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Generalized Co-Semisimple Modules

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We call an R-module M *generalized co-semisimple* (*GCO-module*) if every singular simple R-module is M-injective or M-projective. The generalized V-modules of Hirano [12] are special cases of GCO-modules. For ${}_RR$ the two notions coincide and define the GV-rings of Ramamurthi-Rangaswamy [15]. An s-unital ring is a left GCO-module exactly if it is a left V'-ring in the sense of Tominaga [19].

Let $\sigma[M]$ denote the category of R-modules whose objects are submodules of M-generated modules. A module is *M-singular* if it is of the form L/K with L in $\sigma[M]$ and $K \trianglelefteq L$. After recording general properties of these modules in section 1 we characterize GCO-modules M in section 2 by the condition that every simple M-singular module is M-injective or, equivalently, $M/Soc\, M$ is co-semisimple and every simple M-singular submodule of M is a direct summand. A whole list of characterizing properties of GCO-modules is given (in 2.2, 2.3) which extends corresponding results for GV-modules in [12], GV-rings in [15] and left s-unital V'-rings in [19].

A finitely generated self-projective module M over a commutative ring is a GCO-modules if it is co-semisimple or regular in $\sigma[M]$ or, equivalently, if its endomorphism ring has the corresponding property (see 2.4).

In section 3 we are concerned with finiteness conditions on GCO-modules. We obtain that a GCO-module M has *acc* on essential submodules if and only if for every essential submodule $K \subset M$ the factor module M/K has finite uniform dimension (see 3.2). This implies related results on GV-modules in [29] and [22]. Self-projective GCO-modules M with *acc* on essential submodules are characterized by the facts that M has no M-singular submodules and that $M/Soc\, M$ is noetherian and co-semisimple (see 3.5).

Special cases of GCO-modules with *acc* on essential submodules are modules M for which *every self-injective M-singular module is M-injective* or for which *every M-singular module is M-injective.* Finitely generated self-projective modules of the latter type are ***hereditary in*** $\sigma[M]$ (see 3.6, 3.8 and 3.10).

Finally we characterize (in 3.11) finitely generated self-projective GCO-modules M with *dcc* on essential submodules by the properties that $M/SocM$ *is semisimple and* $Soc\, M$ *is M-projective* (or M *has no M-singular submodule*).

For $M = R$ the last mentioned results yield assertions on QI-rings and SI-rings (e.g. Faith [7], Goodearl [10]).

We close with some remarks about the connection of our techniques with torsion theoretic generalizations of V-rings as considered in Takehana [18], Varadarajan [23], Ahsan-Enochs [2] and Page-Yousif [14].

1 M-singular modules

Let R be an associative ring with unity and *R-Mod* the category of unital left R-modules. For $M \in$ *R-Mod* we denote by $\sigma[M]$ the full subcategory of *R-Mod* whose objects are submodules of M-generated modules.

Soc M and *Rad M* denote the *socle* and the *radical* of the module M respectively. If $K \subset M$ is a proper essential submodule we write $K \trianglelefteq M$. The kernel of a homomorphism f is denoted by $Ke\, f$. Morphisms are written on the opposite side of the scalars. For basic definitions see [1, 27].

The R-module M is called *co-semisimple* or *V-module*, if every simple module (in $\sigma[M]$) is M-injective ([8, 19, 24]). It is known that submodules, factor modules and direct sums of co-semisimple modules are again co-semisimple (e.g. [27, § 23]).

Let M and N be R-modules. N is called *singular in* $\sigma[M]$ or *M-singular* if $N \simeq L/K$ for an $L \in \sigma[M]$ and $K \trianglelefteq L$ (see [26]).

By definition every M-singular module belongs to $\sigma[M]$. For $M = R$ the notion *R-singular* is identical to the usual definition of *singular* for modules.

Every M-singular module is of course R-singular but R-singular modules need not be M-singular (see [26, § 2]).

It is elementary to see that the class of all M-singular modules is closed under submodules, homomorphic images and direct sums (e.g. [11, 1.1], [27, 17.3]). Hence every module $N \in \sigma[M]$ contains a *largest M-singular submodule* which we denote by $Z_M(N)$. In our notation $Z(N) = Z_R(N)$ is just the largest singular submodule of N and $Z_M(N) \subset Z(N)$. For a non-projective simple R-module M we always have $Z(M) = M$ but $Z_M(M) = 0$.

The following basic observations on M-singular modules will be useful:

1.1 Proposition. *Let M be an R-module.*

(1) A simple R-module E is M-singular or M-projective.

(2) If $Z_M(M) \cap Soc\, M = 0$ then $Soc\, M$ is projective in $\sigma[M]$.

(3) If $Z_M(M) \cap Rad\, M = 0$ then every M-singular simple submodule is a direct summand of M.

Proof: (1) A simple module which does not belong to $\sigma[M]$ is trivially M-projective. Assume the simple module $E \in \sigma[M]$ is not M-singular and consider an exact sequence

$$0 \longrightarrow K \longrightarrow L \longrightarrow E \longrightarrow 0$$

in $\sigma[M]$. By assumption the maximal submodule $K \subset L$ is not essential and hence is a direct summand in L, i.e. the sequence splits and E is a projective object in $\sigma[M]$.

(2) is an immediate consequence of (1).

(3) (see [12, 3.15]) Let E be an M-singular simple submodule in M. By hypothesis $E \not\subset Rad\, M$ and hence there is a maximal submodule $L \subset M$ with $E \cap L = 0$. This implies that E is a direct summand.

1.2 Proposition. *Let M be an R-module.*

(1) Every M-singular module is submodule of an M-generated M-singular module.

(2) Every finitely generated M-singular module belongs to $\sigma[M/L]$ for some $L \trianglelefteq M$.

(3) $\{M/K \mid K \trianglelefteq M\}$ is a generating set for the M-generated M-singular modules.

Proof: (1) Consider $L \in \sigma[M]$ and $K \trianglelefteq L$. The M-injective hull $\hat{L}$ of L is M-generated and

$$L/K \subset \hat{L}/K, \quad K \trianglelefteq L \trianglelefteq \hat{L}.$$

(2) A finitely generated M-singular module is of the form N/K with a finitely generated $N \in \sigma[M]$ and $K \trianglelefteq N$. N is an essential submodule of a finitely M-generated module $\widetilde{N}$, i.e. we have an epimorphism $\varphi : M^k \to \widetilde{N}$, $k \in \mathbb{N}$ (compare (1)), and $U := (N)\varphi^{-1}$ and $V := (K)\varphi^{-1}$ are essential submodules of M^k.

With the canonical inclusions $\varepsilon_i : M \to M^k$ we get that $L := \bigcap_{i \leq k}(V)\varepsilon^{-1}$ is an essential submodule of M and L^k lies in the kernel of the composed map

$$U \xrightarrow{\varphi} N \longrightarrow N/K.$$

This implies $N/K \in \sigma[M/L]$.

(3) is an immediate consequence of (2).

1.3 Proposition. *Let M be an R-module.*

(1) For every module $N \in \sigma[M]$ we have $N/Soc\,N \in \sigma[M/Soc\,M]$.

(2) Every (simple) M-singular R-module belongs to $\sigma[M/Soc\,M]$.

(3) Every simple module in $\sigma[M/Soc\,M]$ is M-singular.

Proof: (1) Let $\widehat{N}$ denote the M-injective hull of $N \in \sigma[M]$. Since $\widehat{N}$ is M-generated there is an epimorphism $\varphi : M^{(\Lambda)} \to \widehat{N}$ for a suitable index set Λ. Obviously $L := (N)\varphi^{-1}$ is an essential submodule of $M^{(\Lambda)}$ and hence $Soc\,M^{(\Lambda)} \subset L$. The kernel of the composed map

$$L \xrightarrow{\varphi} N \longrightarrow N/Soc\,N$$

contains the socle of $M^{(\Lambda)}$ and this implies $N/Soc\,N \in \sigma[M/Soc\,M]$.

(2) Consider an M-singular module N/K with $K \trianglelefteq N$ and $N \in \sigma[M]$. Then $Soc\,N \subset K$ and N/K is a factor module of $N/Soc\,N$ which belongs to $\sigma[M/Soc\,M]$ by (1).

(3) Since every simple module E in $\sigma[M/Soc\,M]$ is a factor module of an essential submodule of $M/Soc\,M$ we find an essential submodule $N \subset M$ with an epimorphism

$$\varphi : \quad N \longrightarrow N/Soc\,M \longrightarrow E.$$

In case E is not M-singular φ splits and $N \simeq E' \oplus Ke\,\varphi$. But $E' \simeq E$ is in the socle of N and hence also in $Ke\,\varphi$, a contradiction.

1.4 Proposition. *Let M be an R-module, N an M-singular module and $f \in Hom(M,N)$.*

(1) If M is self-projective and $(M)f$ finitely generated then $Ke\,f \trianglelefteq M$.

(2) If M is projective in $\sigma[M]$ then $Ke\,f \trianglelefteq M$.

Proof: (1) Under the given conditions we may assume $(M)f = L/K$ with $L \in \sigma[M]$ finitely generated and $K \trianglelefteq L$. Since M is self-projective it is also L-projective and the diagram with the canonical projection p

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow_f & & \\ L & \xrightarrow{p} & L/K & \longrightarrow & 0 \end{array}$$

can be completed to a commutative diagram by $g : M \to L$.
Then $Ke\,f = (K)g^{-1}$ is essential in M.

(2) The arguments in (1) apply without finiteness condition.

2 Generalized co-semisimple modules

We call an R-module M *generalized co-semisimple* or *GCO-module* if every singular simple R-module is M-injective or M-projective.

M is a *generalized V-module* or *GV-module* in the sense of Hirano [12] if every singular simple module is M-injective. Obviously, GV-modules are also GCO-modules. The definition of GCO-modules only refers to properties with respect to the module M whereas the definition of GV-modules also refers to the ring R (*R-singular*). After 2.2 we will give an example of a GCO-module which is not a GV-module.

Since a singular simple R-module cannot be R-projective a ring R is a *left GV-ring*, i.e. ${}_RR$ is a GV-module (see [15, 12]), if and only if ${}_RR$ is a GCO-module.

The equivalence of the first three conditions in our next result was shown in Yousif [29, Lemma 4]. The remaining assertions easily follow from 1.3:

2.1 $M/Soc\, M$ **co-semisimple.**

For an R-module M the following conditions are equivalent:

(a) $M/Soc\, M$ *is co-semisimple;*

(b) M/K *is co-semisimple for every* $K \trianglelefteq M$*;*

(c) *every* $K \trianglelefteq M$ *is an intersection of maximal submodules;*

(d) *for every* $N \in \sigma[M]$ *the module* $N/Soc\, N$ *is co-semisimple;*

(e) *every M-singular simple module is* $M/Soc\, M$*-injective;*

(f) *every M-singular module is co-semisimple.*

For $M = R$ the equivalence $(a) \Leftrightarrow (d)$ yields Proposition 2.1 in Baccella [4].

2.2 Characterization of GCO-modules.

For an R-module M the following conditions are equivalent:

(a) *M is a GCO-module;*

(b) *every M-singular simple module is M-injective;*

(c) *for every module* $N \in \sigma[M]$ *we have* $Z_M(N) \cap Rad\, N = 0$*;*

(d) *for every simple module* $E \in \sigma[M]$ *with M-injective hull* $\widehat{E}$ *we have* $Z_M(\widehat{E}) \cap Rad(\widehat{E}) = 0$*;*

(e) M/K *is co-semisimple for every* $K \trianglelefteq M$ *and* $Z_M(M) \cap Rad\, M = 0$*;*

(f) $M/Soc\, M$ *is co-semisimple and* $Z_M(M) \cap Rad\, M = 0$*;*

(g) $M/Soc\, M$ *is co-semisimple and every M-singular simple submodule of M is a direct summand;*

(h) $M/Soc\, M$ *is co-semisimple and every finitely generated submodule of* $Z_M(M) \cap Soc\, M$ *is a direct summand in M;*

(i) *every module in* $\sigma[M]$ *is a GCO-module.*

Proof: $(a) \Rightarrow (b)$ A simple singular module in $\sigma[M]$ cannot be M-projective and hence has to be M-injective by (a).

$(b) \Rightarrow (a)$ Let E be a simple singular R-module. If E is M-singular then it is M-injective by (b). Otherwise it is M-projective according to 1.1.

$(b) \Rightarrow (c)$ (compare [12, 3.15], [23, 2.1]): For a module $N \in \sigma[M]$ assume $0 \neq m \in Z_M(N) \cap Rad\, N$. By Zorn's Lemma there is a submodule $L \subset N$ which is maximal with respect to $m \notin L$. Then $(Rm + L)/L \trianglelefteq N/L$. Since $(Rm + L)/L$ is an M-singular simple module it is M-injective by (b) and hence isomorphic to N/L. This implies that L is a maximal submodule of N with $m \notin L$ which contradicts the choice of m.

$(c) \Rightarrow (d)$ is trivial.

$(d) \Rightarrow (b)$ Let E be a simple module with M-injective hull $\widehat{E}$ in $\sigma[M]$ and $Z_M(\widehat{E}) \cap Rad(\widehat{E}) = 0$. If E is M-singular then $Z_M(\widehat{E}) \neq 0$. Since $\widehat{E}$ is uniform this implies $Rad(\widehat{E}) = 0$ and $\widehat{E}$ is cogenerated by simple modules, i.e. $E = \widehat{E}$ and E is M-injective.

$(b) \Rightarrow (e)$ If $K \trianglelefteq M$ then every simple module in $\sigma[M/K]$ is M-singular and hence M-injective and M/K-injective, i.e. M/K is co-semisimple. The condition $Z_M(M) \cap Rad\, M = 0$ was shown in $(b) \Rightarrow (c)$.

$(e) \Leftrightarrow (f)$ is a consequence of 2.1. $(f) \Rightarrow (g)$ is shown in 1.1.

$(g) \Rightarrow (b)$ (see [12, 3.15]) Consider the diagram with exact line

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \longrightarrow & M \\ & & \downarrow_f & & \\ & & E & & \end{array}$$

with an M-singular simple E and $L \trianglelefteq M$. Set $K := Ke\, f$.

If K is not essential in L then $L \simeq E' \oplus K$ where E' ($\simeq E$) is an M-singular simple submodule and hence a direct summand of M. This yields the desired commutative extension of the above diagram.

If $K \trianglelefteq L$ then $K \trianglelefteq M$. Since M/K is co-semisimple (see 2.1) the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & L/K & \longrightarrow & M/K \\ & & \downarrow_f & & \\ & & E & & \end{array}$$

can be completed in the desired way.

$(b) \Rightarrow (h) \Rightarrow (g)$ is readily verified.

$(i) \Leftrightarrow (a)$ For every $N \in \sigma[M]$ the N-singular modules in $\sigma[N]$ are also M-singular and hence the assertion is clear.

Example: *GCO-modules need not be GV-modules.*
Let S be a left GV-ring which is not a left V-ring (see [4] for examples) and R the ring of lower triangular $(2,2)$-matrices. Then the map

$$R = \begin{pmatrix} S & 0 \\ S & S \end{pmatrix} \longrightarrow S, \qquad \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \longmapsto a,$$

is a surjective ring homomorphism whose kernel is essential as left ideal in R. Hence every (simple) S-module is singular as R-module and all S-singular modules are S-injective, i.e. S is a GCO-module over R. Since not every R-singular simple module is S-injective (S is not a left V-ring) S is not a GV-module over R.

For self-projective modules the properties (g) and (h) in 2.2 can be expressed differently:

2.3 Self-projective GCO-modules.

For a self-projective R-module M the following assertions are equivalent:

(a) *M is a GCO-module;*

(b) *$M/Soc\, M$ is co-semisimple and $Soc\, M$ is M-projective;*

(c) *$M/Soc\, M$ is co-semisimple and $Z_M(M) \cap Soc\, M = 0$.*

Proof: (a) $\Rightarrow$ (c) By 2.2 (h) every finitely generated submodule of $Z_M(M) \cap Soc\, M$ is M-projective and M-singular and hence zero.

(c) $\Leftrightarrow$ (b) follows from 1.1.

(c) $\Rightarrow$ (a) is a special case of 2.2.

Our Theorem 2.2 extends the characterizations of GV-rings in [15, Theorem 3.3] and [23, Theorem 4.2] as well as the characterization of GV-modules in

[12, Theorem 3.15] to GCO-modules. For $M = R$ the description of self-projective GCO-modules in 2.3 yields Theorem 2.2 in [4].

It was pointed out after Corollary 4 in [21] that the left s-unital rings T (without unity) of Tominaga [19] are self-generators as left modules and the category of all s-unital left T-modules is just $\sigma[_TT]$. Now it is easy to see that a left s-unital ring T is a *left V'-ring* in the sense of [19] if and only if $_TT$ is a GCO-module. Hence 2.2 implies and extends the characterizations of these rings in Theorem 4 of [19].

We already observed that for a ring R the module $_RR$ is a GV-module exactly if it is a GCO-module. Hence a commutative ring R is (von Neumann) regular if $_RR$ is a GCO-module (e.g. [15, Theorem 3.6]). This can be extended to certain modules over commutative rings and their endomorphism rings.

Recall the a finitely generated self-projective module M is *regular in* $\sigma[M]$ if every finitely generated submodule is a direct summand in M (e.g. [27, 37.4]).

2.4 Self-projective modules over commutative rings.
For a finitely generated and self-projective module M over a commutative ring R and $S = End(M)$ the subsequent assertions are equivalent:

(a) M is a GCO-module;

(b) M is co-semisimple;

(c) M is regular in $\sigma[M]$;

(d) ${}_S S$ is a GCO-module;

(e) ${}_S S$ is co-semisimple;

(f) S is (von Neumann) regular;

(g) $\overline{R} = R/An(M)$ is (von Neumann) regular.

Proof: Under the given conditions $\sigma[M] = \overline{R}$-*Mod*, M is a projective generator in $\overline{R}$-*Mod*, and the functor $Hom(M, -) : \sigma[M] \to S$-*Mod* is an equivalence. Hence the assertions are readily verified (see [27, 37.11], [24, Theorem 1.8]).

The equivalences in 2.4 can be applied to establish a similar result for *locally projective* modules as given in [12, Theorem 4.8].

3 GCO-modules with finiteness conditions

Recall that a module is *locally noetherian* if its finitely generated submodules are noetherian.

3.1 $M/Soc\, M$ locally noetherian and co-semisimple.
For an R-module M the following conditions are equivalent:

(a) $M/Soc\, M$ is co-semisimple and locally noetherian;

(b) every M-singular module in $\sigma[M]$ is co-semisimple and locally noetherian;

(c) every M-singular semisimple module in $\sigma[M]$ is $M/Soc\, M$-injective.

Proof: $(a) \Rightarrow (b)$ follows immediately from 1.3.

$(b) \Rightarrow (a)$ Consider a finitely generated submodule $N/Soc M \subset M/Soc M$ (with $Soc M \subset N \subset M$). Then for every $K \trianglelefteq N$ the factor module N/K is noetherian and co-semisimple according to (b) and we know from [20, Lemma 2] that $N/Soc M$ is noetherian. Hence $M/Soc M$ is co-semisimple and locally noetherian.

$(a) \Rightarrow (c)$ The simple M-singular modules are $M/Soc M$-injective by 2.1. Since $M/Soc M$ is locally noetherian the direct sum of of these modules is also $M/Soc M$-injective.

$(c) \Rightarrow (b)$ We obtain from (c) that every M-singular module $N \in \sigma[M]$ is co-semisimple and every semisimple module in $\sigma[N]$ is N-injective. This implies that every finitely generated M-singular module is noetherian.

3.2 GCO-modules with acc on essential submodules.
For an R-module M the following conditions are equivalent:

(a) M is a GCO-module with acc on essential submodules;

(b) M is a GCO-module and M/K has finite uniform dimension for every $K \trianglelefteq M$;

(c) $M/Soc M$ is co-semisimple noetherian and $Z_M(M) \cap Rad M = 0$.

Proof: $(a) \Leftrightarrow (b)$ is obtained with the same proof as Theorem 2 in [22].
$(a) \Leftrightarrow (c)$ This equivalence follows from 2.2.

The modules in the preceding theorem are obviously noetherian if they have finitely generated socles. Applying 3.2 we can extend the characterization of noetherian GV-modules in [22, Corollary 3] to GCO-modules:

3.3 Noetherian GCO-modules.
For a GCO-module M the following assertions are equivalent:

(a) M is noetherian;

(b) M has Krull dimension;

(c) every factor module of M has finite uniform dimension;

(d) M has acc on essential submodules and $Soc M$ is finitely generated.
In this case $Z_M(M) \cap Soc M$ is a direct summand in M.

The next result is motivated by Lemma 2.13 in Page-Yousif [14]:

3.4 Proposition. *Let M be a self-projective R-module with $M/Soc\,M$ finitely generated. Assume that $Z_M(M) \cap Rad\,M = 0$ and that M satisfies acc on essential kernels of endomorphisms. Then $Z_M(M) = 0$.*

Proof: Consider $f \in Hom(M, Z_M(M))$. We know from 1.1 that the simple M-singular submodules of M are direct summands and hence

$Z_M(M) \cap Soc\,M = 0$ and $Soc\,M \subset Ke\,f$.

This implies that $(M)f$ is finitely generated and by 1.4 $Ke\,f \trianglelefteq M$ and

$$Ke\,f \subset Ke\,f^2 \subset Ke\,f^3 \subset \cdots$$

is an ascending chain of essential kernels of endomorphisms. By our chain condition we find a $k \in I\!N$ with $Ke\,f^k = Ke\,f^{2k}$. Assume $f^k \neq 0$. Then there is an $m \in M$ with $0 \neq (m)f^k \in Ke\,f^k$. This implies $(m)f^{2k} = 0$ and hence $(m)f^k = 0$, a contradiction. Therefore $Hom(M, Z_M(M))$ is a nil left ideal in $End(M)$ and hence contained in the Jacobson radical of $End(M)$, i.e. (see [27, 22.2])

$$Hom(M, Z_M(M)) \subset Jac(End(M)) \subset Hom(M, Rad\,M).$$

By our assumptions this means $Hom(M, Z_M(M)) = 0$.

$Z_M(M) \cap Rad\,M = 0$ also tells us that $Z_M(M)$ is not a small submodule of M if it is not zero. In this case there is a non-trivial submodule $L \subset M$ with $Z_M(M) + L = M$ and we get an epimorphism $Z_M(M) \to M/L$. Now the self-projectivity of M implies that $Hom(M, Z_M(M)) \neq 0$ which contradicts our above observation.

Our next theorem extends the characterization of left GV-rings with *acc* on essential left ideals in [14, Corollary 2.16] from rings to self-projective modules:

3.5 Self-projective GCO-modules with acc on essentials.
For a self-projective R-module M the following assertions are equivalent:

(a) M is a GCO-module and $M/Soc\,M$ is locally noetherian;

(b) $M/SocM$ is co-semisimple, locally noetherian and $Z_M(M) \cap SocM = 0$ (or: $Soc\,M$ is M-projective);

(c) every M-singular semisimple module is M-injective.

If $M/Soc\,M$ is finitely generated, then there are also equivalent:

(d) M is a GCO-module with acc on essential submodules;

(e) $M/Soc\,M$ is co-semisimple noetherian and $Z_M(M) = 0$.

In this case the endomorphism ring of the M-injective hull $\widehat{M}$ of M is (von Neumann) regular and left self-injective.

Proof: $(a) \Leftrightarrow (b)$ This is evident from 2.3.

$(c) \Rightarrow (a)$ Of course, (c) implies that M is a GCO-module. The finiteness condition is obtained in 3.1.

$(b) \Rightarrow (c)$ We know from 3.1 that every M-singular semisimple module F is $M/Soc\,M$-injective. We have to show that F is even M-injective:

For an essential submodule $K \trianglelefteq M$ let $f : K \to F$ be a homomorphism. Then $Ke\,f$ is an essential submodule of K: Assume there is a non-zero $L \subset K$ with $L \cap Ke\,f = 0$. Then the restriction map $f|_L : L \to F$ is a monomorphism and hence $L \subset Z_M(M) \cap Soc\,M = 0$, a contradiction. Hence $Soc\,M \subset Ke\,f$ and we get the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & K/Soc\,M & \longrightarrow & M/Soc\,M \\ & & \downarrow_f & & \\ & & F & & \end{array}$$

which can be completed in the desired way by 3.1.

$(a) \Leftrightarrow (d)$ Since $M/Soc\,M$ is finitely generated this assertion follows from [20, Lemma 2].

$(b) \Leftrightarrow (e)$ is a consequence of 3.4.

For modules M with $Z_M(M) = 0$ it was shown in Theorem 3.4 of [26] that $End(\widehat{M})$ has the indicated properties.

A ring R for which all self-injective modules are R-injective is called *QI-ring* (in [5]). We call an R-module M a *QI-module* if every self-injective module in $\sigma[M]$ is M-injective.

Evidently, a QI-module is locally noetherian and co-semisimple.

The TQI-rings in [2] can be described as rings whose singular self-injective modules are R-injective (see [14, Corollary 3.3]). Generalizing this notion we look at modules M for which *all self-injective M-singular modules are M-injective.* These are obviously special GCO-modules for which $M/Soc\,M$ is locally noetherian. Our techniques provide straightforward proofs for characterizing properties of these modules:

3.6 Self-injective M-singular modules are M-injective.

Let M be a self-projective R-module and $M/Soc\,M$ finitely generated. Then the following assertions are equivalent:

(a) every self-injective M-singular module is M-injective;

(b) M/K is a QI-module for every $K \trianglelefteq M$ and $Z_M(M) = 0$;

(c) $Z_M(M) = 0$ and $M/Soc\,M$ is a QI-module.

Proof: $(a) \Rightarrow (b)$ If $K \trianglelefteq M$ then every self-injective module in $\sigma[M/K]$ is M-singular and hence M-injective. Also, every M-singular semisimple module is M-injective and hence $Z_M(M) = 0$ by 3.5.

$(b) \Rightarrow (c)$ $\overline{M} = M/Soc\,M$ is self-projective and noetherian co-semisimple by 3.2. Hence it is a generator in $\sigma[\overline{M}]$ and this category is equivalent to the category *S-Mod* with $S = End(\overline{M})$.

Condition (b) also implies that every $\overline{M}$-singular self-injective module in $\sigma[\overline{M}]$ is $\overline{M}$-injective and therefore the ring S is left noetherian co-semisimple (e.g. [27, 23.8]) and every singular self-injective left S-module is S-injective. By Corollary 9 in [6] this implies that S is a left QI-ring. Again referring to the equivalence between $\sigma[\overline{M}]$ and *S-Mod* we deduce that $\overline{M}$ is a QI-module.

$(c) \Rightarrow (a)$ Since every M-singular self-injective module $N \in \sigma[M]$ belongs to $\sigma[M/Soc\,M]$ it is clear that N is $M/Soc\,M$-injective.

Applying the condition $Z_M(M) = 0$ we can use the proof $(b) \Rightarrow (c)$ in 3.5 to show that N is also M-injective.

Considering further finiteness conditions we get as a special case of 3.1:

3.7 Every M-singular module semisimple.

The following conditions are equivalent for an R-module M:

(a) every M-singular module is semisimple;

(b) every (cyclic) M-singular module is $M/Soc\,M$-injective;

(c) M/K is semisimple for every $K \trianglelefteq M$;

(d) $M/SocM$ is co-semisimple and locally noetherian and $Soc(M/K) \neq 0$ for every $K \trianglelefteq M$.

If $M/Soc\,M$ is finitely generated then (a)-(d) are also equivalent to:

(e) $M/SocM$ is co-semisimple and M/K is finitely cogenerated for every $K \trianglelefteq M$.

Proof: $(a) \Rightarrow (b)$ Applying 3.1 we get from (a) that every M-singular semisimple module is $M/Soc\,M$-injective.

$(b) \Rightarrow (c)$ Assume that every cyclic M-singular module is $M/Soc\,M$-injective and $K \trianglelefteq M$. Then M/K has the property that every quotient of a cyclic submodule is M/K-injective. By Corollary 2 of Osofsky-Smith [13] this implies that M/K is semisimple.

$(c) \Rightarrow (a)$ We know from 1.3 that every finitely generated M-singular module is contained in $\sigma[M/K]$ for a suitable $K \trianglelefteq M$ and hence is semisimple by (c).

$(b) \Rightarrow (d)$ is a consequence of 3.1 and the equivalences already shown.

$(d) \Rightarrow (c)$ Since M-singular semisimple modules are $M/Soc\,M$-injective by 3.1 it is easily seen from (d) that M/K has an essential and M/K-injective socle, i.e. it is semisimple.

$(c) \Leftrightarrow (e)$ is obvious by 2.1.

We now want to see how self-projective modules of the above type are related to GCO-modules. This leads us to a generalization of the description of SI-rings in Corollary 2.20 of [14]:

3.8 Every M-singular module M-injective.
For a self-projective R-module M the following conditions are equivalent:

(a) every M-singular module is semisimple, $Z_M(M) = 0$ and $M/Soc\,M$ is finitely generated;

(b) every (cyclic) M-singular module is M-injective and $M/Soc\,M$ is finitely generated;

(c) M/K is finitely generated semisimple for every $K \trianglelefteq M$ and $Z_M(M) = 0$;

(d) M is a GCO-module with acc on essential submodules and $Soc(M/K) \neq 0$ for every $K \trianglelefteq M$;

(e) M is a GCO-module and M/K is finitely cogenerated for every $K \trianglelefteq M$;

(f) $M/Soc\,M$ is a finitely generated QI-module, $Z_M(M) = 0$ and $Soc(M/K) \neq 0$ for every $K \trianglelefteq M$.

Proof: The equivalences from (a) to (e) are readily obtained combining the assertions in 3.6 and 3.7.

$(b) \Rightarrow (f) \Rightarrow (d)$ is easily derived from 3.6 and 3.5.

3.9 Self-projective GCO-modules with $M/Rad\, M$ semisimple.
For a self-projective R-module M with $M/Rad\, M$ semisimple the subsequent properties are equivalent:

(a) M is a GCO-module;

(b) every (cyclic) M-singular module is M-injective.

Proof: $(b) \Rightarrow (a)$ is trivial (see 3.7).

$(a) \Rightarrow (b)$ We know from 2.2 that $Z_M(M) \cap Rad\, M = 0$. Since $M/Rad\, M$ is semisimple this implies that $Z_M(M)$ is semisimple. By 2.2 the simple summands of $Z_M(M)$ are direct summands in M and hence M-projective, i.e. $Z_M(M) = 0$.

For every $K \trianglelefteq M$ we have $Rad\, M/K = 0$. Hence M/K is generated by $M/Rad\, M$ and therefore semisimple. Now we know from 3.7 that every M-singular module is $M/Soc\, M$-injective. Since $Z_M(M) = 0$ we can conclude as in $(b) \Rightarrow (c)$ of 3.5 that the M-singular modules are even M-injective.

A corresponding result for locally projective GV-modules was shown (with a similar proof) in Proposition 2.9 of Yousif [28]. For $M = R$ the above result implies that semilocal TQI-rings are SI-rings, a refinement of Theorem 4 in Ahsan-Enochs [2].

An R-module M is called ***hereditary in*** $\sigma[M]$ if every submodule of M is projective in $\sigma[M]$. It was shown in Satz 2.6 of [25] that M is hereditary in $\sigma[M]$ if and only if every factor module of an M-injective module in $\sigma[M]$ is again M-injective (also [27, 39.6]). This definition generalizes the ***hereditary modules*** of Shrikhande [17] which are R-modules whose submodules are projective in R-*Mod*.

According to Yousif [28] ***SI-modules M*** are defined by the property that all singular left R-modules are M-injective. Since M-singular modules are singular these modules have the property that all M-singular modules are M-injective (compare 3.8). Hence 3.8 is an extension of the description of locally projective SI-modules in [28, Proposition 2.4].

It was proved in Proposition 1.11 of [14] that projective SI-modules are hereditary in the sense of Shrikhande. Essentially the same proof yields a more general result:

3.10 Proposition.

Let M be an R-module which is projective in $\sigma[M]$ and assume that all M-singular modules are M-injective. Then M is hereditary in $\sigma[M]$.

Proof: (compare [14, 1.11]) Let L be an M-injective module in $\sigma[M]$ and $N \subset L$. There is an M-injective hull $\widehat{N}$ of N in L and we have $L = \widehat{N} \oplus K$ for an M-injective submodule $K \subset L$. Since $L/N \simeq \widehat{N}/N \oplus K$ and the M-singular module $\widehat{N}/N$ is M-injective by assumptions we see that L/N is also M-injective. Hence M is hereditary in $\sigma[M]$ by our remark above.

This Proposition tells us that finitely generated modules M of the type considered in 3.8 are hereditary in $\sigma[M]$. As a very special case, every finitely generated, self-projective QI-module M for which $Soc\, M/K \neq 0$ for $K \trianglelefteq M$ is hereditary in $\sigma[M]$. For $M = R$ this means that a left QI-ring with the *restricted left socle condition* is left hereditary. This was proved in Theorem 18 of Faith [7].

Finally we want to consider the *descending chain condition (dcc)* on essential submodules in connection with GCO-modules.

3.11 GCO-modules with dcc on essential submodules.

For an R-module M with $M/Soc\, M$ finitely generated the subsequent conditions are equivalent:

(a) M is a GCO-module with dcc on essential submodules;

(b) $M/Soc\, M$ is semisimple and $Z_M(M) \cap Rad\, M = 0$.

If M is self-projective then (a) and (c) are are also equivalent to:

(c) every (cyclic) M-singular module is M-injective and $Soc\, M \trianglelefteq M$;

(d) $M/Soc\, M$ is semisimple and $Z_M(M) = 0$;

(e) $M/Soc\, M$ is semisimple and $Soc\, M$ is M-projective.

Proof: $(a) \Leftrightarrow (b)$ It was observed in [3, Proposition 2] that a module M has *dcc* on essential submodules if and only if $M/Soc\, M$ is artinian.

The other implications are a straightforward application of 3.5 and 3.8.

For $M = R$ we obtain from the above results the characterization of SI-rings with essential socle in [4, Lemma 2.6].

Remarks: Let $\mathcal{T}$ be a hereditary (pre-)torsion class in *R-Mod.* Following Takehana [18] we may call R a *left* $\mathcal{T}$-V-*ring* if every simple $\mathcal{T}$-torsion module is $\mathcal{T}$-injective. Since every hereditary pretorsion class $\mathcal{T}$ is of the form $\sigma[N]$ for a suitable R-module N the ring R is a $\mathcal{T}$-V-ring exactly if this N is co-semisimple. This was observed in the introduction of Garcia Hernández-Gómez Pardo [9].

In Varadarajan [23] left $\mathcal{T}$-V-rings are studied for *stable* hereditary torsion classes $\mathcal{T}$ (i.e. $\mathcal{T}$ is closed under R-injective hulls). In this case $\mathcal{T}$-*injective* implies R-*injective* (see [23, Remark 1.1]) and hence $\mathcal{T}$-V-rings are just the rings R for which simple $\mathcal{T}$-torsion modules are R-injective.

As a special case of this setting in Ahsan-Enochs [2] rings are investigated for which the self-injective modules in the *Goldie torsion class* are R-injective. We extend some of their results in 3.6 and 3.9.

Investigations in Page-Yousif [14] can also be interpreted in this setting. For example, in Proposition 2.15 they describe rings for which semisimple modules in a certain hereditary pretorsion class $\mathcal{T}$ are $\mathcal{T}$-injective. Their condition that the ideal in R which defines $\mathcal{T}$ is pure as a right ideal ensures that again $\mathcal{T}$-*injective* implies R-*injective.*

Our methods allow to study more generally injectivity properties of simple or semisimple modules in hereditary pretorsion classes in any category $\sigma[M]$. In fact, for the class of M-singular modules this is done in section 2 and 3.

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