A CATEGORICAL APPROACH TO ALGEBRAS AND COALGEBRAS

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ABSTRACT. Algebraic and coalgebraic structures are often handled independently. In this survey we want to show that they both show up naturally when introducing them from a categorical point of you. Azumaya, Frobenius, separable, and Hopf algebras are obtained when both notions are combined. The starting point and guiding lines for this approach are given by *adjoint pairs of functors* and their elementary properties.

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1. INTRODUCTION

In the last decades categorical techniques turned out to be very effective in algebra and representation theory. Hereby, it was a key observation that module theory of an algebra A over a field K is essentially the theory of the functor

$$A \otimes_K - : \mathbb{M}_K \to \mathbb{M}_K,$$

an endofunctor of the category of K-vector spaces. An algebra A is defined by K-linear maps multiplication $A \otimes_K A \to A$ and unit $e : K \to A$, subject to associativity and unitality conditions. Left A-modules are given by a K-vector space V with K-linear maps $\varrho : A \otimes_K V \to V$, also subject to associativity and unitality conditions. Together with A-linear maps, this yields the category $_A\mathbb{M}$ of left A-modules, The tensor product with product

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$(A \otimes_K V, m \otimes V)$ is a left A-module and this leads to the free and forgetful functors

$$\phi_A: \quad \mathbb{M}_K \to {}_A\mathbb{M}, \ V \mapsto (A \otimes V, m \otimes V),$$
$$U_A: \quad {}_A\mathbb{M} \to \mathbb{M}_K, \ (M, \rho) \mapsto M,$$

and the bijection

$$\mathrm{Hom}_A(A \otimes_K V, M) \to \mathrm{Hom}_K(V, U_A(M)),$$

that is, the functor U_A is right adjoint to $A \otimes_K -$.

These basic structures can be defined for arbitrary categories $\mathbb{A}$, replacing $\mathbb{M}_K$, and any functor $F : \mathbb{A} \to \mathbb{A}$, replacing $A \otimes_K - : \mathbb{M}_K \to \mathbb{M}_K$. Multiplication and unit are replaced by natural transformations, $m : FF \to F$ and $\eta : 1 \to F$, satisfying the respective associativity and unitality conditions. This gives F a *monad* structure.

An F-module is an object $V \in \mathbb{A}$ with a morphism $\varrho : F(V) \to V$ and for any $W \in \mathbb{A}$, $(F(W), m_W)$ is an example for this. Morphisms of F-modules are morphisms from $\mathbb{A}$ respecting the module structures and they yield the category $\mathbb{A}_F$ of F-modules with free and forgetful functors

$$\phi_F: \quad \mathbb{A} \to \mathbb{A}_F, \ V \mapsto (F(V), m_V),$$
$$U_F: \quad \mathbb{A}_F \to \mathbb{A}, \ (M, \varrho) \mapsto M,$$

and the bijection

$$\mathrm{Mor}_F(F(V), M) \quad \to \quad \mathrm{Mor}_{\mathbb{A}}(V, U_F(M)),$$
$$F(V) \xrightarrow{f} M \quad \mapsto \quad V \xrightarrow{\eta_V} F(V) \xrightarrow{f} M,$$

shows that the functor U_F is right adjoint to ϕ_F.

This shows that structures from module theory can be formulated in great generality. As we will see, if the functor $F : \mathbb{A} \to \mathbb{A}$ has a right adjoint $G : \mathbb{A} \to \mathbb{A}$, then the monad structure on F provides G with the structure of a *comonad*. Thus this approach leads naturally to comonads (coalgebras, bocses) and comodules and we will highlight this interplay. Notice that the categorical tools developed are also successfully applied in theoretical computer science and logic (e.g. [25], [37]).

2. Category theory

The idea that the role of elements in algebraic structures should be taken over by homomorphisms came up from the beginning of the last century. It has finally been poured into a solid frame 1945 by Samual Eilenberg and Saunders MacLane in the seminal paper [6]. For convenience and to fix notation we recall the basic notions and refer to [11] for more details.

2.1. **Categories.** A *category* $\mathbb{A}$ consists of a class of objects, and

(i) for each pair of objects A, A' of objects, there is a set of morphisms $\mathrm{Mor}_{\mathbb{A}}(A, A')$, usually denoted by arrows $A \to A'$,

(ii) for any triple A, A', A'' of objects there is a composition of morphisms,

$$\mathrm{Mor}_\mathbb{A}(A, A') \times \mathrm{Mor}_\mathbb{A}(A', A'') \to \mathrm{Mor}_\mathbb{A}(A, A'')$$

which is associative in an obvious sense,

(iii) for any object A, $\mathrm{Mor}_\mathbb{A}(A, A)$ contains an identity morphism 1_A, leaving any composition with it unchanged.

2.2. Functors.

A *covariant functor* $F : \mathbb{A} \to \mathbb{B}$ between categories sends

(i) an object A from $\mathbb{A}$ to an object $F(A)$ in $\mathbb{B}$,

(ii) a morphism $f : A \to A'$ to $F(f) : F(A) \to F(A')$ in $\mathbb{B}$,

(iii) a composit fg of morphisms in $\mathbb{A}$ to $F(fg) = F(f)F(g)$ in $\mathbb{B}$,

(iv) the identity 1_A for A in $\mathbb{A}$, to the identy of $F(1_A)$ in $\mathbb{B}$, $F(1_A) = 1_{F(A)}$.

It follows from this definition that F induces a set map

$$\Phi_F : \mathrm{Mor}_\mathbb{A}(A, A') \to \mathrm{Mor}_\mathbb{B}(F(A), F(A')),$$

and F is called *faithful* if Φ_F is injective, and *full* if Φ_F is surjective.

F defines an *equivalence of categories* provided there exists a functor $G : \mathbb{B} \to \mathbb{A}$ such that FG and GF both yield the respective identities.

A fully faithful functor $F : \mathbb{A} \to \mathbb{B}$ induces an equivalence between $\mathbb{A}$ and a full subcategory of $\mathbb{B}$, the image of F.

2.3. Natural transformations.

Given two functors $F, G : \mathbb{A} \to \mathbb{B}$ between categories, a *natural transformation* $\psi : F \to G$ is given by a family of morphisms $\psi_A : F(A) \to G(A)$, $A \in \mathbb{A}$, with commutative diagrams, for any morphism $h : A \to A'$ in $\mathbb{A}$,

$$
\begin{array}{ccc}
A & F(A) \xrightarrow{\psi_A} G(A) \\
\Big\downarrow h & F(h)\Big\downarrow \qquad \Big\downarrow G(h) \\
A', & F(A') \xrightarrow{\psi_{A'}} G(A').
\end{array}
$$

ψ is called a *(natural) isomorphism* if all ψ_A are isomorphisms in $\mathbb{B}$.

2.4. Separable functors.

A functor $F : \mathbb{A} \to \mathbb{B}$ is said to be *separable* if, for any $A, A' \in \mathbb{A}$, the canonical map Φ_F (from 2.2) is a split monomorphism, that is, there is a map

$$\Psi_F : \mathrm{Mor}_\mathbb{B}(F(A), F(A')) \to \mathrm{Mor}_\mathbb{A}(A, A'),$$

natural in A, A', with $\Psi_F \cdot \Phi_F$ the identity on $\mathrm{Mor}_\mathbb{A}(A, A')$.

Clearly, for a separable functor F, Φ_F is always injective, and every fully faithful functor is separable. A survey on separability in algebra and category theory is given in [36].

It turned out that the following notion introduced and described by S. Eilenberg and J.C. Moore in [5], is a milestone in category theory.

2.5. Adjoint pair of functors. A pair of (covariant) functors $F : \mathbb{A} \to \mathbb{B}$, $G : \mathbb{B} \to \mathbb{A}$ between any categories $\mathbb{A}$, $\mathbb{B}$, is said to be *adjoint*, we write $F \dashv G$, provided there is a bijection

$$\alpha_{A,B} : \mathrm{Mor}_{\mathbb{B}}(F(A), B) \to \mathrm{Mor}_{\mathbb{A}}(A, G(B)),$$

natural in $A \in \mathbb{A}$ and $B \in \mathbb{B}$. Such a bijection can be described by natural transformations, called *unit* and *counit*,

$$\eta : 1 \to GF, \quad \varepsilon : FG \to 1,$$

satisfying the *triangualar identities*

$$F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F = 1_F, \quad G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G = 1_G.$$

They are obtained as images of the identities of $F(A)$ and $G(B)$, respectively, in the defining bijection.

Adjointness of *contravariant functors* $\mathbb{A} \to \mathbb{B}$ is defined by considering relating them with covariant functors between opposite categories.

2.6. Remark. The notion of adjointness can be weakened in various ways. For example, instead of being invertible one may require α to be *regular*, that is, there exists $\beta_{A,B} : \mathrm{Mor}_{\mathbb{A}}(A, G(B)) \to \mathrm{Mor}_{\mathbb{B}}(F(A), B)$, natural in $A \in \mathbb{A}$ and $B \in \mathbb{B}$, such that $\alpha\beta\alpha = \alpha$ (and $\beta\alpha\beta = \beta$). This yields a weaker form of the triangular identities (see [34], [18]).

A short argument shows that for adjoint functors $F \dashv G$,

- F preserves epimorphisms and coproducts,
- G preserves monomorphisms and products.

2.7. Properties of units and counits. Let $F \dashv G : \mathbb{B} \to \mathbb{A}$ be an adjoint pair of functors (notation from 2.5).

(1) ε is an isomorphism if and only if G is a fully faithful functor: G yields an equivalence between $\mathbb{B}$ and the image of G.

(2) η is an isomorphism if and only if F is a fully faithful functor: F yields an equivalence between $\mathbb{A}$ and the image of F.

(3) ε and η are isomorphisms if F and G both are fully faithful: F and G determine an equivalence between $\mathbb{A}$ and $\mathbb{B}$.

(4) ε is a split epimorphism if and only if G is a separable functor.

(5) η is a split monomorphism if and only if F is a separable functor.

(6) η is an extremal epimorphism and ε an extremal monomorphism: the categories $\mathbb{A}_{GF}$ and $\mathbb{B}^{FG}$ (see 4.2) are equivalent ($*$-functors, [4]).

2.8. Rings and modules. We follow the notation from [26]. For associative rings R and S, denote by $_R\mathbb{M}$ and $\mathbb{M}_S$ the category of left and right modules, respectively. Then any bimodule $_RP_S$ induces an adjoint pair of functors

$$_RP \otimes_S - : {}_S\mathbb{M} \to {}_R\mathbb{M}, \quad \mathrm{Hom}_R(P, -) : {}_R\mathbb{M} \to {}_S\mathbb{M}$$

with bijection, counit, and unit

$$\mathrm{Hom}_R(P \otimes_S Y, X) \simeq \mathrm{Hom}_S(Y, \mathrm{Hom}_R(P, X)),$$

$$\varepsilon_X : M \otimes_S \mathrm{Hom}_R(P, X) \to X, \quad m \otimes f \mapsto f(m)$$
$$\eta_Y : Y \to \mathrm{Hom}_R(P, P \otimes_S Y), \quad y \mapsto [m \mapsto m \otimes y].$$

Denote by $\mathrm{Gen}(P)$ ($\mathrm{Pres}(P)$) the category of P-generated (P-presented) R-modules. Clearly the image of $P \otimes_S -$ is contained in $\mathrm{Pres}(P)$.

Choose an (injective) cogenerator Q in $_R\mathbb{M}$ and put $U = \mathrm{Hom}_R(P, Q)$. For the S-module U, denote by $\mathrm{Cog}(U)$ ($\mathrm{Cop}(U)$) the category of U-cogenerated (U-copresented) S-modules. The image of $\mathrm{Hom}_R(P, -)$ is contained in $\mathrm{Cog}(U)$. If R is a cogenerator in $_R\mathbb{M}$ we can choose $U = P^*$.

(i) ε isomorphism: P is a generator in $_R\mathbb{M}$ and the categories $_R\mathbb{M}$ and $\mathrm{Cog}(U)$ are equivalent.

(ii) η isomorphism: $P \otimes_S -$ is faithful on the image of $\mathrm{Hom}_R(P, -)$ and the categories $_S\mathbb{M}$ and $\mathrm{Pres}(P)$ are equivalent (Sato equivalence, [22, Theorem 2.1]).

(iii) η and ε are isomorphisms: the categories $_R\mathbb{M}$ and $_S\mathbb{M}$ are equivalent; P is a finitely generated projective generator in $_R\mathbb{M}$ and $S = \mathrm{End}_R(P)$ (Morita equivalence, [20]).

(iv) ε monomorph and η epimorph: the categories $\mathrm{Gen}(P)$ and $\mathrm{Cog}(U)$ are equivalent ($*$-modules, e.g. [23, Theorem 1.3], [4], [27], [28]).

3. Monads and comonads

Monads and comonads on categories are modelled after the algebras and coalgebras on vector spaces.

3.1. Monads and their modules. A *monad* on any category $\mathbb{A}$ is an endofunctor $T : \mathbb{A} \to \mathbb{A}$ with natural transformations, *product* and *unit*,

$$m : TT \to T, \quad \eta : 1_\mathbb{A} \to T,$$

subject to associativity and unitality conditions (as for algebras).

T-modules are objects $A \in \mathbb{A}$ with a morphism $\varrho : T(A) \to A$ subject to associativity and unitality conditions (as for modules over rings).

Morphisms between T-modules (A, ϱ) and (A', ϱ') (or *T-morphisms*) are morphims $f : A \to A'$ in $\mathbb{A}$ with commutative diagram

$$
\begin{array}{ccc}
T(A) & \xrightarrow{T(f)} & T(A') \\
\varrho \downarrow & & \downarrow \varrho' \\
A & \xrightarrow{f} & A'.
\end{array}
$$

The category determined by T-modules and their morphisms is called the **Eilenberg-Moore category** - or just the *module category* - of the monad (T, m, η) and we denote it by $\mathbb{A}_T$.

For any $A \in \mathbb{A}$, $T(A)$ has a T-module structure by $m_A : TT(A) \to T(A)$ and this leads to the *free functor* ϕ_T which allows for a right adjoint, the *forgetful functor*,

$$\phi_T : \mathbb{A} \to \mathbb{A}_T, \quad A \mapsto (T(A), m_A),$$
$$U_T : \mathbb{A}_T \to \mathbb{A}, \quad (A, \varrho_A) \mapsto A.$$

The adjunction $\phi_T \dashv U_T$ is given by the bijection, for $A \in \mathbb{A}$, $B \in \mathbb{A}_T$,

$$(3.1) \qquad \mathrm{Mor}_{\mathbb{A}_T}(\phi_T(A), B) \xrightarrow{\simeq} \mathrm{Mor}_{\mathbb{A}}(A, U_T(B)), \quad f \mapsto f\,\eta_A.$$

Putting $B = \phi_T(A')$ with $A' \in \mathbb{A}$ and applying $U_T \phi_T = T$, we obtain

$$(3.2) \qquad \mathrm{Mor}_{\mathbb{A}_T}(\phi_T(A), \phi_T(A')) \xrightarrow{\simeq} \mathrm{Mor}_{\mathbb{A}}(A, T(A')).$$

Define a new category $\widetilde{\mathbb{A}}_T$ with the objects of $\mathbb{A}$ but choosing

$$\mathrm{Mor}_{\widetilde{\mathbb{A}}_T}(A, A') = \mathrm{Mor}_{\mathbb{A}}(A, T(A'))$$

and defining the composition of $f : A \to T(A')$ and $g : A' \to T(A'')$ by

$$g \diamond f : A \xrightarrow{f} T(A') \xrightarrow{T(g)} TT(A'') \xrightarrow{m_{A''}} T(A').$$

This is known as the **Kleisli category** of the monad T and the bijection (3.1) shows that $\widetilde{\mathbb{A}}_T$ is isomorphic to the full subcategory of the Eilenberg-Moore category $\mathbb{A}_T$ determined by the objects $T(A)$, $A \in \mathbb{A}$ (free T-modules).

Reversing the arrows in the definitions around monads yields

3.2. Comonads and their comodules. A *comonad* on a category $\mathbb{A}$ is an endofunctor $S : \mathbb{A} \to \mathbb{A}$ with natural transformations, *coproduct* and *counit*,

$$\delta : S \to SS, \quad \varepsilon : S \to 1_{\mathbb{A}},$$

subject to coassociativity and counitality conditions (dual to monad case).

S-comodules are objects $A \in \mathbb{A}$ with a morphism $\omega : A \to S(A)$ subject to coassociativity and counitality conditions.

Morphisms between comodules (A, ω) and (A', ω') (or *S-morphisms*), are morphisms $g : A \to A'$ in $\mathbb{A}$ with commutative diagrams

$$
\begin{array}{ccc}
A & \xrightarrow{\ \ g\ \ } & A' \\
{\scriptstyle \omega}\Big\downarrow & & \Big\downarrow{\scriptstyle \omega'} \\
S(A) & \xrightarrow{\ S(g)\ } & S(A').
\end{array}
$$

The category formed by the S-comodules and their morphisms is called the **Eilenberg-Moore category** - or just the *comodule category* - of the comonad (S, δ, ε) and we denote it by $\mathbb{M}^S$.

For any $A \in \mathbb{A}$, the structure map $\delta_A : S(A) \to SS(A)$ makes $S(A)$ an S-comodule. This yields the *free functor* ϕ^S which is right adjoint to the forgetful functor U^S,

$$\phi^S : \mathbb{A} \to \mathbb{A}^S, \ A \mapsto (S(A), \delta_A),$$
$$U^S : \mathbb{A}^S \to \mathbb{A}, \ (A, \omega) \mapsto A.$$

The adjunction $U^S \dashv \phi^S$ is given by the bijection, $B \in \mathbb{A}^S$, $A' \in \mathbb{A}$,

$$(3.3) \qquad \mathrm{Mor}^S(B, S(A')) \xrightarrow{\cong} \mathrm{Mor}_{\mathbb{A}}(U^S(B), A'), \quad h \mapsto \varepsilon_{A'} \, h.$$

For $B = S(A)$ with $A \in \mathbb{A}$, using $U^S \phi^S = S$, we obtain

$$(3.4) \qquad \mathrm{Mor}^S(S(A), S(A')) \xrightarrow{\cong} \mathrm{Mor}_{\mathbb{A}}(S(A), A').$$

Define a new category $\widetilde{\mathbb{A}}^S$ with the objects of $\mathbb{A}$ but choosing

$$\mathrm{Mor}_{\widetilde{\mathbb{A}}^S}(A, A') := \mathrm{Mor}_{\mathbb{A}}(S(A), A')$$

and take as composition, of $h : S(A) \to A'$ and $k : S(A') \to A''$,

$$k \diamond h : S(A) \xrightarrow{\delta_A} SS(A) \xrightarrow{S(h)} S(A') \xrightarrow{k} A'.$$

This is known as the **Kleisli category** of the comonad S. The bijection (3.4) shows that $\widetilde{\mathbb{A}}^S$ is isomorphic to the full subcategory of the Eilenberg-Moore category $\mathbb{A}^S$ determined by the objects $S(A)$, $A \in \mathbb{A}$ ((co)free S-comodules).

3.3. Rings and modules. For a ring R, any (R, R)-bimodule A defines an endofunctor $A \otimes_R - : {}_R\mathbb{M} \to {}_R\mathbb{M}$. A is called an R-**ring** if this functor allows for a monad structure. If R is commutative and $ra = ar$ for $a \in A$, $r \in R$, an R-ring is called an R-**algebra**.

An (R, R)-bimodule C is called an R-**coring** provided the functor $C \otimes_R -$ allows for a comonad structure. If R is commutative and $rc = cr$ for $c \in C$, $r \in R$, an R-coring is called an R-**coalgebra**.

Notice that not every monad or comonad on ${}_R\mathbb{M}$ can be represented by a tensor functor. For an extensive treatment of corings refer to [3].

4. Adjoints and (co)monads

The notion of adjoints and (co)monads are intimately related. Using naturality of the transformations involved it is straightforward to show:

4.1. From adjoints to (co)monads. Let $F \dashv G : \mathbb{B} \to \mathbb{A}$ be an adjoint pair of functors with unit $\eta : 1_{\mathbb{A}} \to GF$ and counit $\varepsilon : FG \to 1_{\mathbb{B}}$.

(i) $T := GF : \mathbb{A} \to \mathbb{A}$ is an endofunctor and the natural transformations, *product* and *unit*,

$$m : TT = GFGF \xrightarrow{G\varepsilon F} GF = T, \quad \eta : 1_{\mathbb{A}} \to GF = T,$$

make (T, m, η) a **monad** on $\mathbb{A}$.

(ii) $S := FG : \mathbb{B} \to \mathbb{B}$ is an endofunctor and the natural transformations *coproduct* and *counit*,

$$\delta : S = FG \xrightarrow{F\eta G} FGFG = SS, \quad \varepsilon : S = FG \to 1_{\mathbb{B}},$$

make (S, δ, ε) is a **comonad** on $\mathbb{B}$.

Recall that the construction of module and comodule categories in 3.1 and 3.2 show the inverse direction. In fact, these structures were introduced to show that monads as well as comonads can be written as a composition of adjoint functors ([10], [5]).

4.2. From (co)monads to adjoints. *Let $\mathbb{A}$ be any category.*

(i) *For a monad (T, m, η) on $\mathbb{A}$, the category $\mathbb{A}_T$ allows for an adjoint pair of functors $\phi_T \dashv U_T : \mathbb{A}_T \to \mathbb{A}$ with $T = U_T \phi_T$.*

(ii) *For a comonad (S, δ, ε) on $\mathbb{A}$, the category $\mathbb{A}^S$ allows for an adjoint pair of functors $U^S \dashv \phi^S : \mathbb{A} \to \mathbb{A}^S$ with $S = U^S \phi^S$.*

The same assertions hold replacing the Eilenberg-Moore categories by the corresponding Kleisli categories.

The structure of the (co)monads related to an adjunction are strongly influenced by the properties of the unit and counit.

4.3. Separability. Let $F \dashv G : \mathbb{B} \to \mathbb{A}$ be an adjunction as in 4.2 and $T = GF$, $S = FG$ the associated monad and comonad.

(i) Assume G to be a separable functor (see 2.4). Then the monad (T, m, η) allows for a coproduct

$$\delta' : T = GF \xrightarrow{G\varepsilon'F} GFGF = TT$$

with $m \cdot \delta' = 1_T$ and commutative diagrams (Frobenius conditions)

$$(4.1) \qquad \begin{array}{ccc} TT & \xrightarrow{\delta T} & TTT \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle Tm} \\ T & \xrightarrow{\delta} & TT, \end{array} \qquad \begin{array}{ccc} TT & \xrightarrow{T\delta} & TTT \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle mT} \\ T & \xrightarrow{\delta} & TT. \end{array}$$

With these properties, (T, m, η, δ') is (called) a *separable monad*.

(ii) Assume F to be a separable functor (see 2.4). Then the comonad (S, δ, ε) allows for a product

$$m' : SS = FGFG \xrightarrow{F\eta'G} FG = S$$

with $m' \cdot \delta = 1_S$ and commutative diagrams as in (4.1).
$(S, \delta, \varepsilon, m')$ is (called) a *coseparable comonad*.

4.4. Rings and modules. As in 2.8, we consider an (R, S)-bimodule P. The related adjunction yield

$$\text{monad} \quad \text{Hom}_R(P, P \otimes_S -) : {}_S\mathbb{M} \to {}_S\mathbb{M},$$
$$\text{comonad} \quad P \otimes_S \text{Hom}_R(P, -) : {}_R\mathbb{M} \to {}_R\mathbb{M}.$$

Now $P^* := \text{Hom}_R(P, R)$ is an (S, R)-bimodule and hence induces

$$\text{monad} \quad \text{Hom}_S(P^*, P^* \otimes_R -) : {}_R\mathbb{M} \to {}_R\mathbb{M},$$
$$\text{comonad} \quad P^* \otimes_R \text{Hom}_S(P^*, -) : {}_S\mathbb{M} \to {}_S\mathbb{M}.$$

If $_RP$ is finitely generated and projective, $\mathrm{Hom}_R(P,-) \simeq P^* \otimes_R -$ and the comonad $P \otimes_S \mathrm{Hom}_R(P,-) \simeq P \otimes_S P^* \otimes_R -$, that is, $P \otimes_S P^*$ is an R-coring (and $P^* \otimes_R P \simeq S$).

If η is an isomorphism, $P \otimes_S -$ is (left) exact on the image of $\mathrm{Hom}_R(P,-)$ and hence the comonad

$$P \otimes_S \mathrm{Hom}_R(P,-) : {}_R\mathbb{M} \to {}_R\mathbb{M}$$

is a left exact functor commuting with products in $_R\mathbb{M}$. Therefore, by the dual of Watts' theorem (e.g. [8]), it is determined by the image of a cogenerator $Q \in {}_R\mathbb{M}$, that is,

$$
\begin{aligned}
P \otimes_S \mathrm{Hom}_R(P,-) &\simeq \mathrm{Hom}_R(P \otimes_S \mathrm{Hom}_R(P,Q),-) \\
&\simeq \mathrm{Hom}_S(\mathrm{Hom}_R(P,Q), \mathrm{Hom}_R(P,-)).
\end{aligned}
$$

In case R is a cogenerator in $_R\mathbb{M}$, we put $Q = R$ to obtain

$$
\begin{aligned}
P \otimes_S \mathrm{Hom}_R(P,-) &\simeq \mathrm{Hom}_R(P \otimes_S P^*,-) \\
&\simeq \mathrm{Hom}_S(P^*, \mathrm{Hom}_R(P,-)).
\end{aligned}
$$

This is a comonad, thus $P \otimes_S P^* \otimes_R -$ is a monad (by 5.1) which means that $P \otimes_S P^*$ is an R-ring (see 3.3).

If unit η and counit ε are isomorphisms, the comonad $P^* \otimes_R \mathrm{Hom}_S(P^*,-) \simeq P^* \otimes_R P \otimes_S -$, that is, $P^* \otimes_R P$ is an S-coring.

As a special case, we look at a situation studied in representation theory.

4.5. **Nakayama functors.** Let A be a finite dimensional algebra over a field K and A-mod the category of finitely generated left A-modules. It is customary to write $D(-) = \mathrm{Hom}_K(-,K)$, so $A^* = D(A)$, and one gets $\mathrm{Hom}_K(A,-) \simeq D(A) \otimes_K -$. The functor (e.g. [8])

$$\nu = D\,\mathrm{Hom}_A(-,{}_AA) : A\text{-mod} \to A\text{-mod}$$

is called the *Nakayama functor*, and

$$\nu^- = \mathrm{Hom}_A(D(-), A_A) : A\text{-mod} \to A\text{-mod}$$

is said to be the *inverse Nakayama functor*. By the Eilenberg-Watts theorems one gets the natural isomorphisms

$$\nu(-) \simeq D(A) \otimes_A -, \quad \nu^-(-) \simeq \mathrm{Hom}_A(D(A),-),$$

thus there is an adjoint pair of endofunctors to be investigated.

From 2.8 and 4.4 one can see which monad or comonad structures may show up in this setting. The (A-module) structure of $D(A)$ is of course influenced by the structure of A.

5. Adjoint monads and comonads

In 3.2, *comonads* (S, δ, ε) were defined by reversing arrows in the definition of *monads* (T, m, η). This does not mean that the two notions are strictly dual in a categorical way: in our definition we do not have anything like the "dual" of a functor T (or S). In this section we will outline that *adjointness* provides a bijective correspondence between monads and comonads.

5.1. Monads versus comonads *Let $F \dashv G : \mathbb{A} \to \mathbb{A}$ be an adjoint pair of endofunctors with unit $\bar{\eta}$, counit $\bar{\varepsilon}$ and bijection, for $X, Y \in \mathbb{A}$,*

$$\alpha_{X,Y} : \mathrm{Mor}_{\mathbb{A}}(F(X), Y) \to \mathrm{Mor}_{\mathbb{A}}(X, G(Y)),$$

(i) *A monad structure on F induces a comonad structure on G, and vice versa, such that the associated (Eilenberg-Moore) categories $\mathbb{A}_F$ and $\mathbb{A}^G$ are isomorphic.*

(ii) *A comonad structure on F induces a monad structure on G, and vice versa, such that the associated Kleisli categories $\widetilde{\mathbb{A}}_F$ and $\widetilde{\mathbb{A}}^G$ are equivalent*

Proof. (Sketch) (i) Let (F, m, η) be a monad. The adjunction $F \dashv G$ induces the diagram

$$\begin{array}{ccc}
\mathrm{Mor}_{\mathbb{A}}(F(X), Y) & \xrightarrow{\alpha_{X,Y}} & \mathrm{Mor}_{\mathbb{A}}(X, G(Y)) \\
{\scriptstyle \mathrm{Mor}(m_X, Y)}\Big\downarrow & & \Big\downarrow {\scriptstyle \mathrm{Mor}(X, ?)} \\
\mathrm{Mor}_{\mathbb{A}}(FF(X), Y) & \xrightarrow{\simeq} & \mathrm{Mor}_{\mathbb{A}}(X, GG(Y))
\end{array}$$

where the dotted arrow exists by composition of the other maps (α is invertible) and determines a natural transformation $\delta : G \to GG$. A similar argument shows the existence of a counit $\varepsilon : G \to 1_{\mathbb{A}}$. Explicitly we get

$$\delta : G \xrightarrow{\bar{\eta}G} GFG \xrightarrow{G\bar{\eta}FG} GGFFG \xrightarrow{GGmG} GGFG \xrightarrow{GG\bar{\varepsilon}} GG,$$

$$\varepsilon : G \xrightarrow{\eta G} FG \xrightarrow{\bar{\varepsilon}} 1_{\mathbb{A}}.$$

The symmetric construction shows that a comonad structure on G leads to a monad structure on F. The equivalence of categories is given by

$$\mathbb{A}_F \to \mathbb{A}^G, \quad F(A) \xrightarrow{h} A \mapsto A \xrightarrow{\bar{\eta}_A} GF(A) \xrightarrow{G(h)} G(A),$$

$$\mathbb{A}^G \to \mathbb{A}_F, \quad A \xrightarrow{\varrho} G(A) \mapsto F(A) \xrightarrow{F(\varrho)} FG(A) \xrightarrow{\bar{\varepsilon}_A} A.$$

(ii) By arguments symmetric to those in (i), a comodule structure on F induces a module structure on G, and vice versa. The equivalence of Kleisli categories $\widetilde{\mathbb{A}}^F$ and $\widetilde{\mathbb{A}}_G$ follows from the isomorphisms

$$\begin{array}{rl}
\mathrm{Mor}_{\widetilde{\mathbb{A}}^F}(\phi^F(A), \phi^F(A')) & \stackrel{(1)}{\simeq} \quad \mathrm{Mor}_{\mathbb{A}}(F(A), A') \\
& \stackrel{(2)}{\simeq} \quad \mathrm{Mor}_{\mathbb{A}}(A, G(A')) \\
& \stackrel{(3)}{\simeq} \quad \mathrm{Mor}_{\widetilde{\mathbb{A}}_G}(\phi_G(A), \phi_G(A')),
\end{array}$$

where (1) is the isomorphism from (3.4), (2) is the adjunction, and (3) is the isomorphism from (3.2). $\qquad\square$

5.2. Frobenius monads. Let $F : \mathbb{A} \to \mathbb{A}$ be a self-adjoint endofunctor, that is, $F \dashv F$. By subsection 5.1, there are equivalent:

(a) F *has a monad structure* (F, m, η);

(b) F *has a comonad structure* (F, δ, ε);

With this (mutually) induced structures, (F, m, δ) satisfies the Frobenius conditions (4.1) and the categories $\mathbb{A}_F$ and $\mathbb{A}^F$ are isomorphic. These data define a *Frobenius monad*. ([17], [24])

5.3. Rings and modules. Let A be an (R, R)-bimodule and consider the adjoint functor pair of endofunctors (as in 2.8),

$$A \otimes_R -, \ \mathrm{Hom}_R(A, -) : {}_R\mathbb{M} \to {}_R\mathbb{M}.$$

(i) An R-ring (A, m, η), that is, a monad $A \otimes_R -$ on ${}_R\mathbb{M}$, induces a comonad structure on $\mathrm{Hom}_R(A, -)$ with coproduct

$$\mathrm{Hom}_R(A, -) \xrightarrow{\mathrm{Hom}(m, -)} \ \mathrm{Hom}_R(A \otimes_R A, -)$$
$$\simeq \mathrm{Hom}_R(A, \mathrm{Hom}_R(A, -)),$$

and the module category $\mathbb{M}_A$ is isomorphic to the comodule category $\mathbb{M}^{\mathrm{Hom}(A, -)}$.

(ii) A coring (A, δ, ε), that is, a comonad $A \otimes_R -$ on ${}_R\mathbb{M}$, induces a monad structure on $\mathrm{Hom}_R(A, -)$ with product

$$\mathrm{Hom}_R(A, \mathrm{Hom}_R(A, -)) \simeq$$
$$\mathrm{Hom}_R(A \otimes_R A, -) \ \xrightarrow{\mathrm{Hom}(\delta, -)} \mathrm{Hom}_R(A, -),$$

and the *Kleisli categories* are isomorphic by assigning, for $X \in {}_R\mathbb{M}$,

$$\widetilde{\mathbb{M}}^A \to \widetilde{\mathbb{M}}_{\mathrm{Hom}(A, -)}, \quad A \otimes_R X \mapsto \mathrm{Hom}_R(A, X).$$

Note that the objects of $\mathbb{M}_{\mathrm{Hom}(A, -)}$ are also called *contramodules* (e.g. [2, Section 4], [32]).

The R-ring (A, m, η) is called a **Frobenius R-ring** provided $A \otimes_R -$ is a Frobenius monad on ${}_R\mathbb{M}$. This implies that ${}_R A$ is finitely generated and projective and hence $\mathrm{Hom}_R(A, -) \simeq A^* \otimes_R -$, thus $A \simeq A^*$ as left A-modules and $\mathbb{M}_A \simeq \mathbb{M}^A$. For commutative rings R one obtains **Frobenius algebras**.

6. COMPOSITION OF MONADS AND COMONADS

Over a commutative ring R, the tensorproduct of two R-algebras (A, m, e), (B, m', e') can be made an algebra defining a product (writing $\otimes := \otimes_R$)

$$\overline{m}_{AB} : A \otimes B \otimes A \otimes B \xrightarrow{A \otimes \mathrm{tw} \otimes B} A \otimes A \otimes B \otimes B \xrightarrow{m \otimes m'} A \otimes B,$$

with the twist map $\mathrm{tw} : B \otimes A \to A \otimes B, \ b \otimes a \mapsto a \otimes b$.

This map is no longer available if R is not commutative. On the other hand, the product is formally defined replacing tw by any morphism $\tau :$ $B \otimes A \to A \otimes B$, however, it is not clear which properties it has. So one may ask which conditions are to be satisfied by τ to make the product $\overline{m}_{AB}$ associative. These lead to commutativity of diagrams of the form

$$
\begin{array}{ccc}
B \otimes B \otimes A & \xrightarrow{\;\;m' \otimes A\;\;} & B \otimes A \\
{\scriptstyle B \otimes \tau}\downarrow & & \downarrow{\scriptstyle \tau} \\
B \otimes A \otimes B \xrightarrow{\tau \otimes B} A \otimes B \otimes B \xrightarrow{A \otimes m'} A \otimes B,
\end{array}
\qquad
\begin{array}{ccc}
A & \xrightarrow{\;e' \otimes A\;} & B \otimes A \\
& {\scriptstyle A \otimes e'}\searrow & \downarrow{\scriptstyle \tau} \\
& & A \otimes B,
\end{array}
$$

and should induce commutative diagrams of functors, with the forgetful functor $U_B : {}_B\mathbb{M} \to {}_R\mathbb{M}$,

$$
\begin{array}{ccc}
{}_B\mathbb{M} & \xrightarrow{\;A \otimes B \otimes -\;} & {}_B\mathbb{M} \\
{\scriptstyle U_B}\downarrow & & \downarrow{\scriptstyle U_B} \\
{}_R\mathbb{M} & \xrightarrow{\;A \otimes_R -\;} & {}_R\mathbb{M}.
\end{array}
$$

The functor $A \otimes B \otimes -$ is called a lifting of $A \otimes -$ from ${}_R\mathbb{M}$ to ${}_B\mathbb{M}$.

The questions considered above for the functors $A \otimes -$ and $B \otimes -$ can be asked for endofunctors in any category.

6.1. **Liftings of endofunctors.** Let $\mathbb{A}$ be any category with endofunctors $F, G : \mathbb{A} \to \mathbb{A}$. If (F, m, e) is a monad or (G, δ, ε) is a comonad, we get the two diagrams, respectively,

$$
\begin{array}{ccc}
\mathbb{A}_F & \dashrightarrow^{\;\overline{G}\;} & \mathbb{A}_F \\
{\scriptstyle U_F}\downarrow & & \downarrow{\scriptstyle U_F} \\
\mathbb{A} & \xrightarrow{\;G\;} & \mathbb{A},
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{A}^G & \dashrightarrow^{\;\overline{F}\;} & \mathbb{A}^G \\
{\scriptstyle U^G}\downarrow & & \downarrow{\scriptstyle U^G} \\
\mathbb{A} & \xrightarrow{\;F\;} & \mathbb{A}.
\end{array}
$$

If $\overline{G}$ ($\overline{F}$) exists it is called a lifting of G (F) from $\mathbb{A}$ to $\mathbb{A}_F$ ($\mathbb{A}^G$). The following questions come up:

(i) when does a lifting $\overline{G}$ or $\overline{F}$ exist ?

(ii) if F and G are monads, when is $\overline{G}$ a monad ?

(iii) if F and G are comonads, when is $\overline{F}$ a monad ?

(iv) if F is a monad and G is a comonad, when is $\overline{G}$ a comonad, when is $\overline{F}$ a monad ?

All these problems can successfully be handled applying *distributive laws* as introduced and investigated in the 1970's by J. Beck [1] and others (see [31] for an overview and [2], [13], [15], et al. for more details).

Question (ii) above will lead to conditions which make the composition FG a monad on $\mathbb{A}$ (as considered for algebras above) and (iii) describes the corresponding properties of comonads. Of particular interest are the questions in (iv) since they reveal an interesting interplay between monads and comonads.

6.2. Mixed distributive laws. Let (F, m, η) be a monad and (G, δ, ε) a comonad on a category $\mathbb{A}$. A natural transformations $\lambda : FG \to GF$ is called a *mixed distributive law* or *(mixed) entwining* if it induces commutativity of the diagrams

$$
\begin{array}{ccc}
FFG & \xrightarrow{\ m_G\ } & FG \\
{\scriptstyle F\lambda}\downarrow & & \downarrow{\scriptstyle \lambda} \\
FGF \xrightarrow{\lambda_F} GFF \xrightarrow{Gm} & & GF,
\end{array}
\qquad
\begin{array}{ccc}
FG \xrightarrow{F\delta} FGG \xrightarrow{\lambda_G} GFG \\
{\scriptstyle \lambda}\downarrow \qquad\qquad\qquad \downarrow{\scriptstyle G\lambda} \\
GF \xrightarrow{\qquad \delta_F \qquad} GGF,
\end{array}
$$

$$
\begin{array}{ccc}
G & \xrightarrow{\ \eta_G\ } & FG \\
& {\scriptstyle G\eta}\searrow & \downarrow{\scriptstyle \lambda} \\
& & GF,
\end{array}
\qquad
\begin{array}{ccc}
FG & \xrightarrow{\ F\varepsilon\ } & F \\
{\scriptstyle \lambda}\downarrow & \nearrow{\scriptstyle \varepsilon_F} & \\
GF. & &
\end{array}
$$

The following are equivalent (with notation from 6.1):

(a) *G can be lifted from $\mathbb{A}$ to $\overline{G} : \mathbb{A}_F \to \mathbb{A}_F$;*

(b) *F can be lifted from $\mathbb{A}$ to $\overline{F} : \mathbb{A}^G \to \mathbb{A}^G$;*

(c) *there exists a mixed distributive law $\lambda : FG \to GF$.*

Given a monad and a comonad on $\mathbb{A}$, objects can have a module and a comodule structure. An entwining allows to require a compatibility of these structures.

6.3. Mixed modules. Let (F, G, λ) be a mixed entwining and assume $A \in \mathbb{A}$ to be an F-module $\varrho : F(A) \to A$ and a G-comodule $\omega : A \to G(A)$. Then (A, ϱ, ω) is called a *mixed (F, G)-module* if we get commutativity of the diagram

$$
\begin{array}{ccccc}
F(A) & \xrightarrow{\ \varrho\ } & A & \xrightarrow{\ \omega\ } & G(A) \\
{\scriptstyle F(\omega)}\downarrow & & & & \uparrow{\scriptstyle (\varrho)} \\
FG(A) & & \xrightarrow{\qquad \lambda_A \qquad} & & GF(A).
\end{array}
$$

These objects with morphisms that are module as well as comodule morphisms, form a category which we denote by $\mathbb{A}_F^G$.

In 6.1, the lifting to Eilenberg-Moore categories was considered. A corresponding construction for Kleisli categories is the following.

6.4. Extending of endofunctors. Let $\mathbb{A}$ be any category with endofunctors $F, G : \mathbb{A} \to \mathbb{A}$. If (F, m, e) is a monad or (G, δ, ε) is a comonad, we get the two diagrams (with ϕ the free functors), respectively,

$$
\begin{array}{ccc}
\mathbb{A} & \xrightarrow{\ G\ } & \mathbb{A} \\
{\scriptstyle \phi_F}\downarrow & & \downarrow{\scriptstyle \phi_F} \\
\widetilde{\mathbb{A}}_F & \dashrightarrow{\widetilde{G}} & \widetilde{\mathbb{A}}_F,
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{A} & \xrightarrow{\ F\ } & \mathbb{A} \\
{\scriptstyle \phi^G}\downarrow & & \downarrow{\scriptstyle \phi^G} \\
\widetilde{\mathbb{A}}^G & \dashrightarrow{\widetilde{F}} & \widetilde{\mathbb{A}}^G.
\end{array}
$$

$\widetilde{G}$ and $\widetilde{F}$ are called the *extensions* of F and G, respectively. Here again distributive laws apply for further investigation but with the role of monad and comonad interchanged, that is, one needs natural transformations $\sigma :$ $GF \to FG$ inducing commutativity of the corresponding diagrams. Notice that the role of mixed modules (as in 6.3) does not transfer to this situation.

7. Bimonad and Hopf monads

Of special interest are endofunctors B on $\mathbb{A}$ which carry a monad structure (B, m, η) and a comonad structure (B, δ, ε) at the same time. To make these data a **bimonad** we first require the existence of a mixed distributive law $\lambda : BB \to BB$.

The relevant commutative diagrams are either for the monad structure or else for the comonad structure - they do not relate product with coproduct, for example. To connect these we need further compatibility conditions, namely commutativity of the diagrams

$$(7.1) \qquad \begin{array}{ccccc} BB & \xrightarrow{m} & B & \xrightarrow{\delta} & BB \\ {\scriptstyle B\delta}\downarrow & & & & \uparrow{\scriptstyle Bm} \\ BBB & & \xrightarrow{\lambda_B} & & BBB, \end{array}$$

$$(7.2) \qquad \begin{array}{ccc} BB \xrightarrow{B\varepsilon} B & 1_{\mathbb{A}} \xrightarrow{\eta} B & 1_{\mathbb{A}} \xrightarrow{\eta} B \\ {\scriptstyle m}\downarrow \quad \downarrow{\scriptstyle \varepsilon} & {\scriptstyle \eta}\downarrow \quad \downarrow{\scriptstyle B\eta} & {\scriptstyle =}\searrow \quad \downarrow{\scriptstyle \varepsilon} \\ B \xrightarrow{\varepsilon} 1_{\mathbb{A}}, & B \xrightarrow{\delta} BB, & \qquad 1_{\mathbb{A}}, \end{array}$$

where the bottom diagrams mean that η is comonad morphism and ε a monad morphism. Diagram (7.1) guarantees that for each $A \in \mathbb{A}$, $B(A)$ is a mixed (B, B)-module in the sense of 6.3, that is, $B(A) \in \mathbb{A}_B^B$. This determines a functor

$$\phi_B^B : \mathbb{A} \to \mathbb{A}_B^B, \quad A \mapsto BB(A) \xrightarrow{m_A} B(A) \xrightarrow{\delta_A} BB(A).$$

which is full and faithful by the isomorphisms (see 3.4, 3.1)

$$\mathrm{Mor}_B^B(B(A), B(A')) \simeq \mathrm{Mor}_B(B(A), A') \simeq \mathrm{Mor}_{\mathbb{A}}(A, A')$$

A natural transformation $S : B \to B$ is called an **antipode** if it induces commutativity of the diagram

$$(7.3) \qquad \begin{array}{ccccc} B & \xrightarrow{\varepsilon} & 1_{\mathbb{A}} & \xrightarrow{\eta} & B \\ {\scriptstyle \delta}\downarrow & & & & \uparrow{\scriptstyle m} \\ BB & & \underset{BS}{\overset{S_B}{\rightrightarrows}} & & BB \end{array} \quad .$$

A bimonad which allows for an antipode is called a **Hopf monad**. Such an antipode exists if and only if either of the composits

$$BB(A) \xrightarrow{B\delta} BBB \xrightarrow{mB} BB, \quad BB(A) \xrightarrow{\delta B} BBB \xrightarrow{Bm} BB$$

is an isomorphism ([14, 5.5]).

Under certain conditions on the categry $\mathbb{A}$ and the functor B, the existence of an *antipode* S is equivalent to $\phi_B^B : \mathbb{A} \to \mathbb{A}_B^B$ defining an equivalence ([14, 5.6, 6.11]).

7.1. Adjoints of bimonads.

Let (B, m, δ, λ) be a bimonad on $\mathbb{A}$ (as above). Assume B allows for a right adjoint functor $C : \mathbb{A} \to \mathbb{A}$. Then by 5.1, C can be endowed with a comonad as well as a monad structure. Furthermore, there is a mixed distributive law $\lambda' : CC \to CC$ (derived from λ, see [14, 7.4]) which makes C a bimonad.

The bimonad B has an antipode if and only if the associated bimonad C has an antipode. Thus, given an adjoint pair $B \dashv C$ of functors on $\mathbb{A}$, B allows for a Hopf monad structure if and only if so does C.

7.2. Bimonad on SET.

([31, 5.19], [14, 7.9]) For any set G, the cartesian product defines an endofunctor

$$G \times - : \text{SET} \to \text{SET}, \quad A \mapsto G \times A,$$

which is a *comonad* with coproduct $\delta : G \to G \times G$, $g \mapsto (g, g)$, and is a *monad* provided G is a monoid. Then it is a *bimonad* with entwining

$$\lambda : G \times G \to G \times G, \quad (g, h) \mapsto (gh, g).$$

Now $G \times -$ has an antipode, i.e., is a Hopf monad, if and only if the monoid G is in fact a group. By 7.1, this is also equivalent to $\text{Map}(G, -) : \text{SET} \to \text{SET}$, a right adjoint of $G \times -$, being a Hopf monad.

7.3. Bialgebras.

Let R be a commutative ring. An R-module B with an algebra structure (B, m, η) and a coalgebra structure (B, δ, ε) is called a *bialgebra* if δ and ε are algebra morphisms, or, equivalently, m and η are coalgebra morphisms. These conditions require commutativity of the outer path in the diagram

$$
\begin{array}{ccccc}
B \otimes B & \xrightarrow{\;m\;} & B & \xrightarrow{\;\delta\;} & B \otimes B \\
{\scriptstyle B \otimes \delta}\downarrow & & & & \uparrow{\scriptstyle B \otimes m} \\
B \otimes B \otimes B & \dashrightarrow{\;\omega \otimes B\;} & & \dashrightarrow & B \otimes B \otimes B \\
{\scriptstyle \delta \otimes B \otimes B}\downarrow & & & & \uparrow{\scriptstyle m \otimes B \otimes B} \\
B \otimes B \otimes B \otimes B & \xrightarrow{\;B \otimes \text{tw} \otimes B\;} & & & B \otimes B \otimes B \otimes B.
\end{array}
$$

Defining an R-linear map

$$\omega : B \otimes B \xrightarrow{\;\delta \otimes B\;} B \otimes B \otimes B \xrightarrow{\;B \otimes \text{tw}\;} B \otimes B \otimes B \xrightarrow{\;m \otimes B\;} B \otimes B,$$

the condition reduces to commutativity of the upper rectangle. As readily checked, $\omega : B \otimes B \to B \otimes B$ is a mixed distributive law between the monad $B \otimes -$ and the comonad $B \otimes -$. Commutativity of the upper rectangle is just the compatibiity condition which makes B a bialgebra. Thus a bialgebra B corresponds to a monad $B \otimes -$ with the associated monad and comonad

structures and the specific mixed distributive law ω. The mixed (B, B)-modules $\mathbb{M}_B^B$ are called **Hopf modules**.

The composite

$$\overline{\omega} : B \otimes B \xrightarrow{B \otimes \delta} B \otimes B \otimes B \xrightarrow{\mathrm{tw} \otimes B} B \otimes B \xrightarrow{B \otimes m} B \otimes B$$

yields a similar rectangle (sides interchanged). It gives a mixed distributive law between the related monad $- \otimes B$ and the comonad $- \otimes B$. It may also be considered as a mixed distributive law between the comonad $B \otimes -$ and the monad $B \otimes -$ (see 6.4).

An **antipode** for the bialgebra B is an R-linear map $S : B \to B$ leading to commutativity of the diagram (7.3) for the monad $B \otimes -$.

Bialgebras B with an antipode are called **Hopf algebras** and they are characterised by the fact that $\phi_B^B : \mathbb{M}_R \to \mathbb{M}_B^B$ is an equivalence, that is, every Hopf module is of the form $B \otimes X$, for some $X \in \mathbb{M}_R$ (**Fundamental Theorem**).

In case B is a finite dimensional algebra over a field K, $B \otimes_K -$ is left adjoint to the endofunctor $\mathrm{Hom}_K(B, -) \simeq B^* \otimes_K -$, where $B^* = \mathrm{Hom}_K(B, K)$. Hence B is a Hopf algebra if and only if B^* is a Hopf algebra (see 7.1). This is known as the *duality principle* of Hopf algebras.

Hopf algebras were brought to light by Heinz Hopf in his seminal paper [9] in topology (1941). The algebraic essentials of this notion were extracted by Milnor-Moore in [19] (1965). As a result, interest was also directed to the more elementary notions of coalgebras and corings as building blocks for the theory. It took several years until their value for representation theory was unveiled (1980) by A.V. Roiter in [21], there a *coring* is called *bocs*, and the interest is focused on Kleisli categories. In the meantime more attention is paid to coalgebraic aspects of finite dimensional algebras. For this we refer to the recent paper [12] by R. Marczinzik and the references given there.

7.4. Remark. As pointed out in subsection 7.3, the mixed distributive law for a bialgebra was derived from the canonical twist of the tensor product of modules over a commutative ring. This setting was extended to *monoidal braided categories*, where a general version of such a twist map is required. Concentrating on the properties needed, for endofunctors B on any category it is sufficient to define a *local braiding* $\tau : BB \to BB$ to derive the corresponding theory. For details we refer to [14, Section 6] and [16].

7.5. Rings and modules. The preceding sections show that the *coproduct* plays an important role in the structure theory of algebras. Summarising we consider an algebra (A, m, η) over a commutative ring R. Assume A allows for a coproduct $\delta : A \to A \otimes A$. Then A becomes

(1) a *separable algebra* if (A, m, δ) satisfies the Frobenius condition and $m \cdot \delta = 1_A$;

(2) an *Azumaya algebra* if it is separable and $R \simeq C(A)$ (center of A); the categroy of $C(A)$-modules is equivalent to the category ${}_A\mathbb{M}_A$ of (A, A)-bimodules;

(3) a *Frobenius algebra* if (A, m, δ) satisfies the Frobenius condition and (A, δ) has a counit $\varepsilon : A \to R$; every A-module has an A-comodule struture (Frobenius (bi)module, see [35]);

(4) a *Hopf algebra* if (A, m, δ) induce commutativity of (7.1) and (A, δ) has a counit $\varepsilon : A \to R$; the category of R-modules is equivalent to the category $\mathbb{M}_A^A$ of mixed (A, A)-bimodules (e.g. [3]).

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