MONADS AND COMONADS ON MODULE CATEGORIES

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Abstract. Let A be a ring and $\mathbb{M}_A$ the category of right A-modules. It is well known in module theory that any A-bimodule B is an *A-ring* if and only if the functor $-\otimes_A B : \mathbb{M}_A \to \mathbb{M}_A$ is a *monad* (or *triple*). Similarly, an A-bimodule $\mathcal{C}$ is an *A-coring* provided the functor $-\otimes_A \mathcal{C} : \mathbb{M}_A \to \mathbb{M}_A$ is a *comonad* (or *cotriple*). The related categories of *modules* (or *algebras*) of $-\otimes_A B$ and *comodules* (or *coalgebras*) of $-\otimes_A \mathcal{C}$ are well studied in the literature. On the other hand, the right adjoint endofunctors $\mathrm{Hom}_A(B, -)$ and $\mathrm{Hom}_A(\mathcal{C}, -)$ are a comonad and a monad, respectively, but the corresponding (co)module categories did not find much attention so far. The category of $\mathrm{Hom}_A(B, -)$-comodules is isomorphic to the category of B-modules, while the category of $\mathrm{Hom}_A(\mathcal{C}, -)$-modules (called $\mathcal{C}$-*contramodules* by Eilenberg and Moore) need not be equivalent to the category of $\mathcal{C}$-comodules.

The purpose of this paper is to investigate these categories and their relationships based on some observations of the categorical background. This leads to a deeper understanding and characterisations of algebraic structures such as corings, bialgebras and Hopf algebras. For example, it turns out that the categories of $\mathcal{C}$-comodules and $\mathrm{Hom}_A(\mathcal{C}, -)$-modules are equivalent provided $\mathcal{C}$ is a coseparable coring. Furthermore, we describe equivalences between categories of $\mathrm{Hom}_A(\mathcal{C}, -)$-modules and comodules over a coring $\mathcal{D}$ in terms of new Galois properties of bicomodules. Finally, we characterise Hopf algebras H over a commutative ring R by properties of the functor $\mathrm{Hom}_R(H, -)$ and the category of mixed $\mathrm{Hom}_R(H, -)$-bimodules. This generalises in particular the fact that a finite dimensional vector space H is a Hopf algebra if and only if the dual space H^* is a Hopf algebra.

Contents

1. Introduction

The purpose of this paper is to present a categorical framework for studying problems in the theories of rings and modules, corings and comodules, bialgebras and

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(mixed) bimodules and Hopf algebras and Hopf modules. The usefulness of this framework is illustrated by analysing the structure of the category of *contramodules* and the bearing of this structure on the properties of corings and bialgebras.

It is well-known that for a right module V over an R-algebra A, the dual R-module $V^* = \mathrm{Hom}_R(V, R)$ is a left module over A. It is equally well-known that for a right comodule V of an R-coalgebra C, in general V^* is not a C-comodule (left or right). It has already been realised in [10, Chapter IV.5] that to a coalgebra C two (different) representation categories can be associated: the familiar category of C-comodules and the category of C-*contramodules* introduced therein. If V is a C-comodule, then V^* is a C-contramodule.

While comodules of coalgebras (and corings) have been intensively studied, contramodules seem to have been rather neglected. Yet the category of contramodules is as fundamental as that of comodules, and both categories are complementary to each other. To substantiate this claim, one needs to resort to the categorical point of view on corings. An A-coring can be defined as an A-bimodule $\mathcal{C}$ such that the tensor endofunctor $-\otimes_A \mathcal{C}$ on the category of right A-modules $\mathbb{M}_A$ is a comonad or a cotriple. Right $\mathcal{C}$-comodules are the same as *comodules* (or *coalgebras* in category theory terminology) of the comonad $-\otimes_A \mathcal{C}$. On the other hand, the tensor functor $-\otimes_A \mathcal{C}$ has a right adjoint, the Hom-functor $\mathrm{Hom}_A(\mathcal{C}, -)$, which we denote by $[\mathcal{C}, -]$. By purely categorical arguments (see Eilenberg and Moore [11, Proposition 3.1]), the functor $-\otimes_A \mathcal{C}$ is a *comonad* if and only if its *right* adjoint $[\mathcal{C}, -]$ is a *monad*. Thus, $\mathcal{C}$ is an A-coring if and only if $[\mathcal{C}, -]$ is a monad on $\mathbb{M}_A$; right $\mathcal{C}$-contramodules are simply *modules* (or *algebras* in category theory terminology) of this monad. This categorical interpretation explains the way in which contramodules complement comodules (e.g. 3.6, 3.7).

Again purely categorical considerations (see [11]) explain that, while there are two categories of representations of a coring, there is only one category of representations of a ring – the familiar category of modules. More precisely, a ring morphism $A \to B$ can be equivalently described as the monad structure of the tensor functor $-\otimes_A B$ on $\mathbb{M}_A$ associated to an A-bimodule B. With this interpretation, right B-modules are simply modules of the monad $-\otimes_A B$. The right adjoint functor $\mathrm{Hom}_A(B, -)$ is a comonad on $\mathbb{M}_A$ and the category of comodules of $\mathrm{Hom}_A(B, -)$ is *isomorphic* to the category of modules of the monad $-\otimes_A B$. Consequently, there is only one type of representation categories for rings – the category of right (or left) modules over a ring.

The above comments illustrate how the categorical point of view can give significant insight into algebraic structures. There are many constructions developed in category theory that are directly applicable to ring theoretic situations but they seem not to be sufficiently explored. Contramodules of a coring are a good example of this. On one hand, from the category point of view, they are as natural as comodules, on the other hand, their structure was not analysed properly until very recently, when their important role in semi-infinite homology was outlined by Positselski [24]. The main motivation of our paper is a study of contramodules of corings. This aim is achieved by placing it in a broader context: we revisit category theory, more specifically the theory of adjoint comonad-monad pairs, in the context of rings and modules.

We begin by summarising the categorical framework, and then apply it first to rings in module categories, next to corings. In the latter case, we concentrate on properties of the less-known category of contramodules, and derive consequences of the categorical formulation in this context. We analyse functors between categories of comodules and contramodules, and introduce the notion of a $[\mathcal{C},-]$-Galois bicomodule. We then relate the equivalences between categories of $\mathcal{C}$-contramodules and $\mathcal{D}$-comodules to the existence of $[\mathcal{C},-]$-Galois bicomodules. In particular we prove that the existence of the latter is a necessary condition for such an equivalence; see 5.8 and 5.9. We also derive the characterisation of entwining structures as liftings of Hom-functors to module, comodule and contramodule categories; see 6.1. Finally, we study contramodules over corings associated to bialgebras and provide new extensions of the Fundamental Theorem of Hopf algebras (see 7.9). These are achieved by investigating properties of the category of contramodules over the corings associated to a bialgebra B. Again this can be seen as B being a *Galois comodule* with respect to the Hom-functors of the associated corings, and thus indicates the role which is played by adjoint functors (that are not tensor functors) in the description of Hopf algebra dualities.

2. Categorical framework

Our main concern is to apply abstract categorical notions to special situations in module categories. We begin by recalling some basic definitions and properties (e.g. from [11]) to fix notation, and then develop a categorical framework which is later applied to categories of (co)modules.

Throughout, the composition of functors is denoted by juxtaposition, and the usual composition symbol $\circ$ is reserved for natural transformations and morphisms. Given functors F, G and a natural transformation φ, $F\varphi G$ denotes the natural transformation, which, evaluated at an object X gives a morphism obtained by applying F to a morphism provided by the natural transformation φ evaluated at the object GX.

By $\mathbb{A} \simeq \mathbb{B}$ we denote equivalences between categories and $\mathbb{A} \cong \mathbb{B}$ is written for their isomorphisms. The symbol $\cong$ is also used to denote isomorphisms between objects in any category, in particular isomorphisms of modules and (natural) isomorphisms of functors.

2.1. **Adjoint functors.** A pair (L, R) of functors $L : \mathbb{A} \to \mathbb{B}$ and $R : \mathbb{B} \to \mathbb{A}$ between categories $\mathbb{A}$, $\mathbb{B}$ is called an *adjoint pair* if there is a natural isomorphism

$$\mathrm{Mor}_{\mathbb{B}}(L(-), -) \xrightarrow{\cong} \mathrm{Mor}_{\mathbb{A}}(-, R(-)).$$

This can be described by natural transformations *unit* $\eta : I_{\mathbb{A}} \to RL$ and *counit* $\varepsilon : LR \to I_{\mathbb{B}}$ satisfying the *triangular identities* $\varepsilon L \circ L\eta = I_L$ and $R\varepsilon \circ \eta R = I_R$.

2.2. **Natural transformations for adjoints.** For two adjunctions (L, R) and $(\widetilde{L}, \widetilde{R})$ between $\mathbb{A}$ and $\mathbb{B}$, with respective units η, $\widetilde{\eta}$ and counits ε, $\widetilde{\varepsilon}$, there is an isomorphism between the natural transformations (cf. [17], [20])

$$\mathrm{Nat}(L, \widetilde{L}) \to \mathrm{Nat}(\widetilde{R}, R), \quad f \mapsto \bar{f} := R\widetilde{\varepsilon} \circ Rf\widetilde{R} \circ \eta\widetilde{R},$$

with the inverse map

$$\mathrm{Nat}(\widetilde{R}, R) \to \mathrm{Nat}(L, \widetilde{L}), \qquad \bar{f} \mapsto f := \varepsilon\widetilde{L} \circ L\bar{f}\widetilde{L} \circ L\widetilde{\eta}.$$

We say that f and $\bar{f}$ are *mates under the adjunctions* (L, R) *and* $(\widetilde{L}, \widetilde{R})$. For natural transformations $f : L_1 \to L_2$ and $g : L_2 \to L_3$ between left adjoint functors, naturality and the triangle identities imply $\overline{g \circ f} = \bar{f} \circ \bar{g}$. In particular, f is a natural isomorphism if and only if its mate $\bar{f}$ is a natural isomorphism. Moreover, if for an adjunction (L,R), the composites LL_1 (and hence LL_2) are meaningful, then $\overline{Lf} = \bar{f}R$. Similarly, if the composites L_1L (and thus L_2L) are meaningful then $\overline{fL} = R\bar{f}$.

2.3**. Monads on** $\mathbb{A}$**.** A *monad on the category* $\mathbb{A}$ is a triple $\mathbf{F} = (F,m,i)$, where $F : \mathbb{A} \to \mathbb{A}$ is a functor with natural transformations $m : FF \to F$ and $i : I_{\mathbb{A}} \to F$ satisfying associativity and unitality conditions. A *morphism of monads* $(F,m,i) \to (F',m',i')$ is a natural transformation $\varphi : F \to F'$ such that $m' \circ \varphi F' \circ F\varphi = \varphi \circ m$ and $\varphi \circ i = i'$.

An F*-module* is a pair consisting of $A \in \mathrm{Obj}(\mathbb{A})$ and a morphism $\varrho_A : FA \to A$ satisfying $\varrho_A \circ mA = \varrho_A \circ F\varrho_A$ and $\varrho_A \circ iA = I_A$.

Morphisms between F-modules $f : A \to A'$ are morphisms in $\mathbb{A}$ with $\varrho_{A'} \circ Ff = f \circ \varrho_A$ and the *Eilenberg-Moore category of* F*-modules* is denoted by $\mathbb{A}_F$.

For any object A of $\mathbb{A}$, FA is an F-module and this yields the *free functor*

$$\phi_F : \mathbb{A} \to \mathbb{A}_F,\ A \mapsto (FA, mA),$$

which is left adjoint to the *forgetful functor* $U_F : \mathbb{A}_F \to \mathbb{A}$ by the isomorphism

$$\mathrm{Mor}_{\mathbb{A}_F}(\phi_F A, B) \to \mathrm{Mor}_{\mathbb{A}}(A, U_F B), \quad f \mapsto f \circ iA.$$

The full subcategory of $\mathbb{A}_F$ consisting of all free F-modules (i.e. the full subcategory of $\mathbb{A}_F$ generated by the image of ϕ_F) is called the *Kleisli category of* $\mathbf{F}$ and is denoted by $\widetilde{\mathbb{A}}_F$.

2.4**. Comonads on** $\mathbb{A}$**.** A *comonad on* $\mathbb{A}$ is a triple $\mathbf{G} = (G,d,e)$, where $G : \mathbb{A} \to \mathbb{A}$ is a functor with natural transformations $d : G \to GG$ and $e : G \to I_{\mathbb{A}}$ satisfying coassociativity and counitality conditions. A *morphism of comonads* is a natural transformation that is compatible with the coproduct and counit.

A G*-comodule* is an object $A \in \mathbb{A}$ with a morphism $\varrho^A : A \to GA$ compatible with d and e. Morphisms between G-comodules $g : A \to A'$ are morphisms in $\mathbb{A}$ with $\varrho^{A'} \circ g = Gg \circ \varrho^A$ and the *Eilenberg-Moore category of* G*-comodules* is denoted by $\mathbb{A}^G$.

For any $A \in \mathbb{A}$, GA is a G-comodule yielding the *(co)free functor*

$$\phi^G : \mathbb{A} \to \mathbb{A}^G,\ A \mapsto (GA, dA)$$

which is right adjoint to the forgetful functor $U^G : \mathbb{A}^G \to \mathbb{A}$ by the isomorphism

$$\mathrm{Mor}_{\mathbb{A}^G}(B, \phi^G A) \to \mathrm{Mor}_{\mathbb{A}}(U^G B, A), \quad f \mapsto eA \circ f.$$

The full subcategory of $\mathbb{A}^G$ consisting of all (co)free G-comodules (i.e. the full subcategory of $\mathbb{A}^G$ generated by the image of ϕ^G) is called the *Kleisli category of* $\mathbf{G}$ and is denoted by $\widetilde{\mathbb{A}}^G$.

2.5**. (Co)monads related to adjoints.** *Let* $L : \mathbb{A} \to \mathbb{B}$ *and* $R : \mathbb{B} \to \mathbb{A}$ *be an adjoint pair of functors with unit* $\eta : I_{\mathbb{A}} \to RL$ *and counit* $\varepsilon : LR \to I_{\mathbb{B}}$*. Then*

$$\mathbf{F} = (RL,\ RLRL \xrightarrow{R\varepsilon L} RL,\ I_{\mathbb{A}} \xrightarrow{\eta} RL)$$

is a monad on $\mathbb{A}$*. Similarly a comonad on* $\mathbb{B}$ *is defined by*

$$\mathbf{G} = (LR,\ LR \xrightarrow{L\eta R} LRLR,\ LR \xrightarrow{\varepsilon} I_{\mathbb{B}}).$$

As already observed by Eilenberg and Moore in [11], the monad structure of an endofunctor induces a comonad structure on any adjoint endofunctor.

2.6. **Adjoints of monads and comonads.** *Let $L : \mathbb{A} \to \mathbb{A}$ and $R : \mathbb{A} \to \mathbb{A}$ be an adjoint pair of functors.*

(1) *L is a monad if and only if R is a comonad.*
In this case the Eilenberg-Moore categories $\mathbb{A}_L$ and $\mathbb{A}^R$ are isomorphic.

(2) *L is a comonad if and only if R is a monad.*
In this case the Kleisli categories $\widetilde{\mathbb{A}}^L$ and $\widetilde{\mathbb{A}}_R$ are isomorphic to each other.

Proof. (1) The first claim is proven in [11, Proposition 3.1], here it follows from 2.2 (see also [20]). The isomorphism of $\mathbb{A}_L$ and $\mathbb{A}^R$ is stated in [29, p. 3935].

(2) The first claim is proved similarly to the first claim in (1). The isomorphism of $\widetilde{\mathbb{A}}^L$ and $\widetilde{\mathbb{A}}_R$ was observed in [18] and is also stated in [29, p. 3935]. It is provided by the canonical isomorphisms for $A, A' \in \mathbb{A}$,

$$\begin{array}{rcl} \mathrm{Mor}_{\mathbb{A}^L}(\phi^L A, \phi^L A') & \cong & \mathrm{Mor}_{\mathbb{A}}(LA, A') \\ & \cong & \mathrm{Mor}_{\mathbb{A}}(A, RA') \cong \mathrm{Mor}_{\mathbb{A}_R}(\phi_R A, \phi_R A'). \end{array}$$

□

2.7. **Relative projectivity and injectivity.** An object A of a category $\mathbb{A}$ is said to be *projective relative to a functor* $F : \mathbb{A} \to \mathbb{B}$ (or *F-projective* in short) if $\mathrm{Mor}_{\mathbb{A}}(A, f) : \mathrm{Mor}_{\mathbb{A}}(A, X) \to \mathrm{Mor}_{\mathbb{A}}(A, Y)$ is surjective for all those morphisms f in $\mathbb{A}$, for which Ff is a split epimorphism in $\mathbb{B}$. Dually, $A \in \mathbb{A}$ is said to be *injective relative to F* (or *F-injective*) if $\mathrm{Mor}_{\mathbb{A}}(f, A) : \mathrm{Mor}_{\mathbb{A}}(Y, A) \to \mathrm{Mor}_{\mathbb{A}}(X, A)$ is surjective for all such morphisms f in $\mathbb{A}$, for which Ff is a split monomorphism in $\mathbb{B}$.

For an adjunction $(L : \mathbb{A} \to \mathbb{B}, R : \mathbb{B} \to \mathbb{A})$, with unit η and counit ε, an object $A \in \mathbb{A}$ is L-injective if and only if ηA is a split monomorphism in $\mathbb{A}$. Dually, $B \in \mathbb{B}$ is R-projective if and only if εB is a split epimorphism in $\mathbb{B}$.

Recall (e.g. [5, Section 6.5],[14, Chapter 2]) that the *Cauchy completion*, also called *Karoubian closure*, of any category $\mathbb{A}$ is the smallest (unique up to equivalence) category $\overline{\mathbb{A}}$ that contains $\mathbb{A}$ as a subcategory and in which idempotent morphisms split (i.e. can be written as a composite of an epimorphism and its section).

2.8. **Equivalent subcategories.** *Let $\mathbb{A}$ be a category in which idempotent morphisms split. Then for any comonad (L, d, e) on $\mathbb{A}$ with right adjoint monad (R, m, i), there is an equivalence*

$$E : \mathbb{A}^L_{inj} \to \mathbb{A}^{proj}_R,$$

where $\mathbb{A}^L_{inj}$ denotes the full subcategory of $\mathbb{A}^L$ whose objects are injective relative to the forgetful functor $U^L : \mathbb{A}^L \to \mathbb{A}$ and $\mathbb{A}^{proj}_R$ denotes the full subcategory of $\mathbb{A}_R$ whose objects are projective relative to the forgetful functor $U_R : \mathbb{A}_R \to \mathbb{A}$.

Explicitly, for $(A, \varrho^A) \in \mathbb{A}^L_{inj}$, the object $E(A, \varrho^A)$ is given by the equaliser of the parallel morphisms $R\varrho^A$ and $\omega := mLA \circ R\eta A : RA \to RLA$, where η is the unit of the adjunction (L, R).

Proof. By 2.6(2), the Kleisli categories $\tilde{\mathbb{A}}^L$ and $\tilde{\mathbb{A}}_R$ are isomorphic and this isomorphism extends to their Karoubian closures. The Karoubian closure of $\tilde{\mathbb{A}}_R$ is equivalent to the full subcategory of U_R-projective objects of $\mathbb{A}_R$ (see [16], [27, Theorem 2.5]). Dually, the Karoubian closure of $\tilde{\mathbb{A}}^L$ is equivalent to the full subcategory of U^L-injective objects of $\mathbb{A}^L$. This proves the equivalence $\mathbb{A}^L_{inj} \simeq \mathbb{A}^{proj}_R$.

The explicit form of the equivalence functor is obtained by computing the composite of the isomorphism between the Karoubian closures of the Kleisli categories with the equivalences in [27, Theorem 2.5]. This (straightforward) computation yields the equaliser $E(A,\varrho^A) \to RA$ of the identity morphism I_{RA} and the idempotent morphism $R\nu^A \circ \omega : RA \to RA$, where ν^A is a retraction of $\eta(A,\varrho^A) = \varrho^A$ in $\mathbb{A}^L$. This equaliser exists by the assumption that idempotents split in $\mathbb{A}$. Since

$$\omega \circ R\nu^A \circ \omega = R\varrho^A \circ R\nu^A \circ \omega \quad \text{and} \quad R\nu^A \circ R\varrho^A = I_{RA},$$

$E(A,\varrho^A) \to RA$ is also an equaliser of ω and $R\varrho^A$. □

Recall from [23], [26] that a functor $F : \mathbb{B} \to \mathbb{A}$ is said to be *separable* if and only if the transformation $\mathrm{Mor}_{\mathbb{B}}(-,-) \to \mathrm{Mor}_{\mathbb{A}}(F(-),F(-))$, $f \mapsto Ff$, is a split natural monomorphism. Separable functors reflect split epimorphisms and split monomorphisms. Questions related to 2.9(1) are also discussed in [6, Proposition 6.3].

2.9**. Separable monads and comonads.** *Let $\mathbb{A}$ be a category.*

(1) *For a monad (R,m,i) on $\mathbb{A}$, the following are equivalent:*

(a) *m has a natural section $\widehat{m}$ such that*

$$Rm \circ \widehat{m}R = \widehat{m} \circ m = mR \circ R\widehat{m};$$

(b) *the forgetful functor $U_R : \mathbb{A}_R \to \mathbb{A}$ is separable.*

(2) *For a comonad (L,d,e) on $\mathbb{A}$, the following are equivalent:*

(a) *d has a natural retraction $\widehat{d}$ such that*

$$\widehat{d}L \circ Ld = d \circ \widehat{d} = L\widehat{d} \circ dL;$$

(b) *the forgetful functor $U^L : \mathbb{A}^L \to \mathbb{A}$ is separable.*

Proof. (1) By Rafael's theorem [26, Theorem 1.2], U_R is separable if and only if the counit ε_R of the adjunction (ϕ_R, U_R) (see 2.3) is a split natural epimorphism.

(1) (a)⇒(b). A section $\nu : I_{\mathbb{A}_R} \to \phi_R U_R$ of ε_R is given by a morphism

$$\nu(X,\varrho_X) \ : \ X \xrightarrow{iX} RX \xrightarrow{\widehat{m}X} RRX \xrightarrow{R\varrho_X} RX,$$

for any (X,ϱ_X) in $\mathbb{A}_R$. By naturality and the properties of $\widehat{m}$ required in (a), $\nu(X,\varrho_X)$ is an R-module morphism, i.e. $mX \circ R\nu(X,\varrho_X) = \nu(X,\varrho_X) \circ \varrho_X$. Since $\widehat{m}$ is a section of m, $\nu(X,\varrho_X)$ is a section of $\varepsilon_R(X,\varrho_X) = \varrho_X$. In order to see that, use also associativity and unitality of the R-action ϱ_X. The morphism ν is natural, i.e. for $f : (X,\varrho_X) \to (Y,\varrho_Y)$ in $\mathbb{A}_R$, $Rf \circ \nu(X,\varrho_X) = \nu(Y,\varrho_Y) \circ f$. This follows by definition of an R-module morphism and naturality.

(b)⇒(a). A section $\nu : I_{\mathbb{A}_R} \to \phi_R U_R$ of ε_R induces a section of $m = U_R\varepsilon_R\phi_R$ by putting $\widehat{m} := U_R\nu\phi_R$. It obeys the properties in (a) by naturality.

(2) The proof is symmetric to (1). □

2.10**. Separability of adjoints.** *Let $L,R : \mathbb{A} \to \mathbb{A}$ be an adjoint pair of endofunctors with unit $\eta : I_{\mathbb{A}} \to RL$ and counit $\varepsilon : LR \to I_{\mathbb{A}}$.*

If (L, d, e) is a comonad with corresponding monad (R, m, i), then there are pairs of adjoint (free and forgetful) functors (see 2.3, 2.4):

$$\mathbb{A} \xrightarrow{\phi_R} \mathbb{A}_R\,, \quad \mathbb{A}_R \xrightarrow{U_R} \mathbb{A}\,, \quad \textit{with unit } \eta_R \textit{ and counit } \varepsilon_R, \textit{ and}$$

$$\mathbb{A}^L \xrightarrow{U^L} \mathbb{A}\,, \quad \mathbb{A} \xrightarrow{\phi^L} \mathbb{A}^L\,, \quad \textit{with unit } \eta^L \textit{ and counit } \varepsilon^L.$$

(1) *ϕ^L is separable if and only if ϕ_R is separable.*

(2) *U^L is separable if and only if U_R is separable.*

If the properties in part (2) hold, then any object of $\mathbb{A}^L$ is injective relative to U^L and every object of $\mathbb{A}_R$ is projective relative to U_R.

Proof. (1) By Rafael's theorem [26, Theorem 1.2], ϕ^L is separable if and only if $\varepsilon^L = e$ is a split natural epimorphism, while ϕ_R is separable if and only if $\eta_R = i$ is a split natural monomorphism. By construction, i and e are mates under the adjunction (L, R) and the trivial adjunction $(I_\mathbb{A}, I_\mathbb{A})$. That is, $e = \varepsilon \circ Li$ equivalently, $i = Re \circ \eta$. Hence a natural transformation $\widehat{i} : R \to I_\mathbb{A}$ is a retraction of i if and only if its mate $\widehat{e} := \widehat{i}L \circ \eta$ under the adjunctions $(I_\mathbb{A}, I_\mathbb{A})$ and (L, R) is a section of e.

(2) Since d and m are mates under the adjunctions (L, R) and (LL, RR), a natural transformation $\widehat{m}$ satisfies the properties in 2.9(1)(a) if and only if its mate $\widehat{d}$ satisfies the properties in 2.9(2)(a). Thus the claim follows by 2.9.

It remains to prove the final claims. Following 2.7, an L-comodule (A, ϱ^A) is U^L-injective if and only if $\eta^L(A, \varrho^A) = \varrho^A$ is a split monomorphism in $\mathbb{A}^L$. Since ϱ^A is split in $\mathbb{A}$ (by eA) and U^L, being separable, reflects split monomorphisms, any $(A, \varrho^A) \in \mathbb{A}^L$ is U^L-injective. U_R-projectivity of every object of $\mathbb{A}_R$ is proven by a symmetrical reasoning. ☐

2.11**. Lifting of endofunctors.** For a monad F, a comonad G and an endofunctor T on the category $\mathbb{A}$, consider the diagrams with Eilenberg-Moore and Kleisli categories

$$\begin{tikzcd} \mathbb{A}_F \arrow[r, dashed, "\overline{T}"] \arrow[d, "U_F"'] & \mathbb{A}_F \arrow[d, "U_F"] \\ \mathbb{A} \arrow[r, "T"] & \mathbb{A}, \end{tikzcd} \quad \begin{tikzcd} \mathbb{A}^G \arrow[r, dashed, "\widehat{T}"] \arrow[d, "U^G"'] & \mathbb{A}^G \arrow[d, "U^G"] \\ \mathbb{A} \arrow[r, "T"] & \mathbb{A}, \end{tikzcd} \quad \begin{tikzcd} \mathbb{A} \arrow[r, "T"] \arrow[d, "\phi_F"'] & \mathbb{A} \arrow[d, "\phi_F"] \\ \widetilde{\mathbb{A}}_F \arrow[r, dashed, "\widetilde{T}"] & \widetilde{\mathbb{A}}_F, \end{tikzcd} \quad \begin{tikzcd} \mathbb{A} \arrow[r, "T"] \arrow[d, "\phi^G"'] & \mathbb{A} \arrow[d, "\phi^G"] \\ \widetilde{\mathbb{A}}^G \arrow[r, dashed, "\approx{T}"] & \widetilde{\mathbb{A}}^G, \end{tikzcd}$$

where the U's denote the forgetful functors and the ϕ's the free functors. If there exist $\overline{T}$, $\widehat{T}$, $\widetilde{T}$ or $\widetilde{\widetilde{T}}$ making the corresponding diagram commutative, they are called *liftings of T*.

By Power and Watanabe's observation [25], liftings of endofunctors $\mathbb{A} \to \mathbb{A}$ arise as images under a strict monoidal functor, from the monoidal category (vertical subcategory at $\mathbb{A}$ of the 2-category) of (co)monad morphisms, to the monoidal category of endofunctors on $\mathbb{A}$. By their result, liftings to (co)monads are in bijective correspondence with (co)monads in the monoidal category of (co)monad morphisms, that is, with various *distributive laws* [4]:

2.12**. Monad distributive laws.** For monads F and T on a category $\mathbb{A}$, the liftings of T to a monad $\overline{T}$ on $\mathbb{A}_F$ are in bijective correspondence with *monad distributive laws* $\lambda : FT \to TF$, and also with the liftings of F (along ϕ_T) to a monad $\widetilde{F}$ on $\widetilde{\mathbb{A}}_T$.

A monad distributive law $\lambda : FT \to TF$ induces a canonical monad structure on TF and TF-modules are equivalent to $\overline{T}$-modules. For more details we refer, e.g., to [4, p. 120], [32, 4.4], [19].

2.13. **Comonad distributive laws.** If G and T are comonads on $\mathbb{A}$, a natural transformation $\varphi : TG \to GT$ is called a *comonad distributive law* provided T can be lifted to a comonad $\widehat{T}$ (in 2.11), equivalently, F can be lifted along ϕ^T to a comonad $\widetilde{\widetilde{F}}$ on $\widetilde{\mathbb{A}}^T$. In this case $\varphi : TG \to GT$ induces a comonad structure on TG and TG-comodules are equivalent to $\widehat{T}$-comodules. For more details see [1], [4], [32], [22].

2.14. **Mixed distributive laws.** Let $\mathbf{F} = (F, m, i)$ be a monad and $\mathbf{T} = (T, d, e)$ a comonad on $\mathbb{A}$.

A natural transformation $\lambda : FT \to TF$ is called a *mixed distributive law*, or an *entwining*, from F to T, provided

$$T\lambda \circ \lambda T \circ Fd = dF \circ \lambda, \quad eF \circ \lambda = Fe,$$
$$Tm \circ \lambda F \circ F\lambda = \lambda \circ mT, \quad \lambda \circ iT = Ti.$$

These conditions are equivalent to the existence of a comonad lifting $\overline{T} : \mathbb{A}_F \to \mathbb{A}_F$ or also a monad lifting $\widehat{F} : \mathbb{A}^T \to \mathbb{A}^T$ (see [4, p. 133], [32, 5.3, 5.4]).

We call a natural transformation $\psi : TF \to FT$ a *mixed distributive law*, or an *entwining*, from T to F, provided (e.g. [20, 2.4])

$$\psi T \circ T\psi \circ dF = Fd \circ \psi, \quad Fe \circ \psi = eF,$$
$$mT \circ F\psi \circ \psi F = \psi \circ Tm, \quad \psi \circ Ti = iT.$$

These conditions are equivalent to the existence of a comonad lifting $\widetilde{T} : \widetilde{\mathbb{A}}_F \to \widetilde{\mathbb{A}}_F$ or a monad lifting $\widetilde{\widetilde{F}} : \widetilde{\mathbb{A}}^T \to \widetilde{\mathbb{A}}^T$ (see [25, Sec 8]).

2.15. **Mixed bimodules.** With the notation above, let $\lambda : FT \to TF$ be a mixed distributive law. *Mixed bimodules* or λ*-bimodules* are defined as those $A \in \mathrm{Obj}(\mathbb{A})$ with morphisms

$$FA \xrightarrow{h} A \xrightarrow{k} TA$$

such that (A, h) is an F-module and (A, k) is a T-comodule satisfying the pentagonal law

$$\begin{tikzcd} FA \arrow[r, "h"] \arrow[d, "Fk"'] & A \arrow[r, "k"] & TA \\ FTA \arrow[rr, "\lambda A"] & & TFA. \arrow[u, "Th"'] \end{tikzcd}$$

Morphisms between two λ-bimodules, called *bimodule morphisms*, are both F-module and T-comodule morphisms. These notions yield the category of λ-bimodules denoted by $\mathbb{A}_F^T$. It can also be considered as the category of $\overline{T}$-comodules for the comonad $\overline{T} : \mathbb{A}_F \to \mathbb{A}_F$, or the category of $\widehat{F}$-modules for the monad $\widehat{F} : \mathbb{A}^T \to \mathbb{A}^T$.

2.16. **Distributive laws for adjoint functors.** *Let (L, R) be an adjoint pair of endofunctors on a category $\mathbb{A}$ with unit η and counit ε, and F be an endofunctor on $\mathbb{A}$. Consider a natural transformation $\psi : LF \to FL$ and set*

$$\tilde{\psi} = RF\varepsilon \circ R\psi R \circ \eta FR : FR \to RF.$$

(1) *If L and F are monads, then ψ is a monad distributive law if and only if $\tilde{\psi}$ is a mixed distributive law (or entwining).*
(2) *If L is a monad and F is a comonad, then ψ is a mixed distributive law (or entwining) if and only if $\tilde{\psi}$ is a comonad distributive law.*
(3) *If L is a comonad and F is a monad, then ψ is a mixed distributive law (or entwining) if and only if $\tilde{\psi}$ is a monad distributive law.*
(4) *If L and F are comonads, then ψ is a comonad distributive law if and only if $\tilde{\psi}$ is a a mixed distributive law (or entwining).*

Proof. All these claims are easily checked by using that the structure maps of the adjoint monad-comonad (or comonad-monad) pair (L, R) are mates under adjunctions, together with naturality and the triangle identities. Details are left to the reader. $\square$

Combining the correspondences in 2.16(1) and (2) with the isomorphism of module and comodule categories in 2.6(1), further isomorphisms, between categories of mixed bimodules, can be derived.

The following was obtained in cooperation with Bachuki Mesablishvili.

2.17. **Modules and distributive laws.** *Let L be a monad with right adjoint comonad R on a category $\mathbb{A}$.*

(1) *Let G be a comonad with a mixed distributive law $\lambda : LG \to GL$. Then the category of mixed (L, G)-bimodules $\mathbb{A}_L^G$ is isomorphic to the category of GR-comodules $\mathbb{A}^{GR}$ for the composite comonad (see 2.13) defined by the associated comonad distributive law $\tilde{\lambda} : GR \to RG$ (see 2.16(2)).*
(2) *Let F be a monad with a mixed distributive law $\tau : FR \to RF$. Then the category of mixed (F, R)-bimodules $\mathbb{A}_F^R$ is isomorphic to the category of FL-modules $\mathbb{A}_{FL}$ for the composite monad defined by the monad distributive law $\tilde{\tau} : LF \to FL$ (see 2.16(1)).*

Proof. (1) The mixed distributive law $\lambda : LG \to GL$ yields a lifting of G to a comonad $\overline{G}$ on the category $\mathbb{A}_L$ of L-modules. Moreover, it determines a comonad distributive law $\tilde{\lambda} : GR \to RG$ which is equivalent to a lifting of G to a comonad $\widehat{G}$ on the category $\mathbb{A}^R$ of R-comodules. By 2.6(1), there is an isomorphism $K : \mathbb{A}_L \to \mathbb{A}^R$, and this isomorphism obviously 'intertwines' the comonads $\widehat{G}$ and $\overline{G}$. That is, $\widehat{G} = K\overline{G}K^{-1}$ *as comonads*. Thus the isomorphism $K : \mathbb{A}_L \to \mathbb{A}^R$ lifts to an isomorphism between the categories of $\overline{G}$-comodules and $\widehat{G}$-comodules. The lifted isomorphism has the object map and morphism map

$$\begin{aligned} (A, \varrho^A : A \to \overline{G}A) \quad &\mapsto (KA, K\varrho^A : KA \to K\overline{G}A = \widehat{G}KA) \\ \big((A, \varrho^A) \xrightarrow{f} (A', \varrho^{A'})\big) \quad &\mapsto \big((KA, K\varrho^A) \xrightarrow{Kf} (KA', K\varrho^{A'})\big). \end{aligned}$$

The inverse functor has the same form in terms of K^{-1}. By characterisation of $\widehat{G}$-comodules as comodules for the composite comonad GR, and characterisation of $\overline{G}$-comodules as mixed (L, G)-bimodules, we obtain the isomorphism claimed.

(2) is shown similarly to (1). $\square$

2.18. **F-actions on functors.** Let $\mathbb{A}$ and $\mathbb{B}$ be categories. Given a monad $\mathbf{F} = (F, m, i)$ on $\mathbb{A}$, composition by F on the left yields a monad on the category whose objects are the functors $\mathbb{B} \to \mathbb{A}$ and whose morphisms are natural transformations. A module (R, φ) for this monad is called a *(left) F-module functor*. Explicitly, this means a functor $R : \mathbb{B} \to \mathbb{A}$ and a natural transformation $\varphi : FR \to R$ satisfying associativity and unitality conditions (see [20, 3.1]). By 2.3, for any functor $R : \mathbb{B} \to \mathbb{A}$, (FR, mR) is an F-module functor.

2.19. **F-Galois functors.** For a monad $\mathbf{F}$ on a category $\mathbb{A}$ and any functor $R : \mathbb{B} \to \mathbb{A}$, consider the diagram

$$\begin{tikzcd} \mathbb{A}_F \arrow[r, dashed, "\overline{L}"] & \mathbb{B} \arrow[r, dashed, "\overline{R}"] \arrow[d, equal] & \mathbb{A}_F \arrow[d, "U_F"] \\ \mathbb{A} \arrow[u, dashed, "\phi_F"] \arrow[r, dashed, "L"] & \mathbb{B} \arrow[r, "R"] & \mathbb{A}. \end{tikzcd}$$

There exists some functor $\overline{R}$ making the right square commutative if and only if R has an F-module structure $\varphi : FR \to R$ (see 2.18).

If R has a left adjoint $L : \mathbb{A} \to \mathbb{B}$ with unit η, then there is a monad morphism (see [17, Proposition 3.3]),

$$\mathsf{can} : F \xrightarrow{F\eta} FRL \xrightarrow{\varphi L} RL.$$

We call an F-module functor R an *F-Galois functor* if it has a left adjoint and can is an isomorphism.

Consider an F-module functor $R : \mathbb{B} \to \mathbb{A}$ with F-action φ, a left adjoint L, unit η and counit ε of the adjunction. If $\mathbb{B}$ admits coequalisers of the parallel morphisms $L\varrho_X$ and $\varepsilon LX \circ L\varphi LX \circ LF\eta X : LFX \to LX$, for any object (X, ϱ_X) in $\mathbb{A}_F$, then this coequaliser yields the left adjoint $\overline{L}(X, \varrho_X)$ of $\overline{R}$ (see left square). By uniqueness of the adjoint, $\overline{L}\phi_F \cong L$. Denoting the coequaliser natural epimorphism $LU_F \to \overline{L}$ by p, the unit of the adjunction $(\overline{L}, \overline{R})$ is the unique natural morphism $\overline{\eta} : I_{\mathbb{A}_\mathbf{F}} \to \overline{R}\,\overline{L}$ such that $U_F\overline{\eta} = Rp \circ \eta U_F$. The counit is the unique natural morphism $\overline{\varepsilon} : \overline{L}\,\overline{R} \to I_\mathbb{B}$, such that $\overline{\varepsilon} \circ p\overline{R} = \varepsilon$.

If $\mathbb{B}$ has coequalisers of all parallel morphisms, then the following are equivalent (dual to [20, Theorem 3.15]):

(a) R is an F-Galois functor;

(b) the unit of $(\overline{L}, \overline{R})$ is an isomorphism for

- (i) all free F-modules (i.e. modules in the Kleisli category of $\mathbf{F}$), or
- (ii) all U_F-projective F-modules.

From [3], [9] and [2, Theorem 3.14] we recall a result of central importance in our setting in the form it can be found in [15, Theorem 1.7].

2.20. **Beck's theorem.** *Consider a monad F on a category $\mathbb{A}$ and an F-module functor $R : \mathbb{B} \to \mathbb{A}$. Then the induced lifting $\overline{R} : \mathbb{B} \to \mathbb{A}_F$ in 2.19 is an equivalence if and only if the following hold:*

(i) *R is an F-Galois functor,*

(ii) *R reflects isomorphisms,*

(iii) *$\mathbb{B}$ has coequalisers of R-contractible coequaliser pairs and R preserves them.*

The Galois property of a functor also transfers to its adjoint functor.

2.21. **Proposition.** *Consider an adjoint pair (F, G) of endofunctors on a category $\mathbb{A}$. Let $T : \mathbb{B} \to \mathbb{A}$ be a functor which has both a left adjoint L and a right adjoint R.*

(1) *If F is a monad (equivalently, G is a comonad), then T is an F-Galois functor as in 2.19 if and only if it is a G-Galois functor (as in* [20, Definition 3.5]*).*

(2) *If F is a comonad (equivalently, G is a monad), then R is a G-Galois functor as in 2.19 if and only if L is an F-Galois functor (as in* [20, Definition 3.5]*).*

Proof. Denote the unit of the adjunction (F, G) by η and its counit by ε. Denote furthermore the unit and counit of the adjunction (L, T) by η_L and ε_L, respectively, and for the unit and counit of the adjunction (T, R) write η_R and ε_R, respectively.

(1) A bijective correspondence between F-actions φ_T and G-coactions φ^T on T is given by $\varphi^T := G\varphi_T \circ \eta T$. The comonad morphism corresponding to φ^T comes out as

$$\widetilde{\mathsf{can}} \ : \ TR \xrightarrow{\eta TR} GFTR \xrightarrow{G\varphi_T R} GTR \xrightarrow{G\varepsilon_R} G \ .$$

Comparing it with the canonical monad morphism $\mathsf{can} : F \to TL$ in 2.19, they are easily seen to be mates under the adjunctions (F, G) and (TL, TR). That is,

$$\widetilde{\mathsf{can}} = G\varepsilon_R \circ GT\varepsilon_L R \circ G\mathsf{can}TR \circ \eta TR.$$

Thus $\widetilde{\mathsf{can}}$ is an isomorphism if and only if can is an isomorphism.

(2) A bijective correspondence between G-actions $\varphi_R : GR \to R$ and F-coactions $\varphi^L : L \to FL$ is given by

$$\varphi^L := \varepsilon_L FL \circ L\varepsilon_R TFL \circ LG\varphi_R TFL \circ LTG\eta_R FL \circ LT\eta L \circ L\eta_L.$$

The canonical comonad morphism $\widetilde{\mathsf{can}} : LT \to F$ corresponding to φ^L, and the canonical monad morphism $\mathsf{can} : G \to RT$ corresponding to φ_R, turn out to be mates under the adjunctions (LT, RT) and (F, G). That is,

$$\widetilde{\mathsf{can}} = \varepsilon_L F \circ L\varepsilon_R TF \circ LT\mathsf{can}F \circ LT\eta.$$

Thus can is a natural isomorphism if and only if $\widetilde{\mathsf{can}}$ is an isomorphism, as stated. □

3. Rings and corings in module categories

Let A be an associative ring with unit. We first study the relationship between ring extensions of A and monads on the category $\mathbb{M}_A$ of right A-modules.

3.1. ***A*-rings.** A ring B is said to be an *A-ring* provided there is a ring morphism $\iota : A \to B$. Equivalently, B is an A-bimodule with A-bilinear multiplication $\mu : B \otimes_A B \to B$ and unit $\iota : A \to B$ subject to associativity and unitality conditions.

A right *B-module* is a right A-module M with an A-linear map $\varrho_M : M \otimes_A B \to M$ satisfying the associativity and unitality conditions. B-module morphisms $f : M \to N$ are A-linear maps with $f \circ \varrho_M = \varrho_N \circ (f \otimes_A I_B)$. The category of right B-modules is denoted by $\mathbb{M}_B$. It is isomorphic to the module category over the ring B and thus is an abelian category with B as a projective generator.

As an endofunctor on $\mathbb{M}_A$, $- \otimes_A B$ is left adjoint to the endofunctor $\mathrm{Hom}_A(B, -)$.

3.2. **Monad-comonad.** *For an A-bimodule B, the following are equivalent:*

(a) *(B, μ, ι) is an A-ring;*

(b) *$- \otimes_A B : \mathbb{M}_A \to \mathbb{M}_A$ is a monad;*

(c) *$\mathrm{Hom}_A(B, -) : \mathbb{M}_A \to \mathbb{M}_A$ is a comonad.*

Proof. (a)⇔(b) is obvious and (b)⇔(c) follows by 2.6. □

Adjointness of the free and forgetful functors for the monad $- \otimes_A B$ is just the isomorphism

$$\mathrm{Hom}_B(- \otimes_A B, N) \to \mathrm{Hom}_A(-, N), \quad f \mapsto f \circ (- \otimes_A \iota).$$

We write $[B, -] = \mathrm{Hom}_A(B, -)$ for short. Comultiplication and counit of the comonad $[B, -]$ in 3.2(c) are denoted by $[\mu, -]$ and $[\iota, -]$, respectively.

The following was pointed out in [11, p. 397], see [2, Section 3.7]. It is a special case of 2.6(1).

3.3. **B-modules are $\mathrm{Hom}_A(B, -)$-comodules.** *For any A-ring B, the category of right B-modules is isomorphic to the category of $\mathrm{Hom}_A(B, -)$-comodules, that is, there exists an isomorphism*

$$\mathbb{M}_B \xrightarrow{\cong} \mathbb{M}^{[B,-]}.$$

Next we investigate the relationship between a comonad and its right adjoint monad.

3.4. **A-Corings.** An A-coring is an A-bimodule $\mathcal{C}$ with A-bilinear maps, the coproduct $\Delta : \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}$ and the counit $\varepsilon : \mathcal{C} \to A$, subject to coassociativity and counitality conditions. Similar to the characterisation of A-rings in 3.2, we derive from 2.6:

3.5. **Comonad-monad.** *For an A-bimodule $\mathcal{C}$, the following are equivalent:*

(a) $(\mathcal{C}, \Delta, \varepsilon)$ *is an A-coring;*
(b) $- \otimes_A \mathcal{C} : \mathbb{M}_A \to \mathbb{M}_A$ *induces a comonad;*
(c) $\mathrm{Hom}_A(\mathcal{C}, -) : \mathbb{M}_A \to \mathbb{M}_A$ *induces a monad.*

Writing $\mathrm{Hom}_A(\mathcal{C}, -) = [\mathcal{C}, -]$, the related monad is $([\mathcal{C}, -], [\Delta, -], [\varepsilon, -])$.

In the rest of this section $\mathcal{C}$ will be an A-coring. We first recall properties of the category of comodules (e.g. [8]).

3.6. **The category $\mathbb{M}^{\mathcal{C}}$.** *The comodules for the comonad $- \otimes_A \mathcal{C} : \mathbb{M}_A \to \mathbb{M}_A$ are called* right $\mathcal{C}$-comodules *and their category is denoted by $\mathbb{M}^{\mathcal{C}}$.*

(1) $\mathbb{M}^{\mathcal{C}}$ *is an additive category with coproducts and cokernels.*
(2) *The (co)free functor $- \otimes_A \mathcal{C}$ is right adjoint to the forgetful functor.*
(3) *For any monomorphism $f : X \to Y$ in $\mathbb{M}_A$, $f \otimes_A I_{\mathcal{C}} : X \otimes_A \mathcal{C} \to Y \otimes_A \mathcal{C}$ is a monomorphism in $\mathbb{M}^{\mathcal{C}}$.*
(4) *$\mathcal{C}$ is a flat left A-module if and only if monomorphisms in $\mathbb{M}^{\mathcal{C}}$ are injective maps.*

Left comodules of an A-coring $\mathcal{C}$ are defined symmetrically to the right comodules in 3.6, as comodules of the comonad $\mathcal{C} \otimes_A -$ on the category ${}_A\mathbb{M}$. Furthermore, if $\mathcal{C}$ is an A-coring and $\mathcal{D}$ is a B-coring, then we can consider the (composite) comonad $\mathcal{C} \otimes_A - \otimes_B \mathcal{D}$ on the category of (A, B)-bimodules. Its comodules are called $(\mathcal{C}, \mathcal{D})$-*bicomodules*. Equivalently, a $(\mathcal{C}, \mathcal{D})$-bicomodule is a left $\mathcal{C}$-comodule and a right $\mathcal{D}$-comodule such that the right $\mathcal{D}$-coaction is a left $\mathcal{C}$-comodule map. The category of $(\mathcal{C}, \mathcal{D})$-bicomodules is denoted by ${}^{\mathcal{C}}\mathbb{M}^{\mathcal{D}}$.

While $\mathcal{C}$-comodules are well studied in the literature (e.g. [8]), $[\mathcal{C}, -]$-modules have not attracted so much attention so far. They were addressed by Eilenberg-Moore in [10] and [11] as $\mathcal{C}$-*contramodules* and reconsidered recently by Positselski [24] in the context of semi-infinite cohomology.

3.7. **The category** $\mathbb{M}_{[\mathcal{C},-]}$. *The modules for the monad* $[\mathcal{C},-]:\mathbb{M}_A\to\mathbb{M}_A$ *are right* A*-modules* N *with some* A*-linear map* $[\mathcal{C},N]\to N$ *subject to associativity and unitality conditions. Their category is denoted by* $\mathbb{M}_{[\mathcal{C},-]}$.

(1) $\mathbb{M}_{[\mathcal{C},-]}$ *is an additive category with products and kernels.*
(2) *The (free) functor* $[\mathcal{C},-]:\mathbb{M}_A\to\mathbb{M}_{[\mathcal{C},-]}$ *is left adjoint to the forgetful functor.*
(3) *For any epimorphism* $h:X\to Y$ *in* $\mathbb{M}_A$, $[\mathcal{C},h]:[\mathcal{C},X]\to[\mathcal{C},Y]$ *is an epimorphism (not necessarily surjective) in* $\mathbb{M}_{[\mathcal{C},-]}$.
(4) $\mathcal{C}$ *is a projective right* A*-module if and only if epimorphisms in* $\mathbb{M}_{[\mathcal{C},-]}$ *are surjective maps.*

Proof. The proofs are similar to the comodule case. Some of the assertions can also be found in [24]. □

3.8. **Right and left contramodules.** In 3.7, modules of the monad $[\mathcal{C},-]\equiv \mathrm{Hom}_{-,A}(\mathcal{C},-)$ on the category $\mathbb{M}_A$ of right A-modules are considered. Symmetrically, an A-coring $\mathcal{C}$ determines a monad $\mathrm{Hom}_{A,-}(\mathcal{C},-)$ also on the category ${}_A\mathbb{M}$ of left A-modules. Modules for the monad $[\mathcal{C},-]\equiv \mathrm{Hom}_{-,A}(\mathcal{C},-)$ on $\mathbb{M}_A$ are called *right* $\mathcal{C}$-contramodules, and modules for the monad $\mathrm{Hom}_{A,-}(\mathcal{C},-)$ on ${}_A\mathbb{M}$ are called *left* $\mathcal{C}$-contramodules. If not specified otherwise, we mean by contramodules *right* contramodules, throughout.

Following a long-established (co)module-theoretic tradition, we often do not write explicitly the structure morphism $\alpha_M:[\mathcal{C},M]\to M$ for a contramodule (M,α_M). In the same way, the set (or abelian group) of all $[\mathcal{C},-]$-module maps $(M,\alpha_M)\to(N,\alpha_N)$ is denoted by $\mathrm{Hom}_{[\mathcal{C},-]}(M,N)$.

We saw in 3.3 that for any A-ring B, the categories $\mathbb{M}_B$ and $\mathbb{M}^{[B,-]}$ are isomorphic (see also 2.6(1)). In view of the asymmetry of assertions (1) and (2) in 2.6, the corresponding statement for corings is no longer true and we will come back to this question in 4.6. So far we know from 2.6(2):

3.9. **Related Kleisli categories.** *For any* A*-coring* $\mathcal{C}$*, the Kleisli categories of* $-\otimes_A\mathcal{C}$ *and* $[\mathcal{C},-]$ *are isomorphic by the isomorphisms for* $X,Y\in\mathbb{M}_A$,

$$\begin{aligned}\mathrm{Hom}^{\mathcal{C}}(X\otimes_A\mathcal{C},Y\otimes_A\mathcal{C}) &\cong \mathrm{Hom}_A(X\otimes_A\mathcal{C},Y)\\ &\cong \mathrm{Hom}_A(X,[\mathcal{C},Y]) \cong \mathrm{Hom}_{[\mathcal{C},-]}([\mathcal{C},X],[\mathcal{C},Y]).\end{aligned}$$

Recall that for any A-coring $\mathcal{C}$, the right dual $\mathcal{C}^*=\mathrm{Hom}_A(\mathcal{C},A)$ has a ring structure by the convolution product for $f,g\in\mathcal{C}^*$, $f*g=f\circ(g\otimes_A I_{\mathcal{C}})\circ\Delta$ (convention opposite to [8, 17.8]). Similarly a product is defined for the left dual ${}^*\mathcal{C}$.

The relation between $\mathcal{C}$-comodules and modules over the dual ring of $\mathcal{C}$ is well studied (see e.g. [8, Section 19]).

3.10. **The comonads** $-\otimes_A\mathcal{C}$ **and** $[{}^*\mathcal{C},-]$. *The comonad morphism*

$$\alpha: -\otimes_A\mathcal{C}\to \mathrm{Hom}_A({}^*\mathcal{C},-),\quad -\otimes c\mapsto [f\mapsto -f(c)],$$

yields a faithful functor $G_\alpha:\mathbb{M}^{\mathcal{C}}\to\mathbb{M}^{[{}^*\mathcal{C},-]}\cong\mathbb{M}_{{}^*\mathcal{C}}$*, and the following are equivalent:*

(a) α_N *is injective for each* $N\in\mathbb{M}_A$;
(b) G_α *is a full functor;*
(c) $\mathcal{C}$ *is a locally projective left* A*-module.*

If these conditions are satisfied, $\mathbb{M}^{\mathcal{C}}$ *is equal to* $\sigma[\mathcal{C}_{^*\mathcal{C}}]$*, the full subcategory of* $\mathbb{M}_{^*\mathcal{C}}$ *subgenerated by* $\mathcal{C}$*.*

Similar to 3.10, $\mathcal{C}$-contramodules can be related to $\mathcal{C}^*$-modules.

3.11. **The monads** $[\mathcal{C},-]$ **and** $-\otimes_A \mathcal{C}^*$. *The monad morphism*

$$\beta : -\otimes_A \mathcal{C}^* \to \mathrm{Hom}_A(\mathcal{C},-), \quad -\otimes f \mapsto [c \mapsto -f(c)],$$

yields a faithful functor $F_\beta : \mathbb{M}_{[\mathcal{C},-]} \to \mathbb{M}_{\mathcal{C}^*}$*, and the following are equivalent:*

(a) β *is surjective for all* $M \in \mathbb{M}_A$*;*

(b) F_β *is an isomorphism;*

(c) $\mathcal{C}$ *is a finitely generated and projective right* A*-module.*

In general, $\mathcal{C}$ is not a $[\mathcal{C},-]$-module and $[\mathcal{C},A]$ is not a $\mathcal{C}$-comodule. In fact, $[\mathcal{C},A] \in {}^{\mathcal{C}}\mathbb{M}$ holds provided $\mathcal{C}$ is finitely generated and projective as a right A-module.

4. Functors between co- and contramodules

Categories of comodules and contramodules have complementary features. Therefore, it is of interest to find A-corings $\mathcal{C}$ and B-corings $\mathcal{D}$ (over possibly different base rings) such that the category of $\mathcal{D}$-comodules and that of $[\mathcal{C},-]$-modules are equivalent. As we will see in 4.4, functors between these categories are provided by bicomodules. It turns out that the question, when they provide an equivalence, fits the standard problem in (categorical) descent theory.

Since comodules for the trivial B-coring B are simply B-modules, our considerations include the particular case when the category of $[\mathcal{C},-]$-modules is equivalent to the category of B-modules. Dually, when the coring $\mathcal{C}$ is trivial (i.e. equal to A), the problem reduces to a study of equivalences between A-module and $\mathcal{D}$-comodule categories. This question is already discussed in the literature, see e.g. [30], [15].

Throughout this section $\mathcal{C}$ is an A-coring and $\mathcal{D}$ a B-coring for rings A and B. The following observation was made in [24, 5.1.2].

4.1. $[\mathcal{C},-]$**-modules induced by** $\mathcal{C}$**-comodules.** *Let* N *be a* $(\mathcal{C},\mathcal{D})$*-bicomodule with left* $\mathcal{C}$*-coaction* ${}^N\varrho$*. For any* $Q \in \mathbb{M}^{\mathcal{D}}$*, there is an isomorphism*

$$\varphi : \mathrm{Hom}_A(\mathcal{C}, \mathrm{Hom}^{\mathcal{D}}(N,Q)) \to \mathrm{Hom}^{\mathcal{D}}(\mathcal{C}\otimes_A N, Q),\ h \mapsto [c\otimes m \mapsto h(c)(m)],$$

(see e.g. [8, 18.11]*). Then the right* A*-module* $N^Q := \mathrm{Hom}^{\mathcal{D}}(N,Q)$ *is a* $[\mathcal{C},-]$*-module by* α_{N^Q}*:*

$$\mathrm{Hom}_A(\mathcal{C}, \mathrm{Hom}^{\mathcal{D}}(N,Q)) \xrightarrow{\varphi} \mathrm{Hom}^{\mathcal{D}}(\mathcal{C}\otimes_A N, Q) \xrightarrow{\mathrm{Hom}^{\mathcal{D}}({}^N\varrho, Q)} \mathrm{Hom}^{\mathcal{D}}(N,Q).$$

Thus there is a bifunctor $\mathrm{Hom}^{\mathcal{D}}(-,-) : ({}^{\mathcal{C}}\mathbb{M}^{\mathcal{D}})^{op} \times \mathbb{M}^{\mathcal{D}} \to \mathbb{M}_{[\mathcal{C},-]}$*,*

$$(N,Q) \mapsto (N^Q, \alpha_{N^Q}), \quad (f,g) \mapsto \mathrm{Hom}^{\mathcal{D}}(f,g).$$

In a symmetric way, for any left $\mathcal{D}$-comodule Q and a $(\mathcal{D},\mathcal{C})$-bicomodule N (with $\mathcal{C}$-coaction $\varrho^N : N \to N\otimes_A\mathcal{C}$), $\mathrm{Hom}^{\mathcal{D}}(N,Q)$ is a left $\mathcal{C}$-contramodule by $\mathrm{Hom}^{\mathcal{D}}(\varrho^N, Q)$.

If N is just a left $\mathcal{C}$-comodule we tacitly assume $\mathcal{D} = B = \mathrm{End}^{\mathcal{C}}(N)$ to apply the preceding notions and results.

4.2. **Corollary.**

(1) *Let N be a left $\mathcal{C}$-comodule with $B = \mathrm{End}^{\mathcal{C}}(N)$. For any subring $B' \subset B$ and $Q \in \mathbb{M}_{B'}$, $\mathrm{Hom}_{B'}(N, Q)$ is a $[\mathcal{C}, -]$-module.*

(2) *For any $Q \in \mathbb{M}^{\mathcal{C}}$, $\mathrm{Hom}^{\mathcal{C}}(\mathcal{C}, Q)$ is a $[\mathcal{C}, -]$-module.*

4.3. **Contratensor product.** For any $(\mathcal{C}, \mathcal{D})$-bicomodule N, the construction in 4.1 yields a functor

$$\mathrm{Hom}^{\mathcal{D}}(N, -) : \mathbb{M}^{\mathcal{D}} \to \mathbb{M}_{[\mathcal{C},-]},$$

inducing the commutative diagram of (right adjoint) functors

$$\begin{tikzcd} \mathbb{M}^{\mathcal{D}} \arrow[r, "\mathrm{Hom}^{\mathcal{D}}(N,-)"] \arrow[d, equal] & \mathbb{M}_{[\mathcal{C},-]} \arrow[d, "U_{[\mathcal{C},-]}"] \\ \mathbb{M}^{\mathcal{D}} \arrow[r, "\mathrm{Hom}^{\mathcal{D}}(N,-)"'] & \mathbb{M}_A . \end{tikzcd}$$

Since $\mathrm{Hom}^{\mathcal{D}}(N, -) : \mathbb{M}^{\mathcal{D}} \to \mathbb{M}_A$ has the left adjoint $- \otimes_A N$ and $\mathbb{M}^{\mathcal{D}}$ has coequalisers, it follows from 2.19 that $\mathrm{Hom}^{\mathcal{D}}(N, -) : \mathbb{M}^{\mathcal{D}} \to \mathbb{M}_{[\mathcal{C},-]}$ also has a left adjoint which comes out as follows (see [24]).

For any $(\mathcal{C}, \mathcal{D})$-bicomodule $(N, {}^N\varrho, \varrho^N)$ and $[\mathcal{C}, -]$-module (M, α_M), the *contratensor product*, $M \otimes_{[\mathcal{C},-]} N$ is defined as the coequaliser

$$\mathrm{Hom}_A(\mathcal{C}, M) \otimes_A N \rightrightarrows M \otimes_A N \longrightarrow M \otimes_{[\mathcal{C},-]} N,$$

where the coequalised maps are $f \otimes n \mapsto (f \otimes_A I_N) \circ {}^N\varrho(n)$ and $\alpha_M \otimes_A I_N$. Projection of an element $m \otimes n$ to $M \otimes_{[\mathcal{C},-]} N$ is denoted by $m \otimes_{[\mathcal{C},-]} n$.

As a coequaliser of right $\mathcal{D}$-comodule maps, $M \otimes_{[\mathcal{C},-]} N$ is a right $\mathcal{D}$-comodule, and thus defines a functor $- \otimes_{[\mathcal{C},-]} N : \mathbb{M}_{[\mathcal{C},-]} \to \mathbb{M}^{\mathcal{D}}$. Note that this coequaliser splits in $\mathbb{M}^{\mathcal{D}}$ provided (M, α_M) is $U_{[\mathcal{C},-]}$-projective.

4.4. **Functors between comodules and contramodules.** *Any $(\mathcal{C}, \mathcal{D})$-bicomodule N induces an adjoint pair of functors*

$$- \otimes_{[\mathcal{C},-]} N : \mathbb{M}_{[\mathcal{C},-]} \to \mathbb{M}^{\mathcal{D}}, \quad \mathrm{Hom}^{\mathcal{D}}(N, -) : \mathbb{M}^{\mathcal{D}} \to \mathbb{M}_{[\mathcal{C},-]},$$

that is, for $M \in \mathbb{M}_{[\mathcal{C},-]}$ and $P \in \mathbb{M}^{\mathcal{D}}$, there is an isomorphism

$$\mathrm{Hom}^{\mathcal{D}}(M \otimes_{[\mathcal{C},-]} N, P) \cong \mathrm{Hom}_{[\mathcal{C},-]}(M, \mathrm{Hom}^{\mathcal{D}}(N, P)).$$

Conversely, any right adjoint functor $F : \mathbb{M}^{\mathcal{D}} \to \mathbb{M}_{[\mathcal{C},-]}$ is naturally isomorphic to $\mathrm{Hom}^{\mathcal{D}}(N, -)$, for an appropriate $(\mathcal{C}, \mathcal{D})$-bicomodule N.

Proof. In view of the discussion in 2.19, for $[\mathcal{C}, -]$-modules M, the unit of the adjunction is given by

$$\eta_M : M \to \mathrm{Hom}^{\mathcal{D}}(N, M \otimes_{[\mathcal{C},-]} N), \; m \mapsto [n \mapsto m \otimes_{[\mathcal{C},-]} n].$$

Also by 2.19, the counit of the adjunction comes out (and is in particular well defined) as

$$\varepsilon_Q : \mathrm{Hom}^{\mathcal{D}}(N, Q) \otimes_{[\mathcal{C},-]} N \to Q, \quad f \otimes_{[\mathcal{C},-]} n \mapsto f(n),$$

for all right $\mathcal{D}$-comodules Q.

Conversely, assume that $F : \mathbb{M}^{\mathcal{D}} \to \mathbb{M}_{[\mathcal{C},-]}$ has a left adjoint. Then so does the composite $F' := U_{[\mathcal{C},-]} \circ F : \mathbb{M}^{\mathcal{D}} \to \mathbb{M}_A$, in light of 3.7. Hence it follows by [30, Theorem 3.2] that there exists an $(A, \mathcal{D})$-bicomodule N such that F' is naturally isomorphic to $\mathrm{Hom}^{\mathcal{D}}(N,-)$. Moreover, by construction, for any $Q \in \mathbb{M}^{\mathcal{D}}$, $\mathrm{Hom}^{\mathcal{D}}(N,Q)$ is a $[\mathcal{C},-]$-module via some action $\kappa_Q : \mathrm{Hom}_A(\mathcal{C}, \mathrm{Hom}^{\mathcal{D}}(N,Q)) \to \mathrm{Hom}^{\mathcal{D}}(N,Q)$, and for $q \in \mathrm{Hom}^{\mathcal{D}}(Q,Q')$, $\mathrm{Hom}^{\mathcal{D}}(N,q)$ is a morphism of $[\mathcal{C},-]$-modules. This amounts to saying that $\kappa_{(-)}$ is a natural transformation $\mathrm{Hom}^{\mathcal{D}}(\mathcal{C} \otimes_A N, -) \to \mathrm{Hom}^{\mathcal{D}}(N,-)$. Therefore, it follows by the Yoneda Lemma that there is an $(A,\mathcal{D})$-bicomodule map $\tau : N \to \mathcal{C} \otimes_A N$, such that $\kappa_Q = \mathrm{Hom}^{\mathcal{D}}(\tau, Q)$, for $Q \in \mathbb{M}^{\mathcal{D}}$. Unitality and associativity of the action κ_Q, for any $Q \in \mathbb{M}^{\mathcal{D}}$, imply counitality and coassociativity of the left $\mathcal{C}$-coaction τ, respectively. □

Consider a $(\mathcal{C},\mathcal{D})$-bicomodule N, over an A-coring $\mathcal{C}$ and a B-coring $\mathcal{D}$. A $[\mathcal{C},-]$-module map $g : (L,\alpha_L) \to (M,\alpha_M)$ is said to be ${}_{[\mathcal{C},-]}N$*-pure* provided the sequence

$$0 \longrightarrow \ker g \otimes_{[\mathcal{C},-]} N \longrightarrow L \otimes_{[\mathcal{C},-]} N \xrightarrow{g \otimes_{[\mathcal{C},-]} I_N} M \otimes_{[\mathcal{C},-]} N\ ,$$

is exact (in $\mathbb{M}_B$).

4.5**. Some tensor relations.** *Let $(N, {}^N\varrho)$ be a left $\mathcal{C}$-comodule. Then:*

(1) *For any right A-module X, $\mathrm{Hom}_A(\mathcal{C}, X) \otimes_{[\mathcal{C},-]} N \cong X \otimes_A N$.*

(2) *If (M, ϱ^M) is right $\mathcal{C}$-comodule for which the map*

$$\gamma : \mathrm{Hom}_A(\mathcal{C}, M) \to \mathrm{Hom}_A(\mathcal{C}, M \otimes_A \mathcal{C}),\ f \mapsto \varrho^M \circ f - (f \otimes_A I_{\mathcal{C}}) \circ \Delta,$$

is ${}_{[\mathcal{C},-]}N$-pure, then $\mathrm{Hom}^{\mathcal{C}}(\mathcal{C}, M) \otimes_{[\mathcal{C},-]} N$ is isomorphic to the cotensor product $M \otimes^{\mathcal{C}} N$.

Proof. (1) This is outlined in [24, 5.1.1].

(2) Consider the commutative diagram in $\mathbb{M}_B$ (for $B = \mathrm{End}^{\mathcal{C}}(N)$).

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{Hom}^{\mathcal{C}}(\mathcal{C}, M) \otimes_{[\mathcal{C},-]} N & \longrightarrow & \mathrm{Hom}_A(\mathcal{C}, M) \otimes_{[\mathcal{C},-]} N & \xrightarrow{\gamma \otimes_{[\mathcal{C},-]} I_N} & \mathrm{Hom}_A(\mathcal{C}, M \otimes_A \mathcal{C}) \otimes_{[\mathcal{C},-]} N \\
& \downarrow{\scriptstyle \vartheta} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
0 \longrightarrow & M \otimes^{\mathcal{C}} N & \longrightarrow & M \otimes_A N & \longrightarrow & M \otimes_A \mathcal{C} \otimes_A N,
\end{array}$$

Since γ is ${}_{[\mathcal{C},-]}N$-pure, and $\mathrm{Hom}^{\mathcal{C}}(\mathcal{C}, M) = \ker \gamma$, the top row is exact. The bottom row is the defining exact sequence of the cotensor product (see e.g. [8, 21.1]). The vertical isomorphisms are obtained from part (1). Thus there is an isomorphism $\vartheta : \mathrm{Hom}^{\mathcal{C}}(\mathcal{C}, M \otimes_{[\mathcal{C},-]} N) \to M \otimes^{\mathcal{C}} N$ extending the diagram commutatively. □

From previous considerations we obtain the following result by Positselski.

4.6**. Correspondence of categories.**

(1) *For any A-coring $\mathcal{C}$, there is an adjoint pair of functors*

$$- \otimes_{[\mathcal{C},-]} \mathcal{C} : \mathbb{M}_{[\mathcal{C},-]} \to \mathbb{M}^{\mathcal{C}}, \quad \mathrm{Hom}^{\mathcal{C}}(\mathcal{C}, -) : \mathbb{M}^{\mathcal{C}} \to \mathbb{M}_{[\mathcal{C},-]}.$$

(2) *On the classes of free objects, the functors in (1) restrict to the following maps. For any $X \in \mathbb{M}_A$,*

$$X \otimes_A \mathcal{C} \mapsto \mathrm{Hom}^{\mathcal{C}}(\mathcal{C}, X \otimes_A \mathcal{C}) \cong \mathrm{Hom}_A(\mathcal{C}, X),$$
$$\mathrm{Hom}_A(\mathcal{C}, X) \mapsto \mathrm{Hom}_A(\mathcal{C}, X) \otimes_{[\mathcal{C},-]} \mathcal{C} \cong X \otimes_A \mathcal{C}.$$

Thus the functors in part (1) restrict to inverse isomorphisms between the Kleisli subcategories *of* $\mathbb{M}^{\mathcal{C}}$ *and* $\mathbb{M}_{[\mathcal{C},-]}$.

(3) *There is an equivalence*

$$\mathrm{Hom}^{\mathcal{C}}(\mathcal{C}, -) : \mathbb{M}^{\mathcal{C}}_{inj} \to \mathbb{M}^{proj}_{[\mathcal{C},-]},$$

where $\mathbb{M}^{\mathcal{C}}_{inj}$ *denotes the full subcategory of* $\mathbb{M}^{\mathcal{C}}$ *of objects relative injective to the forgetful functor* $\mathbb{M}^{\mathcal{C}} \to \mathbb{M}_A$*, and* $\mathbb{M}^{proj}_{[\mathcal{C},-]}$ *the full subcategory of* $\mathbb{M}_{[\mathcal{C},-]}$ *of objects relative projective to the forgetful functor* $\mathbb{M}_{[\mathcal{C},-]} \to \mathbb{M}_A$.

Proof. This is shown in [24, Theorem in 5.3]. Here (1) follows by putting $\mathcal{D} = \mathcal{C}$ in 4.4 and considering $\mathcal{C}$ as a $(\mathcal{C}, \mathcal{C})$-bicomodule, see 4.2. Claim (2) (cf. 3.9) is obtained by applying 2.6(2) to the adjoint comonad-monad functor pair $(-\otimes_A \mathcal{C}, \mathrm{Hom}_A(\mathcal{C}, -))$. Part (3) follows from 2.8. Note that the equaliser in the more general situation of 2.8 yields here the equivalence functor $E(M, \varrho^M) = \mathrm{Hom}^{\mathcal{C}}(\mathcal{C}, M)$, for any $(M, \varrho^M) \in \mathbb{M}^{\mathcal{C}}_{inj}$, as stated. □

Recall that an A-coring $\mathcal{C}$ is said to be a *coseparable coring* if its coproduct is a split monomorphism of $\mathcal{C}$-bicomodules. Equivalently, there is an A-bimodule map $\delta : \mathcal{C} \otimes_A \mathcal{C} \to A$ such that $\delta \circ \Delta = \varepsilon$ and

$$(I_{\mathcal{C}} \otimes_A \delta) \circ (\Delta \otimes_A I_{\mathcal{C}}) = (\delta \otimes_A I_{\mathcal{C}}) \circ (I_{\mathcal{C}} \otimes_A \Delta).$$

Such a map δ is called a *cointegral* (e.g. [8, 26.1]). Equivalently, coseparable corings can be described by separable functors as follows.

4.7. **Coseparable corings.** *For* $\mathcal{C}$ *the following are equivalent.*

(a) $\mathcal{C}$ *is a coseparable coring;*

(b) *the forgetful functor* $U^{\mathcal{C}} : \mathbb{M}^{\mathcal{C}} \to \mathbb{M}_A$ *is separable;*

(c) *the forgetful functor* $U_{[\mathcal{C},-]} : \mathbb{M}_{[\mathcal{C},-]} \to \mathbb{M}_A$ *is separable.*

If these assertions hold then, in particular, any $\mathcal{C}$*-comodule is* $U^{\mathcal{C}}$*-injective and any* $[\mathcal{C}, -]$*-module is* $U_{[\mathcal{C},-]}$*-projective.*

Proof. Equivalence (a)⇔(b) is quoted from [8, 26.1]. It can be derived alternatively from 2.9(2). Equivalence (b)⇔(c) and the final claims follow by 2.10(2). □

Combining 4.7 with 4.6 we obtain:

4.8. **Comodules and contramodules of coseparable corings.** *For a coseparable coring* $\mathcal{C}$*, the category* $\mathbb{M}^{\mathcal{C}}_{inj}$ *coincides with* $\mathbb{M}^{\mathcal{C}}$ *and* $\mathbb{M}^{proj}_{[\mathcal{C},-]}$ *is equal to* $\mathbb{M}_{[\mathcal{C},-]}$*. Thus there is an equivalence*

$$\mathrm{Hom}^{\mathcal{C}}(\mathcal{C}, -) : \mathbb{M}^{\mathcal{C}} \to \mathbb{M}_{[\mathcal{C},-]}.$$

This equivalence between comodules and contramodules for coseparable corings plays an important role in the characterisation of categories of Hopf (contra)modules in 7.6.

5. Galois bicomodules

In this section we analyse when a comodule category is equivalent to a contramodule category. Any such equivalence is necessarily given by functors associated to a bicomodule. The latter must possess additional properties.

5.1. **$[\mathcal{C},-]$-Galois bicomodules.** *For a $(\mathcal{C},\mathcal{D})$-bicomodule $(N, {}^N\varrho, \varrho^N)$, the commutative diagram in 4.3 yields a canonical monad morphism (by 2.19)*

$$\mathsf{can}^N : \mathrm{Hom}_A(\mathcal{C},-) \to \mathrm{Hom}^{\mathcal{D}}(N, -\otimes_A N), \quad f \mapsto (f \otimes_A I_N) \circ {}^N\varrho.$$

Let η denote the unit of the adjunction $(-\otimes_{[\mathcal{C},-]} N, \mathrm{Hom}^{\mathcal{D}}(N,-))$ in 4.4.

The following statements are equivalent:

(a) *The natural transformation can^N is an isomorphism;*

(b) *$\eta_{\mathrm{Hom}_A(\mathcal{C},Q)}$ is an isomorphism, for all $Q \in \mathbb{M}_A$;*

(c) *η_M is an isomorphism, for all $U_{[\mathcal{C},-]}$-projective $M \in \mathbb{M}_{[\mathcal{C},-]}$.*

If these conditions hold, then $\mathrm{Hom}^{\mathcal{D}}(N,-) : \mathbb{M}^{\mathcal{D}} \to \mathbb{M}_A$ is a $[\mathcal{C},-]$-Galois functor and we call $N \in {}^{\mathcal{C}}\mathbb{M}^{\mathcal{D}}$ a *$[\mathcal{C},-]$-Galois bicomodule.*

If N is just a left $\mathcal{C}$-comodule we tacitly take $\mathcal{D} = B = \mathrm{End}^{\mathcal{C}}(N)$ and call N a *$[\mathcal{C},-]$-Galois left comodule.*

Symmetrically to the above considerations, any right adjoint functor from the category of left comodules of a B-coring $\mathcal{D}$ to the category of left contramodules of an A-coring $\mathcal{C}$ is naturally isomorphic to $\mathrm{Hom}^{\mathcal{D}}(N,-)$, for some $(\mathcal{D},\mathcal{C})$-bicomodule N. In analogy with 5.1, also *$\mathrm{Hom}_{A,-}(\mathcal{C},-)$-Galois $(\mathcal{D},\mathcal{C})$-bicomodules* and in particular *$\mathrm{Hom}_{A,-}(\mathcal{C},-)$-Galois right $\mathcal{C}$-comodules* can be defined.

Studying $[\mathcal{C},-]$-Galois bicomodules we are on the one side interested in there own structural properties and on the other side also in conditions which make the related functors fully faithful.

5.2. **$-\otimes_{[\mathcal{C},-]} N$ fully faithful.** *Let N be a $(\mathcal{C},\mathcal{D})$-bicomodule. Then the functor $-\otimes_{[\mathcal{C},-]} N : \mathbb{M}_{[\mathcal{C},-]} \to \mathbb{M}^{\mathcal{D}}$ is fully faithful if and only if*

(i) *N is a $[\mathcal{C},-]$-Galois bicomodule and*

(ii) *for any $[\mathcal{C},-]$-module M, the functor $\mathrm{Hom}^{\mathcal{D}}(N,-) : \mathbb{M}^{\mathcal{D}} \to \mathbb{M}_A$ preserves the coequaliser*

$$\mathrm{Hom}_A(\mathcal{C},M) \otimes_A N \rightrightarrows M \otimes_A N \longrightarrow M \otimes_{[\mathcal{C},-]} N\ ,$$

defining the contratensor product (cf. 4.3).

Proof. Since in $\mathbb{M}^{\mathcal{D}}$ any parallel pair of morphisms has a coequaliser, the claim follows by (the dual version of) [15, Theorem 2.6]. □

5.3. **Corollary.** *Let $N \in {}^{\mathcal{C}}\mathbb{M}^{\mathcal{D}}$ be a $[\mathcal{C},-]$-Galois bicomodule. If the functor $\mathrm{Hom}^{\mathcal{D}}(N,-) : \mathbb{M}^{\mathcal{D}} \to \mathbb{M}_A$ preserves coequalisers, then $-\otimes_{[\mathcal{C},-]} N : \mathbb{M}_{[\mathcal{C},-]} \to \mathbb{M}^{\mathcal{D}}$ is fully faithful and $\mathcal{C}$ is a projective right A-module.*

Proof. It follows immediately by 5.2 that $-\otimes_{[\mathcal{C},-]} N : \mathbb{M}_{[\mathcal{C},-]} \to \mathbb{M}^{\mathcal{D}}$ is fully faithful. The left adjoint functor $-\otimes_A N$ always preserves cokernels and $\mathrm{Hom}^{\mathcal{D}}(N,-)$ does so by hypothesis. Thus their composite $\mathrm{Hom}^{\mathcal{D}}(N, -\otimes_A N) : \mathbb{M}_A \to \mathbb{M}_A$ preserves cokernels, i.e. epimorphisms. Since can^N in 5.1 is assumed to be an isomorphism, we conclude that also the functor $\mathrm{Hom}_A(\mathcal{C},-)$ preserves epimorphisms, i.e. $\mathcal{C}$ is projective as a right A-module. □

For our investigation it is of interest to extend the notion of Galois comodules from [31, 4.1] to bicomodules.

5.4. **$\mathcal{D}$-Galois bicomodules.** For any $(\mathcal{C},\mathcal{D})$-bicomodule N, the left adjoint functor $-\otimes_{[\mathcal{C},-]} N : \mathbb{M}_{[\mathcal{C},-]} \to \mathbb{M}_B$ is a left $-\otimes_B \mathcal{D}$-comodule functor (in the sense of [20, 3.3]) by the coaction

$$-\otimes_{[\mathcal{C},-]} \varrho^N : -\otimes_{[\mathcal{C},-]} N \to -\otimes_{[\mathcal{C},-]} N \otimes_B \mathcal{D}.$$

We call N a *$\mathcal{D}$-Galois bicomodule* if $-\otimes_{[\mathcal{C},-]} N : \mathbb{M}_{[\mathcal{C},-]} \to \mathbb{M}_B$ is a $-\otimes_B \mathcal{D}$-Galois functor (in the sense of [20, 3.3]), that is, if the comonad morphism

$$\mathrm{Hom}_B(N,-)\otimes_{[\mathcal{C},-]} N \xrightarrow{I_{\mathrm{Hom}_B(N,-)}\otimes_{[\mathcal{C},-]}\varrho^N} \mathrm{Hom}_B(N,-)\otimes_{[\mathcal{C},-]} N \otimes_B \mathcal{D} \xrightarrow{\varepsilon\otimes_B I_{\mathcal{D}}} -\otimes_B \mathcal{D}$$

is an isomorphism.

For a right $\mathcal{D}$-comodule N, one can put $\mathcal{C} = A = \mathrm{End}^{\mathcal{D}}(N)$. In this way we re-obtain the usual notion of a $\mathcal{D}$-Galois right comodule in [31, 4.1].

The $\mathcal{D}$-Galois property of a $(\mathcal{D},\mathcal{C})$-bicomodule is defined symmetrically by the Galois property of the induced functor between the category of left B-modules and the category of left $\mathcal{C}$-contramodules. In the particular case of a left $\mathcal{D}$-comodule N, it reduces to the usual notion of a $\mathcal{D}$-Galois left comodule in [8] by putting $A = \mathcal{C} = \mathrm{End}^{\mathcal{D}}(N)$.

5.5. $\mathrm{Hom}^{\mathcal{D}}(N,-)$ **fully faithful.** *Let N be a $(\mathcal{C},\mathcal{D})$-bicomodule. Then the functor $\mathrm{Hom}^{\mathcal{D}}(N,-) : \mathbb{M}^{\mathcal{D}} \to M_{[\mathcal{C},-]}$ is fully faithful if and only if*

(i) *N is a $\mathcal{D}$-Galois bicomodule and*

(ii) *the functor $-\otimes_{[\mathcal{C},-]} N : \mathbb{M}_{[\mathcal{C},-]} \to \mathbb{M}_B$ preserves the equaliser*

$$\mathrm{Hom}^{\mathcal{D}}(N,Q) \longrightarrow \mathrm{Hom}_B(N,Q) \underset{\omega}{\overset{\mathrm{Hom}_B(N,\varrho^Q)}{\rightrightarrows}} \mathrm{Hom}_B(N, Q\otimes_B \mathcal{D}),$$

for any right $\mathcal{D}$-comodule (Q,ϱ^Q), where $\omega(f) = (f\otimes_B I_{\mathcal{D}})\circ \varrho^N$.

Proof. This follows again by [15, Theorem 2.6]. □

5.6. **Corollary.** *Let $N \in {}^{\mathcal{C}}\mathbb{M}^{\mathcal{D}}$ be a $\mathcal{D}$-Galois bicomodule. If the functor $-\otimes_{[\mathcal{C},-]} N : \mathbb{M}_{[\mathcal{C},-]} \to \mathbb{M}_B$ preserves equalisers, then $\mathrm{Hom}^{\mathcal{D}}(N,-) : \mathbb{M}^{\mathcal{D}} \to \mathbb{M}_{[\mathcal{C},-]}$ is fully faithful and $\mathcal{D}$ is a flat left B-module.*

Proof. (Compare with [31, 4.8]). The first assertion follows immediately from 5.5. In particular, this means that N is a generator in $\mathbb{M}^{\mathcal{D}}$. Moreover, there is a natural isomorphism $-\otimes_B \mathcal{D} \cong \mathrm{Hom}_B(N,-)\otimes_{[\mathcal{C},-]} N$, where the right adjoint functor $\mathrm{Hom}_B(N,-)$ always preserves kernels and by assumption so does $-\otimes_{[\mathcal{C},-]} N$. This implies that $-\otimes_B \mathcal{D} : \mathbb{M}_B \to \mathbb{M}_B$ preserves kernels, i.e. monomorphisms, hence $\mathcal{D}$ is flat as a left B-module. □

Recall from [8, 19.19] that, for a left $\mathcal{C}$-comodule $(N, {}^N\varrho)$ which is finitely generated and projective as a left A-module, the left dual ${}^*N = \mathrm{Hom}_{A,-}(N,A)$ carries a canonical right $\mathcal{C}$-comodule structure, via

$${}^*N \longrightarrow \mathrm{Hom}_{A,-}(N,\mathcal{C}) \cong {}^*N \otimes_A \mathcal{C}, \quad g \mapsto (I_{\mathcal{C}}\otimes_A g)\circ {}^N\varrho.$$

In what follows, $[\mathcal{C},-]$-Galois and $\mathcal{C}$-Galois properties of a finitely generated projective comodule are compared.

5.7. **$[\mathcal{C},-]$-Galois comodules and $\mathcal{C}$-Galois comodules.** *Let N be a left $\mathcal{C}$-comodule finitely generated and projective as a left A-module. The following assertions are equivalent.*

(a) *N is a $\mathrm{Hom}_{-,A}(\mathcal{C},-)$-Galois left comodule;*
(b) *N is a $\mathcal{C}$-Galois left comodule;*
(c) *${}^*N$ is a $\mathrm{Hom}_{A,-}(\mathcal{C},-)$-Galois right comodule;*
(d) *${}^*N$ is a $\mathcal{C}$-Galois right comodule.*

Proof. (b)⇔(d) is proven in [7, p. 514].

(a)⇔(d). Put $B = \mathrm{End}^{\mathcal{C}}(N)$ and consider the (A,B)-bimodule N and the (B,A)-bimodule ${}^*N$. The stated equivalence follows by applying 2.21(2) to the adjoint comonad-monad pair $(-\otimes_A\mathcal{C}, \mathrm{Hom}_A(\mathcal{C},-))$ and the functor $-\otimes_A N \cong \mathrm{Hom}_A({}^*N,-) : \mathbb{M}_A \to \mathbb{M}_B$, possessing the right adjoint $\mathrm{Hom}_B(N,-)$ and the left adjoint $-\otimes_B {}^*N$.

(b)⇔(c) is proven similarly to (a)⇔(d). □

Sufficient and necessary conditions for the equivalence between a comodule and a contramodule category are obtained by applying Beck's theorem; see 2.20.

5.8. **Equivalences.** *For an A-coring $\mathcal{C}$ and a B-coring $\mathcal{D}$, the following assertions are equivalent.*

(a) *The categories $\mathbb{M}_{[\mathcal{C},-]}$ and $\mathbb{M}^{\mathcal{D}}$ are equivalent;*
(b) *there exists a $(\mathcal{C},\mathcal{D})$-bicomodule N with the properties:*
   (i) *N is a $[\mathcal{C},-]$-Galois bicomodule,*
   (ii) *the functor $\mathrm{Hom}^{\mathcal{D}}(N,-) : \mathbb{M}^{\mathcal{D}} \to \mathbb{M}_A$ reflects isomorphisms,*
   (iii) *the functor $\mathrm{Hom}^{\mathcal{D}}(N,-) : \mathbb{M}^{\mathcal{D}} \to \mathbb{M}_A$ preserves $\mathrm{Hom}^{\mathcal{D}}(N,-)$-contractible coequalisers.*
(c) *there exists a $(\mathcal{C},\mathcal{D})$-bicomodule N with the properties:*
   (i) *N is a $\mathcal{D}$-Galois bicomodule,*
   (ii) *the functor $-\otimes_{[\mathcal{C},-]} N : \mathbb{M}_{[\mathcal{C},-]} \to \mathbb{M}_B$ reflects isomorphisms,*
   (iii) *the functor $-\otimes_{[\mathcal{C},-]} N : \mathbb{M}_{[\mathcal{C},-]} \to \mathbb{M}_B$ preserves $-\otimes_{[\mathcal{C},-]} N$-contractible equalisers.*

Proof. (a)⇔(b). By 4.4, any equivalence functor $\mathbb{M}^{\mathcal{D}} \to \mathbb{M}_{[\mathcal{C},-]}$ is naturally isomorphic to $\mathrm{Hom}^{\mathcal{D}}(N,-)$, for some $(\mathcal{C},\mathcal{D})$-bicomodule N. By Beck's theorem 2.20, $\mathrm{Hom}^{\mathcal{D}}(N,-) : \mathbb{M}^{\mathcal{D}} \to \mathbb{M}_{[\mathcal{C},-]}$ is an equivalence if and only if the conditions in part (b) hold.

(a)⇔(c) is shown with similar arguments. □

5.9. **Equivalence for abelian categories.** *For a $(\mathcal{C},\mathcal{D})$-bicomodule N, the following are equivalent.*

(a) *$\mathrm{Hom}^{\mathcal{D}}(N,-) : \mathbb{M}^{\mathcal{D}} \to \mathbb{M}_{[\mathcal{C},-]}$ is an equivalence, $\mathcal{C}$ is a projective right A-module and $\mathcal{D}$ is a flat left B-module;*
(b) *$\mathcal{D}$ is flat as a left B-module and N is a $[\mathcal{C},-]$-Galois bicomodule and a projective generator in $\mathbb{M}^{\mathcal{D}}$;*
(c) *$\mathcal{C}$ is projective as a right A-module and N is a $\mathcal{D}$-Galois bicomodule and the functor $-\otimes_{[\mathcal{C},-]} N : \mathbb{M}_{[\mathcal{C},-]} \to \mathbb{M}_B$ is left exact and faithful.*

Proof. (a)⇒(b). By Theorem 5.8, N is a $[\mathcal{C},-]$-Galois bicomodule. Being an equivalence, $\mathrm{Hom}^{\mathcal{D}}(N,-):\mathbb{M}^{\mathcal{D}}\to\mathbb{M}_{[\mathcal{C},-]}$ is faithful. Since the forgetful functor from $\mathbb{M}_{[\mathcal{C},-]}$ to $\mathbb{M}_A$ (or to $\mathbb{M}_{\mathbb{Z}}$ or Set) is faithful, so is the composite $\mathrm{Hom}^{\mathcal{D}}(N,-):\mathbb{M}^{\mathcal{D}}\to\mathrm{Set}$. This proves that N is a generator in $\mathbb{M}^{\mathcal{D}}$. Finally, $U_{[\mathcal{C},-]}:\mathbb{M}_{[\mathcal{C},-]}\to\mathbb{M}_A$ is right exact by 3.7(4). Since $\mathrm{Hom}^{\mathcal{D}}(N,-):\mathbb{M}^{\mathcal{D}}\to\mathbb{M}_{[\mathcal{C},-]}$ is an equivalence, this implies that also $\mathrm{Hom}^{\mathcal{D}}(N,-):\mathbb{M}^{\mathcal{D}}\to\mathbb{M}_A$ is right exact, by commutativity of the diagram in 4.3. By flatness of $\mathcal{D}$ as a left B-module, this implies projectivity of N (cf. [8, 18.20]).

(b)⇒(a). By the hypothesis, the functor $\mathrm{Hom}^{\mathcal{D}}(N,-):\mathbb{M}^{\mathcal{D}}\to\mathbb{M}_A$ preserves coequalisers and reflects isomorphisms. Thus $\mathrm{Hom}^{\mathcal{D}}(N,-):\mathbb{M}^{\mathcal{D}}\to\mathbb{M}_{[\mathcal{C},-]}$ is an equivalence by Theorem 5.8 and $\mathcal{C}$ is a projective right A-module by Corollary 5.3.

(a)⇒(c). If $\mathrm{Hom}^{\mathcal{D}}(N,-):\mathbb{M}^{\mathcal{D}}\to\mathbb{M}_{[\mathcal{C},-]}$ is an equivalence, then so is its left adjoint $-\otimes_{[\mathcal{C},-]}N$. Thus N is a $\mathcal{D}$-Galois bicomodule by 5.8. The functor $-\otimes_{[\mathcal{C},-]}N:\mathbb{M}_{[\mathcal{C},-]}\to\mathbb{M}_B$ is equal to the composite of the equivalence $-\otimes_{[\mathcal{C},-]}N:\mathbb{M}_{[\mathcal{C},-]}\to\mathbb{M}^{\mathcal{D}}$ and the forgetful functor $\mathbb{M}^{\mathcal{D}}\to\mathbb{M}_B$. The forgetful functor is faithful and also left exact by the flatness of the left B-module $\mathcal{D}$. Thus the functor $-\otimes_{[\mathcal{C},-]}N:\mathbb{M}_{[\mathcal{C},-]}\to\mathbb{M}_B$ is also faithful and left exact.

(c)⇒(a). Since $\mathcal{C}$ is a projective right A-module, $\mathbb{M}_{[\mathcal{C},-]}$ is abelian. Hence faithfulness of $-\otimes_{[\mathcal{C},-]}N:\mathbb{M}_{[\mathcal{C},-]}\to\mathbb{M}_B$ implies that it reflects isomorphisms. Since it also preserves equalisers by assumption, it follows by Theorem 5.8 that $-\otimes_{[\mathcal{C},-]}N:\mathbb{M}_{[\mathcal{C},-]}\to\mathbb{M}^{\mathcal{D}}$ is an equivalence, with inverse $\mathrm{Hom}^{\mathcal{D}}(N,-)$. The left B-module $\mathcal{D}$ is flat by Corollary 5.6. □

In the rest of the section we study the particular case of a trivial B-coring $\mathcal{D}=B$. That is, the situation when the category of contramodules of a coring $\mathcal{C}$ is equivalent to that of modules over a ring B.

5.10. **Lemma.** [12, Proposition 2.5] *Let N be an (A,B)-bimodule which is finitely generated and projective as an A-module. Consider the comatrix coring $\mathcal{C}:=N\otimes_B {}^*N$ and denote by T the ring of endomorphisms of N as a left $\mathcal{C}$-comodule. Then $N\cong N\otimes_B T$ via the right T-action on N.*

The next result may be seen as a counterpart to the Galois comodule structure theorem [8, 18.27], [30, Corollary 3.7].

5.11. **Theorem.** *Let $N\in{}^{\mathcal{C}}\mathbb{M}$ be a $[\mathcal{C},-]$-Galois comodule over an A-coring $\mathcal{C}$, put $T=\mathrm{End}^{\mathcal{C}}(N)$ and assume T to be a B-ring for some ring B. Assume that N is a projective generator of right B-modules. Then the following hold.*

(1) $-\otimes_{[\mathcal{C},-]}N:\mathbb{M}_{[\mathcal{C},-]}\to\mathbb{M}_B$ *is an equivalence.*
(2) $\mathcal{C}$ *is a projective right A-module.*
(3) *N is a finitely generated and projective left A-module.*
(4) *$\mathcal{C}$ is isomorphic to the comatrix A-coring $N\otimes_B {}^*N$.*
(5) *B is isomorphic to T.*
(6) *If, in addition, $\mathcal{C}$ is a generator of right A-modules, then N is a faithfully flat left A-module.*

Proof. Assertions (1) and (2) are immediate by 5.9.

(3) Since $-\otimes_{[\mathcal{C},-]}N$ is an equivalence, it has a left adjoint $\mathrm{Hom}_B(N,-):\mathbb{M}_B\to\mathbb{M}_{[\mathcal{C},-]}$. The free functor $\mathrm{Hom}_A(\mathcal{C},-)$ has a left adjoint $-\otimes_{[\mathcal{C},-]}\mathcal{C}:\mathbb{M}_{[\mathcal{C},-]}\to\mathbb{M}_A$ by 4.3.

Hence also the composite functor, that is naturally isomorphic to $-\otimes_A N : \mathbb{M}_A \to \mathbb{M}_B$ by 4.5(1), has a left adjoint. This proves that N is a finitely generated and projective left A-module.

(4) By part (3),

$$\mathrm{Hom}_B(N, -\otimes_A N) \cong \mathrm{Hom}_B(N, \mathrm{Hom}_A({}^*N, -)) \cong \mathrm{Hom}_A(N \otimes_B {}^*N, -).$$

Composing this natural isomorphism with the canonical monad morphism can^N (at $\mathcal{D} = B$), it yields a monad isomorphism $\mathrm{Hom}_A(\mathcal{C}, -) \cong \mathrm{Hom}_A(N \otimes_B {}^*N, -)$. By Yoneda's Lemma this proves $\mathcal{C} \cong N \otimes_B {}^*N$.

(5) The composite of the forgetful functor $\mathrm{Hom}_T(T, -) : \mathbb{M}_T \to \mathbb{M}_B$ and $\mathrm{Hom}_B(N, -) : \mathbb{M}_B \to \mathbb{M}_A$ is naturally isomorphic to

$$\mathrm{Hom}_B(N, \mathrm{Hom}_T(T, -)) \cong \mathrm{Hom}_T(N \otimes_B T, -) \cong \mathrm{Hom}_T(N, -),$$

where the last isomorphism follows by part (4) and Lemma 5.10. The forgetful functor $\mathbb{M}_T \to \mathbb{M}_B$ reflects isomorphisms. Since N is a generator in $\mathbb{M}_B$ by assumption, the (fully faithful) functor $\mathrm{Hom}_B(N, -) : \mathbb{M}_B \to \mathbb{M}_A$ reflects isomorphisms too. Hence also the composite $\mathrm{Hom}_T(N, -) : \mathbb{M}_T \to \mathbb{M}_A$ reflects isomorphisms. The forgetful functor $\mathbb{M}_T \to \mathbb{M}_B$ has a right adjoint (the coinduction functor $\mathrm{Hom}_B(T, -)$) hence it preserves coequalisers. Since N is a projective right B-module by assumption, $\mathrm{Hom}_B(N, -) : \mathbb{M}_B \to \mathbb{M}_A$ preserves coequalisers too. Hence also the composite $\mathrm{Hom}_T(N, -) : \mathbb{M}_T \to \mathbb{M}_A$ preserves coequalisers. The equivalence functor $-\otimes_{[\mathcal{C},-]} N : \mathbb{M}_{[\mathcal{C},-]} \to \mathbb{M}_B$ factorises through $-\otimes_{[\mathcal{C},-]} N : \mathbb{M}_{[\mathcal{C},-]} \to \mathbb{M}_T$ and the forgetful functor $\mathbb{M}_T \to \mathbb{M}_B$. Thus the forgetful functor is full (and obviously faithful). This implies that $-\otimes_{[\mathcal{C},-]} N : \mathbb{M}_{[\mathcal{C},-]} \to \mathbb{M}_T$ is fully faithful, hence the corresponding canonical monad morphism

$$\mathrm{Hom}_A(\mathcal{C}, -) \to \mathrm{Hom}_T(N, -\otimes_A N), \quad f \mapsto [\, n \mapsto (f \otimes_A I_N) \circ {}^N\varrho(n)],$$

is a natural isomorphism by Theorem 5.2. So we conclude by Theorem 5.8 that $-\otimes_{[\mathcal{C},-]} N : \mathbb{M}_{[\mathcal{C},-]} \to \mathbb{M}_T$ is an equivalence and so is the forgetful functor $\mathbb{M}_T \to \mathbb{M}_B$. This proves the isomorphism of algebras $T \cong B$.

(6) N is a flat left A-module by part (3). Hence it suffices to show that, under the assumptions made, $-\otimes_A N : \mathbb{M}_A \to \mathbb{M}_B$ is a faithful functor, so it reflects both monomorphisms and epimorphisms. Recall that, by 4.5(1), $-\otimes_A N : \mathbb{M}_A \to \mathbb{M}_B$ is naturally isomorphic to the composite of the free functor $\mathrm{Hom}_A(\mathcal{C}, -) : \mathbb{M}_A \to \mathbb{M}_{[\mathcal{C},-]}$ and the equivalence $-\otimes_{[\mathcal{C},-]} N : \mathbb{M}_{[\mathcal{C},-]} \to \mathbb{M}_B$. By assumption, $\mathrm{Hom}_A(\mathcal{C}, -) : \mathbb{M}_A \to \mathbb{M}_A$ is faithful. Then also $\mathrm{Hom}_A(\mathcal{C}, -) : \mathbb{M}_A \to \mathbb{M}_{[\mathcal{C},-]}$ is faithful, what completes the proof. □

Note that $\mathcal{C}$ is a generator of right A-modules (as required in Theorem 5.11 (6)), for example, provided the counit of $\mathcal{C}$ is an epimorphism.

6. Contramodules and entwining structures

As recalled in 2.14, lifting of a monad $\mathbf{F}$ on a category $\mathbb{A}$ to a monad on the category $\mathbb{A}^G$ for a comonad $\mathbf{G}$, or lifting of a comonad $\mathbf{G}$ to a comonad on the category $\mathbb{A}_F$ for a monad $\mathbf{F}$, are both equivalent to the existence of a mixed distributive law (entwining) between $\mathbf{F}$ and $\mathbf{G}$. Combining this general fact with properties of module categories, we obtain a description of entwinings between A-rings and A-corings (A

is an associative ring with unit). Recall that a *(left) entwining map* between an A-ring B and an A-coring $\mathcal{C}$ is an A-bimodule morphism $\psi : B \otimes_A \mathcal{C} \to \mathcal{C} \otimes_A B$ which respects (co)multiplications and (co)units. Similarly, (right) entwining maps $\lambda : \mathcal{C} \otimes_A B \to B \otimes_A \mathcal{C}$ are defined (e.g. [8, Chapter 5]). Note that a left entwining structure is the same as a mixed distributive law in the bicategory of algebras – bimodules – bimodule maps, in the same sense as distributive laws in any bicategory discussed in [28, Section 6]. A right entwining structure can be described as a mixed distributive law in the bicategory with opposite horizontal composition.

The following theorem is a consequence of monoidal equivalences between the category of A-bimodules; the category of left adjoint endofunctors on the category of left A-modules; and the category of right adjoint endofunctors on the category of right A-modules, the latter considered with the *opposite* composition of natural transformations.

6.1. **Entwining maps.** *For all A-rings B and A-corings $\mathcal{C}$, the following assertions are equivalent.*

(a) *There is an entwining map $\psi : B \otimes_A \mathcal{C} \to \mathcal{C} \otimes_A B$;*

(b) *the monad $B \otimes_A -$ on ${}_A\mathbb{M}$ has a lifting to a monad on ${}^{\mathcal{C}}\mathbb{M}$;*

(c) *the comonad $\mathcal{C} \otimes_A -$ on ${}_A\mathbb{M}$ has a lifting to a comonad on ${}_B\mathbb{M}$;*

(d) *the monad $\mathrm{Hom}_A(\mathcal{C}, -)$ on $\mathbb{M}_A$ has a lifting to a monad on $\mathbb{M}_B$;*

(e) *the comonad $\mathrm{Hom}_A(B, -)$ on $\mathbb{M}_A$ has a lifting to a comonad on $\mathbb{M}_{[\mathcal{C},-]}$.*

Proof. (a)⇔(b) and (a)⇔(c). An entwining map ψ determines a mixed distributive law $\Psi := \psi \otimes_A - : B \otimes_A \mathcal{C} \otimes_A - \to \mathcal{C} \otimes_A B \otimes_A -$. Conversely, if $\Psi : B \otimes_A \mathcal{C} \otimes_A - \to \mathcal{C} \otimes_A B \otimes_A -$ is a mixed distributive law, then $\psi := \Psi_A$ is an entwining map.

(a)⇔(d) and (a)⇔(e). An entwining map ψ determines a mixed distributive law $\widetilde{\Psi}$:

$$\begin{array}{ccc} \mathrm{Hom}_A(\mathcal{C}, \mathrm{Hom}_A(B, -)) \cong & & \cong \mathrm{Hom}_A(B, \mathrm{Hom}_A(\mathcal{C}, -)) \\ \mathrm{Hom}_A(\mathcal{C} \otimes_A B, -) & \xrightarrow{\mathrm{Hom}_A(\psi,-)} & \mathrm{Hom}_A(B \otimes_A \mathcal{C}, -). \end{array}$$

On the other hand, by the Yoneda Lemma, any mixed distributive law $\widetilde{\Psi} : \mathrm{Hom}_A(\mathcal{C} \otimes_A B, -) \to \mathrm{Hom}_A(B \otimes_A \mathcal{C}, -)$ is of this form. □

By [8, 18.28], part (c) of 6.1 implies that under the equivalent conditions of 6.1, $\mathcal{C} \otimes_A B$ is a B-coring, cf. [8, 32.6]. Its contramodules can be described as follows.

6.2. **$[\mathcal{C} \otimes_A B, -]$-modules.** *Let B be an A-ring and $\mathcal{C}$ an A-coring with an entwining map $\psi : B \otimes_A \mathcal{C} \to \mathcal{C} \otimes_A B$. Then the following structures on a right B-module M are equivalent.*

(a) *A module structure map $\varrho_M : \mathrm{Hom}_B(\mathcal{C} \otimes_A B, M) \to M$;*

(b) *a B-linear module structure map $\alpha_M : \mathrm{Hom}_A(\mathcal{C}, M) \to M$ (where $fb = \sum f(-^{\psi})b_{\psi}$, for $f \in \mathrm{Hom}(\mathcal{C}, M)$, $b \in B$, hence B-linearity means $\alpha_M(f)b = \sum \alpha_M\big(f(-^{\psi})b_{\psi}\big)$, with notation $\psi(b \otimes_A c) = \sum c^{\psi} \otimes_A b_{\psi}$);*

(c) *a module structure for the monad $\mathrm{Hom}_A(\mathcal{C}, -)$ on $\mathbb{M}_B$;*

(d) *a comodule structure for the comonad $\mathrm{Hom}_A(B, -)$ on $\mathbb{M}_{[\mathcal{C},-]}$.*

Proof. (a)⇔(b). The isomorphism $\mathrm{Hom}_B(\mathcal{C} \otimes_A B, M) \cong \mathrm{Hom}_A(\mathcal{C}, M)$ of right A-modules induces an isomorphism

$$\xi : \mathrm{Hom}_A(\mathrm{Hom}_A(\mathcal{C}, M), M) \to \mathrm{Hom}_A(\mathrm{Hom}_B(\mathcal{C} \otimes_A B, M), M).$$

As easily checked, $\xi(\alpha_M)$ belongs to $\mathrm{Hom}_B(\mathrm{Hom}_B(\mathcal{C} \otimes_A B, M), M)$ if and only if α_M satisfies the B-linearity condition in (b). Associativity and unitality of a $[\mathcal{C} \otimes_A B, -]$-action $\xi(\alpha_M)$ are equivalent to analogous properties of the $[\mathcal{C}, -]$-action α_M.

Equivalences (b)⇔(c) and (b)⇔(d) follow by 2.15 (cf. [32, 5.7]). □

In light of 6.1, the following describes a special case of 2.16 and 2.17.

6.3. **Distributive laws for rings and corings.** *Let B be an A-ring and $\mathcal{C}$ an A-coring over any ring A.*

(1) $\lambda : \mathcal{C} \otimes_A B \to B \otimes_A \mathcal{C}$ *is an entwining map if and only if*

$$\tilde{\lambda} : \mathrm{Hom}_A(B, -) \otimes_A \mathcal{C} \to \mathrm{Hom}_A(B, - \otimes_A \mathcal{C}),\ f \otimes c \mapsto (f \otimes_A I_{\mathcal{C}}) \circ \lambda(c \otimes -),$$

is a comonad distributive law. Then $\mathrm{Hom}_A(B, -) \otimes_A \mathcal{C}$ *is a comonad on* $\mathbb{M}_A$ *and the category of its comodules is isomorphic to the category of λ-bimodules (i.e. usual entwined modules, cf. 2.15).*

(2) $\psi : B \otimes_A \mathcal{C} \to \mathcal{C} \otimes_A B$ *is an entwining map if and only if*

$$\tilde{\psi} : \mathrm{Hom}_A(\mathcal{C}, -) \otimes_A B \to \mathrm{Hom}_A(\mathcal{C}, - \otimes_A B),\ g \otimes a \mapsto (g \otimes_A I_B) \circ \psi(a \otimes -),$$

is a monad distributive law. Then $\mathrm{Hom}_A(\mathcal{C}, - \otimes_A B)$ *is a monad on* $\mathbb{M}_A$ *and the category of its modules is isomorphic to the category of $\tilde{\psi}$-bimodules (cf. 2.15).*

Note that for a commutative ring R, any R-algebra A and R-coalgebra C are entwined by the twist maps $C \otimes_R A \to A \otimes_R C$ and $A \otimes_R C \to C \otimes_R A$. Applying 6.3 to these particular entwinings, we conclude that the canonical natural transformations

$$\mathrm{Hom}_R(A, -) \otimes_R C \to \mathrm{Hom}_R(A, - \otimes_R C), \qquad f \otimes c \mapsto f(-) \otimes c, \quad \text{and}$$
$$\mathrm{Hom}_R(C, -) \otimes_R A \to \mathrm{Hom}_R(C, - \otimes_R A), \qquad g \otimes a \mapsto g(-) \otimes a,$$

yield a comonad distributive law and a monad distributive law, respectively.

7. Bialgebras and bimodules

There are many equivalent characterisations of bialgebras and Hopf algebras. A bialgebra over a commutative ring R can be seen as an R-module that is both an algebra and a coalgebra entwined in a certain way. In category theory terms, bialgebra is defined as an R-module such that the tensor functor $- \otimes_R B$ is a bimonad on $\mathbb{M}_R$. Associated to a bialgebra B, there is a category of Hopf modules, whose objects are B-modules with a compatible B-comodule structure. A Hopf algebra can be characterised as a bialgebra B such that the functor $- \otimes_R B$ is an equivalence between the categories of R-modules and Hopf B-modules. In this section we supplement this description of bialgebras and Hopf algebras by the equivalent description in terms of properties of the Hom-functor $[B, -]$, and hence in terms of contramodules.

Throughout, R is a commutative ring. The unit element of a (bi)algebra B is denoted by 1_B. For the coproduct Δ of a bialgebra B, if applied to an element $b \in B$, we use Sweedler's index notation $\Delta(b) = b_{\underline{1}} \otimes b_{\underline{2}}$, where implicit summation is understood.

7.1. **Bialgebras.** Let B be an R-module which is both an R-algebra $\mu : B\otimes_R B \to B$, $\iota : R \to B$, and an R-coalgebra $\Delta : B \to B \otimes_R B$, $\varepsilon : B \to R$. Based on the canonical twist $\mathsf{tw} : B \otimes_R B \to B \otimes_R B$, we obtain the R-module maps

$$\begin{aligned} \psi_r &= (I_B \otimes_R \mu) \circ (\mathsf{tw} \otimes_R I_B) \circ (I_B \otimes_R \Delta) : \quad B \otimes_R B \to B \otimes_R B, \\ \psi_l &= (\mu \otimes_R I_B) \circ (I_B \otimes_R \mathsf{tw}) \circ (\Delta \otimes_R I_B) : \quad B \otimes_R B \to B \otimes_R B. \end{aligned}$$

Evaluated on elements, $\psi_r(a \otimes b) = b_{\underline{1}} \otimes ab_{\underline{2}}$ and $\psi_l(a \otimes b) = a_{\underline{1}}b \otimes a_{\underline{2}}$.

To make B a *bialgebra*, μ and ι must be coalgebra maps (equivalently, Δ and ε are to be algebra maps) with respect to the obvious product and coproduct on $B \otimes_R B$ (induced by tw). The compatibility between multiplication and comultiplication can be expressed by commutativity of the diagram

$$\begin{tikzcd} B \otimes_R B \arrow[r, "\mu"] \arrow[d, "\Delta\otimes_R I_B"'] & B \arrow[r, "\Delta"] & B \otimes_R B \\ B \otimes_R B \otimes_R B \arrow[rr, "I_B\otimes_R\psi_r"] & & B \otimes_R B \otimes_R B, \arrow[u, "\mu\otimes_R I_B"'] \end{tikzcd}$$

or, equivalently, by its symmetrical counterpart $(I_B \otimes_R \mu) \circ (\psi_l \otimes_R I_B) \circ (I_B \otimes_R \Delta) = \Delta \circ \mu$. Given an R-bialgebra B, we sometimes write $\underline{B}$ when our focus is on the algebra structure and $\overline{B}$ when focussing on the coalgebra part.

For a bialgebra B, both maps ψ_r and ψ_l are (right, respectively, left) entwining maps between the algebra B and the coalgebra B. Going to the functor level it turns out that ψ_r yields a mixed distributive law from the monad $- \otimes_R \underline{B}$ to the comonad $- \otimes_R \overline{B}$ (equivalently, from the comonad $\overline{B} \otimes_R -$ to the monad $\underline{B} \otimes_R -$), while ψ_l induces a mixed distributive law from the monad $\underline{B} \otimes_R -$ to the comonad $\overline{B} \otimes_R -$ (equivalently, from the comonad $- \otimes_R \overline{B}$ to the monad $- \otimes_R \underline{B}$).

The entwining maps $\psi_r : \overline{B} \otimes_R \underline{B} \to \underline{B} \otimes_R \overline{B}$ and $\psi_l : \underline{B} \otimes_R \overline{B} \to \overline{B} \otimes_R \underline{B}$ determine $\underline{B}$-corings $B \otimes_R B$, denoted by $B \otimes^r_R B$ and $B \otimes^l_R B$, respectively.

The following is obtained by applying 2.15 to the mixed distributive law $- \otimes_R \psi_r$ from the monad $- \otimes_R \underline{B}$ to the comonad $- \otimes_R \overline{B}$.

7.2. **B-Hopf modules.** *Let B be an R-bialgebra and consider the $\underline{B}$-coring $B \otimes^r_R B$. The following structures on a right $\underline{B}$-module M are equivalent:*

(a) *A right $B \otimes^r_R B$-comodule structure map $\varrho^M : M \to M \otimes_B (B \otimes^r_R B)$;*

(b) *a right $\underline{B}$-linear $\overline{B}$-comodule structure map $\alpha^M : M \to M \otimes_R B$, (where $\underline{B}$-linearity means commutativity of the diagram*

$$\begin{tikzcd} M \otimes_R B \arrow[r, "\alpha_M"] \arrow[d, "\alpha^M\otimes_R I_B"'] & M \arrow[r, "\alpha^M"] & M \otimes_R B \\ M \otimes_R B \otimes_R B \arrow[rr, "I_M\otimes_R\psi_r"] & & M \otimes_R B \otimes_R B, \arrow[u, "\alpha_M\otimes_R I_B"'] \end{tikzcd}$$

where $\alpha_M : M \otimes_R B \to M$ denotes the $\underline{B}$-action on M);

(c) *a comodule structure for the comonad $- \otimes_R B$ on $\mathbb{M}_{\underline{B}}$;*

(d) *a module structure for the monad $- \otimes_R B$ on $\mathbb{M}^{\overline{B}}$.*

A right $\underline{B}$-module M with these equivalent properties is called a *B-Hopf module.* Morphisms of B-Hopf modules are $B \otimes^r_R B$-comodule maps. Equivalently, they are

$\underline{B}$-module as well as $\overline{B}$-comodule maps. The category of right B-Hopf modules is denoted by $\mathbb{M}_{\underline{B}}^{\overline{B}}$. By the above considerations, it is isomorphic to $\mathbb{M}^{B\otimes_R^r B}$.

Based on the mixed distributive law $\psi_l \otimes_R -$ from the monad $\underline{B}\otimes_R -$ to the comonad $\overline{B}\otimes_R -$, left B-Hopf modules are defined symmetrically. Note that a bialgebra B is both a left and a right B-Hopf module.

From 6.2 we obtain:

7.3. **$[B,-]$-Hopf modules.** *Let B be an R-bialgebra and consider the $\underline{B}$-coring $B\otimes_R^l B$. Then the following structures on a right $\underline{B}$-module M are equivalent.*

(a) *A $[B\otimes_R^l B,-]$-module structure map $\varrho_M : \mathrm{Hom}_B(B\otimes_R^l B, M) \to M$;*

(b) *a $\underline{B}$-linear $[\overline{B},-]$-module structure map $\alpha_M : \mathrm{Hom}_R(B,M)\to M$ (i.e. $\alpha_M(f)b = \sum \alpha_M\big(f(b_{\underline{1}}-)b_{\underline{2}}\big)$ for $f\in \mathrm{Hom}_R(B,M)$, $b\in B$);*

(c) *a module structure for the monad $\mathrm{Hom}_R(B,-)$ on $\mathbb{M}_B$;*

(d) *a comodule structure for the comonad $\mathrm{Hom}_R(B,-)$ on $\mathbb{M}_{[B,-]}$.*

A right $\underline{B}$-module M with these equivalent properties is called a $[B,-]$-*Hopf module* or *right Hopf contramodule for B*. Morphisms of $[B,-]$-Hopf modules are $B\otimes_R^l B$-contramodule maps. Equivalently, they are $\underline{B}$-module as well as $\overline{B}$-contramodule maps. The category of $[B,-]$-Hopf modules is denoted by $\mathbb{M}_{[\overline{B},-]}^{[\underline{B},-]}$. By the above considerations, it is isomorphic to $\mathbb{M}_{[B\otimes_R^l B,-]}$.

Based on ψ_r, left Hopf contramodules for B are defined symmetrically.

Applying 6.3, the following alternative description of Hopf modules is obtained.

7.4. **Distributive laws for bialgebras.** *Let B be an R-bialgebra. Then:*

(1) *The entwining ψ_r in 7.1 induces a comonad distributive law*

$$\mathrm{Hom}_R(\underline{B},-)\otimes_R \overline{B} \to \mathrm{Hom}_R(\underline{B}, -\otimes_R \overline{B}),\quad f\otimes b\mapsto \sum f((-)_{\underline{1}})\otimes b(-)_{\underline{2}}.$$

Hence $\mathrm{Hom}_R(\underline{B},-)\otimes_R \overline{B}$ is a comonad on $\mathbb{M}_R$. The category of its comodules is isomorphic to the category of B-Hopf modules.

(2) *The entwining ψ_l in 7.1 induces a monad distributive law*

$$\mathrm{Hom}_R(\overline{B},-)\otimes_R \underline{B} \to \mathrm{Hom}_R(\overline{B}, -\otimes_R \underline{B}),\quad f\otimes b\mapsto \sum f(b_{\underline{1}}-)\otimes b_{\underline{2}}.$$

Hence $\mathrm{Hom}_R(\overline{B},-\otimes_R \underline{B})$ is a monad on $\mathbb{M}_R$. The category of its modules is isomorphic to the category of $[B,-]$-Hopf modules.

7.5. **Hopf algebras.** An R-bialgebra $(H,\mu,\iota,\Delta,\varepsilon)$ is said to be a *Hopf algebra* if there is an R-module map $S: H\to H$, called the *antipode*, such that

$$\mu\circ(I_H\otimes_R S)\circ\Delta = \iota\circ\varepsilon = \mu\circ(S\otimes_R I_H)\circ\Delta.$$

If the antipode exists, then it is unique and it is an anti-algebra and anti-coalgebra map.

The unit $\iota: R\to H$ of an R-bialgebra $(H,\mu,\iota,\Delta,\varepsilon)$ determines the Sweedler H-coring $H\otimes_R H$; see [8, 25.1]. A bialgebra H is known to be a Hopf algebra if and only if the H-coring map

$$H\otimes_R H\to H\otimes_R^r H,\qquad a\otimes b\mapsto \sum ab_{\underline{1}}\otimes b_{\underline{2}},$$

is an isomorphism, and if and only if the H-coring map

$$H \otimes_R H \to H \otimes^l_R H, \qquad a \otimes b \mapsto \sum a_{\underline{1}} \otimes a_{\underline{2}} b,$$

is an isomorphism; see e.g. [8, 15.5]. Thus in particular, for a Hopf algebra H, the H-corings $H \otimes^r_R H$ and $H \otimes^l_R H$ are mutually isomorphic.

7.6. **Hopf algebras and coseparability.** *Let H be an R-Hopf algebra.*

(1) *The H-coring $H \otimes^r_R H$ is coseparable.*
(2) *The following functor is an equivalence:*
$$\mathrm{Hom}^{H\otimes^r_R H}(H \otimes^r_R H, -) : \mathbb{M}^{H\otimes^r_R H} \to \mathbb{M}_{[H\otimes^r_R H,-]}.$$
(3) *The category of H-Hopf modules (in 7.2) and the category of $[H,-]$-Hopf modules (in 7.3) are equivalent.*

Proof. (1) For any R-bialgebra $(H, \mu, \iota, \Delta, \varepsilon)$, the inclusion $\iota : R \to H$ is split by the R-(bi)module map ε. Consequently, the corresponding Sweedler coring $H \otimes_R H$ is coseparable; see [8, 26.9]. Since, for a Hopf algebra H, the Sweedler H-coring $H \otimes_R H$ is isomorphic to $H \otimes^r_R H$ (see 7.5), the assertion (1) follows.

(2) In view of (1), this is a special case of 4.8.

(3) The category of H-Hopf modules is isomorphic to $\mathbb{M}^{H\otimes^r_R H}$ and the category of $[H,-]$-Hopf modules is isomorphic to $\mathbb{M}_{[H\otimes^l_R H,-]}$. So the claim follows by coring isomorphism $H \otimes^r_R H \cong H \otimes^l_R H$ in 7.5 and part (2). □

The final aim of this section is to characterise Hopf algebras via their induced (co)monads. The following notions were introduced in [32] and [20]. Note that these terms have different meanings in Moerdijk [21] and Bruguières-Virelizier [6].

7.7. **Bimonads and Hopf monads.** A *bimonad* on a category $\mathbb{A}$ is a functor $F : \mathbb{A} \to \mathbb{A}$ with a monad structure $\underline{F} = (F, m, i)$ and a comonad structure $\overline{F} = (F, d, e)$ subject to the compatibility conditions

(i) e is a monad morphism $\underline{F} \to I_{\mathbb{A}}$;
(ii) i is a comonad morphism $I_{\mathbb{A}} \to \overline{F}$;
(iii) there is a mixed distributive law $\Psi : \underline{F}\overline{F} \to \overline{F}\underline{F}$, satisfying
$$d \circ m = Fm \circ \Psi F \circ Fd.$$

A bimonad (F, m, i, d, e) is called a *Hopf monad* if there exists a natural transformation $S : F \to F$, called the *antipode*, such that

$$m \circ SF \circ d = i \circ e = m \circ FS \circ d.$$

A class of examples of bimonads is provided by the following construction in [20, Proposition 6.3] (see also [13]). Let F be a functor $\mathbb{A} \to \mathbb{A}$ allowing a monad structure $\underline{F} = (F, m, i)$ as well as a comonad structure $\overline{F} = (F, d, e)$. Consider a *double entwining* τ, i.e. a natural transformation $FF \to FF$, which is an entwining both in the sense $\underline{F}\overline{F} \to \overline{F}\underline{F}$ and also $\overline{F}\underline{F} \to \underline{F}\overline{F}$. The functor F is called a τ*-bimonad* provided that the above conditions (i) and (ii) hold and in addition

$$mF \circ FFm \circ F\tau F \circ dFF \circ Fd = d \circ m.$$

By [20, Proposition 6.3], a τ-bimonad F is a bimonad with respect to the mixed distributive law $\Psi := mF \circ F\tau \circ dF$.

A τ-bimonad with an antipode is called a *τ-Hopf monad*.

As described in [20], if a τ-bimonad F has a left or right adjoint G, then the mates under the adjunction of the structure maps of the monad and comonad F, equip G with a comonad and a monad structure, respectively. Moreover, the mate $\bar\tau$ of τ under the adjunction is a double entwining for G, and G is a $\bar\tau$-bimonad. If F is a τ-Hopf monad, then G is a $\bar\tau$-Hopf monad.

7.8**. The bimonad $-\otimes_R B$.** For an R-bialgebra $(B,\mu,\iota,\Delta,\varepsilon)$, the functor $-\otimes_R B : \mathbb{M}_R \to \mathbb{M}_R$ is a tw-bimonad, hence a bimonad with respect to the mixed distributive law

$$(-\otimes_R I_B \otimes_R \mu)\circ(-\otimes_R \mathsf{tw}\otimes_R I_B)\circ(-\otimes_R I_B\otimes_R \Delta) = -\otimes_R \psi_r.$$

By duality, $\mathrm{Hom}_R(B,-)$ is a $\overline{\mathsf{tw}}$-bimonad, with coproduct $[\mu,-]$ and counit $[\iota,-]$ in 3.2, product $[\Delta,-]$ and unit $[\varepsilon,-]$ in 3.5, where

$$\overline{\mathsf{tw}} : \mathrm{Hom}_R(B,\mathrm{Hom}_R(B,-)) \to \mathrm{Hom}_R(B,\mathrm{Hom}_R(B,-))$$

is given by switching the arguments. Thus $\mathrm{Hom}_R(B,-) : \mathbb{M}_R \to \mathbb{M}_R$ is a bimonad with respect to the mixed distributive law $\mathrm{Hom}_R(\psi_l,-)$:

$$\begin{array}{ccc} \mathrm{Hom}_R(B,\mathrm{Hom}_R(B,-)) \cong & & \cong \mathrm{Hom}_R(B,\mathrm{Hom}_R(B,-)) \\ \mathrm{Hom}_R(B\otimes_R B,-) & \xrightarrow{\mathrm{Hom}_R(\psi_l,-)} & \mathrm{Hom}_R(B\otimes_R B,-). \end{array}$$

A motivating example of a (tw-)Hopf monad in [20] is the functor $-\otimes_R H : \mathbb{M}_R \to \mathbb{M}_R$, induced by a Hopf algebra H.

Summarising the preceding observations we obtain the following.

7.9**. Characterisations of Hopf algebras.** *For an R-bialgebra $(H,\mu,\iota,\Delta,\varepsilon)$, the following assertions are equivalent.*

(a) *H is a Hopf algebra;*

(b) *the map $\gamma : H\otimes_R H \xrightarrow{\Delta\otimes_R I_H} H\otimes_R H\otimes_R H \xrightarrow{I\otimes_R\mu} H\otimes_R H$ is an isomorphism;*

(c) *H is an $H\otimes^r_R H$-Galois right (equivalently, left) comodule;*

(d) *H is an $H\otimes^l_R H$-Galois right (equivalently, left) comodule;*

(e) *$-\otimes_R H$ is a tw-Hopf monad on $\mathbb{M}_R$;*

(f) *for the $\overline{\mathsf{tw}}$-bimonad $[H,-] = \mathrm{Hom}_R(H,-)$, the natural transformation*

$$[\gamma,-] : [H,[H,-]] \xrightarrow{[H,[\mu,-]]} [H,[H,[H,-]]] \xrightarrow{[\Delta,[H,-]]} [H,[H,-]]$$

is an isomorphism;

(g) *$\mathrm{Hom}_R(H,-)$ is a $\overline{\mathsf{tw}}$-Hopf monad on $\mathbb{M}_R$;*

(h) *$-\otimes_R H : \mathbb{M}_R \to \mathbb{M}^{\overline{H}}_{\underline{H}}$ is an equivalence;*

(i) *$\mathrm{Hom}_R(H,-) : \mathbb{M}_R \to M^{[\underline{H},-]}_{[\overline{H},-]}$ is an equivalence;*

(j) *H is a $\mathrm{Hom}_{-,H}(H\otimes^r_R H,-)$-Galois left comodule (equivalently, a $\mathrm{Hom}_{H,-}(H\otimes^r_R H,-)$-Galois right comodule);*

(k) *H is a $\mathrm{Hom}_{-,H}(H\otimes^l_R H,-)$-Galois left comodule (equivalently, a $\mathrm{Hom}_{H,-}(H\otimes^l_R H,-)$-Galois right comodule).*

Proof. (a)-(d) and (h) are standard equivalent characterisations of Hopf algebras, see e.g. [8, 15.2 and 15.5].

(a)⇔(e)⇔(f)⇔(g) is proven in [20], (c)⇔(j) and (d)⇔(k) follow by 5.7, while (i)⇒(k) follows by 5.8 (a)⇒(b)(i).

(h)⇒(i). There is a sequence of equivalences,

$$\mathbb{M}_{\underline{H}}^{\overline{H}} \cong \mathbb{M}^{H\otimes^r_R H} \simeq \mathbb{M}_{[H\otimes^r_R H,-]} \cong \mathbb{M}_{[H\otimes^l_R H,-]} \cong \mathbb{M}^{[\underline{H},-]}_{[\overline{H},-]},$$

cf. 7.2, 7.6, 7.5 and 7.3 (note that (h)⇒(a)). Combining this composite with the equivalence in part (h), we obtain an equivalence functor

$$\mathrm{Hom}^{H\otimes^l_R H}(H \otimes^l_R H, - \otimes_R H) : \mathbb{M}_R \to \mathbb{M}^{[\underline{H},-]}_{[\overline{H},-]}.$$

We claim that the functor in part (i) is naturally isomorphic to this equivalence, hence it is an equivalence, too.

The equivalence in part (h) gives rise to an R-module isomorphism

$$\mathrm{Hom}^{H\otimes^r_R H}(H \otimes^r_R H, M \otimes_R H) \to \mathrm{Hom}_R(H, M),\ \Psi \mapsto (I_M \otimes_R \varepsilon) \circ \Psi(- \otimes_R \iota),$$

for any R-module M, that is natural in M. Using the coring isomorphism $H \otimes^r_R H \cong H \otimes^l_R H$ in 7.5, we can transfer it to a natural isomorphism

$$\beta_M : \mathrm{Hom}^{H\otimes^l_R H}(H \otimes^l_R H, M \otimes_R H) \to \mathrm{Hom}_R(H, M),\ \Phi \mapsto (I_M \otimes_R \varepsilon) \circ \Phi \circ \Delta.$$

An easy computation shows that β_M is a morphism of $[H \otimes^l_R H, -]$-modules, what completes the proof. □

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