G}} : \mathbb{A}^{\mathcal{G}} \to \mathbb{A},\ (a, \theta) \mapsto a.$$

2.2. **$\mathcal{G}$-comodule functors.** For a comonad $\mathcal{G} = (G, \delta, \varepsilon)$ on $\mathbb{A}$, a functor $F : \mathbb{B} \to \mathbb{A}$ is a *left $\mathcal{G}$-comodule* if there exists a natural transformation $\alpha_F : F \to GF$ inducing commutativity of the diagrams

$$\begin{tikzcd} F \arrow[r, "\alpha_F"] \arrow[dr, equal] & GF \arrow[d, "\varepsilon F"] \\ & F, \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "\alpha_F"] \arrow[d, "\alpha_F"'] & GF \arrow[d, "\delta F"] \\ GF \arrow[r, "G\alpha_F"'] & GGF. \end{tikzcd}$$

Symmetrically, one defines right $\mathcal{G}$-comodules.

2.3. **$\mathcal{G}$-comodules and adjoint functors.** Consider a comonad $\mathcal{G} = (G, \delta, \varepsilon)$ on $\mathbb{A}$ and an adjunction $F \dashv R : \mathbb{A} \to \mathbb{B}$ with counit $\sigma : FR \to 1$.

There exist bijective correspondences (see [3]) between

- functors $K : \mathbb{B} \to \mathbb{A}^{\mathcal{G}}$ with commutative diagrams

$$\begin{tikzcd} \mathbb{B} \arrow[r, "K"] \arrow[dr, "F"'] & \mathbb{A}^{\mathcal{G}} \arrow[d, "U^{\mathcal{G}}"] \\ & \mathbb{A}; \end{tikzcd}$$

- left $\mathcal{G}$-comodule structures $\alpha_F : F \to GF$ on F;
- comonad morphisms from the comonad generated by the adjunction $F \dashv R$ to the comonad $\mathcal{G}$;
- right $\mathcal{G}$-comodule structures $\beta_R : R \to RG$ on R.

In this case, $K(b) = (F(b), \alpha_b)$ for some morphism $\alpha_b : F(b) \to GF(b)$, and the collection $\{\alpha_b,\, b \in \mathbb{B}\}$ constitutes a natural transformation $\alpha_F : F \to GF$ making F into a $\mathcal{G}$-comodule. Conversely, if $(F, \alpha_F : F \to GF)$ is a $\mathcal{G}$-comodule, then $K : \mathbb{B} \to \mathbb{A}^{\mathcal{G}}$ is defined by $K(b) = (F(b), (\alpha_F)_b)$.

For any left $\mathcal{G}$-comodule structure $\alpha_F : F \to GF$, the composite

$$t_K : FR \xrightarrow{\alpha_F R} GFR \xrightarrow{G\sigma} G$$

is a comonad morphism from the comonad generated by the adjunction $F \dashv R$ to the comonad $\mathcal{G}$. Then the corresponding right $\mathcal{G}$-comodule structure $\beta_R : R \to RG$ on R is the composite $R \xrightarrow{\eta R} RFR \xrightarrow{Rt_K} RG$.

Conversely, given a right $\mathcal{G}$-comodule structure $\beta_R : R \to RG$ on R, then the comonad morphism $t_K : FR \to \mathcal{G}$ is the composite

$$FR \xrightarrow{F\beta_R} FRG \xrightarrow{\sigma G} G,$$

while the corresponding left $\mathcal{G}$-comodule structure $\alpha_F : F \to GF$ on F is the composite $F \xrightarrow{F\eta} FRF \xrightarrow{t_K F} GF$.

The following result gives a necessary and sufficient condition on a functor $K : \mathbb{B} \to \mathbb{A}^{\mathcal{G}}$ with $U^{\mathcal{G}}K = F$ to be an equivalence of categories.

2.4. **Theorem.** (*see* [11, Theorem 4.4]) *A functor* $K : \mathbb{B} \to \mathbb{A}^{\mathcal{G}}$ *with* $U^{\mathcal{G}}K = F$ *is an equivalence of categories if and only if*

(i) *the functor* F *is comonadic, and*
(ii) t_K *is an isomorphism of comonads.*

We shall need the following result, the dual version of E. Dubuc's theorem [4].

2.5. **J. Dubuc's Adjoint Triangle Theorem.** *For categories* $\mathbb{A}$, $\mathbb{B}$ *and* $\mathbb{C}$, *let* $\eta, \sigma : F \dashv R : \mathbb{A} \to \mathbb{B}$ *and* $\eta', \sigma' : F' \dashv R' : \mathbb{A} \to \mathbb{C}$ *be adjunctions and let* $K : \mathbb{B} \to \mathbb{C}$ *be such that* $F = F'K$. *Define*

$$\beta_0 : KR \xrightarrow{\eta' KR} R'F'KR = R'FR \xrightarrow{R'\sigma} R'.$$

If $\mathbb{B}$ *has equalisers of coreflexive pairs and the functor* F' *is of descent type, then* K *has a right adjoint* $\overline{K}$ *which can be calculated as the equaliser*

$$\begin{tikzcd}
\overline{K} \arrow[r] & RF' \arrow[rr, "RF'\eta'"] \arrow[dr, "\eta RF'"'] & & RF'R'F' \\
& & RFRF' = RF'KRF'. \arrow[ur, "RF'\beta_0 F'"'] &
\end{tikzcd}$$

2.6. **Right adjoint of** K. Now fix a functor $K : \mathbb{B} \to \mathbb{A}^{\mathcal{G}}$ with $U^{\mathcal{G}}K = F$ and suppose that the category $\mathbb{B}$ has equalisers of coreflexive pairs. Then $U^{\mathcal{G}}\beta_0 = t_K$ and thus $\beta_R = U^{\mathcal{G}}\beta_0 \cdot \eta R$. It then follows from Theorem 2.5 that the functor K has a right adjoint $\overline{K}$ which is determined by the equaliser diagram

$$\overline{K} \xrightarrow{\ i\ } RU^{\mathcal{G}} \underset{\beta_R U^{\mathcal{G}}}{\overset{RU^{\mathcal{G}}\eta^{\mathcal{G}}}{\rightrightarrows}} RGU^{\mathcal{G}} = RU^{\mathcal{G}}\phi^{\mathcal{G}}U^{\mathcal{G}}, \tag{2.1}$$

where $\eta^{\mathcal{G}} : 1 \to \phi^{\mathcal{G}}U^{\mathcal{G}}$ is the unit of the adjunction $U^{\mathcal{G}} \dashv \phi^{\mathcal{G}}$.

An easy inspection shows that the value of $\overline{K}$ at $(a, \theta) \in \mathbb{A}^{\mathcal{G}}$ is given by the equaliser diagram

$$\overline{K}(a, \theta) \xrightarrow{i_{(a,\theta)}} R(a) \underset{(\beta_R)_a}{\overset{R(\theta)}{\rightrightarrows}} RG(a)\,. \tag{2.2}$$

3. Distributive laws

Distributive laws were introduced by J. Beck in [1]. Here we are mainly interested in the following case (e.g. [6] or [17, 5.3]).

3.1. **Mixed distributive laws.** Let $\mathcal{T} = (T, m, e)$ be a monad and $\mathcal{G} = (G, \delta, \varepsilon)$ a comonad on the category $\mathbb{A}$. A natural transformation

$$\lambda : TG \to GT$$

is said to be a *mixed distributive law* or a *(mixed) entwining* provided it induces commutative diagrams

$$\begin{tikzcd} TTG \arrow[rr, "mG"] \arrow[d, "T\lambda"'] & & TG \arrow[d, "\lambda"] \\ TGT \arrow[r, "\lambda T"'] & GTT \arrow[r, "Gm"'] & GT, \end{tikzcd} \qquad \begin{tikzcd} TG \arrow[r, "T\delta"] \arrow[d, "\lambda"'] & TGG \arrow[r, "\lambda G"] & GTG \arrow[d, "G\lambda"] \\ GT \arrow[rr, "\delta T"'] & & GGT, \end{tikzcd}$$

$$\begin{tikzcd} G \arrow[r, "eG"] \arrow[dr, "Ge"'] & TG \arrow[d, "\lambda"] \\ & GT, \end{tikzcd} \qquad \begin{tikzcd} TG \arrow[r, "T\varepsilon"] \arrow[d, "\lambda"'] & T \\ GT. \arrow[ur, "\varepsilon T"'] & \end{tikzcd}$$

Recall (for example, from [18]) that if $\mathcal{T}$ is a monad and $\mathcal{G}$ is a comonad on a category $\mathbb{A}$, then there are bijective correspondences between

- mixed distributive laws $\lambda : TG \to GT$;
- comonads $\widehat{\mathcal{G}} = (\widehat{G}, \widehat{\delta}, \widehat{\varepsilon})$ on $\mathbb{A}_{\mathcal{T}}$ that extend $\mathcal{G}$ in the sense that $U_{\mathcal{T}}\widehat{G} = GU_{\mathcal{T}}$, $U_{\mathcal{T}}\widehat{\varepsilon} = \varepsilon U_{\mathcal{T}}$, and $U_{\mathcal{T}}\widehat{\delta} = \delta U_{\mathcal{T}}$;
- monads $\widehat{\mathcal{T}} = (\widehat{T}, \widehat{m}, \widehat{e})$ on $\mathbb{A}^{\mathcal{G}}$ that extend $\mathcal{T}$ in the sense that $U^{\mathcal{G}}\widehat{T} = TU^{\mathcal{G}}$, $U^{\mathcal{G}}\widehat{e} = eU^{\mathcal{G}}$, and $U^{\mathcal{G}}\widehat{m} = mU^{\mathcal{G}}$.

From the definitions

$$\widehat{G}(a,h) = (G(a), G(h)\cdot\lambda_a),\ \ \widehat{\varepsilon}_{(a,h)} = \varepsilon_a,\ \ \widehat{\delta}_{(a,h)} = \delta_a, \text{ for } (a,h) \in \mathbb{A}_{\mathcal{T}},$$

$$\widehat{T}(a,\theta) = (T(a), \lambda_a \cdot T(\theta)),\ \ \widehat{e}_{(a,\theta)} = e_a,\quad \widehat{m}_{(a,\theta)} = m_a, \text{ for } (a,\theta) \in \mathbb{A}^{\mathcal{G}},$$

it follows that for a mixed distributive law $\lambda : TG \to GT$ one may assume

$$(\mathbb{A}^{\mathcal{G}})_{\widehat{\mathcal{T}}} = (\mathbb{A}_{\mathcal{T}})^{\widehat{\mathcal{G}}}.$$

We write $\mathbb{A}^{\mathcal{G}}_{\mathcal{T}}(\lambda)$ for this category whose objects, called TG-*bimodules* in [6], are triples (a,h,θ), where $(a,h) \in \mathbb{A}_{\mathcal{T}}$, $(a,\theta) \in \mathbb{A}^{\mathcal{G}}$ with commuting diagram

$$\begin{tikzcd} T(a) \arrow[r, "h"] \arrow[d, "T(\theta)"'] & a \arrow[r, "\theta"] & G(a) \\ TG(a) \arrow[rr, "\lambda_a"'] & & GT(a). \arrow[u, "G(h)"'] \end{tikzcd}$$

Morphisms in this category are morphisms in $\mathbb{A}$ which are $\mathcal{T}$-module as well as $\mathcal{G}$-comodule morphisms.

3.2. **Entwined monads and comonads.** Let $\mathcal{T} = (T, m, e)$ be a monad, $\mathcal{G} = (G, \delta, \varepsilon)$ a comonad on $\mathbb{A}$, and consider an entwining $\lambda : TG \to GT$ from $\mathcal{T}$ to $\mathcal{G}$. Denote by $\widehat{\mathcal{T}} = (\widehat{T}, \widehat{m}, \widehat{e})$ the monad on $\mathbb{A}^{\mathcal{G}}$ lifting $\mathcal{T}$ and by $\widehat{\mathcal{G}} = (\widehat{G}, \widehat{\delta}, \widehat{\varepsilon})$ the comonad on $\mathbb{A}_{\mathcal{T}}$ lifting $\mathcal{G}$.

Suppose there exists a functor $K : \mathbb{A} \to (\mathbb{A}_{\mathcal{T}})^{\widehat{\mathcal{G}}}$ with commutative diagram

$$\begin{tikzcd} \mathbb{A} \arrow[r, "K"] \arrow[dr, "\phi_{\mathcal{T}}"'] & (\mathbb{A}_{\mathcal{T}})^{\widehat{\mathcal{G}}} \arrow[d, "U^{\widehat{\mathcal{G}}}"] \\ & \mathbb{A}_{\mathcal{T}} \end{tikzcd} \tag{3.1}$$

and consider the corresponding right $\widehat{\mathcal{G}}$-comodule structure on $U_{\mathcal{T}}$ as in Section 2.3,

$$\beta = \beta_{U_{\mathcal{T}}} : U_{\mathcal{T}} \to U_{\mathcal{T}}\widehat{G} = GU_{\mathcal{T}}.$$

Then, for any $(a, h) \in \mathbb{A}_{\mathcal{T}}$, the (a, h)-component $\beta_{(a,h)} = (\beta_{U_{\mathcal{T}}})_{(a,h)}$ of β is a morphism $a \to G(a)$ in $\mathbb{A}$. Assuming that $\mathbb{A}$ admits coreflexive equalisers, we obtain by Section 2.6 that the functor K admits a right adjoint $\overline{K}$ whose value at $((a, h), \theta) \in (\mathbb{A}_{\mathcal{T}})^{\widehat{\mathcal{G}}}$ appears as the equaliser

$$\overline{K}((a,h),\theta) \xrightarrow{i_{((a,h),\theta)}} a \underset{\beta_{(a,h)}}{\overset{\theta}{\rightrightarrows}} G(a)\,. \tag{3.2}$$

Consider now the left $\widehat{\mathcal{G}}$-comodule structure $\alpha = \alpha_{\phi_{\mathcal{T}}} : \phi_{\mathcal{T}} \to \widehat{G}\phi_{\mathcal{T}}$ on $\phi_{\mathcal{T}}$ induced by the commutative diagram (3.1). As shown in [13, Theorem 2.4], for any $(a, h) \in \mathbb{A}_{\mathcal{T}}$, the component $(t_K)_{(a,h)}$ of the comonad morphism $t_K : \phi_{\mathcal{T}}U_{\mathcal{T}} \to \widehat{G}$, corresponding to Diagram (3.1), is the composite

$$T(a) \xrightarrow{T(\beta_{(a,h)})} TG(a) \xrightarrow{\lambda_a} GT(a) \xrightarrow{G(h)} G(a)\,. \tag{3.3}$$

4. Grouplike morphisms

Let $\mathcal{G} = (G, \delta, \varepsilon)$ be a comonad on a category $\mathbb{A}$. By [13, Definition 3.1], a natural transformation $g : 1 \to G$ is called a *grouplike morphism* provided it is a comonad morphism from the identity comonad to $\mathcal{G}$, that is, it induces commutative diagrams

$$\begin{tikzcd} 1 \arrow[r, "g"] \arrow[dr, "="'] & G \arrow[d, "\varepsilon"] \\ & 1, \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "g"] \arrow[dr, "gg"'] & G \arrow[d, "\delta"] \\ & GG. \end{tikzcd}$$

The dual notion is that of an *augmentation of a monad* $\mathcal{T} = (T, m, e)$ on $\mathbb{A}$, that is, a monad morphism $T \to 1$.

Let $\mathcal{T} = (T, m, e)$ and $\mathcal{G} = (G, \delta, \varepsilon)$ be given on $\mathbb{A}$ with an entwining $\lambda : TG \to GT$. If $\mathcal{G}$ has a grouplike morphism $g : 1 \to G$, then the above conditions guarantee that the morphisms $(g_a : a \to G(a))_{(a,h)\in\mathbb{A}_{\mathcal{T}}})$ form the components of a right $\widehat{\mathcal{G}}$-comodule structure $\beta = \beta_{U_{\mathcal{T}}} : U_{\mathcal{T}} \to U_{\mathcal{T}}\widehat{G}$ on the functor $U_{\mathcal{T}} : \mathbb{A}_{\mathcal{T}} \to \mathbb{A}$. Observing that in the diagram

$$\begin{tikzcd}
T(a) \arrow[r, "T(g_a)"] \arrow[d, "T(e_a)"'] & TG(a) \arrow[r, "\lambda_a"] \arrow[d, "TG(e_a)"'] & GT(a) \arrow[d, "GT(e_a)"'] \arrow[dr, equal] & \\
TT(a) \arrow[r, "T(g_{(T(a)})"'] & TGT(a) \arrow[r, "\lambda_{T(a)}"'] & GTT(a) \arrow[r, "G(m_a)"'] & GT(a)
\end{tikzcd}$$

- the left-hand square commutes by naturality of g,
- the right-hand square commutes by naturality of λ,
- the triangle commutes since e is the unit for the monad $\mathcal{T}$,

and recalling that α is the composite $\phi_{\mathcal{T}} \xrightarrow{\phi_{\mathcal{T}}\eta_{\mathcal{T}}} \phi_{\mathcal{T}} U_{\mathcal{T}} \phi_{\mathcal{T}} \xrightarrow{t_K \phi_{\mathcal{T}}} \widehat{G}\phi_{\mathcal{T}}$, one concludes by (3.3) that

$$\text{for every } a \in \mathbb{A},\ \alpha_a = \lambda_a \cdot T(g_a). \tag{4.1}$$

This leads to a functor

$$K_g : \mathbb{A} \to (\mathbb{A}_{\mathcal{T}})^{\widehat{\mathcal{G}}}, \quad a \longmapsto ((T(a), m_a), \lambda_a \cdot T(g_a)),$$

and the commutative diagram

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "K_g"] \arrow[dr, "\phi_{\mathcal{T}}"'] & (\mathbb{A}_{\mathcal{T}})^{\widehat{\mathcal{G}}} \arrow[d, "U^{\widehat{\mathcal{G}}}"] \\
& \mathbb{A}_{\mathcal{T}}.
\end{tikzcd} \tag{4.2}$$

In this case we say that the comparison functor K_g is induced by the grouplike morphism $g : 1 \to G$.

Specialising now Theorem 2.4 to the present situation gives

4.1. **Theorem.** *Let $\mathcal{T} = (T, m, e)$ be a monad and $\mathcal{G} = (G, \delta, \varepsilon)$ a comonad on $\mathbb{A}$ with an entwining $\lambda : TG \to GT$. If $g : 1 \to G$ is a grouplike morphism of the comonad $\mathcal{G}$, then the induced functor $K_g : \mathbb{A} \to (\mathbb{A}_{\mathcal{T}})^{\widehat{\mathcal{G}}}$ is an equivalence of categories if and only if*

(i) *the functor $\phi_{\mathcal{T}}$ is comonadic, and*
(ii) *the composite*

$$T(a) \xrightarrow{T(g_a)} TG(a) \xrightarrow{\lambda_a} GT(a) \xrightarrow{G(h)} G(a) \tag{4.3}$$

is an isomorphism for every $(a, h) \in \mathbb{A}_{\mathcal{T}}$.

4.2**. Remark.** It follows from [15, Theorem 2.12] that condition (ii) of Theorem 4.1 is equivalent to saying that the composite

$$TT(a) \xrightarrow{T(g_{T(a)})} TGT(a) \xrightarrow{\lambda_{T(a)}} GTT(a) \xrightarrow{G(m_a)} GT(a)$$

is an isomorphism for every $a \in \mathbb{A}$.

5. Compatible entwinings

Let $\underline{\mathcal{H}} = (H, m, e)$ be a monad, $\overline{\mathcal{H}} = (H, \delta, \varepsilon)$ a comonad on $\mathbb{A}$, and let $\lambda : HH \to HH$ be an entwining from the monad $\underline{\mathcal{H}}$ to the comonad $\overline{\mathcal{H}}$. The datum $(\underline{\mathcal{H}}, \overline{\mathcal{H}}, \lambda)$ is called a *monad-comonad triple.* The objects of the category $\mathbb{A}^{\overline{\mathcal{H}}}_{\underline{\mathcal{H}}}(\lambda)$ are called *(mixed) λ-bimodules.*

5.1. **Lemma.** *The triple $(H(a), m_a, \delta_a)$ is a λ-bimodule for all $a \in \mathbb{A}$ if and only if we have a commutative diagram*

(5.1)
$$\begin{tikzcd}
HH \arrow[r, "m"] \arrow[d, "H\delta"'] & H \arrow[r, "\delta"] & HH \\
HHH \arrow[rr, "\lambda H"'] & & HHH. \arrow[u, "Hm"']
\end{tikzcd}$$

In this case, there is a functor

$$K : \mathbb{A} \to (\mathbb{A}_{\underline{\mathcal{H}}})^{\widehat{\overline{\mathcal{H}}}}, \quad a \longmapsto ((H(a), m_a), \delta_a),$$

satisfying $\phi_{\underline{\mathcal{H}}} = U^{\widehat{\overline{\mathcal{H}}}} K$.

5.2**. Definitions.** Given a monad-comonad triple $(\underline{\mathcal{H}}, \overline{\mathcal{H}}, \lambda)$, the entwining $\lambda : HH \to HH$ is said to be *compatible* provided Diagram (5.1) is commutative; then $(\underline{\mathcal{H}}, \overline{\mathcal{H}}, \lambda)$ is said to be a *compatible monad-comonad triple.*

The triple $(\underline{\mathcal{H}}, \overline{\mathcal{H}}, \lambda)$ is called a *bimonad* if it is a compatible triple with additional commutative diagrams (see [14, Definition 4.1])

(5.2)
$$\begin{tikzcd}
HH \arrow[r, "H\varepsilon"] \arrow[d, "m"'] \arrow[dr, phantom, "(i)"] & H \arrow[d, "\varepsilon"] \\
H \arrow[r, "\varepsilon"] & 1,
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "e"] \arrow[d, "e"'] \arrow[dr, phantom, "(ii)"] & H \arrow[d, "\delta"] \\
H \arrow[r, "He"'] & HH,
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "e"] \arrow[dr, bend right, "="'] \arrow[dr, phantom, "(iii)"] & H \arrow[d, "\varepsilon"] \\
 & 1.
\end{tikzcd}$$

Notice that for any monad-comonad triple $(\underline{\mathcal{H}}, \overline{\mathcal{H}}, \lambda)$, to say that Diagram (5.2)(i) commutes amounts to saying that $\varepsilon : H \to 1$ is an augmentation of the monad $\underline{\mathcal{H}}$, while to say that Diagram (5.2)(ii) commutes amounts to saying that $e : 1 \to H$ is a grouplike morphism of the comonad $\overline{\mathcal{H}}$. Thus, for any bimonad $(\underline{\mathcal{H}}, \overline{\mathcal{H}}, \lambda)$, e is a grouplike morphism of the comonad $\overline{\mathcal{H}}$ and ε is an augmentation of the monad $\underline{\mathcal{H}}$.

5.3. **Proposition.** *Let $(\underline{\mathcal{H}}, \overline{\mathcal{H}}, \lambda)$ be a compatible monad-comonad triple. If $\delta \cdot e = He \cdot e$ (i.e. $e : 1 \to H$ is a grouplike morphism of $\overline{\mathcal{H}}$), then $\delta = \lambda \cdot He$ and the comparison functor K in* Lemma 5.1 *is induced by the grouplike morphism e, that is $K = K_e$ (see* Diagram (4.2)*).*

Proof. Assume that $\delta \cdot e = He \cdot e$ and that λ is compatible. Then, in the diagram

$$\begin{tikzcd}
H \arrow[r, "He"] \arrow[d, "He"'] & HH \arrow[r, "m"] \arrow[d, "H\delta"] & H \arrow[r, "\delta"] & HH \\
HH \arrow[r, "HHe"] \arrow[drr, "\lambda"'] & HHH \arrow[rr, "\lambda H"] & & HHH \arrow[u, "Hm"'] \\
& & HH \arrow[ur, "HHe"'] &
\end{tikzcd},$$

the rectangles commute. Since the triangle is also commutative by naturality of composition and since $m \cdot He = 1$, it follows that $\delta = \lambda \cdot He$. From Section 4 and Relation (4.1), we conclude that the comparison functor K is induced by the grouplike morphism e, that is $K = K_e$. □

5.4. **Remark.** Note that if $\varepsilon \cdot m = \varepsilon \cdot H\varepsilon$ (i.e. $\varepsilon : H \to 1$ is an augmentation of $\underline{\mathcal{H}}$) and λ is compatible, then post-composing Diagram (5.1) with the morphism $H\varepsilon$ implies $m = H\varepsilon \cdot \lambda$.

In the next propositions we do not require a priori λ to be a compatible entwining.

5.5. **Proposition.** *Let $(\underline{\mathcal{H}}, \overline{\mathcal{H}}, \lambda)$ be a monad-comonad triple.*

(i) *If $\delta = \lambda \cdot He$, then $\delta \cdot e = He \cdot e$;*

(ii) *if $m = H\varepsilon \cdot \lambda$, then $\varepsilon \cdot m = \varepsilon \cdot H\varepsilon$.*

Moreover, if one of these conditions is satisfied, then $\varepsilon \cdot e = 1$, provided that $e : 1 \to H$ is a (componentwise) monomorphism or ε is a (componentwise) epimorphism.

Proof. (i) Assume $\delta = \lambda \cdot He$. Since $He \cdot e = eH \cdot e$ (by naturality) and $\lambda \cdot eH = He$ (see Section 3.1),

$$\delta \cdot e = \lambda \cdot He \cdot e = \lambda \cdot eH \cdot e = He \cdot e.$$

(ii) Assume $m = H\varepsilon \cdot \lambda$. Since $\varepsilon \cdot H\varepsilon = \varepsilon \cdot \varepsilon H$ and $\varepsilon H \cdot \lambda = H\varepsilon$ (see Section 3.1),

$$\varepsilon \cdot m = \varepsilon \cdot H\varepsilon \cdot \lambda = \varepsilon \cdot H\varepsilon \cdot \lambda = \varepsilon \cdot H\varepsilon.$$

To show the final claim, observe that $\delta = \lambda \cdot He$ implies

$$1 = \varepsilon H \cdot \delta = \varepsilon H \cdot \lambda \cdot He = H\varepsilon \cdot He,$$

and $m = H\varepsilon \cdot \lambda$ implies

$$1 = m \cdot eH = H\varepsilon \cdot \lambda \cdot eH = H\varepsilon \cdot He,$$

so in both cases, $1 = H\varepsilon \cdot He$. Naturality of e and ε imply commutativity of the diagrams, respectively,

$$\begin{tikzcd} H \arrow[r, "eH"] \arrow[d, "\varepsilon"'] & HH \arrow[d, "H\varepsilon"] \\ 1 \arrow[r, "e"'] & H, \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, "He"] \arrow[d, "\varepsilon"'] & HH \arrow[d, "\varepsilon H"] \\ 1 \arrow[r, "e"'] & H. \end{tikzcd}$$

From the left-hand diagram one gets

$$e = H\varepsilon \cdot He \cdot e = H\varepsilon \cdot eH \cdot e = e \cdot \varepsilon \cdot e,$$

thus if e is a (componentwise) monomorphism, $\varepsilon \cdot e = 1$, while the right-hand diagram implies

$$\varepsilon = \varepsilon \cdot H\varepsilon \cdot He = \varepsilon \cdot e \cdot \varepsilon$$

and hence $\varepsilon \cdot e = 1$ provided ε is a (componentwise) epimorphism. □

5.6. **Lemma.** *Let $(\underline{\mathcal{H}}, \overline{\mathcal{H}}, \lambda)$ be a monad-comonad triple. If*

$$m = H\varepsilon \cdot \lambda \quad \textit{or} \quad \delta = \lambda \cdot He,$$

then λ is compatible, that is, Diagram (5.1) *is commutative.*

Proof. If $\delta = \lambda \cdot He$, the triangle is commutative in the diagram

$$\begin{tikzcd} HH \arrow[r, "H\delta"] \arrow[d, "HHe"'] & HHH \arrow[r, "\lambda H"] & HHH \arrow[d, "Hm"] \\ HHH \arrow[ur, "H\lambda"] \arrow[r, "mH"'] & HH \arrow[r, "\lambda"] & HH, \end{tikzcd}$$

whereas the trapezium is commutative by the entwining property of λ. The left path of the outer diagram is

$$\lambda \cdot mH \cdot HHe = \lambda \cdot He \cdot m = \delta \cdot m.$$

This shows that Diagram (5.1) is commutative.

In a similar way the claim for $m = H\varepsilon \cdot \lambda$ is proved. □

To sum up, combining Propositions 5.3 and 5.5, Remark 5.4 and Lemma 5.6 yields

5.7. **Proposition.** *Let $(\underline{\mathcal{H}}, \overline{\mathcal{H}}, \lambda)$ be a monad-comonad triple.*

(1) *$\delta = \lambda \cdot He$ if and only if λ is compatible and $\delta \cdot e = He \cdot e$;*

(2) *$m = H\varepsilon \cdot \lambda$ if and only if λ is compatible and $\varepsilon \cdot m = \varepsilon \cdot H\varepsilon$;*

(3) *if $\delta = \lambda \cdot He$, $m = H\varepsilon \cdot \lambda$, and $e : 1 \to H$ is a (componentwise) monomorphism or ε is a (componentwise) epimorphism, then $(\underline{\mathcal{H}}, \overline{\mathcal{H}}, \lambda)$ is a bimonad (see* Definitions 5.2*).*

If $(\underline{\mathcal{H}}, \overline{\mathcal{H}}, \lambda)$ is a monad-comonad triple such that $\delta = \lambda \cdot He$, then $(\underline{\mathcal{H}}, \overline{\mathcal{H}}, \lambda)$ is a compatible monad-comonad triple by Lemma 5.6, and hence, by Proposition 5.3, the assignment $a \longmapsto (H(a), m_a, \delta_a)$ yields the functor $K_e : \mathbb{A} \to \mathbb{A}^{\overline{\mathcal{H}}}_{\underline{\mathcal{H}}}(\lambda)$ with commutative diagram

$$\begin{tikzcd} \mathbb{A} \arrow[r, "K_e"] \arrow[dr, "\phi_{\underline{\mathcal{H}}}"'] & \mathbb{A}^{\overline{\mathcal{H}}}_{\underline{\mathcal{H}}}(\lambda) = (\mathbb{A}_{\underline{\mathcal{H}}})^{\widehat{\overline{\mathcal{H}}}} \arrow[d, "U^{\widehat{\overline{\mathcal{H}}}}"] \\ & \mathbb{A}_{\underline{\mathcal{H}}} . \end{tikzcd}$$

Recall from [14] that a bimonad $\mathcal{H}$ is said to be a *Hopf monad* provided it has an *antipode*, i.e., there exists a natural transformation $S : H \to H$ such that $m \cdot HS \cdot \delta = e \cdot \varepsilon = m \cdot SH \cdot \delta$.

5.8. **Theorem.** *Let $(\underline{\mathcal{H}}, \overline{\mathcal{H}}, \lambda)$ be a monad-comonad triple on a Cauchy complete category $\mathbb{A}$. Assume that $\delta = \lambda \cdot He$ and $e : 1 \to H$ is a (componentwise) monomorphism. Then the following are equivalent:*

(a) $K_e : \mathbb{A} \to \mathbb{A}^{\overline{\mathcal{H}}}_{\underline{\mathcal{H}}}(\lambda)$ *is an equivalence of categories;*

(b) *the composite* $H(a) \xrightarrow{\delta_a} HH(a) \xrightarrow{H(h)} H(a)$ *is an isomorphism for every* $(a, h) \in \mathbb{A}_{\underline{\mathcal{H}}}$*;*

(c) *the composite* $HH \xrightarrow{\delta H} HHH \xrightarrow{Hm} HH$ *is an isomorphism.*

If, in addition, $\varepsilon : H \to 1$ *is an augmentation of the monad* $\underline{\mathcal{H}}$*, then* $\mathcal{H}$ *is a Hopf monad.*

Proof. Since $\delta = \lambda \cdot He$, (a)$\Rightarrow$(b) is trivial by Theorem 4.1, while (b) and (c) are equivalent by Remark 4.2.

Given (c), it follows from Theorem 4.1 that K is an equivalence of categories if and only if the functor $\phi_{\underline{\mathcal{H}}}$ is comonadic. But by [12, Corollary 3.17] this is always the case, since $e : 1 \to H$ is a monomorphism and hence $\varepsilon \cdot e = 1$ by Proposition 5.5. This proves the implication (c)$\Rightarrow$(a).

Finally, if $\varepsilon : H \to 1$ is an augmentation of the monad $\underline{\mathcal{H}}$, then $\varepsilon \cdot m = \varepsilon \cdot H\varepsilon$, and since $(\underline{\mathcal{H}}, \overline{\mathcal{H}}, \lambda)$ is compatible, $m = H\varepsilon \cdot \lambda$ by Proposition 5.7. Since $\delta = \lambda \cdot He$, $\delta \cdot e = He \cdot e$ again by Proposition 5.7. Thus $\mathcal{H}$ is a bimonad and it now follows from [15, 3.1] that $\mathcal{H}$ is a Hopf monad. $\square$

6. Generalised bialgebras

In this section, we apply our results in the context of operads to recover results of J.-L. Loday on generalised bialgebras in [8]. The Leitmotiv of the section is that a (co)operad is a particular type of (co)monad. Let $\mathbf{k}$ denote a field and $\mathbb{A}$ the category of $\mathbf{k}$-vector spaces.

6.1. **Schur functors.** An $\mathbb{S}$-*module* $\mathscr{M}$ in $\mathbb{A}$ (or *vector species*) is a collection of objects $\mathscr{M}(n)$, for $n \geq 0$, together with an action of the symmetric group

S_n. To an $\mathbb{S}$-module $\mathscr{M}$ one associates the functor

$$F_{\mathscr{M}}: \begin{array}{ccc} \mathbb{A} & \longrightarrow & \mathbb{A} \\ V & \mapsto & \bigoplus_{n>0} \mathscr{M}(n) \otimes_{\mathbf{k}[S_n]} V^{\otimes n} \end{array} .$$

Such a functor is called a *Schur functor*. A. Joyal proved in [7] that for two $\mathbb{S}$-modules $\mathscr{M}$ and $\mathscr{N}$, the composite $F_{\mathscr{M}} \cdot F_{\mathscr{N}}$ is a Schur functor of the form $F_{\mathscr{M}\circ\mathscr{N}}$ with $\mathscr{M} \circ \mathscr{N}$ being the $\mathbb{S}$-module defined by

$$(\mathscr{M} \circ \mathscr{N})(n) = \bigoplus_{k>0, i_1+\cdots+i_k=n} \mathscr{M}(k) \otimes_{\mathbf{k}[S_k]} \mathrm{Ind}_{S_{i_1}\times\cdots\times S_{i_k}}^{S_n} \mathscr{N}(i_1) \otimes \cdots \otimes \mathscr{N}(i_k),$$

where $\mathrm{Ind}_{S_{i_1}\times\cdots\times S_{i_k}}^{S_n}$ denotes the induced representation functor. The product $\circ$ is called the *plethysm* of $\mathbb{S}$-modules, and the category of $\mathbb{S}$-modules, together with the plethysm is a monoidal category. The unit for the plethysm is the $\mathbb{S}$-module

$$1(n) = \begin{cases} \mathbf{k}, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

If $\mathscr{N}(0) = 0$, then the plethysm has a nice expression (see [5, Lemma 1.3.9]). Let $X_{n,k}$ be the set of surjections $f : \{1, \ldots, n\} \to \{1, \ldots, k\}$ such that for any $1 \le i < j \le k$ the smallest element of the set $f^{-1}(i)$ is smaller than the smallest element of the set $f^{-1}(j)$. Then

$$(6.1) \quad (\mathscr{M} \circ \mathscr{N})(n) = \bigoplus_{k>0, f\in X_{n,k}} \mathscr{M}(k) \otimes \mathscr{N}(|f^{-1}(1)|) \otimes \cdots \otimes \mathscr{N}(|f^{-1}(k)|).$$

For our purpose, we will always assume that any $\mathbb{S}$-module $\mathscr{M}$ is *reduced*, that is, it satisfies $\mathscr{M}(0) = 0$ and $\mathscr{M}(1) = \mathbf{k}$.

We denote by $e_{\mathscr{M}} : 1 \to F_{\mathscr{M}}$ the natural transformation which maps V to the summand V of $F_{\mathscr{M}}(V)$ and by $\varepsilon_{\mathscr{M}} : F_{\mathscr{M}} \to 1$ the projection of $F_{\mathscr{M}}(V)$ to the summand V, thus $\varepsilon_{\mathscr{M}} \cdot e_{\mathscr{M}} = 1$.

6.2**. Operads, cooperads.** A *(reduced) operad* $\mathscr{A}$ in $\mathbb{A}$ is a monoid in the monoidal category of (reduced) $\mathbb{S}$-modules. This amounts to saying that the functor $F_{\mathscr{A}}$ is the functor part of a monad $\mathcal{T}_{\mathscr{A}} = (F_{\mathscr{A}}, m_{\mathscr{A}}, e_{\mathscr{A}})$.

An *algebra over an operad* $\mathscr{A}$, or $\mathscr{A}$*-algebra*, is a $\mathcal{T}_{\mathscr{A}}$-module. Hence, the free $\mathscr{A}$-algebra generated by a vector space V is nothing but

$$(T_{\mathscr{A}}(V), (m_{\mathscr{A}})_V, (e_{\mathscr{A}})_V).$$

A *(reduced) cooperad* $\mathscr{C}$ in $\mathbb{A}$ is a comonoid in the monoidal category of $\mathbb{S}$-modules. This amounts to saying that the functor $F_{\mathscr{C}}$ is the functor part of a comonad $\mathcal{G}_{\mathscr{C}} = (F_{\mathscr{C}}, \delta_{\mathscr{C}}, \varepsilon_{\mathscr{C}})$.

A *coalgebra over a cooperad* $\mathscr{C}$, or $\mathscr{C}$*-coalgebra*, is a $\mathcal{G}_{\mathscr{C}}$-comodule.

With our definitions and assumptions, any coalgebra over a reduced cooperad $\mathscr{C}$ is naturally conilpotent.

We assume from now on that either the action of the symmetric group is free or the field $\mathbf{k}$ has characteristic 0. We assume that all the operads and cooperads considered are reduced.

6.3. **Proposition.** *If $\mathscr{A}$ is a reduced operad, then $\varepsilon_{\mathscr{A}}$ is an augmentation for the monad $\mathcal{T}_{\mathscr{A}}$. If $\mathscr{C}$ is a reduced cooperad then $e_{\mathscr{C}}$ is a grouplike morphism for the comonad $\mathcal{G}_{\mathscr{C}}$.*

Proof. The unit for the plethysm forms a (co)operad and the associated (co)monad is the identity functor. Let $m : \mathscr{A} \circ \mathscr{A} \to \mathscr{A}$ denote the operad composition. For every $n \geq 1$, one has to prove commutativity of the diagram

$$\begin{tikzcd}
(\mathscr{A} \circ \mathscr{A})(n) \arrow[r, "\mathscr{A} \circ \varepsilon_{\mathscr{A}}"] \arrow[d, "m"'] & \mathscr{A}(n) \arrow[d, "\varepsilon_{\mathscr{A}}"] \\
\mathscr{A}(n) \arrow[r, "\varepsilon_{\mathscr{A}}"] & 1(n).
\end{tikzcd}$$

If $n > 1$, then the diagram commutes because the top and bottom compositions vanish. If $n = 1$, since $\mathscr{A}(1) = \mathbf{k}$ then $(\mathscr{A} \circ \mathscr{A})(1) = \mathbf{k} \otimes \mathbf{k} = \mathbf{k}$ and m is the identity as well as $\mathscr{A} \circ \varepsilon_{\mathscr{A}}$ and $\varepsilon_{\mathscr{A}}$. So the diagram is commutative. Furthermore, as pointed out in Section 6.1, $\varepsilon_{\mathscr{A}} \cdot e_{\mathscr{A}} = 1$. A similar proof shows that $e_{\mathscr{C}}$ is a grouplike morphism for the comonad $\mathcal{G}_{\mathscr{C}}$. ☐

6.4**. Distributive laws and generalised bialgebras.** Let $\mathscr{A}$ be an operad and $\mathscr{C}$ be a cooperad in $\mathbb{A}$.

(H0) A distributive law between $\mathscr{A}$ and $\mathscr{C}$ is a morphism of $\mathbb{S}$-modules $\mathscr{A} \circ \mathscr{C} \to \mathscr{C} \circ \mathscr{A}$ satisfying some relations (see [8, Section 2.1]) which amount to saying that the corresponding natural transformation

$$\lambda : F_{\mathscr{A} \circ \mathscr{C}} = F_{\mathscr{A}} F_{\mathscr{C}} \longrightarrow F_{\mathscr{C}} F_{\mathscr{A}} = F_{\mathscr{C} \circ \mathscr{A}}$$

is an entwining (see Section 3.1). If such an entwining exists, we say, as in [8], that *Hypothesis* (H0) *is satisfied.* Under this hypothesis, an object (V, h, θ) in $(\mathbb{A}_{\mathcal{T}_{\mathscr{A}}})^{\widehat{\mathcal{G}}_{\mathscr{C}}}$ is called a $(\mathscr{C}, \mathscr{A})$*-bialgebra.*

(H1) Assume that there is a map $\alpha : \mathscr{A} \to \mathscr{C} \circ \mathscr{A}$ making $\mathscr{A}$ a left $\mathscr{C}$-comodule, that is, every free $\mathscr{A}$-algebra is endowed with a structure of a $\mathscr{C}$-coalgebra. This amounts to saying that there is a functor $K : \mathbb{A} \longrightarrow (\mathbb{A}_{\mathcal{T}_{\mathscr{A}}})^{\widehat{\mathcal{G}}_{\mathscr{C}}}$ such that Diagram (3.1) is commutative. If such a functor exists, we say, as in [8], that *Hypothesis* (H1) *is satisfied.* The corresponding left $\mathcal{G}_{\mathscr{C}}$-comodule structure on $\mathcal{T}_{\mathscr{A}}$ is given by $\alpha : F_{\mathscr{A}} \to F_{\mathscr{C}} F_{\mathscr{A}}$.

At the level of $\mathbb{S}$-modules one gets that $\alpha_1 : \mathscr{A}(1) = \mathbf{k} \to (\mathscr{C} \circ \mathscr{A})(1) = \mathbf{k}$ is the identity, because $(\varepsilon_{\mathscr{C}} \circ \mathscr{A}) \cdot \alpha = 1$, so that

$$\alpha \cdot e_{\mathscr{A}} = e_{\mathscr{C}} F_{\mathscr{A}} \cdot e_{\mathscr{A}} = F_{\mathscr{C}} e_{\mathscr{A}} \cdot e_{\mathscr{C}},$$

that is, commutativity of the diagram

$$\begin{tikzcd}
 & F_{\mathscr{A}} \arrow[dr, "\alpha"] & \\
1 \arrow[ur, "e_{\mathscr{A}}"] \arrow[r, "e_{\mathscr{A}}"'] \arrow[dr, "e_{\mathscr{C}}"'] & F_{\mathscr{A}} \arrow[r, "e_{\mathscr{C}}T"'] & F_{\mathscr{C}}F_{\mathscr{A}} \\
 & F_{\mathscr{C}} \arrow[ur, "F_{\mathscr{C}}e_{\mathscr{A}}"'] &
\end{tikzcd} .$$

As a consequence, if $(V,h) \in \mathbb{A}_{\mathcal{T}_{\mathscr{A}}}$, then

$$F_{\mathscr{C}}(h) \cdot \alpha_V \cdot (e_{\mathscr{A}})_V = F_{\mathscr{C}}(h) \cdot (F_{\mathscr{C}}e_{\mathscr{A}})_V \cdot (e_{\mathscr{C}})_V = (e_{\mathscr{C}})_V,$$

and since the (V,h)-component $\beta_{(V,h)}$ of the right $\mathcal{G}_{\mathscr{C}}$-comodule structure on $\beta : U_{\mathcal{T}_{\mathscr{A}}} \to U_{\mathcal{T}_{\mathscr{A}}}\widehat{F_{\mathscr{C}}}$ is just the composite $F_{\mathscr{C}}(h) \cdot \alpha_V \cdot (e_{\mathscr{A}})_V$, we get

$$\beta_{(V,h)} = (e_{\mathscr{C}})_V. \tag{6.2}$$

Thus, β is defined by the grouplike morphism $e_{\mathscr{C}} : 1 \to F_{\mathscr{C}}$ and hence the comparison functor $K : \mathbb{A} \longrightarrow (\mathbb{A}_{\mathcal{T}_{\mathscr{A}}})^{\widehat{\mathcal{G}}_{\mathscr{C}}}$ is induced by this grouplike morphism, i.e., $K = K_{e_{\mathscr{C}}}$ (see Diagram (4.2)). So we can apply the results of the previous sections to the present setting, in particular, Relation (4.1) gives

$$\alpha = \lambda \cdot F_{\mathscr{A}}e_{\mathscr{C}}. \tag{6.3}$$

Now assume the Hypotheses (H0) and (H1) to hold. Consider the $\mathscr{C}$-comodule map $\varphi : \mathscr{A} \to \mathscr{C}$ induced by the projection $\varepsilon_{\mathscr{A}} : \mathscr{A} \to 1$. Since $\varphi = (\mathscr{C} \circ \varepsilon_{\mathscr{A}}) \cdot \alpha$, where $\alpha : \mathscr{A} \to \mathscr{C} \circ \mathscr{A}$ is the $\mathscr{C}$-comodule morphism of Hypothesis (H1), according to Formula (6.1),

$$\alpha(\mu) = \varphi_n(\mu) \otimes 1^{\otimes n} + \sum_{k<n, f\in X_{n,k}} c^{\mu}_{k,f} \otimes \alpha^{\mu}_{1,f} \otimes \cdots \otimes \alpha^{\mu}_{k,f}, \ \forall \mu \in \mathscr{A}(n), \tag{6.4}$$

where φ_n is the component of φ on $\mathscr{A}(n)$, $c^{\mu}_{k,f} \in \mathscr{C}(k)$, and $\alpha^{\mu}_{i,f} \in \mathscr{A}(|f^{-1}(i)|)$.

(H2iso) When φ is an isomorphism, we say, as in [8], that *Hypothesis* (H2iso) *is satisfied.*

In the sequel we will be interested in the link between φ and the comonad morphism $t : \phi_{\mathcal{T}_{\mathscr{A}}} U_{\mathcal{T}_{\mathscr{A}}} \longrightarrow \widehat{F}_{\mathscr{C}}$ as in Section 2.3. Recall that for every module $(V,h) \in \mathbb{A}_{\mathcal{T}_{\mathscr{A}}}$, $t_{(V,h)}$ is the composite

$$F_{\mathscr{A}}(V) \xrightarrow{\alpha_V} F_{\mathscr{C}}F_{\mathscr{A}}(V) \xrightarrow{F_{\mathscr{C}}h} F_{\mathscr{C}}(V).$$

6.5. **Lemma.** *Assume Hypotheses* (H0) *and* (H1) *to hold. Then the map* φ *is an isomorphism if and only if* t *is an isomorphism.*

Proof. We use the natural arity-grading on $\mathbb{S}$-modules. Given $\mu \in \mathscr{A}(n), \underline{v} \in V^{\otimes n}$, one has

$$t_{(V,h)}(\mu \otimes \underline{v}) = \\ \varphi_n(\mu) \otimes \underline{v} + \sum_{k<n, f \in X_{n,k}} c^{\mu}_{k,f} \otimes h(\alpha^{\mu}_{1,f} \otimes \underline{v}^f_1) \otimes \cdots \otimes h(\alpha^{\mu}_{k,f} \otimes \underline{v}^f_k),$$

where $\underline{v}^f = \underline{v}^f_1 \otimes \cdots \otimes \underline{v}^f_k \in V^{\otimes |f^{-1}(1)|} \otimes \cdots \otimes V^{\otimes |f^{-1}(k)|}$ is obtained from $\underline{v}$ by permuting its variables according to the preimages of f. This is a triangular system with dominant coefficient φ_n. As a consequence, we get that if φ is an isomorphism so is $t_{(V,h)}$. The converse is immediate because $\varphi_V = t_{(V,(\varepsilon_{\mathscr{A}})_V)}$ for all $V \in \mathbb{A}$. □

6.6**. The primitive part of a $(\mathscr{C}, \mathscr{A})$-bialgebra.** Since the category of $\mathbf{k}$-vector spaces admits equalisers, under Hypotheses (H0) and (H1), the functor K admits a right adjoint $\overline{K}$ whose value at $((H,h),\theta) \in (\mathbb{A}_{\mathcal{T}_{\mathscr{A}}})^{\widehat{\mathcal{G}_{\mathscr{C}}}}$ appears as the equaliser

$$\overline{K}((H,h),\theta) \xrightarrow{i_{((H,h),\theta)}} H \underset{(e_{\mathscr{C}})_H}{\overset{\theta}{\rightrightarrows}} \mathscr{C}(H)\,.$$

As a consequence,

$$\overline{K}((H,h),\theta) = \{x \in H, \theta(x) = 1 \otimes x\},$$

and thus $\overline{K}((H,h),\theta)$ is just the *primitive part* $\mathrm{Prim}H$ of the $(\mathscr{C}, \mathscr{A})$-bialgebra (H,h,θ) in the sense of [8].

We are now in the position to state and prove our main result.

6.7**. Rigidity Theorem.** ([8, Theorem 2.3.7]) *Let $\mathscr{A}$ be a reduced operad, $\mathscr{C}$ a reduced cooperad, and $\mathcal{T}_{\mathscr{A}} = (F_{\mathscr{A}}, m, e_{\mathscr{A}})$ and $\mathcal{G}_{\mathscr{C}} = (F_{\mathscr{C}}, \delta, \varepsilon_{\mathscr{C}})$ the corresponding monad and comonad on $\mathbb{A}$. Suppose that Hypotheses* (H0), (H1) *and* (H2iso) *are fulfilled. Then the comparison functor*

$$K_{e_{\mathscr{C}}} : \mathbb{A} \longrightarrow (\mathbb{A}_{\mathcal{T}_{\mathscr{A}}})^{\widehat{\mathcal{G}_{\mathscr{C}}}}$$

is an equivalence of categories. Hence, in particular, any $(\mathscr{C}, \mathscr{A})$-bialgebra (H,h,θ) is a free $\mathscr{A}$-algebra and a cofree conilpotent $\mathscr{C}$-coalgebra generated by $\mathrm{Prim}H$.

Proof. Since Hypothesis (H2iso) is satisfied, it follows from Lemma 6.5 that $t_{(V,h)}$ is an isomorphism for all $(V,h) \in \mathbb{A}_{F_{\mathscr{A}}}$. Moreover, since $\varepsilon_{\mathscr{A}} \cdot e_{\mathscr{A}} = 1$ and $\mathbb{A}$ is clearly Cauchy complete, the functor $\phi_{\mathcal{T}_{\mathscr{A}}} : \mathbb{A} \to \mathbb{A}_{\mathcal{T}_{\mathscr{A}}}$ is comonadic by [12, Corollary 3.17]. Applying now Theorem 4.1 yields the result. □

6.8**. Remark.** In [8], for the proof of this theorem J.-L. Loday builds idempotents to produce a projection onto the primitive part. An advantage of our proof is that it does not need such a construction.

The following corollary is a special case of the Rigidity Theorem which does not need verification of Hypothesis (H2iso). Although we assume the operad $\mathscr{A}$ and the cooperad $\mathscr{C}$ to have the same underlying functor, one has to verify that the map $\varphi : \mathscr{A} \to \mathscr{C}$ is an isomorphism. Indeed, Hypothesis (H2iso) implies that the underlying functors of $\mathscr{A}$ and $\mathscr{C}$ are isomorphic, so the latter assumption is weaker than (H2iso).

6.9. **Corollary.** *Let $\mathscr{M}$ be an $\mathbb{S}$-module carrying a structure of an operad $\mathscr{A} = (\mathscr{M}, m, e_{\mathscr{M}})$, a structure of a cooperad $\mathscr{C} = (\mathscr{M}, \delta, \varepsilon_{\mathscr{M}})$, and let*

$$\lambda : \mathscr{M} \circ \mathscr{M} \to \mathscr{M} \circ \mathscr{M}$$

be an entwining between $\mathscr{A}$ and $\mathscr{C}$. If one of the three equivalent conditions

$$\text{(i) } \lambda \textit{ is compatible}, \quad \text{(ii) } \delta = \lambda \cdot (\mathscr{M} \circ e_{\mathscr{M}}), \quad \text{(iii) } m = (\mathscr{M} \circ \varepsilon_{\mathscr{M}}) \cdot \lambda,$$

holds, then the compatible monad-comonad triple $(\mathcal{T}_{\mathscr{A}}, \mathcal{G}_{\mathscr{C}}, \lambda)$ is a Hopf monad. Moreover, any $(\mathscr{C}, \mathscr{A})$-bialgebra is a free $\mathscr{A}$-algebra and a cofree conilpotent $\mathscr{C}$-coalgebra.

Proof. Denote by $\mathscr{H}$ the monad-comonad triple $(\mathcal{T}_{\mathscr{A}}, \mathcal{G}_{\mathscr{C}}, \lambda)$. By Proposition 6.3, the triple satisfies Relations (5.2), and since $e_{\mathscr{M}}$ is a componentwise monomorphism, $\mathscr{H}$ is a bimonad by Proposition 5.7. Thus there is a comparison functor

$$K : \mathbb{A} \to (\mathbb{A}_{\mathcal{T}_{\mathscr{A}}})^{\widehat{\mathcal{G}_{\mathscr{C}}}}, \quad V \longmapsto ((\mathscr{M}(V), m_V), \delta_V),$$

and $K = K_{e_{\mathscr{M}}}$.

By Theorem 5.8, the functor K is an equivalence of categories if and only if the composite

$$\mathscr{M}(V) \xrightarrow{\delta_V} (\mathscr{M} \circ \mathscr{M})(V) \xrightarrow{\mathscr{M}(h)} \mathscr{M}(V)$$

is an isomorphism for every $(V, h) \in \mathbb{A}_{\mathcal{T}_{\mathscr{A}}}$. Now $\mathscr{M}(h) \cdot \delta_V = t_{(V,h)}$, where $t_{(V,h)}$ is the (V, h)-component of the comonad morphism $t : \phi_{\mathcal{T}_{\mathscr{A}}} U_{\mathcal{T}_{\mathscr{A}}} \to \widehat{F_{\mathscr{C}}}$ induced by K. It follows that $\varphi_V = t_{(V,\varepsilon_V)} = \mathscr{M}(\varepsilon_V) \cdot \delta_V = 1$ for every $V \in \mathbb{A}$. Thus φ is an isomorphism and so t is also an isomorphism by Lemma 6.5. Hence K is an equivalence of categories. It now follows from [15, 3.1] that $\mathcal{H}$ is a Hopf monad. Furthermore, the Rigidity Theorem applies to our case because (H2iso) is satisfied. □

6.10. **Example.** As an example consider the case of infinitesimal bialgebras. The functor $V \mapsto \mathscr{A}(V) = \oplus_n V^{\otimes n}$ forms a monad $\mathcal{T} = (\mathscr{A}, m, e)$ for the concatenation product. One can formulate this as

$$\begin{array}{rcccc} m_V : & \mathscr{A}_1\mathscr{A}_2(V) & \to & \mathscr{A}(V) \\ & \otimes_1 & \mapsto & \otimes \\ & \otimes_2 & \mapsto & \otimes \\ & v & \mapsto & v \end{array} \qquad \begin{array}{rccc} e_V : & V & \to & \mathscr{A}(V) \\ & v & \mapsto & v\,, \end{array}$$

where $\mathscr{A}_1$ denotes the "first copy" of $\mathscr{A}$. It reads like this: any word in $\mathscr{A}_1\mathscr{A}_2(V)$ is composed with letters in $\{\otimes_1, \otimes_2, v \in V\}$ and the map indicates how it acts on letters.

The functor $\mathscr{A}$ forms a comonad $\mathcal{G} = (\mathscr{A}, \delta, \varepsilon)$ with the deconcatenation

$$\begin{array}{cccc} \delta_V : & \mathscr{A}(V) & \to & \mathscr{A}_1\mathscr{A}_2(V) \\ & \otimes & \mapsto & \otimes_1 + \otimes_2 \\ & v & \mapsto & v \end{array} \qquad \begin{array}{cccc} \varepsilon_V : & \mathscr{A}(V) & \to & V \\ & \otimes & \mapsto & 0 \\ & v & \mapsto & v\,. \end{array}$$

The infinitesimal distributive law reads

$$\begin{array}{cccc} \lambda_V : & \mathscr{A}_1\mathscr{A}_2(V) & \to & \mathscr{A}_1\mathscr{A}_2(V) \\ & \otimes_1 & \mapsto & \otimes_1 + \otimes_2 \\ & \otimes_2 & \mapsto & \otimes_1 \\ & v & \mapsto & v\,. \end{array}$$

As easily seen, m is associative, δ is coassociative, and λ is an entwining.

As an example, we check one of the diagrams for λ (see Section 3.1),

$$\begin{tikzcd} \mathscr{A}_1\mathscr{A}_2\mathscr{A}_3 \arrow[rr, "m\mathscr{A}"] \arrow[d, "\mathscr{A}\lambda"'] & & \mathscr{A}_1\mathscr{A}_2 \arrow[d, "\lambda"] \\ \mathscr{A}_1\mathscr{A}_2\mathscr{A}_3 \arrow[r, "\lambda\mathscr{A}"'] & \mathscr{A}_1\mathscr{A}_2\mathscr{A}_3 \arrow[r, "\mathscr{A}m"'] & \mathscr{A}_1\mathscr{A}_2\,. \end{tikzcd}$$

The top arrows send $\otimes_1 \mapsto \otimes_1 \mapsto \otimes_1 + \otimes_2$, $\otimes_2 \mapsto \otimes_1 \mapsto \otimes_1 + \otimes_2$ and $\otimes_3 \mapsto \otimes_2 \mapsto \otimes_1$, while the lower maps send $\otimes_1 \mapsto \otimes_1 \mapsto \otimes_1 + \otimes_2 \mapsto \otimes_1 + \otimes_2$, $\otimes_2 \mapsto \otimes_2 + \otimes_3 \mapsto \otimes_1 + \otimes_3 \mapsto \otimes_1 + \otimes_2$ and $\otimes_3 \mapsto \otimes_2 \mapsto \otimes_1 \mapsto \otimes_1$, proving commutativity of this diagram.

We have clearly $\delta = \lambda \cdot (\mathscr{A} \circ e)$ and $m = (\mathscr{A} \circ \varepsilon) \cdot \lambda$ and hence Corollary 6.9 applies. Hereby we recover the Rigidity Theorem of J.-L. Loday and M. Ronco for infinitesimal bialgebras which says that any infinitesimal bialgebra is freely and cofreely generated by its primitive part (see [9, Theorem 2.6]).

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References

[1] Beck, J., *Distributive laws*, in Seminar on Triples and Categorical Homology Theory, B. Eckmann (ed.), Springer LNM **80**, 119–140 (1969).

[2] Brzeziński, T. and Majid, Sh., *Comodule bundles*, Commun. Math. Phys. **191**, No.2, 467–492 (1998).

[3] Gómez-Torrecillas, J., *Comonads and Galois corings*, Appl. Categ. Struct. **14**(5-6), 579–598 (2006).

[4] Dubuc, E., *Adjoint triangles*, Rep. Midwest Category Semin. 2, Springer Lect. Notes Math. **61**, 69–91 (1968).
[5] Fresse, B., *Koszul duality of operads and homology of partition posets*, in "Homotopy theory and its applications (Evanston, 2002)", Contemp. Math. **346**, Amer. Math. Soc., 115–215 (2004).
[6] Fox, T. and Markl, M., *Distributive laws, bialgebras, and cohomology*, in Operads: Proceedings of Renaissance Conferences (Hartford, CT/Luminy, 1995), Contemp. Math. **202**, Amer. Math. Soc., Providence, RI, 167-205 (1997).
[7] Joyal, A., *Foncteurs analytiques et espèces de structures*, in Combinatoire énumérative (Montreal, Que., 1985), Springer LNM **1234**, 126–159 (1986).
[8] Loday, J.-L., *Generalized bialgebras and triples of operads*, Astérisque **320**, Paris, Soc. Math. France (2008).
[9] Loday, J.-L. and Ronco, M., *On the structure of cofree Hopf algebras*, J. Reine Angew. Math. **592**, 123–155 (2006).
[10] Loday, J.-L. and Vallette, B., *Algebraic operads*, Grundlehren Math. Wiss. **346**, Springer, Heidelberg (2012).
[11] Mesablishvili, B., *Entwining structures in monoidal categories*, J. Algebra **319**, 2496–2517 (2008).
[12] Mesablishvili, B., *Monads of effective descent type and comonadicity,* Theory Appl. Categ. **16**, 1–45 (2006).
[13] Mesablishvili, B. and Wisbauer, R., *Galois functors and entwining structures*, J. Algebra **324**, 464–506 (2010).
[14] Mesablishvili, B. and Wisbauer, R., *Bimonads and Hopf monads on categories*, J. K-Theory **7**(2), 349–388 (2011).
[15] Mesablishvili, B. and Wisbauer, R., *Notes on bimonads and Hopf monads*, Theory Appl. Categ. **26**, 281–303 (2012).
[16] Turi, D. and Plotkin, G., *Towards a mathematical operational Semantics*, Proceedings 12th Ann. IEEE Symp. on Logic in Computer Science, LICS'97, Warsaw, Poland (1997).
[17] Wisbauer, R., *Algebras versus coalgebras*, Appl. Categor. Struct. **16**(1-2), 255–295 (2008).
[18] Wolff, H., *V-Localizations and V-monads*, J. of Algebra **24**, 405–438 (1973).

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