ADVANCES IN MATHEMATICS (China)

MODULES AND STRUCTURE PRESHEAVES FOR ARBITARY RINGS

Robert Wisbauer

University of Düsseldorf, West Germany

Basic techniques in the classical theory of commutative associative rings A with unity are:

(i) The homological characterization of A in the category of A-modules.

(ii) The rings of quotients of prime or semiprime rings A.

(iii) The localization A_P at a prime ideal P and the structure sheaf on the prime spectrum of A.

All these parts of structure theory are closely related to each other in this case.

While technique (i) was successfully applied to non-commutative rings, the parts (ii) and (iii) do not allow a satisfying extension to this more general situation. Moreover, in the special cases which admit corresponding constructions, the interplay between the different points usually gets lost.

Replacing the category of left A-modules by a suitable subcategory $\sigma[A]$ of bimodules over an arbitary (non-associative) ring A we get a natural extension of all three techniques under consideration which preserves relationships known from the classical situation.

Part (i) and (ii) of the resulting theory can be found in the author's papers [9] - [13].

§ I of this paper recalls some topics in the theory of associative commutative rings. In the subsequent §§ II, III, IV we look at these topics in different extensions of the classical theory (left modules, bimodules, $\sigma[A]$).

This is an extended version of talks given at Beijing Normal University and Fudan University in autumn 1989.

I. Associative Commutative Rings

Assume A to be an associative, commutative ring with unit.

(I.i) The category A-MOD of (left) A-modules has kernels and cokernels, infinite direct sums and direct limits. This type of category is called Grothendieck category. In addition, A-MOD has a finitely generated, projective generator ${}_AA$. Let us quote elementary examples of homological classification for A:

THEOREM For the ring A the following assertions are equivalent:

(1)(a) A is a field;

(b) ${}_AA$ is a simple module;

(c) every A-module is isomorphic to $A^{(\Lambda)}$ for suitable Λ (i.e. every A-module is free).

(2)(a) A is a finite direct product of fields;

(b) ${}_AA$ is a semisimple module;

(c) every module is projective (injective) in A-MOD.

(3)(a) ${}_AA$ is hereditary;

(b) every factor module of an injective module is injective in A-MOD.

(4)(a) A is (von Neumann) regular;

(b) every module is flat in A-MOD;

(c) every simple module is injective in A-MOD.

(I.ii) Let A be semiprime.

First we can build the classical ring of quotients A_S with respect to the (multiplicative) subset $S \subset A$ of all non-zero divisors.

On the other hand, look at the torsion theory which is defined by the injective hull $\hat{A}$ of A in A-MOD. The quotient module of A with respect to this torsion theory is actually a ring and is called the <u>complete ring</u> of quotients $Q(A)$. Since A is semiprime, the singular submodule of A is zero, which is in this case equivalent to the fact that $\mathrm{Hom}_A(A/I,\hat{A}) = 0$ for every essential ideal I in A (i.e. I is a <u>rational</u> ideal). This implies that $Q(A)$ is a (von Neumann) regular ring which is injective as A-module and as $Q(A)$-module. Moreover, $Q(A) \cong \hat{A}$ as A-module and $Q(A) = A_S$. These facts can be found in Lambek [6].

Now, if A is prime, in the above setting $Q(A)$ becomes the quotient field of A.

(I.iii) For a prime ideal P in A we have the equivalent characterizations

(α) if for ideals I,J in A $IJ \subset P$, then $I \subset P$ or $J \subset P$.

(β) if for elements a,b in A $ab \in P$, then $a \in P$ or $b \in P$.

It follows from (β) that $A\backslash P$ is a multiplicative subset of A and one can form the ring of fractions A_P with respect to $A\backslash P$. Also, to every A-module M one associates a module of fractions M_P which is an A_P-module and $M_P \cong M \otimes A_p$.

The set of all prime ideals of A endowed with the Zariski topology is called the <u>spectrum of A</u>, <u>$\mathrm{Spec}(A)$</u>. In this topology the closed sets are the sets

$$V(I) = \{P \in \mathrm{Spec}(A) \;/\; I \subset P\},\ I \subset A,$$

and the <u>special open sets</u>

$$D(c) = \{P \in \mathrm{Spec}(A) \;/\; c \notin P\}, \quad c \in A,$$

form a basis for the open sets.

Assigning for every $D(c)$ to A the ring of fractions A_c and to an A-module the module of fractions M_c (with respect to $\{1,c,c^2, \ldots\}$) one obtains a sheaf of rings and modules over Spec(A). Hereby to every $P \in \mathrm{Spec}(A)$ the ring A_P is associated.

There is a different approach to the same structure sheaf: For a prime ideal P in A consider the torsion theory cogenerated by the injective hull $\widehat{A/P}$ of A/P in A-MOD. The quotient module of A with respect to this torsion theory is a ring and is equal to A_P as considered above.

For an open subset $X \subset \mathrm{Spec}(A)$ we take the torsion theory defined by $\prod_{P \in X} \widehat{A/P}$. Associating to every A-module the quotient module with respect to this torsion theory we end up with the same structure sheaf as above.

References for most of these results are the books of Stenström [8], or Lafon [5].

II. Non-Commutative Rings and left Modules

Now let A be an associative ring with unity.

(II.i) The category A-MOD of left A-modules is again a Grothendieck category with a finitely generated projective generator ${}_A A$. However, while the kernels of ring homomorphisms are two sided ideals, the kernels of morphisms in A-MOD are only left ideals.

Nevertheless, the homological classification of rings is very effective and is one of the major tools in non-commutative ring theory.

Looking at the Theorem in (I.i) we see that (1) remains valid if we replace "field" by "skew field" in (a). The equivalences in (2) are still true if we replace "fields" by "matrix rings" over skew-fields in (a). (3) is literally correct in the non-commutative case. In (4) (a) and (b) remain equivalent while (c) characterizes a different class of rings (V-rings).

It should be pointed out that there is no counterpart to (1) for the characterization of simple rings (rings without non-trivial ideals) by properties of A-MOD.

(II.ii) In contrast to the well-behaved non-commutative module theory the theory of quotients of semiprime commutative rings does not allow a straightforward generalization to non-commutative rings. None of the techniques given in (I.ii) lead to satisfying results for a general non-commutative A:

The construction of the ring of fractions with respect to a multiplicative subset of A is only possible under additional assumptions (Ore conditions, e.g. [8]).

The complete ring of quotients has no longer such nice properties (e.g. needs not to be regular), since the essential left ideals need not to be rational in A, i.e. A may have a non-zero singular left ideal. Only for certain classes of rings the two constructions coincide and have nice properties, e.g. for Goldie rings (see [8]).

(II.iii) A prime ideal P in a non-commutative A is defined by property (α) in (I.iii). Since this is no longer equivalent to (β), $A\backslash P$ is not multiplicative and the construction of a ring of fraction A_P as considered in (I.iii) is not possible (see also (II.ii)).

Of course, one can use the torsion theory defined by the injective hull $\widehat{A/P}$ in A-MOD to get a ring of quotients with respect to P. However, $\widehat{A/P}$ is not even indecomposable and the outcoming theory seems to be of little use.

The spectrum of A with the Zariski topology can be defined as in the commutative case but the special open sets D(c) (in (I.iii)) no longer form a basis for the open sets. For the reasons just mentioned we do not get a structure sheaf on Spec(A) in the general case with the techniques considered so far.

III. Non-Commutative Rings and Bimodules

Let A be again an associative ring with unity and center C. Connected to the ring A we also have the category of (A,A)-bimodules which can be presented as left $A\otimes_C A^\circ$-modules. $A\otimes_C A^\circ$-MOD obviously is a candidate for generalizing commutative module theory. Observe that the endomorphisms ring of A as an $A\otimes_C A^\circ$-bimodule is isomorphic to the center C.

(III.i) Homological characterization of A in $A\otimes_C A^\circ$-MOD in the sense of the theorem in (I.i) is hardly possible. For example, if A is a simple ring or even a skew-field, the modules in $A\otimes_C A^\circ$-MOD need not to be isomorphic to $A^{(\Lambda)}$. An interesting property of A with respect to this category is

THEOREM For the ring A the following properties are equivalent:

(a) A is projective in $A\otimes_C A^\circ$-MOD;

(b) A is a generator in $A\otimes_C A^\circ$-MOD;

(c) A is a finitely generated projective C-module and $A\otimes_C A^\circ \cong \mathrm{End}\,({}_C A)$;

(d) A is an Azumaya algebra (central and separable).

For an Azumaya algebra $A\otimes_C A^\circ$-MOD is equivalent to C-MOD and the homological characterization of A as bimodule can be done in C-MOD.

(III.ii) The properties of a semiprime ring A as bimodule are closer to the commutative situation than its properties as left modules. Martindale's construction of the central closure of an arbitary (non-associative) semiprime ring can be seen in this context. We shall give this interpretation in (IV.ii). There we also will see the connection to the techniques used by Delale [2] for localization of bimodules and rings.

(III.iii) Delale's ideas were extended by F. Ostaeyen and A. Verschoren. Using the relation $A\otimes_C A^\circ\text{-MOD} \subset A\text{-MOD}$ they studied the interplay between localizations in both categories and introduced relative localization.

With their structure sheaf over certain non-commutative rings (e.g. noetherian PI-rings) they obtained deeper results for an algebraic geometry over these rings (see [7]). Some of their basic definitions are special instances of the situation presented in (IV.iii).

IV. General Rings and Bimodules

In the attempt to find modules or bimodules for arbitary rings one usually tried to imitate the commutative case in the sense that one expected the outcoming module category to be a Grothendieck category with a finitely generated projective generator. For example, for alternative or Jordan algebras A the modules over the enveloping Algebra $U(A)$ were studied (e.g. Jacobson [4]). However, the relation between A and $U(A)$ is not very strong and $U(A)$-MOD is not suitable for an embracing homological characterization of A. As indicated in (III.i) one has the same problem for associative rings and bimodules.

We know from category theory that Grothendieck categories (even without a projective generator!) have many of the properties we need for our purposes, e.g. injective hulls and localization techniques (Gabriel [3]). So our suggestion is to attach to every ring A the smallest (sub-) category of bimodules which still is a Grothendieck category.

For associative rings A this would be a subcategory of $A \otimes_C A°$-MOD, for alternative rings A a subcategory of $U(A)$-MOD. Since A usually is not a faithful module over $A \otimes_C A°$ (resp. $U(A)$) we coinsider A as a module over the factor ring of $A \otimes_C A°$ (resp. $U(A)$) by the ideal annihilating A, which is just the <u>multiplication</u> ring (algebra) of A:

For an arbitrary ring A the left and right multiplication with $a \in A$, $L_a : x \to ax$, $R_a : x \to xa$ for $x \in A$, are elements of $\mathrm{End}_{\mathbb{Z}}(A)$. The <u>multiplication</u> ring $M(A)$ of A is the subring of $\mathrm{End}_{\mathbb{Z}}(A)$ generated by the set $\{L_a, R_a, \mathrm{id} \mid a \in A\}$.

A is a faithful $M(A)$-module and $\mathrm{End}_{M(A)}(A)$ is called the <u>centroid $c(A)$</u> of A. If A contains a unit element then $c(A)$ is isomorphic to the center C of A.

Let $\sigma[A]$ denote the full subcategory of $M(A)$-MOD whose objects are submodules of A-generated modules.

If A is finitely generated as module over its centroid by $a_1, \ldots, a_k$, then the map $\varepsilon : M(A) \to A^k$, $\mu \to \mu(a_1, \ldots, a_k)$ is monomorphic, hence $M(A) \in \sigma[A]$ and $\sigma[A] = M(A)$-MOD.

(IV.i.) The category $\sigma[A]$ is a Grothendieck category. The homological characterization of A in this category extends the commutative case in an interesting way. Some differences in the formulations arise from the fact that in general A is not projective and not a generator in $\sigma[A]$.

A ring A with unity is <u>projective</u> in $\sigma[A]$ if and only if for every ideal $I \subset A$ the center of the ring A/I is isomorphic to $C/C \cap I$. It is a <u>self generator</u>, if for every ideal $I \subset A$ we have $(I \cap C)A = I$. In case A is projective and self generator then A is a projective generator in $\sigma[A]$ and the category $\sigma[A]$ is equivalent to C-MOD. If, in addition, A is a finitely generated module over C, then A is a (non-associative) Azumaya algebra (see [9], also Azumaya [1] and Delale [2] for the associative case). With results from [10] and [12] we now get the following generalization of the theorem in (I.i):

<u>THEOREM</u> For an arbitrary ring A the following assertions are equivalent:

(1)(a) A is a simple ring;

(b) ${}_{M(A)}A$ is a simple module;

(c) every module in $\sigma[A]$ is isomorphic to a direct sum $A^{(\Lambda)}$.

(2)(a) A is a direct sum of simple rings;

(b) ${}_{M(A)}A$ is a semisimple module;

(c) every module is projective (injective) in $\sigma[A]$.

(3)(a) A is hereditary in $\sigma[A]$ (every ideal of A is projective in $\sigma[A]$);

(b) A is projective in $\sigma[A]$ and every factor module of an injective module is injective in $\sigma[A]$.

(4)(a) A is a biregular ring and finitely presented in $\sigma[A]$;

(b) A is finitely presented and the finitely presented modules are projective in $\sigma[A]$;

(c) A is finitely presented and every module is flat in $\sigma[A]$.

(d) A is finitely generated and projective in $\sigma[A]$ and every simple module is injective in $\sigma[A]$.

(IV.ii) Let A be an arbitrary semiprime ring. Consider the injective hull $\hat{A}$ of A in $\sigma[A]$ and the torsion theory cogenerated by $\hat{A}$ in $\sigma[A]$. It is immediately seen that for every essential ideal $I \subset A$ $\mathrm{Hom}_{M(A)}(A/I, \hat{A}) = 0$, i.e. every essential ideal is "rational" in A. Hence the quotient module of A itself with respect to this torsion theory is equal to $\hat{A}$. As an injective module in $\sigma[A]$ $\hat{A}$ is generated by A and we have

$$\hat{A} = \mathrm{Trace}(A,\hat{A}) = A\,\mathrm{Hom}_{M(A)}(A,\hat{A}) = A\,\mathrm{End}_{M(A)}(\hat{A}).$$

Since $A^2(\alpha\beta-\beta\alpha) = 0$ for every $\alpha,\beta \in \mathrm{End}_{M(A)}(\hat{A})$ and (the ideal generated by) A^2 is essential in A, we get $\alpha\beta - \beta\alpha = 0$, i.e. $\mathrm{End}_{M(A)}(\hat{A})$ is a commutative ring. This gives us a chance to make $\hat{A}$ into a ring: For $a\alpha$, $b\beta \in \hat{A}$ with $a,b \in A$, $\alpha,\beta \in \mathrm{End}(\hat{A})$ we define

$$(a\alpha)(b\beta) := (ab)\alpha\beta.$$

To extend this to a multiplication in $\hat{A}$ (by linearity) we have to show that $\Sigma\, a_i\alpha_i = 0$ ($a_i \in A$, $\alpha_i \in \mathrm{End}_{M(A)}(\hat{A})$) implies $(\Sigma\, a_i\alpha_i)(b\beta) = 0$ and $(b\beta)(\Sigma a_i\alpha_i) = 0$. While the first property follows immediately from the equalities

$$(\Sigma a_i\alpha_i)(b\beta) = \Sigma(a_ib)\alpha_i\beta = ((\Sigma a_i\alpha_i)b)\beta = 0,$$

we need the commutativity of $\mathrm{End}_{M(A)}(\hat{A})$ to show

$$(b\beta)(\Sigma a_i\alpha_i) = \Sigma(ba_i)\beta\alpha_i = \Sigma(ba_i)\alpha_i\beta = b(\Sigma a_i\alpha_i)\beta = 0.$$

So our multiplication in $\hat{A}$ is well-defined and $\hat{A}$ is a semi-prime ring which is injective in $\sigma[A]$ by construction and is also injective in the new category $\sigma[\hat{A}]$. It is equal to Martindale's <u>central closure</u> of A. The centroid of the ring $\hat{A}$ is isomorphic to $\mathrm{End}_{M(A)}(\hat{A})$ and module theory tells us that this is a (von Neumann) regular, self-injective (and commutative!) ring which contains the centroid of A.

For a prime ring A with unity the construction above leads to a prime ring $\hat{A}$ whose center is a (possibly transcendental) extension field of the center of A.

If A is an associative prime PI-ring with unity the central closure $\hat{A}$ becomes a simple algebra which has finite dimension over its center. The simplicity of $\hat{A}$ is a consequence of the fact that in the rings considered every non-zero ideal has non-zero intersection with the center. This also implies that the center of $\hat{A}$ is just the quotient field of the center of A and thus $\hat{A}$ can be obtained by constructing the ring of fractions of A with respect to all non-zero central elements. Similar phenomenos also occur in other classes of rings, e.g. for alternative, non-associative prime rings A with $3A \neq 0$. More details about this theory can be found in [13].

(IV.iii) The definition of a prime ideal P in an arbitrary ring A is the same as (α) in (I.iii). The spectrum Spec(A) with the Zariski topology can be introduced as in the classical case. If A is a self generator as M(A)-module the special open sets D(c) form a basis for the open sets in the Zariski topology.

We now can imitate the second approach to a structure sheaf for commutative rings (see (I.iii)): For a prime ideal P in A the module A/P is uniform and its injective hull $\widehat{A/P}$ in $\sigma[A]$ is indecomposable. Let T denote the torsion class defined by $\widehat{A/P}$ in $\sigma[A]$ and $T(N)$ the torsion-submodule of $N \in \sigma[A]$.

The <u>T-injective hull</u> $E_T(N)$ of N is an essential, T-injective extension of N such that $E_T(N)/N \in T$. The T-injective hull of $N/T(N)$ usually is called the <u>quotient module</u> of N (with respect to T). For our purposes we need a slight variation of this concept: For every A-generated module N in $\sigma[A]$ set

$$Q_P(N) := \mathrm{Trace}(A,\ E_T(N/T(N))).$$

Now abstract torsion theory gives us for A-generated modules N, U the fundamental identification

$$\mathrm{Hom}_{M(A)}(U,\ Q_P(N)) = \mathrm{Hom}_{M(A)}(Q_P(U),\ Q_P(N)).$$

Setting $U = A$ we obtain the relations

$$Q_P(A) = \mathrm{Trace}(A,\ Q_P(A)) = A\ \mathrm{Hom}_{M(A)}(A,\ Q_P(A)) = A\ \mathrm{End}_{M(A)}(Q_P(A)).$$

Using the special form $\widehat{A/P}$ of the module defining our torsion theory one can show that $\mathrm{End}_{M(A)}(Q_P(A))$ is a commutative ring. Hence, similar to the construction of the central closure in (IV.ii), we may define a ring structure on $Q_P(A)$. Again from the above equality we obtain for an A-generated $N \in \sigma[A]$:

$$Q_P(N) = A\ \mathrm{Hom}_{M(A)}(A,\ Q_P(N)) = A\ \mathrm{Hom}_{M(A)}(Q_P(A),\ Q_P(N)).$$

Consequently, the elements in $Q_P(N)$ are sums of elements $d\psi$ with $d \in A$ and $\psi \in \mathrm{Hom}_{M(A)}(Q_P(A),\ Q_P(N))$. We can (try to) multiply it with $a\alpha \in Q_P(A)$, $a \in A$, $\alpha \in \mathrm{End}_{M(A)}(Q_P(A))$ by setting

$$(a\alpha)(d\psi) := (ad)\alpha\psi, \quad (d\psi)(a\alpha) := (da)\alpha\psi.$$

As before, we have to refer to the special type of our defining module $\widehat{A/P}$

to prove that these operations are well-defined and $Q_P(N)$ becomes a $Q_P(A)$-generated bimodule over $Q_P(A)$.

Thus, for every $P \in \mathrm{Spec}(A)$ we can associate to A a quotient ring $Q_P(A)$ and to every A-generated module N a quotient module $Q_P(N)$ which is generated by $Q_P(A)$ as $Q_P(A)$-bimodule.

For an open set $X \subset \mathrm{Spec}(A)$ we take the product $\prod_{P \in X} \widehat{A/P}$ in $\sigma[A]$. The torsion theory defined by this module provides a quotient ring of A and a quotient module for every A-generated module with the desired properties to lead to a presheaf of rings and modules over $\mathrm{Spec}(A)$.

These constructions generalize ideas in Delale [2] and result from cooperation with P. Jara Martinez.

References

[1] Azumaya, G., Separable Rings, J. Algebra 63 (1980), 1-14.

[2] Delale, J.P., Sur le spectre d'un anneau non commutatif
Thèse, Université Paris Sud (Orsay), 1974.

[3] Gabriel, P., Des Catégories Abéliennes,
Bull. Soc. math. France 90 (1962), 323-448.

[4] Jacobson, N., Structure of Alternative and Jordan Bimodules,
Osaka Math. J. 6 (1954), 1-71.

[5] Lafon, J.P., Algèbre commutative, Hermann, Paris 1977.

[6] Lambek, J., Lectures on Rings and Modules,
Blaisdell, Waltham-Toronto-London 1966.

[7] Van Ostaeyen, F., Verschoren, A., Non-Commutative Algebraic Geometry,
Springer LNM 887 (1981).

[8] Stenström, B., Rings of Quotients,
Springer, Berlin-Heidelberg-New York 1975.

[9] Wisbauer, R., Zentrale Bimoduln und separable Algebren,
Arch. Math. 30 (1978), 129-137.

[10] Wisbauer, R., Nonassociative left regular and biregular rings,
J. Pure Appl. Algebra 10 (1977), 215-226.

[11] Wisbauer, R., Co-semisimple modules and nonassociative V-rings,
Comm. Algebra 5 (1977), 1193-1209.

[12] Wisbauer, R., Erbliche Moduln und nichtassoziative Ringe,
Comm. Alg. 7 (1979), 47-77.

[13] Wisbauer, R., Localization of modules and the central closure of rings,
Comm. Algebra 9 (1981), 1455-1493.