# Unitary Strongly Prime Rings and Related Radicals

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**Abstract**

A *unitary strongly prime* ring is defined as a prime ring whose central closure is simple with identity element. The class of unitary strongly prime rings is a special class of rings and the corresponding radical is called the *unitary strongly prime radical*. In this paper we prove some results on unitary strongly prime rings. The results are applied to study the unitary strongly prime radical of a polynomial ring and also $R$-disjoint maximal ideals of polynomial rings over $R$ in a finite number of indeterminates. From this we get relations between the Brown-McCoy radical and the unitary strongly prime radical of polynomial rings. In particular, the Brown-McCoy radical of $R[X]$ is equal to the unitary strongly prime radical of $R[X]$ and also equal to $\mathcal{S}(R)[X]$, where $\mathcal{S}(R)$ denotes the unitary strongly prime radical of $R$, when $X$ is an infinite set of either commuting or non-commuting indeterminates. For a PI ring $R$ this holds for any set $X$.

# Introduction

Throughout this paper rings are associative but do not necessarily have an identity element. Recall that the Brown-McCoy radical $U(R)$ of a ring $R$ is defined as the intersection of all the ideals $M$ of $R$ such that $R/M$ is a simple ring with identity. In [14] Krempa proved that the Brown-McCoy radical of a polynomial ring $R[x]$ in one indeterminate $x$ is equal to $\mathcal{I}(R)[x]$, where $\mathcal{I}(R) = U(R[x]) \cap R$.

A description of the ideal $\mathcal{I}(R)$ was given by Puczyłowski and Smoktunowicz ([19], Corollary 4). Namely, let $\mathcal{P}$ be the class of all prime rings with large center, i.e., prime rings $R$ such that for every non-zero ideal $I$ of $R$, $I \cap Z(R) \neq 0$, where $Z(R)$ denotes the center of $R$. Then for every ring $R$, $\mathcal{I}(R)$ is equal to the intersection of all the ideals $I$ of $R$ such that $R/I \in \mathcal{P}$.

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Köthe's problem [13] (is the sum of two nil left ideals nil?) is an outstanding open problem in basic ring theory. There are several equivalent formulations of this problem. In particular, Krempa proved that the problem is equivalent to asking whether the polynomial ring $R[x]$ in one indeterminate $x$ over a nil ring $R$ must be Jacobson radical. An interesting discussion on approximations of solutions of the problem is given in the introduction of [1]. In particular, in [19, Corollary 3(ii)] the authors proved that for any nil ring $R$, $R[x]$ is a Brown-McCoy radical ring, i.e., $R[x]$ cannot be mapped onto a ring with identity element. This result has been improved in [1], where the authors showed: if $R$ is nil, then $R[x]$ cannot be mapped onto a ring with an idempotent element. It remains as an open problem whether a polynomial ring in two or more commuting indeterminates over a nil ring must be Brown-McCoy radical ([19], Question 1, (a)).

*Symmetric strongly prime rings* are defined as prime rings with simple central closures ([20], Chap. 35). For rings with identities this notion and the related radical have been studied in [12]. Here we adapt this definition to rings without identity element in such a way that the corresponding radical will be a special radical which is very useful in our context. We say that a ring $R$ is *unitary strongly prime* (*u-strongly prime*, for short) if $R$ is prime and the central closure $RC$ of $R$ is a simple ring with unit. Of course, if $R$ itself has an identity then this is equivalent to the definition used in [12].

In this paper we study u-strongly prime rings, and the u-strongly prime and Brown-McCoy radicals of polynomial rings. In particular, a good part is devoted to study maximal ideals of polynomial rings in several indeterminates.

In Section 1 we recall some prerequisites. In Section 2 basic properties of u-strongly prime rings, mainly concerning centred extensions of rings, are given. In Section 3 we introduce the u-strongly prime radical of a ring. The main result of this section states that the u-strongly prime radical $\mathcal{S}(R[X])$ of a polynomial ring in any set of either commuting or non-commuting indeterminates $X$ is equal to $\mathcal{S}(R)[X]$, where $\mathcal{S}$ denotes the u-strongly prime radical.

As mentioned before, we have some information about factor rings of polynomial rings in one indeterminate (see also [4, 6, 7]). However, not much is known about factor rings of polynomial rings in several indeterminates. In Section 4 we consider the question whether there exists an integer $n$ and an $R$-disjoint ideal $M$ of a polynomial ring in $n$ indeterminates $R[x_1, ..., x_n]$, such that $R[x_1, ..., x_n]/M$ is a simple ring with identity. The main result states that this is the case if and only if $R$ is a u-strongly prime ring with non-zero pseudo-radical.

In the last Section 5 we consider the Brown-McCoy radical of a polynomial ring. We show that for any infinite set of either commuting or non-commuting indeterminates $X$ we have $U(R[X]) = \mathcal{S}(R)[X]$. This is also true for a finite set $X$, provided $R$ is a PI ring. Finally, we raise some open questions related with our investigations.

# 1 Prerequisites

Let $R$ be a semiprime ring. The self-injective hull of $R$ considered as $(R, R)$-bimodule, endowed with a canonical ring structure, is called the *central closure* of $R$ (see [20], Sect. 32). Equivalently, the central closure of $R$ may be considered as the subring of the Martindale right (left) ring of quotients $Q = Q_r(R)$ (or $Q_l(R)$) of $R$ generated by $R$ and the center $C(R)$ of $Q$, which is called the *extended centroid* of $R$.

The *symmetric ring of quotients* of $R$ is the subring of $Q$ defined as

$$Q_s = \{q \in Q \mid qJ \cup Jq \subseteq R \text{ for some } 0 \neq J \lhd R\}.$$

Clearly $C(R) \subseteq Q_s$ and hence the central closure of $R$ is also a subring of $Q_s$ ([2], Chap. 2).

Throughout this paper, for a prime ring $R$ we denote by $C(R)$ (or just $C$) the extended centroid of $R$ and by $RC$ the central closure of $R$. As a basic property we recall that for any ideal $I$ of $R$ we have $C(R) = C(I)$ ([2], Corollary 2.1.12 and Proposition 2.2.2).

Assume that $\phi : R \to S$ is a monomorphism of rings. Then $S$ becomes a canonical $R$-bimodule. In this paper we say that $\phi$ is a *centred monomorphism* if there exists a surjective ring homomorphism $\Phi : R{<}X{>} \to S$ such that $\Phi\mid_R = \phi$, where $R{<}X{>}$ denotes a free ring over $R$ in $X$, a set of indeterminates.

If $R$ has an identity element, then the definition agrees with the usual definition: we may consider $\Phi((X)) \subseteq S$ as a set of $R$-centralizing generators, where $(X)$ denotes the monoid generated by the set $X$ (cf. [5], [12]).

For basic notions and terminology on radicals we refer the reader to [3].

Let $\mathcal{A}$ be a class of rings such that every non-zero ideal of a ring in $\mathcal{A}$ can be homomorphically mapped onto some non-zero ring of $\mathcal{A}$. Then $\mathcal{A}$ determines a so called *upper radical* property, which we denote by $\mathcal{A}$ again. Thus the rings in $\mathcal{A}$ are all semi-simple rings with respect to this upper radical, and $\mathcal{A}$ is the largest radical for which this happens.

Recall that a class of prime rings $\mathcal{A}$ is said to be a *special class* if for any essential ideal $I$ of a ring $R$, $I$ belongs to $\mathcal{A}$ if and only if $R$ is in $\mathcal{A}$.

Any special class of rings $\mathcal{A}$ determines an upper radical. This radical contains the prime radical and is *hereditary*, i.e., for any ring $R$ and ideal $I$ of $R$, the $\mathcal{A}$-radical of $I$ is equal to the intersection $\mathcal{A}(R) \cap I$, where $\mathcal{A}(R)$ denotes the $\mathcal{A}$-radical of $R$. Moreover, $\mathcal{A}(R)$ is equal to the intersection of all ideals $P$ of $R$ such that $R/P \in \mathcal{A}$ ([3], Ch. 7).

Assume that $R$ is prime. We will consider the ring obtained from $R$ by adjoining an identity, defined as usual in the following way ([10], 2.17, Ex. 5): Consider $R$ as an algebra over the ring of integers $\mathbb{Z}$ and put $T = R \oplus \mathbb{Z}$ with the operations:

$$(a, n) + (b, m) = (a + b, n + m) \text{ and } (a, n)(b, m) = (ab + ma + nb, nm),$$

for $(a, n), (b, m) \in T$. The *natural extension* of $R$ to a ring with identity $R^{\#}$ is defined as the ring $T/Ann_T(R)$, where $Ann_T(R) = \{t \in T \mid Rt = 0\}$ is an ideal of $T$. Since $R$ is prime, $Ann_T(R) \cap R = 0$ and hence we may consider $R \subseteq R^{\#}$. It follows that $R^{\#}$ is prime with unit and $R$ is an essential ideal of $R^{\#}$.

The multiplication ring $M(R)$ of $R$ is defined as the subring of $End_{\mathbb{Z}}R$, acting from the left on $R$, generated as a ring by all the left and right multiplications $l_a$ and $r_b$, where $a, b \in R^{\#}$, and $l_a x = ax$, $r_b x = xb$, for $x \in R$. So each $\lambda \in M(R)$ is of the form $\lambda = \sum_k l_{a_k} r_{b_k}$, where $a_k, b_k \in R^{\#}$, and $\lambda x = \sum_k a_k x b_k$, $x \in R$.

A finite subset $A = \{a_1, ..., a_n\} \subseteq R$ is called an *insulator*, if

$$Ann_{M(R)}\{a_1, ..., a_n\} \subseteq Ann_{M(R)}\{1_{R^{\#}}\},$$

i.e., if $\lambda a_1 = ... = \lambda a_n = 0$, implies $\lambda 1 = 0$. By Proposition 2.6 of [12], a finite subset $A = \{a_1, ..., a_n\}$ of a prime ring $R$ is an insulator if and only if $1 \in AC$, where $C = C(R^{\#}) = C(R)$, i.e., there exist $c_1, ..., c_n \in C$ such that $a_1c_1 + ... + a_nc_n = 1$.

Recall that a ring $R$ is said to be *right strongly prime* if, for any non-zero ideal $I$ of $R$, there exists a finite set $F \subseteq I$ such that $Ann_r(F) = \{r \in R \mid Fr = 0\}$ is zero [9]. This set $F$ is called a *right insulator*. It is well-known that the class of all right strongly prime rings is a special class of rings [8] and the upper radical determined by this class is called the *right strongly prime radical*. It is also known that this notion is not symmetric, i.e., the right and left strongly prime radicals do not coincide (see 5.4 of [16]).

On the other hand, the notion of symmetric strongly prime rings mentioned in the introduction is left-right symmetric.

We point out that the definition which will be used here in general does not coincide with any of the above notions. We will slightly modify the definition of the symmetric case such that the resulting class of prime rings will be a special class which is not the case for the class of all symmetric strongly prime rings.

# 2 U-strongly Prime Rings

Basic to our investigations is the following

**2.1 Definition.** A prime ring $R$ is said to be *unitary strongly prime* (*u-strongly prime*, for short) if $RC$ is a simple ring with identity element.

Note that the definition is an extension of the one used in [12] for rings with identity element. Thus a ring $R$ with identity is symmetric strongly prime if and only if $R$ is u-strongly prime.

We denote by $\mathcal{S}$ the class of all u-strongly prime rings and by $\mathcal{S}'$ the class of all symmetric strongly prime rings. We have the following.

**2.2 Proposition.** *The class $\mathcal{S}$ is a special class of rings and the class $\mathcal{S}'$ is not special.*

**Proof.** Every ring in $\mathcal{S}$ is prime. If $R \in \mathcal{S}$ and $I$ is a non-zero ideal of $R$, then $I$ is prime and $RC$ is a simple ring with identity. Also $IC(I) = IC(R)$ is an ideal of $RC$ and hence $IC(I) = RC$ is simple with identity. This shows that $I$ is u-strongly prime.

Now, assume that $I$ is u-strongly prime and an essential ideal of some ring $R$. Then $R$ is a prime ring, $IC(I)$ is a simple ring with identity and an essential ideal of the ring $RC(R) = RC(I)$. Since the class of simple rings with identity is a special class of rings ([3], Theorem 62) it follows that $RC$ is also simple with identity. This proves the first part.

Assume that $\mathcal{S}'$ is a special class of rings and let $R$ be any prime simple ring without identity element. Then $R = RC$ is a strongly prime ring and the symmetric Martindale ring of quotients $Q_s$ of $R$ is a prime ring. For any $q \in Q_s$ there exists a non-zero ideal $H$ of $R$ with $qH \cup Hq \subseteq R$. Since $R$ is simple, $H = R$ and it follows that $R$ is an ideal of $Q_s$. Consequently $Q_s$ is strongly prime since $\mathcal{S}'$ is (assumed to be) special and $R$ is the minimal ideal of $Q_s$.

Take an element $c \in C(Q_s)$, the extended centroid of $Q_s$. Then there exists a non-zero ideal $J$ of $Q_s$ with $cJ \cup Jc \subseteq Q_s$. Since $R \subseteq J$ and $R = R^2$ we can easily see that $cR \cup Rc \subseteq R$. Hence $c \in C(R)$ and consequently $R = RC(R) = RC(Q_s) = Q_sC(Q_s)$, because $Q_sC(Q_s)$ is a simple ring. This contradicts the fact that $Q_s$ has an identity element. The proof is complete. ■

**2.3 Corollary.** *The class $\mathcal{S}$ is the largest special class of rings $\mathcal{A}$ which is contained in $\mathcal{S}'$ and satisfies the property: if $R \in \mathcal{A}$, then $RC \in \mathcal{A}$.*

**Proof.** Assume that $\mathcal{A} \subseteq \mathcal{S}'$ is a special class of rings and satisfies the above property. If there exists $R \in \mathcal{A} \setminus \mathcal{S}$, then $RC \in \mathcal{A}$ is a prime simple ring without identity element. Using the same arguments as in the proof of the second part of Proposition 2.2 starting with $RC$ we reach a contradiction. ■

Now we partially extend Theorem 2.1 of [12] to rings without identity element.

**2.4 Proposition.** *For any ring $R$ the following conditions are equivalent:*

(1) *$R$ is u-strongly prime.*

(2) *$R$ is prime and $R^{\#}$ is strongly prime.*

(3) *There exists a centred monomorphism $\phi : R \to S$, where $S$ is a simple ring with identity.*

(4) *There exists a centred monomorphism $\phi : R \to S$, where $S$ is a ring with identity, with the property: for each non-zero ideal $I$ of $R$, its extension in $S$, $I^e = SIS$, is equal to $S$.*

(5) *$R$ is prime and any non-zero ideal of $R$ contains an insulator.*

**Proof.** (3)⇒(4) is a tautology and (2) ⇒ (1) holds by Proposition 2.2.

(1) ⇒ (3). If $R$ is prime and $RC$ is simple with identity we have an obvious centred monomorphism $R \to RC$ as in (3).

(4) ⇒ (2). Assume that $\phi : R \to S$ is a centred monomorphism as in (4). Note that if $I$ is an ideal of $R$, then using the above notation we have $I^e = \Phi(I{<}X{>}) = IS = SI$. Then for non-zero ideals $I$ and $J$ of $R$ we must have $IJ \neq 0$ and so $R$ is prime. Now we consider the natural extension of $R$ to a prime ring with identity $R^{\#}$, as above.

We extend $\phi$ to a homomorphism $\psi : T \to S$ by $\psi(a,k) = \phi(a) + k1_S$, for $(a,k) \in R \oplus \mathbb{Z} = T$, and $Ker\psi \cap R = Ker\phi = 0$. Thus $Ker\psi \subseteq Ann_T(R)$. Conversely, if $(a,k) \in Ann_T(R)$ we have that $R(a,k) = 0$. It follows that $S\phi(R)(\phi(a) + k1_S) = 0$ and since $S\phi(R) = S$ we obtain $\psi(a,k) = 0$. Consequently $Ker\psi = Ann_T(R)$. Thus $\psi$ induces a monomorphism of rings from $R^{\#}$ into $S$ which we denote by $\phi$ again.

Also, the identity of $S$ can be written as $1 = \sum_i \Phi(w_i r_i)$, where $r_i \in R$ and $w_i \in (X)$, $i = 1, ..., n$. For any element $w \in (X)$ we can define a mapping

$$w' : R \oplus \mathbb{Z} \to S \text{ by } w'(r,k) = \Phi(wr) + k\sum_i \Phi(ww_i r_i).$$

It can easily be seen that if $(r,k) \in Ann_R(T)$, then $\phi(R)(\Phi(wr) + k\sum_i \Phi(ww_i r_i)) = 0$ and since $1 \in SR = S\phi(R)$ it follows that $\Phi(wr) + k\sum_i \Phi(ww_i r_i) = 0$. Hence $w'$ can be considered as a bimodule homomorphism from $R^{\#}$ into $S$ which we denote by $w$ again. Thus $\{w(1_{R^{\#}}) \mid w \in (X)\}$ is a set of $R^{\#}$-centralizing generators of $S$ in the usual sense. Consequently $R^{\#}$ is u-strongly prime by Theorem 2.1 of [12].

(1) ⇒ (5). If $RC$ is a simple ring with identity and $I$ is a non-zero ideal of $R$, then $IC = RC$ and so $1 \in IC$. Hence $I$ contains an insulator.

(5) ⇒ (1). If $H$ is a non-zero ideal of $RC$, then $I = H \cap R$ is a non-zero ideal of $R$. Thus $I$ contains an insulator and consequently $1 \in IC$. This shows that $H = RC$ has an identity element. Consequently, $R$ is u-strongly prime. ■

The following is an easy consequence.

**2.5 Corollary.** *Assume that $\phi : R \to S$ is a centred monomorphism, where $S$ is a u-strongly prime ring. Then $R$ is u-strongly prime.*

**Proof.** By assumption there exists a centred monomorphism $\rho : S \to S'$, where $S'$ is a simple ring with identity. Then it is easy to see that the composition $\rho \circ \phi : R \to S'$ is a centred monomorphism. The result follows by Proposition 2.4. ■

If $R$ is prime and has an identity element, there is a one-to-one correspondence between the set of all $R$-disjoint prime ideals of $R{<}X{>}$ and the set of all $RC$-disjoint prime ideals of $RC{<}X{>}$ ([5], Theorems 2.15 and 5.3). In the proof of the next lemma we will use similar arguments as in [5] for rings without identity element.

An ideal $P$ of a ring $R$ is said to be *u-strongly prime* if the factor ring $R/P$ is a u-strongly prime ring [12]. U-strongly prime ideals have a nice behaviour concernig centred extensions. In fact, we have the following.

**2.6 Lemma.** *For a centred monomorphism $\phi : R \to S$ of rings we have:*

*(i) If $P$ is a u-strongly prime ideal of $S$, then $\phi^{-1}(P)$ is a u-strongly prime ideal of $R$.*

*(ii) If $I$ is a u-strongly prime ideal of $R$ and $P$ is an ideal of $S$ which is maximal with respect to the condition $\phi^{-1}(P) = I$, then $P$ is a u-strongly prime ideal of $S$.*

**Proof.** (i) By factoring out the ideals $P$ and $\phi^{-1}(P) \cap R$ from $S$ and $R$, respectively, the statement (i) can easily be reduced to Corollary 2.5.

(ii) By factoring out the ideals $I$ and $P$ from $R$ and $S$, respectively, we may assume that $P = 0$, $\phi^{-1}(J) \neq 0$ for any non-zero ideal of $J$ of $S$, $R$ is u-strongly prime and $S$ is prime. We have to show that $S$ is u-strongly prime.

By assumption, there exists a surjective ring homomorphism $R{<}X{>} \to S$, for some $X$. Thus $S \simeq R{<}X{>}/M$, for some ideal $M$ of $R{<}X{>}$ which is maximal with respect to the property $M \cap R = 0$.

The ideal $M$ can be extended to an ideal $M^*$ of $RC{<}X{>}$ which is a maximal $RC$-disjoint ideal. It is not hard to check that

$$M^* = \{f \in RC{<}X{>} \mid \text{there exist } 0 \neq F, H \lhd R \text{ with } FfH \subseteq M\},$$

gives the required extension (see [5]). To show that $M^*$ is an ideal of $RC{<}X{>}$ with $M^* \cap R{<}X{>} = M$ we have only to prove that for any $w \in (X)$ and $f \in M^*$ we have $wf, fw \in M^*$, since the rest is clear. Assume that $FfH \subseteq M$, for $0 \neq F, H \lhd R$. Write the identity of $RC$ as $1 = \sum_i r_i c_i$, $r_i \in R$, $c_i \in C$, and take a non-zero ideal $I$ of $R$ with $c_i I \subseteq R$, for all $i$. Then

$$FfwIH^2 \subseteq \sum_i Ffwr_ic_iIH^2 \subseteq FfwH^2 \subseteq FfHwH \subseteq MwH \subseteq M.$$

This shows that $fw \in M^*$ and similarly we obtain $wf \in M^*$.

Since $RC$ is simple with identity, $M^*$ is a maximal ideal. Also we have a natural inclusion $R{<}X{>}/M \to RC{<}X{>}/M^*$ which is clearly a centred monomorphism. Consequently $R{<}X{>}/M$ is a u-strongly prime ring. ■

# 3 The U-strongly Prime Radical

For the rest of the paper $\mathcal{S}$ denotes the class of all u-strongly prime rings as well as the upper radical determined by the class $\mathcal{S}$ [3]. By Proposition 2.2, the radical $\mathcal{S}$ is a special radical and for every ring $R$, $\mathcal{S}(R)$ is equal to the intersection of all ideals $P$ of $R$ such that $R/P \in \mathcal{S}$. This radical is called the u-strongly prime radical of $R$ and was introduced in [12] for rings with identity elements. Since every simple ring with identity is in $\mathcal{S}$, the u-strongly prime radical is contained in the Brown-McCoy radical.

Let $\mathcal{P}$ be the class of all non-zero prime rings with large center. The class $\mathcal{P}$ is also a special class of rings [19]. For any ring $R$ the Brown-McCoy radical, $U(R[x])$, of a polynomial ring in one indeterminate $x$, is equal to $\mathcal{I}(R)[x]$, where $\mathcal{I}$ denotes the upper radical determined by the class $\mathcal{P}$ ([19], Corollary 4). We have the following.

**3.1 Lemma** $\mathcal{P} \subseteq \mathcal{S}$. *In particular, for any ring $R$ we have $\mathcal{S}(R) \subseteq \mathcal{I}(R)$.*

**Proof.** If $R$ is in $\mathcal{P}$, then $R$ is prime and for any non-zero ideal $H$ of $RC$ we have $H \cap R \neq 0$. Thus there exists an element $c$ in $H \cap Z(R)$ and this element is invertible in $C$. Hence $1 = c^{-1}c \in H$, consequently $RC$ is simple with identity. ■

To prove the next theorem we need the following.

**3.2 Remark** The result proved by Krempa and mentioned in the introduction holds in general: if $X$ is a set of either commuting or non-commuting indeterminates, then $U(R[X]) = (U(R[X]) \cap R)[X]$ ([18], 1.6 and [11], Corollary 13).

**3.3 Theorem** *Let $R$ be a ring and $X$ any set of either commuting or non-commuting indeterminates. Then $\mathcal{S}(R[X]) = \mathcal{S}(R)[X]$.*

**Proof.** Let $P$ be any u-strongly prime ideal of $R[X]$. By (i) of Lemma 2.6, $\mathcal{S}(R) \subseteq P \cap R$. Also, for any $w \in (X)$ we have $(P \cap R)wR[X] \subseteq (P \cap R)R[X] \subseteq P$ and so $\mathcal{S}(R)[X] \subseteq (P \cap R)[X] \subseteq P$. Hence $\mathcal{S}(R)[X] \subseteq \mathcal{S}(R[X])$.

Conversely, take a u-strongly prime ideal $P$ of $R$. Then $P[X]$ is an ideal of $R[X]$ with $P[X] \cap R = P$. Factoring out $P$ and $P[X]$ from $R$ and $R[X]$, respectively, we may assume that $P = 0$ and $R$ is u-strongly prime.

Since $RC$ is simple with identity, Remark 3.2 implies that the intersection of all maximal ideals of $RC[X]$ is zero. Also, for any maximal ideal $M$ of $RC[X]$, $M \cap R[X]$ is a u-strongly prime ideal of $R[X]$, since $R[X]/M \cap R[X] \to RC[X]/M$ is a centred monomorphism. This proves that $\mathcal{S}(R[X]) = 0$.

The argument shows that, in general, $\mathcal{S}(R[X]) \subseteq P[X]$, for any u-strongly prime ideal $P$ of $R$. Consequently $\mathcal{S}(R[X]) \subseteq \mathcal{S}(R)[X]$, and the proof is complete. ■

Lemma 2.6 has another application. Recall that given a class of prime rings $\mathcal{A}$, a ring $R$ is said to be an $\mathcal{A}$-Jacobson ring if every prime ideal of $R$ is an intersection of ideals $P$ with $R/P \in \mathcal{A}$ [7]. If $R$ is a ring with identity which is an $\mathcal{A}$-Jacobson ring, then the polynomial ring $R[x]$ is also an $\mathcal{A}$-Jacobson ring, provided the class $\mathcal{A}$ satisfies the following condition: if a ring $R$ is in $\mathcal{A}$ and $P$ is an ideal of $R[x]$ which is maximal with respect to the condition $P \cap R = 0$, then $R[x]/P \in \mathcal{A}$ ([7], Theorem 5). Thus condition (ii) of Lemma 2.6 immediately gives the following.

**3.4 Corollary** *Let $R$ be a ring with identity. If $R$ if an $\mathcal{S}$-Jacobson ring, then so is the polynomial ring $R[x]$.*

# 4 Maximal Ideals of Polynomial Rings

For any ring $R$ and cardinal number $\alpha$ we denote by $R[X_\alpha]$ the polynomial ring over $R$ in a set $X_\alpha$ of $\alpha$ commuting indeterminates.

Given a ring $R$, the *pseudo-radical* $ps(R)$ of $R$ is defined as the intersection of all non-zero prime ideals of $R$. It was proved in [6], Corollary 2.2 that if $R$ is a ring with identity and there exists a maximal ideal of $R[x]$ which is $R$-disjoint, then $ps(R)$ is non-zero. More generally, for rings with identity it was proved in Corollary 2 of [19] that $R[x]$ contains a maximal ideal which is $R$-disjoint if and only if $R \in \mathcal{P}$ and $ps(R)$ is non-zero, where $\mathcal{P}$ is the class of prime rings with large center, as in Section 3.

Now we extend Corollary 2.2 of [6]. For this we use the following result that has been proved in [7], Lemma 3, for rings with identity element. The proof in the general case is the same.

Given a non-zero $R$-disjoint ideal $I$ of the polynomial ring $R[x]$, we denote by $\tau(I)$ the ideal of $R$ consisting of all the leading coefficients of all polynomials of minimal degree in $I$, including zero. Of course, if $I \neq 0$ then $\tau(I) \neq 0$.

**4.1 Lemma** *Assume that $P$ is a non-zero $R$-disjoint prime ideal of $R[x]$ and $Q \neq 0$ is a prime ideal of $R$. If $\tau(P) \not\subseteq Q$, then $(P + Q[x]) \cap R = Q$.*

**Proof.** See Lemma 3 of [7]. ■

The following observation will be helpful.

**4.2 Lemma** *Assume that $P$ is a prime ideal of the polynomial ring $R[x]$ and $ps(R[x]/P) \neq 0$. Then $ps(R/P \cap R) \neq 0$.*

**Proof.** By factoring out convenient ideals we may assume that $P \cap R = 0$ and we have to prove that $ps(R) \neq 0$. By the way of contradiction, suppose there exists a family $\mathcal{F} = (P_i)_{i \in \Lambda}$ of non-zero prime ideals of $R$ such that $\bigcap_i P_i = 0$. If $P = 0$ we immediately obtain the contradiction $\bigcap_i P_i[x] = 0$. So we also may assume that $P$ is maximal among $R$-disjoint ideals (Corollary 2.13 of [17]).

It follows that there is a subfamily $(P_j)_{j \in \Gamma} \subseteq \mathcal{F}$ such that $\tau(P) \not\subseteq P_j$, for every $j \in \Gamma$, and $\bigcap_{j \in \Gamma} P_j = 0$. For any $j \in \Gamma$ we have that $(P + P_j[x]) \cap R = P_j$, by Lemma 4.1. Take an ideal $L_j$ of $R[x]$ which contains $P + P_j[x]$ and is maximal with respect to the property $L_j \cap R = P_j$. Then $L_j$ is prime and $(\bigcap_{j \in \Gamma} L_j) \cap R =$

$\bigcap_{j\in\Gamma} P_j = 0$, consequently $\bigcap_{j\in\Gamma} L_j = P$, by maximality of $P$. This gives the contradiction $ps(R[x]/P) = 0$. The proof is complete. ∎

Now we are in a position to prove an extension of Corollary 2.2 of [6].

**4.3 Corollary** *Assume that $n \geq 1$ is a natural number and there exists a maximal ideal $M$ of $R[X_n]$ which is $R$-disjoint. Then $ps(R) \neq 0$.*

**Proof.** The proof for a single indeterminate is the same as given in Corollary 2.2 of [6]. This implies that the pseudo-radical of $R[X_{n-1}]/M \cap R[X_{n-1}]$ is non-zero, where $R[X_{n-1}]$ is the ring of polynomials obtained by forgetting one indeterminate from $X_n$. Now we apply Lemma 4.2 repeatedly to complete the proof. ∎

Next we need the following.

**4.4 Lemma** *Assume that $R$ is a prime ring and $I$ is a non-zero ideal of $R$. Then $ps(R) \neq 0$ if and only if $ps(I) \neq 0$.*

**Proof.** Suppose that $ps(R) \neq 0$. If $Q$ is a non-zero prime ideal of $I$, then $J = (R^{\#}QR^{\#})^3$ is a non-zero ideal of $R$ with $J \subseteq Q$. Take an ideal $L \supseteq J$ of $R$ which is maximal with respect to the condition $L \cap I \subseteq Q$. Then $L$ is prime and consequently $L \supseteq ps(R)$. Hence $Q \supseteq ps(R) \cap I \neq 0$. This shows that $ps(I) \neq 0$.

Conversely, by the way of contradiction, suppose that $ps(I) \neq 0$ and there exists a family $(P_i)_{i\in\Lambda}$ of non-zero prime ideals of $R$ with $\bigcap_i P_i = 0$. Then there exists a subfamily $(P_j)_{j\in\Gamma}$ of the above such that $I \not\subseteq P_j$, for any $j \in \Gamma$, and $\bigcap_{j\in\Gamma} P_j = 0$. Since $R$ is prime, $P_j \cap I$ is a non-zero prime ideal of $I$ and so $P_j \cap I \supseteq ps(I) \neq 0$, for all $j \in \Gamma$. This contradiction completes the proof. ∎

**4.5 Remark** If there exists an ideal $M$ of $R[X_n]$ such that $R[X_n]/M$ is a simple ring with identity and $M \cap R = 0$, then $R$ is u-strongly prime and $ps(R) \neq 0$ (Proposition 2.4 and Corollary 4.3). This type of u-strongly prime rings are very important in our study.

The subclass of $\mathcal{S}$ consisting of all u-strongly prime rings $R$ with $ps(R) \neq 0$ will be denoted by $\mathcal{S}_1$ and we put $\mathcal{S}_2 = \mathcal{S} \setminus \mathcal{S}_1$. Using Lemma 4.4 it is easy to show that both classes $\mathcal{S}_1$ and $\mathcal{S}_2$ are special classes of rings. However, while the class $\mathcal{S}_1$ is relevant in the computation of the u-strongly prime radical and the Brown-McCoy radical of polynomial rings, the class $\mathcal{S}_2$ can be ignored. Concerning the last one we have the following.

**4.6 Proposition** *Assume that $R \in \mathcal{S}_2$. Then for any $0 \neq a \in R$ there exists a natural number $n$ and an ideal $M$ of $R[X_n]$ such that $R[X_n]/M$ is simple with identity and $a \notin M \cap R$.*

**Proof.** Since $0 \neq a \in R^{\#}$ and $\mathcal{S}(R^{\#}) = 0$, by Theorem 3.2 of [12] there exists a natural number $n$ and elements $a_1, ..., a_n \in (a)$ such that the ideal of $R^{\#}[x_1, ..., x_n]$ generated by the polynomial $f = a_1x_1 + ... + a_nx_n - 1$ is a proper ideal, where $(a)$ denotes the ideal of $R^{\#}$ generated by $a$. Take a maximal ideal $N$ of $R^{\#}[x_1, ..., x_n]$ containing $f$. If $a \in N$, then $a_1, ..., a_n \in N$ and so $1 \in N$, a contradiction. Thus $a \notin N$.

Now $R[x_1, ..., x_n]$ is an ideal of $R^{\#}[x_1, ..., x_n]$ and hence $R[x_1, ..., x_n]/M$ is a non-zero ideal of $R^{\#}[x_1, ..., x_n]/N$, where $M = N \cap R[x_1, ..., x_n]$. Therefore $R[x_1, ..., x_n]/M = R^{\#}[x_1, ..., x_n]/N$ is simple with identity and $a \notin M \cap R$. ∎

Note that in Proposition 4.6 the ring $R/M \cap R \in \mathcal{S}_1$, by Remark 4.5. This immediately gives the following.

**4.7 Corollary** *Any ring in $\mathcal{S}_2$ is a sub-direct product of rings from $\mathcal{S}_1$. In particular, the u-strongly prime radical of any ring $R$ is equal to the intersection of all ideals $P$ of $R$ with $R/P \in \mathcal{S}_1$.*

The classification of u-strongly prime rings induces a partition of $\mathcal{P}$ into subclasses $\mathcal{P}_1$ and $\mathcal{P}_2$ in an obvious way, i.e., $R \in \mathcal{P}_1$ if and only if $R \in \mathcal{P} \cap \mathcal{S}_1$. Now we prove the following extension of Corollary 2 of [19].

**4.8 Theorem** *For any ring $R$, the following conditions are equivalent:*

*$(i)$ There exists an $R$-disjoint ideal $M$ of $R[x]$ such that $R[x]/M$ is simple with identity.*

*$(ii)$ $R \in \mathcal{P}_1$.*

*$(iii)$ $R$ is prime and $ps(R) \cap Z(R) \neq 0$.*

*$(iv)$ $R \in \mathcal{S}$ and there exists $c \in C$ such that $RC = R[c]$.*

**Proof.** $(i) \Rightarrow (ii)$ follows from Corollary 4.3 and ([19], Corollary 1, (ii)). Also $(ii) \Rightarrow (iii)$ is clear by definition of $\mathcal{P}_1$.

$(iii) \Rightarrow (i), (iv)$. Take a non-zero element $d \in ps(R) \cap Z(R)$. Then $c = d^{-1} \in C$ and so $1 \in RC$. We define a ring homomorphism $\phi : R^{\#}[x] \to RC$ by $\phi(r) = r1$, $r \in R^{\#}$, and $\phi(x) = c$. Hence $f = dx - 1 \in Ker(\phi)$ and $Ker(\phi)$ is an $R^{\#}$-disjoint ideal which is (essentially) closed (in the sense of [4], Sect. 1): in fact, if $g \in R[x]$ and $gH \subseteq Ker(\phi)$, $0 \neq H \lhd R^{\#}$, then $\phi(g)H = 0$ and since $RC$ is prime and $0 \neq HC \lhd RC$ we obtain $g \in Ker(\phi)$. Consequently $Ker(\phi) = [f]$ is a maximal ideal of $R^{\#}[x]$, by Proposition 2.3 of [6], where $[f]$ denotes the smallest closed ideal of $R^{\#}[x]$ containing $f$ (the principal closed ideal determined by $f$, according to [4]).

Thus $R[c] = Im(\phi) \simeq R^{\#}[x]/[f]$ is a simple ring with identity. Now we show that $Im(\phi) = RC$. In fact, if $a \in C$ there exists a non-zero ideal $H$ of $R$ such that $aH \subseteq R$. Note that $H \lhd R^{\#}$ and $H[c]$ is a non-zero ideal of $Im(\phi)$. Then there exist $h_0, ..., h_n \in H$ such that $1 = \sum_i h_ic^i$. Take $b = \sum_i ah_ic^i \in Im(\phi)$. It is easy to check that for any $h \in H$ we have $bh = ah$. Therefore $a = b \in Im(\phi)$ and so $RC = R[c]$ is simple with identity.

Finally, note that $R[x]/[f] \cap R[x]$ is a non-zero ideal of $R^{\#}[x]/[f]$. Consequently $R[x]/[f] \cap R[x] = R^{\#}[x]/[f]$ is simple with identity. Thus (i) holds.

$(iv) \Rightarrow (i)$. Define $\phi : R^{\#}[x] \to RC$ by $\phi(r) = r1$, $r \in R^{\#}$, and $\phi(x) = c$. Then $Ker(\phi)$ is an ideal of $R^{\#}[x]$ such that $R^{\#}[x]/Ker(\phi) \simeq Im(\phi) = R[c] = RC$ is simple with identity. Thus $M = Ker(\phi) \cap R[x]$ is an $R$-disjoint ideal of $R[x]$ as required in (i). The proof is complete. ■

If $R \in \mathcal{P}_2$, then $\mathcal{I}(R) = 0$. However there is no $R$-disjoint maximal ideal of $R[x]$ and so the intersection of all the ideals $P$ of $R$ with $R/P \in \mathcal{P}_1$ must be zero. Hence, as in Corollary 4.7 we have the following.

**4.9 Corollary** *Any ring in $\mathcal{P}_2$ is a sub-direct product of rings from $\mathcal{P}_1$. In particular, for any ring $R$ the ideal $\mathcal{I}(R)$ is equal to the intersection of all the ideals $P$ of $R$ with $R/P \in \mathcal{P}_1$.*

To prove an extension of Theorem 4.8 to polynomial rings in several indeterminates we first show two lemmas.

If an ideal $I$ of a prime ring $R$ contains an insulator $\{a_1, ..., a_n\}$, then we have $a_1c_1 + ... + a_nc_n = 1$, for some $c_i \in C$. If $\{a_1, ..., a_n\}$ is not a $C$-independent set, then some $a_i$ is a linear combination of the others with coefficients in $C$. Thus the relation above can always be reduced to a relation, $a_1c_1' + ... + a_sc_s' = 1$ say, where $\{a_1, ..., a_s\}$ is linearly independent over $C$.

**4.10 Lemma** *Let $R$ be a prime ring and assume that $ps(R)$ contains an insulator $\{a_1, ..., a_n\}$ which is linearly independent over $C$. Then there exists a maximal ideal $M$ of $R^{\#}[x_1, ..., x_n]$ which is $R^{\#}$-disjoint. Moreover, $M$ is the smallest (essentially) closed ideal of $R^{\#}[x_1, ..., x_n]$ containing the polynomial $f = a_1x_1 + ... + a_nx_n - 1$. In particular, $R[x_1, ..., x_n]/M \cap R[x_1, ..., x_n]$ is simple with identity.*

**Proof.** By assumption there exist $c_1, ..., c_n \in C$ such that $\sum_{i=1}^{n} a_ic_i = 1$. We define $\phi : R^{\#}[x_1, ..., x_n] \to RC$ by $\phi(r) = r1$, $r \in R^{\#}$, and $\phi(x_i) = c_i$, $i = 1, ..., n$. Put $M = Ker(\phi)$, which is clearly $R^{\#}$-disjoint and contains $(f)$, the ideal generated by $f$. Also, the same argument used in Theorem 4.8 shows that $M$ is an $R^{\#}$-closed ideal.

Put $P_i = M \cap R^{\#}[x_1, ..., x_i]$, $1 \le i \le n$. By Theorem 2.3.3 of [2] there exists $\lambda \in M(R)$ such that $\lambda(a_1) \neq 0$ and $\lambda(a_i) = 0$, for $i \ge 2$. Thus we have $g = \lambda(a_1)x_1 - \lambda(1) \in P_1$. Also, $P_1$ is an $R^{\#}$-closed ideal of $R^{\#}[x_1]$ since it is an intersection of a closed ideal with $R^{\#}[x_1]$. It follows from Corollary 1.7 and 1.9, (ii), of [4] that $P_1$ is prime and maximal with respect to the condition $P_1 \cap R^{\#} = 0$. Furthermore, since $g \in (f)$ and is a polynomial of minimal degree in $P_1$ we have $P_1 = [g]_{R^{\#}[x_1]} \subseteq [f]$, where $[g]_{R^{\#}[x_1]}$ denotes the closure of $(g)$ in $R^{\#}[x_1]$ and $[f]$ the $R^{\#}$-closure of $(f)$ in $R^{\#}[x_1, ..., x_n]$.

By induction we assume that $P_l$ is maximal with respect to the condition $P_l \cap R^{\#} = 0$ (so it is prime) and $P_l \subseteq [f]$, and put $T = R^{\#}[x_1, ..., x_l]/P_l$.

Using the same argument as above we can show that there exists a polynomial $\mu(a_{l+1})x_{l+1} - \mu(1) \in P_{l+1}$, where $\mu(a_{l+1}) \neq 0$, for some $\mu \in M(R)$. Hence $P_{l+1} \supset P_l[x_{l+1}]$, because $\mu(a_{l+1}) \notin P_l$, and so $K = P_{l+1}/P_l[x_{l+1}]$ is a non-zero ideal of $T[x_{l+1}]$ which is $T$-disjoint. Take a polynomial $h \in R^{\#}[x_1, ..., x_{l+1}]$ and suppose that $F$ is a non-zero ideal of $T$ such that $\bar{h}F \subseteq K$, where $\bar{h}$ is the coset $h + P_l[x_{l+1}] \in T[x_{l+1}]$. Note that $J = F \cap R^{\#} \neq 0$ and $hJ \subseteq P_{l+1} \subseteq M$. Since $M$ is closed it follows that $h \in P_{l+1}$ and hence $\bar{h} \in K$. The argument shows that $K$ is a $T$-closed ideal of $T[x_{l+1}]$. Again the above quoted results of [4] imply that $K$ is maximal with respect to $K \cap T = 0$. Therefore $P_{l+1}$ is prime and maximal with respect to $P_{l+1} \cap R^{\#}[x_1, ..., x_l] = P_l$.

Consequently, $P_{l+1}$ is maximal with respect to $P_{l+1} \cap R^{\#} = 0$. Also, we obtain again as above that $P_{l+1} \subseteq [f]$, since $P_{l+1}$ contains a linear polynomial in $x_{l+1}$ with coefficients in $T$ which is contained in $(f)$.

The inductive argument shows that $M$ is maximal with respect to $M \cap R^{\#} = 0$ and that $M \subseteq [f]$. Also, $M$ is closed and $f \in M$, so $M = [f]$. If $N$ is a maximal ideal of $R^{\#}[x_1, ..., x_n]$ such that $N \supset M$ we have $N' = N \cap R^{\#} \neq 0$ and, since $N'$ is prime, $a_i \in N'$ for all $i$. Hence $1 = \sum_i a_i x_i - f \in N$, a contradiction. Consequently $M$ is a maximal ideal, as required.

Finally, $S = R[x_1, ..., x_n]/M \cap R[x_1, ..., x_n]$ is a non-zero ideal of - hence equal to - $R^{\#}[x_1, ..., x_n]/M$ and therefore $S$ is simple with identity. The proof is complete. ■

**4.11 Lemma** *Under the assumptions of Lemma 4.10, there exist $c_1, ..., c_n \in C$ such that $RC = R[c_1, ..., c_n]$ is simple with identity.*

**Proof.** Using the same notation as above we have $R^{\#}[x_1, ..., x_n]/M \simeq Im(\phi) \subseteq RC$. Thus $Im(\phi) = R[c_1, ..., c_n]$ is simple with identity. So it is enough to show that $C \subseteq Im(\phi)$. The proof is done as in Theorem 4.8: take any $c \in C$ and let $H$ be a non-zero ideal of $R$ with $cH \subseteq R$. Then $H[c_1, ..., c_n] = Im(\phi)$ and so we can write the identity as $1 = \sum_j h_j d_j$, for some $h_j \in H$ and $d_j \in C$ which are products of the above $c_i$, $i = 1, ..., n$. Put $l = \sum_j c h_j d_j \in Im(\phi)$. It is easy to check that $lh = ch$ for every $h \in H$. Consequently $c = l \in Im(\phi)$. ■

Corresponding to Theorem 4.8 we obtain the following characterization for rings in $\mathcal{S}_1$.

**4.12 Theorem** *For any ring $R$ the following conditions are equivalent:*

*$(i)$ There exists $n \geq 1$ and an $R$-disjoint ideal $M$ of $R[X_n]$ such that $R[X_n]/M$ is simple with identity.*

*$(ii)$ $R \in \mathcal{S}_1$.*

*$(iii)$ $R$ is prime and $ps(R)$ contains an insulator.*

*$(iv)$ $R \in \mathcal{S}$ and for some $n \geq 1$, there exist $c_1, ..., c_n \in C$ such that $RC = R[c_1, ..., c_n]$.*

**Proof.** $(i) \Rightarrow (ii)$ is already known (Remark 4.5).

$(ii) \Rightarrow (iii)$ is a consequence of Proposition 2.4.

$(iii) \Rightarrow (i), (iv)$ follow by Lemmas 4.10 and 4.11.

$(iv) \Rightarrow (i)$ Assume that $(iv)$ holds. Then $\phi : R^{\#}[x_1, ..., x_n] \to RC$ defined by $\phi(r) = r1$, $r \in R^{\#}$, and $\phi(x_i) = c_i$, for $i = 1, ..., n$, is a surjective homomorphism and so $N = Ker(\phi)$ is a maximal ideal of $R^{\#}[x_1, ..., x_n]$ with $N \cap R^{\#} = 0$. Thus $(i)$ follows by taking $M = N \cap R[x_1, ..., x_n]$. ∎

If $R$ is a simple ring without identity element, then $R[x]$ is a Brown-McCoy radical ring ([19], Corollary 3, (i)). The following extends this result.

**4.13 Corollary** *If $R$ is simple without identity element, then $R[X_n]$ is Brown-McCoy radical, for any $n \geq 1$.*

**Proof.** If, for some $n$, $R[X_n]$ is not Brown-McCoy radical, then by Theorem 4.12, $R \in \mathcal{S}_1$ and so $RC = R$ is simple with identity, a contradiction. ∎

Example 4.2 of [6] gives a subdirectly irreducible ring $R$ with idempotent heart (the intersection of all non-zero two-sided ideals of $R$) such that $R[x]$ cannot be mapped onto a simple ring with identity. We can give here the following more general example.

**4.14 Example** Let $R$ be any subdirectly irreducible ring with idempotent heart $H$. Then $R$ is prime and $ps(R) = H \neq 0$. But $R[X_n]$ does not have an $R$-disjoint ideal $M$ such that $R[X_n]/M$ is simple with identity, for any $n$.

In fact, if there exists an $R$-disjoint ideal $M$ of $R[X_n]$ such that $R[X_n]/M$ is simple with identity, then $H[X_n]/M \cap H[X_n]$ is also simple with identity. This contradicts Corollary 4.13 since $H$ is a simple ring.

# 5 Brown-McCoy radical of polynomial rings

For any cardinal $\alpha$ there exists an ideal $\mathcal{I}_\alpha = \mathcal{I}_\alpha(R)$ of $R$ such that the Brown-McCoy radical $U(R[X_\alpha])$ is equal to $\mathcal{I}_\alpha[X_\alpha]$ (Remark 3.2). If we consider a single indeterminate $x$, then the ideal $\mathcal{I}_1$ defined here coincides with the ideal $\mathcal{I}$ defined in [19] and already mentioned before.

For $\beta \geq \alpha$ we have $\mathcal{I}_\beta \subseteq \mathcal{I}_\alpha$ since every ideal $M$ of $R[X_\alpha]$ such that $R[X_\alpha]/M$ is a simple ring with identity can easily be extended to an ideal $M'$ of $R[X_\beta]$ such that the factor ring $R[X_\beta]/M'$ is also a simple ring with identity. Also, since the Brown-McCoy radical of any ring contains the u-strongly prime radical, it follows from Theorem 3.3 that $\mathcal{S}(R) \subseteq \mathcal{I}_\alpha$, for any cardinal $\alpha$.

Now we prove the following.

**5.1 Theorem** *For any ring $R$ and infinite set $X$ of either commuting or non-commuting indeterminates we have $U(R[X]) = \mathcal{S}(R[X]) (= \mathcal{S}(R)[X])$.*

**Proof.** One inclusion is clear. To prove the other inclusion take an ideal $P$ of $R$ such that $R/P \in \mathcal{S}_1$ and consider the ideal $I = U((R/P)[X]) \cap R/P$. By Remark 3.2 $U((R/P)[X]) = I[X]$, and we show that $I = 0$. This gives $U(R[X]) \subseteq P[X]$ and so $U(R[X]) \subseteq \mathcal{S}(R)[X]$ by Corollary 4.7.

For the rest of the proof we may assume $P = 0$. If $X$ is a commuting set we have that $I \subseteq \mathcal{I}_n(R)$, for every finite cardinal $n$, and it follows by Theorem 4.12 that $\mathcal{I}_m(R) = 0$, for some $m$. The result follows in this case.

Now assume that $X$ is any set of indeterminates and denote by $R[Y]$ the factor ring of $R[X]$ by the ideal generated by all the elements of the type $xy - yx$, for $x, y \in X$. Thus $R[Y]$ is a polynomial ring in an infinite set of commuting indeterminates. Since $I[X]$ is a Brown-McCoy radical ring its image $I[Y]$ in $R[Y]$ is also Brown-McCoy radical and an ideal of $R[Y]$. Consequently $I = 0$ by the commuting case. ■

There are some interesting open problems related with the questions considered in this paper. By Theorem 5.1, for a ring $R$ and any infinite cardinal $\alpha$, we have a chain $\mathcal{I}_1(R) \supseteq \mathcal{I}_2(R) \supseteq ... \supseteq \mathcal{I}_\alpha(R) = \mathcal{S}(R)$. We could not find an answer to

**Question 1** *Is there a ring $R$ for which the above sequence is not constant?*

In the remaining part we make some remarks concerning this and other related questions.

For $n \geq 1$, let $\mathcal{M}_n$ be the class of all rings $R$ such that there exists an $R$-disjoint ideal $M$ of $R[X_n]$ such that $R[X_n]/M$ is simple with identity.

**5.2 Proposition** *For any natural number $n$, $\mathcal{M}_n$ is an special class of rings with $\mathcal{P}_1 = \mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \mathcal{M}_3 \subseteq ... \subseteq \mathcal{S}_1 = \bigcup_n \mathcal{M}_n$.*

**Proof.** Every ring in $\mathcal{M}_n$ is clearly prime. If $R \in \mathcal{M}_n$ and $0 \neq I \lhd R$, then $R[X_n]/M$ is simple with identity, for some $R$-disjoint ideal $M$ of $R[X_n]$. Also $I[X_n]/M \cap I[X_n]$ is a non-zero ideal of $R[X_n]/M$. It follows that $I \in \mathcal{M}_n$.

Now, assume $I$ is an essential ideal of $R$ and $I \in \mathcal{M}_n$. Thus there exists an $I$-disjoint ideal $N$ of $I[X_n]$ such that $I[X_n]/N$ is simple with identity. Note that $N$ is an ideal of $I^{\#}[X_n]$. In fact, let $f \in I[X_n]$ denote an element such that the coset $\bar{f} \in I[X_n]/N$ is the identity element. If $g \in N$ and $w$ is any monomial in the indeterminates from $X_n$, we have $gwf - gw \in N$ and $gwf \in N$, consequently $gw \in N$.

Note that $I$ is prime. So we can extend $N$ to an ideal of $R[X_n]$ by defining

$$M = \{h \in R[X_n] \mid FhH \subseteq N, \text{ for some } 0 \neq F, H \lhd I\}.$$

Using the fact that $N$ is an ideal of $I^{\#}[X_n]$ it is easy to see that $M$ is an $R$-disjoint ideal of $R[X_n]$ such that $M \cap I[X_n] \supseteq N$ and, by maximality of $N$ we have $M \cap I[X_n] = N$. Also, if $K \supseteq M$ is an ideal of $R[X_n]$ which is maximal with respect to the condition $K \cap I[X_n] = N$, then $K$ is prime and for any $g \in K$

we have $IgI \subseteq K \cap I[X_n] = N$. Hence $g \in M$ and so $K = M$. It follows that $I[X_n]/N$ is an ideal of the prime ring $R[X_n]/M$ and consequently $R[X_n]/M$ is simple with identity, because the class of simple rings with identity is a special class of rings. This gives $R \in \mathcal{M}_n$.

By Theorem 4.8, $\mathcal{P}_1 = \mathcal{M}_1$. Also, by an easy argument already used in the beginning of this section, if $m \geq n$ we have $\mathcal{M}_n \subseteq \mathcal{M}_m$. Finally, Theorem 4.12 completes the proof. ■

The upper radical determined by the class $\mathcal{M}_n$ is a special radical and clearly equal to $\mathcal{I}_n$. Actually we do not know whether (all) the classes $\mathcal{M}_i$, $i = 1, 2, ...$, defined above are different, and this question is obviously connected with Question 1.

If for any ideal $P$ of a ring $R$, $R/P \in \mathcal{M}_n$ implies that $R/P \in \mathcal{M}_{n-1}$, then $\mathcal{I}_n(R) = \mathcal{I}_{n-1}(R)$. In particular, we have the following extension of a result which is well-known for commutative rings.

**5.3 Corollary** *Assume that $R$ is a PI ring. Then for every $n \geq 1$, we have $\mathcal{I}_n(R) = \mathcal{S}(R)$ coincides with the intersection of all ideals $P$ of $R$ such that $R/P$ is a prime ring with non-zero pseudo-radical.*

*In particular, for any (possible finite) set $X$ of, either commuting or non-commuting, indeterminates we have $U(R[X]) = \mathcal{S}(R)[X]$.*

**Proof.** The first part follows from the well-known fact that a prime PI ring ring has always large center. The second part follows by Theorem 5.1, when $X$ is an infinite set, and from the first part when $X$ is a finite commuting set. Finally, if $X$ is any finite set the proof can be completed using a similar argument as in the last part of the proof of Theorem 5.1. ■

Any prime PI ring has large center and is always u-strongly prime. The question of whether a u-strongly prime ring has always large center was raised by K. Beidar (private communication). It seems that this question is still open for u-strongly prime rings with non-zero pseudo-radical. Of course, a positive answer to this question would imply that Corollary 5.3 will be true for any ring, and our Question 1 will have a negative answer. Moreover, in this case we will have $\mathcal{I}_n(R) = \mathcal{S}(R)$, for any ring $R$.

To prove that the last relation holds it would be enough to give a positive answer to the following

**Question 2** *Is it true that if $R \in \mathcal{S}_1$, then $ps(R)$ contains a non-zero central element?*

Now we prove the following result concernig these questions.

**5.4 Proposition** *Assume that $\mathcal{A}$ is a class of rings which is closed under taking homomorphic images. Then the following conditions are equivalent:*

*(i) For any ring $R \in \mathcal{A}$ we have $\mathcal{I}_1(R) = \mathcal{S}(R)$.*

($ii$) *For any ring $R \in \mathcal{A}$ and any set $X$ of either commuting or non-commuting indeterminates we have $U(R[X]) = \mathcal{S}(R)[X]$.*

($iii$) *Question 2 has a positive answer for any ring $R \in \mathcal{A}$.*

*If the equivalent conditions above are satified, then the following condition ($iv$) also holds, and the converse is true provided that $\mathcal{A}$ is closed under taking polynomial extensions.*

($iv$) *If $R \in \mathcal{A} \cap \mathcal{M}_2$, then $R \in \mathcal{M}_1$.*

**Proof.** The equivalence between ($i$) and ($ii$) is easy to prove.

($i$) $\Rightarrow$ ($iii$). Assume $R \in \mathcal{A} \cap \mathcal{S}_1$. Then $\mathcal{I}_1(R) = \mathcal{S}(R) = 0$ and $ps(R) \neq 0$. Hence there exists an $R$-disjoint ideal $M$ of $R[x]$ such that $R[x]/M$ is simple with identity and so, by Theorem 4.8, $ps(R)$ contains a central element.

($iii$) $\Rightarrow$ ($i$), ($iv$). Assume $R \in \mathcal{A}$ and let $P$ be an ideal of $R$ such that $R/P \in \mathcal{S}_1$. Then $ps(R/P)$ contains a central element, and by Theorem 4.8 $R/P \in \mathcal{P}_1$. Hence ($i$) follows from Corollaries 4.7 and 4.9. The argument also shows that if $R \in \mathcal{A} \cap \mathcal{S}_1$, then $R \in \mathcal{M}_1$ and so ($iv$) holds.

Finally, we prove ($iv$) $\Rightarrow$ ($iii$), provided that $\mathcal{A}$ is closed under taking polynomial extensions. Let $R \in \mathcal{A}$ and assume that $R \in \mathcal{S}_1$. Then there exists $n \geq 1$ such that $R \in \mathcal{M}_n$. For $n = 1, 2$ there is nothing to show, so consider the case $n \geq 3$. Now for some $R$-disjoint ideal $M$ of $R[x_1, ..., x_n]$, $R[x_1, ..., x_n]/M$ is simple with identity. This shows that

$$T = R[x_1, ..., x_{n-2}]/M \cap R[x_1, ..., x_{n-2}] \in \mathcal{M}_2 = \mathcal{M}_1.$$

Repeating the same argument we obtain $R \in \mathcal{M}_1 = \mathcal{P}_1$. Theorem 4.8 completes the proof. ■

It is not known whether a polynomial ring in two or more indeterminates over a nil ring $R$ must be Brown-McCoy radical [19], Question 1(a). On the other hand, it is also an open problem whether the upper nil radical of a ring is contained in the strongly prime radical ([12], Problem). These two questions are related:

**5.5 Proposition** *The following conditions are equivalent:*

($i$) *For any ring $R$, the upper nil radical is contained in the u-strongly prime radical of $R$.*

($ii$) *If $R$ is a nil ring, then a polynomial ring over $R$ in any finite number of commuting indeterminates is a Brown-McCoy radical ring.*

**Proof.** ($i$) $\Rightarrow$ ($ii$) Assume that $R$ is a nil ring and for some $n \geq 1$, there exists an ideal $M$ of $R[X_n]$ such that $R[X_n]/M$ is simple with identity. Then $M \cap R$ is a strongly prime ideal of $R$ and so the upper nil radical $Nil(R)$ of $R$ must be contained in $M \cap R$. This is a contradiction to $Nil(R) = R$. Hence $R[X_n]$ is Brown-McCoy radical for any $n$.

$(ii) \Rightarrow (i)$ Assume that there exists an ideal $P$ of $R$ such that $R/P \in \mathcal{S}_1$ and $Nil(R) \not\subseteq P$. Then the factor ring $R/P$ is strongly prime and has a non-zero nil ideal $I/P$. We may assume that $P = 0$. Hence $I$ is a nil ring which belongs to $\mathcal{S}_1$. Thus, by Theorem 4.12, there exist an integer $n$ and an ideal $M$ of $I[X_n]$ such that $I[X_n]/M$ is simple with identity. This contradicts the fact that $I[X_n]$ must be Brown-McCoy radical by $(ii)$.

If $R$ is a prime ring and $ps(R)$ contains an insulator of cardinality $n$ which is a linearly independent set over $C$, then, by Lemma 4.10, there exists an $R$-disjoint ideal $M$ of $R[X_n]$ such that $R[X_n]/M$ is simple with identity, i.e., $R \in \mathcal{M}_n$. We end the paper with the following question which has an affirmative answer when $n = 1$.

**Question 3** *Is it true that if $R \in \mathcal{M}_n$, then there exists an insulator in $ps(R)$ of cardinality $n$?*

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