Hopf monads on categories

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Abstract. Generalising modules over associative rings, the notion of *modules for an endofunctor* of any category is well established and useful in large parts of mathematics including universal algebra. Similarly, comodules over coalgebras are the model for *comodules for an endofunctor* and they are of basic importance. Compatibility conditions between endofunctors can be described by *distributive laws.* We use these ingredients to define bimonads and Hopf monads on arbitrary categories thus making these notions accessible to universal algebra.

1. Introduction

The language of category theory is a universal tool in various parts of mathematics like algebra, topology, logic, universal algebra and computer science. Many notions were introduced by transfer from algebra. Associative algebras were generalised to *monads* on arbitrary categories, and coassociative coalgebras (or corings) lead to *comonads* on categories. Related to these the categories of *modules* for a monad and *comodules* for a comonad are studied. It was observed by Beck [**Beck**], van Osdol [**Osdol**], and others that the compatibility between monads and comonads can be controlled by *distributive laws.* These ideas again showed up recently in papers from theoretical computer science (e.g. Turi and Plotkin [**TuPl**]).

The purpose of this talk is to recall the fundamental terminology from algebra in a form which makes it quite obvious how to transfer them to general categories. In particular we will focus on bialgebras and Hopf algebras and their interpretation as *bimonads* on categories. Most of the generalisations of the classical situation were formulated for categories with a tensor product, i.e., monoidal categories (e.g. Moerdijk [**Moer**], Bruguières-Virelizier [**BruVir**]). We want to avoid any conditions on the base category. This is possible by posing all requirements on the functors and for this we exploit the fact that the endofunctors do carry a monoidal structure.

For more details on the subject the reader is referred to Mesablishvili [**Mes, MesWis**], Škoda [**SkoDis, SkoNon**], and [**Wis**] and the literature cited there.

2. Preliminaries

In this section we recall the basic definitions for modules and comodules. Throughout R will be a commutative associative ring with identity.

2.1. **Algebras.** An algebra over a ring R is an R-module A with linear maps

$$\mu : A \otimes_R A \to A, \quad \eta : R \to A,$$

the *multiplication* and *unit*, inducing commutative diagrams

$$\begin{array}{ccc} A \otimes_R A \otimes_R A & \xrightarrow{I\otimes\mu} & A \otimes_R A \\ {\scriptstyle \mu\otimes I}\downarrow & & \downarrow{\scriptstyle \mu} \\ A \otimes_R A & \xrightarrow{\mu} & A, \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\eta\otimes I} & A \otimes_R A \\ {\scriptstyle I\otimes\eta}\downarrow & \overset{I}{\searrow} & \downarrow{\scriptstyle \mu} \\ A \otimes_R A & \xrightarrow{\mu} & A \end{array}$$

2.2. ***A*-modules.** An R-module M is said to be a *left A-module* provided there is an *R-linear map*

$$\rho_M : A \otimes_R M \to M, \qquad a \otimes m \mapsto am,$$

with commutative diagrams

$$\begin{array}{ccc} A \otimes_R A \otimes_R M & \xrightarrow{I\otimes\rho_M} & A \otimes_R M \\ {\scriptstyle \mu\otimes I}\downarrow & & \downarrow{\scriptstyle \rho_M} \\ A \otimes_R M & \xrightarrow{\rho_M} & M \end{array}, \qquad \begin{array}{ccc} M & \xrightarrow{\eta\otimes I} & A \otimes_R M \\ & \underset{=}{\searrow} & \downarrow{\scriptstyle \rho_M} \\ & & M \end{array},$$

Homomorphisms between modules are maps which respect the structural maps and this can be expressed as follows:

2.3. **Module homomorphisms.** Given left A-modules M and N, an *A-module (homo)morphism* is an R-linear map $f : M \to N$ with a commutative diagram

$$\begin{array}{ccc} A \otimes_R M & \xrightarrow{I\otimes f} & A \otimes_R N \\ {\scriptstyle \rho_M}\downarrow & & \downarrow{\scriptstyle \rho_N} \\ M & \xrightarrow{f} & N\,. \end{array}$$

2.4. **Category of left A-modules $_A\mathbf{M}$.** The left A-modules as objects and the A-module homomorphisms as morphisms form a category, the category of left A-modules which we denote by $_A\mathbf{M}$.

$_A\mathbf{M}$ is an abelian category with products and coproducts, kernels and cokernels. This follows partly from properties of the base category, the R-modules, and partly from properties of the functor $A \otimes_R -$.

Reversing the arrows in the above diagrams we arrive at the notion of coalgebras and comodules. Notice that this is not a proper "dualisation process" in the categorical sense because the tensor product (which has a considerable influence on the resulting constructions) is maintained.

2.5. **Coalgebras.** A *coalgebra* over a ring R is an R-module C with linear maps

$$\Delta : C \to C \otimes_R C, \quad \varepsilon : C \to R,$$

the *comultiplication* and the *counit*, inducing commutative diagrams

$$
\begin{array}{ccc}
C & \xrightarrow{\Delta} & C \otimes_R C \\
{\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle I\otimes\Delta} \\
C \otimes_R C & \xrightarrow{\Delta\otimes I} & C \otimes_R C \otimes_R C\,,
\end{array}
\qquad
\begin{array}{ccc}
C & \xrightarrow{\Delta} & C \otimes_R C \\
{\scriptstyle \Delta}\downarrow & \searrow^{I} & \downarrow{\scriptstyle \varepsilon\otimes I} \\
C \otimes_R C & \xrightarrow[I\otimes\varepsilon]{} & C\,.
\end{array}
$$

It is fairly obvious how comodules are to be defined:

2.6. C**-comodules.** A *left C-comodule* is an R-module M with an R-linear map

$$\rho^M : M \longrightarrow C \otimes_R M,$$

inducing commutative diagrams

$$
\begin{array}{ccc}
M & \xrightarrow{\rho^M} & C \otimes_R M \\
{\scriptstyle \rho^M}\downarrow & & \downarrow{\scriptstyle \Delta\otimes I} \\
C \otimes_R M & \xrightarrow{I\otimes\rho^M} & C \otimes_R C \otimes_R M,
\end{array}
\qquad
\begin{array}{ccc}
M & \xrightarrow{\rho^M} & C \otimes_R M\;. \\
 & \searrow^{=} & \downarrow{\scriptstyle \varepsilon\otimes I} \\
 & & M
\end{array}
$$

2.7. C**-comodule morphisms.** Given left C-comodules M and N, a *C-comodule morphism* is an R-linear map $f : M \to N$ with a commutative diagram

$$
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
{\scriptstyle \rho^M}\downarrow & & \downarrow{\scriptstyle \rho^M} \\
C \otimes_R M & \xrightarrow{I\otimes f} & C \otimes_R N.
\end{array}
$$

2.8. **Category of left C-comodules $^C\mathbf{M}$.** The left C-comodules and comodule homomorphisms form the *category of left C-comodules* which we denote by $^C\mathbf{M}$.

$^C\mathbf{M}$ is an additive category with coproducts and cokernels. However, the existence of kernels is not always guaranteed. Moreover, monomorphisms need not be injective maps. This is a consequence of the fact that the functor $C \otimes_R -$ need not be left exact.

Given an algebra and a coalgebra the question arises if there is a reasonable way to express *compatibility* of the two structures.

2.9. **Entwining algebras and coalgebras.** Given an R-algebra (A, μ, η) and an R-coalgebra (C, Δ, ε), an R-linear map

$$\psi : C \otimes_R A \to A \otimes_R C$$

is called an *entwining map* (in [**BrzMaj**]) provided it implies commutativity of the diagrams

$$
\begin{array}{ccccccc}
C\otimes A\otimes A & & \xrightarrow{I\otimes\mu} & C\otimes A & \xrightarrow{\Delta\otimes I} & C\otimes C\otimes A & \xrightarrow{I\otimes\psi} C\otimes A\otimes C \\
{\scriptstyle \psi\otimes I}\downarrow & & & \downarrow{\scriptstyle \psi} & & & \downarrow{\scriptstyle \psi\otimes I} \\
A\otimes C\otimes A & \xrightarrow{I\otimes\psi} & A\otimes A\otimes C \xrightarrow{\mu\otimes I} & A\otimes C & & \xrightarrow{I\otimes\Delta} & A\otimes C\otimes C,
\end{array}
$$

$$\begin{array}{ccccc} C & \xrightarrow{I\otimes\eta} & C\otimes A & \xrightarrow{\varepsilon\otimes I} & A \\ & \searrow^{\eta\otimes I} & \downarrow{\scriptstyle\psi} & \nearrow_{I\otimes\varepsilon} & \\ & & A\otimes C. & & \end{array}$$

A comultiplication can be defined on $A \otimes_R C$ giving it the structure of an A-*coring* (e.g. [**BrzWis**, Section 32]).

2.10. **Entwined modules.** Given an entwined pair (A, C, ψ) of an algebra and a coalgebra, let M be an R-module with an

A-module structure $\varrho_M : M \otimes_R A \to M$ and a

C-comodule structure $\varrho^M : M \to M \otimes_R C$.

Then M is an *entwined module* if the diagram

$$\begin{array}{ccccc} M\otimes A & \xrightarrow{\varrho_M} & M & \xrightarrow{\varrho^M} & M\otimes C \\ {\scriptstyle\varrho^M\otimes I_A}\downarrow & & & & \uparrow{\scriptstyle\varrho_M\otimes I} \\ M\otimes C\otimes A & & \xrightarrow{I\otimes\psi} & & M\otimes A\otimes C, \end{array}$$

is commutative (e.g. [**BrzWis**, 32.4]).

Morphisms between entwined module M and N are maps $M \to N$ which are A-module as well as C-comodule morphisms.

The resulting *category of entwined modules* is an additive category with coproducts and cokernels but not necessarily with kernels (see 2.8). It can be identified with the category of comodules over the coring $A \otimes_R C$ (see 2.9).

Considering the case $B = A = C$ in 2.9 we obtain:

2.11. **Bialgebras.** An R-*bialgebra* is an R-module B that is an

algebra $\mu : B \otimes_R B \to B, \quad \eta : R \to B,$ and a

coalgebra $\Delta : B \to B \otimes_R B, \quad \varepsilon : B \to R,$

such that

Δ and ε are algebra morphisms, or equivalently,

μ and η are coalgebra morphisms.

In this case, multiplication and comultiplication on $B \otimes_R B$ are derived from the canonical twist map $\mathsf{tw} : B \otimes_R B \to B \otimes_R B,\ a \otimes b \mapsto b \otimes a$, which also induces an entwining map

$$\psi : B \otimes_R B \to B \otimes_R B, \quad a \otimes b \mapsto (1 \otimes a)\Delta(b).$$

and with this the compatibility conditions can be formulated as in 2.9.

Note that the ordinary twist map can more generally be replaced by some *braiding map* (e.g. [**Wis**, 5.16]). Not all entwining maps are derived from a braiding.

2.12. **Bimodules.** A right *(mixed) B-bimodule* (or *Hopf module*) over a bialgebra B is an R-module M which is a B-module and a B-comodule

$$\rho_M : M \otimes_R B \to M, \quad \rho^M : M \to M \otimes_R B,$$

such that

$$\rho^M(mb) = \rho^M(m) \cdot \Delta(b), \quad \text{for } b \in B,\ m \in M.$$

Similar to the situation in 2.11, this compatibility condition can be expressed by the entwining induced by the twist map (or a braiding map). The related category of right mixed B-bimodules, denoted by $\mathbf{M}_B^B$, has the bimodules as objects and morphisms are maps which are module and comodule morphisms. B induces a fully faithful functor

$$\phi_B^B : {}_R\mathbf{M} \to \mathbf{M}_B^B, \quad M \mapsto M \otimes_R B.$$

2.13. **Hopf algebras.** An R-bialgebra B is a *Hopf algebra* if the functor ϕ_B^B is an equivalence.

Traditionally Hopf algebras are characterised by the existence of

2.14. **Antipodes.** An R-linear map $S : B \to B$ is called an *antipode* if it induces commutativity of the diagram

$$\begin{array}{ccccc} B & \xrightarrow{\varepsilon} & R & \xrightarrow{\eta} & B \\ {\scriptstyle\Delta}\downarrow & & & & \uparrow{\scriptstyle\mu} \\ B \otimes_R B & & \underset{I\otimes S}{\overset{S\otimes I}{\rightrightarrows}} & & B \otimes_R B. \end{array}$$

The antipode S can also be characterised as the inverse of the identity map with respect to the *convolution product* in $\mathrm{End}_R(B)$ (e.g. [**BrzWis**, Section 15]).

2.15. **Fundamental Theorem.** *For a bialgebra B, the following assertions are equivalent:*

(a) *B has an antipode;*

(b) *the functor $\phi_B^B : {}_R\mathbf{M} \to \mathbf{M}_B^B$ is an equivalence.*

3. Monads and comonads in general categories

The notions considered in the preceding section are written in a way which allows a straightforward transfer to arbitrary categories. Little knowledge is needed from category theory and for the convenience of the reader we repeat the basic facts.

3.1. **Categories $\mathbb{A}$.** A category $\mathbb{A}$ consists of classes of *objects* $\mathrm{Obj}(\mathbb{A})$, *morphism sets* $\mathrm{Mor}(\mathbb{A})$, such that for any objects A, B, C we have

- $\mathrm{Mor}_{\mathbb{A}}(A, B) \cap \mathrm{Mor}_{\mathbb{A}}(A', B') = \emptyset$ for $(A, B) \neq (A', B')$;
- *composition maps*
$$\mathrm{Mor}_{\mathbb{A}}(A, B) \times \mathrm{Mor}_{\mathbb{A}}(B, C) \to \mathrm{Mor}_{\mathbb{A}}(A, C);$$
- *identity morphisms* $I_A \in \mathrm{Mor}_{\mathbb{A}}(A, A)$.

Two categories may be related by

3.2. **Functors.** A *covariant functor* $F : \mathbb{A} \to \mathbb{B}$ consists of assignments

$$\begin{array}{lrcl} \mathrm{Obj}(\mathbb{A}) \to \mathrm{Obj}(\mathbb{B}), & A & \mapsto & F(A) \\ \mathrm{Mor}(\mathbb{A}) \to \mathrm{Mor}(\mathbb{B}), & f : A \to A' & \mapsto & F(f) : F(A) \to F(A') \end{array}$$

such that for $g : A'' \to A$ and $f : A \to A'$,

$$F(fg) = F(f)F(g) \quad \text{and} \quad F(I_A) = I_{F(A)}.$$

Contravariant functors reverse the composition of maps. Here we will only be concerned with covariant functors.

The connection between two functors from a category $\mathbb{A}$ to a category $\mathbb{B}$ are described by

3.3. **Natural transformations.** A *natural transformation* $\alpha : F \to G$ between functors $F, G : \mathbb{A} \to \mathbb{B}$ is given by morphisms

$$\alpha_A : F(A) \to G(A) \text{ in } \mathbb{B},\ A \in \mathbb{A},$$

such that $f : A \to A'$ in $\mathbb{A}$ induces the commutative diagram in $\mathbb{B}$

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(A') \\ {\scriptstyle \alpha_A}\downarrow & & \downarrow{\scriptstyle \alpha_{A'}} \\ G(A) & \xrightarrow{G(f)} & G(A')\,. \end{array}$$

3.4. **Adjoint functors.** Given two categories $\mathbb{A}$ and $\mathbb{B}$, a functor $L : \mathbb{A} \to \mathbb{B}$ is said to be *left adjoint* to a functor $R : \mathbb{B} \to \mathbb{A}$ if there are natural isomorphisms (in $A \in \mathrm{Obj}(\mathbb{A})$ and $B \in \mathrm{Obj}(\mathbb{B})$)

$$\vartheta_{A,B} : \mathrm{Mor}_{\mathbb{B}}(L(A), B) \to \mathrm{Mor}_{\mathbb{A}}(A, R(B)).$$

Associated to such functors are natural transformations

$$\textit{unit}\ \eta : I_{\mathbb{A}} \to RL \quad \text{and} \quad \textit{counit}\ \varepsilon : LR \to I_{\mathbb{B}}.$$

Because of its importance we have a look at the functor which was indispensable for our Section 2.

3.5. **Tensor functor.** For any R-algebra A we have the *tensor functor*

$$\begin{array}{rrcl} A \otimes_R - : & {}_R\mathbf{M} & \longrightarrow & {}_R\mathbf{M}, \\ \textit{objects} & M & \mapsto & A \otimes_R M, \\ \textit{morphisms} & f : M \to N & \mapsto & I \otimes f : A \otimes_R M \to A \otimes_R N. \end{array}$$

It is left adjoint to the functor $\mathrm{Hom}_R(A, -) : {}_R\mathbf{M} \to {}_R\mathbf{M}$ by the canonical isomorphisms for R-modules M, N,

$$\mathrm{Hom}_R(A \otimes_R M, N) \to \mathrm{Hom}_R(M, \mathrm{Hom}_R(A, N)).$$

The functor $A \otimes_R -$ has the property that its composition $A \otimes_R A \otimes_R -$ is related with $A \otimes_R -$: the multiplication induces a natural transformation

$$\mu \otimes - : A \otimes_R A \otimes_R - \to A \otimes_R -.$$

Also the associativity conditions in 2.1 give rise to natural transformations of the compositions of $A \otimes_R -$. This leads to the following definition for endofunctors:

3.6. **Monads.** A *monad* is a triple $\mathbb{F} = (F, \mu, \eta)$, where $F : \mathbb{A} \to \mathbb{A}$ is a functor and

$$\mu : FF \to F, \quad \eta : I_{\mathbb{A}} \to F,$$

are natural transformations with commutative diagrams

$$\begin{array}{ccc} FFF & \xrightarrow{\mu F} & FF \\ {\scriptstyle F\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ FF & \xrightarrow[\mu]{} & F\,, \end{array} \qquad \begin{array}{ccc} F & \xrightarrow{\eta F} & FF \\ {\scriptstyle F\eta}\downarrow & \searrow^{=} & \downarrow{\scriptstyle \mu} \\ FF & \xrightarrow[\mu]{} & F\,. \end{array}$$

Of course, $A \otimes_R - : {}_R\mathbf{M} \to {}_R\mathbf{M}$ is a monad if and only if A is an associative R-algebra with unit.

The definitions of A-modules and their morphisms are generalised to

3.7. **F-modules and their morphisms.** Given an endofunctor $F : \mathbb{A} \to \mathbb{A}$, an object $A \in \mathrm{Obj}(\mathbb{A})$ is an *F-module* provided there is a morphism

$$\varrho_A : F(A) \to A \quad \text{in} \quad \mathbb{A}.$$

A morphism $f : A \to A'$ in $\mathbb{A}$ between F-modules is an *F-module morphism* if it induces commutativity of the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(A') \\ {\scriptstyle \varrho_A}\downarrow & & \downarrow{\scriptstyle \varrho_{A'}} \\ A & \xrightarrow{f} & A'. \end{array}$$

Obviously, the composition of two F-module morphisms is again of this type and thus we have the *category of F-modules* which we denote by $\mathbb{A}_F$.

Note that F-modules are defined for *any* functors $F : \mathbb{A} \to \mathbb{A}$. So, for example, any R-module N gives rise to a functor $N \otimes_R - : {}_R\mathbf{M} \to {}_R\mathbf{M}$. In 2.2, for associative algebras A we put more conditions on the A-modules. Similarly, the modules over any monad should be compatible with the defining properties of the monad:

3.8. **Modules for monads.** Given a monad $\mathbb{F} = (F, \mu, \eta)$ on a category $\mathbb{A}$, an **$\mathbb{F}$-module** is an object $A \in \mathrm{Obj}(\mathbb{A})$ with a morphism

$$\varrho_A : F(A) \to A$$

inducing commutative diagrams

$$\begin{array}{ccc} FF(A) & \xrightarrow{\mu_A} & F(A) \\ {\scriptstyle F\varrho_A}\downarrow & & \downarrow{\scriptstyle \varrho_A} \\ F(A) & \xrightarrow[\varrho_A]{} & A\,, \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\eta_A} & F(A) \\ & {\scriptstyle I_A}\searrow & \downarrow{\scriptstyle \varrho_A} \\ & & A\,. \end{array}$$

In particular, for any $A \in \mathrm{Obj}(\mathbb{A})$, $F(A)$ is an $\mathbb{F}$-module by

$$\mu_A : FF(A) \to F(A).$$

This yields the *free functor*

$$\phi_{\mathbb{F}} : \mathbb{A} \to \mathbb{A}_{\mathbb{F}}, \quad A \mapsto F(A),$$

which is *left adjoint* to the forgetful functor $U_F : \mathbb{A}_{\mathbb{F}} \to \mathbb{A}$ by the bijection

$$\mathrm{Mor}_{\mathbb{A}_F}(F(A), B) \to \mathrm{Mor}_{\mathbb{A}}(A, U_F(B)), \quad f \mapsto f \circ \eta_A.$$

Although the modules for a monad $\mathbb{F}$ are fairly close to the modules over an associative unital algebra, there are many properties of the category of A-modules which are not shared by all $\mathbb{F}$-modules. This depends on the special properties of $A \otimes_R -$: it is a right exact functor which preserves direct sums and cokernels. This implies, for example, that $A \otimes_R R$, the image of R, is a (projective) generator in ${}_A\mathbf{M}$.

The notions of coalgebras and comodules as considered in 2.5 and 2.6 are the blueprint for the introduction of comonads and their comodules.

3.9. **Comonads.** A *comonad* on a category $\mathbb{A}$ is a triple $\mathbb{G} = (G, \delta, \varepsilon)$, where $G : \mathbb{A} \to \mathbb{A}$ is a functor and

$$\delta : G \to GG, \quad \varepsilon : G \to I_{\mathbb{A}},$$

are natural transformations with commuting diagrams

$$\begin{array}{ccc} G & \xrightarrow{\delta} & GG \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle G\delta} \\ GG & \xrightarrow{\delta_G} & GGG\,, \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\delta} & GG \\ {\scriptstyle\delta}\downarrow & \searrow^{=} & \downarrow{\scriptstyle\varepsilon_G} \\ GG & \xrightarrow[G\varepsilon]{} & G. \end{array}$$

3.10. ***G*-comodules and their morphisms.** A *G-comodule* for any functor $G : \mathbb{A} \to \mathbb{A}$ is an $A \in \mathrm{Obj}(\mathbb{A})$ with a morphism in $\mathbb{A}$,

$$\varrho^A : A \to G(A).$$

A *G-comodule morphism* between G-comodules A and A' is a morphism $f : A \to A'$ in $\mathbb{A}$ with a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & A' \\ {\scriptstyle\varrho^A}\downarrow & & \downarrow{\scriptstyle\varrho^{A'}} \\ G(A) & \xrightarrow{G(f)} & G(A')\,. \end{array}$$

The G-comodules with the morphisms defined above form a category which we denote by $\mathbb{A}^G$.

3.11. **Comodules for comonads.** A $\mathbb{G}$-comodule is an object $A \in \mathrm{Obj}(\mathbb{A})$ with a morphism

$$\varrho^A : A \to G(A) \text{ in } \mathbb{A}$$

and commutative diagrams

$$\begin{array}{ccc} A & \xrightarrow{\varrho^A} & G(A) \\ {\scriptstyle\varrho^A}\downarrow & & \downarrow{\scriptstyle\delta_A} \\ G(A) & \xrightarrow{G\varrho^A} & GG(A)\,, \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\varrho^A} & G(A) \\ & {\scriptstyle I_A}\searrow & \downarrow{\scriptstyle\varepsilon_A} \\ & & A. \end{array}$$

For any object $A \in \mathrm{Obj}(\mathbb{A})$, $G(A)$ is a comodule canonically and thus we have the *free functor*

$$\phi^G : \mathbb{A}^G \to \mathbb{A}^G, \quad A \mapsto G(A),$$

which is right adjoint to the forgetful functor $U^G : \mathbb{A}^G \to \mathbb{A}$ by the bijection

$$\mathrm{Mor}_{\mathbb{A}^G}(B, G(A)) \to \mathrm{Mor}_{\mathbb{A}}(U^G(B), A), \quad f \mapsto \varepsilon_A \circ f.$$

As observed for A-modules and $\mathbb{F}$-modules, in general the category of C-comodules and $\mathbb{G}$-comodules may differ considerably depending on the properties of the comonad $\mathbb{G}$.

Having transferred algebras and coalgebras to monads and comonads on arbitrary categories the question arises how to express the compatibility conditions as considered in 2.9 for monads and comonads. The key to this is provided by Johnstone's lifting theorem from [**John**]. The resulting diagrams are called *distributive laws* (e.g. [**Beck**], [**Osdol**]).

3.12. **Lifting of endofunctors.** Let F and G be endofunctors of the category $\mathbb{A}$. *Liftings* of a functor $T : \mathbb{A} \to \mathbb{A}$ are functors

$$\overline{T} : \mathbb{A}_F \to \mathbb{A}_F \quad \text{and} \quad \widehat{T} : \mathbb{A}^G \to \mathbb{A}^G$$

with commutative diagrams

$$\begin{array}{ccc} \mathbb{A}_F & \xrightarrow{\overline{T}} & \mathbb{A}_F \\ {\scriptstyle U_F}\downarrow & & \downarrow{\scriptstyle U_F} \\ \mathbb{A} & \xrightarrow{T} & \mathbb{A}, \end{array} \qquad \begin{array}{ccc} \mathbb{A}^G & \xrightarrow{\widehat{T}} & \mathbb{A}^G \\ {\scriptstyle U^G}\downarrow & & \downarrow{\scriptstyle U^G} \\ \mathbb{A} & \xrightarrow{T} & \mathbb{A}. \end{array}$$

Such liftings need not always exist and we are asking under which conditions they do exist. We first consider the case of monads.

3.13. **Lifting of monads.** *For a monad $\mathbb{F} = (F, \mu, \eta)$ on $\mathbb{A}$, the liftings*

$$\overline{T} : \mathbb{A}_{\mathbb{F}} \to \mathbb{A}_{\mathbb{F}} \quad \textit{of} \quad T : \mathbb{A} \to \mathbb{A}$$

are in bijective correspondence *with the natural transformations*

$$\lambda : FT \to TF$$

with commutative diagrams

$$\begin{array}{ccccc} FFT & & \xrightarrow{\mu_T} & & FT \\ {\scriptstyle F\lambda}\downarrow & & & & \downarrow{\scriptstyle \lambda} \\ FTF & \xrightarrow{\lambda_F} & TFF & \xrightarrow{T\mu} & TF, \end{array} \qquad \begin{array}{ccc} T & \xrightarrow{\eta_T} & FT \\ & {\scriptstyle T\eta}\searrow & \downarrow{\scriptstyle \lambda} \\ & & TF. \end{array}$$

Knowing about the existence of a lifting we still do not know which properties it has. So we may ask when it is a monad.

3.14. **Lifting of monads to monads.** *If $\mathbb{T} = (T, \mu', \eta')$ is a monad, then the lifting*

$$\overline{T} : \mathbb{A}_{\mathbb{F}} \to \mathbb{A}_{\mathbb{F}} \ \textit{of} \ \ T : \mathbb{A} \to \mathbb{A}$$

with natural transformation

$$\lambda : FT \to TF$$

is a monad *if and only if we have commutative diagrams*

$$\begin{array}{ccccc} FTT & & \xrightarrow{F\mu'} & & FT \\ {\scriptstyle \lambda_T}\downarrow & & & & \downarrow{\scriptstyle \lambda} \\ TFT & \xrightarrow{T\lambda} & TTF & \xrightarrow{\mu'_F} & TF, \end{array} \qquad \begin{array}{ccc} F & \xrightarrow{F\eta'} & FT \\ & {\scriptstyle \eta'_F}\searrow & \downarrow{\scriptstyle \lambda} \\ & & TF. \end{array}$$

The entwining structures considered in 2.9 correspond to the

3.15. **Mixed distributive laws.** Given a monad $\mathbb{F} = (F, \mu, \eta)$ and a comonad $\mathbb{G} = (G, \delta, \varepsilon)$ on the category $\mathbb{A}$, a natural transformation

$$\lambda : FG \to GF$$

is called a *mixed distributive law* or an *entwining* provided it induces commutative diagrams

$$\begin{array}{ccccc} FFG & \xrightarrow{\mu_G} & & & FG \\ {\scriptstyle F\lambda}\downarrow & & & & \downarrow{\scriptstyle \lambda} \\ FGF & \xrightarrow{\lambda_F} & GFF & \xrightarrow{G\mu} & GF, \end{array} \qquad \begin{array}{ccccc} FG & \xrightarrow{F\delta} & FGG & \xrightarrow{\lambda_G} & GFG \\ {\scriptstyle \lambda}\downarrow & & & & \downarrow{\scriptstyle G\lambda} \\ GF & & \xrightarrow{\delta_F} & & GGF, \end{array}$$

$$\begin{array}{ccc} G & \xrightarrow{\eta_G} & FG \\ & {\scriptstyle G\eta}\searrow & \downarrow{\scriptstyle \lambda} \\ & & GF, \end{array} \qquad \begin{array}{ccc} FG & \xrightarrow{F\varepsilon} & F \\ {\scriptstyle \lambda}\downarrow & \nearrow{\scriptstyle \varepsilon_F} & \\ GF. & & \end{array}$$

3.16. **Mixed bimodules.** For a monad $\mathbb{F} = (F, \mu, \eta)$ and comonad $\mathbb{G} = (G, \delta, \varepsilon)$ on $\mathbb{A}$ with an entwining $\lambda : FG \to GF$, *mixed bimodules* are defined as those $A \in \mathrm{Obj}(\mathbb{A})$ with morphisms

$$F(A) \xrightarrow{h} A \xrightarrow{k} G(A)$$

such that (A, h) is an $\mathbb{F}$-module and (A, k) is a $\mathbb{G}$-comodule satisfying the pentagonal law

$$\begin{array}{ccccc} F(A) & \xrightarrow{h} & A & \xrightarrow{k} & G(A) \\ {\scriptstyle F(k)}\downarrow & & & & \uparrow{\scriptstyle G(h)} \\ FG(A) & & \xrightarrow{\lambda_A} & & GF(A). \end{array}$$

A morphism $f : A \to A'$ between two mixed bimodules is a *bimodule morphism* provided it is both an F-module and a G-comodule morphism. These notions yield the category of mixed bimodules which we denote by $\mathbb{A}_F^G$.

We are now prepared to formulate the conditions on bialgebras from 2.11 for endofunctors.

3.17. **Mixed bimonads and bimodules.** An endofunctor $B : \mathbb{A} \to \mathbb{A}$ is called a *(mixed) bimonad* if it is a

- monad $\mathbb{B} = (B, \mu, \eta)$ and a comonad $\mathbb{B} = (B, \delta, \varepsilon)$
- with an entwining functorial morphism $\psi : BB \to BB$
  inducing commutativity of the diagram

$$\begin{array}{ccccc} BB & \xrightarrow{\mu} & B & \xrightarrow{\delta} & BB \\ {\scriptstyle B\delta}\downarrow & & & & \uparrow{\scriptstyle B\mu} \\ BBB & & \xrightarrow{\psi_B} & & BBB. \end{array}$$

For a bimonad B, *mixed B-bimodules* are defined as B-modules and B-comodules A satisfying the pentagonal law

$$\begin{array}{ccccc} B(A) & \xrightarrow{\varrho_A} & A & \xrightarrow{\varrho^A} & B(A) \\ {\scriptstyle B(\varrho^A)}\downarrow & & & & \uparrow{\scriptstyle B(\varrho_A)} \\ BB(A) & & \xrightarrow{\psi_A} & & BB(A). \end{array}$$

Taking as morphisms $A \to A'$ between bimodules the morphisms which are B-module and B-comodule morphisms, we obtain the *category of mixed B-bimodules* denoted by $\mathbb{A}_B^B$.

The diagram for mixed bimonads implies that for any $A \in \mathrm{Obj}(\mathbb{A})$, $B(A)$ is a mixed B-bimodule and thus we get a functor

$$\phi_B^B : \mathbb{A} \to \mathbb{A}_B^B, \quad A \mapsto B(A),$$

which is full and faithful by the functorial isomorphisms for $A, A' \in \mathrm{Obj}(\mathbb{A})$,

$$\mathrm{Mor}_B^B(B(A), B(A')) \simeq \mathrm{Mor}_B(B(A), A') \simeq \mathrm{Mor}_{\mathbb{A}}(A, A').$$

As for bialgebras, we may consider the case when the functor ϕ_B^B is an equivalence of categories. Furthermore, we may define an

3.18. **Antipode.** An *antipode* for a mixed bimonad $B : \mathbb{A} \to \mathbb{A}$ is a natural transformation $S : B \to B$ with commutative diagram

$$\begin{array}{ccccc} B & \xrightarrow{\varepsilon} & I & \xrightarrow{\eta} & B \\ {\scriptstyle\delta}\downarrow & & & & \uparrow{\scriptstyle\mu} \\ BB & & \underset{BS}{\overset{S_B}{\rightrightarrows}} & & BB. \end{array}$$

3.19. **Hopf bimonads.** For a bimonad B on a category $\mathbb{A}$ one can require:

- B has an antipode;
- the functor $\phi_B^B : \mathbb{A} \to {}^B_B\mathbb{A}$ is an equivalence.

The two conditions are equivalent provided $\mathbb{A}$ has equalisers and colimits and B preserves colimits.

For a deeper study of these conditions the reader is referred to [**MesWis**].

We conclude by recalling a familiar example of a non-additive Hopf monad on the category of sets (e.g. [**Wis**, 5.19]).

3.20. **Bimonads on Set.**

- Endofunctor $G \times - : \mathbf{Set} \to \mathbf{Set}$, $A \mapsto G \times A$
- $G \times -$ monad, G is monoid
- $G \times -$ comonad, $\delta : G \to G \times G$, $g \mapsto (g, g)$
- entwining morphism $\psi : G \times G \to G \times G$, $(g, h) \mapsto (gh, g)$

Hopf monads on Set. *For a set G the following assertions are equivalent:*

(a) *$G \times -$ is a bimonad and $\phi_G^G : \mathbf{Set} \to \mathbf{Set}_G^G$ is an equivalence;*

(b) *$G \times -$ is a bimonad with antipode $S : G \times - \to G \times -$;*

(c) *G is a group.*

Here the antipode is given by the map

$$s : G \to G, \quad g \mapsto g^{-1}.$$

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