BIMONADS AND HOPF MONADS ON CATEGORIES

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Abstract. The purpose of this paper is to develop a theory of bimonads and Hopf monads on arbitrary categories thus providing the possibility to transfer the essentials of the theory of Hopf algebras in vector spaces to more general settings. There are several extensions of this theory to *monoidal* categories which in a certain sense follow the classical trace. Here we do not pose any conditions on our base category but we do refer to the monoidal structure of the category of endofunctors on any category $\mathbb{A}$ and by this we retain some of the combinatorial complexity which makes the theory so interesting. As a basic tool we use distributive laws between monads and comonads (entwinings) on $\mathbb{A}$: we define a *bimonad* on $\mathbb{A}$ as an endofunctor B which is a monad and a comonad with an entwining $\lambda : BB \to BB$ satisfying certain conditions. This λ is also employed to define the category $\mathbb{A}_B^B$ of (mixed) B-bimodules. In the classical situation, an entwining λ is derived from the twist map for vector spaces. Here this need not be the case but there may exist special distributive laws $\tau : BB \to BB$ satisfying the Yang-Baxter equation (*local prebraidings*) which induce an entwining λ and lead to an extension of the theory of *braided Hopf algebras*.

An antipode is defined as a natural transformation $S : B \to B$ with special properties. For categories $\mathbb{A}$ with limits or colimits and bimonads B preserving them, the existence of an antipode is equivalent to B inducing an equivalence between $\mathbb{A}$ and the category $\mathbb{A}_B^B$ of B-bimodules. This is a general form of the *Fundamental Theorem* of Hopf algebras.

Finally we observe a nice symmetry: If B is an endofunctor with a right adjoint R, then B is a (Hopf) bimonad if and only if R is a (Hopf) bimonad. Thus a k-vector space H is a Hopf algebra if and only if $\mathrm{Hom}_k(H, -)$ is a Hopf bimonad. This provides a rich source for Hopf monads not defined by tensor products and generalises the well-known fact that a finite dimensional k-vector space H is a Hopf algebra if and only if its dual $H^* = \mathrm{Hom}_k(H, k)$ is a Hopf algebra. Moreover, we obtain that any set G is a group if and only if the functor $\mathrm{Map}(G, -)$ is a Hopf monad on the category of sets.

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Contents

1. Introduction

The theory of algebras (monads) as well as of coalgebras (comonads) is well understood in various fields of mathematics as algebra (e.g. [8]), universal algebra (e.g. [12]), logic or operational semantics (e.g. [27]), theoretical computer science (e.g. [22]). The relationship between monads and comonads is controlled by *distributive laws* introduced in the seventies by Beck (see [2]). In algebra one of the fundamental notions emerging in this context are the Hopf algebras. The definition is making heavy use of the tensor product and thus generalisations of this theory were mainly considered for *monoidal categories*. They allow readily to transfer formalisms from the category of vector spaces to the more general settings (e.g. Bespalov and Brabant [3] and [20]).

A Hopf algebra is an algebra as well as a coalgebra. Thus one way of generalisation is to consider distinct algebras and coalgebras and some relationship between them. This leads to the theory of *entwining structures* and *corings* over associative rings (e.g. [8]) and one may ask how to formulate this in more general categories. The definition of *bimonads* on a monoidal category as monads whose functor part is comonoidal by Bruguières and Virelizier in [7, 2.3] may be seen as going in this direction. Such functors are called *Hopf monads* in Moerdijk [21] and *opmonoidal monads* in McCrudden [17, Example 2.5]. In 2.2 we give more details of this notion.

Another extension of the theory of corings are the *generalised bialgebras* in Loday in [15]. These are Schur functors (on vector spaces) with a monad structure (operads) and a specified coalgebra structure satisfying certain compatibility conditions [15, 2.2.1]. While in [15] use is made of the canonical twist map, it is stressed in [7] that the theory is built up without reference to any braiding. More comments on these constructions are given in 2.3.

The notions of monads and comonads were formulated in the setting of 2-catgeories in Lack and Street [14]. Based on this point of view, elements of a Hopf theory were developed (e.g. [9]) and eventually a formal Hopf algebra theory in (monoidal) Gray categories was presented in López Franco [16]. Similar to Loday's approach, the central notions are built up on the tensor product in the base category. This allows for a rich structure theory and a deep study of Hopf algebras and comprises (co)quasi-Hopf algebras over base fields.

The purpose of the present paper is somewhat different. Our intention is to formulate the essentials of the classical theory of Hopf algebras for any (not necessarily monoidal) category, thus making it accessible to a wide field of applications. We employ the fact that the category of endofunctors (with the Godement product as composition) always has a tensor product given by composition of natural transformations but no tensor product is required for the base category.

Compatibility between monads and comonads are formulated as distributive laws whose properties are recalled in Section 2. In Section 3, general categorical notions are presented and *Galois functors* are defined and investigated.

As suggested in [29, 5.13], we define a *bimonad* $\mathbf{H} = (H, m, e, \delta, \varepsilon)$ on any category $\mathbb{A}$ as an endofunctor H with a monad and a comonad structure satisfying certain compatibility conditions (see 4.1). The latter do not refer to any braiding but in special cases they can be derived from a *local prebraiding* $\tau : HH \to HH$ (see 6.3). In this case the bimonad shows the characteristics of *braided bialgebras* (Section 6).

Related to a bimonad H there is the (Eilenberg-Moore) category $\mathbb{A}_H^H$ of bimodules with a comparison functor $K_H : \mathbb{A} \to \mathbb{A}_H^H$. An *antipode* is defined as a natural transformation $S : H \to H$ satisfying $m \cdot SH \cdot \delta = e \cdot \varepsilon = m \cdot HS \cdot \delta$. It exists if and only if the natural transformation $\gamma := Hm \cdot \delta H : HH \to HH$ is an isomorphism. If the category $\mathbb{A}$ admits limits or colimits and H preserves them, the existence of an antipode is equivalent to the comparison functor being an equivalence (see 5.6). This is a general form of the Fundamental Theorem for Hopf algebras. Corresponding theorems are provided in Bruguières-Virelizier [7] and Loday [15] as well as in López Franco [16].

Of course, bialgebras and Hopf algebras over commutative rings R provide the prototypes for this theory: on R-Mod, the category of R-modules, one considers the endofunctor $B \otimes_R$

$-$: R-Mod$\to$ R-Mod where B is an R-module with algebra and coalgebra structures, and an entwining derived from the twist map (braiding) $M \otimes_R N \to N \otimes_R M$ (e.g. [5, Section 8]).

More generally, for a comonad H, the entwining $\lambda : HH \to HH$ may be derived from a *local prebraiding* $\tau : HH \to HH$ (see 6.7) and then results similar to those known for braided Hopf algebras are obtained. In particular, the composition HH is again a bimonad (see 6.8) and, if $\tau^2 = 1$, an *opposite bimonad* can be defined (see 6.10).

In case a bimonad H on $\mathbb{A}$ has a right (or left) adjoint endofunctor R, then R is again a bimonad and has an antipode (or local prebraiding) if and only if so does H (see 7.5). In particular, for R-modules B, the functor $\mathrm{Hom}_R(B, -)$ is right adjoint to $B \otimes_R -$ and hence B is a Hopf algebra if and only if $\mathrm{Hom}_R(B, -)$ is a Hopf monad. This provides a rich source for examples of Hopf monads not defined by a tensor product and extends a symmetry principle known for finite dimensional Hopf algebras (see 7.8). We close with the observation that a set G is a group if and only if the endofunctor $\mathrm{Map(G}, -)$ is a Hopf monad on the category of sets (see 7.9).

Note that the pattern of our definition of bimonads resembles the definition of *Frobenius monads* on any category by Street in [24]. Those are monads $\mathbf{T} = (T, \mu, \eta)$ with natural transformations $\varepsilon : T \to I$ and $\rho : T \to TT$, subject to suitable conditions, which induce a comonad structure $\delta = T\mu \cdot \rho T : T \to TT$ and product and coproduct on T satisfy the compatibility condition $T\mu \cdot \delta T = \delta \cdot \mu = \mu T \cdot T\delta$.

2. Distributive laws

Distributive laws between endofunctors were studied by Beck [2], Barr [1] and others in the seventies of the last century. They are a fundamental tool for us and we recall some facts needed in the sequel. For more details and references we refer to [29].

2.1. Entwining from monad to comonad. Let $\mathbf{T} = (T, m, e)$ be a monad and $\mathbf{G} = (G, \delta, \varepsilon)$ a comonad on a category $\mathbb{A}$. A natural transformation $\lambda : TG \to GT$ is called a *mixed distributive law* or *entwining* from the monad $\mathbf{T}$ to the comonad $\mathbf{G}$ if it induces commutativity of the diagrams

$$\begin{array}{ccc} & G & \\ {}^{eG}\swarrow & & \searrow^{Ge} \\ TG & \xrightarrow{\lambda} & GT, \end{array} \qquad \begin{array}{ccc} TG & \xrightarrow{\lambda} & GT \\ & {}_{T\varepsilon}\searrow \quad \swarrow_{\varepsilon T} & \\ & T & \end{array},$$

$$\begin{array}{ccccc} TG & \xrightarrow{T\delta} & TGG & \xrightarrow{\lambda G} & GTG \\ {\scriptstyle\lambda}\downarrow & & & & \downarrow{\scriptstyle G\lambda} \\ GT & & \xrightarrow{\quad\delta T\quad} & & GGT, \end{array} \qquad \begin{array}{ccccc} TTG & \xrightarrow{T\lambda} & TGT & \xrightarrow{\lambda T} & GTT \\ {\scriptstyle mG}\downarrow & & & & \downarrow{\scriptstyle Gm} \\ TG & & \xrightarrow{\quad\lambda\quad} & & GT. \end{array}$$

It is shown in [30] that for an arbitrary mixed distributive law $\lambda : TG \to GT$ from a monad $\mathbf{T}$ to a comonad $\mathbf{G}$, the triple $\widehat{\mathbf{G}} = (\widehat{G}, \widehat{\delta}, \widehat{\varepsilon})$, is a comonad on the category $\mathbb{A}_{\mathbf{T}}$ of $\mathbf{T}$-modules (also called $\mathbf{T}$-algebras), where for any object (a, h_a) of $\mathbb{A}_{\mathbf{T}}$,

$$\bullet\ \widehat{G}(a, h_a) = (G(a), G(h_a) \cdot \lambda_a), \quad \bullet\ (\widehat{\delta})_{(a,h_a)} = \delta_a, \quad \bullet\ (\widehat{\varepsilon})_{(a,h_a)} = \varepsilon_a.$$

$\widehat{\mathbf{G}}$ is called the *lifting of* $\mathbf{G}$ corresponding to the mixed distributive law λ.

Furthermore, the triple $\widehat{\mathbf{T}} = (\widehat{T}, \widehat{m}, \widehat{e})$ is a monad on the category $\mathbb{A}^{\mathbf{G}}$ of $\mathbf{G}$-comodules, where for any object (a, θ_a) of the category $\mathbb{A}^{\mathbf{G}}$,

$$\bullet\ \widehat{T}(a, \theta_a) = (T(a), \lambda_a \cdot T(\theta_a)), \quad \bullet\ (\widehat{m})_{(a,\theta_a)} = m_a, \quad \bullet\ (\widehat{e})_{(a,\theta_a)} = e_a.$$

This monad is called the *lifting of* $\mathbf{T}$ corresponding to the mixed distributive law λ. One has an isomorphism of categories

$$(\mathbb{A}^G)_{\widehat{T}} \simeq (\mathbb{A}_T)^{\widehat{G}},$$

and we write $\mathbb{A}_T^G(\lambda)$ for this category. An object of $\mathbb{A}_T^G(\lambda)$ is a triple (a, h_a, θ_a), where $(a, h_a) \in \mathbb{A}_T$ and $(a, \theta_a) \in \mathbb{A}^G$ with commuting diagram

(2.1)
$$\begin{array}{ccccc} T(a) & \xrightarrow{h_a} & a & \xrightarrow{\theta_a} & G(a) \\ {\scriptstyle T(\theta_a)}\downarrow & & & & \uparrow{\scriptstyle G(h_a)} \\ TG(a) & & \xrightarrow[\lambda_a]{} & & GT(a). \end{array}$$

We consider two examples of entwinings which may (also) be considered as generalisations of Hopf algebras. They are different from our approach and we will not refer to them later on.

2.2. Opmonoidal functors. Let $(\mathbb{V}, \otimes, \mathbb{I})$ be a strict monoidal category. Following McCrudden [17, Example 2.5], one may call a monad (T, μ, η) on $\mathbb{V}$ *opmonoidal* if there exist morphisms

$$\theta : T(\mathbb{I}) \to \mathbb{I} \quad \text{and} \quad \chi_{X,Y} : T(X \otimes Y) \to T(X) \otimes T(Y),$$

the latter natural in $X, Y \in \mathbb{V}$, which are compatible with the tensor structure of $\mathbb{V}$ and the monad structure of T.

Such functors can also be characterised by the condition that the tensor product of $\mathbb{V}$ can be lifted to the category of T-modules (e.g. [29, 3.4]). They were introduced and named *Hopf monads* by Moerdijk in [21, Definition 1.1] and called *bimonads* by Bruguières and Virelizier in [7, 2.3]. It is mentioned in [7, Example 2.8] that Szlachányi's bialgebroids in [25] may be interpreted in terms of such "bimonads". It is preferable to use the terminology from [17] since these functors are neither bimonads nor Hopf monads in a strict sense but rather an entwining (as in 2.1) between the monad T and the comonad $T(\mathbb{I}) \otimes -$ on $\mathbb{V}$:

Indeed, the compatibility conditions required in the definitions induce a coproduct $\chi_{\mathbb{I},\mathbb{I}} : T(\mathbb{I}) \to T(\mathbb{I}) \otimes T(\mathbb{I})$ with counit $\theta : T(\mathbb{I}) \to \mathbb{I}$. Moreover, the relation between χ and μ (e.g. (15) in [7, 2.3]) lead to the commutative diagram (using $X \otimes \mathbb{I} = X$)

$$\begin{array}{ccccc} TT(X) & \xrightarrow{\mu} & T(X) & \xrightarrow{\chi_{\mathbb{I},X}} & T(\mathbb{I}) \otimes T(X) \\ {\scriptstyle T(\chi_{\mathbb{I},X})}\downarrow & & & & \uparrow{\scriptstyle T(\mathbb{I})\otimes \mu_X} \\ T(T(\mathbb{I}) \otimes T(X)) & \xrightarrow{\chi_{T(\mathbb{I}),T(X)}} & TT(\mathbb{I}) \otimes TT(X) & \xrightarrow{\mu_{\mathbb{I}} \otimes TT(X)} & T(\mathbb{I}) \otimes TT(X) \end{array}$$

This shows that $T(X)$ is a mixed $(T, T(\mathbb{I}) \otimes -)$-bimodule for the entwining map

$$\lambda = (\mu_{\mathbb{I}} \otimes T(-)) \circ \chi_{T(\mathbb{I}),-} : T(T(\mathbb{I}) \otimes -) \to T(\mathbb{I}) \otimes T(-).$$

The *antipode* of a classical Hopf algebra H is defined as a special endomorphism of H. Since opmonoidal monads T relate two distinct functors it is not surprising that the notion of an antipode can not be transferred easily to this situation and the attempt to do so leads to an "apparently complicated definition" in [7, 3.3 and Remark 3.5]. Hereby the base category $\mathcal{C}$ is required to be *autonomous*.

2.3. Generalized bialgebras and Hopf operads. The *generalised bialgebras* over fields as defined in Loday [15, Section 2.1] are similar to the mixed bimodules (see 2.1): they are vector spaces which are modules over some *operad* $\mathcal{A}$ (Schur functors with multiplication and unit) and comodules over some coalgebras $\mathcal{C}^c$, which are linear duals of some operad $\mathcal{C}$.

Similar to the opmonoidal monads, the coalgebraic structure is based on the tensor product (of vector spaces). The Hypothesis (H0) in [15] resembles the role of the entwining λ in 2.1. The Hypothesis (H1) requires that the free $\mathcal{A}$-algebra is a $(\mathcal{C}^C, \mathcal{A})$-bialgebra: this is similar to the condition on an A-coring C, A an associative algebra, to have a C-comodule structure (equivalently the existence of a group-like element, e.g. [8, 28.2]). The condition (H2iso) plays the role of the canonical isomorphism defining *Galois corings* and the *Galois Coring Structure Theorem* [8, 28.19] may be compared with the *Rigidity Theorem* [15, 2.3.7]. The latter can be considered as a generalisation of the Hopf-Borel Theorem (see [15, 4.1.8]) and of the Cartier-Milnor-Moore Theorem (see [15, 4.1.3]). In [15, 3.2], *Hopf operads* are defined in the sense of Moerdijk [21] and thus the coalgebraic part is dependent on the tensor product. This is only a sketch of the similarities between Loday's setting and our approach here. It will be interesting to work out the relationship in more detail.

Similar to 2.1 we will also need the notion of mixed distributive laws from a comonad to a monad.

2.4. **Entwining from comonad to monad.** A natural transformation $\lambda : GT \to TG$ is a *mixed distributive law* from a comonad $\mathbf{G}$ to a monad $\mathbf{T}$, also called an *entwining* of $\mathbf{G}$ and $\mathbf{T}$, if the diagrams

$$
\begin{array}{ccc}
 & G & \\
{\scriptstyle Ge}\swarrow & & \searrow{\scriptstyle eG} \\
GT & \xrightarrow{\ \lambda\ } & TG\,,
\end{array}
\qquad
\begin{array}{ccc}
GT & \xrightarrow{\ \lambda\ } & TG \\
{\scriptstyle \varepsilon T}\searrow & & \swarrow{\scriptstyle T\varepsilon} \\
 & T & ,
\end{array}
$$

$$
\begin{array}{ccccc}
GTT & \xrightarrow{\lambda T} & TGT & \xrightarrow{T\lambda} & TTG \\
{\scriptstyle Gm}\downarrow & & & & \downarrow{\scriptstyle mG} \\
GT & & \xrightarrow{\quad\lambda\quad} & & TG,
\end{array}
\qquad
\begin{array}{ccccc}
GGT & \xrightarrow{G\lambda} & GTG & \xrightarrow{\lambda G} & TGG \\
{\scriptstyle \delta T}\uparrow & & & & \uparrow{\scriptstyle T\delta} \\
GT & & \xrightarrow{\quad\lambda\quad} & & TG
\end{array}
$$

are commutive.

For convenience we recall the distributive laws between two monads and between two comonads (e.g. [2], [1], [29, 4.4 and 4.9]).

2.5. **Monad distributive.** Let $\mathbf{F} = (F, m, e)$ and $\mathbf{T} = (T, m', e')$ be monads on the category $\mathbb{A}$. A natural transformation $\lambda : FT \to TF$ is said to be *monad distributive* if it induces commutativity of the diagrams

$$
\begin{array}{ccc}
 & T & \\
{\scriptstyle e_T}\swarrow & & \searrow{\scriptstyle Te} \\
FT & \xrightarrow{\ \lambda\ } & TF,
\end{array}
\qquad
\begin{array}{ccc}
 & F & \\
{\scriptstyle Fe'}\swarrow & & \searrow{\scriptstyle e'_F} \\
FT & \xrightarrow{\ \lambda\ } & TF.
\end{array}
$$

$$
\begin{array}{ccccc}
FFT & & \xrightarrow{\quad m_T\quad} & & FT \\
{\scriptstyle F\lambda}\downarrow & & & & \downarrow{\scriptstyle \lambda} \\
FTF & \xrightarrow{\lambda_F} & TFF & \xrightarrow{Tm} & TF,
\end{array}
\qquad
\begin{array}{ccccc}
FTT & & \xrightarrow{\quad Fm'\quad} & & FT \\
{\scriptstyle \lambda_T}\downarrow & & & & \downarrow{\scriptstyle \lambda} \\
TFT & \xrightarrow{T\lambda} & TTF & \xrightarrow{m'_F} & TF.
\end{array}
$$

In this case $\lambda : FT \to TF$ induces a canonical monad structure on TF.

2.6. **Comonad distributive.** Let $\mathbf{G} = (G, \delta, \varepsilon)$ and $\mathbf{T} = (T, \delta', \varepsilon')$ be comonads on the category $\mathbb{A}$. A natural transformation $\varphi : TG \to GT$ is said to be *comonad distributive* if it induces the commutative diagrams

$$\begin{array}{ccc} TG & \xrightarrow{\varphi} & GT \\ & {\scriptstyle T\varepsilon}\searrow \quad \swarrow{\scriptstyle \varepsilon_T} & \\ & T & \end{array}, \qquad \begin{array}{ccc} TG & \xrightarrow{\varphi} & GT \\ & {\scriptstyle \varepsilon'_G}\searrow \quad \swarrow{\scriptstyle G\varepsilon'} & \\ & G & \end{array},$$

$$\begin{array}{ccccc} TG & \xrightarrow{T\delta} & TGG & \xrightarrow{\varphi G} & GTG \\ {\scriptstyle \varphi}\downarrow & & & & \downarrow{\scriptstyle G\varphi} \\ GT & & \xrightarrow{\quad \delta_T \quad} & & GGT, \end{array} \qquad \begin{array}{ccccc} TG & \xrightarrow{\delta'_G} & TTG & \xrightarrow{T\varphi} & TGT \\ {\scriptstyle \varphi}\downarrow & & & & \downarrow{\scriptstyle \varphi_T} \\ GT & & \xrightarrow{\quad G\delta' \quad} & & GTT. \end{array}$$

In this case $\varphi{:}TG \to GT$ induces a canonical comonad structure on TG.

3. Actions on functors and Galois functors

The language of modules over rings can also be used to describe actions of monads on functors. Doing this we define Galois functors and to characterise those we investigate the relationships between categories of relative injective objects.

3.1. **T-actions on functors.** Let $\mathbb{A}$ and $\mathbb{B}$ be categories. Given a monad $\mathbf{T} = (T, m, e)$ on $\mathbb{A}$ and any functor $L : \mathbb{A} \to \mathbb{B}$, we say that L is a *(right)* $\mathbf{T}$*-module* if there exists a natural transformation $\alpha_L : LT \to L$ such that the diagrams

$$(3.1) \qquad \begin{array}{ccc} L & \xrightarrow{Le} & LT \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \alpha_L} \\ & & L, \end{array} \qquad \begin{array}{ccc} LTT & \xrightarrow{Lm} & LT \\ {\scriptstyle \alpha_L T}\downarrow & & \downarrow{\scriptstyle \alpha_L} \\ LT & \xrightarrow[\alpha_L]{} & L \end{array}$$

commute. It is easy to see that (T, m) and (TT, Tm) both are $\mathbf{T}$-modules.

Similarly, given a comonad $\mathbf{G} = (G, \delta, \varepsilon)$ on $\mathbb{A}$, a functor $K : \mathbb{B} \to \mathbb{A}$ is a *left* $\mathbf{G}$*-comodule* if there exists a natural transformation $\beta_K : K \to GK$ for which the diagrams

$$\begin{array}{ccc} K & \xrightarrow{\beta_K} & GK \\ & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \varepsilon K} \\ & & K, \end{array} \qquad \begin{array}{ccc} K & \xrightarrow{\beta_K} & GK \\ {\scriptstyle \beta_K}\downarrow & & \downarrow{\scriptstyle \delta K} \\ GK & \xrightarrow[G\beta_K]{} & GGK \end{array}$$

commute.

Given two $\mathbf{T}$-modules (L, α_L), $(L', \alpha_{L'})$, a natural transformation $g : L \to L'$ is called $\mathbf{T}$*-linear* if the diagram

$$(3.2) \qquad \begin{array}{ccc} LT & \xrightarrow{gT} & L'T \\ {\scriptstyle \alpha_L}\downarrow & & \downarrow{\scriptstyle \alpha_{L'}} \\ L & \xrightarrow[g]{} & L' \end{array}$$

commutes.

3.2. **Lemma.** *Let* (L, α_L) *be a* $\mathbf{T}$*-module. If* $f, f' : TT \to L$ *are* $\mathbf{T}$*-linear morphisms from the* $\mathbf{T}$*-module* (TT, Tm) *to the* $\mathbf{T}$*-module* (L, α_L) *such that* $f \cdot Te = f' \cdot Te$*, then* $f = f'$*.*

Proof. Since $f \cdot Te = f' \cdot Te$, we have $\alpha_L \cdot fT \cdot TeT = \alpha_L \cdot f'T \cdot TeT$. Moreover, since f and f' are both $\mathbf{T}$-linear, we have the commutative diagrams

$$\begin{array}{ccc} TTT & \xrightarrow{fT} & LT \\ {\scriptstyle Tm}\downarrow & & \downarrow{\scriptstyle \alpha_L} \\ TT & \xrightarrow{f} & L, \end{array} \qquad \begin{array}{ccc} TTT & \xrightarrow{f'T} & LT \\ {\scriptstyle Tm}\downarrow & & \downarrow{\scriptstyle \alpha_L} \\ TT & \xrightarrow{f'} & L. \end{array}$$

Thus $\alpha_L \cdot fT = f \cdot Tm$ and $\alpha_L \cdot f'T = f' \cdot Tm$, and we have $f \cdot Tm \cdot TeT = f' \cdot Tm \cdot TeT$. It follows - since $Tm \cdot TeT = 1$ - that $f = f'$. □

3.3. **Left G-comodule functors.** Let $\mathbf{G}$ be a comonad on a category $\mathbb{A}$, let $U^G : \mathbb{A}^G \to \mathbb{A}$ be the forgetful functor and write $\phi^G : \mathbb{A} \to \mathbb{A}^G$ for the cofree G-comodule functor. Fix a functor $F : \mathbb{B} \to \mathbb{A}$, and consider a functor $\overline{F} : \mathbb{B} \to \mathbb{A}^G$ making the diagram

(3.3)
$$\begin{array}{ccc} \mathbb{B} & \xrightarrow{\overline{F}} & \mathbb{A}^G \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle U^G} & \\ & \mathbb{A} & \end{array}$$

commutative. Then $\overline{F}(b) = (F(b), \alpha_{F(b)})$ for some $\alpha_{F(b)} : F(b) \to GF(b)$. Consider the natural transformation

$$\bar{\alpha}_F : F \to GF, \tag{3.4}$$

whose b-component is $\alpha_{F(b)}$. It should be pointed out that $\bar{\alpha}_F$ makes F a left $\mathbf{G}$-comodule, and it is easy to see that there is a one to one correspondence between functors $\overline{F} : \mathbb{B} \to \mathbb{A}^G$ making the diagram (3.3) commute and natural transformations $\bar{\alpha}_F : F \to GF$ making F a left $\mathbf{G}$-comodule.

The following is an immediate consequence of (the dual of) [10, Propositions II,1.1 and II,1.4]:

3.4. **Theorem.** *Suppose that F has a right adjoint $R : \mathbb{A} \to \mathbb{B}$ with unit $\eta : 1 \to RF$ and counit $\varepsilon : FR \to 1$. Then the composite*

$$t_{\overline{F}} : FR \xrightarrow{\bar{\alpha}_F R} GFR \xrightarrow{G\varepsilon} G$$

is a morphism from the comonad $\mathbf{G}' = (FR, F\eta R, \varepsilon)$ generated by the adjunction $\eta, \varepsilon : F \dashv R : \mathbb{A} \to \mathbb{B}$ to the comonad $\mathbf{G}$. Moreover, the assignment

$$\overline{F} \longrightarrow t_{\overline{F}}$$

yields a one to one correspondence between functors $\overline{F} : \mathbb{B} \to \mathbb{A}^G$ making the diagram (3.3) commutative and morphisms of comonads $t_{\overline{F}} : \mathbf{G}' \to \mathbf{G}$.

3.5. **Definition.** We say that a left $\mathbf{G}$-comodule $F : \mathbb{B} \to \mathbb{A}$ with a right adjoint $R : \mathbb{B} \to \mathbb{A}$ is $\mathbf{G}$-*Galois* if the corresponding morphism $t_{\overline{F}} : FR \to \mathbf{G}$ of comonads on $\mathbb{A}$ is an isomorphism.

As an example, consider an A-coring $\mathcal{C}$, A an associative ring, and any right $\mathcal{C}$-comodule P with $S = \mathrm{End}^{\mathcal{C}}(P)$. Then there is a natural transformation

$$\tilde{\mu} : \mathrm{Hom}_A(P, -) \otimes_S P \to - \otimes_A \mathcal{C}$$

and P is called a *Galois comodule* provided $\tilde{\mu}_X$ is an isomorphism for any right A-module X, that is, the functor $- \otimes_S P : \mathbb{M}_S \to \mathbb{M}^{\mathcal{C}}$ is a $- \otimes_A \mathcal{C}$-Galois comodule (see [28, Definiton 4.1]).

3.6. **Right adjoint functor of $\overline{F}$.** When the category $\mathbb{B}$ has equalisers, the functor $\overline{F}$ has a right adjoint, which can be described as follows: Writing β_R for the composite

$$R \xrightarrow{\eta R} RFR \xrightarrow{Rt_{\overline{F}}} RG,$$

it is not hard to see that the equaliser $(\overline{R}, \overline{e})$ of the following diagram

$$RU^G \underset{\beta_R U^G}{\overset{RU^G \eta_G}{\rightrightarrows}} RGU^G = RU^G \phi^G U^G,$$

where $\eta_G : 1 \to \phi^G U^G$ is the unit of the adjunction $U^G \dashv \phi^G$, is right adjoint to $\overline{F}$.

3.7. **Adjoints and monads.** For categories $\mathbb{A}$, $\mathbb{B}$, let $L : \mathbb{A} \to \mathbb{B}$ be a functor with right adjoint $R : \mathbb{B} \to \mathbb{A}$. Let $\mathbf{T} = (T, m, e)$ be a monad on $\mathbb{A}$ and suppose there exists a functor $\overline{R} : \mathbb{B} \to \mathbb{A}_T$ yielding the commutative diagram

$$\begin{array}{ccc} \mathbb{B} & \xrightarrow{\overline{R}} & \mathbb{A}_T \\ & {\scriptstyle R}\searrow \quad \swarrow{\scriptstyle U_T} & \\ & \mathbb{A}. & \end{array}$$

Then $\overline{R}(b) = (R(b), \beta_b)$ for some $\beta_b : TR(b) \to R(b)$ and the collection $\{\beta_b,\, b \in \mathbb{B}\}$ constitutes a natural transformation $\beta_{\overline{R}} : TR \to R$. It is proved in [10] that the natural transformation

$$t_{\overline{R}} : \; T \xrightarrow{T\eta} TRL \xrightarrow{\beta L} RL$$

is a morphism of monads. By the dual of [20, Theorem 4.4], we obtain:

The functor $\overline{R}$ is an equivalence of categories if and only if the functor R is monadic and $t_{\overline{R}}$ is an isomorphism of monads.

4. Bimonads

The following definition was suggested in [29, 5.13]. For monoidal categories similar conditions were considered by Takeuchi [26, Definition 5.1] and in [20]. Notice that the term *bimonad* is used with a different meaning by Bruguières and Virelizier (see 2.2).

4.1. **Definition.** A *bimonad* $\mathbf{H}$ on a category $\mathbb{A}$ is an endofunctor $H : \mathbb{A} \to \mathbb{A}$ which has a monad structure $\underline{H} = (H, m, e)$ and a comonad structure $\overline{H} = (H, \delta, \varepsilon)$ such that

(i) $\varepsilon : H \to 1$ is a morphism from the monad $\underline{H}$ to the identity monad;
(ii) $e : 1 \to H$ is a morphism from the identity comonad to the comonad $\overline{H}$;
(iii) there is a mixed distributive law $\lambda : HH \to HH$ from the monad $\underline{H}$ to the comonad $\overline{H}$ yielding the commutative diagram

(4.1)
$$\begin{array}{ccccc} HH & \xrightarrow{m} & H & \xrightarrow{\delta} & HH \\ {\scriptstyle H\delta}\downarrow & & & & \uparrow{\scriptstyle Hm} \\ HHH & & \xrightarrow[\lambda H]{} & & HHH. \end{array}$$

Note that the conditions (i), (ii) just mean commutativity of the diagrams

(4.2)
$$\begin{array}{ccc} HH & \xrightarrow{H\varepsilon} & H \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ H & \xrightarrow{\varepsilon} & 1, \end{array} \qquad \begin{array}{ccc} 1 & \xrightarrow{e} & H \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle \delta} \\ H & \xrightarrow[eH]{} & HH \end{array}, \qquad \begin{array}{ccc} 1 & \xrightarrow{e} & H \\ & {\scriptstyle =}\searrow & \downarrow{\scriptstyle \varepsilon} \\ & & 1. \end{array}$$

4.2. **Hopf modules.** Given a bimonad $\mathbf{H} = (\underline{H}, \overline{H}, \lambda)$ on $\mathbb{A}$, the objects of $\mathbb{A}^{\overline{H}}_{\underline{H}}(\lambda)$ are called *mixed H-bimodules* or *H-Hopf modules.* By 2.1, they are triples (a, h_a, θ_a), where $(a, h_a) \in \mathbb{A}_{\underline{H}}$ and $(a, \theta_a) \in \mathbb{A}^{\overline{H}}$ with commuting diagram

$$\begin{array}{ccccc} \underline{H}(a) & \xrightarrow{h_a} & a & \xrightarrow{\theta_a} & \overline{H}(a) \\ {\scriptstyle \underline{H}(\theta_a)}\downarrow & & & & \uparrow{\scriptstyle \overline{H}(h_a)} \\ \underline{H}\overline{H}(a) & & \xrightarrow[\lambda_a]{} & & \overline{H}\underline{H}(a). \end{array} \tag{4.3}$$

The morphisms in $\mathbb{A}^{\overline{H}}_{\underline{H}}(\lambda)$ are morphisms in $\mathbb{A}$ which are $\underline{H}$-monad as well as $\overline{H}$-comonad morphisms.

4.3. **Comparison functors.** Given a bimonad $\mathbf{H} = (\underline{H} = (H, m, e), \overline{H} = (H, \delta, \varepsilon), \lambda)$ on a category $\mathbb{A}$, the mixed distributive law λ induces functors

$$K_{\underline{H}} : \mathbb{A} \to (\mathbb{A}_{\underline{H}})^{\widehat{\overline{H}}}, \quad a \mapsto ((H(a), m_a), \delta_a),$$
$$K_{\overline{H}} : \mathbb{A} \to (\mathbb{A}^{\overline{H}})_{\widehat{\underline{H}}}, \quad a \mapsto ((H(a), \delta_a), m_a),$$

where $\widehat{\overline{H}}$ is the lifting of the comonad $\overline{H}$ and $\widehat{\underline{H}}$ is the lifting of the monad $\underline{H}$ corresponding to the mixed distributive law λ. Moreover, there are commutative diagrams

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{K_{\underline{H}}} & (\mathbb{A}_{\underline{H}})^{\widehat{\overline{H}}} \\ & {\scriptstyle \phi_{\underline{H}}}\searrow & \downarrow{\scriptstyle U^{\widehat{\overline{H}}}} \\ & & \mathbb{A}_{\underline{H}}, \end{array} \qquad \begin{array}{ccc} \mathbb{A} & \xrightarrow{K_{\overline{H}}} & (\mathbb{A}^{\overline{H}})_{\widehat{\underline{H}}} \\ & {\scriptstyle \phi^{\overline{H}}}\searrow & \downarrow{\scriptstyle U_{\widehat{\underline{H}}}} \\ & & \mathbb{A}^{\overline{H}}. \end{array} \tag{4.4}$$

(i) The functor $\phi_{\underline{H}}$. The forgetful functor $U_{\underline{H}} : \mathbb{A}_{\underline{H}} \to \mathbb{A}$ is right adjoint to the free functor $\phi_{\underline{H}}$ and the unit $\eta_{\underline{H}} : 1 \to U_{\underline{H}}\phi_{\underline{H}}$ of this adjunction is the natural transformation $e : 1 \to H$. Since $\varepsilon : H \to 1$ is a morphism from the monad $\underline{H}$ to the identity monad, $\varepsilon \cdot e = 1$, thus e is a split monomorphism.

The adjunction $\phi_{\underline{H}} \dashv U_{\underline{H}}$ generates the comonad $\phi_{\underline{H}}U_{\underline{H}}$ on $\mathbb{A}_{\underline{H}}$. Recall that for any $(a, h_a) \in \mathbb{A}_{\underline{H}}$, $\phi_{\underline{H}}U_{\underline{H}}(a, h_a) = (H(a), m_a)$ and $\widehat{\overline{H}}(a, h_a) = (H(a), H(h_a) \cdot \lambda_a)$.

As pointed out in [20], for any object b of $\mathbb{A}$, $K_{\underline{H}}(b) = (H(b), \alpha_{H(b)})$ for some $\alpha : H(b) \to HH(b)$, thus inducing a natural transformation

$$\alpha_{K_{\underline{H}}} : \phi_{\underline{H}} \to \widehat{\overline{H}}\phi_{\underline{H}},$$

whose component at $b \in \mathbb{A}$ is $\alpha_{H(b)}$, we may choose it to be just δ_b, and we have a morphism of comonads

$$t_{K_{\underline{H}}} : \ \phi_{\underline{H}}U_{\underline{H}} \xrightarrow{\alpha_{K_{\underline{H}}}U_{\underline{H}}} \widehat{\overline{H}}\phi_{\underline{H}}U_{\underline{H}} \xrightarrow{\widehat{\overline{H}}\varepsilon_{\underline{H}}} \widehat{\overline{H}},$$

where $\varepsilon_{\underline{H}}$ is the counit of the adjunction $\phi_{\underline{H}} \dashv U_{\underline{H}}$, and since $(\varepsilon_{\underline{H}})_{(a,h_a)} = h_a$, we see that for all $(a, h_a) \in \mathbb{A}_{\underline{H}}$, $(t_{K_{\underline{H}}})_{(a,h_a)}$ is the composite

$$H(a) \xrightarrow{\delta_a} HH(a) \xrightarrow{H(h_a)} H(a). \tag{4.5}$$

(ii) The functor $\phi^{\overline{H}}$. The cofree $\overline{H}$-comodule functor $\phi^{\overline{H}}$ has the forgetful functor $U^{\overline{H}} : \mathbb{A}^{\overline{H}} \to \mathbb{A}$ as a left adjoint. The unit $\eta : 1 \to \phi^{\overline{H}} U^{\overline{H}}$ and counit $\sigma : U^{\overline{H}} \phi^{\overline{H}} \to 1$ of the adjunction $U^{\overline{H}} \dashv \phi^{\overline{H}}$ are given by the formulas:

$$\eta_{(a,\theta_a)} = \theta_a : (a, \theta_a) \to \phi^{\overline{H}} U^{\overline{H}}(a, \theta_a) = (H(a), \delta_a)$$

and

$$\sigma_a = \varepsilon_a : H(a) = U^{\overline{H}} \phi^{\overline{H}}(a) \to a.$$

Since ε is a split epimorphism, it follows from [19, Corollary 3.17] that, when $\mathbb{A}$ is Cauchy complete, the functor $\phi^{\overline{H}}$ is monadic.

Since $K_{\overline{H}}(a) = ((H(a), \delta_a), m_a)$, it is easy to see that the a-component of

$$\alpha_{K_{\overline{H}}} : \widehat{\underline{H}} K_{\overline{H}} \to K_{\overline{H}}$$

is just the morphism $m_a : HH(a) \to H(a)$, and we have a monad morphism

$$t_{K_{\overline{H}}} : \widehat{\underline{H}} \xrightarrow{\widehat{\underline{H}}\eta} \underline{H} \phi^{\overline{H}} U^{\overline{H}} \xrightarrow{\alpha_{K_{\overline{H}}} U^{\underline{H}}} \phi^{\overline{H}} U^{\overline{H}}.$$

It follows that for any $(a, \theta_a) \in \mathbb{A}^{\overline{H}}$, $(t_{K_{\overline{H}}})_{(a,\theta_a)}$ is the composite

$$H(a) \xrightarrow{H(\theta_a)} HH(a) \xrightarrow{m_a} H(a)\,. \tag{4.6}$$

Commutativity of the diagram (4.1) induces a functor

$$K_H : \mathbb{A} \to \mathbb{A}^{\overline{H}}_{\underline{H}}(\lambda), \quad a \mapsto (H(a), m_a, \delta_a).$$

We know that the categories $(\mathbb{A}_{\underline{H}})^{\widehat{\overline{H}}}$, $(\mathbb{A}^{\overline{H}})_{\widehat{\underline{H}}}$ and $\mathbb{A}^{\overline{H}}_{\underline{H}}(\lambda)$ are isomorphic. This allows us to identify, module these isomorphisms, the functors $K_{\overline{H}}$, $K_{\underline{H}}$ and K_H. As noticed in [30, 5.13], the comparison functor K_H (and hence also $K_{\underline{H}}$ and $K^{\overline{H}}$) is full and faithful by the isomorphism

$$\mathrm{Mor}^{\overline{H}}_{\underline{H}}(\underline{H}(a), \overline{H}(b)) \to \mathrm{Mor}_{\mathbb{A}}(a, b), \quad f \mapsto \varepsilon_b \circ f \circ e_a.$$

4.4. **The comparison functor as reflection and coreflection.** *Let $\boldsymbol{H} = (H, m, e, \delta, \varepsilon, \lambda)$ be a bimonad on an arbitrary category $\mathbb{A}$ with comparison functor*

$$K_H : \mathbb{A} \to \mathbb{A}^{\overline{H}}_{\underline{H}}(\lambda).$$

(1) *If $\mathbb{A}$ admits coequalisers, K_H makes $\mathbb{A}$ (isomorphic to) a reflective subcategory of $\mathbb{A}^{\overline{H}}_{\underline{H}}(\lambda)$.*

(2) *If $\mathbb{A}$ admits equalisers, K_H makes $\mathbb{A}$ (isomorphic to) a coreflective subcategory of $\mathbb{A}^{\overline{H}}_{\underline{H}}(\lambda)$.*

Proof. (1) Since the functor K_H is full and faithful, it suffices to show that it has a left adjoint. By assumption the category $\mathbb{A}$ admits coequalisers. Since the functor $U_{\widehat{\underline{H}}} : (\mathbb{A}^{\overline{H}})_{\widehat{\underline{H}}} \to \mathbb{A}^{\overline{H}}$ is (pre)monadic and the functor $\phi^{\overline{H}}$ has a left adjoint, one can apply the Adjoint Triangle Theorem of Dubuc (see, for example, [10]) to the second commutative diagram of (4.4) to conclude that the functor $K_{\overline{H}} : \mathbb{A} \to (\mathbb{A}^{\overline{H}})_{\widehat{\underline{H}}}$ (and hence also K_H) has a left adjoint, proving that $\mathbb{A}$ is (isomorphic to) a reflective subcategory of the category $\mathbb{A}^{\overline{H}}_{\underline{H}}(\lambda)$.

(2) Similar to the above arguments, apply the dual of Dubuc's Adjoint Triangle Theorem to the first commutative diagram of (4.4). □

Recall that a morphism $q : a \to a$ in a category $\mathbb{A}$ is an *idempotent* when $qq = q$, and an idempotent q is said to *split* if q has a factorization $q = i \cdot \bar{q}$ with $\bar{q} \cdot i = 1$. This happens if and only if the equaliser $i = \mathrm{Eq}(1_a, q)$ exists or - equivalently - the coequaliser $\bar{q} = \mathrm{Coeq}(1_a, q)$ exists (e.g. [6, Proposition 1]). The category $\mathbb{A}$ is called *Cauchy complete* provided every idempotent in $\mathbb{A}$ splits.

4.5. **The comparison functor as an equivalence.** *Let $\mathbb{A}$ be a Cauchy complete category. For a bimonad $\mathbf{H} = (\underline{H} = (H, m, e), \overline{H} = (H, \delta, \varepsilon), \lambda)$, the following are equivalent:*

(a) $K_H : \mathbb{A} \to \mathbb{A}^{\overline{H}}_{\underline{H}}(\lambda)$, $a \to (H(a), \delta_a, m_a)$, *is an equivalence of categories;*

(b) $t_{K_{\underline{H}}} : \phi_{\underline{H}} U_{\underline{H}} \to \widehat{\overline{H}}$ *is an isomorphism of comonads;*

(c) *for any* $(a, h_a) \in \mathbb{A}_{\underline{H}}$*, the composite* $H(h_a) \cdot \delta_a$ *is an isomorphism;*

(d) $t_{K_{\overline{H}}} : \widehat{\underline{H}} \to \phi^{\overline{H}} U^{\overline{H}}$ *is an isomorphism of monads;*

(e) *for any* $(a,\, \theta_a) \in \mathbb{A}^{\overline{H}}$ *, the composite* $m_a \cdot H(\theta_a)$ *is an isomorphism.*

Proof. We still identify the functors $K_{\underline{H}}$, $K_{\overline{H}}$ and K_H.

(a)⇔(b) Since $\mathbb{A}$ is Cauchy complete and since the unit $\eta_{\underline{H}} : 1 \to U_{\underline{H}} \phi_{\underline{H}}$ of the adjunction $\phi_{\underline{H}} \dashv U_{\underline{H}}$ is a split monomorphism, the functor $\phi_{\underline{H}}$ is comonadic by the dual of [18, Theorem 6]. Now, by [20, Theorem 4.4.], $K_{\underline{H}}$ is an equivalence if and only if $t_{K_{\underline{H}}}$ is an isomorphism.

(b)⇔(c) and (d)⇔(e). By 4.3, the morphisms in (b) come out as the morphisms in (c), and the morphisms in (d) are just those in (e).

(a)⇔(d) Since ε is a split epimorphism, it follows from [19, Corollary 3.17] that (since $\mathbb{A}$ is Cauchy complete) the functor $\phi^{\overline{H}}$ is monadic and hence K is an equivalence by 3.7. □

5. Antipode

We consider a bimonad $\mathbf{H} = (H, m, e, \delta, \varepsilon, \lambda)$ on any category $\mathbb{A}$.

5.1. **Canonical maps.** Define the composites

$$\begin{aligned} \gamma :\ & HH \xrightarrow{\delta H} HHH \xrightarrow{Hm} HH, \\ \gamma' :\ & HH \xrightarrow{H\delta} HHH \xrightarrow{mH} HH. \end{aligned} \tag{5.1}$$

In the diagram

$$\begin{array}{ccccc} HHH & \xrightarrow{\delta HH} & HHHH & \xrightarrow{HmH} & HHH \\ {\scriptstyle Hm}\downarrow & & \downarrow{\scriptstyle HHm} & & \downarrow{\scriptstyle Hm} \\ HH & \xrightarrow{\delta H} & HHH & \xrightarrow{Hm} & HH, \end{array}$$

the left square commutes by naturality of δ, while the right square commutes by associativity of m. From this we see that γ is left $\underline{H}$-linear as a morphism from (HH, Hm) to itself. A similar diagram shows that γ' is right $\underline{H}$-linear as a morphism from (HH, mH) to itself. Moreover, in the diagram

$$\begin{array}{ccccc} H & \xrightarrow{He} & HH & \xrightarrow{\delta H} & HHH \\ {\scriptstyle \delta}\downarrow & & & \nearrow{\scriptstyle HHe} & \downarrow{\scriptstyle Hm} \\ HH & & = & & HH, \end{array}$$

the top triangle commutes by functoriality of composition, while the bottom triangle commutes because $m \cdot He = 1$. Drawing a similar diagram for $H\delta$ and mH, we obtain

$$\gamma \cdot He = \delta, \quad \gamma' \cdot eH = \delta. \tag{5.2}$$

5.2. **Definition.** A natural transformation $S : H \to H$ is said to be

- a *left antipode* if $m \cdot (SH) \cdot \delta = e \cdot \varepsilon$;
- a *right antipode* if $m \cdot (HS) \cdot \delta = e \cdot \varepsilon$;
- an *antipode* if it is a left and a right antipode.

A bimonad **H** is said to be a *Hopf monad* provided it has an antipode.

Following the pattern of the proof of [8, 15.2] we obtain:

5.3. **Proposition.** *We refer to the notation in 5.1.*

(1) *If γ has an $\overline{H}$-linear left inverse, then* **H** *has a left antipode.*

(2) *If γ' has an $\overline{H}$-linear left inverse, then* **H** *has a right antipode.*

Proof. (1) Suppose there exists an **H**-linear morphism $\beta : HH \to HH$ with $\beta \cdot \gamma = 1$. Consider the composite

$$S : H \xrightarrow{He} HH \xrightarrow{\beta} HH \xrightarrow{\varepsilon H} H.$$

We claim that S is a left antipode of **H**. Indeed, in the diagram

$$\begin{array}{ccccccccc}
H & \xrightarrow{\delta} & HH & \xrightarrow{HeH} & HHH & \xrightarrow{\beta H} & HHH & \xrightarrow{\varepsilon HH} & HH \\
 & & & \searrow^{=} & \downarrow{\scriptstyle Hm} & (1) & \downarrow{\scriptstyle Hm} & (2) & \downarrow{\scriptstyle m} \\
 & & & & HH & \xrightarrow[\beta]{} & HH & \xrightarrow[\varepsilon H]{} & H,
\end{array}$$

the *triangle* commutes since e is the unit for the monad $\underline{H}$, *rectangle* (1) commutes by $\overline{H}$-linearity of β, and *rectangle* (2) commutes by naturality of ε. Thus

$$m \cdot SH \cdot \delta = m \cdot \varepsilon HH \cdot \beta H \cdot HeH \cdot \delta = \varepsilon H \cdot \beta \cdot \delta,$$

and using (5.2), we have

$$\varepsilon H \cdot \beta \cdot \delta = \varepsilon H \cdot \beta \cdot \gamma \cdot He = \varepsilon H \cdot He = e \cdot \varepsilon.$$

Therefore S is a left antipode of **H**.

(2) Denoting the left inverse of γ' by β', it is shown along the same lines that $S' = H\varepsilon \cdot \beta' \cdot eH$ is a right antipode. □

5.4. **Lemma.** *Suppose that γ is an epimorphism. If $f, g : H \to H$ are two natural transformations such that*

$$m \cdot fH \cdot \delta = m \cdot gH \cdot \delta \quad \text{or} \quad m \cdot Hf \cdot \delta = m \cdot Hg \cdot \delta,$$

then $f = g$.

Proof. Assume $m \cdot fH \cdot \delta = m \cdot gH \cdot \delta$. Since $\gamma \cdot He = \delta$ by (5.2), we have

$$m \cdot fH \cdot \gamma \cdot He = m \cdot gH \cdot \gamma \cdot He,$$

and, since γ is also $\underline{H}$-linear, it follows by Lemma 3.2 that

$$m \cdot fH \cdot \gamma = m \cdot gH \cdot \gamma.$$

But γ is an epimorphism by our assumption, thus

$$m \cdot fH = m \cdot gH.$$

By naturality of $e : 1 \to H$, we have the commutative diagrams

$$\begin{array}{ccc} H & \xrightarrow{f} & H \\ {\scriptstyle He}\downarrow & & \downarrow{\scriptstyle He} \\ HH & \xrightarrow{fH} & HH, \end{array} \qquad \begin{array}{ccc} H & \xrightarrow{g} & H \\ {\scriptstyle He}\downarrow & & \downarrow{\scriptstyle He} \\ HH & \xrightarrow{gH} & HH. \end{array}$$

Thus, since $m \cdot He = 1$,

$$f = m \cdot He \cdot f = m \cdot fH \cdot He = m \cdot gH \cdot He = m \cdot He \cdot g = g.$$

If $m \cdot Hf \cdot \delta = m \cdot Hg \cdot \delta$ similar arguments apply. □

5.5. **Characterising Hopf monads.** *Let* $\mathbf{H} = (H, m, e, \delta, \varepsilon, \lambda)$ *be a bimonad. The following are equivalent:*

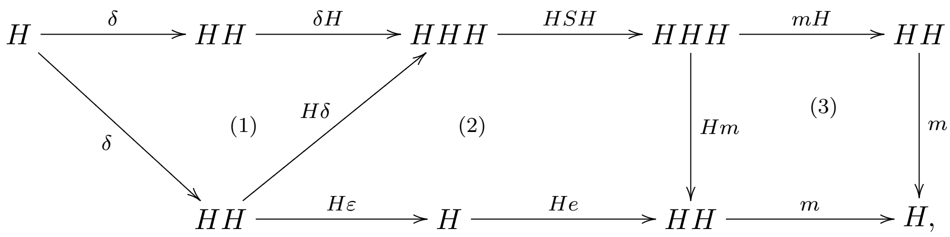
(a) $\gamma = Hm \cdot \delta H : HH \to HH$ *is an isomorphism;*

(b) $\gamma' = mH \cdot H\delta : HH \to HH$ *is an isomorphism;*

(c) $\mathbf{H}$ *has an antipode.*

Proof. (c)⇒(a) The proof for [20, Proposition 6.10] applies almost literally.

(a)⇒(c) Write $\beta : HH \to HH$ for the inverse of γ. Since γ is $\underline{H}$-linear, it follows that β also is $\underline{H}$-linear. Then, by Proposition 5.3, $S = \varepsilon H \cdot \beta \cdot He$ is a left antipode of $\mathbf{H}$. We show that S is also a right antipode of $\mathbf{H}$. In the diagram



- (1) commutes by coassociativity of δ,
- (2) commutes because S is a left antipode of $\mathbf{H}$,
- (3) commutes by associativity of m.

Since $m \cdot He = 1 = m \cdot eH$ and $H\varepsilon \cdot \delta = 1 = \varepsilon H \cdot \delta$, it follows that

$$\begin{aligned} m \cdot (m \cdot HS \cdot \delta)H \cdot \delta \;&=\; m \cdot mH \cdot HSH \cdot \delta H \cdot \delta \;=\; m \cdot He \cdot H\varepsilon \cdot \delta \\ &=\; m \cdot eH \cdot \varepsilon H \cdot \delta \;=\; m \cdot ((e \cdot \varepsilon)H) \cdot \delta. \end{aligned}$$

γ being an epimorphism, Lemma 5.4 implies $m \cdot HS \cdot \delta = e \cdot \varepsilon$, proving that S is also a right antipode of $\mathbf{H}$.

(b)⇔(c) can be shown in a similar way. □

Combining 5.5 and 4.5, we get:

5.6. **Antipode and equivalence - 1.** *Let* $\mathbf{H} = (H, m, e, \delta, \varepsilon, \lambda)$ *be a bimonad on a category* $\mathbb{A}$ *and assume that* $\mathbb{A}$ *admits colimits or limits and* H *preserves them. Then the following are equivalent:*

(a) $\mathbf{H}$ *has an antipode;*

(b) $\gamma = Hm \cdot \delta H : HH \to HH$ *is an isomorphism;*

(c) $\gamma' = mH \cdot H\delta : HH \to HH$ *is an isomorphism;*

(d) $K_H : \mathbb{A} \to \mathbb{A}^{\overline{H}}_{\underline{H}}(\lambda),\ a \to (H(a), \delta_a, m_a)$, *is an equivalence.*

Proof. (a)⇔(b)⇔(c) (in any category) is shown in 5.5.

If $\mathbb{A}$ admits colimits and H preserves them, then we claim that (b) and (d) are equivalent. Indeed, since H preserves colimits, the category $\mathbb{A}_{\underline{H}}$ admits colimits and the functor $U_{\underline{H}} : \mathbb{A}_{\underline{H}} \to \mathbb{A}$ creates them (see, for example, [23]). Thus

- the functor $\phi_{\underline{H}} U_{\underline{H}}$ preserves colimits;
- any functor $L : \mathbb{B} \to \mathbb{A}_{\underline{H}}$ preserves colimits if and only if the composite $U_{\underline{H}} L$ does; so, in particular, the functor $\widehat{\overline{H}}$ preserves colimits, since $U_{\underline{H}} \widehat{\overline{H}} = H U_{\underline{H}}$ and since the functor $H U_{\underline{H}}$, being the composite of two colimit-preserving functors, is colimit-preserving.

The full subcategory of $\mathbb{A}_{\underline{H}}$ given by the free $\underline{H}$-modules is dense and since the functors $\phi_{\underline{H}} U_{\underline{H}}$ and $\widehat{\overline{H}}$ both preserve colimits, it follows from [23, Theorem 17.2.7] that the natural transformation (see 4.5)

$$t_{K_{\underline{H}}} : \phi_{\underline{H}} U_{\underline{H}} \to \widehat{\overline{H}}$$

is an isomorphism if and only if its restriction to the free $\underline{H}$-modules is so; i.e. if $(t_{K_{\underline{H}}})_{\phi_{\underline{H}}(a)}$ is an isomorphism for all $a \in \mathbb{A}$. But since $\phi_{\underline{H}}(a) = (H(a), m_a)$, $t_{K_{\underline{H}}}$ is an isomorphism if and only if the composite

$$HH(a) \xrightarrow{\delta_{H(a)}} HHH(a) \xrightarrow{H(m_a)} HH(a)$$

is an isomorphism for all $a \in \mathbb{A}$, that is, the isomorphism

$$\gamma : \; HH \xrightarrow{\delta H} HHH \xrightarrow{Hm} HH \; .$$

Now, if instead $\mathbb{A}$ admits limits and H preserves them, then (c) and (d) are equivalent. Indeed, since the functor H preserves limits, the category $\mathbb{A}^{\overline{H}}$ admits and the functor $U^{\overline{H}}$ creates limits. Since $\phi^{\overline{H}}$, being right adjoint, preserves limits, the functor $\phi^{\overline{H}} U^{\overline{H}}$ also preserves limits. Moreover, since the monad $\widehat{\underline{H}}$ is a lifting of the monad $\underline{H}$ along the functor $U^{\overline{H}}$, $U^{\overline{H}} \widehat{\underline{H}} = \underline{H} U^{\overline{H}}$, implying that the functor $\widehat{\underline{H}}$ also preserves limits. Now, since the full subcategory of $\mathbb{A}^{\overline{H}}$ spanned by cofree $\overline{H}$-comodules is codense, it follows from the dual of [23, Theorem 17.2.7] that the natural transformation $t_{K_{\overline{H}}}$ (see 4.5) is an isomorphism if and only if its restriction to free $\overline{H}$-comodules is so. But for any $a \in \mathbb{A}$, $(t_{K_{\overline{H}}})_{(H(a), \delta_a)} = m_{H(a)} \cdot H(\delta_a)$. Thus $t_{K_{\overline{H}}}$ is an isomorphism if and only if the composite γ' is an isomorphism. □

6. Local prebraidings for Hopf monads

For any category $\mathbb{A}$ we now fix a system $\mathbf{H} = (H, m, e, \delta, \varepsilon)$ consisting of an endofunctor $H : \mathbb{A} \to \mathbb{A}$ and natural transformations $m : HH \to H$, $e : 1 \to H$, $\delta : H \to HH$ and $\varepsilon : H \to 1$ such that the triple $\underline{H} = (H, m, e)$ is a monad and the triple $\overline{H} = (H, \delta, \varepsilon)$ is a comonad on $\mathbb{A}$.

6.1. **Double entwinings.** A natural transformation $\tau : HH \to HH$ is called a *double entwining* if

(i) τ is a mixed distributive law from the monad $\underline{H}$ to the comonad $\overline{H}$;

(ii) τ is a mixed distributive law from the comonad $\overline{H}$ to the monad $\underline{H}$.

These conditions are obviously equivalent to

(iii) τ is a monad distributive law for the monad $\underline{H}$;

(iv) τ is a comonad distributive law for the comonad $\overline{H}$.

Explicitely (i) encodes the identities

$$He = \tau \cdot eH \tag{6.1}$$

$$H\varepsilon = \varepsilon H \cdot \tau \tag{6.2}$$

$$\delta H \cdot \tau = H\tau \cdot \tau H \cdot H\delta \tag{6.3}$$

$$\tau \cdot mH = Hm \cdot \tau H \cdot H\tau, \tag{6.4}$$

and (ii) is equivalent to the identities

$$eH = \tau \cdot He \tag{6.5}$$

$$\varepsilon H = H\varepsilon \cdot \tau \tag{6.6}$$

$$H\delta \cdot \tau = \tau H \cdot H\tau \cdot \delta H \tag{6.7}$$

$$\tau \cdot Hm = mH \cdot H\tau \cdot \tau H. \tag{6.8}$$

6.2. **τ-bimonad.** Let $\tau : HH \to HH$ be a double entwining. Then $\mathbf{H}$ is called a *τ-bimonad* provided the diagram

$$\begin{array}{ccccc} HH & \xrightarrow{m} & H & \xrightarrow{\delta} & HH \\ \Big\downarrow{\scriptstyle \delta\delta} & & & & \Big\uparrow{\scriptstyle mm} \\ HHHH & & \xrightarrow[H\tau H]{} & & HHHH \end{array} \tag{6.9}$$

is commutative, that is

$$\delta \cdot m = mm \cdot H\tau H \cdot \delta\delta = Hm \cdot mHH \cdot H\tau H \cdot HH\delta \cdot \delta H,$$

and also the following diagrams commute

$$\begin{array}{ccc} HH & \xrightarrow{H\varepsilon} & H \\ {\scriptstyle m}\Big\downarrow & & \Big\downarrow{\scriptstyle \varepsilon} \\ H & \xrightarrow{\varepsilon} & 1, \end{array} \qquad \begin{array}{ccc} 1 & \xrightarrow{e} & H \\ {\scriptstyle e}\Big\downarrow & & \Big\downarrow{\scriptstyle \delta} \\ H & \xrightarrow[eH]{} & HH, \end{array} \qquad \begin{array}{ccc} 1 & \xrightarrow{e} & H \\ & \searrow^{=} & \Big\downarrow{\scriptstyle \varepsilon} \\ & & 1. \end{array} \tag{6.10}$$

6.3. **Proposition.** *Let $\mathbf{H}$ be a τ-bimonad. Then the composite*

$$\tilde{\tau} : HH \xrightarrow{\delta H} HHH \xrightarrow{H\tau} HHH \xrightarrow{mH} HH$$

is a mixed distributive law from the monad $\underline{H}$ to the comonad $\overline{H}$. Thus $\mathbf{H}$ is a bimonad (as in 4.1) with mixed distributive law $\tilde{\tau}$.

Proof. We have to show that $\tilde{\tau}$ satisfies

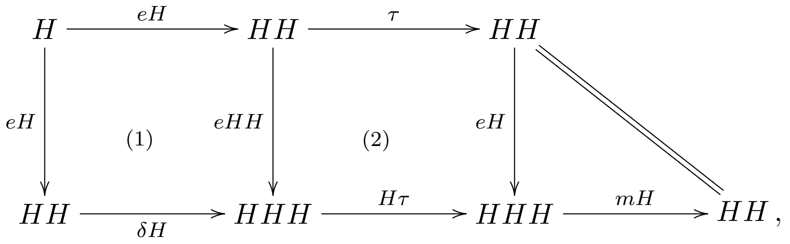
$$He = \tilde{\tau} \cdot eH \tag{6.11}$$

$$H\varepsilon = \varepsilon H \cdot \tilde{\tau} \tag{6.12}$$

$$\delta H \cdot \tilde{\tau} = H\tilde{\tau} \cdot \tilde{\tau} H \cdot H\delta \tag{6.13}$$

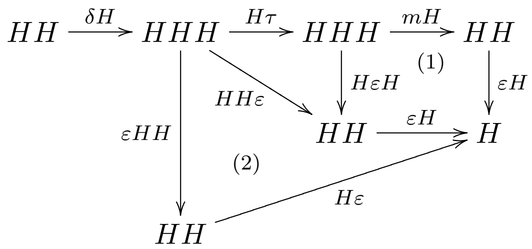
$$\tilde{\tau} \cdot mH = Hm \cdot \tilde{\tau} H \cdot H\tilde{\tau} \tag{6.14}$$

Consider the diagram



which is commutative since *square* (1) commutes by (6.10), *square* (2) commutes by functoriality of composition, the *triangle* commutes since e is the identity of the monad $\underline{H}$. Thus $\tilde{\tau} \cdot eH = mH \cdot H\tau \cdot \delta H \cdot eH = \tau \cdot eH$, and (6.1) implies $\tilde{\tau} \cdot eH = He$, showing (6.11).

Consider now the diagram



in which *square* (1) commutes because ε is a morphism of monads and thus $\varepsilon \cdot m = \varepsilon \cdot H\varepsilon$, the *triangle* commutes because of (6.2), *diagram* (2) commutes because of functoriality of composition.

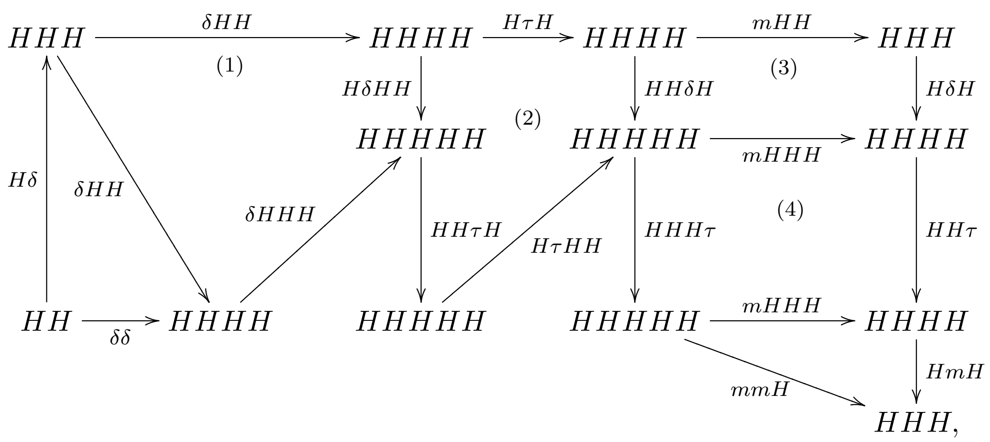
Thus $\varepsilon H \cdot \tilde{\tau} = \varepsilon H \cdot mH \cdot H\tau \cdot \delta H = H\varepsilon \cdot \varepsilon HH \cdot \delta H = H\varepsilon$, showing (6.12).

Constructing suitable commutative diagram we can show

$$\begin{aligned}
\tilde{\tau} \cdot mH &= mH \cdot H\tau \cdot \delta H \cdot mH \\
&= mH \cdot HHm \cdot HmHH \cdot HH\tau H \cdot H\tau HH \cdot HHH\tau \cdot \delta\delta H, \\
Hm \cdot \tilde{\tau} H \cdot H\tilde{\tau} &= Hm \cdot mHH \cdot H\tau H \cdot \delta HH \cdot HmH \cdot HH\tau \cdot H\delta H \\
&= mH \cdot HHm \cdot HmHH \cdot HH\tau H \cdot H\tau HH \cdot HHH\tau \cdot \delta\delta H.
\end{aligned}$$

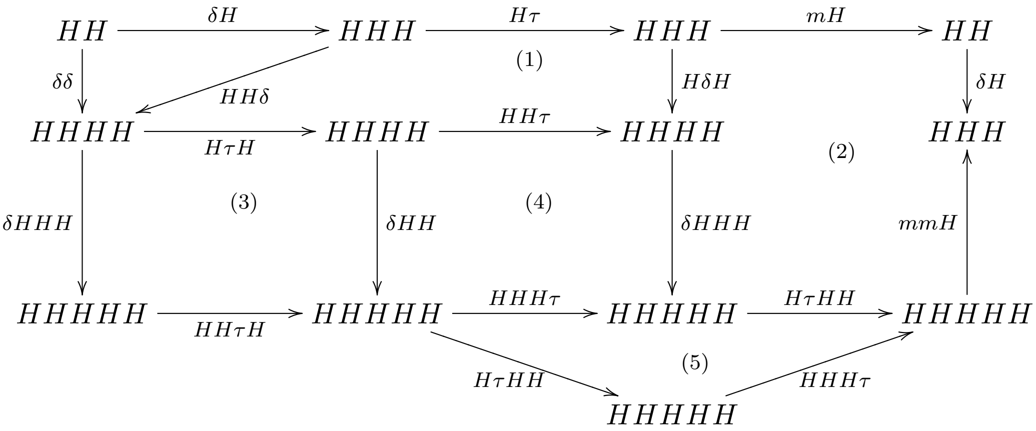
Comparing these two identities we get the condition (6.14).

To show that (6.13) also holds, consider the diagram



in which the *triangles* and *diagrams* (1) and (3) commute by functoriality of composition; *diagram* (2) commutes by (6.7); *diagram* (4) commutes by naturality of m.

Finally we construct the diagram



in which *diagram* (1) commutes by (6.3), *diagram* (2) commutes by (6.9) because $\delta HHH \cdot H\delta H = \delta\delta H$, the *triangle* and *diagrams* (3), (4) and (5) commute by functoriality of composition.

It now follows from the commutativity of these diagrams that

$$\begin{aligned} \delta H \cdot \tilde{\tau} &= \delta H \cdot mH \cdot H\tau \cdot \delta H \\ &= mmH \cdot HHH\tau \cdot H\tau HH \cdot HH\tau H \cdot \delta HHH \cdot \delta\delta \\ &= (HmH \cdot HH\tau \cdot H\delta H) \cdot (mHH \cdot H\tau H \cdot \delta HH) \cdot H\delta \\ &= H\tilde{\tau} \cdot \tilde{\tau} H \cdot H\delta. \end{aligned}$$

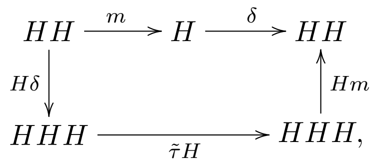
Therefore $\tilde{\tau}$ satisfies the conditions (6.11)-(6.14) and hence is a mixed distributive law from the monad $\underline{H}$ to the comonad $\overline{H}$. □

6.4. **Corollary.** *In the situation of the previous proposition, if* $(a, \theta_a) \in \mathbb{A}^{\overline{H}}$*, then* $(H(a), \theta_{H(a)}) \in \mathbb{A}^{\overline{H}}$*, where* $\theta_{H(a)}$ *is the composite*

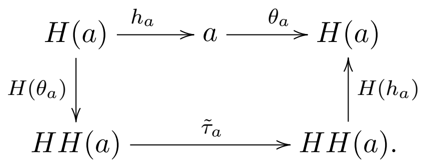
$$H(a) \xrightarrow{H(\theta_a)} HH(a) \xrightarrow{\delta_{H(a)}} HHH(a) \xrightarrow{H\tau_a} HHH(a) \xrightarrow{m_{H(a)}} HH(a)\ .$$

Proof. Write $\widehat{\underline{H}}$ for the monad on the category $\mathbb{A}^{\overline{H}}$ that is the lifting of $\underline{H}$ corresponding to the mixed distributive law $\tilde{\tau}$. Since $\theta_{H(a)} = \tilde{\tau}_a \cdot H(\theta_a)$, it follows that $(H(a), \theta_{H(a)}) = \widehat{\underline{H}}(a, \theta_a)$, and thus $(H(a), \theta_{H(a)})$ is an object of the category $\mathbb{A}^{\overline{H}}$. □

6.5. **τ-Bimodules.** Given the conditions of Proposition 6.3, we have the commutative diagram (see (4.1))

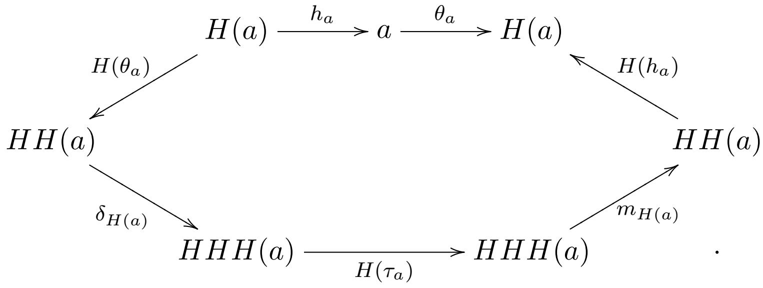


and thus H is a bimonad by the entwining $\tilde{\tau}$ and the mixed bimodules are objects a in $\mathbb{A}$ with a module structure $h_a : H(a) \to a$ and a comodule structure $\theta_a : a \to H(A)$ with a commutative diagram



By definition of $\tilde{\tau}$, commutativity of this diagram is equivalent to the commtativity of

(6.15)



A morphism $f : (a, h_a, \theta_a) \to (a', h_{a'}, \theta_{a'})$ is a morphism $f : a \to a'$ such that $f \in \mathbb{A}^{\overline{H}}$ and $f \in \mathbb{A}_{\underline{H}}$.

We denote the category $\mathbb{A}^{\overline{H}}_{\underline{H}}(\tilde{\tau})$ by $\mathbb{A}^{H}_{H}$.

6.6. **Antipode of a τ-bimonad.** *Let $\mathbf{H} = (H, m, e, \delta, \varepsilon)$ be a τ-bimonad with an antipode S where $\tau : HH \to HH$ is a double entwining. Then*

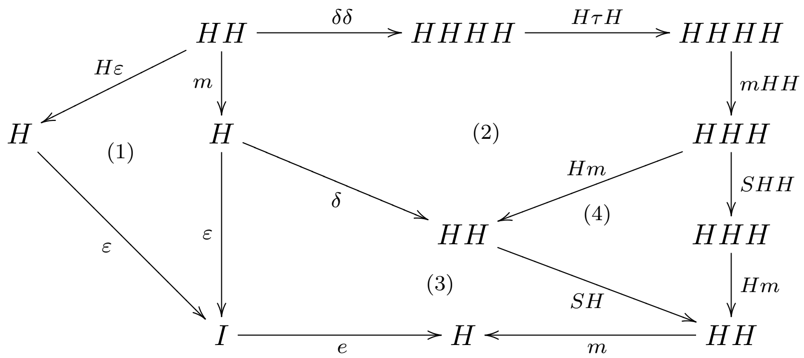
$$S \cdot m = m \cdot SS \cdot \tau \quad \text{and} \quad \delta \cdot S = \tau \cdot SS \cdot \delta. \tag{6.16}$$

If $\tau \cdot HS = SH \cdot \tau$ and $\tau \cdot SH = HS \cdot \tau$, then $S : H \to H$ is a monad as well as a comonad morphism.

Proof. Since $(HH, H\tau H \cdot \delta, \varepsilon\varepsilon)$ is a comonad and (H, m, e) is a monad, the collection $\mathrm{Nat}(HH, H)$ of all natural transformations from HH to H forms a semigroup with unit $e \cdot \varepsilon\varepsilon$ and with product

$$f * g : HH \xrightarrow{\delta\delta} HHHH \xrightarrow{H\tau H} HHHH \xrightarrow{fg} HH \xrightarrow{m} H\,.$$

Consider now the diagram



in which the diagrams (1),(2) and (3) commute because H is a bimonad, while diagram (4) commutes by naturality. It follows that

$$m \cdot Hm \cdot SHH \cdot mHH \cdot H\tau H \cdot \delta\delta = e \cdot \varepsilon \cdot H\varepsilon = \varepsilon\varepsilon \cdot e.$$

Thus $S \cdot m = m^{-1}$ in $\mathrm{Nat}(HH, H)$. Furthermore, by (a somewhat tedious) computation we can show

$$m \cdot Hm \cdot HHS \cdot HSH \cdot H\tau \cdot mHH \cdot H\tau H \cdot \delta\delta = e \cdot \varepsilon \cdot H\varepsilon = e \cdot \varepsilon\varepsilon.$$

This shows that $m \cdot SS \cdot \tau = m^{-1}$ in $\mathrm{Nat}(HH, H)$. Thus $m \cdot SS \cdot \tau = S \cdot m$.

To prove the formula for the coproduct consider $\mathrm{Nat}(H, HH)$ as a monoid with unit $ee \cdot \varepsilon$ and the convolution product for $f, g \in \mathrm{Nat}(H, HH)$ given by

$$f * g : H \xrightarrow{\delta} HH \xrightarrow{fH} HHH \xrightarrow{HHg} HHHH \xrightarrow{mm} HH\,.$$

By computation we get

$$\begin{array}{rcl}(\delta \cdot S) * \delta & = & eH \cdot e \cdot \varepsilon = ee \cdot \varepsilon, \\ \delta * (\tau \cdot SS \cdot \delta) & = & He \cdot e \cdot \varepsilon = ee \cdot \varepsilon.\end{array}$$

Thus $(\delta \cdot S) * \delta = 1$ and $\delta * (\tau \cdot SS \cdot \delta) = 1$, and hence $\delta \cdot S = \tau \cdot SS \cdot \delta$.

Now assume $\tau \cdot HS = SH \cdot \tau$ and $\tau \cdot SH = HS \cdot \tau$. Then we have

$$SS \cdot \tau = SH \cdot HS \cdot \tau = SH \cdot \tau \cdot SH = \tau \cdot HS \cdot SH = \tau \cdot SS, \text{ thus}$$

$$S \cdot m = m \cdot SS \cdot \tau = m \cdot \tau \cdot SS = m' \cdot SS.$$

Moreover, since $m \cdot He = 1$, we have

$$S \cdot e = m \cdot He \cdot S \cdot e \overset{\text{nat}}{=} m \cdot SH \cdot He \cdot e \overset{(6.10)}{=} m \cdot SH \cdot \delta \cdot e \overset{\text{antip.}}{=} e \cdot \varepsilon \cdot e \overset{(6.10)}{=} e\,.$$

Hence S is a monad morphism from (H, m, e) to $(H, m \cdot \tau, e)$.

For the coproduct, $SS \cdot \tau = \tau \cdot SS$ implies

$$\delta \cdot S = \tau \cdot SS \cdot \delta = SS \cdot \tau \cdot \delta = SS \cdot \delta'.$$

Furthermore,

$$\varepsilon \cdot S = \varepsilon \cdot S \cdot H\varepsilon \cdot \delta \overset{\text{nat}}{=} \varepsilon \cdot H\varepsilon \cdot SH \cdot \delta \overset{(6.10)}{=} \varepsilon \cdot m \cdot SH \cdot \delta \overset{\text{antip.}}{=} \varepsilon \cdot e \cdot \varepsilon \overset{(6.10)}{=} \varepsilon.$$

This shows that S is a comonad morphism from (H, δ, ε) to $(H, \tau \cdot \delta, \varepsilon)$. □

It is readily checked that for a bimonad H, the composite HH is again a comonad as well as a monad. However, the compatibility between these two structures needs an additional property of the double entwining τ. This will also help to construct a bimonad "opposite" to H.

6.7. **Local prebraiding.** Let $\tau : HH \to HH$ be a natural transformation. τ is said to satisfy the *Yang-Baxter equation (YB)* if it induces commutativity of the diagram

$$\begin{array}{ccccc} HHH & \xrightarrow{\tau H} & HHH & \xrightarrow{H\tau} & HHH \\ {\scriptstyle H\tau}\downarrow & & & & \downarrow{\scriptstyle \tau H} \\ HHH & \xrightarrow[\tau H]{} & HHH & \xrightarrow[H\tau]{} & HHH\,. \end{array}$$

τ is called a *local prebraiding* provided it is a double entwining (see 6.1) and satisfies the Yang-Baxter equation.

6.8. **Doubling a bimonad.** *Let* $\mathbf{H} = (H, m, e, \delta, \varepsilon)$ *be a* τ*-bimonad where* $\tau : HH \to HH$ *is a local prebraiding. Then* $\mathbf{HH} = (HH, \bar{m}, \bar{e}, \bar{\delta}, \bar{\varepsilon})$ *is a* $\bar{\tau}$*-bimonad with* $\bar{e} = ee$, $\bar{\varepsilon} = \varepsilon\varepsilon$,

$$\bar{m} :\ HHHH \xrightarrow{H\tau H} HHHH \xrightarrow{mm} HH\,,$$

$$\bar{\delta} :\ HH \xrightarrow{\delta\delta} HHHH \xrightarrow{H\tau H} HHHH$$

and double entwining

$$\bar{\tau} :\ HHHH \xrightarrow{H\tau H} HHHH \xrightarrow{\tau HH} HHHH \xrightarrow{HH\tau} HHHH \xrightarrow{H\tau H} HHHH\,.$$

Proof. We already know that $(HH, \bar{m}, \bar{e})$ is a monad and that $(HH, \bar{\delta}, \bar{\varepsilon})$ is a comonad. First we have to show that $\bar{\tau}$ is a mixed distributive law from the monad $(HH, \bar{m}, \bar{e})$ to the comonad$(HH, \bar{\delta}, \bar{\varepsilon})$, that is

$$HH\bar{e} = \bar{\tau} \cdot \bar{e}HH, \quad HH\bar{\varepsilon} = \bar{\varepsilon}HH \cdot \bar{\tau},$$
$$HH\bar{m} \cdot \bar{\tau}HH \cdot HH\bar{\tau} = \bar{\tau} \cdot \bar{m}HH,$$
$$HH\bar{\tau} \cdot \bar{\tau}HH \cdot HH\bar{\delta} = \bar{\delta}HH \cdot \bar{\tau}.$$

The first two equalities can be verified by placing the composites in suitable commutative diagrams. The second two identities are obtained by lengthy standard computations (as known for classical Hopf algebras).

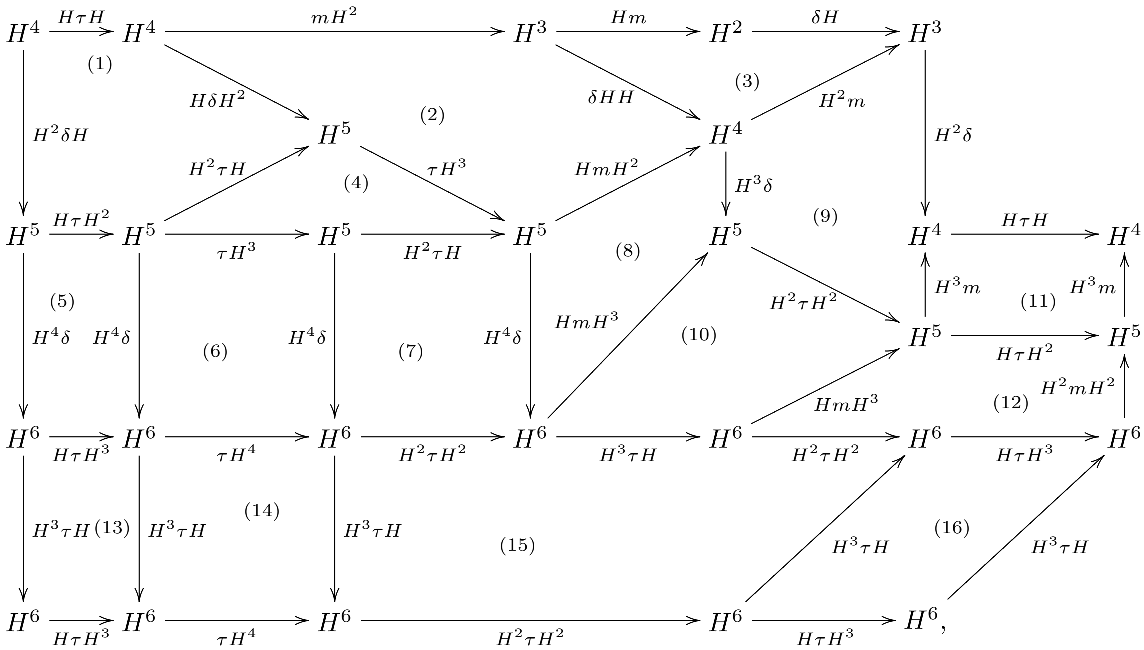
It remains to show that $(HH, \bar{m}, \bar{e}, \bar{\delta}, \bar{\varepsilon})$ satisfies the conditions of Definition 4.1 with respect to $\bar{\tau}$. Again

$$\begin{array}{rcl} \bar{\varepsilon} \cdot \bar{m} & = & \varepsilon \cdot H\varepsilon \cdot HH\varepsilon\varepsilon \;=\; \bar{\varepsilon} \cdot HH\bar{\varepsilon}, \text{ and} \\ \bar{\delta} \cdot \bar{e} & = & HHee \cdot He \cdot e = HH\bar{e}\bar{e} \end{array}$$

are shown by standard computations and

$$\bar{\varepsilon}\bar{e} = \varepsilon \cdot \varepsilon H \cdot eH \cdot e \overset{(4.2)}{=} \varepsilon \cdot e = 1.$$

To show that $(HH, \bar{m}, \bar{e}, \bar{\delta}, \bar{\varepsilon}, \bar{\tau})$ satisfies (4.1), consider the diagram



in which *diagram* (1) commutes because τ is a mixed distributive law and thus

$$H\tau \cdot \tau H \cdot H\delta = \delta H \cdot \tau,$$

the *diagrams* (2) and (9) commute by (4.1), the *diagrams* (3)-(8), (10), (11), (13), (14) and (16) commute by naturality, *diagram* (12) commutes because τ is a mixed distributive law (hence $Hm \cdot \tau H \cdot H\tau = \tau \cdot mH$), *diagram* (15) commutes by 6.7. By commutativity of the whole diagram,

$$\begin{array}{rl} \bar{\delta} \cdot \bar{m} & = H\tau H \cdot H^2\delta \cdot \delta H \cdot Hm \cdot mH^2 \cdot H\tau H \\ & = H^2m \cdot H^2mH^2 \cdot H^3\tau H \cdot H\tau H^3 \cdot H^2\tau H^2 \cdot \tau H^4 \cdot H\tau H^3 \cdot H^3\tau H \cdot H^4\delta \cdot H^2\delta H \\ & = HH\bar{\delta} \cdot \bar{\tau}HH \cdot HH\bar{m}, \end{array}$$

and hence $\mathbf{HH} = (HH, \bar{m}, \bar{e}, \bar{\delta}, \bar{\varepsilon})$ is a $\bar{\tau}$-bimonad. □

6.9. Opposite monad and comonad. *Let $\tau : HH \to HH$ be a natural transformation satisfying the Yang-Baxter equation.*

1. *If (H, m, e) is a monad and τ is monad distributive, then $(H, m \cdot \tau, e)$ is also a monad and τ is monad distributive for it.*
2. *If (H, δ, ε) is a comonad and τ is comonad distributive, then $(H, \tau \cdot \delta, \varepsilon)$ is also a comonad and τ is comonad distributive for it.*

Proof. (1) To show that $m \cdot \tau$ is associative construct the diagram

$$\begin{array}{ccccccc}
HHH & & \xrightarrow{\tau H} & & HHH & \xrightarrow{mH} & HH \\
 & & & & \downarrow{\scriptstyle H\tau} & & \\
\downarrow{\scriptstyle H\tau} & & (1) & & HHH & (2) & \downarrow{\scriptstyle \tau} \\
 & & & & \downarrow{\scriptstyle \tau H} & & \\
HHH & \xrightarrow{\tau H} & HHH & \xrightarrow{H\tau} & HHH & \xrightarrow{Hm} & HH \\
\downarrow{\scriptstyle Hm} & & (3) & & \downarrow{\scriptstyle mH} & (4) & \downarrow{\scriptstyle m} \\
HH & & \xrightarrow{\tau} & & HH & \xrightarrow{m} & H,
\end{array}$$

where the *rectangle* (1) is commutative by the YB-condition, (2) and (3) are commutative by the monad distributivity of τ, and the *square* (4) is commutative by associativity of m. Now commutativity of the outer diagram shows associativity of $m \cdot \tau$.

From 2.5 we know that $\tau \cdot eH = He$ and $\tau \cdot He = eH$ and this implies that e is also the unit for $(H, m \cdot \tau, e)$.

The two pentagons for monad distributivity of τ for $(H, m \cdot m, e)$ can be read from the above diagram by combining the two top rectangles as well as the two left hand rectangles.

(2) The proof is dual to the proof of (1). □

6.10. Opposite bimonad. *Let $\tau : HH \to HH$ be a local prebraiding with $\tau^2 = I$ and let $\mathbf{H} = (H, m, e, \delta, \varepsilon)$ be a τ-bimonad on $\mathbb{A}$. Then:*

1. *$\mathbf{H}' = (H, m \cdot \tau, e, \tau \cdot \delta, \varepsilon)$ is also a τ-bimonad.*
2. *If $\mathbf{H}$ has an antipode S with $\tau \cdot HS = SH \cdot \tau$ and $\tau \cdot SH = HS \cdot \tau$, then S is a τ-bimonad morphism between the τ-bimonads $\mathbf{H}$ and $\mathbf{H}'$.*
   In this case S is an antipode for $\mathbf{H}'$.

Proof. (1) By (1), (2) in 6.9, τ is a (co)monad distributive law from the (co)monad H to the (co)monad H', and $\varepsilon' \cdot e' = \varepsilon \cdot e = 1$ by (6.10). Moreover,

$$\varepsilon' \cdot m' = \varepsilon \cdot m \cdot \tau \overset{(6.10)}{=} \varepsilon \cdot H\varepsilon \cdot \tau \overset{2.4}{=} \varepsilon \cdot \varepsilon H = \varepsilon \cdot H\varepsilon = \varepsilon' \cdot H\varepsilon', \quad \text{and}$$

$$\delta' \cdot e' = \tau \cdot \delta \cdot e \overset{(6.10)}{=} \tau \cdot eH \cdot e \overset{2.1}{=} He \cdot e = eH \cdot e = e'H \cdot e'.$$

To prove compatibility for $\mathbf{H}'$ we have to show the commutativity of the diagram

(6.17)
$$\begin{array}{ccccc}
HH & \xrightarrow{m'} & H & \xrightarrow{\delta'} & HH \\
\downarrow{\scriptstyle \delta'\delta'} & & & & \uparrow{\scriptstyle m'm'} \\
HHHH & & \xrightarrow[H\tau H]{} & & HHHH.
\end{array}$$

For this standard computations (from Hopf algebras) apply.

(2) By 6.6, S is a τ-bimonad morphism from the τ-bimonad $\mathbf{H}$ to the τ-bimonad $\mathbf{H}'$. To show that S is an antipode for $\mathbf{H}'$ we need the equalities

$$m' \cdot SH \cdot \delta' = e' \cdot \varepsilon' = e \cdot \varepsilon \quad \text{and} \quad m' \cdot HS \cdot \delta' = e' \cdot \varepsilon' = e \cdot \varepsilon.$$

Since $\tau \cdot SH = HS \cdot \tau$, we have

$$m' \cdot SH \cdot \delta' = m \cdot \tau \cdot SH \cdot \tau \cdot \delta = m \cdot HS \cdot \tau \cdot \tau \cdot \delta \overset{\tau^2=1}{=} m \cdot HS \cdot \delta = e \cdot \varepsilon.$$

Since $\tau \cdot HS = SH \cdot \tau$, we have

$$m' \cdot HS \cdot \delta' = m \cdot \tau \cdot HS \cdot \tau \cdot \delta = m \cdot SH \cdot \tau \cdot \tau \cdot \delta \overset{\tau^2=1}{=} m \cdot SH \cdot \delta = e \cdot \varepsilon.$$

□

As we have seen in Theorem 5.6, the existence of an antipode for an bimonad $\mathbf{H}$ on a category $\mathbb{A}$ is equivalent to the comparison functor being an equivalence provided $\mathbb{A}$ admits limits or colimits and H preserves them. It is shown in [3, Theorem 3.5.2] (see also [4, Lemma 4.2]) that in a braided monoidal category the existence of an antipode implies that the comparison functor is an equivalence provided idempotents split in this category. As conjectured in [29, Remarks 5.18], we are able to generalize this to Hopf monads on arbitrary Cauchy complete categories whose entwining map is derived from a local prebraiding.

6.11. **Antipode and equivalence - 2.** *Let $\tau : HH \to HH$ be a local prebraiding and let $\mathbf{H} = (H, m, e, \delta, \varepsilon)$ be a τ-bimonad on a Cauchy complete category $\mathbb{A}$. Consider the category of bimodules*

$$\mathbb{A}^H_H = \mathbb{A}^{\overline{H}}_{\underline{H}}(\tilde{\tau}),$$

where $\tilde{\tau} = mH \cdot H\tau \cdot \delta H$ (see 6.3, 6.5). Then the comparison functor $K_H : \mathbb{A} \to \mathbb{A}^H_H$ is an equivalence of categories if and only if $\mathbf{H}$ has an antipode.

Proof. Since $(H(a), m_a, \delta_a) \in \mathbb{A}^{\overline{H}}_{\underline{H}}(\tilde{\tau})$ for all $a \in \mathbb{A}$, one direction is clear from 4.5. For the converse, it suffices – since the comparison functor K_H is full and faithful (see 4.3)– to show that the functor K_H is surjective on objects. So consider an arbitrary object $(a, h_a, \theta_a) \in \mathbb{A}^H_H$ and write q_a for the composition

$$a \xrightarrow{\theta_a} H(a) \xrightarrow{S_a} H(a) \xrightarrow{h_a} a,$$

where $S : H \to H$ is an antipode of the bimonad $\mathbf{H}$. We are going to prove that

$$e_a \cdot q_a = \theta_a \cdot q_a. \tag{6.18}$$

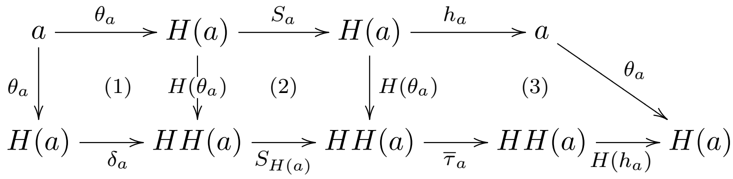
Considering the diagram

$$\begin{array}{ccccc}
H(a) & \xrightarrow{S_a} & H(a) & \xrightarrow{h_a} & a \\
\downarrow{\scriptstyle e_{H(a)}} & & \downarrow{\scriptstyle e_{H(a)}} & & \downarrow{\scriptstyle e_a} \\
HH(a) & \xrightarrow[H(S_a)]{} & HH(a) & \xrightarrow[H(h_a)]{} & H(a)
\end{array}$$

in which both squares commute by naturality of $e : 1 \to H$, one sees that

$$e_a \cdot h_a \cdot S_a \cdot \theta_a = H(h_a) \cdot H(S_a) \cdot e_{H(a)} \cdot \theta_a.$$

Next, considering the diagram



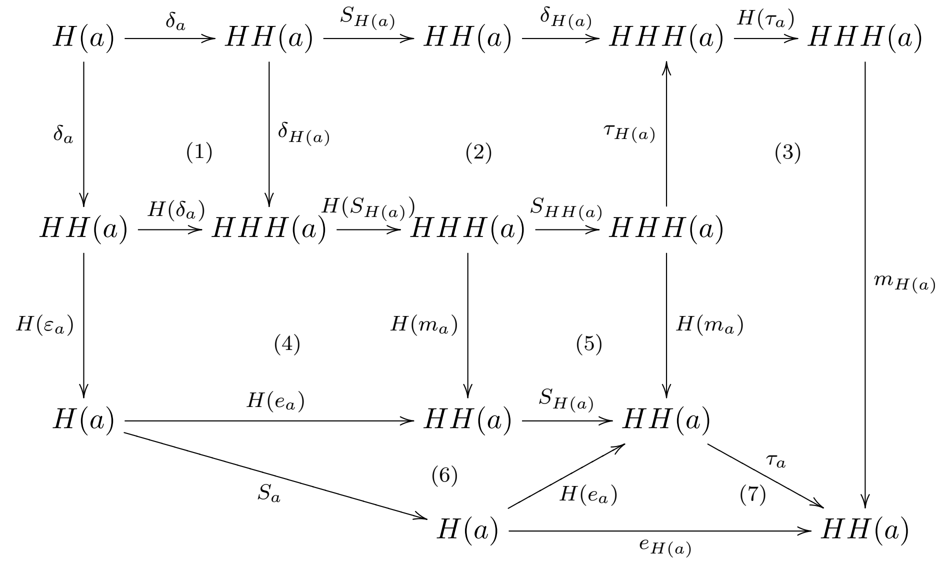
in which

- diagram (1) commutes since $(a, \theta_a) \in \mathbb{A}^{\overline{H}}$,
- diagram (2) commutes because of naturality of $S : H \to H$,
- diagram (3) commutes since $(a, h_a, \theta_a) \in \mathbb{A}^{H}_{H}$,

we get

$$\theta_a \cdot h_a \cdot S_a \cdot \theta_a = H(h_a) \cdot \overline{\tau}_a \cdot S_{H(a)} \cdot \delta_a \cdot \theta_a.$$

Now, the diagram



is commutative since

- diagram (1) commutes since the triple (H, δ, ε) is a comonad,
- diagram (2) commutes by 6.6,
- diagram (3) commutes since $\mathbf{H}$ is a τ-bimonad,
- diagram (4) commutes since S is an antipode,
- diagram (5) commutes by naturality of composition,
- diagram (6) commutes by naturality of $S : H \to H$,
- diagram (7) commutes since τ is an entwining.

Since $e_{H(a)} \cdot S_a = H(S_a) \cdot e_{H(a)}$ by naturality of $e : 1 \to H$ and since $H(\varepsilon_a) \cdot \delta_a = 1$, it follows from the commutativity of the diagram that

$$\begin{aligned} \theta_a \cdot h_a \cdot S_a \cdot \theta_a &= H(h_a) \cdot \overline{\tau}_a \cdot S_{H(a)} \cdot \delta_a \cdot \theta_a \\ &= H(h_a) \cdot m_{H(a)} \cdot H(\tau_a) \cdot \delta_{H(a)} \cdot S_{H(a)} \cdot \delta_a \cdot \theta_a \\ &= H(h_a) \cdot H(S_a) \cdot e_{H(a)} \cdot \theta_a. \end{aligned}$$

Thus $e_a \cdot q_a = \theta_a \cdot q_a$. Dually, one can prove that $q_a \cdot \varepsilon_a = q_a \cdot h_a$. Moreover, it follows – since $S_a \cdot e_a = e_a$ (see the proof of theorem 6.6) and $h_a \cdot e_a = 1$ – that

$$q_a \cdot q_a = h_a \cdot S_a \cdot \theta_a \cdot q_a = h_a \cdot S_a \cdot e_a \cdot q_a = h_a \cdot e_a \cdot q_a = q_a.$$

Thus $q_a \cdot q_a = q_a$, and since idempotents split in $\mathbb{A}$, there exist an object $\overline{a}$ and morphisms $i_a : \overline{a} \to a$ and $\overline{q}_a : a \to \overline{a}$ with $q_a = i_a \cdot \overline{q}_a$. Then $(\overline{a}, i_a)$ is an equaliser of the pair (e_a, θ_a) and $(\overline{a}, \overline{q}_a)$ is a coequaliser of the pair (ε_a, h_a).

For any $(a, h_a, \theta_a) \in \mathbb{A}_H^H$, write α_a for the composite $h_a \cdot H(i_a) : H(\bar{a}) \to a$. We claim that α_a is a morphism in $\mathbb{A}_H^H$ from $K_{\underline{H}}(\bar{a}) = (H(\bar{a}), m_{\bar{a}}, \delta_{\bar{a}})$ to (a, h_a, θ_a). Indeed, we have

$$\begin{array}{rcl} \alpha_a \cdot m_{\bar{a}} & = & h_a \cdot H(i_a) \cdot m_{\bar{a}} \\ \text{naturality} & = & h_a \cdot m_a \cdot H^2(i_a) \\ (a,h_a) \in \mathbb{A}_{\underline{H}} & = & h_a \cdot H(h_a) \cdot H^2(i_a) = h_a \cdot H(H(h_a) \cdot i_a) = h_a \cdot H(\alpha_a), \end{array}$$

and this just means that α_a is a morphism in $\mathbb{A}_{\underline{H}}$ from $(H(\bar{a}), m_{\bar{a}})$ to (a, h_a).

Next - using (6.15) and (6.18) - we compute

$$\theta_a \cdot \alpha_a = H(\alpha_a) \cdot \delta_{\bar{a}}.$$

Thus, α_a is a morphism in $\mathbb{A}^{\overline{H}}$ from $(H(\bar{a}), \delta_{\bar{a}})$ to (a, δ_a), and hence α_a is a morphism in $\mathbb{A}_H^H$ from $K_{\underline{H}}(\bar{a}) = (H(\bar{a}), m_{\bar{a}}, \delta_{\bar{a}})$ to (a, h_a, θ_a).

Similarly it is proved that the composite $\beta_a = H(\bar{q}_a) \cdot \theta_a : a \to H(\bar{a})$ is a morphism in $\mathbb{A}^{\overline{H}}$ from (a, h_a, δ_a) to $(H(\bar{a}), m_{\bar{a}}, \delta_{\bar{a}})$ and a further calculation yields

$$\alpha_a \cdot \beta_a = 1_a \text{ and } \beta_a \cdot \alpha_a = 1_{H(\bar{a})}.$$

Hence we have proved that for any $(a, h_a, \theta_a) \in \mathbb{A}_H^H$, α_a is an isomorphism in $\mathbb{A}_H^H$. Thus the functor $K_{\underline{H}}$ is surjective on objects. This completes the proof. □

For an example, let $\mathcal{V} = (\mathbb{V}, \otimes, I, \sigma)$ be a braided monoidal category and $\mathbf{H} = (H, m, e, \delta, \varepsilon)$ a bialgebra in $\mathcal{V}$. Then

$$(H \otimes -, m \otimes -, e \otimes -, \delta \otimes -, \varepsilon \otimes -, \tau = \sigma_{H,H} \otimes -)$$

is a bimonad on $\mathbb{V}$, and it is easy to see that the category $\mathbb{V}_H^H$ of Hopf modules is just the category $\mathbb{V}_{\underline{H\otimes -}}^{\overline{H\otimes -}}(\bar{\tau}) = \mathbb{V}_{H\otimes -}^{H\otimes -}$.

6.12. **Theorem.** *Let $\mathcal{V} = (\mathbb{V}, \otimes, I, \sigma)$ be a braided monoidal category such that idempotents split in $\mathbb{V}$. Then for any bialgebra $\mathbf{H} = (H, m, e, \delta, \varepsilon)$ in $\mathcal{V}$, the following are equivalent:*

(a) $\mathbf{H}$ *has an antipode;*

(b) *the comparison functor*

$$K_H : \mathbb{V} \to \mathbb{V}_H^H, \quad V \mapsto (H \otimes V, m \otimes V, \delta \otimes V), \quad f \mapsto H \otimes f,$$

is an equivalence of categories.

7. Adjoints of bimonads

This section deals with the transfer of properties of monads and comonads to adjoint (endo-)functors. The relevance of this interplay was already observed by Eilenberg and Moore in [11]. An effective formalism to handle this was developed for adjunctions in 2-categories and is nicely presented in Kelly and Street [13]. For our purpose we only need this for the 2-category of categories. For convenience we recall basic facts of this situation here.

7.1. **Adjunctions.** Let $L : \mathbb{A} \to \mathbb{B}$, $R : \mathbb{B} \to \mathbb{A}$ be an adjoint pair of functors with unit and counit η, ε, and $L' : \mathbb{A}' \to \mathbb{B}'$, $R' : \mathbb{B}' \to \mathbb{A}'$ be an adjoint pair of functors with unit and counit η', ε'. Given any functors $F : \mathbb{A} \to \mathbb{A}'$ and $G : \mathbb{B} \to \mathbb{B}'$, there is a bijection between natural transformations

$$\alpha : L'F \to GL \quad \text{and} \quad \overline{\alpha} : FR \to R'G$$

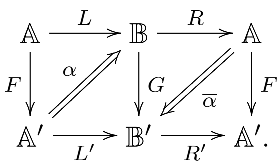
where $\overline{\alpha}$ is obtained as the composite

$$FR \xrightarrow{\eta' FR} R'L'FR \xrightarrow{R'\alpha R} R'GLR \xrightarrow{R'G\varepsilon} R'G,$$

and α is given as the composite

$$L'F \xrightarrow{L'F\eta} L'FRL \xrightarrow{L'\overline{\alpha}L} L'R'GL \xrightarrow{\varepsilon' GF} GL.$$

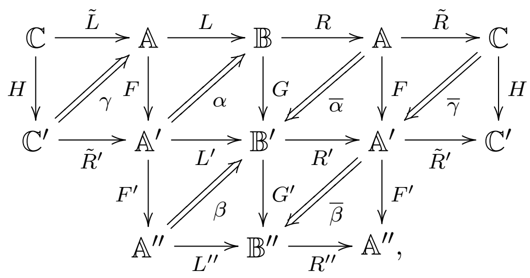
In this situation, α and $\overline{\alpha}$ are called *mates* under the given adjunction and this is denoted by $a \dashv \overline{\alpha}$. It is nicely displayed in the diagram



Given further

(i) adjunctions $\tilde{L} : \mathbb{C} \to \mathbb{A}$, $\tilde{R} : \mathbb{A} \to \mathbb{C}$ and $\tilde{L}' : \mathbb{C}' \to \mathbb{A}'$, $\tilde{R}' : \mathbb{A}' \to \mathbb{C}'$ and a functor $H : \mathbb{C} \to \mathbb{C}'$, or

(ii) an adjunction $L'' : \mathbb{A}'' \to \mathbb{B}''$, $R'' : \mathbb{B}'' \to \mathbb{A}''$ and functors $F' : \mathbb{A}' \to \mathbb{A}''$ and $G' : \mathbb{B}' \to \mathbb{B}''$, we get the diagram



yielding the mates

$$(M1) \quad L''F'F \xrightarrow{\beta F} G'L'F \xrightarrow{G'\alpha} G'GL \quad \dashv \quad F'FG \xrightarrow{F'\overline{\alpha}} F'R'G \xrightarrow{\overline{\beta}G} R''G'G,$$

$$(M2) \quad L'\tilde{L}'H \xrightarrow{L'\beta} L'F\tilde{L} \xrightarrow{\alpha\tilde{L}} LG\tilde{L} \quad \dashv \quad H\tilde{R}R \xrightarrow{\overline{\beta}G} \tilde{R}'R'G \xrightarrow{\tilde{R}'\overline{\beta}} \tilde{R}'R'G.$$

7.2. **Properties of mates.** *Let $L, L' : \mathbb{A} \to \mathbb{B}$ be functors with right adjoints R, R', respectively, and $\alpha : L' \to L$ a natural transformation.*

(i) *If $L'' : \mathbb{A} \to \mathbb{B}$ is a functor with right adjoint R'' and $\beta : L'' \to L'$ a natural transformation, then*

$$\alpha \cdot \beta \dashv \overline{\beta} \cdot \overline{\alpha}.$$

(ii) *If $\tilde{L} : \mathbb{C} \to \mathbb{A}$ is a functor with right adjoint $\tilde{R}$, then*

$$(\alpha_{L'} : L'\tilde{L} \to L\tilde{L}) \dashv (\tilde{R}\overline{\alpha} : \tilde{R}R \to \tilde{R}R').$$

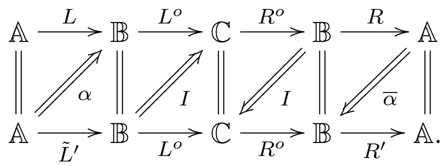
(iii) *If $L^o : \mathbb{B} \to \mathbb{C}$ is a functor with right adjoint R^o, then*

$$(L^o\alpha : L^oL' \to L^oL) \dashv (\overline{\alpha}R^o : RR^o \to R'R^o).$$

Proof. (i) is a special case of 7.1(M1).

(ii) follows from 7.1(M2) by putting $\mathbb{A}' = \mathbb{A}$, $\mathbb{B}' = \mathbb{B}$, $\mathcal{C}' = \mathcal{C}$ and $H' = H$.

(iii) is derived by applying 7.1 to the diagram



□

As observed by Eilenberg and Moore in [11, Section 3], for a left adjoint endofunctor which is a monad, the right adjoint (if it exists) is a comonad (and vice versa). The techniques outlined above provide a convenient and effective way to describe this transition and to prove related properties. Recall that for any endofunctor $L : \mathbb{A} \to \mathbb{A}$ with right adjoint R, for a positive integer n, the powers L^n have the right adjoints R^n.

7.3. **Adjoints of monads and comonads.** *Let $L : \mathbb{A} \to \mathbb{A}$ be an endofunctor with right adjoint R.*

(1) *If $\underline{L} = (L, m_L, e_L)$ is a monad, then $\overline{R} = (R, \delta_R, \varepsilon_R)$ is a comonad, where $m_L \dashv \delta_R$ and $e_L \dashv \varepsilon_R$.*
(2) *If $\overline{L} = (L, \delta_L, \varepsilon_L)$ is a comonad, then $\underline{R} = (R, m_R, e_R)$ is a monad where $\delta_L \dashv m_R$, $\varepsilon_L \dashv e_R$.*

Proof. (1) Since $e_L \dashv \varepsilon_R$ and $m_L \dashv \delta_R$, it follows from 7.2 (ii) and (iii) that

$$Le_L \dashv \varepsilon_R R, \quad e_L L \dashv R\varepsilon_R, \quad m_L L \dashv R\delta_R, \quad Lm_L \dashv \delta_R R.$$

Applying 7.2 (i) now yields

$$\begin{array}{ll} m_L \cdot Le_L \dashv \varepsilon_R R \cdot \delta_R, & m_L \cdot e_L L \dashv R\varepsilon_R \cdot \delta_R, \\ m_L \cdot m_L L \dashv R\delta_R \cdot \delta_R, & m_L \cdot Lm_L \dashv \delta_R R \cdot \delta_R. \end{array}$$

Since $\underline{L}$ is a monad we have $m_L \cdot e_L L = m_L \cdot Le_L = I$ and $m_L \cdot m_L L = m_L \cdot Lm_L$, implying

$$\varepsilon_R R \cdot \delta_R = R\varepsilon_R \cdot \delta_R = I \quad \text{and} \quad R\delta_R \cdot \delta_R = \delta_R R \cdot \delta_R.$$

This shows that $\overline{R} = (R, \delta_R, \varepsilon_R)$ is a comonad.

The proof of (2) is similar. □

The methods under consideration also apply to the natural transformations $LL \to LL$ which were basic for the definition and investigation of bimonads in previous sections. The following results were obtained in cooperation with Gabriella Böhm and Tomasz Brzeziński.

7.4. **Adjointness and distributive laws.** *Let $L : \mathbb{A} \to \mathbb{A}$ be an endofunctor with right adjoint R and a natural transformation $\lambda_L : LL \to LL$. Then the mate $\lambda_R : RR \to RR$ has the following properties:*

(1) *$L\lambda_L \dashv \lambda_R R$ and $\lambda_L L \dashv R\lambda_R$.*
(2) *λ_L satisfies the Yang-Baxter equation if and only if λ_R does.*
(3) *$\lambda_L^2 = I$ if and only if $\lambda_R^2 = I$.*
(4) *If $\underline{L} = (L, m_L, e_L)$ is a monad and λ_L is monad distributive, then λ_R is comonad distributive for the comonad $\overline{R} = (R, \delta_R, \varepsilon_R)$.*
(5) *If $\overline{L} = (L, \delta_L, \varepsilon_L)$ is a comonad and λ_L is comonad distributive, then λ_R is monad distributive for the comonad $\underline{R} = (R, m_R, e_R)$.*

Proof. (1) follows from 7.2, (ii) and (iii). The remaining assertions follow by (1) and the identities in the proof of 7.3. □

Recall from Definition 4.1 that a *bimonad* H is a monad and a comonad with compatibility conditions involving an entwining $\lambda_H : HH \to HH$.

7.5. Adjoints of bimonads. *Let* $\mathbf{H}$ *be a monad* $\underline{H} = (H, m_H, e_H)$ *and a comonad* $\overline{H} = (H, \delta_H, \varepsilon_H)$ *on the category* $\mathbb{A}$. *Then a right adjoint* R *of* H *induces a monad* $\underline{R} = (R, m_R, e_R)$ *and a comonad* $\overline{R} = (R, \delta_R, \varepsilon_R)$ *(see 7.3) and*

(1) $\mathbf{H} = (\underline{H}, \overline{H})$ *is a bimonad with entwining* $\lambda_H : \underline{H}\overline{H} \to \overline{H}\underline{H}$ *if and only if* $\mathbf{R} = (\underline{R}, \overline{R})$ *is a bimonad with entwining* $\lambda_R : \overline{R}\underline{R} \to \underline{R}\overline{R}$.

(2) $\mathbf{H} = (\underline{H}, \overline{H})$ *is a bimonad with entwining* $\lambda'_H : \overline{H}\underline{H} \to \underline{H}\overline{H}$ *if and only if* $\mathbf{R} = (\underline{R}, \overline{R})$ *is a bimonad with entwining* $\lambda'_R : \underline{R}\overline{R} \to \overline{R}\underline{R}$.

(3) *If* $\mathbf{H} = (\underline{H}, \overline{H}, \lambda_H)$ *is a bimonad with antipode, then* $\mathbf{R} = (\underline{R}, \overline{R}, \lambda_R)$ *is a bimonad with antipode (Hopf monad).*

Proof. (1) With arguments similar to those in the proof of 7.4 we get that λ_R is an entwining from $\overline{R}$ to $\underline{R}$. It remains to show the properties required in Definition 4.1. From 7.2(i) we know that

$$\begin{array}{c} \varepsilon_H \cdot H\varepsilon_H \dashv e_R R \cdot e_R, \quad \varepsilon_H \cdot m_H \dashv \delta_R \cdot e_R, \\ \delta_H \cdot e_H \dashv \varepsilon_R \cdot m_R, \quad e_H H \cdot e_H \dashv \varepsilon_R \cdot R\varepsilon_R, \quad \varepsilon_H \cdot e_H \dashv \varepsilon_R \cdot e_R. \end{array}$$

Thus the equalities

$$\varepsilon_H \cdot H\varepsilon_H = \varepsilon_H \cdot m_H, \quad \delta_H \cdot e_H = \varepsilon_H H \cdot e_H, \quad \varepsilon_H \cdot e_H = I$$

hold if and only if

$$e_R R \cdot e_R = \delta_R \cdot e_R, \quad \varepsilon_R \cdot m_R = \varepsilon_R \cdot R\varepsilon_R, \quad \varepsilon_R \cdot e_R = I.$$

The transfer of the compatibility between product and coproduct 4.1 is seen from the corresponding diagrams

$$\begin{array}{ccccc} HH & \xrightarrow{m_H} & H & \xrightarrow{\delta_H} & HH \\ {\scriptstyle H\delta_H}\downarrow & & & & \uparrow{\scriptstyle Hm_H} \\ HHH & & \xrightarrow[\lambda_H H]{} & & HHH, \end{array} \qquad \begin{array}{ccccc} RR & \xleftarrow{\delta_R} & R & \xleftarrow{m_R} & RR \\ {\scriptstyle m_R R}\uparrow & & & & \downarrow{\scriptstyle \delta_R R} \\ RRR & & \xleftarrow[R\lambda_R]{} & & RRR. \end{array}$$

The proof of (2) is similar.

(3) By 5.5, the existence of an antipode is equivalent to the bijectivity of the morphism

$$\gamma_H = Hm_H \cdot \delta_H H : HH \to HH.$$

Since $\delta_H H \dashv Rm_R$ and $Hm_H \dashv \delta_R R$, γ_H is an isomorphism if and only if $\gamma_R = Rm_R \cdot \delta_R R$ is an isomorphism. □

Functors with right (resp. left) adjoints preserve colimits (resp. limits) and thus 5.6 and 7.5 imply:

7.6. Hopf monads with adjoints. *Assume the category* $\mathbb{A}$ *to admit limits or colimits. Let* $\mathbf{H} = (H, m_H, e_H, \delta_H, \varepsilon_H, \lambda_H)$ *be a bimonad on* $\mathbb{A}$ *with a right adjoint bimonad* $\mathbf{R} = (R, m_R, e_R, \delta_R, \varepsilon_R, \lambda_R)$. *Then the following are equivalent:*

(a) *the comparison functor* $K_H : \mathbb{A} \to \mathbb{A}^{\overline{H}}_{\underline{H}}(\lambda_H)$ *is an equivalence;*

(b) *the comparison functor* $K_R : \mathbb{A} \to \mathbb{A}^{\overline{R}}_{\underline{R}}(\lambda_R)$ *is an equivalence;*

(c) $\mathbf{H}$ *has an antipode;*

(d) $\mathbf{R}$ *has an antipode.*

Finally we observe that local prebraidings are also tranferred to the adjoint functor.

7.7. Adjointness of τ-bimonads. *Let H be a monad $\underline{H} = (H, m_H, e_H)$ and a comonad $\overline{H} = (H, \delta_H, \varepsilon_H)$ on the category $\mathbb{A}$ with a right adjoint R.*

If $\mathbf{H} = (\underline{H}, \overline{H})$ is a bimonad with double entwining $\tau_H : HH \to HH$, then $\mathbf{R} = (\underline{R}, \overline{R})$ is a bimonad with double entwining $\tau_R : RR \to RR$.

Moreover, τ_H satisfies the Yang-Baxter equation if and only if so does τ_R.

Proof. Most of the assertions follow immediately from 7.4 and 7.5.

It remains to verify the compatibility condition 6.9. For this observe that from 7.2(i) we get

$$\delta_H \delta_H \dashv m_H m_H, \quad H\tau_H H \dashv R\tau_R R, \quad m_H m_H \dashv \delta_R \delta_R,$$

and hence

$$m_H m_H \cdot \tau_H H \cdot \delta_H \delta_H \dashv m_R m_R \cdot R\tau_R R \cdot \delta_R \delta_R \text{ and } \delta_H \cdot m_M \dashv \delta_R \cdot m_R.$$

It follows that $\mathbf{H}$ satisfies 6.9 if and only if so does $\mathbf{R}$. □

7.8. Dual Hopf algebras. Let B be a module over a commutative ring R. B is a Hopf algebra if and only if the endofunctor $B \otimes_R -$ on the category of R-modules is a Hopf monad. By 7.5, $B \otimes_R -$ is a bimonad (with antipode) if and only if its right adjoint functor $\mathrm{Hom}_R(B, -)$ is a bimonad (with antipode). This situation is considered in more detail in [5].

If B is finitely generated and projective as an R-module and $B^* = \mathrm{Hom}_R(B, R)$, then $\mathrm{Hom}_R(B, -) \simeq B^* \otimes_R -$ and we obtain the familiar result that B is a Hopf algebra if and only if B^* is.

7.9. Characterisations of groups. For any set G, the endofunctor $G \times - : \mathbf{Set} \to \mathbf{Set}$ is a Hopf bimonad on the category of sets if and only if G has a group structure (e.g. [29, 5.20]). Since the functor $\mathrm{Map(G, -)}$ is right adjoint to $G \times -$, it follows from 7.6 that a set G is a group if and only if the functor $\mathrm{Map(G, -)} : \mathbf{Set} \to \mathbf{Set}$ is a Hopf monad.

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References

[1] Barr, M., *Composite cotriples and derived functors*, in: Sem. Triples Categor. Homology Theory, Springer LN Math. 80, 336-356 (1969)

[2] Beck, J., *Distributive laws*, in: Seminar on Triples and Categorical Homology Theory, B. Eckmann (ed.), Springer LNM 80, 119-140 (1969)

[3] Bespalov, Y. and Drabant, B., *Hopf (bi-)modules and crossed modules in braided monoidal categories*, J. Pure Appl. Algebra 123(1-3), 105-129 (1998)

[4] Bespalov, Y., Kerler, Th., Lyubashenko V. and Turaev, V., *Integrals for braided Hopf algebras*, J. Pure Appl. Algebra 148(2), 113-164 (2000)

[5] Böhm, G., Brzeziński, T. and Wisbauer, R., *Monads and comonads in module categories*, J. Algebra 322, 1719-1747 (2009)

[6] Borceux, F. and Dejean, D., *Cauchy completion in category theory*, Cah. Topol. Géom. Différ. Catégoriques 27, 133-146 (1986)

[7] Bruguières, A. and Virelizier, A., *Hopf monads*, Adv. Math. 215(2), 679-733 (2007)

[8] Brzeziński, T. and Wisbauer, R., *Corings and Comodules*, London Math. Soc. Lecture Note Series 309, Cambridge University Press (2003)

[9] Day, B., McCrudden, P. and Street, R., *Dualizations and antipodes*, Appl. Categ. Struct. 11(3), 229-260 (2003)

[10] Dubuc, E., *Kan extensions in enriched category theory*, Lecture Notes in Mathematics 145, Berlin-Heidelberg-New York: Springer-Verlag (1970)
[11] Eilenberg, S. and Moore, J.C., *Adjoint functors and triples*, Ill. J. Math. 9, 381-398 (1965)
[12] Gumm, H.P., *Universelle Coalgebra*, in: Allgemeine Algebra, Ihringer, Th., Berliner Stud. zur Math., Band 10, 155-207, Heldermann Verlag (2003)
[13] Kelly, G.M. and Street, R., *Review of the elements of 2-categories*, Category Sem., Proc., Sydney 1972/1973, Lect. Notes Math. 420, 75-103 (1974)
[14] Lack, S. and Street, R., *The formal theory of monads. II*, J. Pure Appl. Algebra 175(1-3), 243-265 (2002)
[15] Loday, J.-L., *Generalized bialgebras and triples of operads*, Astérisque 320 (2008), arXiv:math/0611885
[16] López Franco, I., *Formal Hopf algebra theory* I *: Hopf modules for pseudomonoids*, J. Pure Appl. Algebra 213, 1046-1063 (2009)
[17] McCrudden, P., *Opmonoidal monads*, Theory Appl. Categ. 10, 469-485 (2002)
[18] Mesablishvili, B., *Descent in categories of (co)algebras*, Homology, Homotopy and Applications 7, 1-8 (2005)
[19] Mesablishvili, B., *Monads of effective descent type and comonadicity*, Theory Appl. Categ. 16, 1-45 (2006)
[20] Mesablishvili, B., *Entwining Structures in Monoidal Categories*, J. Algebra 319(6), 2496-2517 (2008)
[21] Moerdijk, I., *Monads on tensor categories*, J. Pure Appl. Algebra 168(2-3), 189-208 (2002)
[22] Power, J. and Watanabe, H., *Combining a monad and a comonad*, Theor. Comput. Sci. 280(1-2), 137-162 (2002)
[23] Schubert, H., *Categories*, Berlin-Heidelberg-New York, Springer-Verlag (1972)
[24] Street, R., *Frobenius monads and pseudomonoids*, J. Math. Phys. 45(10), 3930-3948 (2004)
[25] Szlachányi, K., *Adjointable monoidal functors and quantum groupoids*, Caenepeel, S. (ed.) et al., Hopf algebras in noncommutative geometry and physics, Proc. conf. on Hopf algebras and quantum groups, Brussels 2002, Marcel Dekker LN PAM 239, 291-307 (2005)
[26] Takeuchi, M., *Survey of braided Hopf algebras*, in: New trends in Hopf algebra theory, Proc. Coll. Quantum Groups and Hopf Algebras, La Falda, Argentina 1999, Andruskiewitsch, N. et al.(ed.), Providence, RI: American Math. Soc., Contemp. Math. 267, 301-323 (2000)
[27] Turi, D. and Plotkin, G., *Towards a mathematical operational semantics*, Proceedings 12th Ann. IEEE Symp. on Logic in Computer Science, LICS'97, Warsaw, Poland (1997)
[28] Wisbauer, R., *On Galois comodules*, Commun. Algebra 34(7), 2683-2711 (2006)
[29] Wisbauer, R., *Algebras versus coalgebras*, Appl. Categor. Struct. 16(1-2), 255-295 (2008)
[30] Wolff, H., *V-Localizations and V-monads*. J. Algebra 24, 405-438 (1973)

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