Modules and Algebras

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Bimodule Structure and

Group Actions on Algebras

Robert Wisbauer
University of Düsseldorf

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Contents

Preface

In the theory of commutative associative algebras A, three of the most important techniques are:

(i) the homological characterization of A in the category of A-modules,

(ii) forming the ring of quotients for prime or semiprime A and

(iii) localization at prime ideals of A.

While (i) can be successfully transferred to non-commutative associative algebras A using left (or right) A-modules, techniques (ii) and (iii) do not allow a satifactory extension to the non-commutative setting. However, some of the basic results in (ii) and (iii) remain true if the algebras considered are reasonably close to commutative rings. This closeness is attained by rings with *polynomial identities (PI-rings)* and such rings have been studied in great detail by many authors. They behave like commutative rings, e.g., in a semiprime PI-ring the non-zero ideals intersect with the centre non-trivially. In particular, the maximal quotient ring of a prime PI-algebra is obtained by central localization.

One of the reasons that one-sided module theory cannot imitate the commutative case completely is that one-sided ideals are no longer kernels of ring homomorphisms. To remedy this, one might consider the R-algebra A as an (A, A)-bimodule which is tantamount to studying the structure of A as an $A \otimes_R A^o$-module (where A^o denotes the opposite ring). Obviously the $A \otimes_R A^o$-submodules of A are precisely the ideals in A, i.e., the kernels of ring homomorphisms, giving us an analogue of the commutative case. However, in general A is neither projective nor a generator in the category $A \otimes_R A^o$-*Mod* of $A \otimes_R A^o$-modules and hence homological characterizations of A, which prove to be so useful in one-sided module theory, are not possible using this technique. Indeed, the case where A is projective in $A \otimes_R A^o$-*Mod* is of special interest. Such algebras are called *separable* R-algebras (see [58]). Central R-algebras of this type are in fact generators in $A \otimes_R A^o$-*Mod* and are named *Azumaya algebras.* Consequently, only the bimodule structure of both separable and Azumaya algebras has been the focus of much attention.

Even more serious problems occur in the attempt to make module theory accessible to non-associative algebras. By definition, a basic property of bimodules M over an associative algebra A is the associativity condition:

$$a(mb) = (am)b, \text{ for all } m \in M, \ a, b \in A.$$

This no longer makes sense for non-associative A (since A itself does not satisfy this condition). Following a suggestion of Eilenberg (in [120]) one might try (and many authors have) to replace this identity by suitable identities characterizing the variety of algebras under consideration. For example, a bimodule M over an alternative algebra A should satisfy the conditions (with $(-,-,-)$ denoting the associator):

$$(a,m,b) = -(m,a,b) = (b,a,m) = -(b,m,a), \text{ for all } m \in M,\ a,b \in A.$$

These modules are in fact the modules over the *universal enveloping algebra* over A (and thus depend on the identities of A). For this type of investigation we refer to [166] and [41]. Looking for a more effective module theory for non-associative algebras, J.M. Osborn writes in the introduction to [215]: *One of the most disconserting characteristics of the Eilenberg theory when applied to a particular variety is that the worst-behaved bimodules occur over rings that are the best-behaved in the variety. ... It is our feeling that the module theory used to obtain structure theory ought to be independent of which variety the ring is thought of belonging to.*

One of the (artificial) handicaps in looking for modules for non-associative algebras was the desire to get a full module category, i.e., a Grothendieck category with a finitely generated projective generator. This requirement turns out to be too restrictive and – for many purposes – superfluous. Indeed, it was already known from Gabriel's fundamental paper ([140], 1962) that most of the localization techniques are available in any Grothendieck category (since it has enough injectives). Consequently, we may meet our localization needs by finding a Grothendieck category related to A which does not necessarily have a projective generator. On the other hand, for a homological characterization of an algebra A it is essential that the objects in the category used are closely related to A. Both objectives are met in the following construction.

For any R-algebra A we consider the multiplication algebra $M(A)$, i.e., the R-subalgebra of $End_R(A)$ generated by left and right multiplications by elements of A and the identity map of A. Then A is a left module over the unital associative algebra $M(A)$ and we denote by $\sigma[A]$ the smallest full Grothendieck subcategory of $M(A)$-*Mod* containing A. The objects of $\sigma[A]$ are just the $M(A)$-modules which are submodules of A-generated modules. This category is close enough to A to reflect (internal) properties of A (see (i)) and rich enough for the constructions necessary for (ii) and (iii). Moreover, the construction is independent of the variety to which A belongs. (This will be studied in detail in the second part of this monograph.) In particular it extends the study of bimodules over associative algebras by M. Artin [57] and Delale [115] to arbitrary algebras.

It is easy to see that for A finitely generated as an R-module, $\sigma[A] = M(A)$-*Mod*. Moreover, if A is associative and commutative with unit then $\sigma[A] = A$-*Mod*. Hence $\sigma[A]$ generalizes the module theory over associative commutative rings. To measure how close an algebra A is to an associative commutative algebra, we consider its behaviour as an $M(A)$-module instead of the aforementioned polynomial identities. For example, we may ask if $\sigma[A] = M(A)$-*Mod*, or if $Hom_{M(A)}(A,U) \neq 0$ for non-zero ideals $U \subset A$. The conditions which determine when these occur do not depend on associativity. For associative (and some non-associative) prime algebras they are conveniently equivalent to A satisfying a polynomial identity.

The use of the multiplication algebra to investigate (bimodule) properties of any algebra is by no means new. A footnote in Albert [42] says: *The idea of studying these relations was suggested to both Jacobson and the author (Albert) by the lectures of H. Weyl on Lie Algebras which were given in Fine Hall in 1933.* One of the early results in this context is the observation that a finite dimensional algebra A over a field is a direct sum of simple algebras if and only if the same is true for $M(A)$ (see Jacobson [165], 1937; Albert [43], 1942). Later, in Müller [208] separable algebras A over a ring R were defined by the R-separability of $M(A)$, provided A is finitely generated and projective as an R-module. A wider and more effective application of module theory to the structure theory of algebras was made possible through the introduction of the category $\sigma[A]$. In particular, in this case A need not be finitely generated as an R-module (which would imply $\sigma[A] = M(A)$-*Mod*).

The category $\sigma[A]$ is a special case of the following more general situation. Let M be any module over any associative ring A and denote by $\sigma[M]$ the smallest full subcategory of A-*Mod* which is a Grothendieck category. Its objects are just the submodules of M-generated modules. Clearly, for $M = A$ we have $\sigma[M] = A$-*Mod*.

Many results and constructions in A-*Mod* can be transferred to $\sigma[M]$ and this is done in [11, 40]. In the first part (Chaps. 2, 3) of this monograph we will recall some of these results and introduce new ones which are of particular interest for the applications we have in mind. For example, we consider generating and projectivity properties of M in $\sigma[M]$ and also special torsion theories in $\sigma[M]$.

The first application is – as mentioned above – the investigation of any algebra A as a module over its multiplication $M(A)$.

For another application we will use our setting to study the action of a group G on any algebra A. In the case A is unital and associative we may consider A as a left module over the skew group algebra $A'G$. The endomorphism ring of this module is the fixed ring A^G and applying our techniques we obtain relations between properties of ${}_{A'G}A$ and A^G.

For arbitrary A with unit we observe that the action of G on A can be extended to an action on $M(A)$. This allows us to consider A as a module over the skew group algebra $M(A)'G$. The endomorphism ring of this module consists of the fixed elements of the centroid.

Introduction

As pointed out in the preface, the structure of algebras can be studied by methods of associative module theory. The purpose of this monograph is to give an up-to-date account of this theory. We begin in Chapter 1 by presenting those topics of the associative theory which are of relevance to the bimodule structure of algebras.

In Chapter 2 we collect results on modules M over associative algebras A and the related category $\sigma[M]$, a full subcategory of the category of all left A-modules whose objects are submodules of M-generated modules. In addition to information taken from the monographs [40] and [11] new notions are introduced which will be helpful later on. This leads on to Chapter 3 where we outline the localization theory in $\sigma[M]$.

If A is an associative algebra over an associative, commutative ring R, every left A-module M is also an R-module and there is an interplay between the properties of the A-module M and the R-module M. In particular the tensor product of A-modules can be formed over R. This facilitates the study of localization of A and M with respect to multiplicative subsets of R. These techniques are considered in Chapter 4. Then they are applied in Chapter 5 to obtain local-global characterizations of various module properties.

Next, in what follows, A will be a not necessarily associative R-algebra. In Chapter 6 some radicals for such algebras are considered. In Chapters 7 to 9 the module theory presented earlier will be applied to the subcategory $\sigma[A]$ of $M(A)$-*Mod*, where $M(A)$ denotes the multiplication algebra of A. Generating and projectivity properties of A as an $M(A)$-module are considered and *Azumaya rings* are characterized as projective generators in $\sigma[A]$, whereas *Azumaya algebras* are (projective) generators in $M(A)$-*Mod*. The effectiveness of localization in $\sigma[A]$ for semiprime algebras A is based on the observation that such algebras are *non-A-singular* as $M(A)$-modules. This yields in particular an interpretation of Martindale's *central closure* of A as the injective envelope of A in $\sigma[A]$.

In Chapter 10 the module theoretic results are used to study the action of groups on any algebra A. If A is associative and unital we consider A as an algebra over the skew group ring $A'G$. In particular we ask when ${}_{A'G}A$ is a self-generator or when it is self-projective and what are the properties of the fixed ring A^G (which is isomorphic to the endomorphism ring of ${}_{A'G}A$). This generalizes the case when A is a projective generator in $A'G$-*Mod*, a property which has attracted interest in connection with Galois theory for rings.

For arbitrary A with unit we use the fact that the action of G on A induces an action on the multiplication algebra $M(A)$. Hence we have A as module over the skew group algebra $M(A)'G$ whose endomorphism ring consists of the fixed elements of the centre. As a counterpart to the central closure of prime rings, we obtain a central quotient ring for G-semiprime algebras.

Throughout the monograph R will denote a commutative associative ring with unit. A will be any R-algebra which in some chapters is assumed to be associative or to have a unit. For the general properties of $\sigma[A]$ associativity of A is of no importance. However, any (polynomial) identity on A may imply special properties of A as an $M(A)$-module.

For the readers convenience, each section begins with a listing of its paragraph titles. Most of the sections are ended by exercises which are intended to point out further relationships and to draw attention to related results in the literature.

I wish to express my sincere thanks to all the colleagues and friends who helped to write this book. In particular I want to mention Toma Albu, John Clark, Maria José Arroyo Paniagua, José Ríos Montes and my students for their interest and the careful reading of the manuscript. Several interesting results around semiprime rings were obtained in cooperation with Kostja Beidar and Miguel Ferrero. Moreover I am most indebted to Bernd Wilke for the present form of Chapter 10 and to Vladislav Kharchenko for many helpful comments.

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Robert Wisbauer

Notation

R	commutative associative ring with unit
$\mathcal{M}$	set of maximal ideals of R
$B(R)$	Boolean ring of idempotents of R
$\mathcal{X}$	set of maximal ideals of $B(R)$
A	associative or non-associative R-algebra
$M(A)$ / $M^*(A)$	multiplication algebra / ideal of A
$C(A)$ / $Z(A)$	centroid / centre of A
$Z_A(M)$	centre of an $M(A)$-module M
$\sigma[A]$	subcategory of $M(A)$-*Mod* subgenerated by A
$\widehat{A}$	injective hull of A in $\sigma[A]$
$BMc\,A$	Brown-McCoy radical of A
$Jac\,A$	Jacobson radical of A
$Q_{\max}(A)$	maximal left quotient ring of A
$A[G]$	group algebra of a group G over A
$A'G$	skew group algebra of a group G acting on A
A^G	fixed ring of the group G acting on A
tr_G	trace map $A \to A^G$
acc / dcc	ascending / descending chain condition
$E(N)$	injective hull of a module N
$\sigma[M]$	subcategory of A-*Mod* subgenerated by a module M
$\widehat{N}$, $I_M(N)$	M-injective hull of a module $N \in \sigma[M]$
$Im\,f$ / $Ke\,f$	image / kernel of a map f
$Tr(U, L)$	trace of a module U in L
$An_A(M)$	annihilator of an A-module M
$K \subset M$	K is a subset/submodule or equal to M
$K \trianglelefteq M$	K is an essential submodule of M
$K \ll M$	K is a superfluous submodule of M
$Soc\,M$	socle of M
$Rad\,M$	radical of M
M_x	Pierce stalks of a module M
$\varinjlim M_i$	direct limit of modules M_i
$\mathcal{S}_M$ / $\mathcal{S}$	class of singular modules in $\sigma[M]$ / in A-*Mod*
$\mathcal{S}^2_M$ / $\mathcal{S}^2$	Goldie torsion class in $\sigma[M]$ / in A-*Mod*

Chapter 1

Basic notions

1 Algebras

1.Algebra. 2.Algebra morphism. 3.Category of R-algebras. 4.Products of algebras. 5.Ideals. 6.Properties of ideals. 7.Factorization Theorem. 8.Isomorphism Theorems. 9.Embedding into algebras with units. 10.A as (subdirect) product of rings. 11.Nucleus and centre.

R will always denote an associative and commutative ring with unit 1. For module theory over associative rings we mainly refer to [40]. The basic facts can also be found in Anderson-Fuller [1], Kasch [21], Faith [12, 13], Pierce [32], Bourbaki [6].

1.1 Algebra. Definition.

A unital R-module A with an R-bilinear map $\mu : A \times A \to A$ is called an *R-algebra*. μ can also be described by an R-linear map $\mu' : A \otimes_R A \to A$ and is said to be the *multiplication on A*.

An R-submodule $B \subset A$ is a *subalgebra* if $\mu(B \times B) \subset B$.

Writing $\mu(a,b) = ab$ for $a, b \in A$, we deduce from the above definition the following rules for all $a, b, c \in A, r \in R$:

$$\begin{aligned} a(b+c) &= ab + ac, \\ (b+c)a &= ba + ca, \\ r(ab) &= (ra)b = a(rb). \end{aligned}$$

These conditions are in fact sufficient for a map $A \times A \to A, (a,b) \mapsto ab$, to be R-bilinear and hence may be used for the definition of an R-algebra.

A is called *associative* if $(ab)c = a(bc)$ for all $a, b, c \in A$.

Algebras over $R = \mathbb{Z}$ are called (non-associative) *rings*. In fact any R-algebra is a $\mathbb{Z}$-algebra, i.e., a ring.

An element $e \in A$ is called a *left (right) unit* if $ea = a$ ($ae = a$) for all $a \in A$. $e \in A$ is a *unit* if it is a left and right unit. If A has such an element we call A an *algebra with unit* or a *unital algebra*. Of course, there can be at most one unit in A.

For non-empty subsets $I, J \subset A$ we define:

$$\begin{aligned} I + J &:= \{a + b \mid a \in I,\ b \in J\} \subset A, \\ IJ &:= \{\textstyle\sum_{i \leq k} a_i b_i \mid a_i \in I,\ b_i \in J, k \in \mathbb{N}\} \subset A. \end{aligned}$$

1.2 Algebra morphism. Definition.

Let A, B be R-algebras. A map $f : A \to B$ is called an *(algebra) homomorphism* or *(algebra) morphism* if f is R-linear (R-module morphism) and

$$f(ab) = f(a)f(b) \text{ for all } a, b, \in A.$$

Such an f is called an *isomorphism* if it is injective and surjective.

In case A and B are algebras with units, then f is said to be *unital* if it maps the unit of A to the unit of B. It is easily seen that any surjective algebra morphism between unital algebras is unital.

1.3 Category of R-algebras.

The class of R-algebras as objects and the algebra homomorphisms as morphisms form a category which we denote by *R-Alg*.

Observe that *R-Alg* is not an additive category since the sum of two algebra morphisms need not be an algebra morphism.

1.4 Products of algebras. Definitions.

Consider a family $\{A_\lambda\}_\Lambda$ of R-algebras.

(1) Defining the operations by components, the cartesian product $\prod_\Lambda A_\lambda$ becomes an R-algebra, called the *(algebra) product* of the A_λ's, and the canonical projections and injections (as defined for R-modules)

$$\pi_\mu : \prod_\Lambda A_\lambda \to A_\mu, \qquad \varepsilon_\mu : A_\mu \to \prod_\Lambda A_\lambda,$$

are algebra morphisms. $\prod_\Lambda A_\lambda$ has a unit if and only if every A_λ has a unit.

(2) The *weak (algebra) product* $\bigoplus_\Lambda A_\lambda$ of the A_λ's is defined as subalgebra of $\prod_\Lambda A_\lambda$ consisting of elements with only finitely many non-zero components.

(3) A *subdirect product* of the A_λ's is a subalgebra $B \subset \prod_\Lambda A_\lambda$ for which the restrictions of the projections to B, $\pi_\mu|_B : B \to A_\mu$, are surjective for every $\mu \in \Lambda$.

It is easy to check that $\prod_\Lambda A_\lambda$ together with the projections π_λ form a *product in the category R-Alg* (e.g., [40, 9.1]):

For every family of algebra morphisms $\{f_\lambda : X \to A_\lambda\}_\Lambda$ there is a unique algebra morphism $f : X \to \prod_\Lambda A_\mu$ making the following diagram commutative for any $\lambda \in \Lambda$:

$$\begin{array}{ccc} \prod_\Lambda A_\mu & \xrightarrow{\pi_\lambda} & A_\lambda \\ {}_f\nwarrow & & \nearrow_{f_\lambda} \\ & X & \end{array} .$$

The weak product $\bigoplus_\Lambda A_\lambda$ is the coproduct in the category of R-modules but not the coproduct in the category *R-Alg*. It is a special case of a subdirect product. For infinite sets Λ the weak product cannot have a unit.

1.5 Ideals. Definitions. A subset $I \subset A$ of an R-algebra A is called a

left (right) algebra ideal	if I is an R-submodule and $AI \subset I$ (resp. $IA \subset I$),
left (right) ring ideal	if I is a $\mathbb{Z}$-submodule and $AI \subset I$ (resp. $IA \subset I$),
algebra (ring) ideal	if I is a left and right algebra (ring) ideal.

If no ambiguity arises we will just say *ideal* instead of *algebra* or *ring ideal.* Let us collect some basic statements about ideals:

1.6 Properties of ideals. *Let A be an R-algebra.*

(1) Any intersection of (left, right) ideals is again a (left, right) ideal.

(2) A^2 is an ideal in A.

(3) $IJ \subset I \cap J$ for any ideals $I, J \subset A$ but IJ need not be an ideal.

(4) Every (left, right) algebra ideal is a (left, right) ring ideal in A.

(5) If A has a unit, every (left, right) ring ideal is a (left, right) algebra ideal in A.

Proof. (5) Let $e \in A$ denote the unit of A and I a (left, right) ring ideal. It is to show that I is an R-module. For any $r \in R$ we have $re \in A$ and $rI = (re)I \subset I$.

The other assertions are evident. $\square$

The *kernel* of an algebra morphism $f : A \to B$ is defined as

$$Ke\, f = \{a \in A \mid f(a) = 0\}.$$

Obviously, $Ke\, f$ is an algebra ideal. On the other hand, for every algebra ideal $I \subset A$ an algebra structure on the factor module A/I is obtained with the multiplication

$$(a + I)(b + I) = ab + I \text{ for } a, b \in A.$$

Then I is the kernel of the algebra morphism defined by the projection $p_I : A \to A/I$ and we get the

1.7 Factorization Theorem.

Assume $f : A \to B$ is an an algebra morphism and I an ideal of A with $I \subset Ke\, f$. Then there is a unique algebra morphism $\bar{f} : A/I \to B$ making the following diagram commutative:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "p_I"'] & & B \\ & A/I \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

If $I = Ke\, f$, then $\bar{f}$ is injective. If f is surjective, then $\bar{f}$ is also surjective.

From this we derive the

1.8 Isomorphism Theorems.

(1) Let I, J be ideals in the algebra A with $I \subset J$. Then J/I is an ideal in the algebra A/I and there is an algebra isomorphism

$$(A/I) \,/\, (J/I) \;\simeq\; A/J.$$

(2) Let B be a subalgebra and I an ideal in the algebra A. Then I is an ideal in $B + I$, $I \cap B$ is an ideal in B, and there is an algebra isomorphism

$$(B + I) \,/I \simeq B \,/\, (I \cap B).$$

Every R-algebra can be embedded into an R-algebra with unit. We can achieve this with the construction of the *Dorroh overring* (see [40, 1.5]):

1.9 Embedding into algebras with units. *Let A be an R-algebra.*

(1) The R-module $R \times A$ becomes an R-algebra with unit $(1, 0)$ by defining

$$(s, a)(t, b) = (st, sb + ta + ab) \text{ for } s, t \in R,\ a, b \in A.$$

This algebra is called the Dorroh overring *and is denoted by A^*.*

(2) The map

$$\varepsilon : A \to A^*,\ a \mapsto (0, a),$$

is an injective algebra morphism and $\varepsilon(A)$ is an algebra ideal in A^*.

(3) If B is an R-algebra with unit e and $f : A \to B$ an algebra morphism, then there is a unique unital algebra morphism $f^* : A^* \to B$ with $f = f^*\varepsilon$, namely

$$f^*(s, a) = se + f(a) \text{ for } (s, a) \in A^*.$$

Since ideals are just kernels of morphisms we obtain by the universal property of products:

1.10 A as (subdirect) product of rings. *Let A be an R-algebra.*

(1) For any family of ideals $\{I_\lambda\}_\Lambda$ in A the canonical mappings $p_{I_\lambda} : A \to A/I_\lambda$ yield an algebra morphism

$$\gamma : A \to \prod\nolimits_\Lambda A/I_\lambda,\ a \mapsto (a + I_\lambda)_\Lambda,$$

with $Ke\,\gamma = \bigcap_\Lambda I_\lambda$. Hence γ is injective if and only if $\bigcap_\Lambda I_\lambda = 0$. In this case A is a subdirect product of the algebras A/I_λ.

(2) Chinese Remainder Theorem.
If A has a unit, for ideals $I_1, \cdots, I_n$ in A, the following are equivalent:
(a) The canonical map $\kappa : A \to \prod_{i\le n} A/I_i$ is surjective (and injective);
(b) for every $j \le n$, $I_j + \bigcap_{i\neq j} I_i = A$ (and $\bigcap_{i\le n} I_i = 0$).

For the proof we refer to [40, 3.12 and 9.12].

1.11 Nucleus and centre. Definitions. Let A be an R-algebra. We define the

$$\begin{array}{rl} \textit{associator} & (a,b,c) = (ab)c - a(bc), \\ \textit{commutator} & [a,b] = ab - ba \quad \text{for } a,b,c \in A; \\ \textit{left nucleus of } A, & N_l(A) = \{c \in A \mid (c,a,b) = 0 \text{ for all } a,b \in A\}, \\ \textit{right nucleus of } A, & N_r(A) = \{c \in A \mid (a,b,c) = 0 \text{ for all } a,b \in A\}, \\ \textit{middle nucleus of } A, & N_m(A) = \{c \in A \mid (a,c,b) = 0 \text{ for all } a,b \in A\}, \\ \textit{nucleus of } A, & N(A) = N_l(A) \cap N_m(A) \cap N_r(A), \\ \textit{centre of } A, & Z(A) = \{c \in N(A) \mid [c,a] = 0 \text{ for all } a \in A\}. \end{array}$$

By definition, A is an associative algebra if and only if $(a,b,c) = 0$ for all $a,b,c \in A$, and A is commutative if and only $[a,b] = 0$ for all $a,b \in A$.

It is easy to check that the left, right and middle nucleus all are associative subalgebras of A. The centre of A is a commutative and associative subalgebra containing at least the zero of A and may be described by

$$Z(A) = \{c \in A \mid (a,b,c) = (a,c,b) = [a,c] = 0 \text{ for all } a,b \in A\}.$$

If A has a unit e, then obviously $e \in Z(A)$.

Any surjective algebra morphism maps the (left, right) nucleus into the (left, right) nucleus and the centre into the centre of the image.

2 Multiplication algebras

1.Multiplication algebra. 2.Change of base ring. 3.Morphisms and multiplication algebras. 4.Module structures on A. 5.Associative algebras. 6.Centroid. 7.Properties of the centroid. 8.Connection between centre and centroid. 9.Central algebras. 10.Centroid and $M^*(A)$. 11.A finitely generated as module. 12.Module finite algebras. 13.Exercises.

To any algebra A there is a closely related associative algebra which provides a good deal of information about the structure of A. In this section we introduce this algebra and consider A as a module over it.

2.1 Multiplication algebra. Definitions.

Let A be an R-algebra. The left and right multiplication by any $a \in A$,

$$L_a : A \to A, \quad x \mapsto ax,$$
$$R_a : A \to A, \quad x \mapsto xa,$$

define R-endomorphisms of A, i.e., $L_a, R_a \in End(_RA)$.

The R-subalgebra of $End(_RA)$ generated by all left multiplications in A (and the identity map id_A) is called the *left multiplication ideal (algebra) of* A, i.e.,

$$\begin{aligned} L^*(A) &= < \{L_a \,|\, a \in A\} >, \\ L(A) &= < \{L_a \,|\, a \in A\} \cup \{id_A\} >\subset End(_RA). \end{aligned}$$

Similarly the *right multiplication ideal* and *algebra*, $R^*(A)$ and $R(A)$, are defined.

The R-subalgebra of $End(_RA)$ generated by all left and right multiplications (and id_A) is called the *multiplication ideal (algebra)* of A, i.e.,

$$\begin{aligned} M^*(A) &= < \{L_a, R_a \,|\, a \in A\} >, \\ M(A) &= < \{L_a, R_a \,|\, a \in A\} \cup \{id_A\} >\subset End(_RA). \end{aligned}$$

Obviously, $L(A)$, $R(A)$ and $M(A)$ are associative R-algebras with unit, and $M^*(A)$ is an ideal in $M(A)$.

By definition, $M(A) = R \cdot id_A + M^*(A)$ and we have the elementary equalities

$$M(A)A = A \quad \text{ and } M^*(A)A = A^2.$$

Observe that the definition of $M(A)$ depends on the ring R. To indicate this we sometimes write $M(_RA)$ or $M^*(_RA)$ to avoid confusion. However, the definition of $M^*(A)$ is in fact independent of R:

2.2 Change of base ring.

Let S be an associative and commutative R-algebra with unit. Let A be an S-algebra. Then A is also an R-algebra and

(1) $M^*({}_RA)$ *is an* S*-algebra;*

(2) $M^*({}_RA) = M^*({}_SA)$*;*

(3) $M({}_SA) = S \cdot M({}_RA)$.

Proof. (1) This follows from $sL_a = L_{sa}$ and $sR_a = R_{sa}$ for any $s \in S$, $a \in A$.

(2) and (3) are immediate consequences of (1). $\square$

2.3 Morphisms and multiplication algebras.

(1) Any surjective algebra morphism $f : A \to B$ *induces an surjective algebra morphisms*

$$f_m : M^*(A) \to M^*(B).$$

(2) For any finite family of R*-algebras* $\{A_1, \dots, A_n\}$*,*

$$M^*(\prod_{i=1}^m A_i) \simeq \prod_{i=1}^n M^*(A_i).$$

Proof. (1) For $a \in A$ put $f_m(L_a) := L_{f(a)}$ and $f_m(R_a) := R_{f(a)}$.

We have to verify that this definition extends to $M^*(A)$. Assume $\nu \in M^*(A)$ to be a linear combination of a composition of left and right multiplications in A representing the zero map, i.e., $\nu a = 0$ for all $a \in A$. Since f is an algebra morphism, $0 = f(\nu a) = f_m(\nu)f(a)$. f being surjective this implies $f_m(\nu) = 0$.

(2) By (1), for every $j \leq n$ the canonical algebra morphism $\prod_{i=1}^n A_i \to A_j$ yields an algebra morphism $M^*(\prod_{i=1}^n A_i) \to M^*(A_j)$. Now the isomorphism stated is obtained by the universal property of the product of algebras. $\square$

2.4 Module structures on A**.**

Consider the R*-algebra* A *as a left module over the* R*-algebra* $End({}_RA)$*. Then* A *may be regarded as a faithful left module over the rings* $L(A)$*,* $R(A)$*,* $M(A)$ *and* $M^*(A)$*.*

(1) The $L(A)$*-submodules are the left algebra ideals of* A*,*
the $R(A)$*-submodules are the right algebra ideals of* A*.*

(2) The $M(A)$*-submodules are the (two-sided) algebra ideals of* A*,*
the $M^*(A)$*-submodules are the ring ideals of* A*.*

(3) For any subset $X \subset A$ *the algebra ideal in* A *generated by* X *can be written as* $M(A)X$*.*

For associative algebras A we observe close connections between A and $M(A)$. Let A^o denote the opposite algebra of A, i.e., the algebra with the same additive group as A but with reversed multiplication. Right modules over A may be considered as left modules over A^o.

2.5 Associative algebras.

Let A be an associative R-algebra. Then A is a left module over the R-algebra $A \otimes_R A^o$ by defining

$$(a \otimes b)x = axb \text{ for } a, b, x \in A.$$

There is a surjective algebra morphism

$$\rho : A \otimes_R A^o \to M^*(A),\ a \otimes b \mapsto L_a R_b.$$

(1) ρ is injective if and only if A is a faithful $A \otimes_R A^o$-module.

(2) If A has a unit $\rho : A \otimes_R A^o \to M(A)$ is surjective.

(3) If A is generated as an R-module by $a_1, \ldots, a_n$, then $M^*(A)$ is generated as R-module by

$$\{ L_{a_i}, R_{a_i}, L_{a_i} R_{a_j} \mid i, j \in \{1, \ldots, n\} \},$$

and $M(A)$ is generated by the union of this set with $\{id_A\}$.

Studying modules the endomorphism rings play an important part:

2.6 Centroid.

Let A be any R-algebra. The endomorphism ring of the $M(A)$-module A is called the *centroid* $C(A)$ of A, i.e., $C(A) = End_{M(A)}(A)$.

Writing morphisms of left modules on the right, A is a right module over $C(A)$ and

$$\begin{aligned} C(A) &= \{\varphi \in End({}_{\mathbb{Z}}A) \mid \varphi\psi = \psi\varphi, \text{ for all } \psi \in M(A)\} \\ &= \{\varphi \in End({}_R A) \mid (ab)\varphi = a(b\varphi) = (a\varphi)b, \text{ for all } a, b \in A\}. \end{aligned}$$

2.7 Properties of the centroid.

Let A be an R-algebra with centroid $C(A)$.

(1) For any $\alpha, \beta \in C(A)$, $A^2(\alpha\beta - \beta\alpha) = 0$.

(2) If $Hom_{M(A)}(A/A^2, A) = 0$, then $C(A)$ is a commutative algebra.

(3) If $I^2 = I$ for every ideal $I \subset A$, then $C(A)$ is a regular ring.

(4) If $C(A)$ is not commutative, then there exists a non-zero ideal $I \subset A$ with $IA = AI = 0$.

(5) If $C(A)$ is a division ring, then $C(A)$ is commutative.

Proof. (1) For any $a, b \in A$,

$$(ab)\alpha\beta = [(a\alpha)b]\beta = (a\alpha)(b\beta) = [a(b\beta)]\alpha = (ab)\beta\alpha.$$

(2) By (1), for $\alpha, \beta \in C(A)$ the commutator $[\alpha, \beta]$ annihilates A^2 and hence has to be zero by the condition in (2), i.e., $C(A)$ is commutative.

(3) We have to show that for every $\alpha \in C(A)$, $Im\,\alpha$ and $Ke\,\alpha$ are direct summands in A (see [40, 37.7]).

By assumption, $A\alpha = (A\alpha)(A\alpha) = (A^2)\alpha^2 = A\alpha^2$. Hence for every $a \in A$, there exists $b \in A$ such that $(a)\alpha = (b)\alpha^2$. So $a = (b)\alpha + (a - (b)\alpha)$ with $(a - (b)\alpha)\alpha = 0$, implying $A = Im\,\alpha + Ke\,\alpha$.

Since $A\alpha \cap Ke\,\alpha = (A\alpha \cap Ke\,\alpha)^2 = 0$, $A = Im\,\alpha \oplus Ke\,\alpha$.

(4) We know by (2) that there exists a non-zero $\alpha \in C(A)$ with $0 = (A^2)\alpha = A(A)\alpha = (A)\alpha A$ and $(A)\alpha$ is a non-zero ideal in A.

(5) Without restriction let A be a faithful R-module. Assume $C(A)$ to be a non-commutative division ring. Since every non-zero map in $C(A)$ is invertible, it follows from the proof of (4) that $A^2 = 0$. This implies $M(A) = R \cdot id_A$ and $R \subset End({}_RA) = C(A)$.

Since $C(A)$ is a division ring, R is an integral domain contained in the centre of $C(A)$, and the quotient field Q of R is also contained in the centre of $C(A)$. We conclude $End({}_RA) \simeq End({}_QA)$, which is a division ring only if $dim_Q A = 1$ and so $C(A) \simeq Q$, i.e., $C(A)$ is commutative. □

2.8 Connection between centre and centroid.
Let A be an R-algebra with centre $Z(A)$ and centroid $C(A)$.

(1) There exists an R-algebra morphism

$$\nu : Z(A) \to C(A), \; a \mapsto L_a.$$

(2) If A is a faithful $Z(A)$-module, then ν is injective.

(3) $Z(A)$ is a right $C(A)$-module.

(4) If A has a unit, then ν is an isomorphism.

Proof. (1) and (2) are easily verified.

(3) For any $a \in Z(A)$ and $\gamma \in C(A)$, $(a)\gamma \in Z(A)$.

(4) For the unit $e \in A$, the map $C(A) \to Z(A), \gamma \mapsto (e)\gamma$, is inverse to ν. □

2.9 Central algebras. *Let A be an R-algebra with centroid $C(A)$.*

(1) For any $r \in R$ the multiplication $L_r : A \to A, x \mapsto rx$, belongs to $C(A)$ and there is a ring morphism

$$\varphi : R \to C(A), \quad r \mapsto L_r.$$

(2) φ is injective if and only if A is a faithful R-module.

(3) In case φ is an isomorphism, A is called a central *R-algebra.*

(4) If A is a central R-algebra, every algebra ideal is a fully invariant $M(A)$-submodule of A.

(5) If $C(A)$ is commutative then A is a central $C(A)$-algebra.

(6) If A has a unit e, then φ may be replaced by $R \to Z(A), r \mapsto re$.

Proof. (5) By commutativity of $C(A)$, any $\gamma \in C(A)$ is in fact a $C(A)$-module homomorphism. From this it follows that $C(A)$ is the centroid of A as $C(A)$-algebra (see 2.6).

The other assertions are obvious. □

The relationship between $M^*(A)$ and $C(A)$ is displayed in

2.10 Centroid and $M^*(A)$.

Let A be any R-algebra with centroid $C(A)$ and centre $Z(A)$.

(1) Assume $C(A)$ is commutative. Then:
  - *(i) $M^*(A)$ is a $C(A)$-algebra;*
  - *(ii) $M^*({}_RA) = M^*({}_{C(A)}A)$;*
  - *(iii) $M({}_{C(A)}A) = C(A) \cdot M({}_RA)$;*
  - *(iv) $C(A)$ is isomorphic to the centre of $M(A)$.*

(2) If A has a unit the centre of $M(A)$ is isomorphic to $Z(A)$.

Proof. (1) If $C(A)$ is commutative A is a $C(A)$-algebra and the assertion follows from 2.2. (*ii*) tells us that the multiplication ideal of A as an R-algebra is equal to the multiplication ideal of A as a $C(A)$-algebra.

(iv) Since $M(A)$ has id_A as unit we conclude from (1) that $C(A) \simeq C(A)id_A$ is a subalgebra of $M(A)$ contained in the centre of $M(A)$.

Clearly every element of the centre of $M(A)$ belongs to $C(A)$ by definiton and so we can identify these two algebras.

(2) For unital algebras A, $C(A) = Z(A)$ by 2.8. □

For example, let A be a finite dimensional algebra over an algebraically closed field K. If A has no nilpotent elements, then $M^*(A) = M(A)$. This is shown in Farrand-Finston [127].

From 2.8 we see in particular that the centroid of an algebra with unit is isomorphic to a subalgebra of A, namely the centre of A. A more general situation is described in

2.11 A finitely generated as module.
Let A be an R-algebra with $M(A)$ and $C(A)$ defined as above.

(1) If A is finitely generated as an $M(A)$-module, then $C(A)$ is isomorphic to a $C(A)$-submodule of a finite direct sum A^n, $n \in I\!N$.

(2) If A is finitely generated as right a $C(A)$-module, then $M(A)$ is isomorphic to an $M(A)$-submodule of a finite direct sum A^k, $k \in I\!N$.

Proof. (1) Assume A is generated as an $M(A)$-module by $a_1, \ldots, a_n$. Then the map

$$C(A) \to A^n, \quad \gamma \mapsto (a_1, \ldots, a_n)\gamma = (a_1\gamma, \ldots, a_n\gamma),$$

is a $C(A)$-monomorphism: Assume $a_i\gamma = 0$ for $i = 1, \ldots, n$. Then for any $a = \sum \nu_i a_i$ in A, $\nu_i \in M(A)$, we have

$$a\gamma = (\sum \nu_i a_i)\gamma = \sum \nu_i(a_i\gamma) = 0,$$

i.e., $\gamma = 0$ and the map is injective.

(2) Starting with a generating set $b_1, \ldots, b_k$ of A as $C(A)$-module, the proof is symmetric to the above argument. □

Recall that $A \simeq M(A) \simeq C(A)$ for any associative commutative ring A with unit. This is a special case of the following situation:

2.12 Module finite algebras.
Let A be an R-algebra which is finitely generated as an R-module. Then:

(1) $C(A) \subset A^n_{C(A)}$ and $M(A) \subset {}_{M(A)}A^k$ for some $n, k \in I\!N$.

(2) If R is a noetherian ring, then the algebras $M(A)$ and $C(A)$ are finitely generated as R-modules. A, $M(A)$ and $C(A)$ have the ascending chain condition on (left, right) ideals.

(3) If R is an artinian ring, then the algebras A, $M(A)$ and $C(A)$ satisfy the descending chain condition on (left,right) ideals.

Proof. (1) The given condition implies that A is both finitely generated as $M(A)$-module and as $C(A)$-module and the assertion follows from 2.11.

(2) A^n is a noetherian R-module for any $n \in I\!N$. By (1), this implies that $M(A)$ and $C(A)$ are noetherian R-modules. Since all (left) ideals are in particular R-submodules we have the ascending chain condition for (left) ideals in all these algebras.

(3) An artinian ring R is also noetherian and finitely generated modules over artinian rings are again artinian. Hence the assertion follows from (2). □

2.13 Exercises.

(1) Let A be an R-algebra with right nucleus $N_r(A)$. Prove that

$$\delta : N_r(A) \to End({}_{L(A)}A),\ a \mapsto R_a,$$

is a ring morphism. If A has a right unit then δ is surjective; if A has a unit then δ is an isomorphism.

(2) [135] Let A be an R-algebra with multiplication algebra $M(A)$.

For any ideal $I \subset A$ put $(I : A) := \{\mu \in M(A) \mid \mu A \subset I\}$;

for a left ideal $\mathcal{I} \subset M(A)$ put $\mathcal{I}(A) := \sum_{\alpha\in\mathcal{I}} \alpha A$.

Show for ideals $I, J \subset A$ and left ideals $\mathcal{I}, \mathcal{J} \subset M(A)$:

(i)	*$(I : A)$ is a two-sided ideal in $M(A)$;*	*$\mathcal{I}(A)$ is a two-sided ideal in A;*
(ii)	*$I \subset J$ implies $(I : A) \subset (J : A)$;*	*$\mathcal{I} \subset \mathcal{J}$ implies $\mathcal{I}(A) \subset \mathcal{J}(A)$;*
(iii)	*$(I : A)(A) \subset I$;*	*$(\mathcal{I}(A) : A) \supset \mathcal{I}$;*
(iv)	*$(I : A) = ((I : A)(A) : A)$;*	*$\mathcal{I}(A) = (\mathcal{I}(A) : A)(A)$;*
(v)	*$M(A/I) \simeq M(A)/(I : A)$.*	

(3) Let A be an algebra over a field K. Prove ([222]):
(i) For a maximal ideal $\mathcal{I} \subset M(A)$ there are three possibilities:
(α) $\mathcal{I}(A) = A$, (β) $\mathcal{I} = M^*(A)$, (γ) $\mathcal{I}(A)$ is a maximal ideal in A.
(ii) For maximal ideals $\mathcal{I}_1, \dots, \mathcal{I}_k$ in $M(A)$, $\bigcap_{i=1}^k \mathcal{I}_i(A) = (\bigcap_{i=1}^k \mathcal{I}_i)(A)$.

(4) Let A be an R-algebra such that $M(A)$ is a division algebra. Show that A is commutative and associative and $M(A)$ is commutative.

(5) Consider a four dimensional algebra A with basis a_1, a_2, a_3, a_4 over a field K given by the multiplication table ([135])

	a_1	a_2	a_3	a_4
a_1	0	0	a_1	0
a_2	0	0	0	0
a_3	0	0	0	0
a_4	0	0	0	a_2

Show that

$$M(A) = \{ \begin{pmatrix} b & 0 & a & 0 \\ 0 & b & 0 & c \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix} \mid a, b, c \in K\},$$

$$C(A) = \{ \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & \beta & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix} \mid \alpha, \beta \in K \},$$

and hence $C(A) \not\subset M(A)$.

References: Courter [111], Farrand [126], Farrand-Finston [127], Finston [135], Müller [208], Pritchard [222], Röhrl [229], Schafer [37], Wisbauer [267, 268].

3 Identities for algebras

1.Alternative algebras. 2.Moufang identities. 3.Bruck-Kleinfeld map. 4.Artin's Theorem. 5.Multiplications in alternative algebras. 6.Generating sets in alternative algebras. 7.Module finite alternative algebras. 8.Simple alternative algebras. 9.Jordan algebras. 10.Special Jordan algebras. 11.Identity. 12.Generating sets in Jordan algebras. 13.Module finite Jordan algebras. 14.Lie algebras. 15.Generating sets in Lie algebras. 16.Special Lie algebras. 17.Malcev algebras. 18.Simple Malcev algebras. 19.Exercises.

In this section we study classes of algebras defined by some identities replacing associativity. One of our purposes is to show how such identities can be used to get information about the multiplication algebra.

3.1 Alternative algebras. Definition. An algebra A is called

left alternative if $a^2b = a(ab)$,
right alternative if $ab^2 = (ab)b$,
flexible if $a(ba) = (ab)a$ for all $a, b \in A$,
alternative if it is left and right alternative (and flexible).

Using the associator defined in 1.11 we can express these identies by

$$(a, a, b) = 0, \quad (a, b, b) = 0, \quad (a, b, a) = 0.$$

Hence alternative algebras are characterized by the fact that the associator yields an alternating map

$$(-, -, -) : A \times A \times A \to A.$$

The identities above are of degree 2 in a or b. We can derive multilinear identities from these by *linearization*. In our case this can be achieved by replacing a by $a + d$ in the first identity

$$(a + d)^2b = (a + d)[(a + d)b].$$

Evaluating this equality and referring again to the first identity in the form $a^2b = a(ab)$ and $d^2b = d(db)$ we obtain the multilinear identity

$$(ad + da)b = a(db) + d(ab).$$

Putting $b = a$ we obtain a new relation for left alternative rings

$$(ad + da)a = a(da) + da^2.$$

Now applying the second identiy (right alternative) we have $(ad)a = a(da)$ and thus:

Left and right alternative implies flexible.

In an alternative algebra we also have

$$[a(ab)]b - a[(ab)b] = a^2b^2 - a^2b^2 = 0,$$

which means $(a, ab, b) = 0$ for all $a, b \in A$.

Linearizing with respect to b, i.e., replacing b by $b + c$, we derive

$$\begin{aligned} 0 &= (a, ab, c) + (a, ac, b) = -(ab, a, c) - (a, b, ac) \\ &= -[(ab)a]c + (ab)(ac) - (ab)(ac) + a[b(ac)] \\ &= a[b(ac)] - [(ab)a]c. \end{aligned}$$

This identity is named after *Ruth Moufang* who studied alternative fields in the context of geometric investigations (see [207]). By similar computations we obtain further identities of this kind:

3.2 Moufang identities.

In an alternative algebra we have the

left Moufang identity	$(aba)c = a[b(ac)]$,
right Moufang identity	$c(aba) = [(ca)b]a$,
middle Moufang identity	$a(bc)a = (ab)(ca)$.

Using the associator we have for all $a, b, c \in A$:

$$\begin{aligned} (a, b, ca) &= a(b, c, a), \\ (ab, c, a) &= (a, b, c)a, \\ (b, a^2, c) &= a(b, a, c) + (b, a, c)a. \end{aligned}$$

It was observed in Bruck-Kleinfeld [92] that the following map is helpful for proving our next theorem:

3.3 Bruck-Kleinfeld map.

Let A be an alternative algebra. For the map

$$f : A^4 \to A, \quad (a, b, c, d) \mapsto (ab, c, d) - (b, c, d)a - b(a, c, d),$$

we have
(1) $f(a, b, c, d) = -f(d, a, b, c)$ *and*
(2) $f(a, b, c, d) = 0$ *in case any two arguments coincide.*

Proof. It is easy to verify that for a, b, c, d in any algebra A,

$$(ab, c, d) - (a, bc, d) + (a, b, cd) = a(b, c, d) + (a, b, c)d.$$

For alternative algebras this yields

$$(ab, c, d) - (bc, d, a) + (cd, a, b) = a(b, c, d) + (a, b, c)d.$$

Forming the map

$$\begin{aligned} F(a,b,c,d) &:= f(a,b,c,d) - f(b,c,d,a) + f(c,d,a,b) \\ &= [a,(b,c,d)] - [b,(c,d,a)] + [c,(d,a,b)] - [d,(a,b,c)], \end{aligned}$$

we conclude from above

$$0 = F(a,b,c,d) + F(b,c,d,a) = f(a,b,c,d) + f(d,a,b,c),$$

proving (1).

By definition, $f(a,b,c,d) = -f(a,b,d,c)$ and $f(a,b,c,c) = 0$. This together with (1) yields (2). $\square$

3.4 Artin's Theorem.
An algebra A is alternative if and only if any subalgebra of A generated by two elements is associative.

Proof. (1) (compare [92]) One implication is clear. Now assume that A is alternative. For $D := \{a,b\}$ we define in obvious notation

$$\begin{aligned} K &:= \{k \in A \mid (D,D,k) = 0\}, \\ M &:= \{m \in K \mid (D,m,K) = 0 \text{ and } mK \subset K\}, \\ S &:= \{s \in M \mid (s,M,K) = 0\}. \end{aligned}$$

It suffices to show that S is an associative subalgebra and $D \subset S$.

Obviously, K, M and S are R-submodules of A and $(D,D,K) = 0$. Since D contains only two distinct elements, we have $(D,D,D) = 0$ and by 3.3, $f(D,K,D,D) = 0$. This yields $(D,D,DK) = 0$ and so

$$DK \subset K, \quad D \subset M \text{ and } \quad D \subset S.$$

Since $S \subset M \subset K$ this means $(S,S,S) = 0$, i.e., S is associative.

It remains to show that $SS \subset S$. From $(S,M,K) = 0$ and $S \subset M$ we derive $(K,S,S) = 0 = (MK,S,S)$. By 3.3,

$$f(s,s',m,k) = f(m,k,s,s') \text{ for all } s,s' \in S,\ m \in M,\ k \in K.$$

This yields $(SS,M,K) = 0$, in particular, $(D,SS,K) = -(SS,D,K) = 0$. We have $SS \subset K$ and $(SS)K = S(SK) \subset K$. Hence we conclude $SS \subset M$ and $SS \subset S$. $\square$

A different proof, using an induction argument, is given in [244].

The following generalized form of Artin's Theorem is proved in [92]:

Let A be an alternative algebra and $a,b,c \in A$ with $(a,b,c) = 0$. Then the subalgebra of A generated by $\{a,b,c\}$ is associative.

We see from Artin's theorem that, in particular, in an alternative algebra every subalgebra generated by *one* element is associative. Algebras with this property are called *power-associative*.

We are now going to investigate the effect of identities of an algebra on the multiplication algebra. For this we consider left and right multiplications in alternative algebras:

3.5 Multiplications in alternative algebras.

For any elements a, b in an alternative algebra A we have:

$$\begin{array}{rl} \textit{left alternative} & L_{a^2} = L_a L_a, \\ \textit{right alternative} & R_{a^2} = R_a R_a, \\ \textit{flexible} & R_a L_a = L_a R_a, \\ \textit{left Moufang identity} & L_{aba} = L_a L_b L_a, \\ \textit{right Moufang identity} & R_{aba} = R_a R_b R_a, \\ \textit{middle Moufang identities} & R_a L_a L_b = L_{ab} R_a, \\ & R_a L_a R_b = R_{ba} L_a. \end{array}$$

Recalling that the associator is alternating we derive from this

$$L_{ab} = L_a L_b - [L_b, R_a], \quad R_{ab} = R_b R_a - [L_a, R_b],$$
$$L_a L_b + L_b L_a = L_{ab+ba}, \quad R_a R_b + R_b R_a = R_{ab+ba}.$$

These identities show, for example, that R_{ab} and L_{ab} belong to the subalgebra generated by L_a, L_b, R_a and R_b and we conclude:

3.6 Generating sets in alternative algebras.

Let A be an alternative R-algebra.

(1) *If A is generated by $\{a_\lambda\}_\Lambda$ as an R-algebra, then $M(A)$ is generated as an R-algebra by*

$$\{L_{a_\lambda}, R_{a_\lambda}, id_A \mid \lambda \in \Lambda\}.$$

(2) *If A is finitely generated as an R-algebra, then $M(A)$ is also finitely generated as an R-algebra.*

We have seen in 2.5 that for an associative module finite R-algebra the multiplication algebra is also module finite. This remains true for alternative algebras:

3.7 Module finite alternative algebras.

Let A be an alternative R-algebra which is finitely generated as an R-module. Then $M(A)$ is also finitely generated as an R-module.

Proof. Assume $a_1, \ldots, a_n$ to generate A as an R-module. We want to show that $M(A)$ is generated as an R-module by monomials of the form

$$L_{a_1}^{\varepsilon_1} L_{a_2}^{\varepsilon_2} \cdots L_{a_n}^{\varepsilon_n} R_{a_1}^{\delta_1} R_{a_2}^{\delta_2} \cdots R_{a_n}^{\delta_n}, \quad \varepsilon_i, \delta_i \in \{0,1\},$$

where, by convention, $L_a^0 = id_A = R_a^0$ for any $a \in A$.

First observe that every $a \in A$ can be written as a finite sum $a = \sum r_i a_i$ with $r_i \in R$ and hence $L_a = \sum r_i L_{a_i}$. Therefore it suffices to show that every product of L_{a_j}'s and R_{a_i}'s can be represented in the given form. This can be achieved by substituting according to 3.5 (with appropriate $s_k, t_l \in R$):

(i) $R_{a_i} L_{a_j} = L_{a_j} R_{a_i} + L_{a_i a_j} - L_{a_i} L_{a_j}$;

(ii) $L_{a_i} L_{a_j} = -L_{a_j} L_{a_i} + L_{a_i a_j + a_j a_i} = -L_{a_j} L_{a_i} + \sum s_k L_{a_k}$ if $i > j$;

(iii) $L_{a_i^k} = L_{a_i}^k = \sum t_l L_{a_l}$ (by Artin's Theorem).

Similar operations can be applied to the R_{a_i}'s. □

The structure theory of alternative algebras is closely related to associative algebras partly due to the following observation (by Kleinfeld [180], see [41, Chap. 7]):

3.8 Simple alternative algebras.

Let A be a simple alternative algebra which is not associative. Then the centre of A is a field and A is a Cayley-Dickson algebra (of dimension 8) over its centre.

Interest in alternative rings arose first in axiomatic geometry. For an incidence plane in which Desargues' theorem holds the coordinate ring is an associative division ring. A weaker form of Desargues' theorem, the *Satz vom vollständigen Vierseit*, is equivalent to the coordinate ring being an alternative division ring (see [206]).

The motivation for the investigation of the next type of algebras came from quantum mechanics. In 1932 the physicist *Pascual Jordan* draw attention to this class of algebras. Later on they turned out to be also useful in analysis. For example, in [177] the reader may find an account of their role for the description of bounded symmetric domains.

3.9 Jordan algebras. Definition.

An algebra A over a ring R with 2 invertible in R is called a *(linear) Jordan algebra* if it is commutative and, for all $a, b, \in A$,

$$a(a^2 b) = a^2(ab) \quad \text{(Jordan identiy)}.$$

In a commutative algebra A the left multiplication algebra $L(A)$ coincides with the multiplication algebra $M(A)$ and the Jordan identity can also be written in the following forms

$$(a, b, a^2) = 0, \quad L_a L_{a^2} = L_{a^2} L_a, \quad [L_a, L_{a^2}] = 0.$$

The condition $\frac{1}{2} \in R$ is not necessary for the definition of a Jordan algebra. However, especially for producing new identities by linearization it is quite often useful to divide by 2. Hence it makes sense to include the condition $\frac{1}{2} \in R$ already in the definition. Over rings with 2 not invertible it is preferable to consider *quadratic Jordan algebras* instead of linear Jordan algebras (e.g., Jacobson [20]). Important examples of Jordan algebras can be derived from associative algebras:

3.10 Special Jordan algebras.

Let $(B, +, \cdot)$ be an associative algebra over the ring R with 2 invertible. Then the new product for $a, b \in B$,

$$a \times b = \frac{1}{2}(a \cdot b + b \cdot a),$$

turns $(B, +, \times)$ into a Jordan algebra.

Algebras isomorphic to a subalgebra of an algebra of type $(B, +, \times)$ are called special Jordan algebras*.*

Proof. Obviously, the new multiplication is commutative. Denoting by L_a the left multiplication $b \mapsto a \times b$, it is easy to check that

$$L_a L_{a^2}(b) = L_{a^2} L_a(b) \text{ for all } a, b \in B.$$

$\square$

Not every Jordan algebra is a special Jordan algebra.

Consider a Jordan algebra A. In the Jordan identity $[L_a, L_{a^2}] = 0$ the element a occurs with degree 3. Let us try to gain linear identities from this.

Replacing a by $a + rb$ with $r \in R, b \in A$, we obtain

$$\begin{aligned} 0 &= [L_{a+rb}, L_{(a+rb)^2}] \\ &= r(2[L_a, L_{ab}] + [L_b, L_{a^2}]) + r^2(2[L_b, L_{ab}] + [L_a, L_{b^2}]). \end{aligned}$$

Putting $r = 1$ and $r = \frac{1}{2}$ and combining the resulting relations we derive

$$[L_b, L_{a^2}] + 2[L_a, L_{ab}] = 0.$$

Again linearizing by replacing a by $a + c$ we obtain

$$[L_a, L_{bc}] + [L_b, L_{ac}] + [L_c, L_{ab}] = 0.$$

Applying this to an $x \in A$ we have

$$a[(bc)x] + b[(ac)x] + c[(ab)x] = (bc)(ax) + (ac)(bx) + (ab)(cx).$$

Interpreting this as a transformation on a for fixed $b, c, x \in A$ and renaming our elements we conclude

3.11 Identity. *In any Jordan algebra we have the identity*

$$L_{(ab)c} = L_{ab}L_c + L_{bc}L_a + L_{ca}L_b - L_aL_cL_b - L_bL_cL_a.$$

This shows that $L_{(ab)c}$ belongs to the subalgebra of $M(A)$ generated by left multiplications with one, or products of two, of the elements $\{a, b, c\}$. Hence, similar to the alternative case considered in 3.6, we have the following relationship between A and $M(A)$:

3.12 Generating sets in Jordan algebras.

Let A be a Jordan algebra over the ring R in which 2 is invertible.

(1) If A is generated by $\{a_\lambda\}_\Lambda$ as an R-algebra, then $M(A)$ is generated as an R-algebra by

$$\{L_{a_\lambda a_\mu}, L_{a_\lambda}, id_A \mid \lambda, \mu \in \Lambda\}.$$

(2) If A is finitely generated as R-algebra, then $M(A)$ is also finitely generated as an R-algebra.

Proof. Every $a \in A$ is a linear combination of finite product of a_λ's. By 3.11, L_a is a linear combination of products of $L_{a_\lambda a_\mu}$'s and L_{a_ν}'s and hence belongs to the subalgebra generated by these elements. □

3.13 Module finite Jordan algebras.

Let A be a Jordan algebra over the ring R with $\frac{1}{2} \in R$. If A is finitely generated as an R-module, then $M(A)$ is also finitely generated as an R-module.

Proof. Let the R-module A be generated by $a_1, \ldots, a_n$. Every element in $M(A)$ is a linear combination of products of L_{a_i}'s. We show that the finitely many products of the form

$$L_{a_1}^{\varepsilon_1} L_{a_2}^{\varepsilon_2} \cdots L_{a_n}^{\varepsilon_n} L_{a_{\sigma(1)}}^{\delta_1} L_{a_{\sigma(2)}}^{\delta_2} \cdots L_{a_{\sigma(n)}}^{\delta_n}, \quad \varepsilon_i \in \{0, 2\},\ \delta_i \in \{0, 1\},\ \sigma \in \mathcal{S}_n,$$

where $\mathcal{S}_n$ denotes the group of permutations of n elements, generate the R-module $M(A)$.

We see from 3.11 that a product of the form $L_{a_i}L_{a_j}L_{a_k}$ can be replaced by $L_{a_k}L_{a_j}L_{a_i}$ plus expressions of lower degree in the L_{a_i}'s. Hence, if L_{a_i} occurs several times in a product, we finally arrive at partial products of the form $L_{a_i}^2$, $L_{a_i}L_{a_j}L_{a_i}$ or $L_{a_i}^3$.

Again referring to 3.11, the last two expressions can be replaced by formulas of lower degree and the $L_{a_i}^2$ can be collected at the left side with increasing indices. □

Remark. The linear factors could be arranged such that the next but one factors have a higher index. In fact, by careful analysis one obtains that $M(A)$ can be generated as an R-module by $\leq \binom{2n+1}{n}$ elements (compare Theorem 13 in Chapter II of [19]).

The interest in the next class of algebras - named after the mathematician *Sophus Lie* - stems from their relationship to topological groups (*Lie groups*). For a detailled study of this interplay see, for example, [34].

3.14 Lie algebras. Definiton.

An algebra A over a ring R is called a *Lie algebra* if for all $a, b, c \in A$,

$$a^2 = 0 \quad \text{and} \quad a(bc) + b(ca) + c(ab) = 0 \text{ (Jacobi identity).}$$

These properties imply in particular

$$0 = (a + b)^2 = ab + ba, \text{ i.e., } ab = -ba \text{ (anti-commutativity).}$$

Hence the multiplication algebra $M(A)$ coincides with the left multiplication algebra $L(A)$ and the Jacobi identity can be written as

$$L_{ab} = L_a L_b - L_b L_a, \text{ or } [L_a, L_b] = L_{ab}.$$

From this the following is obvious:

3.15 Generating sets in Lie algebras.

Let A be a Lie algebra which is generated by $\{a_\lambda\}_\Lambda$ as an R-algebra. Then $M(A)$ is generated as an R-algebra by

$$\{L_{a_\lambda}, id_A \mid \lambda \in \Lambda\}.$$

By our next observations Lie algebras are intimately related to associative algebras.

3.16 Special Lie algebras.

Let $(B, +, \cdot)$ be an associative algebra over the ring R. Then the new product for $a, b \in B$,

$$[a, b] = a \cdot b - b \cdot a,$$

turns $(B, +, [\ ,\ ])$ into a Lie algebra.

Algebras isomorphic to a subalgebra of an algebra of type $(B, +, [\ ,\ ])$ are called special Lie algebras.

Proof. It is straightforward to verify the identities required. □

It follows from the *Poincare-Birkhoff-Witt Theorem* that every Lie algebra A over R which is free as an R-module is in fact a special Lie algebra (e.g., [4]).

Let M be any finitely generated free R-module, i.e., $M \simeq R^n$, for $n \in I\!N$. Then the Lie algebra $(End_R(M), +, [\ ,\ ])$ is denoted by $gl(M,R)$ or $gl(n,R)$ (the *general linear group*). Subalgebras of $gl(n,R)$ are called *linear Lie algebras*. The matrices with trace zero form such a subalgebra (denoted by $sl(n,R)$). Other subalgebras of $gl(n,R)$ are (upper) triangular matrices (with trace zero, or diagonal zero) and skew symmetric matrices.

For any non-associative R-algebras $(A, +, \cdot)$, the commutator defines a new R-algebra $(A, +, [\ ,\ ])$ which is usually denoted by $A^{(-)}$. Obviously this is always an anti-commutative algebra but other identities depend on properties of $(A, +, \cdot)$.

In particular, the algebra A is called *Lie admissible* if $A^{(-)}$ is a Lie algebra.

As noticed in 3.16, any associative algebra is Lie admissible. An algebra A is called *left symmetric* ([155]) if

$$(a, b, c) = (b, a, c) \quad \text{for all} \quad a, b, c \in A.$$

Left symmetric algebras are Lie admissible (see Exercise (7)). See [61] for a connection between left symmetric products and flat structures on a Lie algebra.

Notice that alternative algebras need not be Lie admissible. For an alternative algebra A, the algebra $A^{(-)}$ satisfies identities which define a new class of algebras containing all Lie algebras:

3.17 Malcev algebras. Definiton.

An R-algebra A is called a *Malcev algebra* if for all $a, b, c, d \in A$,

$$a^2 = 0 \quad \text{and} \quad (ab)(ac) = ((ab)c)a + ((bc)a)a + ((ca)a)b.$$

Notice that the characterizing identity is quadratic in a. If 2 is invertible in R, Malcec algebras can be characterized by a multilinear identity (in 4 variables, see Exercise (6)).

Every Lie algebra is a Malcev algebra. The structure of Malcev algebras is closely related to Lie algebras based on the following fact (from [133]):

3.18 Simple Malcev algebras.

Let A be a central simple Malcev R-algebra which is not a Lie algebra, and assume R is a field of characteristic $\neq 2, 3$. Then A is of the form $D^{(-)}/R$, where D is a Cayley-Dickson algebra over R.

An R-algebra A is called *Malcev admissible* if the attached algebra $A^{(-)}$ is a Malcev algebra. As mentioned above, every alternative algebra is Malcev admissible (see [29, Proposition 1.4]). For a detailled study of these algebras we refer to [29].

3.19 Exercises.

(1) Let A be an alternative algebra. Prove that the product of any two ideals is again an ideal in A.

(2) For an alternative algebra A, consider the map (see 3.3)

$$f : A^4 \to A,\ (a,b,c,d) \mapsto (ab,c,d) - (b,c,d)a - b(a,c,d).$$

Prove:

(i) $3f(a,b,c,d) = [a,(b,c,d)] - [b,(c,d,a)] + [c,(d,a,b)] - [d,(a,b,c)]$;

(ii) $f(a,b,c,d) = ([a,b],c,d) + ([c,d],a,b)$.

(3) Let A be an R-module over a ring R, with 2 invertible in R.

Assume $\beta : A \times A \to R$ to be a symmetric bilinear form and $e \in A$ satisfying $\beta(e,e) = 1$. Prove that the product of $a, b \in A$,

$$a \cdot b := \beta(e,a)b + \beta(e,b)a - \beta(a,b)e,$$

turns A into a Jordan algebra with unit e.

(4) Consider an eight dimensional algebra A with basis $1, a_1, a_2, \ldots, a_7$, over a field K with char $K \neq 2$, given by the multiplication table ([41])

	a_1	a_2	a_3	a_4	a_5	a_6	a_7
a_1	α	a_3	αa_2	a_5	αa_4	$-a_7$	$-\alpha a_6$
a_2	$-a_3$	β	$-\beta a_1$	a_6	a_7	βa_4	βa_5
a_3	$-\alpha a_2$	βa_1	$-\alpha\beta$	a_7	αa_6	$-\beta a_5$	$-\alpha\beta a_4$
a_4	$-a_5$	$-a_6$	$-a_7$	γ	$-\gamma a_1$	$-\gamma a_2$	$-\gamma a_3$
a_5	$-\alpha a_4$	$-a_7$	$-\alpha a_6$	γa_1	$-\alpha\gamma$	γa_3	$\alpha\gamma a_2$
a_6	a_7	$-\beta a_4$	βa_5	γa_2	$-\gamma a_3$	$-\beta\gamma$	$-\beta\gamma a_1$
a_7	αa_6	$-\beta a_5$	$\alpha\beta a_4$	γa_3	$-\alpha\gamma a_2$	$\beta\gamma a_1$	$\alpha\beta\gamma$

with non-zero $\alpha, \beta, \gamma \in K$. Show that A is an alternative central simple algebra which is not associative (Cayley-Dickson algebra).

(5) Let A be any R-algebra. An R-linear map $D : A \to A$ is called a derivation *if*

$$D(ab) = D(a)b + aD(b),\ \text{for any } a, b \in A.$$

Show that the set of derivations of A is a subalgebra of the Lie algebra $End_R(A)^{(-)}$.

(6) Let A be an anti-commutative R-algebra and assume 2 to be invertible in R. Put

$$J(a,b,c) := (ab)c + (bc)a + (ca)b \qquad \text{(Jacobian)}.$$

Prove that the following are equivalent ([238]):

(a) A is a Malcev algebra;

(b) $J(a,b,ac) = J(a,b,c)a$, for all $a,b,c \in A$;

(c) $J(a,ab,c) = J(a,b,c)a$, for all $a,b,c \in A$;

(d) $(ab)(cd) = a((db)c) + d((bc)a) + b((ca)d) + c((ad)b)$, for all $a,b,c,d \in A$.

(7) Show that an algebra A is Lie admissible if and only if for all $a,b,c \in A$,

$$(a,b,c) + (b,c,a) + (c,a,b) - (a,c,b) - (c,b,a) - (b,a,c) = 0\,.$$

References: Bakhturin [3], Baues [61], Bruck-Kleinfeld [92], Filippov [133, 134], Grabmeier-Wisbauer [147], Helmstetter [155], Jacobson [19, 20], Kaup [177], Kleinfeld [180], Moufang [207], Myung [29], Schafer [37], Smiley [244], Wisbauer [267], Zhevlakov-Slinko-Shestakov-Shirshov [41].

Chapter 2

Modules over associative algebras

In this chapter A will always denote an associative R-algebra with unit.

4 Generalities

1.Modules over algebras. 2.Bimodules. 3.The category $\sigma[M]$. 4.Properties of $\sigma[M]$.

Any module over an R-algebra may be considered as R-module. Related to this there are some canonical operations we want to describe.

4.1 Modules over algebras.
A unital left module over A is an abelian group $(M, +)$ with a ring homomorphism $\varphi : A \to End(_{\mathbb{Z}}M)$. As usual we write $\varphi(a)m = am$ for $a \in A$, $m \in M$ (e.g. [40, 6.1]). Together with the map $R \to A$, $r \mapsto r1$, we have a ring homomorphism

$$R \to A \to End(_{\mathbb{Z}}M),$$

which makes M a (left) R-module. We write morphisms of left modules on the right, i.e., we consider M as right module over $End(_AM)$.

The multiplication with elements of R, $L_r : M \to M$, $m \mapsto rm$ obviously yields an A-endomorphism and we obtain a ring homomorphism

$$\alpha : R \to End(_AM), \quad r \mapsto L_r,$$

whose image is in fact in the centre of $End(_AM)$ and hence turns $End(_AM)$ into an R-algebra. α makes M a right R-module in a canonical way and $rm = mr$ for all $r \in R$, $m \in M$.

Obviously α maps idempotents of R to central idempotents of $End(_AM)$. We say that α is *epic on central idempotents* if every central idempotent in $End(_AM)$ is in the image of α. For example, this is the case if α is an isomorphism between R and the centre of $End(_AM)$.

From the above it is obvious that a unital R-module M is a module over the R-algebra A if and only if there exists an R-algebra morphism

$$\varphi : A \to End(_RM).$$

In this case the R-module structure of M defined by $R \to A$, $r \mapsto r1$, coincides with the given R-module structure of M.

4.2 Bimodules.

Assume A and B are unital associative R-algebras. An abelian group M is called an (A, B)-*bimodule* if M is a left A-module, a right B-module and

$$a(mb) = (am)b \text{ and } rm = mr, \text{ for all } a \in A,\ b \in B,\ m \in M,\ r \in R.$$

In this case right multiplication of M with elements of B yields A-endomorphisms of M, and left multiplication of M with elements of A yields B-endomorphisms of M.

For a left A-module M, denote $B = End({}_AM)$. Then M is a right B-module, in fact M is an (A, B)-bimodule or - equivalently - a left module over the R-algebra $A \otimes_R B^o$. $End({}_{A\otimes B^o}M)$ is equal to the centre of B, since

$$\begin{aligned} End({}_{A\otimes B^o}M) &= \{f \in End({}_RM) \,|\, (a \otimes b)(m)f = (amb)f \text{ for } m \in M, a \in A, b \in B\} \\ &= \{f \in B \,|\, (m)bf = (mb)f = (m)fb \text{ for } m \in M, b \in B\} \ = Z(B)\,. \end{aligned}$$

The $A \otimes_R B^o$-submodules are the fully invariant submodules of ${}_AM$, and the direct $A \otimes_R B^o$-summands U of M are of the form $U = Me$, for a central idempotent $e \in B$.

The category of all unital left A-modules will be denoted by A-*Mod*. Let M and N be A-modules. N is called M-*generated* if it is a homomorphic image of a direct sum of copies of M.

4.3 The category $\sigma[M]$.

For any A-module M, we denote by $\sigma[M]$ the full subcategory of A-*Mod* whose objects are submodules of M-generated modules.

$\sigma[M]$ is the smallest subcategory of A-*Mod* which contains M and is closed under direct sums, factor modules and submodules.

So internal properties of the module M are closely related to properties of the category $\sigma[M]$. This is the starting point for *homological characterizations* of modules. In particular, there are enough injectives in $\sigma[M]$ and hence localizaton techniques known from module (or Grothendieck) categories apply.

For the convenience of the reader we shall collect old and new results about the category $\sigma[M]$ in the next sections. Here we recall some basic facts:

4.4 Properties of $\sigma[M]$. *Let M be an A-module.*

(1) Morphisms in $\sigma[M]$ have kernels and cokernels.

(2) For any family $\{N_\lambda\}_\Lambda$ in $\sigma[M]$, the coproduct (direct sum) $\bigoplus_\Lambda N_\lambda$ in A-Mod belongs to $\sigma[M]$ and is the coproduct in $\sigma[M]$.

(3) The finitely generated (cyclic) submodules of $M^{(\mathbb{N})}$ form a set of generators in $\sigma[M]$. The direct sum of these modules is a generator in $\sigma[M]$.

(4) Objects in $\sigma[M]$ are finitely (co-) generated in $\sigma[M]$ if and only if they are finitely (co-) generated in A-Mod.

(5) For a family $\{N_\lambda\}_\Lambda$ in $\sigma[M]$, let $\prod_\Lambda N_\lambda$ denote the product in A-Mod (cartesian product). Then

$$\prod_\Lambda^M N_\lambda := Tr(\sigma[M], \prod_\Lambda N_\lambda)$$

is the product of $\{N_\lambda\}_\Lambda$ in $\sigma[M]$.

(6) Every simple module in $\sigma[M]$ is a subfactor of M.

(7) If M is a faithful A-module and finitely generated as $End_A(M)$-module, then $\sigma[M] = A$-Mod.

Proof. See 15.1, 15.4 and the proof of 18.5 in [40]. □

5 Projectivity and generating

1.Projective modules. 2.Self-generators. 3.M as a generator in $\sigma[M]$. 4.Density Theorem. 5.Generators in A-Mod. 6.Examples. 7.Intrinsically projective modules. 8.Properties of intrinsically projective modules. 9.Ideal modules. 10.Self-progenerator. 11.Properties of the trace. 12.The socle of a module. 13.Exercises.

5.1 Projective modules.
Let M and P be A-modules. P is said to be *M-projective* if every diagram in *A-Mod* with exact row

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow & & \\ M & \xrightarrow{g} & N & \longrightarrow & 0 \end{array}$$

can be extended commutatively by some morphism $P \to M$.

So, P is M-projective if and only if $Hom_A(P,-)$ is exact with respect to all exact sequences $0 \to K \to M \to N \to 0$ in *A-Mod*.

If M is M-projective, then it is also called *self-* (or *quasi-) projective.*

Assume M is self-projective and $K \subset M$ is a fully invariant submodule, i.e., for any $f \in End({}_AM)$, $Kf \subset K$. Then M/K is also self-projective.

A finitely generated self-projective module M is also N-projective for every $N \in \sigma[M]$, i.e., it is projective in $\sigma[M]$ (see [40, Section 18]).

A submodule K of M is *superfluous* or *small* in M, written $K \ll M$, if, for every submodule $L \subset M$, $K + L = M$ implies $L = M$.

Let $N \in \sigma[M]$. By a *projective cover of N in $\sigma[M]$* we mean an epimorphism $\pi : P \to N$ with P projective in $\sigma[M]$ and $Ke\,\pi \ll P$.

Definitions. M is called a *self-generator* if it generates all its submodules. A module $N \in \sigma[M]$ is a *generator in $\sigma[M]$* if N generates all modules in $\sigma[M]$.

5.2 Self-generators.
For an A-module M with $B = End({}_AM)$, the following are equivalent:

(a) M is a self-generator;

(b) for every A-submodule $U \subset M$, $U = MHom_A(M,U)$;

(c) the map $U \mapsto Hom_A(M,U)$ from submodules of M to left ideals of B is injective.

Assume M is a self-generator. Then:

(1) For any $m \in M$, $\beta \in End(M_B)$, there exists $a \in A$ with $\beta(m) = am$.
Hence A-submodules of M are precisely its $End(M_B)$-submodules.

(2) If $K \subset M$ is a fully invariant submodule, then M/K is also a self-generator.

Proof. The equivalences are easily derived from the fact that a submodule $U \subset M$ is M-generated if and only if $U = MHom_A(M, U)$.

(1) follows from the proof of the Density Theorem (e.g., [40, 15.7]).

(2) A submodule in M/K is of the form L/K, for some submodule L with $K \subset L \subset M$. By assumption, there exists an epimorphism $M^{(\Lambda)} \to L$. Since every component of this map is an endomorphism of M, we see that the submodule $K^{(\Lambda)} \subset M^{(\Lambda)}$ is contained in the kernel of the epimorphism

$$M^{(\Lambda)} \to L \to L/K.$$

Hence L/K is generated by M/K. □

The following properties of generators are shown in [40, 15.5 and 15.9].

5.3 M as a generator in $\sigma[M]$.

For an A-module M with $B = End(_AM)$, the following are equivalent:

(a) M is a generator in $\sigma[M]$;

(b) the functor $Hom_A(M, -) : \sigma[M] \to B$-Mod is faithful;

(c) M generates every (cyclic) submodule of $M^{(\mathbb{N})}$;

(d) $M^{(\mathbb{N})}$ is a self-generator;

(e) for every (cyclic) submodule $U \subset M^{(\mathbb{N})}$, $U = MHom_A(M, U)$.

If M is a generator in $\sigma[M]$, then M is a flat $End(_AM)$-module.

There is another important property of generators and cogenerators (see 6.3) in $\sigma[M]$ shown in [40, 15.7 and 15.8]:

5.4 Density Theorem.

Let M be an A-module with $B = End(_AM)$ having one of the following properties:

(i) M is a generator in $\sigma[M]$, or

(ii) for any cyclic submodule $U \subset M^n$, $n \in \mathbb{N}$, the factor module M^n/U is cogenerated by M. Then:

(1) For any finitely many $m_1, \ldots, m_n$ in M and $\alpha \in End(M_B)$, there exists $a \in A$ with $\alpha(m_i) = am_i$ for all $i = 1, \ldots, n$.

In this case $A/An_A(M)$ is said to be a *dense subring of $End(M_B)$*.

(2) Every A-submodule of $M^{(\Lambda)}$ is an $End(M_B)$-submodule of $M^{(\Lambda)}$.

(3) If M is finitely generated over $End(_AM)$, then $A/An_A(M) \simeq End(M_B)$.

Generators in the full module category have additional chacterizations:

5.5 Generators in A-*Mod*.

Let M be an A-module and $B = End(_AM)$.

(1) *M is a generator in A-Mod if and only if*

  (i) *M_B is finitely generated and B-projective;*

  (ii) *$A \simeq End(M_B)$.*

(2) *Assume M is a generator in A-Mod and B is commutative. Then M is a finitely generated and projective A-module.*

Proof. (1) See, for example, [40, 18.8].

(2) By (1), M_B is finitely generated, projective and faithful. Since B is commutative this implies that M_B is a generator in *Mod*-B (see [40, 18.11]). Now applying (1) to the module M_B, we conclude that M is finitely generated and projective as module over $A \simeq End(M_B)$. □

Studying the relationship between submodules of M and left ideals of the endomorphism ring a weak form of projectivity (introduced in Brodskii [90]) turns out to be of interest:

An A-module M is said to be *intrinsically projective* if every diagram with exact row

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow & & \\ M^n & \longrightarrow & N & \longrightarrow & 0\,, \end{array}$$

where $n \in I\!N$ and $N \subset M$, can be extended commutatively by some $M \to M^n$.

M is called *semi-projective* if the above condition (only) holds for $n = 1$. As easily seen, M is semi-projective if and only if for every cyclic left ideal $I \subset End(_AM)$, $I = Hom_A(M, MI)$ (see [40], before 31.10).

Of course every self-projective module is intrinsically projective. However there are also other types of examples:

5.6 Examples. Let M be an A-module with $B = End(_AM)$.

(1) If kernels of endomorphisms of M are M-generated and B is a left PP ring, then M is semi-projective.

(2) If M_B is flat and B is left semihereditary, then M is intrinsically projective.

(3) If B is a regular ring, then M is intrinsically projective.

Proof. (1) For any $f \in B$, Bf is projective, and by [40, 39.10], $Tr(M, Kef)$ is a direct summand in M. By our condition $Tr(M, Kef) = Kef$.

(2) Since M_B is flat, the kernel of any $g : M^k \to M^k$, $k \in I\!N$, is M-generated (see [40, 15.9]). Since B is left semihereditary, $End({}_AM^k)$ is left PP (semihereditary) by [40, 39.13] and we see from (1) that $Ke\,g$ is a direct summand in M^k.

From this we conclude that the exact sequence in the diagram for the definition of intrinsically projective is in fact splitting. Hence M is intrinsically projective.

(3) This is a special case of (2). □

Now we turn to general properties of the modules just introduced.

5.7 Intrinsically projective modules.

For an A-module M with $B = End({}_AM)$, the following are equivalent:

(a) M is intrinsically projective;

(b) $I = Hom_A(M, MI)$ for every finitely generated left ideal $I \subset B$.

If ${}_AM$ is finitely generated, then $(a), (b)$ are equivalent to:

(c) $I = Hom_A(M, MI)$ for every left ideal $I \subset B$;

(d) the map $I \mapsto MI$ from left ideals in B to submodules of M is injective.

Proof. $(a) \Rightarrow (b)$ Consider a left ideal $I \subset B$ generated by $\gamma_1, \dots, \gamma_k \in B$. From the exact sequence

$$M^k \xrightarrow{\sum \gamma_i} MI \longrightarrow 0$$

the functor $Hom_A(M, -)$ yields the sequence

$$Hom_A(M, M^k) \xrightarrow{\sum Hom(M,\gamma_i)} Hom_A(M, MI) \longrightarrow 0,$$

which is also exact by (a). Since $B^k \simeq Hom_A(M, M^k)$, this means that $\gamma_1, \dots, \gamma_k$ generate $Hom_A(M, MI)$ as B-module. Hence $I = Hom_A(M, MI)$.

$(b) \Rightarrow (a)$ Consider an exact sequence of A-modules

$$M^k \xrightarrow{f} U \longrightarrow 0 \text{ with } k \in I\!N,\ U \subset M.$$

With the canonical injections $\varepsilon_i : M \to M^k$, we form the left ideal $I = \sum B\varepsilon_i f \subset B$. Then $U = MI$ and the lower row in the following diagram is exact

$$\begin{array}{ccccc}
Hom_A(M, M^k) & \xrightarrow{Hom(M,f)} & Hom_A(M, U) & & \\
\downarrow\simeq & & \downarrow\simeq & & \\
B^k & \xrightarrow{\sum \varepsilon_i f} & I & \longrightarrow & 0.
\end{array}$$

From this we see that $Hom_A(M, f)$ is epic, i.e., M is intrinsically projective.

$(a) \Rightarrow (c)$ Assume M is intrinsically projective and finitely generated. Then it is easy to see that $Hom_A(M, -)$ is exact on all exact sequences

$$M^{(\Lambda)} \longrightarrow U \longrightarrow 0\,, \text{ where } U \subset M, \ \Lambda \text{ any set.}$$

Hence the assertion is shown with the same proof as $(a) \Rightarrow (b)$.

$(c) \Rightarrow (a)$ and $(c) \Rightarrow (d)$ are obvious.

$(d) \Rightarrow (c)$ This is an immediate consequence of the equality $MI = MHom_A(M, MI)$, for any left ideal $I \subset B$. □

5.8 Properties of intrinsically projective modules.

Let M be an intrinsically projective A-module, which is finitely generated by elements $m_1, \ldots, m_k \in M$, and denote $B = End(_AM)$. Then the map

$$B \to M^k, \quad s \mapsto (m_1, \ldots, m_k)s,$$

is a monomorphism of right B-modules.

(1) For every left ideal $I \subset B$, $(m_1, \ldots, m_k)B \cap M^k I = (m_1, \ldots, m_k)I$.

(2) For every proper left ideal $I \subset B$, $MI \neq M$.

(3) If M_B is flat, then $(m_1, \ldots, m_k)B$ is a pure submodule of M_B^k.

(4) If M_B is projective, then B is isomorphic to a direct summand of M_B^k and hence M_B is a generator in Mod-B.

Proof. (1) Certainly the right hand side is contained in the left hand side.

Assume $(m_1, \ldots, m_k)f \in M^k I = (MI)^k$ for $f \in B$. Then $Mf = \sum_i Am_i f \subset MI$ and hence $f \in Hom(M, MI) = I$ (since M is intrinsically projective).

(2) This is an obvious consequence of (1)

(3) If M_B is flat, then M_B^k is also flat and by (1), $(m_1, \ldots, m_k)B \simeq B$ is a pure submodule of M_B^k (e.g., [40, 36.6]).

(4) Since M_B^k is projective, the factor module $M_B^k/(m_1, \ldots, m_k)B$ is pure-projective (see [40, 34.1]) and hence projective by (3). So B is isomorphic to a direct summand of M_B^k. □

Definitions. Let A, B denote associative unital R-algebras. An (A, B)-bimodule M is said to be a *B-ideal module* if the map $I \mapsto MI$ defines a bijection between the left ideals of B and the A-submodules of M (with inverse $K \mapsto \{b \in B \mid Mb \subset K\}$).

An A-module M is called *ideal module* if - considered as $(A, End(_AM))$-module - M is an $End(_AM)$-ideal module, i.e., if the map $I \mapsto MI$ is a bijection between the left ideals of $End(_AM)$ and the A-submodules of M (with inverse $K \mapsto Hom_A(M, K)$).

For these modules we have interesting characterizations (compare [276, Proposition 2.6], [139, Lemma 2.4]):

5.9 Ideal modules.

Let A, B denote associative unital R-algebras and M an (A,B)-bimodule, which is finitely generated as A-module.

(1) The following are equivalent:

(a) M is a B-ideal module;

(b) every submodule of ${}_AM$ is of the form MI for some left ideal $I \subset B$, and M_B is faithfully flat.

For $B = End({}_AM)$, (a)-(b) are equivalent to:

(c) M is intrinsically projective and a self-generator.

(2) If M is a B-ideal module then for any multiplicative subset $S \subset R$, the (AS^{-1}, BS^{-1})-bimodule MS^{-1} is a BS^{-1}-ideal module (see 16.5).

(3) Assume $B = End({}_AM)$ and M is an ideal module. Then:

(i) B is isomorphic to a pure submodule of M_B^k, $k \in \mathbb{N}$.

(ii) If B is right perfect, then M_B is a generator in Mod-B and ${}_AM$ is a projective generator in $\sigma[M]$.

Proof. (1) $(a) \Rightarrow (b)$ We show that M_B is faithfully flat. Without restriction, for any $f \in B$ we may assume $f \in End({}_AM)$ and we have the exact commutative diagram with canonical map μ_f

$$\begin{array}{ccccccccc} & & M \otimes_B Hom_A(M, Ke\,f) & \rightarrow & M \otimes_B B & \rightarrow & M \otimes_B Bf & \rightarrow & 0 \\ & & \downarrow & & \downarrow_{\simeq} & & \downarrow_{\mu_f} & & \\ 0 & \rightarrow & Ke\,f & \rightarrow & M & \rightarrow & Mf & \rightarrow & 0 \end{array}.$$

Since $Ke\,f$ is M-generated the first vertical map is epic and hence μ_f is an isomorphism.

Using the relation $M(I \cap J) = MI \cap MJ$ for left ideals I, J of B, it is straightforward to show by induction (on the number of generators of I as B-module) that $\mu_I : M \otimes_B I \rightarrow MI$ is an isomorphism for every finitely generated left ideal $I \subset B$ and therefore M_B is flat. Since $MI \neq M$ for every proper left ideal $I \subset B$, the B-module M_B is faithfully flat.

$(b) \Rightarrow (a)$ We prove that $I \mapsto MI$ is injective. Assume $MI = MJ$ for left ideals $I, J \subset B$. Without restriction we may assume $J \subset I$. Then $M \otimes_B I/J \simeq MI/MJ = 0$ and hence $I = J$ since M_B is faithfully flat.

$(a) \Leftrightarrow (c)$ follows immediately from 5.2 and 5.7.

(2) Assume M is a B-ideal module. Every AS^{-1}-submodule $U \subset MS^{-1}$ is of the form $U'S^{-1}$ for some submodule $U' \subset M$ (see 16.5). Since $U' = MI$ for some left ideal $I \subset B$, we conclude

$$U = U'S^{-1} = MIS^{-1} = MS^{-1}IS^{-1}.$$

Since M_B is a faithfully flat B-module, MS^{-1} is a faithfully flat BS^{-1}-module. Now the assertion follows from (1).

(3) (i) By (1), M_B is a flat module and the assertion follows from 5.8(3).

(ii) Since B is right perfect, M_B is projective (by (1)), and M_B is a generator in *Mod-B* by 5.8(4). Hence M is (finitely generated and) projective over its biendomorphism ring $End(M_B)$ (see 5.5). Since the A-module structure of M is identical to its $End(M_B)$-module structure (see 5.2) we conclude that M is self-projective as an A-module and, by [40, 18.5], M is a projective generator in $\sigma[M]$. □

Definitions. A finitely generated, projective generator N in $\sigma[M]$ is called a *progenerator* in $\sigma[M]$. If M is a progenerator in $\sigma[M]$ we call it a *self-progenerator*. Of course, every self-progenerator is an ideal module.

The importance of this notion is clear by the following characterizations:

5.10 Self-progenerator.

For a finitely generated A-module M with $B = End({}_AM)$, the following are equivalent:

(a) M is a self-progenerator;

(b) M is a generator in $\sigma[M]$ and M_B is faithfully flat;

(c) (i) for every left ideal $I \subset B$, $MI \neq M$ and
(ii) for every finitely M-generated A-module U, the canonical map $M \otimes_B Hom_A(M,U) \to U$ is injective (bijective);

(d) there are functorial isomorphisms
(i) $id_{B-Mod} \simeq Hom_A(M, M \otimes_B -)$ and
(ii) $M \otimes_B Hom_A(M,-) \simeq id_{\sigma[M]}$;

(e) $Hom_A(M,-) : \sigma[M] \to B$-Mod is an equivalence of categories.

Proof. $(a) \Leftrightarrow (b) \Leftrightarrow (e)$ follow from [40, 18.5 and 46.2].

For any A-submodule $K \subset M^n$, we put $U = M^n/K$ and form the commutative diagram with exact lower row,

$$\begin{array}{ccccccccc} & & M \otimes_B Hom_A(M,K) & \to & M \otimes_B Hom_A(M,M^n) & \to & M \otimes_B Hom_A(M,U) & \to & 0 \\ & & \downarrow \varphi_1 & & \downarrow\simeq & & \downarrow \varphi_2 & & \\ 0 & \to & K & \to & M^n & \to & U & \to & 0. \end{array}$$

$(a) \Rightarrow (c)$ Since M is self-projective, the upper row is also exact, and by the Kernel Cokernel Lemma, φ_2 is injective if and only if φ_1 is surjective.

M_B being flat, property (i) is equivalent to M_B being faithfully flat. We know this from $(a) \Leftrightarrow (b)$.

$(c) \Rightarrow (b)$ Since φ_2 is an isomorphism, the upper row is exact. Again by the Kernel Cokernel Lemma, φ_1 is surjective and M is a generator in $\sigma[M]$.

$(d) \Leftrightarrow (e)$ This characterizes $Hom_B(M, -)$ to be an equivalence (e.g., [40, Section 46]). $\square$

Let $\mathcal{U}$ be a class of A-modules and N an A-module. N is said to be *$\mathcal{U}$-generated*, if there exists an epimorphism $\bigoplus_\Lambda U_\lambda \to N$ with $U_\lambda \in \mathcal{U}$.
The sum of all $\mathcal{U}$-generated submodules of N is called the *trace of $\mathcal{U}$ in N*,

$$Tr(\mathcal{U}, N) = \sum\{Im\, h \mid h \in Hom(U, N),\ U \in \mathcal{U}\}.$$

If $\mathcal{U} = \{U\}$ we simply write $Tr(U, L) = Tr(\{U\}, L)$.

From [40, 13.5] we obtain:

5.11 Properties of the trace.

Let U and N be A-modules.

(1) $Tr(U, N)$ is the largest submodule of N generated by U.

(2) $N = Tr(U, N)$ if and only if N is U-generated.

(3) $Tr(U, N)$ is an $End(_AN)$-submodule of N.

As a special case we consider the trace of all simple modules in M.

5.12 The socle of a module.

The *socle* of an A-module M is defined as the sum of all simple (minimal) submodules of M and we have the characterization (e.g., [40, 21.1 and 21.2])

$$\begin{aligned} Soc(M) &= \sum\{K \subset M \mid K \text{ is a simple submodule in } M\} \\ &= \bigcap\{L \subset M \mid L \text{ is an essential submodule in } M\}. \end{aligned}$$

and the following properties:

(1) For any morphism $f : M \to N$, $Soc(M)f \subset Soc(N)$.

(2) For any submodule $K \subset M$, $Soc(K) = K \cap Soc(M)$.

(3) $Soc(M) \trianglelefteq M$ if and only if $Soc(K) \neq 0$ for every non-zero submodule $K \subset M$.

(4) $Soc(M)$ is also an $End(_AM)$-submodule of M (fully invariant).

(5) $Soc(\bigoplus_\Lambda M_\lambda) = \bigoplus_\Lambda Soc(M_\lambda)$.

5.13 Exercises.

(1) Let M be a finitely generated A-module which is a generator in $\sigma[M]$. Assume that $End({}_AM)$ is a left semihereditary ring. Prove that M is self-projective (and hence a progenerator in $\sigma[M]$).

(2) Let M, N be A-modules and $B = End({}_AM)$.
N is called restricted M-projective *if the functor $Hom_A(N, -)$ is exact on exact sequences of the form*

$$M^n \to L \to 0, \text{ where } L \subset M^k,\ n, k \in I\!N.$$

Prove:

(i) If M_B is FP-injective and N is restricted M-projective, then $Hom_A(N, M)_B$ is FP-injecticve.

(ii) If $Hom_A(N, M)_B$ is FP-injecticve, then N is restricted M-projective.

(iii) The following are equivalent:
(a) Every A-module is restricted M-projective;
(b) M_B is flat and B is left semihereditary.

References: Brodskii [90], Fuller [139], Wisbauer [40, 275].

6 Injectivity and cogenerating

1.Injective modules. 2.Socles of weakly injective modules. 3.Self-cogenerators. 4.Weak cogenerators. 5.Quasi-Frobenius modules. 6.M as a cogenerator in $\sigma[M]$. 7.Cogenerator with commutative endomorphism ring. 8.Self-projective cogenerators. 9.Semisimple modules. 10.Noetherian QF modules. 11.Examples. 12.Intrinsically injective modules. 13.Finitely cogenerated modules. 14.Radical of modules. 15.Quasi-regularity. 16.The Jacobson radical. 17.Relation between radicals. 18.Co-semisimple modules. 19.Exercises.

6.1 Injective modules.

Let U and M be A-modules. U is said to be *M-injective* if every diagram in A-*Mod* with exact row

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \xrightarrow{f} & M \\ & & \downarrow & & \\ & & U & & \end{array}$$

can be extended commutatively by a morphism $M \to U$.

Clearly U is M-injective if and only if $Hom_A(-, U)$ is exact with respect to all exact sequences $0 \to K \to M \to N \to 0$.

If U is M-injective, then it is also N-injective for every $N \in \sigma[M]$ (see [40, Section 16]). In case U belongs to $\sigma[M]$, it is M-injective if and only if every exact sequence $0 \to U \to L \to N \to 0$ in $\sigma[M]$ splits, i.e., U is *injective in* $\sigma[M]$. As a consequence we observe that every injective module in $\sigma[M]$ is M-generated.

M is said to be *self-injective* (or *quasi-injective*) if it is M-injective.

A submodule $K \subset M$ is called *essential* or *large* in M if, for every non-zero submodule $L \subset M$, $K \cap L \neq 0$. Then M is an *essential extension* of K and we write $K \trianglelefteq M$.

A monomorphism $f : L \to M$ is called *essential* if $Im\, f \trianglelefteq M$.

Consider $N \in \sigma[M]$. An injective module E in $\sigma[M]$ together with an essential monomorphism $\varepsilon : N \to E$ is called an *injective hull of N in* $\sigma[M]$ or an *M-injective hull of N* and is usually denoted by $\widehat{N}$. Every module N in $\sigma[M]$ has an injective hull $\widehat{N}$ in $\sigma[M]$. Injective hulls are unique up to isomorphism.

An A-module U is called *weakly M-injective* if every exact diagram in A-*Mod*,

$$\begin{array}{ccccc} 0 & \longrightarrow & K & \longrightarrow & M^{(\mathbb{N})} \\ & & \downarrow & & \\ & & U & & , \end{array}$$

with K *finitely generated*, can be extended commutatively by a morphism $M^{(\mathbb{N})} \to U$.

Any direct sum of weakly M-injective modules is again weakly M-injective. Weakly M-injective modules in $\sigma[M]$ are M-generated (see [40, 16.10 and 16.11]).

Weakly R-injective modules are also called *FP-injective*.

6.2 Socles of weakly injective modules.

For a weakly M-injective A-module M with $B = End({}_AM)$,

$$Soc\,_AM \subset Soc\,M_B.$$

Proof. Let $m \in Soc\,_AM$ with Am a simple A-module. Then mB is a simple B-module. Indeed, take any $f \in B$ with $mf \neq 0$. Then for each $g \in B$ we have the diagram with exact row,

$$\begin{array}{ccccc} 0 & \longrightarrow & Am & \xrightarrow{f} & M \\ & & \downarrow g & & \\ & & M & & , \end{array}$$

which can be extended commutatively by some $t \in B$, and $mg = mft \in mf\,B$. Hence mB is simple and $Soc\,_AM$ is a sum of simple B-modules. $\square$

Definitions. An A-module is said to be *cogenerated by the A-module M* if it can be embedded into a product of copies of M.

M is called a *self-cogenerator* if it cogenerates all its factor modules. $N \in \sigma[M]$ is a *cogenerator in* $\sigma[M]$ if it cogenerates all modules in $\sigma[M]$.

M is said to be a *weak cogenerator (in $\sigma[M]$)* if for every finitely generated submodule $K \subset M^n$, $n \in I\!N$, the factor module M^n/K is cogenerated by M.

For any submodule $U \subset M$, we identify

$$Hom_A(M/U, M) = \{f \in End({}_AM) \,|\, Uf = 0\}.$$

This a right ideal in $End({}_AM)$.

For any subset $I \subset End({}_AM)$, we use the notation

$$Ke\,I = \bigcap\{Ke\,f \,|\, f \in I\}.$$

6.3 Self-cogenerators.

For an A-module M with $B = End({}_AM)$, the following are equivalent:

(a) M is a self-cogenerator;

(b) for every submodule $U \subset M$, $U = Ke\,Hom_A(M/U, M)$;

(c) the map $U \mapsto Hom_A(M/U, M)$ from submodules of M to right ideals of B is injective.

Assume M is a self-cogenerator. Then:

(1) For every $m \in M$ and $\beta \in End(M_B)$, there exists $a \in A$ with $\beta(m) = am$. Hence A-submodules are $End(M_B)$-submodules in M.

(2) If $K \subset M$ is a fully invariant submodule, then K is also a self-cogenerator.

Proof. The equivalences are shown with arguments dual to those used in 5.2.

(1) This is obtained by the proof of the Density Theorem [40, 15.7].

(2) Dualize the proof of 5.2(1). □

From [40, 48.1] we recall the following observation:

6.4 Weak cogenerators.

For any A-module M with $B = End({}_AM)$, the following are equivalent:

(a) M is a weak cogenerator (in $\sigma[M]$);

(b) M_B is weakly M_B-injective, and every finitely generated submodule $K \subset M^n$, $n \in I\!N$, is M-reflexiv.

If ${}_AM$ is a finitely generated weak cogenerator, then M_B is FP-injective.

A weakly M-injective weak cogenerator M is called a *Quasi-Frobenius (QF) module.* For such modules we have the following characterizations:

6.5 Quasi-Frobenius modules.

For a faithful A-module M with $B = End({}_AM)$, the following are equivalent:

(a) ${}_AM$ is a QF-module;

(b) (i) ${}_AM$ is weakly ${}_AM$-injective,
(ii) M_B is weakly M_B-injective and
(iii) A is dense in $End_B(M)$;

(c) M_B is a QF-module and A is dense in $End_B(M)$;

(d) ${}_AM$ and M_B are weak cogenerators in $\sigma[{}_AM]$, resp. $\sigma[M_B]$.

For any QF-module ${}_AM$, $Soc\,{}_AM = Soc\,M_B$.

Proof. The equivalences are given in [40, 48.2].

The final assertion follows from 6.2 applied to ${}_AM$ and M_B. □

The following properties of cogenerators are collected from [40], 14.6, 15.7, 15.9, 17.12 and 47.7:

6.6 M as a cogenerator in $\sigma[M]$.

For an A-module M with $B = End({}_AM)$, the following are equivalent:

(a) M is a cogenerator in $\sigma[M]$;

(b) the functor $Hom_A(-, M) : \sigma[M] \to$ Mod-B is faithful;

(c) M cogenerates injective hulls of simple modules in $\sigma[M]$.

If M is a cogenerator in $\sigma[M]$, then

(1) M is an FP-injective right B-module and

(2) $A/An_A(M)$ is dense in $End(M_B)$ (see 5.4).

We have seen in 5.5 that a generator in *A-Mod* with a commutative endomorphism ring is a projective A-module. For cogenerators with this property we have even stronger consequences:

6.7 Cogenerator with commutative endomorphism ring.

Let M be an A-module and assume $B = End(_AM)$ to be commutative. Choose $\{E_\lambda\}_\Lambda$ as a minimal representing set of simple modules in $\sigma[M]$, and denote by $\widehat{E}_\lambda$ the injective hull of E_λ in $\sigma[M]$. Then the following statements are equivalent:

(a) M is a cogenerator in $\sigma[M]$;

(b) M is self-injective and self-cogenerator;

(c) $M \simeq \bigoplus_\Lambda \widehat{E}_\lambda$;

(d) M is a direct sum of indecomposable modules N which are cogenerators in $\sigma[N]$.

Under these conditions we have:

(1) Every A-submodule of M is fully invariant and $Soc\,_AM = Soc\,M_B$.

(2) For every $\lambda \in \Lambda$, in $\sigma[\widehat{E}_\lambda]$ all simple modules are isomorphic.

(3) $B = \prod_\Lambda B_\lambda$ with local rings $B_\lambda := End_A(\widehat{E}_\lambda) \simeq B_{m_\lambda}$, where m_λ is the kernel of the restriction map $B \to End_A(E_\lambda)$.

(4) If the $\widehat{E}_\lambda$'s are finitely generated A-modules, then M generates all simple modules in $\sigma[M]$.

(5) If M is projective in $\sigma[M]$, then M is a generator in $\sigma[M]$.

(6) If M is finitely generated, then M is finitely cogenerated.

Proof. (1) The equality of the socles is shown in 6.5.

(3) $B = \prod_\Lambda B_\lambda$ is clear by the fact that the $\widehat{E}_\lambda$'s are fully invariant.

Write $B = B_\lambda \oplus C_\lambda$ where $C_\lambda = \prod_{\mu \neq \lambda} B_\mu$. Choose $m_\lambda \subset B$ as indicated in (3). By localizing at m_λ (see Section 17), we get

$$B_{m_\lambda} \simeq (B_\lambda)_{m_\lambda} \oplus (C_\lambda)_{m_\lambda}.$$

For every $t \in B \setminus m_\lambda$, multiplication by t is an automorphism of $\widehat{E}_\lambda$. This implies $B_\lambda \simeq (B_\lambda)_{m_\lambda}$. However, B_{m_λ} is a local ring, hence indecomposable and $(C_\lambda)_{m_\lambda} = 0$.

All the other assertions are shown in [40, 48.16]. □

For self-projective modules we derive further properties.

6.8 Self-projective cogenerators.

Let M be a finitely generated, self-projective A-module and assume $B = End_A(M)$ to be commutative. Then the following are equivalent:

(a) M is a cogenerator in $\sigma[M]$;

(b) M is self-injective, self-generator and finitely cogenerated;

(c) every module which cogenerates M is a generator in $\sigma[M]$;

(d) every cogenerator is a generator in $\sigma[M]$;

(e) M is a self-generator and B is cogenerator in B-Mod (= PF-ring).

Proof. From 6.7 we know that (a) implies that M is finitely cogenerated and hence there are only finitely many simple modules in $\sigma[M]$. Therefore the equivalences (a) - (d) follow from [40, 48.11].

$(b) \Rightarrow (e)$ and $(e) \Rightarrow (a)$ In both cases M is a progenerator in $\sigma[M]$ and so the functor $Hom(M, -) : \sigma[M] \to B\text{-}Mod$ is an equivalence (e.g., [40, 46.2]). □

Important classes of modules can be characterized by properties in $\sigma[M]$:

6.9 Semisimple modules.

For an A-module M, the following properties are equivalent:

(a) M is a (direct) sum of simple modules (= semisimple);

(b) every submodule of M is a direct summand;

(c) every module (in $\sigma[M]$) is M-projective;

(d) every module (in $\sigma[M]$) is M-injective;

(e) every short exact sequence in $\sigma[M]$ splits;

(f) every simple module (in $\sigma[M]$) is M-projective;

(g) every cyclic module (in $\sigma[M]$) is M-injective.

Obviously, any semisimple module M is a projective generator and an injective cogenerator in $\sigma[M]$.

If M is a noetherian, injective cogenerator in $\sigma[M]$, then it is called a noetherian *Quasi-Frobenius* or *QF module*. We give some characterizations of these modules taken from [40, 48.14]:

6.10 Noetherian QF modules.

For a finitely generated A-module M with $B = End({}_A M)$, the following are equivalent:

(a) M is a self-projective noetherian QF module;

(b) M is a noetherian projective cogenerator in $\sigma[M]$;

(c) M is an artinian projective cogenerator in $\sigma[M]$;

(d) M is a projective cogenerator in $\sigma[M]$ and B_B is artinian;

(e) M is an injective generator in $\sigma[M]$ and ${}_B B$ is artinian;

(f) M is a noetherian injective generator in $\sigma[M]$;

(g) M is self-injective and injectives are projective in $\sigma[M]$;

(h) M is a progenerator and projectives are injective in $\sigma[M]$.

We continue to investigate injectivity. Dual to intrinsically projective we define:

An A-module M is said to be *intrinsically injective* if every diagram with exact row,

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \longrightarrow & M^n \\ & & \downarrow & & \\ & & M & & , \end{array}$$

where $n \in I\!N$ and N a factor module of M, can be extended commutatively by some morphism $M^n \to M$.

M is called *semi-injective* if the above condition (only) holds for $n = 1$.

Self-injective modules satisfy the above conditions and dual to 5.6 we get further examples:

6.11 Examples. Let M be an A-module with $B = End({}_A M)$.

(1) If cokernels of endomorphisms are M-cogenerated and B is a right PP ring, then M is semi-injective.

(2) If M_B is FP-injective and B is right semihereditary, then M is intrinsically injective.

(3) If B is a regular ring, then M is intrinsically injective.

Proof. Referring to [40, 39.11 and 47.7] the proof is dual to the proof of 5.6. □

Dual to 5.7 we list some properties which are easy to verify (e.g., [90, Lemma 3], [40, 28.1 and 31.12]):

6.12 Intrinsically injective modules.

Let M be an A-module with $B = End(_AM)$.

(1) The following are equivalent:

(a) *M is intrinsically injective;*

(b) *$I = Hom_A(M/Ke\, I, M)$ for every finitely generated right ideal $I \subset B$;*

(c) *the map $I \mapsto Ke\, I$ from finitely generated right ideals in B to submodules of M is injective.*

(2) The following are also equivalent:

(a) *M is semi-injective;*

(b) *$fB = Hom_A(M/Ke\, f, M)$ for every $f \in B$;*

(c) *the map $I \mapsto Ke\, I$ from cyclic right ideals in B to submodules of M is injective.*

6.13 Finitely cogenerated modules.

An A-module N is called *finitely cogenerated* if for every monomorphism $\psi : N \to \prod_\Lambda U_\lambda$, there is a finite subset $E \subset \Lambda$ such that the canonical map

$$N \xrightarrow{\psi} \prod\nolimits_\Lambda U_\lambda \longrightarrow \prod\nolimits_E U_\lambda$$

is monic. N is finitely cogenerated if and only if $Soc(N)$ is finitely generated and essential in N.

6.14 Radical of modules.

Dual to the socle the *radical* of an A-module M is defined as the intersection of all maximal submodules of M and is characterized by

$$\begin{aligned} Rad(M) &= \bigcap \{K \subset M \mid K \text{ is maximal in } M\} \\ &= \sum \{L \subset M \mid L \text{ is superfluous in } M\}. \end{aligned}$$

It has the following properties:

(1) *$Rad(M/Rad\, M) = 0$ and, for any $f : M \to N$, $(Rad\, M)f \subset Rad\, N$. If $Ke\, f \subset Rad\, M$, then $(Rad\, M)f = Rad(Mf)$.*

(2) *$Rad\, M$ is also an $End(_AM)$-submodule of M (fully invariant).*

(3) *If every proper submodule of M is contained in a maximal submodule, then $Rad\, M \ll M$ (e.g., if M is finitely generated).*

(4) *If $M = \bigoplus_\Lambda M_\lambda$, then $Rad\, M = \bigoplus_\Lambda Rad\, M_\lambda$ and $M/Rad\, M \simeq \bigoplus_\Lambda M_\lambda/Rad\, M_\lambda$.*

(5) *If M is finitely cogenerated and $Rad\, M = 0$, then M is semisimple and finitely generated.*

6.15 Quasi-regularity.

An element r in A is called *left* (*right*) *quasi-regular* if there exists $t \in A$, with $r + t - tr = 0$ (resp. $r + t - rt = 0$).

r is called *quasi-regular* if it is left and right quasi-regular. A *subset* of A is said to be *(left, right) quasi-regular* if every element in it has the corresponding property.

These notions are also of interest for algebras without units.

In algebras with units, the relation $r + t - tr = 0$ is equivalent to the equation $(1 - t)(1 - r) = 1$. Hence in such algebras an element r is left quasi-regular if and only if $(1 - r)$ is *left invertible.*

These definitions also make sense for alternative algebras.

6.16 The Jacobson radical.

The radical of ${}_AA$ is called the *Jacobson radical of* A, i.e.,

$$Jac(A) = Rad({}_AA).$$

This is a two-sided ideal in A and can be described as the

(a) intersection of the maximal left ideals in A (= definition);

(b) sum of all superfluous left ideals in A;

(c) sum of all left quasi-regular left ideals in A;

(d) largest (left) quasi-regular ideal in A;

(e) intersection of the annihilators of simple left A-modules;

(f) intersection of the maximal right ideals in A.

For every A-module M, $Jac(A)M \subset Rad({}_AM)$.

Next we observe a connection between the radicals of R and M:

6.17 Relation between radicals.

Let M be a module over the R-algebra A and assume

(i) M is finitely generated as R-module, or (ii) R is a perfect ring.

Then $Jac(R)M \subset Rad({}_AM)$.

Proof. $\overline{M} := M/Rad({}_AM)$ is a subdirect product of A-modules M/U, with maximal A-submodules $U \subset M$. $Jac(R)(M/U)$ is an A-submodule of the simple A-module M/U and hence $Jac(R)(M/U) = 0$ or $Jac(R)(M/U) = M/U$. If M/U is a finitely generated R-module or $Jac(R)$ is t-nilpotent, the second possibility cannot occur (Nakayama Lemma, e.g., [40, 21.13 and 43.5]).

Hence $Jac(R)$ annihilates $\overline{M}$ which means $Jac(R)M \subset Rad({}_AM)$. $\square$

An A-module M is said to be *co-semisimple* if every simple module in $\sigma[M]$ (or in A-*Mod*) is M-injective. From [40, 23.1] we have:

6.18 Co-semisimple modules.

For an A-module M, the following statements are equivalent:

(a) M is co-semisimple;

(b) every finitely cogenerated module in $\sigma[M]$ is M-injective;

(c) every finitely cogenerated module in $\sigma[M]$ is semisimple;

(d) $\sigma[M]$ has a semisimple cogenerator;

(e) for every module $N \in \sigma[M]$, $Rad(N) = 0$;

(f) for every factor module N of M, $Rad(N) = 0$;

(g) any proper submodule of M is an intersection of maximal submodules.

6.19 Exercises.

(1) Let M be a weakly M-injective A-module and $B = End(_AM)$. Prove that for any finitely generatde submodules $K, L \subset M$,

$$An_B(K \cap L) = An_B(K) + An_B(L).$$

(2) Let M, N be A-modules and $B = End(_AM)$. N is called restricted M-injective *if $Hom_A(-, N)$ is exact on exact sequences*

$$0 \to K \to M^n, \text{ where } K \text{ is finitely } M\text{-generated.}$$

Prove:

(i) If M_B is flat and N is restricted M-injective, then $_BHom_A(M, N)$ is FP-injecticve.

(ii) If $_BHom_A(M, N)$ is FP-injecticve, then N is restricted M-injective.

(iii) The following are equivalent:

(a) Every A-module is restricted M-injective;

(b) M_B is FP-injective and B is right semihereditary.

References: Brodskii [90], Menini-Orsatti [197], Wisbauer [275].

7 Regular modules

1.Finitely presented modules. 2.Purity. 3.Regular modules. 4.Regular algebras. 5.Strongly regular rings. 6.Regular endomorphism rings. 7.Exercises.

For an A-module M, let $\mathcal{P}$ denote a non-empty class of modules in $\sigma[M]$. In [40, Chapter 7] the notion of $\mathcal{P}$-pure exact sequences was introduced:

A short exact sequence in $\sigma[M]$ is called *$\mathcal{P}$-pure in $\sigma[M]$*, if every module P in $\mathcal{P}$ is projective with respect to this sequence.

Of course, the properties of $\mathcal{P}$-pure sequences strongly depend on the choice of the class $\mathcal{P}$.

Except for an exercise on relatively semisimple modules (see 20.13) we will here restrict our attention to the class of finitely presented modules in $\sigma[M]$:

7.1 Finitely presented modules.

A finitely generated module $N \in \sigma[M]$ is *finitely presented in* $\sigma[M]$, if in every exact sequence $0 \to K \to L \to N \to 0$ in $\sigma[M]$ with L finitely generated, the module K is also finitely generated.

$N \in \sigma[M]$ is finitely presented in $\sigma[M]$ if and only if $Hom_A(N, -)$ commutes with direct limits in $\sigma[M]$ (see [40, 25.2]).

For example, every finitely generated M-projective module in $\sigma[M]$ is finitely presented in $\sigma[M]$. Since we will not vary the class $\mathcal{P}$ we delete it in our definitions:

7.2 Purity. An exact sequence

$$(*) \qquad 0 \to K \to L \to N \to 0$$

in $\sigma[M]$ is called *pure in* $\sigma[M]$, if every finitely presented module P in $\sigma[M]$ is projective with respect to this sequence, i.e., if every diagram

$$\begin{array}{ccccccccc} & & & & & & P & & \\ & & & & & & \downarrow & & \\ 0 & \to & K & \to & L & \to & N & \to & 0 \end{array}$$

can be extended commutatively by a morphism $P \to L$. Equivalently, we may demand the following sequence to be exact:

$$0 \to Hom_A(P, K) \to Hom_A(P, L) \to Hom_A(P, N) \to 0.$$

The sequence $(*)$ is *pure in A-Mod* if and only if the sequence

$$0 \to F \otimes_A K \to F \otimes_A L \to F \otimes_A N \to 0$$

is exact, for all (finitely presented) right A-modules F (see [40, 34.5]).

Every direct limit of splitting exact sequences is pure in $\sigma[M]$ and A-*Mod*.

On the other hand, if there is a generating set of finitely presented modules in $\sigma[M]$, then any pure exact sequences in $\sigma[M]$ is a direct limit of splitting sequences in $\sigma[M]$ (see [40, 34.2]).

This always applies in A-*Mod*. Here we still have another characterization for such sequences: $(*)$ is pure in A-*Mod* if and only if every pure injective A-module is injective with respect to $(*)$ (see [40, 34.7]).

A module $X \in \sigma[M]$ is called *pure projective* if X is projective with respect to every pure sequence in $\sigma[M]$; Y is *pure injective* if Y is injective with respect to every pure sequence in $\sigma[M]$. We say that

K is *absolutely pure* in $\sigma[M]$	if for K fixed all sequence $(*)$ are pure in $\sigma[M]$,
L is *regular* in $\sigma[M]$	if for L fixed all sequences $(*)$ are pure in $\sigma[M]$,
N is *flat* in $\sigma[M]$	if for N fixed all sequences $(*)$ are pure in $\sigma[M]$.

Regular modules generalize semisimple modules. They also have a number of interesting properties (see [40, 37.2, 37.3 and 37.4]):

7.3 Regular modules.

For an A-module M, the following are equivalent:

(a) M is regular in $\sigma[M]$;

(b) every short exact sequence in $\sigma[M]$ is pure;

(c) every module in $\sigma[M]$ is flat;

(d) every module in $\sigma[M]$ is absolutely pure;

(e) every finitely presented module is projective in $\sigma[M]$.

If M is finitely presented in $\sigma[M]$, then (a)-(e) are equivalent to:

(f) every module in $\sigma[M]$ is weakly M-injective;

(g) every finitely generated submodule of M is a direct summand in M.

The algebra A is called *(von Neumann) regular* if for every $a \in A$ there exists $b \in A$ with $a = aba$. Putting $M = A$ we obtain from 7.3:

7.4 Regular algebras.

For an algebra A, the following are equivalent:

(a) A is a regular algebra;

(b) ${}_AA$ is regular in A-Mod;

(c) every (cyclic) left A-module is flat;

(d) every left A-module is absolutely pure (= FP-injective);

(e) every finitely presented left A-module is projective.

Also the right hand versions of (b)-(d) apply.

An algebra A is called *reduced* if it has no non-zero nilpotent elements.

A is *strongly regular* if for every $a \in A$, there exists $b \in A$ with $a = a^2b$. This notion is left-right symmetric as can be seen from the following (see [40, 3.11]):

7.5 Strongly regular rings. Characterizations.

For an algebra A with unit, the following are equivalent:

(a) A is strongly regular;

(b) A is regular and reduced;

(c) every principal left (right) ideal is generated by a central idempotent;

(d) A is regular and every left (right) ideal is an ideal.

Notice that many of these definitions and properties also apply to algebras without unit and to alternative algebras.

From [40, 37.7] we recall important properties of regular endomorphism rings.

7.6 Regular endomorphism rings.

Let M be an A-module and $B = End_A(M)$.

(1) For $f \in B$, the following properties are equivalent:
(a) There exists $g \in B$ with $fgf = f$;
(b) $Ke\,f$ and $Im\,f$ are direct summands of M.

(2) B is regular if and only if $Im\,f$ and $Ke\,f$ are direct summands of M, for every $f \in B$.

(3) If B is regular, then every finitely M-generated submodule of M is a direct summand in M.

7.7 Exercises.

(1) Let A be a regular ring. Prove that the following are equivalent:
(a) A is reduced;
(b) A is strongly regular;
(c) every idempotent in A is central;
(d) A is a subdirect product of division rings;
(e) every prime ideal P in A is completely prime ($ab \in P$ implies $a \in P$ or $b \in P$).

(2) Let A be a strongly regular ring. Prove that A is left co-semisimple (every simple left A-module is injective).

References. Wisbauer [40].

8 Lifting and semiperfect modules

1.π-projective modules. 2.Local modules. 3.Direct projective modules. 4.Refinable modules. 5.Properties of refinable modules. 6.Supplemented modules. 7.Properties of (f-) supplemented modules. 8.Finitely lifting modules. 9.Lifting modules. 10.π-projective lifting modules. 11.Decomposition of π-projective supplemented modules. 12.Projective semiperfect modules. 13.Projective f-semiperfect modules. 14.Perfect modules. 15.Uniserial modules.

We begin with weak projectivity conditions which are of interest for the investigations to follow.

Definitons. An A-module M is called *π-projective* if, for any two submodules $U, V \subset M$ with $U + V = M$, the epimorphism $U \oplus V \to M$, $(u, v) \mapsto u + v$, splits.

M is called *direct projective* if, for any direct summand X of M, every epimorphism $M \to X$ splits.

It is easy to see that M is π-projective if and only if for any two submodules $U, V \subset M$ with $U + V = M$, there exists $f \in End(M)$ with

$$Im(f) \subset U \text{ and } Im(id_M - f) \subset V.$$

From [40, 41.14] we recall:

8.1 Properties of π-projective modules.

Assume M to be a π-projective A-module. Then:

(1) *Every direct summand of M is π-projective.*

(2) *If $M = U + V$ and U is a direct summand in M, then there exists $V' \subset V$ with $M = U \oplus V'$.*

(3) *If $M = U \oplus V$, then V is U-projective (and U is V-projective).*

(4) *If $M = U + V$ and U, V are direct summands in M, then $U \cap V$ is also a direct summand in M.*

A module M is called *local* if it has a largest proper submodule. It is obvious that this submodule has to be equal to the radical of M and that $Rad(M) \ll M$. Local modules are trivially π-projective.

8.2 Local modules. Characterizations.

For an non-zero A-module M, the following properties are equivalent:

(a) *M is local;*

(b) *M has a maximal submodule which is superfluous;*

(c) *M is cyclic and every proper submodule is superfluous;*

(d) *M is cyclic and for any $m, k \in M$, $Am = M$ implies $Ak = M$ or $A(m-k) = M$;*

(e) *M is finitely generated and non-zero factor modules of M are indecomposable.*

If M is projective in $\sigma[M]$, then (a)-(c) are equivalent to:

(e) *M is a projective cover of a simple module in $\sigma[M]$;*

(f) *End$({}_AM)$ is a local ring.*

Proof. Most of the assertions follow from [40, 19.7 and 41.4]. It only remains to show

$(c) \Leftrightarrow (d)$ Assume (c) and consider $m, k \in M$ with $Am = M$. Then $Ak + A(m-k) = M$ and we conclude $Ak = M$ or $A(m-k) = M$.

Now assume (d) and $U + V = M$ for submodules $U, V \subset M$. Then $u + v = m$ for some $u \in U$, $v \in V$ and so either u or $m - u = v$ generate M. Hence every proper submodule of M is superfluous. □

For direct projective modules we recall from [40, 41.18 and 41.19]:

8.3 Direct projective modules.

Assume M is a direct projective A-module and $B = End({}_AM)$. Then:

(1) *For direct summands $U, V \subset M$, every epimorphism $U \to V$ splits.*

(2) *For direct summands $U, V \subset M$ with $U + V = M$, $U \cap V$ is a direct summand in U (and M) and $M = U \oplus V'$ for some $V' \subset V$.*

(3) *$K(B) := \{f \in B \mid Im f \ll M\} \subset Jac(B)$.*

(4) *The following assertions are equivalent:*
  (a) *For every $f \in B$, there is a decomposition $M = X \oplus Y$ with $X \subset Im\, f$ and $Y \cap Im\, f \ll Y$;*
  (b) *B is f-semiperfect and $K(B) = Jac(B)$.*

(5) *If M is a local module, then B is a local ring.*

Definitions. An A-module M is called *refinable* (or *suitable*) if for any submodules $U, V \subset M$ with $U + V = M$, there exists a direct summand U' of M with $U' \subset U$ and $U' + V = M$.

M is said to be *strongly refinable* if, in the given situation, there exist $U' \subset U$, $V' \subset V$ of M with $M = U' \oplus V'$.

Notice that a finitely generated module M is (strongly) refinable if the defining conditions are satisfied for finitely generated submodules. So, for example, a finitely generated module M with every finitely generated submodule a direct summand is refinable (7.3).

We say that *direct summands lift modulo a submodule* $K \subset M$ if under the canonical projection $M \to M/K$ every direct summand of M/K is an image of a direct summand of M. The connection between the above notions is evident from the next result.

8.4 Refinable modules.

(1) *For an A-module M, the following are equivalent:*
   (a) M is refinable;
   (b) direct summands lift modulo every submodule of M.

(2) *The following assertions are also equivalent:*
   (a) M is strongly refinable;
   (b) finite decompositions lift modulo every submodule of M.

Proof. (1) $(a) \Rightarrow (b)$ Let $K \subset M$ be a submodule. Assume $M/K = U/K \oplus V/K$ for submodules $K \subset U, V \subset M$. Then $M = U + V$ and there is a direct summand U' of M with $U' \subset U$ and $U' + V = M$. This implies $(U' + K)/K = U/K$.

$(b) \Rightarrow (a)$ Now assume $M = U+V$ and put $K = U \cap V$. Then $M/K = U/K \oplus V/K$ and, by assumption, U/K lifts to a direct summand U' of M. Since $(U' + K)/K = U/K$ we have $U' + K = U$, i.e., $U' \subset U$ and $M = U' + V + K = U' + V$.

(2) The proof for strongly refinable modules is similar to the above. □

8.5 Properties of refinable modules.

Let M be an A-module and $B = End(M)$.

(1) *If M is (strongly) refinable and $X \subset M$ is a fully invariant submodule, then M/X is (strongly) refinable.*

(2) *Every π-projective refinable module is strongly refinable.*

(3) *As an (A, B)-bimodule, M is refinable if and only if it is strongly refinable.*

Proof. (1) Assume $M/X = U/X + V/X$ for submodules $U, V \subset M$ containing X. Then $M = U + V$ and there exists a direct summand U' of M with $U' \subset U$ and $U' + V = M$. Assume $M = U' \oplus W$. Since X is fully invariant we have $X = (X \cap U') \oplus (X \cap W)$ and hence $U'/(X \cap U')$ is a direct summand of M/X with the properties desired.

The same proof shows the assertion for strongly refinable modules.

(2) This follows from 8.1(2).

(3) Assume M is a refinable (A, B)-bimodule and $M = U + V$ with fully invariant submodule $U, V \subset M$. Then there exists a central idempotent $e \in B$, such that

$$M = Me + V \text{ and } Me \subset U.$$

Multiplying the left equality with $id_M - e$, we obtain $M(id_M - e) = V(id_M - e) \subset V$, showing that M is strongly refinable. □

Notice that finitely generated self-projective modules are refinable if and only if they have the *exchange property* (see Nicholson [212, Proposition 2.9]). Additional characterizations of refinable modules will be given in 18.7 and 18.10.

Definitions. Let U be a submodule of the A-module M. A submodule $V \subset M$ is called a *supplement* of U in M if V is minimal with the property $U + V = M$. It is easy to see that V is a supplement of U if and only if

$$U + V = M \text{ and } U \cap V \ll V.$$

We say $U \subset M$ has *ample supplements in* M if for every $V \subset M$ with $U + V = M$, there is a supplement V' of U with $V' \subset V$.

M is called *(amply) supplemented*, if every submodule has (ample) supplements in M. Similarly *(amply) f-supplemented* modules are defined as modules whose finitely generated submodules have (ample) supplements.

Trivially, local modules are (amply) supplemented. Characterizations of these modules are given in [40, 41.6]:

8.6 Supplemented modules.

For a finitely generated A-module M, the following are equivalent:

(a) M is supplemented;

(b) every maximal submodule of M has a supplement in M;

(c) M is an irredundant (finite) sum of local submodules.

From [40, 41.2 and 41.3] we recall:

8.7 Properties of (f-) supplemented modules.

Let M be an A-module.

(1) Assume M is supplemented. Then any finitely M-generated module is supplemented and $M/Rad(M)$ is semisimple.

(2) Assume M is f-supplemented.
(i) If $L \subset M$ is finitely generated or superfluous, then M/L is also f-supplemented.
(ii) If $Rad(M) \ll M$, then finitely generated submodules of $M/Rad(M)$ are direct summands.

Definition. We call an A-module M *(finitely) lifting*, provided for every (finitely generated) submodule $U \subset M$, there exists a direct summand X of M with $X \subset U$ and $U/X \ll M/X$.

From [40, 41.11 and 41.13] we have characterizations for finitely generated modules of this type showing that they are in particular refinable:

8.8 Finitely lifting modules.

For a finitely generated A-module M, the following are equivalent:

(a) *M is a finitely lifting module;*

(b) *for every cyclic (finitely generated) submodule $U \subset M$, there is a decomposition $M = X \oplus Y$, where $X \subset U$ and $U \cap Y \ll Y$;*

(c) *for any cyclic submodule $U \subset M$, there exists an idempotent $e \in End({}_AM)$, such that $Me \subset U$ and $U(1-e) \ll M(1-e)$;*

(d) *M is amply f-supplemented and supplements are direct summands;*

(e) *M is strongly refinable and in $M/Rad(M)$ every finitely generated submodule is a direct summand.*

Proof. The first equivalences come from [40, 41.11 and 41.13].

$(d) \Rightarrow (e)$ This implication is obvious by 8.7.

$(e) \Rightarrow (b)$ For any finitely generated submodule $U \subset M$, there is a decompositon

$$M/Rad(M) = U + Rad(M)/Rad(M) \oplus V/Rad(M),$$

for some submodule $Rad(M) \subset V \subset M$. Since direct summands lift modulo $Rad(M)$, there exists a direct summand $Y' \subset M$ with $Y' + Rad(M) = V$, $M = U + Y'$ and $U \cap Y' \subset Rad(M)$, implying $U \cap Y' \ll Y'$.

M being strongly refinable, there is a decomposition $M = X \oplus Y$ with $X \subset U$ and $Y \subset Y'$. From above we know $U \cap Y \subset U \cap Y' \ll Y'$ implying $U \cap Y \ll Y$. □

Now we characterize lifting modules.

8.9 Lifting modules.

For an A-module M, the following are equivalent:

(a) *M is a lifting module;*

(b) *for every submodule $U \subset M$, there is a decomposition $M = X \oplus Y$, where $X \subset U$ and $U \cap Y \ll Y$;*

(c) *for every submodule $U \subset M$, there exists an idempotent $e \in End({}_AM)$, such that $Me \subset U$ and $U(id_M - e) \ll M(id_M - e)$;*

(d) *M is amply supplemented and every supplement is a direct summand;*

(e) *M is strongly refinable and $M/Rad(M)$ is semisimple.*

Proof. The first equivalences come from [40, 41.11 and 41.12].

$(d) \Rightarrow (e)$ This implication is obvious by 8.7.

$(e) \Rightarrow (b)$ Repeat the proof of the corresponding implication in 8.8. □

π-projective supplemented modules are in fact lifting. From [40, 41.15] we have:

8.10 π-projective lifting modules.
Let M be an A-module and $B = End({}_AM)$.

(1) The following are equivalent:

(a) M is lifting and π-projective;

(b) M is supplemented and π-projective;

(c) M is amply supplemented and the intersection of mutual supplements is zero;

(d) M is lifting and for direct summands $U, V \subset M$ with $M = U + V$, $U \cap V$ is a direct summand;

(e) for any submodules $U, V \subset M$ with $U + V = M$, there exists an idempotent $e \in B$, such that

$$Me \subset U,\ M(id_M - e) \subset V \text{ and } U(id_M - e) \ll M(id_M - e).$$

(2) As an (A, B)-bimodule, M is lifting if and only if it is supplemented and π-projective.

Proof. It remains to show (2). Assume M is lifting and $U, V \subset M$ are submodules with $U + V = M$. By 8.9, there is a central idempotent $e \in B$ with

$$Me \subset U \text{ and } U(id_M - e) \ll M(id_M - e).$$

This implies $M = Me + V$ and $M(id_M - e) = V(id_M - e) \subset V$. So M is π-projective and supplemented by (1). □

For finitely generated modules of the above type the decomposition theorem [40, 41.17] yields:

8.11 Decomposition of π-projective supplemented modules.
Assume M is a finitely generated, π-projective lifting A-module. Then:

(1) $M = \bigoplus_\Lambda L_\lambda$ with local modules L_λ, and

(2) for every direct summand N of M, there exists a subset $\Lambda' \subset \Lambda$, such that $M = (\bigoplus_{\Lambda'} L_\lambda) \oplus N$.

(3) If $M = \sum_\Lambda N_\lambda$ is an irredundant sum with indecomposable N_λ, then $M = \bigoplus_\Lambda N_\lambda$.

Remarks. The notion *lifting modules* is, for example, used in Oshiro [217]. In Zöschinger [288] the same modules are called *strongly complemented* (see [288, Satz 3.1]) and there are various other names used by different authors. Though we will not use this terminology we recall, for the convenience of the reader, some notation applied in the literature (e.g., Mohamed-Müller [27]).

Consider the following conditions on an A-module M:

(D_1) for every submodule $U \subset M$, there is a decomposition $M = X \oplus Y$, where $X \subset U$ and $Y \cap U \ll Y$.

(D_2) If $U \subset M$ is a submodule such that M/U is isomorphic to a direct summand of M, then U is a direct summand of M.

(D_3) If $U,\ V \subset M$ are direct summands with $U + V = M$, then $U \cap V$ is a direct summand of M.

Condition (D_1) characterizes lifting modules (see 8.9), (D_2) is equivalent to direct projective. A module M with (D_1) and (D_2) is called *dual continuous* or *discrete*, and M is called *quasi-discrete* or *quasi-semiperfect* (in [216]) if it satisfies (D_1) and (D_3).

Quasi-discrete modules are just π-projective supplemented modules (e.g., [40, 41.15], [288, Satz 5.1]). The decomposition theorem for these modules mentioned above is originally due to Oshiro [216] and is also proved in [27, Theorem 4.15].

Dual continuous modules are direct projective lifting modules. Some of their properties are described in [27, Lemma 4.27] and [40, 41.18 and 41.19]. In particular they are quasi-discrete.

Dual to the notions considered above there are *extending*, *π-injective* and *continuous modules*. For a detailed account on these we refer to [40], [27] and [11].

Definitions. $N \in \sigma[M]$ is said to be *semiperfect in* $\sigma[M]$ if every factor module of N has a projective cover in $\sigma[M]$.

N is *f-semiperfect in* $\sigma[M]$ if, for every finitely generated submodule $K \subset N$, the factor module N/K has a projective cover in $\sigma[M]$. Obviously, finitely generated, self-projective local modules M are (f-) semiperfect in $\sigma[M]$.

Supplemented and semiperfect modules are closely related ([40, 42.5, 42.12]):

8.12 Projective semiperfect modules.

Assume the A-module M is projective in $\sigma[M]$. Then the following are equivalent:

(a) M is semiperfect in $\sigma[M]$;

(b) M is supplemented;

(c) every finitely M-generated module has a projective cover in $\sigma[M]$;

(d) (i) $M/Rad(M)$ is semisimple and $Rad(M) \ll M$, and
(ii) decompositions of $M/Rad(M)$ lift modulo $Rad(M)$;

(e) M is a direct sum of local modules and $Rad(M) \ll M$.

If M is a finitely generated A-module, (a)-(f) are equivalent to:

(g) $End(_AM)$ is a semiperfect ring.

For f-semiperfect modules we get from [40, 42.10]:

8.13 Projective f-semiperfect modules.

For a finitely generated, self-projective A-module M, the following are equivalent:

(a) *M is f-semiperfect in $\sigma[M]$;*

(b) *M is f-supplemented;*

(c) (i) *$M/Rad(M)$ is regular in $\sigma[M/Rad(M)]$, and*
(ii) *decompositions lift modulo $Rad(M)$.*

M is said to be *perfect in* $\sigma[M]$ if, for any index set Λ, $M^{(\Lambda)}$ is semiperfect in $\sigma[M]$. By [40, 43.8] we have a series of characterizations of these modules.

8.14 Perfect modules.

Let M be a finitely generated, self-projective A-module with endomorphism ring $B = End(_AM)$. The following statements are equivalent:

(a) *M is perfect in $\sigma[M]$;*

(b) *every (indecomposable) M-generated flat module is projective in $\sigma[M]$;*

(c) *$M^{(\mathbb{N})}$ is semiperfect in $\sigma[M]$;*

(d) *$M/Rad(M)$ is semisimple and $Rad\,(M^{(\mathbb{N})}) \ll M^{(\mathbb{N})}$;*

(e) *$B/Jac(B)$ is left semisimple and $Jac(B)$ is right t-nilpotent;*

(f) *B satisfies the descending chain condition for cyclic right ideals;*

(g) *$_BB$ is perfect in B-Mod.*

For $M = A$, the above list characterizes *left perfect* algebras.

An R-module N is called *uniserial* if its submodules are linerarly ordered by inclusion. N is *serial* if it is a direct sum of uniserial modules. Recall from [40, 55.1]:

8.15 Uniserial modules.

For an A-module N the following are equivalent:

(a) *The (cyclic) submodules of N are linearly ordered;*

(b) *for every finitely generated non-zero submodule $K \subset N$, $K/Rad(K)$ is simple;*

(c) *for every factor module L of N, $Soc\,L$ is simple or zero.*

References: Burkholder [102], Fuller [139], Mohamed-Müller [27], Nicholson [212], Oshiro [216, 217]. Wisbauer [40, 276], Zöschinger [288].

Chapter 3

Torsion theories and prime modules

In this chapter A will always denote an associative R-algebra with unit.

9 Torsion theory in $\sigma[M]$

1.Definitions. 2.Pretorsions. 3.Torsion submodules. 4.$\mathcal{T}$-torsionfree modules. 5.Hereditary torsion classes. 6.$\mathcal{T}$-dense submodules. 7.Properties of $\mathcal{T}$-dense submodules. 8.Dense submodules of generators. 9.$(M,\mathcal{T})$-injective modules. 10.$(M,\mathcal{T})$-injective modules and direct sums. 11.$(M,\mathcal{T})$-injective modules. Characterizations. 12.Lemma. 13.Lemma. 14.Faithfully $(M,\mathcal{T})$-injective modules. 15.Properties of quotient modules. 16.Essential submodules $\mathcal{T}$-dense. 17.Quotient modules and direct limits. 18.The quotient functor. 19.$\mathcal{T}$-dense left ideals. 20.Gabriel filter and quotient modules. 21.Exercises.

We recall some notions from torsion theory. These techniques are familiar from A-*Mod* but it is well-known that they also apply to Grothendieck categories. For basic facts we refer to [5] or [39].

9.1 Definitions. Let $M \in A$-*Mod*. A class $\mathcal{T}$ of modules in $\sigma[M]$ is called a

pretorsion class if $\mathcal{T}$ is closed under direct sums and factor modules;

hereditary pretorsion class if $\mathcal{T}$ is closed under direct sums, factor and submodules;

torsion class if $\mathcal{T}$ is closed under direct sums, factors and extensions in $\sigma[M]$;

hereditary torsion class if $\mathcal{T}$ is closed under direct sums, factors, submodules and extensions in $\sigma[M]$;

stable class if $\mathcal{T}$ is closed under essential extensions in $\sigma[M]$.

Notice that for any $M \in A$-*Mod*, $\sigma[M]$ is a hereditary pretorsion class in A-*Mod*, not necessarily closed under extensions. In fact every hereditary pretorsion class in $\sigma[M]$ (or in A-*Mod*) is of type $\sigma[U]$ for some suitable $U \in \sigma[M]$.

9.2 Pretorsions.

Any pretorsion class $\mathcal{T}$ is closed under traces, i.e., for any A-module N,

$$\mathcal{T}(N) := Tr(\mathcal{T}, N) = \sum \{U \subset N \,|\, U \in \mathcal{T}\} \in \mathcal{T}.$$

$\mathcal{T}(N)$ is called the *$\mathcal{T}$-torsion submodule* of N. It is the largest submodule of N belonging to $\mathcal{T}$. It follows from well-known properties of the trace that

$$\text{for any } L \in A\text{-}Mod \text{ and } f \in Hom_A(N, L),\ \ \mathcal{T}(N)f \subset \mathcal{T}(L).$$

Hence there are unique homomorphisms

$$f' := f|_{\mathcal{T}(N)} : \mathcal{T}(N) \to \mathcal{T}(L) \quad \text{and} \quad \overline{f} : N/\mathcal{T}(N) \to L/\mathcal{T}(L).$$

N is called a *$\mathcal{T}$-torsion module* if $N = \mathcal{T}(N)$, i.e., $N \in \mathcal{T}$.

N is said to be *$\mathcal{T}$-torsionfree* if $\mathcal{T}(N) = 0$. This is the case if and only if $Hom_A(L, N) = 0$ for all $L \in \mathcal{T}$.

Properties of $\mathcal{T}$-torsion submodules depend on the properties of the class $\mathcal{T}$.

9.3 Torsion submodules. Properties.

Let $\mathcal{T}$ be a pretorsion class in $\sigma[M]$ and $N \in \sigma[M]$. Then:

(1) If $\mathcal{T}$ is hereditary, for any $K \subset N$, $\mathcal{T}(K) = K \cap \mathcal{T}(N)$.

(2) If $\mathcal{T}$ is a torsion class, $\mathcal{T}(N/\mathcal{T}(N)) = 0$.

Proof. (1) Obviously $\mathcal{T}(K) \subset K \cap \mathcal{T}(N)$.

$\mathcal{T}$ being closed under submodules, $K \cap \mathcal{T}(N) \in \mathcal{T}$. Hence $K \cap \mathcal{T}(N) \subset \mathcal{T}(K)$.

(2) Let $\mathcal{T}(N) \subset U \subset N$ such that $U/\mathcal{T}(N) \in \mathcal{T}$. Consider the exact sequence

$$0 \to \mathcal{T}(N) \to U \to U/\mathcal{T}(N) \to 0.$$

Since $\mathcal{T}$ is closed under extensions $U \in \mathcal{T}$, i.e., $U \subset \mathcal{T}(N)$. □

9.4 $\mathcal{T}$-torsionfree modules. Properties.

Let $\mathcal{T}$ be a hereditary torsion class in $\sigma[M]$. Denote by $\mathcal{F}$ the class of $\mathcal{T}$-torsionfree modules in $\sigma[M]$, i.e., $\mathcal{F} = \{N \in \sigma[M] \,|\, \mathcal{T}(N) = 0\}$.

(1) $\mathcal{F}$ is closed under submodules, isomorphic images, injective hulls (essential extensions) in $\sigma[M]$, and direct products in $\sigma[M]$.

(2) There exists an M-injective module $E \in \sigma[M]$ with the property

$$L \in \mathcal{F} \textit{ if and only if } L \textit{ is cogenerated by } E.$$

Proof. (1) Assume $N \trianglelefteq L$. Since $\mathcal{T}(N) = \mathcal{T}(L) \cap N$ (see 9.3(2)), $\mathcal{T}(L) = 0$ if and only if $\mathcal{T}(N) = 0$. Hence $\mathcal{F}$ is closed under essential extensions.

Let $\{N_\lambda\}_\Lambda$ be any family of modules in $\mathcal{F}$. $Hom_A(L, \prod_\Lambda N_\lambda) \simeq \prod_\Lambda Hom_A(L, N_\lambda)$ implies that $Tr(\sigma[M], \prod_\Lambda N_\lambda)$, which is the direct product of $\{N_\lambda\}_\Lambda$ in $\sigma[M]$, belongs to $\mathcal{F}$.

The other assertions are obvious.

(2) Choose $E = \prod^M \widehat{A/K}$, $K \subset A$ (product in $\sigma[M]$), with $A/K \in \mathcal{F}$ and $\widehat{A/K}$ the M-injective hull of A/K. By (1), $E \in \mathcal{F}$. For any $L \in \mathcal{F}$ and $0 \neq l \in L$, $Al \in \mathcal{F}$. By construction of E, there exists a monomorphism $Al \to E$ and for the diagram

$$\begin{array}{ccccc} 0 & \to & Al & \to & L \\ & & \downarrow & & \\ & & E & & \end{array}$$

we obtain an $f : L \to E$ with $lf \neq 0$. This implies that L is cogenerated by E. $\square$

9.5 Hereditary torsion classes. Characterization.

For a class $\mathcal{T}$ of modules in $\sigma[M]$, the following are equivalent :

(a) $\mathcal{T}$ is a hereditary torsion class in $\sigma[M]$;

(b) there exists an M-injective module $E \in \sigma[M]$ with the property

$$\mathcal{T} = \{N \in \sigma[M] \mid Hom_A(N, E) = 0\}.$$

Proof. $(a) \Rightarrow (b)$ Let $\mathcal{F}$ be the class of $\mathcal{T}$-torsionfree modules and choose E as an M-injective cogenerator of $\mathcal{F}$ (by 9.4). Then for all $N \in \mathcal{T}$, $Hom_A(N, E) = 0$ (since $E \in \mathcal{F}$). If $K \notin \mathcal{T}$, then $0 \neq K/\mathcal{T}(K) \in \mathcal{F}$ and $Hom_A(K/\mathcal{T}(K), E) \neq 0$, i.e., $Hom_A(K, E) \neq 0$.

$(b) \Rightarrow (a)$ Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence in $\sigma[M]$. Then $0 \to Hom_A(N'', E) \to Hom(N, E)_A \to Hom_A(N', E) \to 0$ is also exact and it follows that $\mathcal{T}$ is closed under submodules, factor modules and extensions.

Consider a family $\{N_\lambda\}_\Lambda$ of modules with $Hom_A(N_\lambda, E) = 0$. Then

$$Hom_A(\bigoplus\nolimits_\Lambda N_\lambda, E) \simeq \prod\nolimits_\Lambda Hom_A(N_\lambda, E) = 0.$$

Therefore $\mathcal{T}$ is closed under direct sums. $\square$

Definition. Let $\mathcal{T}$ be a torsion class in $\sigma[M]$ and $N \in \sigma[M]$. A submodule $L \subset N$ is said to be $\mathcal{T}$-*dense* in N, if $N/L \in \mathcal{T}$. The set of $\mathcal{T}$-dense submodules of a module N is denoted by

$$\mathcal{L}(N, \mathcal{T}) := \{L \subset N \mid N/L \in \mathcal{T}\}.$$

9.6 $\mathcal{T}$-dense submodules.

Let M be an A-module and $N \in \sigma[M]$.

(1) *Let $\mathcal{T}$ be a hereditary pretorsion class in $\sigma[M]$. Assume $\mathcal{T}(N) = 0$. Then every $\mathcal{T}$-dense submodule of N is essential in N.*

(2) *Assume $\mathcal{T}$ is a hereditary torsion class in $\sigma[M]$ and $K \subset L \subset N$. Then $K \in \mathcal{L}(N, \mathcal{T})$ if and only if $K \in \mathcal{L}(L, \mathcal{T})$ and $L \in \mathcal{L}(N, \mathcal{T})$.*

Proof. (1) Assume $\mathcal{T}(N) = 0$ and L is $\mathcal{T}$-dense in N. For $K \subset N$ with $K \cap L = 0$, the map $K \to N \to N/L$ is monomorphic, i.e., $K \in \mathcal{T}$ and hence $K \subset \mathcal{T}(N) = 0$.

(2) follows from the exact sequence $0 \to L/K \to N/K \to N/L \to 0$. □

9.7 Properties of $\mathcal{T}$-dense submodules.

Let $\mathcal{T}$ be a hereditary pretorsion class in $\sigma[M]$ and $N \in \sigma[M]$. Then $\mathcal{L} := \mathcal{L}(N, \mathcal{T})$ has the following properties:

(1) *If $K \in \mathcal{L}$ and $K \subset L \subset N$, then $L \in \mathcal{L}$;*

(2) *if $K, L \in \mathcal{L}$ then $K \cap L \in \mathcal{L}$;*

(3) *if $K \in \mathcal{L}$ and $f \in End_A(N)$, then $(K)f^{-1} \in \mathcal{L}$;*

(4) *if $K \subset N$ and N/K is generated by $\{N/L \,|\, L \in \mathcal{L}\}$, then $K \in \mathcal{L}$.*

(5) *Assume that $\mathcal{T}$ is a hereditary torsion class. Let $K \subset N$ and suppose there exists an N-generated $L \in \mathcal{L}$, such that for each $g \in Hom_A(N, L)$, $(K \cap L)g^{-1} \in \mathcal{L}$. Then $K \in \mathcal{L}$.*

Proof. (1)-(4) are obvious.

(5) Let $K, L \subset N$ as desired. By (1), $K + L \in \mathcal{L}$ and by 9.6, it suffices to show $K \in \mathcal{L}(K + L, \mathcal{T})$. For $g \in Hom_A(N, L)$ and the projection $\pi : L \to L/(K \cap L)$, $Ke\, g\pi = (K \cap L)g^{-1} \in \mathcal{L}$ by assumption. Hence $Im\, g\pi \simeq N/Ke\, g\pi \in \mathcal{T}$, implying

$$(K + L)/K \simeq L/(K \cap L) = (L)\pi = (N\, Hom_A(N, L))\pi \in \mathcal{T}. \qquad \square$$

9.8 Dense submodules of generators.

Let $\mathcal{T}$ be a hereditary pretorsion class in $\sigma[M]$.

(1) *Assume $N, L \in \sigma[M]$ and L is N-generated. Then $L \in \mathcal{T}$ if and only if for all $f \in Hom_A(N, L)$, $Ke\, f \in \mathcal{L}(N, \mathcal{T})$.*

(2) *Let $\{G_\lambda\}_\Lambda$ be a family of generators in $\sigma[M]$. Then $L \in \sigma[M]$ belongs to $\mathcal{T}$ if and only if it is generated by factor modules G_λ/K_λ, with $K_\lambda \in \mathcal{L}(G_\lambda, \mathcal{T})$. Hence the class $\mathcal{T}$ is uniquely determined by the sets $\mathcal{L}(G_\lambda, \mathcal{T})$.*

Proof. (1) By assumption, $N\,Hom_A(N, L) = L$ and the assertion follows from 9.7.

(2) is shown similarly. □

Definitions. Let $\mathcal{T}$ be a pretorsion class in $\sigma[M]$. A module $N \in \sigma[M]$ is called *(M,$\mathcal{T}$)-injective*, if N is injective with respect to every exact sequence $0 \to K \to L$ in $\sigma[M]$ with $L/K \in \mathcal{T}$, i.e., if every diagram

$$\begin{array}{ccccc} 0 & \to & K & \to & L \\ & & \downarrow & & \\ & & N & & , \end{array}$$

with exact row and $L/K \in \mathcal{T}$, can be extended commutatively by some $f : L \to N$. N is called *faithfully (M,$\mathcal{T}$)-injective* if such an f is always uniquely determined.

9.9 $(M,\mathcal{T})$-injective modules. Properties.

Let $\mathcal{T}$ be a hereditary pretorsion class in $\sigma[M]$ and $N,\ L \in \sigma[M]$. Assume N is injective with respect to every exact sequence $0 \to K \to L$ in $\sigma[M]$ with $L/K \in \mathcal{T}$. Then:

(1) If P is a factor module of L, then N is injective with respect to every exact sequence $0 \to Q \to P$ with $P/Q \in \mathcal{T}$.

(2) Let $\mathcal{T}$ be a hereditary torsion class. If U is a submodule of L with $L/U \in \mathcal{T}$, then N is also injective with respect to every exact sequence $0 \to K \to U$ with $U/K \in \mathcal{T}$.

Proof. (1) Let $p : L \to P$ be an epimorphism, $\varepsilon_2 : Q \to P$ a monomorphism such that $P/Q \in \mathcal{T}$, and $g \in Hom_A(Q,N)$. Forming a pullback we obtain the commutative diagram with exact rows,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{\varepsilon_1} & L & \longrightarrow & L/K & \longrightarrow & 0 \\ & & \downarrow\varphi & & \downarrow p & & \| & & \\ 0 & \longrightarrow & Q & \xrightarrow{\varepsilon_2} & P & \longrightarrow & P/Q & \longrightarrow & 0 \\ & & \downarrow g & & & & & & \\ & & N & & & & & & . \end{array}$$

By diagram lemmata, φ is epic. Since N is injective with respect to the first row, there exists $f : L \to N$ satisfying $\varphi g = \varepsilon_1 f$.

Since in the given situation the pullback is also a pushout we obtain $h : P \to N$ such that $g = \varepsilon_2 h$.

(2) Under the given conditions a pushout yields the commutative exact diagram

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & K & \to & U & \to & U/K & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & K & \to & L & \to & L/K & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & L/U & = & L/U & & . \end{array}$$

Since $U/K, L/U \in \mathcal{T}$ we obtain, by 9.6, that $L/K \in \mathcal{T}$. So any $K \to N$ extends to L and hence to U. □

9.10 $(M, \mathcal{T})$-injective modules and direct sums.

Let $\mathcal{T}$ be a hereditary pretorsion class in $\sigma[M]$ and $N \in \sigma[M]$. Let $\{L_\lambda\}_\Lambda$ be a family of modules in $\sigma[M]$. Assume N is injective with respect to exact sequences $0 \to K_\lambda \to L_\lambda$, with $L_\lambda/K_\lambda \in \mathcal{T}$. Then N is injective with respect to exact sequences $0 \to K \to \bigoplus_\Lambda L_\lambda$, where $(\bigoplus_\Lambda L_\lambda)/K \in \mathcal{T}$.

Proof. Let $L := \bigoplus_\Lambda L_\lambda$ and $K \subset L$. For any $g : K \to N$, consider the set

$$\mathcal{M} := \{h : U \to N \mid K \subset U \subset L \text{ and } h|_K = g\}.$$

This can be ordered by

$$[h_1 : U_1 \to N] \leq [h_2 : U_2 \to N] \Leftrightarrow U_1 \subset U_2 \text{ and } h_2|_{U_1} = h_1.$$

$\mathcal{M}$ is in fact inductively ordered and hence, by Zorn's Lemma, has a maximal element $h_0 : U_0 \to N$. To show $L = U_0$ it suffices to prove $L_\lambda \subset U_0$ for all $\lambda \in \Lambda$. By assumption, every diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & U_0 \cap L_\lambda & \longrightarrow & L_\lambda \\ & & \downarrow & & \\ & & U_0 & \xrightarrow{h_0} & N \end{array}$$

can be extended commutatively by some homomorphism $h_\lambda : L_\lambda \to N$, since

$$L_\lambda/(L_\lambda \cap K) \simeq (L_\lambda + K)/K \subset L/K \in \mathcal{T},$$

and hence $L_\lambda/(U_0 \cap L_\lambda)$, as an image of $L_\lambda/(L_\lambda \cap K)$, belongs to $\mathcal{T}$. The assignement

$$h^* : U_0 + L_\lambda \to N, \quad u + l_\lambda \mapsto uh_0 + l_\lambda h_\lambda,$$

does not depend on the representation $u + l_\lambda$, since $u + l_\lambda = 0$ implies $u = -l_\lambda \in L_\lambda \cap U_0$, and hence

$$(u + l_\lambda)h^* = uh_0 + l_\lambda h_\lambda = -l_\lambda h_\lambda + l_\lambda h_\lambda = 0.$$

So $h^* : U_0 + L_\lambda \to N$ is a homomorphism which belongs to $\mathcal{M}$ and obviously is greater or equal to $h_0 : U_0 \to N$. By maximality of h_0, we conclude $h^* = h_0$. Hence in particular $U_0 + L_\lambda = U_0$ and $L_\lambda \subset U_0$. □

Definition. Let $\mathcal{T}$ be a hereditary torsion class, and $N \in \sigma[M]$. An $(M, \mathcal{T})$-injective module $E \in \sigma[M]$ with $N \subset E$ is called an *$(M, \mathcal{T})$-injective hull of* N if N is essential and $\mathcal{T}$-dense in E. In this case we denote E by $E_{\mathcal{T}}(N)$.

$E_{\mathcal{T}}(N)$ is uniquely determined up to isomorphism and can be obtained as the submodule E' of the M-injective hull $\widehat{N}$ of N with

$$E'/N = \mathcal{T}(\widehat{N}/N).$$

By construction $E'/N \in \mathcal{T}$ and $\widehat{N}/E'$ is $\mathcal{T}$-torsionfree. If N is $\mathcal{T}$-torsionfree then E' and $\widehat{N}$ are also $\mathcal{T}$-torsionfree. E' is in fact $(M,\mathcal{T})$-injective. This results from the following characterization of $(M,\mathcal{T})$-injective modules.

9.11 $(M,\mathcal{T})$-injective modules. Characterizations.

For a hereditary torsion class $\mathcal{T}$ and $N \in \sigma[M]$, the following are equivalent:

(a) N is $(M,\mathcal{T})$-injective;

(b) every exact sequence $0 \to N \to L \to L/N \to 0$ with $L/N \in \mathcal{T}$ splits;

(c) for a family of generators $\{G_\lambda\}_\Lambda$ of $\sigma[M]$, N is injective with respect to every exact sequence $0 \to K \to G_\lambda$, with $\lambda \in \Lambda$ and $G_\lambda/K \in \mathcal{T}$;

(d) for the M-injective hull $\widehat{N}$ of N, $\widehat{N}/N$ is $\mathcal{T}$-torsionfree.

Proof. $(a) \Rightarrow (b)$ and $(a) \Rightarrow (c)$ are obvious.

$(b) \Rightarrow (a)$ Consider any exact sequence $0 \to B \to C \to C/B \to 0$ with $C/B \in \mathcal{T}$ and $g : B \to N$. Forming a pushout we obtain the commutative exact diagram

$$\begin{array}{ccccccc} 0 & \to & B & \to & C & \to & C/B & \to 0 \\ & & \downarrow g & & \downarrow & & \downarrow \simeq & \\ 0 & \to & N & \to & L & \to & L/N & \to 0. \end{array}$$

Then $L/N \in \mathcal{T}$ and the lower row splits. This yields the morphism $C \to N$ wanted.

$(c) \Rightarrow (a)$ By assumption and 9.10, N is also injective with respect to every exact sequence $0 \to U \to \bigoplus_\Lambda G_\lambda$ with $\bigoplus_\Lambda G_\lambda/U \in \mathcal{T}$.

Let $0 \to K \to L$ denote an exact sequence in $\sigma[M]$ with $L/K \in \mathcal{T}$. L is a factor module of some direct sum of G_λ's and hence (by 9.9(1)), N is injective with respect to any sequence $0 \to K \to L$ with $L/K \in \mathcal{T}$. So N is $(M,\mathcal{T})$-injective.

$(b) \Rightarrow (d)$ Let $\mathcal{T}(\widehat{N}/N) = U/N$. The sequence $0 \to N \to U \to U/N \to 0$ splits by assumption, i.e., N is a direct summand of U. Since N is essential in $\widehat{N}$ this means $U = N$. Hence $\widehat{N}/N$ is $\mathcal{T}$-torsionfree.

$(d) \Rightarrow (b)$ Let $0 \to N \to L \to L/N \to 0$ be an exact sequence in $\sigma[M]$, $L/N \in \mathcal{T}$. We have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & N & \to & L & \to & L/N & \to & 0 \\ & & \| & & \downarrow & & & & \\ 0 & \to & N & \to & \widehat{N} & \to & \widehat{N}/N & \to & 0, \end{array}$$

which extends commutatively by the zero map $L/N \to \widehat{N}/N$ since $\widehat{N}/N$ is $\mathcal{T}$-torsionfree. Now apply the Homotopy Lemma. $\square$

9.12 Lemma. *Let $\mathcal{T}$ be a hereditary torsion class in $\sigma[M], N \in \sigma[M]$ and $E_{\mathcal{T}}(N)$ the $(M,\mathcal{T})$-injective hull of N. Consider the following assertions:*

(1) $E_{\mathcal{T}}(N)$ is M-injective;

(2) for every $K \subset M$ and $f \in Hom_A(K,N)$, there exist $K \subset L \subset M$, where $M/L \in \mathcal{T}$ and $g \in Hom_A(L,N)$ with $g|_K = f$.

Then (1) $\Rightarrow$ (2). If N is $\mathcal{T}$-torsionfree, then (2) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2) By assumption, the diagram

$$\begin{array}{ccccc} 0 & \to & K & \to & M \\ & & \downarrow_f & & \\ & & N & \to & E_{\mathcal{T}}(N) \end{array}$$

can be extended commutatively by some $h : M \to E_{\mathcal{T}}(N)$. Then $Nh^{-1} =: L \supset K$ and h induces a monomorphism $M/L \to E_{\mathcal{T}}(N)/N$, i.e., $M/L \in \mathcal{T}$.

(2) $\Rightarrow$ (1) Assume $\mathcal{T}(N) = 0$. Let $X \subset M$ and $f' \in Hom_A(X, E_{\mathcal{T}}(N))$. Put

$$K := Nf'^{-1} \subset X \text{ and } f := f'|_K \in Hom_A(K,N).$$

By assumption, there exists $K \subset L \subset M$ with $M/L \in \mathcal{T}$ and $g : L \to N$ with $g|_K = f$. So we have the commutative diagram

$$\begin{array}{ccccc} K & \subset & L & \to & M \\ \downarrow_f & & \downarrow_g & & \downarrow_h \\ N & = & N & \to & E_{\mathcal{T}}(N). \end{array}$$

Since $f = f'|_K = g|_K = h|_K$ the difference $h|_X - f' : X \to E_{\mathcal{T}}(N)$ induces a map $X/K \to E_{\mathcal{T}}(N)$. Now $E_{\mathcal{T}}(N)$ is $\mathcal{T}$-torsionfree and $X/K \in \mathcal{T}$ implies that this map is zero, i.e., $f' = h|_X$. Hence $E_{\mathcal{T}}(N)$ is M-injective. □

9.13 Lemma. *Let $\mathcal{T}$ be a hereditary torsion class in $\sigma[M]$. Assume that for every $L \trianglelefteq M$, $M/L \in \mathcal{T}$. Then:*

(1) Every $(M,\mathcal{T})$-injective module $N \in \sigma[M]$ is M-injective.

(2) The $(M,\mathcal{T})$-injective hull of any $N \in \sigma[M]$ is equal to its M-injective hull.

Proof. (1) For $K \subset M$ let K' be a complement of K in M. Then $L := K \oplus K' \trianglelefteq M$ and every $f \in Hom_A(K,N)$ can be extended to $L \trianglelefteq M$. By our assumptions this map can be extended to M.

(2) This is clear by (1). □

9.14 Faithfully (M,$\mathcal{T}$)-injective modules. Characterization.

Let $\mathcal{T}$ be a torsion class in $\sigma[M]$. For any $N \in \sigma[M]$, the following are equivalent:

(a) *N is faithfully $(M,\mathcal{T})$-injective;*

(b) *N is $\mathcal{T}$-torsionfree and $(M,\mathcal{T})$-injective.*

Proof. $(a) \Rightarrow (b)$ The diagram

$$\begin{array}{ccccc} 0 & \to & 0 & \to & \mathcal{T}(N) \\ & & \downarrow & & \\ & & N & & \end{array}$$

can be extended commutatively both by the inclusion map $\mathcal{T}(N) \to N$ and the zero map. By the uniquenes condition the two maps coincide, i.e., $\mathcal{T}(N) = 0$.

$(b) \Rightarrow (a)$ Consider the diagram with exact row,

$$\begin{array}{ccccc} 0 & \to & K & \to & L \\ & & \downarrow & & \\ & & N & & , \end{array}$$

and $L/K \in \mathcal{T}$. Assume $f, g : L \to N$ extend the diagram commutatively. Then $f - g$ induces a homomorphism $h : L/K \to N$. Since N is $\mathcal{T}$-torsionfree, h has to be the zero map, i.e., $f = g$. □

Definition. Let $\mathcal{T}$ be a hereditary torsion class in $\sigma[M]$ and $N \in \sigma[M]$. The $(M,\mathcal{T})$-injective hull of the factor module $N/\mathcal{T}(N)$ is called the *quotient module* of N with respect to $\mathcal{T}$,

$$Q_{\mathcal{T}}(N) := E_{\mathcal{T}}(N/\mathcal{T}(N)).$$

9.15 Properties of quotient modules.

Let $\mathcal{T}$ be a hereditary torsion class in $\sigma[M]$ and $N \in \sigma[M]$.

(1) *$Q_{\mathcal{T}}(N)$ is torsionfree and faithfully $(M,\mathcal{T})$-injective.*

(2) *N is a submodule of $Q_{\mathcal{T}}(N)$ if and only if N is torsionfree.*

(3) *For $N \in \mathcal{T}$, $Q_{\mathcal{T}}(N) = 0$.*

Proof. All these observations are obvious from the definitions. □

9.16 Essential submodules $\mathcal{T}$-dense.

Let $\mathcal{T}$ be a hereditary torsion class in $\sigma[M]$ and $T := End_A(Q_{\mathcal{T}}(M))$. Assume $M \notin \mathcal{T}$ and every essential submodule of M is $\mathcal{T}$-dense in M. Then $Q_{\mathcal{T}}(M)$ is the M-injective hull of $M/\mathcal{T}(M)$ and

(1) *T is a (von Neumann) regular left self-injective ring.*

(2) $M/\mathcal{T}(M)$ *has finite uniform dimension if and only if* T *is left semisimple.*

(3) If M *is uniform and* $\mathcal{T}(M) = 0$*, then* $End_A(\widehat{M})$ *is a division ring.*

Proof. By 9.13, $Q_{\mathcal{T}}(M)$ is the M-injective hull of $M/\mathcal{T}(M)$.

(1) For $f \in Jac(T)$, $K := Kef \trianglelefteq Q_{\mathcal{T}}(M)$. For every $\varphi : M \to Q_{\mathcal{T}}(M)$, $K\varphi^{-1}$ is essential in M and hence $\mathcal{T}$-dense in M by assumption. Since M generates $Q_{\mathcal{T}}(M)$, $Q_{\mathcal{T}}(M)/K \in \mathcal{T}$, i.e.,

$$Q_{\mathcal{T}}(M)/K \simeq Im\, f \subset \mathcal{T}(Q_{\mathcal{T}}(M)) = 0.$$

Hence $f = 0$ and $Jac(T) = 0$ and T is regular. By [40, 22.1], $Jac(T) = 0$ implies that ${}_TT$ is self-injective.

(2) M has finite uniform dimension if and only if T has no infinite set of orthogonal idempotents. Since T is regular this means that it is left semisimple.

(3) For $f \in T$ and $Ke\, f \neq 0$, $Ke\, f \cap M$ is $\mathcal{T}$-dense in M and hence $\mathcal{T}$-dense in $\widehat{M}$. Then $\widehat{M}/Ke\, f \simeq \widehat{M}f \subset \mathcal{T}(\widehat{M}) = 0$, i.e. $f = 0$. Therefore every $0 \neq f \in T$ is monic.

Since $M/K \in \mathcal{T}$ for every $0 \neq K \subset M$, by 9.6, M is uniform and so is $\widehat{M}$. For every $f \in T$, $Imf \simeq \widehat{M}$ is M-injective. Now $Imf \trianglelefteq \widehat{M}$ implies $\widehat{M}f = \widehat{M}$, i.e. f is an isomorphism. □

In studying quotient modules direct limits are very useful. As observed in 9.7, for any hereditary pretorsion class $\mathcal{T}$ and $N \in A\text{-}Mod$, the set of $\mathcal{T}$-dense submodules

$$\mathcal{L}(N, \mathcal{T}) = \{U \subset N \mid N/U \in \mathcal{T}\},$$

is downwards directed with respect to inclusion. For $V \subset U \subset N$, denote the inclusion map by $\varepsilon_{V,U} : V \to U$. For any $L \in \sigma[M]$, there is a canonical $\mathbb{Z}$-homomorphism

$$f_{U,V} : Hom_A(U, L) \to Hom_A(V, L), \quad f \mapsto \varepsilon_{V,U} f.$$

In fact, $(Hom(U, L), f_{U,V}, \mathcal{L}(N, \mathcal{T}))$ is a direct system of $\mathbb{Z}$-modules.

9.17 Quotient modules and direct limits.

Let $\mathcal{T}$ *be a hereditary torsion class in* $\sigma[M]$ *and* N, $L \in \sigma[M]$ *with* $\mathcal{T}(L) = 0$.

(1) $\varinjlim \{Hom_A(U, L) \mid U \in \mathcal{L}(N, \mathcal{T})\} \simeq Hom_A(N, Q_{\mathcal{T}}(L))$.

(2) $Hom_A(N, Q_{\mathcal{T}}(L)) \simeq Hom_A(Q_{\mathcal{T}}(N), Q_{\mathcal{T}}(L))$.

(3) If G *is a generator in* $\sigma[M]$*, then for any* $L \in \sigma[M]$*,* $Q_{\mathcal{T}}(G)$ *generates* $Q_{\mathcal{T}}(L)$.

(4) $\varinjlim \{Hom_A(U, N/\mathcal{T}(N)) \mid U \in \mathcal{L}(N, \mathcal{T})\} \simeq End_A(Q_{\mathcal{T}}(N))$.

Proof. (1) For $U \in \mathcal{L}(N,\mathcal{T})$, every $f \in Hom_A(U,L)$ can be uniquely extended to $\overline{f} \in Hom_A(N, Q_{\mathcal{T}}(L))$:

$$\begin{array}{ccccc} 0 & \to & U & \to & N \\ & & \downarrow f & & \downarrow \overline{f} \\ 0 & \to & L & \to & Q_{\mathcal{T}}(L). \end{array}$$

This yields a $\mathbb{Z}$-morphism $\varphi_U : Hom_A(U,L) \to Hom_A(N, Q_{\mathcal{T}}(L))$, and for $V \subset U$, we have the commutative diagram

$$\begin{array}{ccc} Hom_A(U,L) & \stackrel{\varphi_U}{\to} & Hom_A(N, Q_{\mathcal{T}}(L)) \\ \downarrow f_{UV} & \nearrow \varphi_V & \\ Hom_A(V,L) & & . \end{array}$$

The φ_U form a direct system of homomorphisms, and by the universal property of the direct limit, there exists a homomorphism

$$\varphi : \varinjlim \{Hom_A(U,L) \,|\, U \in \mathcal{L}(N,\mathcal{T})\} \to Hom_A(N, Q_{\mathcal{T}}(L)).$$

Since all φ_U are monic, φ is monic.

We show that φ is epic. Let $h \in Hom_A(N, Q_{\mathcal{T}}(L))$. For $Lh^{-1} \subset N$, $N/Lh^{-1} \subset Q_{\mathcal{T}}(L)/L$, hence $Lh^{-1} = V \in \mathcal{L}(N,\mathcal{T})$. By restriction we obtain $\overline{h} = h|_V \in Hom_A(V,L)$. The unique extension of $\overline{h}$ to $N \to Q_{\mathcal{T}}(L)$ yields h.

(2) For every $f \in Hom_A(N, Q_{\mathcal{T}}(L))$ there is a unique $f' \in Hom_A(Q_{\mathcal{T}}(N), Q_{\mathcal{T}}(L))$ (see 9.14) which makes the diagram commutative:

$$\begin{array}{ccccc} 0 & \to & N/\mathcal{T}(N) & \to & Q_{\mathcal{T}}(N) \\ & & \downarrow f & \swarrow f' & \\ & & Q_{\mathcal{T}}(L) & & \end{array}$$

On the other hand, every $g \in Hom_A(Q_{\mathcal{T}}(N), Q_{\mathcal{T}}(L))$ is uniquely determined by its restriction to $N/\mathcal{T}(N)$.

(3) By (2), we have

$$\begin{aligned} Q_{\mathcal{T}}(L) &= G \cdot Hom_A(G, Q_{\mathcal{T}}(L)) = G \cdot Hom_A(Q_{\mathcal{T}}(G), Q_{\mathcal{T}}(L)) \\ &= Q_{\mathcal{T}}(G) \cdot Hom_A(Q_{\mathcal{T}}(G), Q_{\mathcal{T}}(L)). \end{aligned}$$

(4) follows from (1) and (2). □

9.18 The quotient functor.

Let $\mathcal{T}$ be a hereditary torsion class in $\sigma[M]$ and $N \in \sigma[M]$.

(1) *Any $f : N \to L$ in $\sigma[M]$ induces a unique $Q_{\mathcal{T}}(f) : Q_{\mathcal{T}}(N) \to Q_{\mathcal{T}}(L)$.*

(2) *Assume f is monic. Then $Q_{\mathcal{T}}(f)$ is monic, and $Q_{\mathcal{T}}(f)$ is an isomorphism if and only if $L/Nf \in \mathcal{T}$.*

(3) $Q_{\mathcal{T}} : \sigma[M] \to \sigma[M]$ is a left exact functor.

Proof. (1) $f : N \to L$ in $\sigma[M]$ factors uniquely to $\bar{f} : N/\mathcal{T}(N) \to L/\mathcal{T}(L)$ and $\bar{f}$ is uniquely extended to $Q_{\mathcal{T}}(f) : Q_{\mathcal{T}}(N) \to Q_{\mathcal{T}}(L)$.

(2) If f is monic, the induced map $\overline{f} : N/\mathcal{T}(N) \to L/\mathcal{T}(L)$ is monic (because $\mathcal{T}(L)f^{-1} = \mathcal{T}(N)$). Since $N/\mathcal{T}(N) \trianglelefteq Q_{\mathcal{T}}(N)$, $Q_{\mathcal{T}}(f)$ is also monic.

Assume $L/Nf \in \mathcal{T}$. Then it is easy to see that $Coke\,\overline{f} \in \mathcal{T}$. This implies that $Im\,\overline{f}$ is $\mathcal{T}$-dense in $L/\mathcal{T}(L)$ and $Im\,Q_{\mathcal{T}}(f)$ is $\mathcal{T}$-dense - and hence essential (see 9.6) - in $Q_{\mathcal{T}}(L)$. However, $Im\,Q_{\mathcal{T}}(f) \simeq Q_{\mathcal{T}}(N)$ is $(M, \mathcal{T})$-injective and therefore is a direct summand in $Q_{\mathcal{T}}(L)$. This implies $Im\,Q_{\mathcal{T}}(f) = Q_{\mathcal{T}}(L)$.

Assume $Q_{\mathcal{T}}(f)$ is known to be an isomorphism. By definition of $Q_{\mathcal{T}}(-)$, the projection $p : L \to L/Nf =: \overline{L}$ yields the commutative diagram

$$\begin{array}{ccccc} L & \to & L/\mathcal{T}(L) & \to & Q_{\mathcal{T}}(L) \\ \downarrow p & & \downarrow & & \downarrow Q_{\mathcal{T}}(p) \\ \overline{L} & \to & \overline{L}/\mathcal{T}(\overline{L}) & \to & Q_{\mathcal{T}}(\overline{L}), \end{array}$$

with $Q_{\mathcal{T}}(p) = 0$. This implies $\overline{L}/\mathcal{T}(\overline{L}) = 0$ and $\overline{L} = L/Nf \in \mathcal{T}$.

(3) By (1) it is easy to verify that $Q_{\mathcal{T}}(-)$ is in fact a functor.

For $X \in \sigma[M]$, let $p_X : X \to X/\mathcal{T}(X) =: \overline{X}$ denote the canonical projection. Consider the commutative diagram in $\sigma[M]$ with the first line exact,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{f} & L & \xrightarrow{g} & N & \longrightarrow & 0 \\ & & \downarrow p_K & & \downarrow p_L & & \downarrow p_N & & \\ 0 & \longrightarrow & \overline{K} & \xrightarrow{\overline{f}} & \overline{L} & \xrightarrow{\overline{g}} & \overline{N} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Q_{\mathcal{T}}(K) & \xrightarrow{Q_{\mathcal{T}}(f)} & Q_{\mathcal{T}}(L) & \xrightarrow{Q_{\mathcal{T}}(g)} & Q_{\mathcal{T}}(N) & & . \end{array}$$

We have to show that the lower row is exact. From above we know that $Q_{\mathcal{T}}(f)$ is monic. It remains to prove $Im\,Q_{\mathcal{T}}(f) = Ke\,Q_{\mathcal{T}}(g)$. Since $Q_{\mathcal{T}}(fg) = Q_{\mathcal{T}}(f)Q_{\mathcal{T}}(g) = 0$, $Im\,Q_{\mathcal{T}}(f) \subset Ke\,Q_{\mathcal{T}}(g)$.

Put $U := (\mathcal{T}(N))g^{-1} \subset L$. Obviously $Kf \subset U$ and $U/Kf \in \mathcal{T}$. As shown above, this implies that $Q_{\mathcal{T}}(f) : Q_{\mathcal{T}}(K) \to Q_{\mathcal{T}}(U)$ is an isomorphism. By construction, $Up_L = Ke\,\overline{g} = \overline{L} \cap Ke\,Q_{\mathcal{T}}(g)$ is a $\mathcal{T}$-dense submodule of $Ke\,Q_{\mathcal{T}}(g)$. From this we conclude

$$Ke\,Q_{\mathcal{T}}(g) \subset Q_{\mathcal{T}}(U) = Im\,Q_{\mathcal{T}}(f).$$

□

In A-*Mod* any hereditary torsion class is determined by the dense left ideal in A. In fact we have by 9.7:

9.19 $\mathcal{T}$-dense left ideals. Properties.

Let $\mathcal{T}$ be a hereditary torsion class in A-Mod, and $\mathcal{L} = \{K \subset {}_AA \mid A/K \in \mathcal{T}\}$.

(1) If $K \in \mathcal{L}$ and $K \subset L \subset A$, then $L \in \mathcal{L}$;

(2) if $K, L \in \mathcal{L}$, then $K \cap L \in \mathcal{L}$;

(3) if $K \in \mathcal{L}$, then for every $a \in A$, $(K : a) = \{r \in A \mid ra \in K\} \in \mathcal{L}$;

(4) if $K \subset A$ and there exists $L \in \mathcal{L}$ with $(K : b) \in \mathcal{L}$ for all $b \in L$, then $K \in \mathcal{L}$.

In the following application of 9.17 the importance of $\mathcal{T}$-dense left ideals for forming quotient modules becomes obvious.

9.20 Gabriel filter and quotient modules.

Let $\mathcal{T}$ be a hereditary torsion class in A-Mod and $\mathcal{L} = \{K \subset {}_AA \mid A/K \in \mathcal{T}\}$. Then

(1) For every A-module N, $Q_\mathcal{T}(N) = \varinjlim \{Hom(K, N/\mathcal{T}(N)) \mid K \in \mathcal{L}\}$.

(2) $Q_\mathcal{T}(A) \simeq End_A(Q_\mathcal{T}(A))$, hence a ring structure is defined on $Q_\mathcal{T}(A)$.

(3) $Q_\mathcal{T}(N)$ has a $Q_\mathcal{T}(A)$-module structure extending the A-structure.

(4) $Hom_A(Q_\mathcal{T}(N), Q_\mathcal{T}(L)) = Hom_{Q_\mathcal{T}(A)}(Q_\mathcal{T}(N), Q_\mathcal{T}(L))$.

Proof. (1) By 9.17, $\varinjlim Hom_A(K, N/\mathcal{T}(N)) \simeq Hom_A(A, Q_\mathcal{T}(N)) \simeq_A Q_\mathcal{T}(N)$.

(2) follows from 9.17(2).

(3) Again by 9.17(2), we obtain

$$Q_\mathcal{T}(N) \simeq Hom_A(A, Q_\mathcal{T}(N)) \simeq Hom_A(Q_\mathcal{T}(A), Q_\mathcal{T}(N)).$$

Hence the $End_A(Q_\mathcal{T}(A))$-module structure given on the right side can be transferred to the left side.

(4) We have to show that every A-homomorphism $f : Q_\mathcal{T}(N) \to Q_\mathcal{T}(L)$ is also a $Q_\mathcal{T}(A)$-homomorphism. This follows from the definition of the $Q_\mathcal{T}(A)$-structure of $Q_\mathcal{T}(N)$, resp. $Q_\mathcal{T}(L)$. For $q \in Q_\mathcal{T}(A)$, $n \in Hom_A(Q_\mathcal{T}(A), Q_\mathcal{T}(N)) = Q_\mathcal{T}(N)$:

$$Q_\mathcal{T}(A) \xrightarrow{q} Q_\mathcal{T}(A) \xrightarrow{n} Q_\mathcal{T}(N) \xrightarrow{f} Q_\mathcal{T}(L), \quad (qn)f = q(nf).$$

$\square$

9.21 Exercises.

(1) Let M be an A-module, $U \in \sigma[M]$ and $p : N \to L$ an epimorphism in $\sigma[M]$. U is called pseudo-projective *with respect to p if, for all non-zero $f \in Hom(U, L)$, there*

are $s \in End(U)$ *and* $g : U \to N$ *satisfying* $gp = sf \neq 0$, *i.e., we have the commutative diagram*

$$\begin{array}{ccccc} U & \xrightarrow{s} & U & & \\ \downarrow g & & \downarrow f & & \\ N & \xrightarrow{p} & L & \longrightarrow & 0. \end{array}$$

U *is called* self-pseudo-projective *in* $\sigma[M]$ *if* U *is pseudo-projective with respect to all epimorphisms* $p \in \sigma[M]$ *with* $Ke\,p \in Gen(U)$.

$Gen(U)$ *denotes the class of all* U*-generated modules.*

Prove that the following are equivalent ([78], [279, 3.4]):

(a) U *is self-pseudo-projective in* $\sigma[M]$;
(b) $Gen(U)$ *is a torsion class in* $\sigma[M]$;
(c) every generator in $Gen(U)$ *is U-pseudo-projective in* $\sigma[M]$;
(d) every $N \in \sigma[M]$ *with an exact sequence* $U^{(\Lambda)} \to N \to U^{(\Lambda)} \to 0$ *belongs to* $Gen(U)$.

(2) A family of left ideals $\mathcal{L}$ *of* A *with the properties given in 9.19 is called a* Gabriel filter.

Assume $\mathcal{L}$ *is a Gabriel filter of A. Prove that the class of modules generated by* $\{A/K\}_{K \in \mathcal{L}}$ *is a hereditary torsion class in A-Mod.*

(3) Let M *be an A-module. For* $Q \in \sigma[M]$ *consider*

$$Q\text{-}Inj := \{N \in \sigma[M] \mid Q \text{ is } N\text{-injective}\}.$$

Prove that Q-Inj is a hereditary pretorsion class in $\sigma[M]$.

References. Berning [78], Dung-Huynh-Smith-Wisbauer [11], Gabriel [140], Golan [14], Goldman [144], Hutchinson-Leu [161], Leu [187], Wisbauer [273, 279].

10 Singular pretorsion theory

1.Examples. 2.Non-M-singular modules. 3.Proposition. 4.Proposition. 5.Goldie torsion class. 6.Quotient ring of A. 7.Corollary. 8.Rational submodules. 9.Maximal ring of quotients. 10.Exercises.

In torsion theory in A-*Mod*, an important part is played by the *singular modules*. This notion can be adapted to the category $\sigma[M]$ and many of the interesting properties are preserved.

Definition. Let M and N be A-modules. N is called *singular in* $\sigma[M]$ or M-*singular* if $N \simeq L/K$ for some $L \in \sigma[M]$ and $K \trianglelefteq L$ (see [273]).

In case $M = A$, instead of A-*singular* we just say *singular*. It is well-known that a module N is singular in A-*Mod* if and only if for every $n \in N$, $An_A(n)$ is an essential left ideal in A.

Obviously, every M-singular module is singular but not vice-versa as we will see with the examples below.

We denote by $\mathcal{S}_M$ the class of singular modules in $\sigma[M]$. $\mathcal{S}_M$ is closed under submodules, homomorphic images and direct sums (see [40, 17.3, 17.4]), i.e., it is a hereditary pretorsion class. Hence every module $N \in \sigma[M]$ contains a *largest M-singular submodule*

$$\mathcal{S}_M(N) := Tr(\mathcal{S}_M, N).$$

However, $\mathcal{S}_M$ need not be closed under extensions, i.e., $\mathcal{S}_M(N/\mathcal{S}_M(N))$ might not be zero. In our notation $\mathcal{S}(N) := \mathcal{S}_A(N)$ is just the largest singular submodule of N and $\mathcal{S}_M(N) \subset \mathcal{S}(N)$.

If $\mathcal{T}_A$ is a hereditary pretorsion class in A-*Mod*, then the restriction of $\mathcal{T}_A$ to $\sigma[M]$ yields a (hereditary) pretorsion class $\mathcal{T}_M := \mathcal{T}_A \cap \sigma[M]$ in $\sigma[M]$. If $M \in \mathcal{T}_A$, then $\mathcal{T}_M = \sigma[M]$. Pretorsion classes in $\sigma[M]$ may be induced by different classes in A-*Mod*. In particular, the restriction of the singular torsion class $\mathcal{S}$ in A-*Mod* to $\sigma[M]$ need not yield $\mathcal{S}_M$.

10.1 Examples.

(1) Let M be a simple singular A-module. Then $\sigma[M] \subset \mathcal{S}$ and $\mathcal{S}_M = 0$.

(2) An A-module M is semisimple if and only if $\mathcal{S}_M = 0$.

(3) Let A be an SI-ring (singulars are injective) and choose $M \in A$-*Mod* such that $\mathcal{S} = \sigma[M]$. Then $\mathcal{S}_M = 0$.

(4) Put $\overline{Q} = \mathbb{Q}/\mathbb{Z}$. Then in $\mathbb{Z}$-*Mod*, $\sigma[\overline{Q}] = \mathcal{S}$ and $\sigma[\overline{Q}] = \mathcal{S}_{\overline{Q}}$, i.e., every module in $\sigma[\overline{Q}]$ is $\overline{Q}$-singular. This implies that $\sigma[\overline{Q}]$ has no projectives.

Proof. (1) All modules in $\sigma[M]$ are singular in *A-Mod* and projective in $\sigma[M]$.

(2) If M is semisimple, every module in $\sigma[M]$ is projective and $\mathcal{S}_M = 0$.

Assume $\mathcal{S}_M = 0$. Then M has no non-trivial essential submodules and hence is semisimple.

(3) For an SI-ring, all singular modules in *A-Mod* are semisimple (see [11, 17.4]).

(4) $\sigma[\overline{Q}]$ is the category of torsion $\mathbb{Z}$-modules. To prove that every module in $\sigma[\overline{Q}]$ is $\overline{Q}$-singular it suffices to show this for all cyclic modules $\mathbb{Z}/p^n\mathbb{Z}$, $n \in \mathbb{Z}$, p a prime (a set of generators in $\sigma[\overline{Q}]$). However, this is obvious by

$$\mathbb{Z}/p^n\mathbb{Z} \simeq (\mathbb{Z}/p^{n+1}\mathbb{Z})/(p^n\mathbb{Z}/p^{n+1}\mathbb{Z}) \quad \text{and} \quad p^n\mathbb{Z}/p^{n+1}\mathbb{Z} \trianglelefteq \mathbb{Z}/p^{n+1}\mathbb{Z}.$$

This provides a simple proof for the fact that there are no projective objects in $\sigma[\overline{Q}]$ (see [40, 18.12]). □

If $\mathcal{S}_M(N) = 0$, N is called *non-singular in* $\sigma[M]$ or *non-M-singular*. Obviously, N is non-M-singular if and only if, for any $K \in \sigma[M]$ and $0 \neq f : K \to N$, $Ke\,f$ is not essential in K.

10.2 Non-M-singular modules.

Let M be a module with M-injective hull $\widehat{M}$.

(1) A module N in $\sigma[M]$ with $Hom_A(N, \widehat{M}) = 0$ is M-singular.

(2) Assume $\mathcal{S}_M(M) = 0$. Then:

(i) $N \in \sigma[M]$ is M-singular if and only if $Hom_A(N, \widehat{M}) = 0$.

(ii) The class of M-singular modules is closed under extensions.

(iii) For all $N \in \sigma[M]$, $\mathcal{S}_M(N/\mathcal{S}_M(N)) = 0$.

Proof. (1) Consider $N \in \sigma[M]$ which is not M-singular, i.e., $\mathcal{S}_M(N) \neq N$. Then the M-injective hull $\widehat{N}$ is also not M-singular. Since it is M-generated, there is a morphism $f : M \to \widehat{N}$ whose kernel is not essential in M.

The kernel of the restriction $\bar{f} = f|_{Nf^{-1}} : Nf^{-1} \to N$ again is not an essential submodule and there exists a non-zero submodule $K \subset Nf^{-1}$ with $K \cap Ke\,\bar{f} = 0$. Now $\bar{f}|_K : K \to N$ is monic and the inclusion $K \subset \widehat{M}$ can be extended to a non-zero morphism in $Hom_A(N, \widehat{M})$.

(2) (i) $\mathcal{S}_M(M) = 0$ implies $\mathcal{S}_M(\widehat{M}) = 0$ and hence for every M-singular N in $\sigma[M]$, $Hom_A(N, \widehat{M}) = 0$. The other assertion follows from (1).

(ii) Let $0 \to K \to L \to N \to 0$ be exact in $\sigma[M]$ with K and N both M-singular. By (i), $Hom_A(K, \widehat{M}) = Hom_A(N, \widehat{M}) = 0$. Applying the functor $Hom_A(-, \widehat{M})$ to the exact sequence we see $Hom_A(L, \widehat{M}) = 0$ and L is M-singular by (i).

(iii) This is a consequence of (ii). □

Next we collect some basic information on M-singular modules (see [11, 4.2,4.3]).

10.3 Proposition. *Let M be an A-module.*

(1) Any simple A-module is M-singular or M-projective.

(2) Every M-singular module is an essential submodule of an M-generated M-singular module.

(3) Each finitely generated M-singular module belongs to $\sigma[M/L]$, for some $L \trianglelefteq M$.

(4) $\{M/K \mid K \trianglelefteq M\}$ is a generating set for the M-generated M-singular modules.

Proof. (1) Let E be a simple A-module. If $E \notin \sigma[M]$ then E is trivially M-projective. Assume $E \in \sigma[M]$ is not M-singular and consider an exact sequence

$$0 \longrightarrow K \longrightarrow L \longrightarrow E \longrightarrow 0$$

in $\sigma[M]$. By assumption, the maximal submodule $K \subset L$ is not essential and hence is a direct summand in L, i.e., the sequence splits and E is projective in $\sigma[M]$.

(2) Consider $L \in \sigma[M]$ and $K \trianglelefteq L$. The M-injective hull $\widehat{L}$ of L is M-generated and

$$L/K \subset \widehat{L}/K, \quad K \trianglelefteq L \trianglelefteq \widehat{L}.$$

The inclusion map of L/K into its M-injective hull can be extended to $\widehat{L}/K \to \widehat{L/K}$. The image of this map is an essential M-generated M-singular extension of L/K.

(3) A finitely generated M-singular module is of the form N/K, for a finitely generated $N \in \sigma[M]$ and $K \trianglelefteq N$. N is an essential submodule of a finitely M-generated module $\widetilde{N}$, i.e., there exists an epimorphism $g : M^k \to \widetilde{N}$, $k \in I\!N$ (compare (2)), and $U := (N)g^{-1}$ and $V := (K)g^{-1}$ are essential submodules of M^k.

With the canonical inclusions $\varepsilon_i : M \to M^k$ we get that $L := \bigcap_{i \le k} V\varepsilon_i^{-1}$ is an essential submodule of M and L^k lies in the kernel of the composed map

$$U \xrightarrow{g} N \longrightarrow N/K.$$

This implies $N/K \in \sigma[M/L]$.

(4) is an immediate consequence of (3). $\square$

10.4 Proposition. *Let N be an M-singular module and $f \in Hom_A(M,N)$.*

(1) If M is self-projective and $(M)f$ is finitely generated, then $Ke\, f \trianglelefteq M$.

(2) If M is projective in $\sigma[M]$, then $Ke\, f \trianglelefteq M$.

Proof. (1) Under the given conditions we may assume $(M)f = L/K$ with $L \in \sigma[M]$ finitely generated and $K \trianglelefteq L$. Since M is self-projective it is also L-projective, and the diagram with the canonical projection p,

$$\begin{array}{ccccc} & & M & & \\ & & \downarrow_f & & \\ L & \xrightarrow{p} & L/K & \longrightarrow & 0\,, \end{array}$$

can be extended commutatively by some $g : M \to L$. Then $Ke\, f = (K)g^{-1} \trianglelefteq M$.

(2) The arguments in (1) apply without finiteness condition. $\square$

As mentioned before, the class $\mathcal{S}_M$ is not closed under extensions. We extend $\mathcal{S}_M$ to a torsion class in the following way:

10.5 Goldie torsion class.

Let $\mathcal{S}_M^2$ denote the modules $X \in \sigma[M]$, for which there exists an exact sequence $0 \to K \to X \to L \to 0$, with $K, L \in \mathcal{S}_M$. Then:

(1) $\mathcal{S}_M^2$ is a stable torsion class, called the Goldie torsion class *in $\sigma[M]$.*

(2) For any $N \in \sigma[M]$, $\mathcal{S}_M^2(N) = 0$ if and only if $\mathcal{S}_M(N) = 0$, and

$$\mathcal{S}_M^2(N)/\mathcal{S}_M(N) = \mathcal{S}_M(N/\mathcal{S}_M(N)).$$

(3) For any $N \in \sigma[M]$, $Q_{\mathcal{S}_M^2}(N)$ is M-injective.

(4) $Q_{\mathcal{S}_M^2} : \sigma[M] \to \sigma[M]$ is an exact functor.

Proof. (1) It is straightforward to verify that $\mathcal{S}_M^2$ is closed under submodules, factors and direct summands. To show that it is closed under extensions, consider an exact sequence

$$(*) \quad 0 \to X \to Z \to Y \to 0, \text{ where } X,\, Y \in \mathcal{S}_M^2.$$

Any complement C of X in Z is isomorphic to a submodule of Y and hence $C \in \mathcal{S}_M^2$. Then $X \oplus C \in \mathcal{S}_M^2$ and $X \oplus C \trianglelefteq Z$.

Hence without restriction we may assume in $(*)$ that $X \trianglelefteq Z$ and $Y \in \mathcal{S}_M$. From the proof of (2) we have $\mathcal{S}_M(X) \trianglelefteq X$ and hence $Z \in \mathcal{S}_M^2$.

Now it is obvious that $\mathcal{S}_M^2$ is closed under essential extensions in $\sigma[M]$.

(2) The first assertion is clear since every non-zero module in $\mathcal{S}_M^2$ contains a non-zero submodule of $\mathcal{S}_M$. Obviously, $\mathcal{S}_M^2(N)/\mathcal{S}_M(N) \supset \mathcal{S}_M(N/\mathcal{S}_M(N))$. Consider

$$L \subset \mathcal{S}_M^2(N) \quad \text{with} \quad 0 = \mathcal{S}_M(N) \cap L = \mathcal{S}_M(L).$$

Then $\mathcal{S}_M^2(L) = 0$ and hence $L = 0$. Therefore $\mathcal{S}_M(N) \trianglelefteq \mathcal{S}_M^2(N)$ and hence $\mathcal{S}_M^2(N)/\mathcal{S}_M(N) \subset \mathcal{S}_M(N/\mathcal{S}_M(N))$.

(3) follows from 9.13.

(4) Let $0 \to K \xrightarrow{f} L \xrightarrow{g} N \to 0$ be exact in $\sigma[M]$. We know from 9.18 that

$$0 \longrightarrow Q_{\mathcal{S}_M^2}(K) \xrightarrow{Q_{\mathcal{S}_M^2}(f)} Q_{\mathcal{S}_M^2}(L) \xrightarrow{Q_{\mathcal{S}_M^2}(g)} Q_{\mathcal{S}_M^2}(N)$$

is exact. By (3), $Q_{\mathcal{S}_M^2}(K)$ and $Q_{\mathcal{S}_M^2}(L)$ are M-injective. This implies that $Im\, Q_{\mathcal{S}_M^2}(g)$ is M-injective and hence is a direct summand in $Q_{\mathcal{S}_M^2}(N)$. Since $N/\mathcal{S}_M^2(N) \subset Im\, Q_{\mathcal{S}_M^2}(g)$ is an essential submodule in $Q_{\mathcal{S}_M^2}(N)$, we conclude $Im\, Q_{\mathcal{S}_M^2}(g) = Q_{\mathcal{S}_M^2}(N)$.

This completes the proof that $Q_{\mathcal{S}_M^2}(-)$ is exact. $\square$

The following assertion indicates a connection to the quotient ring of A. We consider the Goldie torsion theory in A-*Mod* and put $\mathcal{S}^2 := \mathcal{S}^2_A$.

10.6 Quotient ring of A.

Let M be an A-module with $A \in \sigma[M]$. Then:

(1) $Q_{\mathcal{S}^2}(A)$ is a ring and $Q_{\mathcal{S}^2}(M)$ is a generator in $Q_{\mathcal{S}^2}(A)$-Mod.

(2) $Q_{\mathcal{S}^2}(M)$ is finitely generated and projective as module over $T := End_A(Q_{\mathcal{S}^2}(M))(= End_{Q_{\mathcal{S}^2}(A)}(Q_{\mathcal{S}^2}(M))$.

(3) $Q_{\mathcal{S}^2}(A) \simeq End(Q_{\mathcal{S}^2}(M)_T)$.

Proof. (1) The ring structure of $Q_{\mathcal{S}^2}(A)$ is given in 9.20.

By assumption, $A \subset M^k$, $k \in \mathbb{N}$. Since $Q_{\mathcal{S}^2}(-)$ is (left) exact, we conclude $Q_{\mathcal{S}^2}(A) \subset Q_{\mathcal{S}^2}(M)^k$. By 9.13, $Q_{\mathcal{S}^2}(A)$ is self-injective and hence is a direct summand in $Q_{\mathcal{S}^2}(M)^k$, i.e., it is generated by $Q_{\mathcal{S}^2}(M)$.

(2) and (3) characterize $Q_{\mathcal{S}^2}(M)$ as generator in $Q_{\mathcal{S}^2}(A)$-*Mod*. □

10.7 Corollary. *Let M be a faithful module with $\mathcal{S}_M(M) = 0$. Then the following are equivalent:*

(a) $A \in \sigma[M]$;

(b) $\widehat{M}$ is finitely generated over $End_A(\widehat{M})$.

Proof. $(a) \Rightarrow (b)$ Apply 10.6.

$(b) \Rightarrow (a)$ ${}_A\widehat{M}$ is a faithful A-module and hence $A \in \sigma[\widehat{M}] = \sigma[M]$ (see [40, 15.4]). □

The hereditary torsion theory in $\sigma[M]$, whose torsionfree class is cogenerated by the M-injective hull $\widehat{M}$ of M, is called the *Lambek torsion theory in* $\sigma[M]$. The torsion class is given by

$$\mathcal{T}_M = \{K \in \sigma[M] \mid Hom_A(K, \widehat{M}) = 0\}.$$

Obviously, M is torsionfree in this torsion theory. In fact, $\mathcal{T}_M$ is the largest torsion classe for which M is torsionfree. $Q_{\mathcal{T}_M}(M)$ is just the $(M, \mathcal{T}_M)$-injective hull of M.

A $\mathcal{T}_M$-dense submodule U of $N \in \sigma[M]$ is said to be *(M-) rational in* N and N is called a *rational extension* of U.

10.8 Rational submodules.

For a submodule $U \subset N$ with $N \in \sigma[M]$, the following are equivalent:

(a) U is rational in N;

(b) for any $U \subset V \subset N$, $Hom_A(V/U, M) = 0$.

Proof. $(a) \Rightarrow (b)$ For $0 \neq f \in Hom_A(V/U, M)$, the diagram

$$\begin{array}{ccccc} 0 & \to & V/U & \to & N/U \\ & & \downarrow_f & & \\ & & M & \to & \widehat{M} \end{array}$$

can be extended commutatively by a non-zero morphism $N/U \to \widehat{M}$.

$(b) \Rightarrow (a)$ Consider a non-zero $g \in Hom_A(N/U, \widehat{M})$ and $Mg^{-1} = V/U$ for some $U \subset V \subset N$. Then the restriction $g|_{V/U} \in Hom_A(V/U, M)$ is not zero. □

10.9 Maximal ring of quotients.

For $M = A$ we have the torsion class in A-*Mod*,

$$\mathcal{T} := \mathcal{T}_A = \{K \in A\text{-}Mod \mid Hom(K, \widehat{A}) = 0\}.$$

This yields the Lambek torsion theory in A-*Mod*. The quotient modul $Q_{\mathcal{T}}(A)$ allows a ring structure for which A is a subring (see 9.20). $Q_{\mathcal{T}}(A)$ is called the *maximal (left) quotient ring* of A, denoted by $Q_{\max}(A)$.

For a non-M-singular M (i.e., a $\mathcal{S}^2_M$-torsionfree M), the Lambek torsion theory in $\sigma[M]$ is closely related to the singular torsion theory. We will investigate these modules in the next section.

10.10 Exercises.

(1) Let M be an A-module. For any two classes of modules $\mathtt{C}$ and $\mathtt{D}$ in $\sigma[M]$, denote by $\mathtt{E}_M(\mathtt{D}, \mathtt{C})$ the class of A-modules N for which there is an exact sequence

$$0 \to C \to N \to D \to 0$$

in $\sigma[M]$, where $C \in \mathtt{C}$ and $D \in \mathtt{D}$.

(i) Let $\mathtt{C}$ and $\mathtt{D}$ be subclasses of $\sigma[M]$. Prove: If $\mathtt{C}$ and $\mathtt{D}$ are closed under submodules (factor modules, direct sums) then $\mathtt{E}_M(\mathtt{D}, \mathtt{C})$ is also closed under submodules (resp. factor modules, direct sums).

(ii) Let $\mathtt{C}$ and $\mathtt{D}$ be subclasses of A-Mod which are closed under isomorphisms. Prove: For any left ideal $I \subset A$, $A/I \in \mathtt{E}_A(\mathtt{D}, \mathtt{C})$ if and only if there exists a left ideal $J \supset I$ of A such that $J/I \in \mathtt{C}$ and $A/J \in \mathtt{D}$.

(2) Let M be a generator in A-Mod which is cogenerated by $\widehat{A}$ (the injective hull in A-Mod). Put $S = End_A(M)$ and $T = End_A(\widehat{M})$. Prove that the following are equivalent ([164, Theorem 3.5]):

(a) T is left self-injective and isomorphic to $Q_{\max}(S)$;

(b) ${}_ST$ *is isomorphic to the injective hull of* ${}_SS$*;*
(c) $Hom_A(\widehat{M}/M, \widehat{M}) = 0$.

(3) Let $Q_{\max}(A)$ *be the maximal left quotient ring of* A*. We ask for properties of* $Q_{\max}(A)$ *as right* A*-module. Prove that the following are equivalent ([194]):*
(a) Every finitely generated submodule of ${}_AQ_{\max}(A)$ *is cogenerated by* A*;*
(b) $Q_{\max}(A)$ *is a rational extension of* A_A.

References. Berning [78], Dung-Huynh-Smith-Wisbauer [11], Izawa [164], Masaike [194], Wisbauer [273, 279].

11 Polyform modules

1.Polyform modules. 2.Properties of $End_A(\widehat{M})$. 3.Monoform modules. 4.Maximal quotients of non-singular rings. 5.Quotient rings of $End_A(M)$. 6.Finite dimensional polyform modules. 7.Goldie's Theorem. 8.Polyform subgenerators. 9.Trace and torsion submodules. 10.Properties. 11.Properties of polyform modules. 12.Bimodule properties of polyform modules. 13.Independence over the endomorphism ring. 14.Self-injective polyform modules. 15.Idempotent closure of polyform modules. 16.Exercises.

A module M is called *polyform* if every essential submodule is rational in M. Our next result shows that this is just another name for a non-M-singular module M.

11.1 Polyform modules. Characterization.
For an A-module M with M-injective hull $\widehat{M}$, the following are equivalent:

(a) Every essential submodule is rational in M;

(b) for any submodule $K \subset M$ and $0 \neq f : K \to M$, $Ke\, f$ is not essential in K;

(c) for any $K \in \sigma[M]$ and $0 \neq f : K \to M$, $Ke\, f$ is not essential in K;

(d) M is non-M-singular (i.e., $\mathcal{S}_M(M) = 0$);

(e) $End_A(\widehat{M})$ is regular.

Proof. $(a) \Rightarrow (e)$ Apply 9.16.

$(a) \Rightarrow (d)$ Assume $0 \neq K \subset M$ is an M-singular submodule. By 10.3, K is contained in an M-generated M-singular submodule $\widetilde{K} \subset \widehat{M}$ and, moreover, $\widetilde{K}$ is generated by $\{M/U \mid U \trianglelefteq M\}$. However, $U \trianglelefteq M$ implies $Hom_A(M/U, \widehat{M}) = 0$. Hence $K \subset \widetilde{K} = 0$.

$(c) \Leftrightarrow (d) \Rightarrow (b)$ are obvious.

$(e) \Rightarrow (b)$ Suppose for some $K \subset M$ and $f : K \to M$ that $Ke\, f \trianglelefteq K$. Then f can be extended to an $\bar{f} \in End_A(\widehat{M})$ with $Ke\, \bar{f} \trianglelefteq \widehat{M}$. $End_A(\widehat{M})$ being regular, $Ke\, \bar{f}$ is a direct summand (see 7.6) and hence $\bar{f}$ and f have to be zero.

$(b) \Rightarrow (a)$ Consider $U \trianglelefteq M$ and $f \in Hom_A(V/U, M)$, for any $U \subset V \subset M$. With the projection $p : V \to V/U$ we obtain $pf \in Hom_A(V, M)$ which has an essential kernel. From (b) we conclude that pf and f are zero. Now apply 10.8. $\square$

11.2 Properties of $End_A(\widehat{M})$.
Let M be a polyform A-module and $T = End_A(\widehat{M})$.

(1) T is regular and left self-injectice.

(2) $End_A(M)$ is subring of T.

(3) *Every monomorphism $f \in End_A(M)$ with $Im\, f \trianglelefteq M$ is invertible in T.*

(4) *M has finite uniform dimension if and only if T is left semisimple.*

Proof. Combine 11.1, 9.18 and 9.16. □

A module M is said to be *monoform* if every submodule is rational in M. Notice that any uniform submodule of a polyform module is monoform. As a special case of 11.1 we obtain:

11.3 Monoform modules. Characterization.
For an A-module M with M-injective hull $\widehat{M}$, the following are equivalent:

(a) *Every non-zero submodule is rational in M (M is monoform);*

(b) *for any submodule $K \subset M$ and $0 \neq f : K \to M$, $Ke\, f = 0$;*

(c) *M is uniform and non-M-singular (i.e., $\mathcal{S}_M(M) = 0$);*

(d) *$End_A(\widehat{M})$ is a division ring.*

Let $E(A)$ denote the A-injective hull of ${}_AA$. Applying the preceding results to $M = A$ we have:

11.4 Maximal quotients of non-singular rings.
Let A be a left non-singular ring.

(1) *$Q_{\max}(A) = End_A(E(A))$ is a selfinjective, regular ring.*

(2) *${}_AA$ has finite uniform dimension if and only if $Q_{\max}(A)$ is left semisimple.*

(3) *Assume A is a subring in any left self-injective ring Q and $A \trianglelefteq {}_AQ$. Then $Q \simeq Q_{\max}(A)$.*

(4) *If ${}_AA$ is uniform then $Q_{\max}(A)$ is a division ring.*

Proof. (1),(2) and (4) follow from 11.2 and 11.1.

(3) Since $A \trianglelefteq {}_AQ$, we may assume ${}_AQ \subset Q_{\max}(A)$. Denote by $\cdot$ the multiplication in Q and by $*$ the multiplication in $Q_{\max}(A)$. We show that $p \cdot q = p * q$ for all $p, q \in Q$.

Consider $K = \{s \in A \,|\, sp \in A\} = Ap^{-1} \trianglelefteq A$. For every $k \in K$,

$$k(p \cdot q - p * q) = (kp)q - (kp)q = 0,$$

i.e., $K(p \cdot q - p * q) = 0$. Since $Q_{\max}(A)$ is a non-singular A-module, this implies $p \cdot q = p * q$. So $Q_{\max}(A)$ is a Q-module and $Q \trianglelefteq Q_{\max}(A)$ is a direct summand, i.e., $Q = Q_{\max}(A)$. □

An overring $Q \supset A$ is called a *classical left quotient ring of A* if every element in A, which is not a zero divisor, is invertible in Q and all elements of Q have the form $s^{-1}t$, where $s, t \in A$ and s is not a zero divisor. For details about these rings we refer to Lambek [183] or Stenström [39].

For a polyform module M, we know from 11.4 that $S = End_A(M)$ is subring of the self-injective, regular ring $T = End_A(\widehat{M})$. In general, T need not to be the maximal quotient ring of S and S need not even be non-singular. This is the case under special conditions.

11.5 Quotient rings of $End_A(M)$.

Let M be a polyform A-module with $S = End_A(M)$.

(1) Assume for all non-zero submodules $N \subset M$, $Hom_A(M,N) \neq 0$. Then:
   (i) $T := End_A(\widehat{M})$ is an essential extension of ${}_SS$.
   (ii) S is left non-singular and $T = Q_{\max}(S)$.

(2) Assume for every $K \trianglelefteq M$ there is a monomorphism $g : M \to K$ with $Mg \trianglelefteq K$. Then $End_A(\widehat{M})$ is a classical left quotient ring of S.

Proof. (1)(i) For a non-zero $f \in End_A(\widehat{M})$, $K = Mf^{-1} \cap M$ is an essential submodule of M. From $Hom_A(M,N) \neq 0$ for all non-zero $N \subset M$, we deduce

$$Tr(M,K) = M\,Hom_A(M,K) \trianglelefteq K \trianglelefteq M.$$

This implies $M\,Hom_A(M,K)f \neq 0$ and we can find some $g \in Hom_A(M,K)$ with $0 \neq gf \in S$. Thus $gf \in S \cap Sf \neq 0$.

(ii) Assume ${}_SS$ has a singular submodule $Sb \neq 0$, $b \in S$. Then $K = An_S(b) \trianglelefteq {}_SS \trianglelefteq {}_ST$. This implies that $TK \subset An_T(b)$ are essential left ideals in T, i.e., Tb is a singular submodule in ${}_TT$. However, T is regular and so $b \in \mathcal{S}({}_TT) = 0$, i.e., ${}_SS$ is non-singular. Therefore $Q_{\max}(S)$ is equal to the S-injective hull $\widehat{S}$ of S.

Since ${}_TT$ is self-injective we can now apply 11.4(3).

(3) We show that any $q \in End_A(\widehat{M})$ has the form $q = s^{-1}t$, for $s, t \in S$: Consider $N := Mq^{-1} \cap M \trianglelefteq M$ and choose a monomorphism $s : M \to N$ with $Ms \trianglelefteq M$. Now (the extension of) s is invertible in $End_A(\widehat{M})$ (see 11.2) and $q = s^{-1}t$. □

Under the general assumptions in 11.5, condition (2) holds for modules with finite uniform dimension and semiprime endomorphism ring. We cite this from [11, 5.19]:

11.6 Finite dimensional polyform modules.

Let M be an A-module with $S = End_A(M)$. Assume $Hom_A(M,N) \neq 0$ for all non-zero $N \subset M$. Then the following assertions are equivalent:

(a) M is polyform with finite uniform dimension and S is semiprime;

(b) *M is polyform with finite uniform dimension and, for every $N \trianglelefteq M$, there is a monomorphism $M \to N$;*

(c) *$End_A(\widehat{M})$ is left semisimple and is the classical left quotient ring of S.*

In this case, for every essential submodule L of M, $LEnd_A(\widehat{M}) = \widehat{M}$.

In case $S = End_A(M)$ is commutative, one of the conditions in 11.6(a) is automatically satisfied: From 11.5 we know that S is non-singular and any non-singular commutative ring is semiprime.

Applied to $M = A$, 11.6 yields the classical (see [11, 5.20])

11.7 Goldie's Theorem. *For a ring A, the following are equivalent:*

(a) *A is semiprime and ${}_AA$ is non-singular with finite uniform dimension;*

(b) *A is semiprime and ${}_AA$ has finite uniform dimension and acc on annihilators;*

(c) *A has a classical left quotient ring which is left semisimple.*

11.8 Polyform subgenerators.

Let M be a polyform A-module, $T = End_A(\widehat{M})$ and suppose $A \in \sigma[M]$. Then:

(1) *A is left non-singular and $Q_{\mathcal{S}}(A) = Q_{\max}(A)$.*

(2) *$\widehat{M}$ is a generator in $Q_{\mathcal{S}}(A)$-Mod.*

(3) *$\widehat{M}_T$ is finitely generated and T-projective, and $Q_{\mathcal{S}}(A) \simeq End_T(\widehat{M})$.*

(4) *If M has finite uniform dimension, then T and $Q_{\max}(A)$ are left semisimple.*

Proof. (1),(4) $A \in \sigma[M]$ implies $A \subset M^k$, for some $k \in I\!N$. Hence ${}_AA$ is non-singular. If M has finite uniform dimension then ${}_AA$ has finite uniform dimension.

(2),(3) Apply 10.6. □

For convenience we fix the following notation.

11.9 Trace and torsion submodules.

Let M be an A-module, $\widehat{M}$ its self-injective hull and $T = End_A(\widehat{M})$. Let $K \subset \widehat{M}$ be a submodule and $L \in \sigma[M]$.

By $\mathcal{T}^K(L)$ we denote the *trace of* $\sigma[K]$ *in* L, i.e.

$$\mathcal{T}^K(L) = \sum\{U \subset L \mid U \in \sigma[K]\}.$$

For cyclic submodules $Aa \subset L$, put $\mathcal{T}^a(L) = \mathcal{T}^{Aa}(L)$.

By $\mathcal{T}_K$ we denote the *hereditary torsion class in* $\sigma[M]$ determined by $\widehat{KT}(\subset \widehat{M})$. Then for $L \in \sigma[M]$, $\mathcal{T}_K(L)$ denotes the corresponding *torsion submodule of* L, i.e.

$$\mathcal{T}_K(L) = \sum\{U \subset L \mid Hom_A(U, \widehat{KT}) = 0\}.$$

In particular, for cyclic modules $K = Aa$, we put $\mathcal{T}_a(L) = \mathcal{T}_{Aa}(L)$.

We list some properties related to these notions.

11.10 Properties. *Let M be an A-module, $K \subset M$ and $T = End_A(\widehat{M})$.*

(1) $\mathcal{T}^K(\widehat{M}) = KT$ is a K-injective module.

(2) $\mathcal{T}^K(M) = \mathcal{T}^K(\widehat{M}) \cap M$.

(3) for any submodule $L \subset \widehat{M}$, $\mathcal{T}^K(L) \cap \mathcal{T}_K(L) = 0$.

(4) For submodules $L, N \subset M$ with $K = L + N$,

$$\mathcal{T}^K(\widehat{M}) = \mathcal{T}^L(\widehat{M}) + \mathcal{T}^N(\widehat{M}) \text{ and } \mathcal{T}_K(M) = \mathcal{T}_L(M) \cap \mathcal{T}_N(M).$$

(5) If K is generated by $a_1, \ldots, a_n \in K$, $\mathcal{T}_K(M) = \bigcap_{i=1}^n \mathcal{T}_{a_i}(M)$.

Proof. (1) KT is injective in $\sigma[K]$ (see [40, 16.8]).

(2) $\mathcal{T}_K$ is a hereditary torsion class.

(3) For $X \subset \mathcal{T}^K(L) \cap \mathcal{T}_K(L)$, $X \subset KT$ and $Hom_A(X, \widehat{KT}) = 0$. This implies $X = 0$.

(4) For $K = L + N$, $KT = LT + NT$. Hence without restriction assume $K = KT$, $L = LT$ and $N = NT$. Clearly $\mathcal{T}_K(M) \subset \mathcal{T}_L(M) \cap \mathcal{T}_N(M)$. Consider an additive complement $N_o \subset N$ of L in K. Then $L \oplus N_o$ is an essential submodule of K and

$$\widehat{K} = \hat{L} \oplus \widehat{N_o} \subset \hat{L} \oplus \widehat{N}.$$

From this we conclude $\mathcal{T}_K(M) \supset \mathcal{T}_L(M) \cap \mathcal{T}_N(M)$.

(5) This is derived from (4) by induction. □

11.11 Properties of polyform modules.

Let M be a polyform A-module, $\widehat{M}$ its M-injective hull, and $T = End_A(\widehat{M})$. Then for any submodule $K \subset M$:

(1) $\mathcal{T}_K(\widehat{M})$ is M-injective.

(2) $\mathcal{T}_{\mathcal{T}_K(\widehat{M})}(\widehat{M}) = \widehat{KT}$.

(3) $\widehat{M} = \mathcal{T}_K(\widehat{M}) \oplus \widehat{KT}$.

(4) $\mathcal{T}_K(M) + \mathcal{T}^K(M) \trianglelefteq M$.

(5) for any $m \in \widehat{M}$, $An_T(m)$ is generated by an idempotent in T and $\widehat{M}$ is a non-singular right T-module.

Proof. Put $L = \mathcal{T}_K(\widehat{M})$.

(1) Since $\widehat{M}$ is polyform, $Hom_A(\hat{L}, \widehat{KT}) = 0$ and hence $\hat{L} \subset L$, i.e., $\hat{L} = L$ is M-injective.

(2) Consider $g \in Hom_A(K, L)$. Since $Ke\,g$ is not essential in K, there exists $0 \neq X \subset K$ with $X \cap Ke\,g = 0$ and $X \simeq (X)g \subset L$, a contradiction. Hence $K \subset \mathcal{T}_L(\widehat{M})$ and $KT \subset \mathcal{T}_L(\widehat{M})$ since $\mathcal{T}_L(\widehat{M})$ is fully invariant.

Now we show that $KT \trianglelefteq \mathcal{T}_L(\widehat{M})$. Assume there is a non-zero submodule $U \subset \mathcal{T}_L(\widehat{M})$ with $U \cap KT = 0$. If $Hom_A(U, \widehat{KT}) = 0$, then $U \subset L$ and
$U \subset \mathcal{T}^L(\widehat{M}) \cap \mathcal{T}_L(\widehat{M}) = 0$, a contradiction.

Hence there is a non-zero $g : U \to \widehat{KT}$. Since $Ke\,g$ is not essential in U, there exists a non-zero submodule $V \subset U$ with $V \cap Ke\,g = 0$. Because of $KT \trianglelefteq \widehat{KT}$, we may assume $V \simeq (V)g \subset KT$. Now for some $t \in T$, we have $V = (V)gt \subset U \cap KT = 0$, a contradiction.

(3) By (1), L and $\mathcal{T}_L(\widehat{M})$ are M-injective. Since $L \cap \mathcal{T}_L(\widehat{M}) = 0$ by definiton, $\widehat{M} = L \oplus \mathcal{T}_L(\widehat{M}) \oplus W$ for some $W \subset \widehat{M}$.

Assume there is a non-zero $h \in Hom_A(W, L)$ and $Q \cap Ke\,h = 0$, for some non-zero $Q \subset W$. Then $Q \simeq (Q)h \subset L$ and there is some $t \in T$ with $Q = (Q)ht \subset W \cap L = 0$. This implies $Hom_A(W, L) = 0$ and $W \subset \mathcal{T}_L(\widehat{M})$.

Hence $W = 0$ and by (2), $\widehat{M} = L \oplus \mathcal{T}_L(\widehat{M}) = L \oplus \widehat{KT}$.

(4) is an immediate consequence of (3).

(5) For $m \in \widehat{M}$ consider $t \in T$ with $(m)t = 0$. Then $(Am)t = 0$ and - $\widehat{M}$ being polyform - $(\widehat{Am})t = 0$. We have $\widehat{M} = \widehat{Am} \oplus U$ for some A-submodule $U \subset \widehat{M}$. For the related projection (idempotent) $g : \widehat{M} \to \widehat{Am}$, $\widehat{M}gt = 0$. This means $gt = 0$ and $t = (1-g)t \in (1-g)T$. Therefore $An_T(m) = (1-g)T$ implying that $\widehat{M}$ is a non-singular T-module. □

As already shown above, the condition on an A-module to be polyform has a strong influence on the structure of its fully invariant submodules. We collect information about this in our next lemma:

11.12 Bimodule properties of polyform modules.

Let M be a polyform A-module, $\widehat{M}$ its M-injective hull and $T = End_A(\widehat{M})$. Denote by C the centre of T (i.e., the endomorphism ring of $\widehat{M}$ as an (A,T)-bimodule). Then:

1. *Every essential (A,T)-submodule of $\widehat{M}$ is essential as an A-submodule.*
2. *$\widehat{M}$ is self-injective and polyform as an (A,T)-bimodule.*
   C is a regular self-injective ring.
3. *For every submodule (subset) $K \subset \widehat{M}$, there exists an idempotent $\varepsilon(K) \in C$, such that $An_C(K) = (1-\varepsilon(K))C$.*
4. *If $K \trianglelefteq L \subset \widehat{M}$, then $\varepsilon(K) = \varepsilon(L)$.*
5. *Every finitely generated C-submodule of $\widehat{M}$ is C-injective.*
6. *If $\widehat{M}$ is a finitely generated (A,T)-module, $\widehat{M}$ is a generator in C-Mod.*

Proof. (1) Let $N \subset \widehat{M}$ be an essential (A,T)-submodule. Then $N \cap \mathcal{T}_N(\widehat{M}) = 0$ implies $\mathcal{T}_N(\widehat{M}) = 0$ and $\widehat{M} = \widehat{NT} = \widehat{N}$ by 11.11.

So $N \trianglelefteq \widehat{M}$ as an A-submodule.

(2) Again let $N \subset \widehat{M}$ be an essential (A,T)-submodule and $h: N \to \widehat{M}$ an (A,T)-morphism. Since $\widehat{M}$ is a self-injective A-module, there is an $f \in T$ which extends h from N to $\widehat{M}$. For any $t \in T$ and $n \in N$,

$$(nt)f - (n)ft = (nt)h - (n)ht = 0.$$

Hence $N \subset Ke(tf - ft)$. By (1), $N \trianglelefteq_A \widehat{M}$, and since $\widehat{M}$ is polyform, $tf - ft = 0$, implying that f is an (A,T)-morphism and $\widehat{M}$ is a self-injective (A,T)-module.

The endomorphism ring of the self-injective (A,T)-module $\widehat{M}$ is the centre of the regular ring T and hence is also regular. So $\widehat{M}$ is a polyform (A,T)-module by 11.1. This in turn implies that C is self-injective.

(3) By 11.11, there is a bimodule decomposition $\widehat{M} = \mathcal{T}_K(\widehat{M}) \oplus \widehat{KT}$. Then the projection $\varepsilon(K): \widehat{M} \to \widehat{KT}$ is an idempotent in C and

$$An_C(K) = An_C(\widehat{KT}) = (1 - \varepsilon(K))C.$$

(4) This property is obvious since $\widehat{M}$ is polyform.

(5) As shown in 11.11, every cyclic C-submodule of $\widehat{M}$ is isomorphic to a direct summand of C and hence is C-injective. Since $\widehat{M}$ is a non-singular C-module, any finite sum of C-injective submodules is again C-injective.

(6) Let $\widehat{M}$ be generated as (A,T)-module by $m_1, \ldots, m_k$. Then the map

$$C \to \widehat{M}^k, \quad c \mapsto (m_1, \ldots, m_k)c,$$

is a monomorphism. Since C is injective, it is a direct summand of $\widehat{M}^k$ and so $\widehat{M}$ is a generator in *C-Mod*. □

For later use we state some linear dependence properties of elements in $\widehat{M}$ with respect to $End_A(\widehat{M})$.

11.13 Independence over the endomorphism ring.

Let M be an A-module, $T = End_A(\widehat{M})$ and $m_1, \ldots, m_n \in \widehat{M}$.

(1) Assume $m_1 \notin \sum_{i=2}^n m_i T$. Then there exists $a \in A$ such that $am_1 \neq 0$ and $am_i = 0$ for $i = 2, \ldots, n$.

(2) Assume and $m_1 T \cap \sum_{i=2}^n m_i T = 0$. Then $An_A(m_2, \ldots, m_n)m_1 \trianglelefteq Am_1$.

(3) If M is polyform and $An_A(m_2, \ldots, m_n)m_1 \trianglelefteq Am_1$, then $m_1 T \cap \sum_{i=2}^n m_i T = 0$.

Proof. Put $U = An_A(m_2, \ldots, m_n)$.

(1) This follows from the proof of (2).

(2) Assume there exists a non-zero submodule $V \subset Am_1$ satisfying $V \cap Um_1 = 0$. Consider the canonical projection $\alpha : V \oplus Um_1 \to V$. M being self-injective, α extends to an endomorphism t of M. From $Am_1t \supset (V + Um_1)t = V \neq 0$ we conclude $m_1t \neq 0$.

We also have $U(m_1t) = (Um_1)t = (Um_1)\alpha = 0$. By [40, 28.1,(4)] (or [169, Corollary 2.2]), this implies

$$m_1t \in \{m \in \widehat{M} \mid Um = 0\} = An_{\widehat{M}} An_A(\sum_{i=2}^{n} m_iT) = \sum_{i=2}^{n} m_iT,$$

a contradiction.

(3) Assume $0 \neq m_1t \in \sum_{i=2}^{n} m_iT$ for some $t \in T$. Then $Um_1t \subset U\sum_{i=2}^{n} m_iT = 0$. Since M is polyform and $Um_1 \trianglelefteq Am_1$ we have $Am_1t = 0$, a contradiction. □

Recall that for $X \subset M$ and $b \in M$, we put $(X : b)_A = \{r \in A \,|\, rb \in X\}$.

11.14 Self-injective polyform modules.

Let M be a self-injective polyform A-module, $T = End_A(\widehat{M})$ and $\Lambda = A \otimes_{\mathbb{Z}} T^o$.

(1) For any submodule $N \subset M$ and $m \in M$, $(\mathcal{T}_N(M) : m)_A = An_A(m\varepsilon(N))$.

(2) For $m_1, \ldots, m_n, m \in M$ and $U = An_A(m_2, \ldots, m_n)$ the following are equivalent:

(a) There exists $h \in \Lambda$ with $hm_1 = m\,\varepsilon(Um_1)$ and $hm_i = 0$ for $i = 2, 3, \ldots, n$;

(b) there exist $r_1, \ldots, r_k \in A$ such that for any $s_1, \ldots, s_n \in A$, the relations

$$\sum_{l=1}^{n} s_lr_jm_l = 0 \ \text{ for } j = 1, \ldots, k, \quad \text{ imply } \ s_1m \in \mathcal{T}_{Um_1}(M).$$

Proof. (1) By definition of the idempotent $\varepsilon(N)$ (see 11.12), $M\varepsilon(N) = \widehat{NT}$ and $M(1 - \varepsilon(N)) = \mathcal{T}_N(M)$. Hence for $r \in A$, $rm \in \mathcal{T}_N(M)$ if and only if $(rm)\varepsilon(N) = 0$. Now our assertion follows from $(rm)\varepsilon(N) = r(m\,\varepsilon(N))$.

(2) $(a) \Rightarrow (b)$ Put $e = \varepsilon(Um_1)$. By (1), $(\mathcal{T}_{Um_1}(M) : m)_A = An_A(me)$.

Choose an element $h = \sum_{j=1}^{k} r_j \otimes t_j$ in Λ with

$$hm_1 = me \ \text{ and } hm_i = 0 \ \text{ for all } i = 2, \ldots, n.$$

Assume for $s_1, \ldots, s_n \in A$, $\sum_{l=1}^{n} s_lr_jm_l = 0$ for $j = 1, \ldots, k$. Then

$$s_1me = s_1hm_1 = \sum_{l=1}^{n} s_lhm_l = \sum_{l=1}^{n} s_l \sum_{j=1}^{k} r_jm_lt_j = \sum_{j=1}^{k}(\sum_{l=1}^{n} s_lr_jm_l)t_j = 0.$$

Hence $s_1 \in An_A(me) = (\mathcal{T}_{Um_1}(M) : m)_A$.

$(b) \Rightarrow (a)$ Put $N = \sum_{i=1}^{n} A(r_1m_i, \ldots, r_km_i) \subset M^k$ and consider the assignement

$$\psi : N \to M, \quad \sum_{i=1}^{n} s_i(r_1m_i, \ldots, r_km_i) \mapsto s_1me.$$

To show that ψ is well-defined assume $\sum_{i=1}^n s_i(r_1 m_i, \ldots, r_k m_i) = 0$.

Then $\sum_{i=1}^n s_i r_j m_i = 0$, for all $j = 1, \ldots, k$.

By assumption, $s_1 \in (\mathcal{T}_{Um_1}(M) : m)_A = An_A(me)$. Hence $s_1 me = 0$ proving that ψ is a well-defined morphism.

M being M-injective, ψ can be extended to a morphism $M^k \to M$, also denoted by ψ. Since $Hom_A(M^k, M) = T^k$ there exist $t_1, \ldots, t_k \in T$ such that

$$(x_1, \ldots, x_k)\psi = \sum_{i=1}^k x_i t_i\,, \quad \text{for all } (x_1, \ldots, x_k) \in M^k.$$

Put $h = \sum_{j=1}^k r_j \otimes t_j$. Then

$$hm_1 = \textstyle\sum_{j=1}^k r_j m_1 t_j = (r_1 m_1, r_2 m_1, \ldots, r_k m_1)\psi = me, \text{ and}$$
$$hm_l = \textstyle\sum_{j=1}^k r_j m_l t_j = (r_1 m_l, r_2 m_l, \ldots, r_k m_l)\psi = 0, \quad \text{for } l = 2, \ldots, n.$$

□

There is an extension of a module M contained in the M-injective hull $\widehat{M}$ which turns out to be of some interest.

Definition. Let M be an A-module, $T = End_A(\widehat{M})$ and B the Boolean ring of all central idempotents of T. Then we call $\widetilde{M} = MB$ the *idempotent closure of M.*

This notion is closely related to the *π-injective hull* of M defined in Goel-Jain [143], which can be written as MU, with U the subring generated by all idempotents in T. Hence if all idempotents in T are central, $\widetilde{M}$ is just the π-injective hull of M.

11.15 Idempotent closure of polyform modules.

We use the above notation. Let M be an A-module with idempotent closure $\widetilde{M}$. Then for every $a \in \widetilde{M}$, there exist $m_1, \ldots, m_k \in M$ and pairwise orthogonal $c_1, \ldots, c_k \in B$ such that $a = \sum_{i=1}^k m_i c_i$.

If M is polyform, there exist pairwise orthogonal $e_1, \ldots, e_k \in B$ such that

(1) $a = \sum_{i=1}^k m_i e_i$;

(2) $e_i = \varepsilon(m_i) e_i$, *for* $i = 1, \ldots, k$;

(3) $\varepsilon(a) = \sum_{i=1}^k e_i$.

Proof. Write $a = \sum_{j=1}^r u_j b_j$, with $u_1, \ldots, u_r \in M$ and $b_1, \ldots, b_r \in B$.

The Boolean subring of B generated by $b_1, \ldots, b_r$ is finite and hence isomorphic to $(\mathbb{Z}_2)^k$ for some $k \in \mathbb{N}$. So it contains a subset of pairwise orthogonal idempotents $c_1, \ldots, c_k$ such that, for all $j \leq r$, $b_j = \sum_{i \in S(j)} c_i$ with $S(j) \subset \{1, \ldots k\}$. Hence

$$a = \sum_{i=1}^k m_i c_i\,, \quad \text{where } m_i = \sum \{u_l \,|\, i \in S(l)\}.$$

Assume M is polyform. Put $e_i = \varepsilon(m_i)c_i\varepsilon(a)$. Since $a\varepsilon(a) = a$ and $m_i\varepsilon(m_i) = m_i$,

$$\sum_{i=1}^{k} m_i e_i = \sum_{i=1}^{k} m_i\varepsilon(m_i)c_i\varepsilon(a) = \sum_{i=1}^{k} m_i c_i\varepsilon(a) = (\sum_{i=1}^{k} m_i c_i)\varepsilon(a) = a.$$

(2) follows from the definition of the idempotents e_i above.

(3) Clearly $\varepsilon(a)e_i = e_i$, for all $i = 1, \ldots, k$, and hence $\varepsilon(a)\sum_{i=1}^{k} e_i = \sum_{i=1}^{k} e_i$. Since $ae_i = m_i e_i$, $a = \sum_{i=1}^{k} m_i e_i = a(\sum_{i=1}^{k} e_i)$. Therefore (see 11.12)

$$id - \sum_{i=1}^{k} e_i \in An_T(a) = (id - \varepsilon(a))T.$$

So $\varepsilon(a)\sum_{i=1}^{k} e_i = \varepsilon(a)$ and $\varepsilon(a) = \sum_{i=1}^{k} e_i$. □

11.16 Exercises.

(1) Let $M = \bigoplus_\Lambda M_\lambda$ be a direct sum of A-modules. Show that the following are equivalent ([284, 3.3]):

(a) M is polyform;

(b) for every $\lambda, \mu \in \Lambda$ and $U \subset M_\lambda$, the kernel of any $f : U \to M_\mu$ is not essential in U.

(2) Show that the following are equivalent for an A-module M:
(a) $Hom_A(M/Soc\,M, \widehat{M}) = 0$;
(b) M is polyform and $Soc\,M \trianglelefteq M$;
(c) the canonical assignement $End_A(Soc\,M) \to End_A(\widehat{M})$ is an isomorphism.

(3) Strongly regular endomorphism ring ([226]).
Prove that for an A-module M and $T = End_A(\widehat{M})$ the following are equivalent:
(a) T is strongly regular;
(b) M is polyform and T is reduced;
(c) every direct summand in $\widehat{M}$ is fully invariant.

(4) Let M be a self-injective polyform A-module and assume every submodule of M contains a uniform submodule. Prove that $End_A(M)$ is a product of full linear rings (endomorphism rings of vector spaces, [200, Theorem 5.12]).

References. Beidar-Wisbauer [75, 76, 77], Dung-Huynh-Smith-Wisbauer [11], Lambek [183], Leu [187], Miyashita [200], Renault [226], Smith [245], Stenström [39], Wisbauer [273, 279], Zelmanowitz [284, 285].

12 Closure operations on modules in $\sigma[M]$

1.Singular closure in A-Mod. 2.Essential closure in non-M-singular modules. 3.Correspondence of closed submodules. 4.Correspondence of singular closed submodules. 5.Closed submodules in self-injective modules. 6.Lemma. 7.Relations with the injective hull. 8.Lemma. 9.Lemma. 10.Correspondences for closed submodules of $M^{(\Lambda)}$. 11.Closed left submodules of $A^{(\Lambda)}$. 12.Corollary. 13.Exercises.

Let M denote any left A-module over the associative ring A with unit.

Definitions. Let $K \subset N$ be modules in $\sigma[M]$. A maximal essential extension of K in N will be called an *essential closure* of K in N. K is said to be *closed in* N if it has no proper essential extension in N.

We define the *M-singular closure* $[K]_N$ of K in N by

$$[K]_N/K = \mathcal{S}_M(N/K).$$

K is said to be *M-singular closed* in N if $K = [K]_N$, i.e., if N/K is non-M-singular.

Notice that forming $[K]_N$ depends on the category $\sigma[M]$ we are working in. Usually it should be clear from the context which closure is meant.

In particular, for $M = A$ the notion A-singular defines precisely those A-modules X for which the annihilator of each element is an essential left ideal in A. Hence we have the following characterization of

12.1 Singular closure in *A-Mod*.

Let $K \subset N$ be left A-modules. Then in A-Mod the singular closure of K in N is

$$[K]_N = \{x \in N \mid Ix \subset K \ \textit{for some essential left ideal } I \subset A\}.$$

It should be observed that, in general, forming the singular closure need not be an idempotent operation on the submodules of N. However, it will be idempotent in the cases we are interested in, for example, if M is polyform or in the following situation:

12.2 Essential closure in non-M-singular modules.

Let $K \subset N$ be non-M-singular modules in $\sigma[M]$. Then for every essential closure $\overline{K}$ in N,

$$\overline{K}/K = \mathcal{S}_M(N/K) = [K]_N/K.$$

This implies that K has a unique essential closure in N.

Proof. Clearly $\overline{K} \subset [K]_N$. Since $[K]_N$ is non-M-singular, any map from $[K]_N$ to an M-singular module has essential kernel. Hence $K \trianglelefteq [K]_N$ and $\overline{K} = [K]_N$. $\square$

The preceding observation implies a close relationship between closed submodules of essential extensions of non-M-singular modules.

Denote by $\mathcal{L}(X)$ the lattice of submodules of any module X.

12.3 Correspondence of closed submodules.
Let $K \trianglelefteq N$ be non-M-singular modules in $\sigma[M]$. Then the mappings

$$\mathcal{L}(K) \to \mathcal{L}(N), \quad U \mapsto [U]_N,$$
$$\mathcal{L}(N) \to \mathcal{L}(K), \quad V \mapsto V \cap K,$$

provide a bijection between closed submodules of K and closed submodules of N.

Proof. For a closed submodule $U \subset K$, $U = K \cap [U]_N$. If $V \subset N$ is a closed submodule, then $V \cap K \trianglelefteq V$ and hence $[V \cap K]_N = V$. Therefore the composition of the two maps yields the identity on closed submodules. □

For polyform modules M, we can relate singular closed submodules of any module $N \in \sigma[M]$ to closed submodules of the non-M-singular module $N/\mathcal{S}_M(N)$:

12.4 Correspondence of singular closed submodules.
Let M be a polyform A-module and $N \in \sigma[M]$. Then the canonical projection

$$p : N \to N/\mathcal{S}_M(N)$$

provides a bijection between singular closed submodules of N and (singular-) closed submodules of $N/\mathcal{S}_M(N)$.

Proof. Since $N/\mathcal{S}_M(N)$ is non-M-singular, its closed submodules coincide with the M-singular closed submodules (by 12.2).

Let $U \subset N$ be singular closed and assume $\mathcal{S}_M(N) \not\subset U$. Then $(U + \mathcal{S}_M(N))/U$ is a non-zero M-singular submodule of N/U, a contradiction.

Now the bijection suggested follows from the canonical isomorphism

$$N/U \simeq (N/\mathcal{S}_M(N))/(U/\mathcal{S}_M(N)).$$

□

Because of the above correspondence we will concentrate our investigation on non-M-singular modules.

The correspondence described in 12.3 has remarkable consequences. They are based on the well-known fact (e.g., [11, Proposition 7.2]):

12.5 Closed submodules in self-injective modules.
In a self-injective module, every closed submodule is a direct summand.

12.6 Lemma. *Let $K \subset N$ be non-M-singular modules in $\sigma[M]$. Let $U \subset K$ be a closed submodule and $\overline{U} \subset N$ its essential closure in N.*

(1) The canonical map $K/U \to N/\overline{U}$ is an essential monomorphism.

(2) If N is M-injective then $N/\overline{U}$ is an M-injective hull on K/U.

(3) If K/U is M-injective then $K/U \simeq N/\overline{U}$.

Proof. (1) The composition of the canonical homomorphisms,

$$K/U \to N/U \to N/\overline{U}\,,$$

is a monomorphism since $K \cap \overline{U} = U$. Assume its image is not essential in $N/\overline{U}$. Then there exists a submodule $\overline{U} \subset V \subset N$ such that

$$((K+\overline{U})/\overline{U}) \cap (V/\overline{U}) = (K \cap V + \overline{U})/\overline{U} = 0,$$

and hence $K \cap V \subset K \cap \overline{U} = U$, implying $V \subset \overline{U}$.

(2) If N is M-injective the closed submodule $\overline{U} \subset N$ is a direct summand (by 12.5) and hence $N/\overline{U}$ is M-injective. Now the assertion is clear by (1).

(3) is obvious by (1). □

For the module M itself we obtain the following properties:

12.7 Relations with the injective hull.
Let M be a polyform module with M-injective hull $\widehat{M}$ and $T := End_A(\widehat{M})$.

(1) There exist bijections between

(i) the closed submodules of M,

(ii) the direct summands of $\widehat{M}$,

(iii) the left ideals which are direct summands of T.

(2) (i) For any essential left ideal $I \trianglelefteq T$, $\widehat{M}I \trianglelefteq \widehat{M}$.
(ii) For every $V \subset \widehat{M}$ with $Tr(\widehat{M}, V) \trianglelefteq \widehat{M}$, $Hom_A(\widehat{M}, V) \trianglelefteq T$.

Proof. (1) This follows from 12.3 and 12.5 and the fact that direct summands in $\widehat{M}$ correspond to left ideals which are direct summands in T.

(2) Assume that $\widehat{M}I$ is not essential in $\widehat{M}$. Then the essential closure $\overline{\widehat{M}I}$ of $\widehat{M}I$ is a proper direct summand in $\widehat{M}$ and $I \subset Hom_A(\widehat{M}, \widehat{M}I) \subset Hom_A(\widehat{M}, \overline{\widehat{M}I})$, which is a proper direct summand in T. This is a contradiction to $I \trianglelefteq T$.

Now let $V \subset \widehat{M}$ such that $Tr(\widehat{M}, V) \trianglelefteq \widehat{M}$. If $Hom_A(\widehat{M}, V)$ is not essential in T, then it is contained in a proper direct summand Te, $e^2 = e \in T$. Then

$$Tr(\widehat{M}, V) = \widehat{M}\,Hom_A(\widehat{M}, V) \subset \widehat{M}e,$$

contradicting $Tr(\widehat{M}, V) \trianglelefteq \widehat{M}$. □

For an infinite index set Λ, the direct sum $\widehat{M}^{(\Lambda)}$ need not be M-injective. Nevertheless we have nice characterizations of its closed submodules.

First we make some technical observations.

12.8 Lemma. *Let M be a polyform A-module with M-injective hull $\widehat{M}$. Then every closed submodule of $\widehat{M}^{(\Lambda)}$ is $\widehat{M}$-generated.*

Proof. Let $U \subset \widehat{M}^{(\Lambda)}$ be a closed submodule, and denote by $\{U_\gamma\}_\Gamma$ the (directed) set of finitely generated submodules of U. Then each U_γ is contained in some finite partial sum of $\widehat{M}^{(\Lambda)}$, which is M-injective and contains the unique essential closure $\overline{U}_\gamma$ (of U_γ in $\widehat{M}^{(\Lambda)}$) as a direct summand. Clearly $\overline{U}_\gamma \subset U$ and $\varinjlim \overline{U}_\gamma = U$. This implies that U is $\widehat{M}$-generated. $\square$

12.9 Lemma. *Let M be a finitely generated polyform A-module with M-injective hull $\widehat{M}$ and $T := End_A(\widehat{M})$. Then:*

(1) For every $f : \widehat{M} \to \widehat{M}^{(\Lambda)}$, $\widehat{M}f$ is contained in a finite partial sum of $\widehat{M}^{(\Lambda)}$, and hence we may identify

$$T^{(\Lambda)} = Hom_A(\widehat{M}, \widehat{M}^{(\Lambda)}).$$

(2) for any $f_1, \ldots, f_n \in Hom_A(\widehat{M}, \widehat{M}^{(\Lambda)})$, $\sum_{i=1}^n \widehat{M}f_i$ is a direct summand in $\widehat{M}^{(\Lambda)}$ and the exact sequence determined by the f_i splits:

$$\widehat{M}^n \to \sum_{i=1}^n \widehat{M}f_i \to 0;$$

(3) for every left T-submodule $X \subset T^{(\Lambda)}$, $Hom_A(\widehat{M}, \widehat{M}X) = X$.

Proof. (1) Since M is finitely generated, for every $f : \widehat{M} \to \widehat{M}^{(\Lambda)}$, we have the following diagram, where $k \in \mathbb{N}$,

$$\begin{array}{ccccc} 0 & \to & M & \to & \widehat{M} \\ & & \downarrow f\,|_M & & \downarrow f \\ & & \widehat{M}^k & \to & \widehat{M}^{(\Lambda)}. \end{array}$$

Since $\widehat{M}^k$ is M-injective we can extend $f\,|_M$ to some $g : \widehat{M} \to \widehat{M}^k$. However, since M is polyform there is a unique extension of $f\,|_M$ from M to $\widehat{M}$. This means $f = g$ and $\widehat{M}f \subset \widehat{M}^k$.

As a consequence, for every $f \in Hom_A(\widehat{M}, \widehat{M}^{(\Lambda)})$ we have in fact, for some $k \in \mathbb{N}$,

$$f \in Hom_A(\widehat{M}, \widehat{M}^k) = T^k,$$

which implies our assertion.

(2) By (1), $\sum_{i=1}^{n} \widehat{M} f_i$ is contained in some finite partial sum $\widehat{M}^k$, $k \in I\!N$. Then we have

$$\widehat{M}^n \xrightarrow{f} \sum_{i=1}^{n} \widehat{M} f_i \subset \widehat{M}^k,$$

where f is determined by the f_i.

Now f may be considered as an endomorphism of $\widehat{M}^{n+k}$. Since $End_A(\widehat{M}^{n+k})$ is regular the image and the kernel of f are direct summands (see 7.6) proving our assertion.

(3) Let $g \in Hom_A(\widehat{M}, \widehat{M}X)$. Then $Mg \subset \sum_{i=1}^{k} \widehat{M} x_i$, for some $x_i \in X$. By (2), $\sum_{i=1}^{k} \widehat{M} x_i$ is M-injective and $g|_M$ can be uniquely extended from M to $\widehat{M}$. Hence we may assume $\widehat{M}g \subset \sum_{i=1}^{k} \widehat{M} x_i$. We describe the situation in the diagram

$$\begin{array}{ccccc} & & \widehat{M} & & \\ & & \downarrow g & & \\ \widehat{M}^k & \to & \sum_{i=1}^{k} \widehat{M} x_i & \to & 0\,. \end{array}$$

By (2), the lower row splits. Hence we have a map $\widehat{M} \to \widehat{M}^k$ which yields a commutative diagram and is determined by some $t_1, \dots, t_k \in T$ satisfying $g = \sum_{i \le k} t_i x_i \in X$.

This proves $Hom_A(\widehat{M}, \widehat{M}X) = X$. □

With these preparations we are able to prove correspondences for closed submodules of infinite direct sums.

12.10 Correspondences for closed submodules of $M^{(\Lambda)}$.

Let M be a finitely generated polyform A-module with M-injective hull $\widehat{M}$. We denote $T := End_A(\widehat{M})$ and identify $T^{(\Lambda)} = Hom_A(\widehat{M}, \widehat{M}^{(\Lambda)})$ (by 12.9).

There are bijective correspondences between

(i) the closed submodules of $M^{(\Lambda)}$,

(ii) the closed submodules of $\widehat{M}^{(\Lambda)}$,

(iii) the closed left T-submodules of $T^{(\Lambda)}$.

For closed submodules $V \subset M^{(\Lambda)}$ and $X \subset T^{(\Lambda)}$, these are given by

$$\begin{array}{ccccc} V & \to & [V]_{\widehat{M}^{(\Lambda)}} & \to & Hom_A(\widehat{M}, [V]_{\widehat{M}^{(\Lambda)}}), \\ M^{(\Lambda)} \cap \widehat{M}X & \leftarrow & \widehat{M}X & \leftarrow & X\,. \end{array}$$

Proof. The correspondence between (i) and (ii) is just a special case of the situation described in 12.3.

Let $U \subset \widehat{M}^{(\Lambda)}$ be a closed submodule. We want to show that $Hom_A(\widehat{M}, U)$ is a closed T-submodule in $T^{(\Lambda)}$. Since T is left non-singular it is to show that any $f \in T^{(\Lambda)}$, which satisfies $If \subset Hom_A(\widehat{M}, U)$ for an essential left ideal $I \subset T$, already belongs to $Hom_A(\widehat{M}, U)$. In fact, this condition implies $\widehat{M}If \subset U$, where $\widehat{M}I \trianglelefteq \widehat{M}$ (by 12.7). Assume that $\widehat{M}f \not\subset U$. Then the map $\widehat{M} \xrightarrow{f} \widehat{M}^{(\Lambda)} \to \widehat{M}^{(\Lambda)}/U$ is non-zero and has essential kernel. This is not possible since $\widehat{M}^{(\Lambda)}/U$ is non-M-singular and we conclude that $f \in Hom_A(\widehat{M}, U)$.

Moreover, $\widehat{M}Hom_A(\widehat{M}, U) = U$ (by 12.8).

Now let $X \subset T^{(\Lambda)}$ be a closed T-submodule and put $U := [\widehat{M}X]_{\widehat{M}^{(\Lambda)}}$. Denote by $\{X_\gamma\}_\Gamma$ the family of finitely generated submodules of X. Assume there is an

$$f \in Hom_A(\widehat{M}, U) \text{ such that } \widehat{M}f \not\subset \widehat{M}X.$$

Then $V := (\widehat{M}X)f^{-1} \trianglelefteq \widehat{M}$. Clearly $(\widehat{M}X)f^{-1} = \bigcup_\Gamma(\widehat{M}X_\gamma)f^{-1}$.

The $\widehat{M}X_\gamma$ are direct summands in $\widehat{M}^{(\Lambda)}$ (by 12.9). So any $(\widehat{M}X_\gamma)f^{-1} \subset \widehat{M}$ is closed and hence a direct summand in $\widehat{M}$. This shows that V is $\widehat{M}$-generated, and by 12.7, $Hom_A(\widehat{M}, V)$ is an essential left ideal in T. By construction and 12.9,

$$Hom_A(\widehat{M}, V)f \subset Hom_A(\widehat{M}, Vf) \subset Hom_A(\widehat{M}, \widehat{M}X) = X,$$

and this implies $f \in X$ (since X is closed in $T^{(\Lambda)}$). □

In particular, the preceding result applies to $A = M$. In this case we detect some more interesting relationships for submodules of free modules. Recall that for a left non-singular ring A with injective hull $E(A)$, the maximal left quotient ring is $Q(A) := End_A(E(A))$ (see 11.4) and 12.10 reads as follows:

12.11 Closed left submodules of $A^{(\Lambda)}$.

Let A be a left non-singular ring with maximal left quotient ring $Q(A)$. There is a bijective correspondence between

(i) the closed left A-submodules of $A^{(\Lambda)}$,

(ii) the closed left A-submodules of $Q(A)^{(\Lambda)}$,

(iii) the closed left $Q(A)$-submodules of $Q(A)^{(\Lambda)}$.

Combining 12.11 with 12.10 we recover some properties of non-singular left A-modules known from localization theory.

12.12 Corollary. *Let A be a left non-singular ring with maximal left ring of quotients $Q(A)$, and let L be any non-singular left A-module. Then:*

(1) L is an essential A-submodule of a $Q(A)$-module $\widetilde{L}$.

(2) If L is a finitely generated A-module, then $\widetilde{L}$ is a finitely generated $Q(A)$-module.

(3) If L is A-injective then it is a $Q(A)$-module.

Proof. L is A-generated and we have the exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & U & \to & A^{(\Lambda)} & \to & L & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \overline{U} & \to & Q(A)^{(\Lambda)} & \to & \widetilde{L} & \to & 0\,, \end{array}$$

where U is a closed submodule of $A^{(\Lambda)}$ (since L is non-singular), and $\overline{U}$ denotes the essential closure of U in $Q(A)^{(\Lambda)}$. Now apply 12.10 and 12.11. □

12.13 Exercises.

Prove that for any A-module M, the following are equivalent ([245, 200]):

(a) every submodule has a unique essential closure in M;

(b) for any submodules $K, L \subset M$, $K \cap L \trianglelefteq K$ implies $L \trianglelefteq K + L$;

(c) if $K \subset M$ is closed, then $K \cap N$ is closed in N for any $N \subset M$;

(d) the intersection of any (two) closed submodules is closed in M;

(e) for any $N \subset M$ and $f \in Hom_A(N, M)$, $N \cap (N)f = 0$ implies that $Ke\, f$ is closed in N.

References. Ferrero-Wisbauer [132], Miyashita [200], Smith [245].

13 Prime modules

1.Prime modules with $(*)$. 2.Proposition. 3.Strongly prime modules. 4.Projective strongly prime modules. 5.Characterization of strongly prime modules. 6.Characterization of left strongly prime rings. 7.Lemma. 8.Strongly prime modules with uniform submodules. 9.Remarks. 10.Exercises.

In this section we consider various conditions on modules M which, for $M = A$ and A commutative, are equivalent to A being a prime ring.

Definitions. An A-module M is called a *prime module* if $A/An_A(M)$ is cogenerated by every $0 \neq K \subset M$. M is called *strongly prime* if for every $0 \neq K \subset M$, $M \in \sigma[K]$.

For a prime module M, $A/An_A(M)$ is a prime ring. The ring A is a prime ring if and only if there exists a faithful (or cofaithful) prime (left) A-module (e.g., ${}_AA$).

There is a property of a module M which allows further conclusions from our primeness conditions:

$(*)$ For any non-zero submodule $K \subset M$, $An_A(M/K) \not\subset An_A(M)$, i.e., there is an $r \in A \setminus An_A(M)$ with $rM \subset K$.

In general, this need not hold for ${}_AA$ if A is not commutative. However, it will hold for our applications to A as a bimodule.

13.1 Prime modules with $(*)$.

Let M be an A-module satisfying $(*)$, $S = End_A(M)$ and $\overline{A} = A/An_A(M)$.

(1) ${}_AM$ is prime if and only if $\overline{A}$ is a prime ring.

(2) If ${}_AM$ is prime, then

(i) M_S is prime (and S is a prime ring);

(ii) ${}_AM$ is polyform;

(iii) $\overline{A} \in \sigma[M]$ if and only if $\widehat{M}_T$ is finitely generated.

Proof. (1) It was already mentioned that ${}_AM$ prime implies $\overline{A}$ prime.

Assume $\overline{A}$ to be a prime ring and K a nonzero submodule of ${}_AM$. By $(*)$, there is a nonzero ideal $I \subset \overline{A}$ with $IM \subset K$. For $J = An_{\overline{A}}(K)$ we have $JI \cdot M \subset J \cdot K = 0$, hence $JI = 0$ and $J = 0$.

(2) Let ${}_AM$ be prime. (i) Assume $Ut = 0$ for an S-submodule U of M_S and $t \in S$. For $I = An_{\overline{A}}(M/AU)$ we get $I \cdot Mt \subset AUt = 0$, implying $Mt = 0$ and $t = 0$.

(ii) Consider submodules $K, L \subset {}_AM$ with $K \subset L$ and $K \trianglelefteq M$.

For $I = An_{\overline{A}}(M/K)$ and $\alpha \in Hom_A(L/K, M)$ we have $I((L/K)\alpha \cap K) = 0$, hence $(L/K)\alpha \cap K = 0$, implying $(L/K)\alpha = 0$ and $\alpha = 0$. So K is rational in M.

(iii) Apply 10.7. $\square$

13.2 Proposition. *For an A-module M satisfying $(*)$, the following are equivalent:*

(a) $_AM$ *is prime and* $Hom_A(M,K) \neq 0$, *for any nonzero submodule* $K \subset M$;

(b) $_AM$ *is cogenerated by any nonzero submodule* $K \subset M$.

Proof. $(b) \Rightarrow (a)$ is evident.

$(a) \Rightarrow (b)$ By 13.1, $S = End_A(M)$ is prime. Hence, for non-zero $K, L \subset M$, we get

$$L \cdot Hom_A(M,K) \supset M \cdot Hom_A(M,L) \cdot Hom_A(M,K) \neq 0,$$

which characterizes (b). □

We now turn to the investigation of the stronger primeness condition.

13.3 Strongly prime modules.

For an A-module M with M-injective hull $\widehat{M}$, the following are equivalent:

(a) M *is strongly prime;*

(b) $\widehat{M}$ *is strongly prime;*

(c) $\widehat{M}$ *is generated by each of its nonzero submodules;*

(d) M *is contained in every non-zero fully invariant submodule of* $\widehat{M}$;

(e) $\widehat{M}$ *has no non-trivial fully invariant submodules;*

(f) for any pretorsion class $\mathcal{T}$ *in* $\sigma[M]$, $\mathcal{T}(M) = 0$ *or* $\mathcal{T}(M) = M$;

(g) for each $m \in M$ *and* $0 \neq b \in M$, *there exist* $r_1, \ldots, r_k \in A$ *such that*

$$An_A(r_1 b, \ldots, r_k b) \subset An_A(m);$$

(h) there exists a module $K \supset M$ *such that for all* $0 \neq m \in M$, $M \subset AmEnd_A(K)$.

Proof. $(a) \Leftrightarrow (b) \Leftrightarrow (c)$ This is obvious since $\widehat{M} \in \sigma[M]$ and injectives in $\sigma[M]$ are generated by any subgenerator.

$(a) \Rightarrow (d)$ For any fully invariant $L \subset \widehat{M}$, $M \subset (L \cap M)End_A(\widehat{M}) \subset L$.

$(d) \Rightarrow (e)$ Let $L \subset \widehat{M}$ be fully invariant and $M \subset L$. Then $\widehat{M} = MEnd_A(\widehat{M}) \subset L$.

$(e) \Rightarrow (f)$ $\mathcal{T}(\widehat{M})$ is fully invariant in $\widehat{M}$.

$(f) \Rightarrow (b)$ For every $0 \neq K \subset \widehat{M}$, $\mathcal{T} := \sigma[K]$ is a hereditary pretorsion class in $\sigma[M]$ and $\mathcal{T}(\widehat{M}) \neq 0$.

$(a) \Leftrightarrow (g)$ The condition given in (g) is equivalent to the existence of a map

$$(Ab)^k \supset A(r_1 b, \ldots, r_k b) \to Am, \quad a(r_1 b, \ldots, r_k b) \mapsto am.$$

This is equivalent to $Am \in \sigma[Ab]$.

$(c) \Rightarrow (h)$ The module $\widehat{M} \supset M$ has the properties stated.

$(h) \Rightarrow (a)$ Clearly $M \in \sigma[Am]$ for all $0 \neq m \in M$. □

It follows from 13.3 that a strongly prime module M is either singular or non-singular in $\sigma[M]$. Since projective modules never are singular, any projective strongly prime module in $\sigma[M]$ is non-M-singular.

13.4 Projective strongly prime modules.

Let M be an A-module which is projective in $\sigma[M]$. Assume M is strongly prime. Then M is polyform and

(1) $S = End_A(M)$ is a left strongly prime ring;

(2) $T = End_A(\widehat{M})$ is the maximal left ring of quotients of S.

Proof. By the preceding remarks, M is polyform (non-M-singular).

(1) Because of projectivity of M, for any left ideal $J \subset S$, $J = Hom_A(M, MJ)$. $\widehat{M}$ being generated by MJ, there exist a submodule $L \subset MJ^{(\Lambda)}$, Λ some index set, and exact sequences

$$L \to M \to 0, \quad Hom_A(M, L) \to Hom_A(M, M) \to 0.$$

Hence

$$S \subset Hom_A(M, L) \subset Hom_A(M, MJ)^{(\Lambda)} = J^{(\Lambda)},$$

showing that S is left strongly prime.

(2) For any non-zero $K \subset M$, $M \in \sigma[K]$. Since M is projective in $\sigma[K]$, $M \subset K^{(\Lambda)}$, for some set Λ. In particular, $Hom_A(M, K) \neq 0$ and 11.5 applies. □

To get more characterizations of strongly prime modules we introduce a new

Definition. A module $N \in \sigma[M]$ is called an *absolute subgenerator in* $\sigma[M]$ if every non-zero submodule $K \subset N$ is a subgenerator in $\sigma[M]$ (i.e., $M \in \sigma[K]$).

It is obvious that every absolute subgenerator is a strongly prime module, and M itself is an absolute subgenerator in $\sigma[M]$ if and only if it is strongly prime. Moreover, if N is an absolute subgenerator in $\sigma[M]$ with $\mathcal{S}_M(N) \neq 0$, then all modules in $\sigma[M]$ are M-singular.

With this notions we have the following

13.5 Characterization of strongly prime modules.

For a polyform A-module M, the following assertions are equivalent;

(a) M is strongly prime;

(b) every module $K \in \sigma[M]$ with $\mathcal{S}_M(K) \neq K$ is a subgenerator;

(c) every non-zero module $K \in \sigma[M]$ with $\mathcal{S}_M(K) = 0$ is an absolute subgenerator;

(d) *for every non-zero* $N \in \sigma[M]$,

$$\mathcal{S}_M(N) = \bigcap \{K \subset N \mid N/K \ \textit{ is an absolute subgenerator in } \sigma[M]\};$$

(e) *there exists an absolute subgenerator in* $\sigma[M]$.

In this case, every projective module in $P \in \sigma[M]$ *is an absolute subgenerator and hence* $\mathcal{S}_M(P) = 0$.

Proof. $(a) \Rightarrow (b)$ Since M is polyform, the M-singular modules $X \in \sigma[M]$ are characterized by the property $Hom_A(X, \widehat{M}) = 0$ (see 10.2). Hence for any $K \in \sigma[M]$ with $\mathcal{S}_M(K) \neq K$, there exists a non-zero homomorphism $f : K \to \widehat{M}$. Since M (and $\widehat{M}$) is strongly prime, the image of f is a subgenerator in $\sigma[M]$ and so is K.

$(b) \Rightarrow (c)$ If $\mathcal{S}_M(N) = 0$, every non-zero submodule of N is non-M-singular.

$(c) \Rightarrow (d)$ Let $K \subset N$ be such that N/K is an absolute subgenerator in $\sigma[M]$. Assume $\mathcal{S}_M(N) \not\subset K$. Then $(K + \mathcal{S}_M(N))/K$ is an M-singular submodule of N/K. This is impossible since not all modules in $\sigma[M]$ are M-singular. So $\mathcal{S}_M(N) \subset K$ and $\mathcal{S}_M(N)$ is contained in the given intersection.

Since M is polyform, $N/\mathcal{S}_M(N)$ is non-M-singular and hence an absolute subgenerator in $\sigma[M]$. This proves our assertion.

$(d) \Rightarrow (e)$ Since $\mathcal{S}_M(M) = 0$ the equality in (d) implies the existence of an absolute subgenerator in $\sigma[M]$.

$(e) \Rightarrow (a)$ Let N be an absolute subgenerator in $\sigma[M]$. Then clearly $\mathcal{S}_M(N) = 0$ and N is a strongly prime module. Now the proof $(a) \Rightarrow (c)$ applies and so M is an absolute subgenerator in $\sigma[M]$.

Every projective module $P \in \sigma[M]$ is isomorphic to a submodule of some $M^{(\Lambda)}$ which is an absolute subgenerator (by (c)). Hence P is also an absolute subgenerator. □

Applying 13.5 to $M = A$, we immediately have the following

13.6 Characterization of left strongly prime rings.

For the ring A *the following properties are equivalent:*

(a) A *is a left strongly prime ring;*

(b) *every left* A*-module which is not singular is a subgenerator in* A*-Mod;*

(c) *every non-singular left* A*-module is an absolute subgenerator in* A*-Mod;*

(d) *for every non-zero left* A*-module* N,

$$\mathcal{S}(N) = \bigcap \{K \subset N \mid N/K \ \textit{ is an absolute subgenerator in } A\text{-}Mod\};$$

(e) *there exists an absolute subgenerator in A-Mod.*

The next lemma is needed for the study of strongly prime modules with uniform submodules.

13.7 Lemma. *Assume M to be a finitely generated, M-projective A-module and $Hom_A(M,K) \neq 0$, for all nonzero $K \subset M$. Then M contains a uniform submodule if and only if $S = End_A(M)$ contains a uniform left ideal.*

Proof. For a uniform submodule $U \subset M$ consider two nonzero left ideals I, J of S contained in $Hom_A(M,U)$. Then $U \supset MI \cap MJ \neq 0$ and

$$I \cap J \supset Hom_A(M, MI \cap MJ) \neq 0,$$

i.e., $Hom_A(M,U)$ is a uniform left ideal of S.

By a similar argument one finds that for a uniform left ideal I of S, MI is a uniform submodule of M. □

13.8 Strongly prime modules with uniform submodules.

For a finitely generated, self-projective and strongly prime A-module M, the following assertions are equivalent:

(a) *${}_AM$ contains a uniform submodule;*

(b) *${}_AM$ has finite uniform dimension;*

(c) *${}_A\widehat{M}$ is a finite direct sum of isomorphic indecomposable modules;*

(d) *$S = End_A(M)$ contains a uniform left ideal;*

(e) *$T = End_A(\widehat{M})$ is a left artinian simple ring.*

Proof. $(a) \Leftrightarrow (b)$ For a uniform submodule $U \subset M$, we have $M \subset U^k$, $k \in I\!N$, and hence M has finite dimension.

$(a) \Leftrightarrow (c)$ From the above argument we obtain $\widehat{M} \subset \widehat{U}^k$ and the assertion follows (since $End_A(\widehat{U})$ is a local ring).

$(a) \Leftrightarrow (d)$ was shown in Lemma 13.7.

$(c) \Rightarrow (e)$ Assume $\widehat{M} = V^k$, $k \in I\!N$, where V is M-injective and uniform. Any submodule of V is essential, hence rational, since $\widehat{M}$ contains no singular submodules. Therefore $End_A(V) =: D$ is a division ring.

Now $T = End_A(V^k) = D^{(k,k)}$ is a matrix ring over D.

$(e) \Rightarrow (b)$ is evident. □

13.9 Remarks. Primeness conditions on modules.

Consider the following properties of an A-module M.

(*i*) $A/An_A(M)$ is cogenerated by every $0 \neq K \subset M$;

(*ii*) for every $0 \neq K \subset M$, $M \in \sigma[K]$;

(*iii*) for every $0 \neq K \subset M$, there is a monomorphism $M \to K^r$, $r \in \mathbb{N}$;

(*iv*) for every $0 \neq K \subset M$, there is a monomorphism $A/An_A(M) \to K^r$, $r \in \mathbb{N}$;

(*v*) M is cogenerated by every $0 \neq K \subset M$;

(*vi*) M is monoform.

For any M, $(iii) \Rightarrow (v) \Rightarrow (i)$, $(iii) \Rightarrow (ii) \Rightarrow (i)$ and $(iv) \Rightarrow (ii)$.

If M is finitely generated and M-projective, then $(ii) \Rightarrow (iii)$.

If M is finitely cogenerated, (ii), (iii) and (v) are equivalent to M being a finite direct sum of isomorphic simple modules.

A direct sum of isomorphic simple modules satisfies (i), (ii), (iii) and (v) but not necessarily (iv) or (vi).

For $M = {}_AA$, (ii), (iii) and (iv) are equivalent and characterize the left *strongly prime rings* or *ATF-rings* as studied by Viola-Prioli [261], Rubin [237], Handelman-Lawrence [150].

For a commutative ring A all these conditions for ${}_AA$ are equivalent to A being a prime ring.

Restricting the above conditions to essential submodules one gets generalizations of commutative semiprime rings. This will be considered in the next section.

Modules with (i) are the *prime modules* already studied in Johnson [168]. Modules with (ii) are called *strongly prime modules* in Beachy [64]. Rings having a faithful module of this type are the *endoprimitive rings* of Desale-Nicholson [118].

Modules with (iii) are called *semicompressible modules* in Beachy-Blair [67]. They are special cases of (ii). Part (1) of 13.4 implies [67, Theorem 3.5]. For $M = A$, $(a) \Leftrightarrow (e)$ in 13.8 yields [67, Theorem 3.3].

As pointed out in Handelman-Lawrence [150], a ring A is left strongly prime if and only if there is a faithful A-module satisfying (iv) (SP-modules in [150]). This is equivalent to the existence of a cofaithful strongly prime A-module (i.e., with (ii)).

Condition (v) is the defining property of *prime modules* in Bican [79]. They are also characterized by the fact that $LHom_A(M,K) \neq 0$ for all nonzero submodules K, L of M.

Modules with condition (vi) (see 11.3) are called *strongly uniform* in Storrer [249]. If M is monoform and $Hom_A(M,K) \neq 0$ for every $0 \neq K \subset M$, then

$$K \cdot End_A(\widehat{M}) \supset M \cdot Hom_A(M,K) \cdot End_A(\widehat{M}) = M \cdot End_A(\widehat{M}) = \widehat{M},$$

and hence M satisfies (ii) (see also 13.3).

For an extensive account and examples for left strongly prime rings we refer to the monograph [16] by Goodearl, Handelman and Lawrence. Notice that a left strongly prime ring need not be right strongly prime (see [150, Example 1]).

13.10 Exercises.
Recall that A denotes an associative algebra with unit.

(1) Let M be a strongly prime A-module with non-zero socle.
Prove that $M \simeq E^{(\Lambda)}$, where E is a simple A-module.

(2) Let M be a uniform and self-injective A-module. Prove that the following are equivalent ([85, Proposition 2.1]):
(a) M is strongly prime;
(b) every cyclic submodule of M is strongly prime;
(c) M has no non-trivial self-injective submodules.

(3) Let A be a left self-injective ring. Prove that A is left strongly prime if and only if A is a simple ring.

(4) Prove that for a ring A, the following are equivalent ([261, 2.1]):
(a) A is left strongly prime;
(b) every finitely generated projective module is a subgenerator in A-Mod;
(c) every non-zero left ideal generates the injective hull of ${}_AA$.

(5) Let A be a left strongly prime ring. Show ([261]):
(i) A is a left non-singular prime ring.
(ii) A is left noetherian if and only if it contains a non-zero noetherian left ideal.
(iii) Any matrix ring $A^{(n,n)}$, $n \geq 1$, is left strongly prime.
(iv) The centre of A is a (strongly) prime ring.
(v) Every self-injective non-singular left A-module is A-injective ([67]).

(6) Let A be a ring such that ${}_AA$ has finite uniform dimension. Prove that the following are equivalent:
(a) A is left strongly prime;
(b) A is a left non-singular prime ring.

(7) Prove that for a left strongly prime ring A, the following are equivalent ([261]):
(a) A is a simple ring;
(b) every left ideal in A is idempotent.

(8) Prove: A regular ring is simple if and only if it is left (right) strongly prime.

(9) Prove that for a ring A, the following are equivalent ([261]):
(a) A is a simple left artinian ring;
(b) A is left strongly prime and $Soc\,{}_AA \neq 0$.

(10) Let A be a ring which is finitely generated as a module over its centre. Prove that the following are equivalent ([237, 1.10]):
(a) A is left strongly prime;
(b) A is a prime ring;
(c) A is right strongly prime.

(11) Prove that for any ring A, the following are equivalent ([230, Proposition 1]):
(a) Every self-injective left A-module is A-injective (A is left QI);
(b) A is left noetherian and any non-zero uniform (injective) A-module is strongly prime.

(12) Prove that for any ring A, the following are equivalent:
(a) A has a faithful strongly prime module;
(b) A has a faithful (self-injective) left module with no non-trivial fully invariant submodules;
(c) A has a left ideal $L \subset A$, such that A/L is faithful and satisfies:
For any $a \in A$, $a' \in A \setminus L$, there exist $r_1, \ldots, r_k \in A$, such that $rr_ia \in L$, for $r \in A$ and all $i \leq k$, implies $ra' \in L$.
Rings with these properties are called left endoprimitive *([118, 2.1]).*

(13) A ring is called left completely torsion free (CTF) *if, for any hereditary torsion classes $\mathcal{T}$ in A-Mod, $\mathcal{T}(A) = 0$ or A .*
Prove that the following are equivalent ([16, 13.1]):
(a) A is left CTF;
(b) every non-zero injective left A-module is faithful;
(c) for each non-zero two-sided ideal $K \subset A$, there exists $k \in K$ such that $Hom_A(Ak, A/K) \neq 0$.

(14) Prove that the following are equivalent ([176, 2.4]):
(a) A is left CTF and $Soc(_AA) \neq 0$;
(b) every non-zero injective left A-module is a cogenerator;
(c) there are only two hereditary torsion classes in A-Mod;
(d) A is isomorphic to some $S^{(n,n)}$, $n \in I\!N$, where S is a local left perfect ring.

(15) Prove for the algebra A ([176]):
(i) The following assertions are equivalent:
(a) Every non-zero left ideal of A is a generator in A-Mod;
(b) every non-zero ideal of A is a generator in A-Mod;
(c) every non-zero submodule of a projective A-module generates A.
Algebras with this properties are called left G-algebras.

(ii) If $Soc_AA \neq 0$ then A is a left G-algebra if and only if it is left artinian and simple.

(iii) Every left G-algebra is left strongly prime. Any left self-injective and left strongly prime algebra is a left G-algebra.

(16) A left A-module M is called coprime *if M is generated by each of its non-zero factor modules. Prove ([79]):*
(i) M is coprime if and only if it is generated by every non-zero cocyclic factor module.
(ii) Every direct sum of copies of a simple module is a coprime module.
(iii) Any coprime module M with $Rad\, M \neq M$ is semisimple.
(iv) Every A-module is coprime if and only if A is isomorphic to a matrix ring over a division ring.

References. Beachy [64], Beidar-Wisbauer [75, 76, 77], Bican-Jambor-Kepka-Nemec [79], Handelman-Lawrence [150], Katayama [176], Rubin [237], Viola-Prioli [261], Wisbauer [274].

14 Semiprime modules

1.Lemma. 2.Properties of trace and torsion submodules. 3.Strongly semiprime modules. 4.Characterizations. 5.SSP and semisimple modules. 6.Properly semiprime modules. 7.Properties of PSP-modules. 8.Corollary. 9.Relation to strongly prime modules. 10.Polyform SSP modules. 11.Polyform PSP modules. 12.Self-injective PSP modules. 13.Self-injective finitely presented PSP modules. 14.Density property of the self-injective hull. 15.Duo endomorphism rings. 16.Projective PSP modules. 17.Projective SSP modules. 18.Torsion submodules of semiprime rings. 19.Left SSP rings. 20.Left PSP rings. 21.Pseudo regular modules. 22.Left fully idempotent rings. 23.Pseudo regular and PSP modules. 24.Remarks. 25.Exercises.

In this section we extend the notion of semiprimeness from (commutative) rings to modules. We refer to 14.24 for a list of various possibilities to do so. In view of our applications we are mainly interested in non-projective modules.

The two following technical lemmas will be crucial for our investigations.

14.1 Lemma. *Let M be an A-module and $K, L \subset M$. The following are equivalent:*

(a) $M/L \in \sigma[K]$;

(b) for any $b \in M$, there exists a finite subset $X \subset K$, with $An_A(X)b \subset L$.

Proof. $(a) \Rightarrow (b)$ Assume $M/L \in \sigma[K]$ and $b \in M$. Then $Ab + L/L \subset M/L$ is a cyclic module in $\sigma[K]$ and hence a factor module of a cyclic submodule of $K^{(\mathbb{N})}$. So there exist $x_1, \ldots, x_k \in K$ and a morphism

$$A(x_1, \ldots, x_k) \to M/L,\ a(x_1, \ldots, x_k) \mapsto a(b + L)/L.$$

This implies $An_A(x_1, \ldots, x_k)b \subset L$.

$(b) \Rightarrow (a)$ Let $b \in M$ and choose $x_1, \ldots, x_k \in K$ with $An_A(x_1, \ldots, x_k)b \subset L$. We define a map as given above and so $(Ab + L)/L \in \sigma[K]$. Hence $M/L \in \sigma[K]$. $\square$

We use the notation for trace and torsion submodules introduced in 11.9.

14.2 Properties of trace and torsion submodules.

Let M be an A-module, $K \subset M$ and $T = End_A(\widehat{M})$. The following are equivalent:

(a) $M/\mathcal{T}_K(M) \in \sigma[K]$;

(b) for any $b \in M$, there exists a finite subset $X \subset K$, with $An_A(X)b \subset \mathcal{T}_K(M)$.

(c) every K-injective, $\mathcal{T}_K$-torsionfree module in $\sigma[M]$ is M-injective and K-generated;

(d) KT is M-injective and $\mathcal{T}^K(M) + \mathcal{T}_K(M) \trianglelefteq M$;

(e) $\widehat{M} = KT \oplus I_M(\mathcal{T}_K(\widehat{M}))$, where I_M denotes the M-injective hull.

Notice that the decomposition of $\widehat{M}$ given in (e) is in A-*Mod*. Though KT obviously is a fully invariant submodule this need not be true for $I_M(\mathcal{T}_K(\widehat{M}))$.

Proof. $(a) \Leftrightarrow (b)$ follows from 14.1.

$(a) \Rightarrow (c)$ Assume $Q \in \sigma[M]$ is K-injective and $\mathcal{T}_K$-torsionfree. For any submodule $L \subset M$, consider a morphism $f : L \to Q$. Since $\mathcal{T}_K(Q) = 0$, f factorizes through $f' : L/\mathcal{T}_K(L) \to Q$. We have the commutative diagram (with canonical mappings)

$$\begin{array}{ccccc} 0 & \to & L & \to & M \\ & & \downarrow & & \downarrow \\ 0 & \to & L/\mathcal{T}_K(L) & \to & M/\mathcal{T}_K(M) \\ & & \downarrow_{f'} & & \\ & & Q & & \end{array}$$

Since Q is K-injective and $M/\mathcal{T}_K(M) \in \sigma[K]$, there exists some $M/\mathcal{T}_K(M) \to Q$ yielding a commutative diagram. Hence Q is M-injective and

$$Q = Tr(M, Q) = Tr(M/\mathcal{T}_K(M), Q) = Tr(K, Q).$$

$(c) \Rightarrow (a)$ Since $I_M(M/\mathcal{T}_K(M))$ is M-injective and $\mathcal{T}_K$-torsionfree, it is K-generated by (c) and so $M/\mathcal{T}_K(M) \in \sigma[K]$.

$(a) \Rightarrow (d)$ As shown above, $I_M(M/\mathcal{T}_K(M)) \in \sigma[K]$. For a complement U of $\mathcal{T}_K(M)$ in M, $U \oplus \mathcal{T}_K(M) \trianglelefteq M$ and U is isomorphic to a submodule of $M/\mathcal{T}_K(M)$. Hence $U \subset \mathcal{T}^K(M)$ and $\mathcal{T}^K(M) + \mathcal{T}_K(M) \trianglelefteq M$.

$(d) \Rightarrow (a)$ Since $\mathcal{T}^K(M) + \mathcal{T}_K(M) \trianglelefteq M$ and $\mathcal{T}^K(M) \cap \mathcal{T}_K(M) = 0$,

$$\mathcal{T}^K(M) \simeq [\mathcal{T}^K(M) + \mathcal{T}_K(M)]/\mathcal{T}_K(M) \trianglelefteq M/\mathcal{T}_K(M)\ .$$

Hence $M/\mathcal{T}_K(M)$ is isomorphic to a submodule of $\widehat{KT} \in \sigma[K]$.

$(d) \Rightarrow (e)$ The assumptions imply $\widehat{M} = \widehat{KT} \oplus I_M(\mathcal{T}_K(M))$. As an injective object, $\widehat{KT} \in \sigma[K]$ is K-generated and hence $\widehat{KT} = KT$.

Since $\mathcal{T}^K(M) \trianglelefteq \mathcal{T}^K(\widehat{M})$, $\widehat{KT} = I_M(\mathcal{T}^K(\widehat{M}))$.

$(e) \Rightarrow (d)$ As a direct summand of $\widehat{M}$, KT is M-injective. Hence $\mathcal{T}^K(M) \trianglelefteq \widehat{KT} = KT$. Since $\mathcal{T}_K(M) \trianglelefteq \mathcal{T}_K(\widehat{M}) \trianglelefteq I_M(\mathcal{T}_K(\widehat{M}))$, we conclude $\mathcal{T}^K(M) + \mathcal{T}_K(M) \trianglelefteq M$. □

Referring to the above relations we define:

14.3 Strongly semiprime modules.

Let M be an A-module and $T = End_A(\widehat{M})$. M is called strongly semiprime (SSP) *if it satisfies the following equivalent conditions for every submodule $K \subset M$:*

(a) $M/\mathcal{T}_K(M) \in \sigma[K]$;

(b) *for any* $b \in M$, *there exists a finite subset* $X \subset K$, *with* $An_A(X)b \subset \mathcal{T}_K(M)$;

(c) *every* K*-injective* $\mathcal{T}_K$*-torsionfree module in* $\sigma[M]$ *is* M*-injective and* K*-generated;*

(d) $\widehat{KT} \in \sigma[K]$ *and* $\mathcal{T}^K(M) + \mathcal{T}_K(M) \trianglelefteq M$;

(e) $\widehat{M} = KT \oplus I_M(\mathcal{T}_K(\widehat{M}))$, *where* I_M *denotes the* M*-injective hull.*

Any SP module is SSP ($\widehat{M} = KT$). Also every semisimple module is SSP. We state some basic important properties.

14.4 Characterizations.

Let M *be an* A*-module,* $S = End_A(M)$ *and* $T = End_A(\widehat{M})$. *Then the following are equivalent:*

(a) ${}_AM$ *is SSP;*

(b) *for any* $N \trianglelefteq {}_AM$, $M \in \sigma[N]$, *and for any* $K \subset {}_AM$, $\mathcal{T}^K(M) + \mathcal{T}_K(M) \trianglelefteq {}_AM$;

(c) ${}_A\widehat{M}$ *is SSP;*

(d) $\widehat{M}$ *is a semisimple* (A,T)*-bimodule.*

Proof. $(a) \Rightarrow (b)$ For $N \trianglelefteq M$, $\mathcal{T}^N(M) \cap \mathcal{T}_N(M) = 0$ implies $\mathcal{T}_N(M) = 0$ and $NT = \widehat{M}$, in particular $M \in \sigma[N]$.

$(b) \Rightarrow (a)$ Let $K \subset M$ be any submodule. Since $\mathcal{T}_K(M) + \mathcal{T}^K(M) \trianglelefteq M$, $M \in \sigma[\mathcal{T}_K(M) + \mathcal{T}^K(M)] = \sigma[M]$ by assumption. So $\mathcal{T}_K(M)+K$ is a subgenerator in $\sigma[M]$ and hence it generates the M-injective module $\widehat{KT}$. However, $Hom_A(\mathcal{T}_K(M), \widehat{KT}) = 0$ implies that $\widehat{KT}$ is K-generated and M is an SSP module.

$(a) \Rightarrow (c)$ For any submodule $N \subset \widehat{M}$, put $K = NT \cap M$. Then $KT \trianglelefteq NT$ and $\mathcal{T}_K(\widehat{M}) = \mathcal{T}_N(\widehat{M})$. Consider any N-injective and $\mathcal{T}_N$-torsionfree module $Q \in \sigma[M]$. Then Q is K-injective and $\mathcal{T}_K$-torsionfree, and hence M-injective and K-generated since M is SSP (cf. 14.3). As easily seen, Q is also N-generated and so $\widehat{M}$ is SSP by 14.3.

$(c) \Rightarrow (a)$ Essential submodules of SSP modules are obviously SSP.

$(c) \Leftrightarrow (d)$ Put $M = \widehat{M}$. Let $U \subset M$ be an essential (A,S)-submodule.

Then $U = \mathcal{T}^U(M)$, and $\mathcal{T}^U(M) \cap \mathcal{T}_U(M) = 0$ implies $\mathcal{T}_U(M) = 0$. We see from 14.3 that $U = M$. So M has no proper essential (A,S)-submodule. Hence it is a semisimple (A,S)-module.

Now assume M is a semisimple (A,S)-module and $K \subset M$ an A-submodule. Then $M = KS \oplus L$ for some fully invariant $L \subset M$. This implies $Hom_A(L, KS) = 0$ and so $L \subset \mathcal{T}_K(M)$. Hence $M/\mathcal{T}_K(M) \in \sigma[M/L] = \sigma[K]$ showing that M is SSP. $\square$

14.5 SSP and semisimple modules.
Let M be an A-module.

(1) *Assume M has essential socle and for every $N \trianglelefteq M$, $M \in \sigma[N]$. Then M is semisimple.*

(2) *M is semisimple if and only if every module in $\sigma[M]$ is SSP.*

Proof. (1) By assumption, $M \in \sigma[Soc(M)]$ and modules in $\sigma[Soc(M)]$ are semisimple.

(2) We see from (1) that every finitely cogenerated module in $\sigma[M]$ is semisimple and hence every simple module in $\sigma[M]$ is M-injective, i.e., M is co-semisimple.

Let N be the sum of all non-isomorphic simple modules in $\sigma[M]$ and consider $L = M \oplus N$. Then $\mathcal{T}_N(L) \subset Rad\,(L) = 0$ (cf. [40], 23.1). Since L is SSP, this implies $L/\mathcal{T}_N(L) \in \sigma[N]$. Hence L and M are semisimple modules. □

Weakening the conditions for strongly semiprime modules we define:

14.6 Properly semiprime modules.
Let M be a left A-module and $T = End_A(\widehat{M})$. We call M properly semiprime (PSP) *if it satisfies the following equivalent conditions:*

(a) *For every element $a \in M$, $M/\mathcal{T}_a(M) \in \sigma[Ra]$;*

(b) *for any $a, b \in M$, there exist $r_1, \ldots, r_n \in A$ such that*

$$An_A(r_1 a, r_2 a, \ldots, r_n a)b \subset \mathcal{T}_a(M);$$

(c) *for every finitely generated submodule $K \subset M$, $M/\mathcal{T}_K(M) \in \sigma[K]$;*

(d) *for any cyclic $K \subset M$, every K-injective $\mathcal{T}_K$-torsionfree module in $\sigma[M]$ is M-injective and K-generated;*

(e) *for any cyclic $K \subset M$, $\widehat{KT} \in \sigma[K]$ and $\mathcal{T}^K(M) + \mathcal{T}_K(M) \trianglelefteq M$;*

(f) *for any cyclic $K \subset M$, $\widehat{M} = KT \oplus I_M(\mathcal{T}_K(\widehat{M}))$.*

Conditions (d)-(f) also hold for finitely generated submodules.

Proof. For the equivalence of (a), (c), (d), (e) and (f) see 14.2. $(c) \Leftrightarrow (b)$ is clear.

$(b) \Rightarrow (c)$ Let $a_1, \ldots, a_k$ be a generating subset of K. By 11.10, $\mathcal{T}_K(M) = \bigcap_{i=1}^k \mathcal{T}_{a_i}(M)$. Hence

$$M/\mathcal{T}_K(M) \subset \bigoplus_{i=1}^k M/\mathcal{T}_{a_i}(M).$$

But $M/\mathcal{T}_{a_i}(M) \in \sigma[Ra_i] \subset \sigma[K]$. Thus $M/\mathcal{T}_K(M) \in \sigma[K]$. □

14.7 Properties of PSP-modules.

Let M be an A-module and $T = End_A(\widehat{M})$.

(1) *Assume M is a PSP-module and $U \subset \widehat{M}$ an (A,T)-submodule of finite uniform dimension. Then U is a semisimple (A,T)-bimodule.*

(2) *Assume $\widehat{M}$ has finite uniform dimension as (A,T)-bimodule. Then M is an SSP-module if and only if M is a PSP-module.*

Proof. (1) First assume U is a uniform (A,T)-bimodule. Let $V \subset U$ be an (A,T)-submodule, and $N \subset M \cap V$ a finitely generated A-submodule. Then NT is an essential (A,T)-submodule of U. $\mathcal{T}^N(U) \cap \mathcal{T}_N(U) = 0$ implies $\mathcal{T}_N(U) = 0$.

From this we deduce $NT \trianglelefteq V \trianglelefteq U$ as A-modules. Since NT is M-injective, we conclude $NT = V = U$ and U is a simple (A,T)-bimodule.

Now assume U has finite uniform dimension as (A,T)-bimodule. Hence there exist uniform submodules $V_i \subset U$, $i = 1, \ldots, n$, such that $\bigoplus_{i=1}^n V_i \trianglelefteq U$ as (A,T)-submodule.

Let N_i be finitely generated submodule of the left A-module $V_i \cap M$ and $N = \bigoplus_{i=1}^n N_i$. As shown above, all $V_i = N_iT$ are simple (A,T)-bimodules. Hence $NT = \sum_{i=1}^n N_iT = \bigoplus_{i=1}^n V_i \trianglelefteq U$ as A-submodule. Therefore $U = NT = \bigoplus_{i=1}^n V_i$ is a finitely generated semisimple (A,T)-bimodule.

(2) If M is SSP then obviously M is PSP.

If M is PSP and $\widehat{M}$ has finite uniform dimension as (A,T)-bimodule, $\widehat{M}$ is a semisimple (A,T)-bimodule by (1) and hence M is SSP by 14.4. $\square$

14.8 Corollary. *For a finitely generated left A-module M and $T = End_A(\widehat{M})$, the following are equivalent:*

(a) *M is an SSP-module;*

(b) *$\widehat{M}$ is (finitely generated and) semisimple as an (A,T)-bimodule;*

(c) *M is PSP and $\widehat{M}$ has finite uniform dimension as an (A,T)-bimodule.*

Proof. Since M is a finitely generated A-module, $\widehat{M} = MT$ is a finitely generated (A,T)-bimodule. Hence the assertions follow from 14.4 and 14.7. $\square$

The next result shows the relation to strongly prime modules.

14.9 Relation to strongly prime modules.

For an A-module M with $T = End_A(\widehat{M})$, the following are equivalent:

(a) *M is strongly prime;*

(b) *M is PSP (or SSP) and $\widehat{M}$ is a uniform (A,T)-bimodule.*

(c) *$\widehat{M}$ is a simple (A,T)-bimodule.*

In particular, for a uniform A-module M, the conditions strongly prime, SSP, and PSP are equivalent.

Proof. $(a) \Rightarrow (b)$ For every submodule $K \subset M$, $KT = \widehat{M}$ and the assertion is clear.

$(b) \Rightarrow (a)$ Assume M is PSP and $K \subset M$ is a finitely generated submodule. By the uniformity condition, $\mathcal{T}^K(\widehat{M}) \cap \mathcal{T}_K(\widehat{M}) = 0$ implies $\mathcal{T}_K(\widehat{M}) = 0$ and $\widehat{M} = KT$. So M is strongly prime.

$(a) \Leftrightarrow (c)$ This follows with the same arguments as applied in the proof of 14.4. □

In general, SSP modules need not be polyform. Modules satisfying both conditions have particularly nice structure properties.

14.10 Polyform SSP modules.
For a polyform A-module M and $T = End_A(\widehat{M})$, the following are equivalent:

(a) M is an SSP-module;

(b) for every $N \trianglelefteq M$, $M \in \sigma[N]$;

(c) for every submodule $K \subset M$, $\widehat{K}T \in \sigma[K]$;

(d) for every submodule $K \subset M$, $\widehat{M} = KT \oplus \mathcal{T}_K(\widehat{M})$.

Proof. The statements follow from 11.11, 14.3 and 14.4. □

We know that a module M is SSP if and only if $\widehat{M}$ is SSP. In general, for a PSP module M, $\widehat{M}$ need not be PSP. For polyform modules the PSP property extends at least to the idempotent closure:

14.11 Polyform PSP modules.
Let M be a polyform A-module, $T = End_A(\widehat{M})$, B the Boolean ring of central idempotents of T, and $\widetilde{M}$ the idempotent closure of M. Then the following are equivalent:

(a) M is a PSP module;

(b) for every $m \in M$, $AmT = \widehat{M}\varepsilon(m)$;

(c) for every finitely generated submodule $K \subset M$, $KT = \widehat{M}\varepsilon(K)$;

(d) $\widetilde{M}$ is a PSP module.

Under the given conditions, $\mathcal{T}_m(\widehat{M}) = \widehat{M}(id - \varepsilon(m))$.

Proof. $(a) \Rightarrow (b)$ By 14.6(f) and 11.11(1), $\widehat{M} = AmT \oplus \mathcal{T}_m(\widehat{M})$. Now (b) follows from the definition of the idempotent $\varepsilon(m)$.

$(b) \Rightarrow (a)$ Since $\widehat{M}\varepsilon(m)$ is M-injective,

$$I_M(\mathcal{T}^m(M)) = I_M(\mathcal{T}^m(\widehat{M})) = I_M(AmT) = AmT \in \sigma[Am].$$

M being polyform, $\mathcal{T}_m(M) + \mathcal{T}^m(M) \trianglelefteq M$ and the assertion follows by 14.6.

$(b) \Leftrightarrow (c)$ This is obvious by 14.6 and 11.12.

$(b) \Rightarrow (d)$ Any $a \in \widetilde{M}$ can be written as $a = \sum_{i=1}^{k} m_i e_i$, with $m_1, \ldots, m_k \in M$ and pairwise orthogonal $e_1, \ldots, e_k \in B$, satisfying

$$\varepsilon(a) = \sum_{i=1}^{k} e_i \text{ and } e_i = \varepsilon(m_i)e_i \text{ for } i = 1, \ldots, k.$$

By (b), $Am_iT = \widehat{M}\varepsilon(m_i)$ for $i = 1, \ldots, k$, and

$$RaT = A(\sum_{i=1}^{k} m_i e_i)T = \sum_{i=1}^{k}(Am_iT)e_i = \sum_{i=1}^{k} \widehat{M}\varepsilon(m_i)e_i = \widehat{M}(\sum_{i=1}^{k} e_i) = \widehat{M}\varepsilon(a).$$

$(d) \Rightarrow (a)$ This is easy to verify. □

Pierce stalks of polyform modules will be considered in 18.16.

By 14.4, any self-injective SSP module is semisimple as a bimodule. For self-injective polyform PSP modules we get a weaker structure theorem:

14.12 Self-injective PSP modules.

Let M be a self-injective polyform A-module and $T = End_A(\widehat{M})$. Denote $\Lambda = A \otimes_{\mathbb{Z}} T^o$ and $C = End_\Lambda(M)$. Then the following conditions are equivalent:

(a) *${}_AM$ is a PSP A-module;*

(b) *every cyclic Λ-submodule of M is a direct summand;*

(c) *every finitely generated Λ-submodule of M is a direct summand;*

(d) *as a Λ-module, M is a selfgenerator;*

(e) *for any $m \in M$ and $f \in End_C(M)$, there exists $h \in \Lambda$ with $f(m) = hm$.*

Proof. Notice that Λ-submodules of M are just fully invariant submodules and C can be identified with the centre of T. Hence C is a commutative regular ring.

$(a) \Leftrightarrow (b) \Leftrightarrow (c)$ is clear by 14.11, $(c) \Rightarrow (d)$ is obvious.

$(d) \Rightarrow (e)$ This follows from the proof of the Density Theorem (e.g., [40, 15.7]).

$(e) \Rightarrow (b)$ Choose any $m \in M$ and $\varepsilon(m) \in C$ as defined in 11.12. Since $mC \simeq \varepsilon(m)C$ is a direct summand, for any $n \in M$, there exists $f \in End_C(M)$ with $f(m) = n\varepsilon(m)$. By (e), $f(m) = hm$ for some $h \in \Lambda$ and hence $M\varepsilon(m) \subset \Lambda m$.

On the other hand, $m = m\varepsilon(m)$ and $\Lambda m \subset (\Lambda M)\varepsilon(m) \subset M\varepsilon(m)$. So $\Lambda m = M\varepsilon(m)$ is a direct summand and the assertion is proved. □

Modules M, whose finitely generated submodules are direct summands, are closely related to M being regular in $\sigma[M]$. In fact, if M is finitely presented in $\sigma[M]$, these two notions coincide (7.3). As a special case we derive from above:

14.13 Self-injective finitely presented PSP modules.

Let M be a self-injective polyform A-module and $T = End_A(\widehat{M})$. Denote $\Lambda = A \otimes_{\mathbb{Z}} T^o$ and $C = End_\Lambda(M)$. Assume M is finitely generated as a Λ-module. Then the following conditions are equivalent:

(a) *M is a PSP A-module and finitely presented in $\sigma[{}_\Lambda M]$;*

(b) *${}_\Lambda M$ is regular and projective in $\sigma[{}_\Lambda M]$;*

(c) *${}_\Lambda M$ is a (projective) generator in $\sigma[{}_\Lambda M]$;*

(d) *for any $m_1, \ldots, m_n \in M$ and $f \in End_C(M)$, there exists $h \in \Lambda$ with $f(m_i) = hm_i$, for $i = 1, \ldots, n$ (density property).*

Proof. Since ${}_\Lambda M$ is finitely generated, M_C is a generator in C-*Mod* by 11.12. So M_C is a faithfully flat C-module.

$(a) \Leftrightarrow (b)$ By 14.12, every finitely generated Λ-submodule of M is a direct summand. Now the assertion follows from 7.3.

$(b) \Rightarrow (c) \Rightarrow (d)$ are obvious (Density Theorem 5.4).

$(d) \Rightarrow (a)$ Since M_C is a generator in C-*Mod*, M is (finitely generated and) projective as an $End_C(M)$-module (5.5).

By the density property, the categories $\sigma[{}_\Lambda M]$ and $\sigma[{}_{End_C(M)}M]$ coincide (5.4). So M is projective (hence finitely presented) in $\sigma[{}_\Lambda M]$.

By 14.12, M is a PSP A-module. □

For modules M with $End_A(\widehat{M})$ commutative we are now able to characterize the density property of $\widehat{M}$ as bimodule.

14.14 Density property of the self-injective hull.

Let M be a polyform A-module, assume $T = End_A(\widehat{M})$ to be commutative and put $\Lambda = A \otimes_{\mathbb{Z}} T$. Then the following are equivalent:

(a) *For any $a_1, \ldots, a_n \in \widehat{M}$ and $f \in End_T(\widehat{M})$, there exists $h \in \Lambda$ with $ha_i = fa_i$ for $i = 1, \ldots, n$.*

(b) *For any $m_1, \ldots, m_n, m \in M$ there exist $r_1, \ldots, r_k \in A$ such that for $s_1, \ldots, s_n \in A$, the relations*

$$\sum_{l=1}^{n} s_l r_j m_l = 0 \quad \textit{for } j = 1, \ldots, k,$$

imply $s_1 m \in \mathcal{T}_{Um_1}(M)$ for $U = An_A(m_2, \ldots, m_n)$.

(c) *M is a PSP module and for any $m_1, \ldots, m_n \in M$, there exist $r_1, \ldots, r_k \in A$ such that for $s_1, \ldots, s_n \in A$ the relations*

$$\sum_{l=1}^{n} s_l r_j m_l = 0 \quad \textit{for } j = 1, \ldots, k,$$

imply $s_1 m_1 \in \mathcal{T}_{Um_1}(M)$ for $U = An_A(m_2, \ldots, m_n)$.

If $\widehat{M}$ is finitely presented in $\sigma[{}_\Lambda \widehat{M}]$, the above are equivalent to:

(d) *$\widehat{M}$ is a PSP A-module.*

Proof. Put $U = An_A(m_2, \ldots, m_n)$.

$(a) \Rightarrow (b)$ Put $N = \sum_{i=1}^{n} m_i T$. By 11.12, $\widehat{M}$ is a non-singular and N is an injective T-module. Since in a non-singular module the intersection of injective submodules is again injective, $K = m_1 T \cap N$ is T-injective. Hence $m_1 T = K \oplus L$ for some submodule $L \subset m_1 T$. By 11.12, $An_T(m_1) = (1 - \varepsilon(m_1))T$. This means that the map

$$m_1 T \to \varepsilon(m_1) T, \quad m_1 t \mapsto \varepsilon(m_1) t,$$

is an isomorphism. So there exist idempotents $u, v \in T$ with the properties

$$(*) \quad uv = 0,\ u + v = \varepsilon(m_1),\ K = m_1 u T \simeq uT \text{ and } L = m_1 v T \simeq vT.$$

Since $m_1 u T = K \subset N$ and $UN = 0$, we have $Um_1 u = 0$ and

$$Um_1 = U[m_1 \varepsilon(m_1)] = U[m_1(u + v)] = Um_1 v.$$

By 11.13, $Um_1 v \trianglelefteq Am_1 v$, and by 11.12, $\varepsilon(Um_1 v) = \varepsilon(Am_1 v) = \varepsilon(m_1 v)$.
The isomorphism $m_1 v T \simeq vT$ implies $\varepsilon(m_1 v) = v$ and hence we have

$$(**) \qquad \varepsilon(Um_1) = v\ .$$

Since $m \in M$ and $\mathcal{T}_{Um_1}(M) = M \cap \mathcal{T}_{Um_1}(\widehat{M})$, by 11.14,

$$(\mathcal{T}_{Um_1}(M) : m)_A = (\mathcal{T}_{Um_1}(\widehat{M}) : m)_A = An_A(mv).$$

As an injective submodule, $m_1 v T \oplus N$ is a direct summand in $\widehat{M}$. Hence there exists a T-endomorphism ψ of $\widehat{M}$ satisfying $\psi N = 0$ and $\psi m_1 v = mv$ (recall $m_1 v T \simeq vT$). Since $m_1 u \in N$, we have $\psi m_1 u = 0$ and $\psi m_1 = \psi[m(u + v)] = mv$.

By assumption, there exists $h \in \Lambda$ satisfying

$$hm_1 = \psi m_1 = mv, \text{ and } hm_i = \psi m_i \text{ for } i = 2, \ldots, n\ .$$

Now the assertion follows from 11.14.

$(b) \Rightarrow (c)$ Putting $m_1 = \cdots = m_n = 0$ we see that M is a PSP module. The second part of the conditions in (c) follows from (b) for $m = m_1$.

$(c) \Rightarrow (a)$ Since $\widehat{M} = MT$ it suffices to show that for any $m_1, \ldots, m_n \in M$ and $f \in End_T(\widehat{M})$, there exists $h \in \Lambda$ such that $fm_i = hm_i$, for all $i = 1, \ldots, n$.

We prove this by induction on the cardinality $|I|$ of minimal subsets $I \subset \{1, \ldots, n\}$ satisfying $\sum_{i \in I} m_i T = \sum_{i=1}^n m_i T$.

Consider the case $|I| = 1$, i.e., $I = \{1\}$. By 14.11, $\widehat{M}\varepsilon(m_1) = \Lambda m_1$. Since

$$fm_1 = f[m_1\varepsilon(m_1)] = (fm_1)\varepsilon(m_1) \in \widehat{M}\varepsilon(m_1) = \Lambda m_1 \ ,$$

$fm_1 = hm_1$ for some $h \in \Lambda$. By assumption, $\sum_{i=1}^n m_i T = m_1 T$, implying $fm_i = hm_i$, for all $i = 1, \ldots, n$.

Now assume $|I| = n$ and consider $N = \sum_{i=2}^n m_i T$. As shown above, there exist idempotents $u, v \in T$ satisfying $(*)$ and $(**)$. By 11.14, there exists $h_1 \in \Lambda$ with

$$h_1 m_1 = m_1 v \ \text{ and } h_1 m_i = 0\,, \ \text{ for } i = 2, \ldots, n \ .$$

Notice that $h_1 m_1 u = (h_1 m_1)u = (m_1 v)u = m_1(vu) = 0$.

By induction hypothesis, there exists $h_2 \in \Lambda$ with

$$h_2 m_i = fm_i\,, \ \text{ for } i = 2, \ldots, n \ .$$

Obviously, $h_2 x = fx$ for any $x \in N$. From $(*)$ we obtain $\quad h_2 m_1 u = fm_1 u$.
Consider the element $m = fm_1 v - h_2 m_1 v$. Clearly $m = mv$ and so $m \in \widehat{M}v$. By $(*)$, $v\varepsilon(m_1) = v$. Now it follows from 14.11 that

$$\Lambda m_1 v = (\Lambda m_1)v = \widehat{M}\varepsilon(m_1)v = \widehat{M}v \ .$$

Hence there exists $h_3 \in \Lambda$ with

$$h_3 m_1 v = m = fm_1 v - h_2 m_1 v \ .$$

Putting $h = h_3 h_1 + h_2$, we have

$$\begin{aligned} hm_1 &= h_3(h_1 m_1 u) + h_3(h_1 m_1 v) + h_2 m_1 u + h_2 m_1 v \\ &= h_3 m_1 v + fm_1 u + h_2 m_1 v \\ &= (fm_1 v - h_2 m_1 v) + fm_1 u + h_2 m_1 v \\ &= fm_1 v + fm_1 u = fm_1 \ , \\ hm_i &= h_3(h_1 m_i) + h_2 m_i = fm_i\,, \ \text{ for } i = 2, \ldots, n \ . \end{aligned}$$

$(a) \Leftrightarrow (d)$ This is clear by 14.13. □

We have seen in 14.12 and 14.13 that self-injective polyform PSP modules have nice structural properties. Hence we may ask for which modules M, the M-injective hull $\widehat{M}$ is a PSP module.

Now we turn to the question which additional conditons on an SSP or PSP module M imply that M is polyform. It is interesting to observe that this is achieved by commutativity conditions on the endomorphism ring as well as by projectivity of the module. Recall that a ring is said to be *(left and right) duo* if all its one-sided ideals are two-sided.

14.15 Duo endomorphism rings.

Let M be an A-module, $T = End_A(\widehat{M})$ and assume, for every $N \trianglelefteq M$, $M \in \sigma[N]$.

(1) $Jac(T) \cap Z(T) = 0$.

(2) Suppose T is a duo ring. Then M is polyform and SSP.

Proof. (2) Assume for $f \in T$, $N = Ke\, f \trianglelefteq \widehat{M}$. Then $M \in \sigma[N]$ which is equivalent to $NT = \widehat{M}$. This implies $\widehat{M}f = (NT)f = (Nf)T = 0$ and hence $f = 0$. So the Jacobson radical of T is zero, i.e., M is polyform. By 14.10, M is SSP.

(1) A similar argument also implies this assertion. □

Projectivity makes any PSP module polyform. This applies in particular for the left module structure of the ring itself.

14.16 Projective PSP modules.

Let M be a PSP module which is projective in $\sigma[M]$. Then:

(1) For any submodules $K, N \subset M$, $\mathcal{T}_N(K)$ is a complement of $\mathcal{T}^N(K)$ in K, hence

$$\mathcal{T}^N(K) + \mathcal{T}_N(K) \trianglelefteq K \text{ and } \quad [\mathcal{T}^N(K) + \mathcal{T}_N(K)]/\mathcal{T}_N(K) \trianglelefteq K/\mathcal{T}_N(K).$$

(2) M is polyform.

Proof. M is projective in $\sigma[M]$ if and only if $M^{(\Lambda)}$ is self-projective, for any set Λ. From this it is obvious that M/X is projective in $\sigma[M/X]$, for every fully invariant submodule $X \subset M$.

(1) First we show $\mathcal{T}^N(K) \neq 0$, for any finitely generated submodules $K, N \subset M$ with $Hom(K, N) \neq 0$. We may assume that there is an epimorphism $f : K \to N$. Then $N \subset \mathcal{T}^K(M)$ and

$$\mathcal{T}_K(M) \cap N = 0 = \mathcal{T}_K(M) \cap K.$$

Put $L = \mathcal{T}_K(M)$ and $\overline{M} = M/L$. There are canonical inclusions $N \subset \overline{M}$ and $K \subset \overline{M}$. Since M is PSP, $\overline{M} \in \sigma[K]$ and so $\sigma[\overline{M}] = \sigma[K]$.

As outlined above, $\overline{M}$ is projective in $\sigma[K]$ and hence is a submodule of $K^{(\Lambda)}$, for some set Λ. Therefore the compostion of the inclusion $N \subset \overline{M}$ with a suitable map $\overline{M} \to K$ yields a non-zero morphism $N \to K$. This means $\mathcal{T}^N(K) \neq 0$.

From 11.10 we know $\mathcal{T}^N(K) \cap \mathcal{T}_N(K) = 0$. We have to show that $\mathcal{T}_N(K)$ is maximal with respect to this property.

For $x \in K \setminus \mathcal{T}_N(K)$, there exists a non-zero $g : Ax \to \widehat{N}$ and $0 \neq (y)g \in N$, for some $y \in Ax$. From the above we know

$$0 \neq \mathcal{T}_N(Ay) \subset \mathcal{T}^N(K) \cap Ax.$$

This shows the maximality of $\mathcal{T}_N(K)$ which implies the assertions (see [40, 140]).

(2) Assume M is not polyform. Then there exist a cyclic submodule $K \subset M$ and a non-zero morphism $f : K \to M$ with $Ke\, f \trianglelefteq K$. Put $N = (K)f$ and $\overline{M} = M/\mathcal{T}_N(M)$.

The map $K \xrightarrow{f} M \to \overline{M}$ factorizes through $\bar{f} : K/\mathcal{T}_N(K) \to \overline{M}$. Since

$$\mathcal{T}^N(K) \cap Ke\, f \trianglelefteq \mathcal{T}^N(K) \text{ and } \mathcal{T}^N(K) \trianglelefteq K/\mathcal{T}_N(K) \text{ by (1)},$$

we observe $Ke\, \bar{f} \trianglelefteq K/\mathcal{T}_N(K)$ and hence $N \simeq Im\, \bar{f}$ is an $\overline{M}$-singular module.

By assumption, $\overline{M} \in \sigma[N] = \sigma[\overline{M}]$. So, in particular, $\overline{M}$ is $\overline{M}$-singular. However, as noted above, $\overline{M}$ is projective in $\sigma[\overline{M}]$ and hence cannot be $\overline{M}$-singular.

Therefore M is polyform. □

14.17 Projective SSP modules.

Let M be projective in $\sigma[M]$ and $T = End_A(\widehat{M})$. Then the following are equivalent:

(a) M is an SSP-module;

(b) for every submodule $K \subset M$, $\widehat{M} = KT \oplus \mathcal{T}_K(\widehat{M})$.

(c) M is polyform and for any $N \trianglelefteq M$, $M \in \sigma[N]$.

Proof. $(a) \Rightarrow (b)$ By 14.16, M is polyform and now apply 14.10.

$(b) \Rightarrow (a)$ The decomposition implies in particular that every fully invariant submodule is a direct summand in $\widehat{M}$ as (A,T)-submodule. Hence $\widehat{M}$ is a semisimple (A,T)-bimodule and M is SSP by 14.4.

$(a) \Leftrightarrow (c)$ By 14.16, M is polyform and the assertion follows from 14.10. □

Definition. A ring A is called *left PSP (SSP)*, if ${}_AA$ is a PSP (SSP) module.

Notice that the definition also applies to rings without units, considering such rings as modules over rings with units in a canonical way. Rings with units are (left) projective and hence are left non-singular if they are left PSP (or SSP) (by 14.16). We will see that they are also semiprime. Before we want to describe the torsion modules related to semiprime rings.

14.18 Torsion submodules of semiprime rings.

Let A be a semiprime ring and $N \subset A$ a left ideal. Then:

(1) $\mathcal{T}_N(A) = An_A(N)$.

(2) $A/\mathcal{T}_N(A) \in \sigma[N]$ *if and only if there exists a finite subset* $X \subset N$ *with* $An_A(X) = An_A(N)$.

Proof. (1) The relation $\mathcal{T}_N(A) \subset An_A(N)$ always holds. Clearly $NAn_A(N)$ is a nilpotent left ideal and hence is zero.

Consider $f \in Hom_A(An_A(N), I_A(N))$ and put $K = (N)f^{-1}$. Then

$$(Kf)^2 \subset N(Kf) = (NK)f \subset (NAn_A(N))f = 0.$$

Since A is semiprime, $Kf = 0$ and so $Im\, f \cap N = 0$, implying $f = 0$. Hence $\mathcal{T}_N(A) \supset An_A(N)$.

(2) Assume $A/\mathcal{T}_N(A) \in \sigma[N]$. By 14.1, there exists a finite subset $X \subset N$ with

$$An_A(X)1 \subset \mathcal{T}_N(A) = An_A(N) \subset An_A(X).$$

Now assume $An_A(X) = An_A(N)$ for some finite $X \subset N$. Then $An_A(X) = \mathcal{T}_N(A)$ by (1), and for any $b \in A$, $An_A(X)b \subset \mathcal{T}_N(A)$. Now apply 14.1. □

Applying our module theoretic results to ${}_AA$, we obtain characterizations of

14.19 Left SSP rings.

For the ring A put $Q := Q_{\max}(A)$. The following are equivalent:

(a) *A is left SSP;*

(b) *for every essential left ideal $N \subset A$, $A \in \sigma[N]$;*

(c) *every essential left ideal $N \subset A$ contains a finite subset X with $An_A(X) = 0$;*

(d) *for every left ideal $I \subset A$, $Q = IQ \oplus \mathcal{T}_I(Q)$;*

(e) *A is semiprime and every left ideal $I \subset A$ contains a finite subset $X \subset I$ with $An_A(X) = An_A(I)$;*

(f) *Q is a semisimple (A, Q)-module.*

If A satisfies these conditions, then Q is left self-injective, von Neumann regular, and a finite product of simple rings.

Proof. $(a) \Rightarrow (b)$ is shown in 14.4.

$(b) \Rightarrow (a)$ Assume for every essential left ideal $N \subset A$, $A \in \sigma[N]$. Any such N is a faithful A-module.

First we show that A is semiprime. For this consider an ideal $I \subset A$ with $I^2 = 0$. Let J be the right annihilator of I, and $L \subset A$ any non-zero left ideal. Obviously, $I \subset J$. Assume $L \cap J = 0$. Then $IL \neq 0$. However, $IL \subset L \cap J = 0$, a contradiction. This implies that J is an essential left ideal in A. By our assumption, J is a faithful left module and $IJ = 0$ means $I = 0$.

In view of 14.4 and 11.11, it remains to show that A is left non-singular

If the left singular ideal $\mathcal{S}(A) \subset A$ is non-zero, $\mathcal{S}(A) \oplus An_A(\mathcal{S}(A))$ is an essential left ideal in A. Hence there are $a_1, \dots, a_k$ with $An_A(a_1, \dots, a_k) = 0$. From this we see that there is a monomorphism $\mathcal{S}(A) \to \mathcal{S}(A)(b_1, \dots, b_r)$, with $b_1, \dots, b_r \in \mathcal{S}(A)$.

The kernel of this map is $\mathcal{S}(A) \cap An_A(b_1) \cap \dots \cap An_A(b_r)$. Since all the $An_A(b_i)$ are essential left ideals in A, this intersection could not be zero, a contradiction. Hence A is left non-singular.

$(b) \Leftrightarrow (c)$ This is obvious by 14.1.

$(a) \Rightarrow (d)$ By 14.16, A is semiprime and left non-singular. Therefore $Q = \widehat{A}$ and $End_A(Q) = Q$. Now the assertion follows from 14.10.

$(d) \Rightarrow (a)$ We show that A is left non-singular. Assume for $a \in A$, $An_A(a)$ is an essential left ideal in A. Since $Q = RaQ \oplus \mathcal{T}_a(Q)$, we have $RaQ = eQ$, for some idempotent $e \in Q$. Write $e = \sum_{i=1}^n r_i a q_i$, with $r_i \in A, q_i \in Q$.

Obviously, $L = \bigcap_{i=1}^n (An_A(a) : r_i)_A$ is an essential left ideal in A and $Le = 0$. Therefore $L \cap Qe = 0$ and also $L \cap (Qe \cap A) = 0$. However, $L \subset A$ is an essential left ideal and $Qe \cap A \neq 0$, a contradiction. Hence A is left non-singular and so $Q = \widehat{A}$ and $End_A(Q) = Q$. Now apply 14.10.

$(a) \Leftrightarrow (e) \Leftrightarrow (f)$ follow from 14.17, 14.3 and 14.4. □

14.20 Left PSP rings.

For a ring A, the following conditions are equivalent:

(a) *A is a left PSP-ring;*

(b) *A is semiprime (and left non-singular) and for every finitely generated left ideal $N \subset A$, $An_A(N) = An_A(X)$, for some finite subset $X \subset N$.*

Proof. $(a) \Rightarrow (b)$ By 14.16, A is left non-singular. To show that A is semiprime, consider a cyclic left ideal $N \in A$ with $N^2 = 0$. By assumption, $A/\mathcal{T}_N(A) \in \sigma[N]$. Since $NK = 0$ for any $K \in \sigma[N]$, in particular $N(A/\mathcal{T}_N(A)) = 0$ and $NA \subset \mathcal{T}_N(A)$. This implies $NA \subset \mathcal{T}^N(A) \cap \mathcal{T}_N(A) = 0$ and $N = 0$. Now refer to 14.18.

$(b) \Rightarrow (a)$ also follows from 14.18. □

Now we consider a condition on modules which is closely related to regularity properties.

14.21 Pseudo regular modules.
Let M be a left A-module, $S = End_A(M)$, $T = End_A(\widehat{M})$ and $D = End_T(\widehat{M})$. Then the following conditions are equivalent:

(a) *For every $m \in M$, there exists $h \in D$, such that $hM \subset AmS$ and $hm = m$;*

(b) *for any (A,S)-submodule $N \subset M$ and $m_1, \ldots, m_k \in N$, there exists $h \in D$, such that $hM \subset N$ and $hm_i = m_i$, for all $i = 1, \ldots, k$.*

A module satisfying these conditions is called pseudo regular.

Proof. $(a) \Rightarrow (b)$ Let $N \subset M$ be an (A,S)-submodule of M and $m_1, \ldots, m_k \in N$. For $k = 1$ there is nothing to show.

Assume the assertion holds for any $k-1$ elements in N. Choose $f \in D$ such that $fm_k = m_k$ and $fM \subset Am_kS \subset N$.

Clearly $m_i - fm_i \in N$, for all $i = 1, \ldots, k-1$, and by hypothesis, there exists $g \in D$ with $gM \subset N$ and

$$g(m_i - fm_i) = m_i - fm_i, \text{ for } i = 1, \ldots, k-1.$$

Put $h = 1 - (1-g)(1-f)$. Then $hm = gm - gfm + fm \in N$, for all $m \in M$, which means $hM \subset N$, and for $i = 1, \ldots, k$,

$$hm_i = m_i - (1-g)(1-f)m_i = m_i - (1-g)(m_i - fm_i) = m_i.$$

$(b) \Rightarrow (a)$ is obvious. □

The next result shows what pseudo regularity means for rings.

14.22 Left fully idempotent rings.
A ring A is left fully idempotent if and only if the left A-module A is pseudo regular.

Proof. Recall $End_A(A) = A$. Assume A is left fully idempotent. Let N be an ideal of A and $m \in N$. Since $(Am)^2 = Am$, there exist $h \in AmA$ such that $hm = m$. Clearly $hA \subset N$ and $h \in A \subset Biend_A(\widehat{A})$. Hence ${}_AA$ is pseudo regular.

Now assume ${}_AA$ to be pseudo regular and $m \in A$. Then there exists $h \in Biend_A(\widehat{A})$ such that $hm = m$ and $hA \subset AmA$. Hence $r = h1 \in AmA$ and $rm = (h1)m = hm = m$. So $m = rm \in (Am)^2$, $(Am)^2 = Am$ and the ring A is left fully idempotent. □

It follows from the above observation that a commutative ring A is von Neumann regular if and only if ${}_AA$ is pseudo regular.

Of particular interest for our investigations is the following relationship:

14.23 Pseudo regular and PSP modules.

Let M be a left A-module, $S = End_A(M)$ and $T = End_A(\widehat{M})$. The following are equivalent:

(a) *For every $m \in M$, there is a central idempotent $e \in T$, with $AmS = Me$;*

(b) *for every $m \in M$, $M = AmS \oplus \mathcal{T}_m(M)$ and $\widehat{M} = AmT \oplus \mathcal{T}_m(\widehat{M})$;*

(c) *for every finitely generated submodule $N \subset M$, there is a central idempotent $e \in T$, with $NS = Me$;*

(d) *for every finitely generated submodule $N \subset M$, $M = NS \oplus \mathcal{T}_N(M)$ and $\widehat{M} = NT \oplus \mathcal{T}_N(\widehat{M})$;*

(e) *M is a pseudo regular PSP-module over A.*

Proof. Denote $D = End_T(\widehat{M})$.

$(a) \Rightarrow (b)$ For $m \in M$ choose a central idempotent $e \in T$, with $AmS = Me$. Since $\widehat{M} = MT$ we have

$$\begin{aligned}\widehat{M} &= MTe \oplus MT(1-e) = MeT \oplus MT(1-e)\\ &= AmST \oplus MT(1-e) = AmT \oplus MT(1-e).\end{aligned}$$

Hence AmT is M-injective and $MTe = AmT = \mathcal{T}_m(\widehat{M}) = I_M(\mathcal{T}_m(\widehat{M}))$ (see 11.10). Since $\widehat{M}$ is M-injective,

$$\begin{aligned}Hom_A(MT(1-e), I_M(\mathcal{T}_m(\widehat{M}))) &= \{t \in T \mid MT(1-e)t \subset I_M(\mathcal{T}_m(\widehat{M}))\}\\ &= \{t \in T \mid MT(1-e)t \subset MTe\} = 0.\end{aligned}$$

So $MT(1-e) \subset \mathcal{T}_m(\widehat{M})$. But $MTe = AmT$, $\widehat{M} = MTe \oplus MT(1-e)$ and

$$MTe \cap \mathcal{T}_m(\widehat{M}) = \mathcal{T}^m(\widehat{M}) \cap \mathcal{T}_m(\widehat{M}) = 0.$$

Hence $\mathcal{T}_m(\widehat{M}) = MT(1-e)$ and $\widehat{M} = AmT \oplus \mathcal{T}_m(\widehat{M})$. Clearly $AmS = Me = M \cap MTe$ and

$$\mathcal{T}_m(M) = M \cap \mathcal{T}_m(\widehat{M}) = M \cap MT(1-e) = M(1-e).$$

So $M = AmT \oplus \mathcal{T}_m(M)$.

$(b) \Rightarrow (a)$ Let e be the projection of $\widehat{M}$ onto AmT along $\mathcal{T}_m(\widehat{M})$. Since both submodules are fully invariant, e is a central idempotent of T. From $AmS \subset AmT$ and $\mathcal{T}_m(M) \subset \mathcal{T}_m(\widehat{M})$, we conclude $Me = AmS$.

$(c) \Leftrightarrow (d)$ can be shown with the above proof.

$(c) \Rightarrow (a)$ is obvious.

$(a) \Rightarrow (e)$ Since $M/\mathcal{T}_m(M) = AmS \in \sigma[Am]$, M is PSP.

Let π be the projection of $\widehat{M}$ onto AmT along $\mathcal{T}_m(\widehat{M})$. Obviously, $\pi \in D$. Since $AmS \subset AmT$ and $\mathcal{T}_m(M) \subset \mathcal{T}_m(\widehat{M})$, $\pi M = AmS$. Hence M is pseudo regular.

$(e) \Rightarrow (c)$ Let $N \subset M$ be a finitely generated submodule. Since M is PSP, $\widehat{M} = NT \oplus I_M(\mathcal{T}_N(\widehat{M}))$.

Suppose $I_M(\mathcal{T}_N(\widehat{M})) \neq \mathcal{T}_N(\widehat{M})$ and consider $x \in I_M(\mathcal{T}_N(\widehat{M})) \setminus \mathcal{T}_N(\widehat{M})$. Then $Hom_A(Ax, NT) \neq 0$ and - by M-injectivity of $\widehat{M}$ - there exists $t \in T$ with $0 \neq xt \in NT$. So $xt = \sum_{i=1}^n m_i s_i$, for some $m_i \in N$, $s_i \in T$. M being pseudo regular, there exists $h \in D$ such that $hM \subset NS$ and $hm_i = m_i$, for $i = 1, \ldots, n$. Clearly,

$$h(xt) = \sum_{i=1}^n h(m_i s_i) = \sum_{i=1}^n (hm_i)s_i = \sum_{i=1}^n m_i s_i = xt.$$

Since $\widehat{M} = MT$, $h\widehat{M} = (hM)T \subset NT$ and $hx \in NT$. Let π be the projection of $\widehat{M}$ onto $I_M(\mathcal{T}_N(\widehat{M}))$ along NT. Then $0 = (hx)\pi = h(x\pi) = hx$ and $xt = h(xt) = (hx)t = 0$, contradicting $xt \neq 0$. Hence $I_M(\mathcal{T}_N(\widehat{M}) = \mathcal{T}_N(\widehat{M})$ and $\widehat{M} = NT \oplus \mathcal{T}_N(\widehat{M})$.

Now consider $\alpha = 1 - \pi : \widehat{M} \to NT$ and $y \in M$. Then $y\alpha = \sum_{i=1}^n n_i t_i$, for some $m_i \in N$, $t_i \in T$. Since M is pseudo regular, there exists $h \in D$, such that $hM \subset NS$ and $hn_i = n_i$, for $i = 1, \ldots, n$. Clearly $h(y\alpha) = y\alpha$ and $hy \in NS \subset NT$. So $y\alpha = h(y\alpha) = (hy)\alpha = hy \in NS$ and

$$y - y\alpha \in \mathcal{T}_N(\widehat{M}) \cap M = \mathcal{T}_N(M).$$

Therefore $M = NS \oplus \mathcal{T}_N(M)$. □

14.24 Remarks. Semiprimeness conditions on modules.

Consider the following properties of an A-module M:

(*i*) $A/An_A(M)$ is cogenerated by every essential submodule of M;

(*ii*) for every $N \trianglelefteq M$, $M \in \sigma[N]$;

(*iii*) for every $N \trianglelefteq M$, $M \subset N^{(\Lambda)}$, for some set Λ;

(*iv*) for every $N \trianglelefteq M$, $A/An_A(M) \subset N^r$, for some $r \in \mathbb{N}$;

(*v*) M is cogenerated by every essential submodule of M;

(*vi*) M is polyform;

(*vii*) for every submodule $K \subset M$, $M/\mathcal{T}_K(M) \in \sigma[K]$;

(*viii*) for every cyclic submodule $K \subset M$, $M/\mathcal{T}_K(M) \in \sigma[K]$.

The conditions (*i*)-(*vi*) stated for *every* submodule were considered in 13.9. (*vii*) is used to define *strongly semiprime modules*, and (*viii*) defines *properly semiprime* *modules.* For any module M, $(iii) \Rightarrow (v) \Rightarrow (i)$, $(iii) \Rightarrow (ii) \Rightarrow (i)$, $(iv) \Rightarrow (i)$ and $(vii) \Rightarrow (ii)$.

For M projective in $\sigma[M]$, $(ii) \Leftrightarrow (iii)$ and (vii) is equivalent to M satisfying both (ii) and (vi) (cf. 14.16, 14.17).

For A commutative and $M = A$, any of the properties (i), (v), (vi) and $(viii)$ characterize A as a semiprime ring (hence $Q_{\max}(A)$ is regular). Properties (ii), (iii), (iv) and (vii) imply that $Q_{\max}(A)$ is a finite product of fields.

For $M = {}_AA$, the conditions (ii), (iii), (iv) and (vii) are equivalent and describe SSP rings (see 14.19). Left ideals $N \subset A$, which contain a finite subset X with $An_A(X) = 0$, are also called *insulated*. Hence A is left SSP if and only if every essential left ideal is insulated. So A is *left strongly semiprime ring* in the sense of Handelman [149, Theorem 1]. Such rings are also investigated in Kutami-Oshiro [182] and generalize *left strongly prime* rings (see 13.9).

14.25 Exercises.

Recall that A is an associative algebra with unit.

(1) The definitions of SSP and PSP modules M refer to the surrounding category. Nevertheless submodules of such modules are of the same type.

Let M be an A-module and $L \subset M$ a submodule. Prove: If M is SSP (resp. PSP) (in $\sigma[M]$), then L is also SSP (resp. PSP) (in $\sigma[L]$).

(2) An A-module M is called weakly semisimple *if it is polyform, every finitely generated submodule has finite uniform dimension, and for every non-zero submodule $N \subset M$, there exists $f \in Hom_A(M, N)$ with $f|_N \neq 0$ (weakly compressible).*

Let M be such a module. Prove ([285, 1.1]):

(i) For any $m \in M$ and $N \trianglelefteq M$, there exists a monomorphism $Am \to N \cap Am$.
(ii) M is strongly semiprime.

(3) Show that for a ring A, the following are equivalent ([285, 5.1, 5.2]):

(a) A has a faithful weakly semisimple left ideal (of finite uniform dimension);
(b) A is semiprime, left non-singular, with a faithful left ideal which contains a (finite) direct sum of uniform left ideals.

(4) A ring is said to be left fully idempotent *if every left ideal is idempotent. Show that for a ring A, the following are equivalent ([77, 4.2]):*

(a) A is biregular;
(b) A is a left fully idempotent left PSP ring;
(c) A is a left PSP ring whose prime factor rings are left fully idempotent;
(d) A is a left PSP ring with all prime ideals maximal.

(5) Let A be a reduced ring. Prove:

(i) For any subset $U \subset A$, the left annihilator $An_A(U)$ of U coincides with the right annihilator of U in A ([2, p. 286]).
(ii) A is left (and right) PSP.

(6) Let A be a reduced ring. Prove that the following are equivalent (see [77, 4.2], [81, Theorem 8]):

(a) A is biregular;
(b) A is left fully idempotent;
(c) all prime ideals in A are maximal.

(7) Let A be a reduced ring. Prove that the following are equivalent ([226]):
(a) $Q_{\max}(A)$ is reduced;
(b) closed left ideals in A are two-sided ideals;
(c) for non-zero elements $x, y \in A$, $Ax \cap Ay = 0$ implies $xy = 0 = yx$.

(8) Let A be a semiprime left non-singular ring. Prove that the following are equivalent ([182, Proposition 2.3]):

(a) $Q_{\max}(A)$ is a direct sum of simple rings;
(b) the set of central idempotents of $Q_{\max}(A)$ is finite;
(c) A contains no infinite direct sum of ideals.

(9) Show that for a ring A, the following are equivalent ([182, Theorem 2.5]):

(a) A is left SSP;
(b) $Q_{\max}(A)$ is a direct sum of simple rings and for every idempotent $e \in Q_{\max}(A)$,

$$AeQ_{\max}(A) = Q_{\max}(A)eQ_{\max}(A).$$

(10) Show that for a ring A, the following are equivalent ([182, Theorem 3.4]):

(a) $A/\mathcal{S}^2(A)$ is left SSP;
(b) every non-singular self-injective left A-module is injective;
(c) any finite direct sum of non-singular self-injective left A-modules is self-injective.

(11) Let A be a semiprime ring. Prove ([149, Lemma 7, Corollary 23]):

(i) If A has no infinite direct sums of ideals, then A satisfies acc and dcc on annihilators of ideals.
(ii) If A has dcc on left annihilators then A is left SSP.

References. Beidar-Wisbauer [75, 76, 77], Birkenmeier-Kim-Park [81], Faith [12], Handelman [149], Kutami-Oshiro [182], Wisbauer [273], Zelmanowitz [285].

Chapter 4

Tensor products

15 Tensor product of algebras

1.Tensor product of algebra morphisms. 2.Properties of the tensor product. 3.Universal property of the tensor product. 4.Scalar extensions. Definition. 5.Algebras over factor rings. 6.Hom-tensor relations for algebras. 7.Tensor product with modules. 8.Tensor product with an algebra. 9.Tensor product for morphisms of modules. 10.Multiplication algebras of tensor products. 11.Multiplication algebras of scalar extensions. 12.Hom-tensor relations for multiplication algebras. 13.Centroid of scalar extensions.

For the definition and basic properties of the tensor product of modules over associative rings we refer to [40, Section 12]. The reader may also consult Anderson-Fuller [1] or Faith [12], for example.

To avoid ambiguity we will sometimes write $f\underline{\otimes}g$ instead of $f \otimes g$ for the tensor product of two R-module morphisms f and g (see [40, 12.3].

Consider two R-algebras A and B, defined by the R-linear maps

$$\mu : A \otimes_R A \to A, \quad \nu : B \otimes_R B \to B.$$

Since factors in a tensor product over commutative rings can be permuted, we get the R-linear map

$$(A \otimes_R B) \otimes_R (A \otimes_R B) \simeq (A \otimes_R A) \otimes_R (B \otimes_R B) \xrightarrow{\mu\otimes\nu} A \otimes_R B\ ,$$

turning $A \otimes_R B$ into an R-algebra (sometimes just denoted by $A \otimes B$).

The multiplication of elements $a_1 \otimes b_1, a_2 \otimes b_2 \in A \otimes_R B$ is given by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1a_2 \otimes b_1b_2.$$

15.1 Tensor product of algebra morphisms.

Let $f : A \to A_1$ and $g : B \to B_1$ be R-algebra morphisms. Then:

(1) There is an algebra morphism

$$h : A \otimes_R B \to A_1 \otimes_R B_1,\ a \otimes b \mapsto f(a) \otimes g(b).$$

(2) *If f and g are surjective, then h is also surjective and*

$$A_1 \otimes_R B_1 \simeq (A \otimes_R B)/(Im(Ke\, f \otimes B) + Im(A \otimes Ke\, g)),$$

where $Im(Ke\, f \otimes B)$ and $Im(A \otimes Ke\, g)$ denote the images of $Ke\, f \otimes B \to A \otimes B$ and $A \otimes Ke\, g \to A \otimes B$.

(3) *If f and g are injective and*
(i) A_1 and B are flat R-modules, or
(ii) f and g are pure as R-module morphisms, or
(iii) A_1 is a flat R-module and f is a pure R-module morphism,
then h is injective.

Proof. (1) Put $h = f\underline{\otimes}g$, the tensor product of f and g as R-module morphism. It remains to show that h is a ring morphism:

$$\begin{aligned} h((a_1 \otimes b_1)(a_2 \otimes b_2)) &= f(a_1 a_2) \otimes g(b_1 b_2) \\ &= (f(a_1) \otimes g(b_1))(f(a_2) \otimes g(b_2)) \\ &= h(a_1 \otimes b_1) h(a_2 \otimes b_2). \end{aligned}$$

(2) Since tensor functors are right exact, we have the exact sequences

$$\begin{gathered} Ke\, f \otimes B \to A \otimes B \to A_1 \otimes B \to 0, \\ A \otimes Ke\, g \to A \otimes B \to A \otimes B_1 \to 0. \end{gathered}$$

Tensoring $A \to A_1$ with B_1 and $B \to B_1$ with A, we obtain the composed surjective algebra morphism

$$A \otimes B \to A \otimes B_1 \to A_1 \otimes B_1$$

with kernel $Im(Ke\, f \otimes B) + Im(A \otimes Ke\, g)$ (see [40, 12.19], [12, 11.3]).

(3) The map h can be decomposed into

$$f\underline{\otimes}id : A \otimes B \to A_1 \otimes B \text{ and } \quad id\underline{\otimes}g : A_1 \otimes B \to A_1 \otimes B_1.$$

Under the given conditions both mappings are injective (see [40, 34.5, 36.5]). □

For example, the conditions (i), (ii) and (iii) are always satisfied for algebras over (von Neumann) regular rings (e.g., [40, 37.6]).

15.2 Properties of the tensor product. *Let A and B be R-algebras.*

(1) If A has a unit e_A, then there is an algebra morphism

$$\varepsilon_B : B \to A \otimes_R B, \quad b \mapsto e_A \otimes b.$$

Assume A is a faithful R-module and, in addition, B is a flat R-module or Re_A is a pure R-submodule of A. Then ε_B is injective.

(2) If e_A and e_B are the units in A and B, then

(i) $e_A \otimes e_B$ is the unit in $A \otimes_R B$,

(ii) $\varepsilon_A(a)\varepsilon_B(b) = \varepsilon_B(b)\varepsilon_A(a)$, for all $a \in A$, $b \in B$, and

$$(\varepsilon_A(A), \varepsilon_B(B), A \otimes_R B) = (A \otimes_R B, \varepsilon_A(A), \varepsilon_B(B)) = 0.$$

(3) $A \otimes_R B \simeq B \otimes_R A$ and, for any R-algebra C,

$$A \otimes_R (B \otimes_R C) \simeq (A \otimes_R B) \otimes_R C.$$

(4) If A and B are associative algebras, then $A \otimes_R B$ is associative.

Proof. (1) It is easily verified that ε_B is an algebra morphism, obtained by tensoring the exact sequence $0 \to Re_A \to A$ with ${}_RB$. If this sequence is pure in R-*Mod* or if ${}_RB$ is flat, the resulting sequence is exact (see 15.1).

(2) (i) is obvious. (ii) By definiton, $\varepsilon_A(a)\varepsilon_B(b) = a \otimes b = \varepsilon_B(b)\varepsilon_A(a)$ and

$$\varepsilon_A(a_1)[\varepsilon_B(b_1)(a_2 \otimes b_2)] = a_1a_2 \otimes b_1b_2 = [\varepsilon_A(a_1)\varepsilon_B(b_1)](a_2 \otimes b_2).$$

(3) follows from commutativity and associativity of tensor products.

(4) is easily verified. □

From the above observations we obtain the

15.3 Universal property of the tensor product.
Let $f : A \to C$ and $g : B \to C$ be unital algebra morphisms such that

$$(f(A), g(B), f(A)g(B)) = 0 = (f(A)g(B), f(A), g(B)) \quad \text{and} \quad [f(A), g(B)] = 0.$$

Then there exists a unique algebra morphism $h : A \otimes_R B \to C$, satisfying

$$h(a \otimes b) = f(a)g(b)\,, \text{ for all } a \in A,\ b \in B.$$

In particular, with notation as in 15.2, $f = h\varepsilon_A$ and $g = h\varepsilon_B$.

Proof. Since f and g are R-module morphisms, the map

$$A \times B \to C, \quad (a,b) \mapsto f(a)g(b),$$

factorizes over $A \otimes_R B$ yielding an R-module morphism h with the desired properties. It remains to verify that h is an algebra morphism. By our assumptions on the associators and commutators of $f(A)$ and $g(B)$, we have

$$\begin{aligned} h[(a_1 \otimes b_1)(a_2 \otimes b_2)] &= f(a_1a_2)g(b_1b_2) &&= [f(a_1)f(a_2)][g(b_1)g(b_2)] \\ &= \{[f(a_1)g(b_1)]f(a_2)\}g(b_2) &&= [f(a_1)g(b_1)][f(a_2)g(b_2)] \\ &= h(a_1 \otimes b_1)h(a_2 \otimes b_2). \end{aligned}$$

□

As a special case, 15.3 implies that in the category of commutative associative unital R-algebras, the tensor product yields the coproduct of two algebras.

An associative commutative R-algebra with unit will be called a *scalar algebra* (over R). For such algebras we define:

15.4 Scalar extensions. Definition.
Let A be an R-algebra and S a scalar algebra over R. Then $A \otimes_R S$ becomes an S-module (by $(a \otimes s)t = a \otimes st$) and the map

$$(A \otimes_R S) \otimes_S (A \otimes_R S) \to A \otimes_R S, \quad (a \otimes s)(b \otimes t) \mapsto ab \otimes st,$$

is in fact an S-linear map, making $A \otimes_R S$ an S-algebra, called the *scalar extension of A by S.*

Of course, for every ideal $I \subset R$ the factor ring R/I is a scalar algebra over R. For this special situation we state:

15.5 Algebras over factor rings. *Let A and B be R-algebras.*

(1) For every ideal $I \subset R$, the product IA is an ideal in A, and

$$A/IA \simeq A \otimes_R (R/I) \quad \textit{as } R/I\textit{-algebras.}$$

(2) For any ideal $I \subset R$ with $IB = 0$,

$$A \otimes_R B \simeq (A/IA) \otimes_{R/I} B \qquad \textit{as } R/I\textit{-algebras.}$$

(3) For any ideals $I, J \subset R$,

$$R/I \otimes_R R/J \simeq R/(I+J).$$

Proof. This is readily derived from 15.1(2). □

We state some relations between *Hom* and tensor products for later use.

15.6 Hom-tensor relations for algebras.

Let A and B be associative unital R-algebras, M, L left A-modules, N a left B-module, and X a left $A \otimes_R B$-module.

With the canonical maps $\varepsilon_B : B \to A \otimes_R B$ and $\varepsilon_A : A \to A \otimes_R B$ (see 15.2) we consider X as left A- and B-module. Then there are canonical isomorphisms

$$\begin{array}{rl} (i) & Hom_{A\otimes B}(M \otimes_R N, X) \to Hom_B(N, Hom_A(M, X)), \\ (ii) & Hom_{A\otimes B}(M \otimes_R B, X) \to Hom_A(M, X), \\ (iii) & Hom_{A\otimes A^o}(A, Hom_R(M, L)) \to Hom_A(M, L). \end{array}$$

Proof. (i) Considering M as an (A, R)-bimodule, the ordinary Hom-tensor isomorphism (e.g., [40, 12.12]) is

$$Hom_A(M \otimes_R N, X) \to Hom_R(N, Hom_A(M, X)), \quad \delta \mapsto [n \mapsto (- \otimes n)\delta],$$

with inverse $\varphi \mapsto [m \otimes n \mapsto (m)(n)\varphi]$.

The $A \otimes B$-module morphisms and the B-module morphisms are subsets of the corresponding R-module morphisms. Hence for the first isomorphism it is only to confirm that the restriction of this isomorphism in fact maps the subsets considered above into each other.

(ii) This isomorphism is obtained from (i) for $N = B$.
With the map $\alpha : M \to M \otimes_R B,\ m \mapsto m \otimes 1_B$, it can be described by $h \mapsto \alpha h$.

(iii) Writing morphisms to the right, $Hom_R(M, L)$ is an A-bimodule by multiplying $f \in Hom_R(M, L)$ in the usual way,

$$[m](afb) = a([bm]f)\,, \quad \text{for } a, b \in A,\ m \in M.$$

Under the canonical isomorphism

$$Hom_A(A, Hom_R(M, L)) \to Hom_R(M, L),\ \varphi \mapsto (1)\varphi,$$

the elements $\psi \in Hom_{A\otimes A^o}(A, Hom_R(M, L))$ are mapped to $Hom_A(M, L)$:

Putting $\psi' = (1)\psi$ we have $a\psi' = \psi' a$, for any $a \in A$, and hence

$$(am)\psi' = (m)\psi' a = (m)a\psi' = a(m\psi'),$$

for all $m \in M$, i.e., $\psi' \in Hom_A(M, L)$. □

Next we observe that some projectivity properties guarantee further Hom-tensor isomorphisms:

15.7 Tensor product with modules.

Let A be an associative unital R-algebra, M and M' left A-modules and Q an R-module. Consider the R-linear map

$$\nu_M : Hom_A(M, M') \otimes_R Q \to Hom_A(M, M' \otimes_R Q), \quad [h \otimes q \mapsto (-)h \otimes q].$$

(1) If Q is a flat R-module and M a finitely generated (finitely presented) A-module, then ν_M is injective (an isomorphism).

(2) ν_M is also an isomorphism in the following cases:

- *(i) M is a finitely generated, M'-projective A-module, or*
- *(ii) M is M'-projective and Q is a finitely presented R-module, or*
- *(iii) Q is a finitely generated projective R-module.*

Proof. (1) It is easy to check that ν_M is an isomorphism for $M = A$ and $M = A^k$, $k \in I\!N$. Since ${}_AM$ is finitely generated, there exists an exact sequence of A-modules $A^{(\Lambda)} \to A^n \to M \to 0$, with Λ an index set, $n \in I\!N$.

The functors $Hom_A(-, M') \otimes_R Q$ and $Hom_A(-, M' \otimes_R Q)$ yield the exact commutative diagram

$$\begin{array}{ccccc}
0 \to Hom_A(M, M') \otimes_R Q & \to Hom_A(A^n, M') \otimes_R Q & \to Hom_A(A^{(\Lambda)}, M') \otimes_R Q \\
\downarrow \nu_M & \downarrow \nu_{A^n} & \downarrow \nu_{A^{(\Lambda)}} \\
0 \to Hom_A(M, M' \otimes_R Q) & \to Hom_A(A^n, M' \otimes_R Q) & \to Hom_A(A^{(\Lambda)}, M' \otimes_R Q).
\end{array}$$

Since ν_{A^n} is an isomorphism, ν_M has to be injective.

If M is finitely presented we can choose Λ to be finite. Then also $\nu_{A^{(\Lambda)}}$ and ν_M are isomorphisms.

(2)(i) From the exact sequence of R-modules $0 \to K \to R^{(\Lambda)} \to Q \to 0$, we construct the commutative diagram with the upper line exact,

$$\begin{array}{ccccccc}
Hom_A(M, M') \otimes_R K & \to & Hom_A(M, M') \otimes_R R^{(\Lambda)} & \to & Hom_A(M, M') \otimes_R Q & \to 0 \\
\downarrow \nu & & \downarrow \simeq & & \downarrow \nu_M & \\
Hom_A(M, M' \otimes_R K) & \to & Hom_A(M, M' \otimes_R R^{(\Lambda)}) & \to & Hom_A(M, M' \otimes_R Q) & \to 0 ,
\end{array}$$

where ν is defined as above replacing Q by K.

Since M is $M' \otimes_R R^{(\Lambda)}$-projective (see 5.1), the lower sequence is also exact and hence ν_M is surjective. By the same argument we obtain that ν is surjective. Now it follows from the Kernel Cokernel Lemma (e.g., [40, 7.15]) that ν_M is injective.

(ii) This statement is obtained from the proof of (1), with Λ a finite set and K a finitely generated R-module.

(iii) The assertion is obvious for $Q = R$ and is easily extended to finitely generated free (projective) modules Q. $\square$

Combining the preceding observations we get

15.8 Tensor product with an algebra.
Let A and B be associative unital R-algebras, M and M' left A-modules. Consider the map

$$Hom_A(M, M') \otimes_R B \to Hom_{A\otimes B}(M \otimes_R B, M' \otimes_R B),\ f \otimes b \mapsto [m \otimes b' \mapsto (m)f \otimes b'b].$$

(1) *If B is a flat R-module and M is a finitely generated (finitely presented) A-module, then the map is injective (an isomorphism).*

(2) *The map is also an isomorphism if*

  (i) *M is a finitely generated, M'-projective A-module, or*
  (ii) *M is M'-projective and B is a finitely presented R-module, or*
  (iii) *B is a finitely generated projective R-module.*

Proof. The map is the composition of maps in 15.7 and 15.6,

$$\begin{array}{rcl} Hom_A(M, M') \otimes_R B & \to & Hom_A(M, M' \otimes_R B) \text{ and} \\ Hom_A(M, M' \otimes_R B) & \to & Hom_{A\otimes B}(M \otimes_R B, M' \otimes_R B). \end{array}$$

□

15.9 Tensor product for morphisms of modules.
Let A and B be associative unital R-algebras, M, M' left A-modules and N, N' left B-modules.

(1) *For $f \in Hom_A(M, M')$ and $g \in Hom_B(N, N')$,*

$$f \underline{\otimes} g \in Hom_{A\otimes B}(M \otimes_R N, M' \otimes_R N').$$

(2) *The mapping $(f, g) \mapsto f \underline{\otimes} g$ induces an R-module morphism*

$$\psi : Hom_A(M, M') \otimes_R Hom_B(N, N') \to Hom_{A\otimes B}(M \otimes_R N, M' \otimes_R N').$$

Assume M and N are finitely generated. Then ψ is an isomorphism if

  (i) *M is M'-projective and N is N'-projective, or*
  (ii) *M and N are projective as A-, resp. B-modules, or*
  (iii) *M is a finitely presented A-module, N and N' are finitely generated, projective B-modules, and B is a flat R-module.*

(3) *$\psi : End_A(M) \otimes_R End_B(N) \to End_{A\otimes B}(M \otimes_R N)$ is an algebra morphism.*

Proof. (1) Just verify that $f \underline{\otimes} g$ is in fact an $A \otimes B$-module morphism.

(2) ψ is well-defined since $(f, g) \mapsto f \underline{\otimes} g$ yields an R-bilinear map.

(i) By 15.6(i) and 15.7$(2.i)$, we have isomorphisms

$$\begin{aligned} Hom_{A\otimes B}(M \otimes N, M' \otimes N') &\simeq Hom_B(N, Hom_A(M, M' \otimes N')) \\ &\simeq Hom_B(N, Hom_A(M, M') \otimes N') \\ &\simeq Hom_A(M, M') \otimes Hom_B(N, N') \,. \end{aligned}$$

(ii) is a special case of (i).

(iii) Since $B \simeq Hom_B(B, B)$ we know from 15.8 that ψ is an isomorphism for $N = N' = B$. Similar to the above argument, this isomorphism can be extended to finitely generated free and projective modules N and N'.

(3) is easily verified. $\square$

Now let us apply the preceding observations to arbitrary algebras as modules over their multiplication algebras.

15.10 Multiplication algebras of tensor products.

Let A and B be R-algebras. With ε denoting the inclusions, the maps defined in 15.1 and 15.9 yield the algebra morphisms

$$\begin{array}{ccc} M(A) \otimes M(B) & & M(A \otimes B) \\ \varepsilon_A \otimes \varepsilon_B \downarrow & & \downarrow \varepsilon_{A\otimes B} \\ End_R(A) \otimes End_R(B) & \xrightarrow{\psi} & End_R(A \otimes B) \end{array} .$$

(1) $Im\, \varepsilon_{A\otimes B} \subset Im\, \psi(\varepsilon_A \otimes \varepsilon_B)$.

(2) $\varepsilon_A \otimes \varepsilon_B$ is injective if one of the following conditions holds:
- *(i) $End(A)$ and $M(B)$ are flat R-modules;*
- *(ii) $M(A) \subset End_R(A)$ and $M(B) \subset End_R(B)$ are pure R-submodules;*
- *(iii) $End(A)$ is a flat R-module and $M(A) \subset End_R(A)$ a pure R-submodule.*

(3) If A and B are finitely generated projective R-modules, then ψ is an isomorphism.

(4) If A and B have units, the diagram is extended commutatively by the surjective algebra morphism

$$h : M(A) \otimes_R M(B) \to M(A \otimes_R B), \quad \mu \otimes \nu \mapsto \mu \underline{\otimes} \nu.$$

The kernel of h is the annihilator of $A \otimes_R B$ in $M(A) \otimes_R M(B)$.

Proof. (1) This follows by the definitions.

(2) is an application of 15.1(3), and (3) is shown in 15.9.

(4) The map is well defined (compare 15.3). Consider $a \otimes b \in A \otimes_R B$ and $L_{a\otimes b} \in M(A \otimes_R B)$. Then h maps $L_a \otimes L_b \in M(A) \otimes_R M(B)$ to $L_a \underline{\otimes} L_b = L_{a\otimes b}$. Similar arguments apply to any right multiplication in $A \otimes_R B$ and hence h is surjective. □

With the same kind of proof we obtain:

15.11 Multiplication algebras of scalar extensions.
Let A be an algebra and S a scalar algebra over R. With ε denoting inclusions, the maps defined in 15.1 and 15.9 yield the commutative diagram of algebra morphisms

$$\begin{array}{ccc} M(A) \otimes_R S & \xrightarrow{\varphi} & M(A \otimes_R S) \\ \varepsilon_A \otimes id_S \downarrow & & \downarrow \varepsilon_{A\otimes S} \\ End_R(A) \otimes_R S & \xrightarrow{\psi} & End_S(A \otimes_R S) \end{array} ,$$

where φ, defined by $\mu \otimes s \mapsto \mu \underline{\otimes} L_s$, is surjective.
The kernel of φ is the annihilator of $A \otimes_R S$ in $M(A) \otimes_R S$.

(1) $\varepsilon_A \otimes id_S$ is injective if
  (i) S is a flat R-module, or
  (ii) $M(A) \subset End_R(A)$ is a pure R-submodule.

(2) If A is a finitely generated, projective R-module, then ψ is an isomorphism.

(3) If S is a flat R-module and A is a finitely generated R-module, then ψ is injective and φ is an isomorphism.

(4) If S is a flat R-module and A is a finitely presented R-module, then ψ and φ are isomorphisms.

Proof. The assertions about φ are easy to verify.
(1) is an application of 15.1(3), and (2) is shown in 15.9.
(3) and (4) follow from 15.8. □

As a special case of the Hom-tensor isomorphism in 15.6 we have:

15.12 Hom-tensor relations for multiplication algebras.
Let A and B be R-algebras with multiplication algebras $M(A)$, $M(B)$, and X a left $M(A) \otimes_R M(B)$-module. Then there is an isomorphism

$$Hom_{M(A)\otimes M(B)}(A \otimes_R B, X) \simeq Hom_{M(B)}(B, Hom_{M(A)}(A, X)).$$

If A and B are unital, then in fact (observe 15.10(4)),

$$Hom_{M(A\otimes B)}(A \otimes_R B, X) \simeq Hom_{M(B)}(B, Hom_{M(A)}(A, X)).$$

In case $B = S$ is a scalar algebra, we have

$$Hom_{M(A\otimes S)}(A \otimes_R S, X) \simeq Hom_{M(A)}(A, X).$$

From the preceding results we get some information about the centroids of scalar extensions:

15.13 Centroid of scalar extensions.
Let A be an algebra and S a scalar algebra over R. Then there is an algebra morphism

$$C(A) \otimes_R S \to C(A \otimes_R S), \quad \gamma \otimes s \mapsto (-)\gamma \otimes s.$$

(1) If S is a flat R-module and A is a finitely generated (finitely presented) $M(A)$-module, then the map is injective (an isomorphism).

(2) The map is also an isomorphism if

(i) A is a finitely generated, self-projective $M(A)$-module, or

(ii) S is a finitely generated, projective R-module.

Proof. Since $C(A \otimes_R S) = End_{M(A\otimes_R S)}(A \otimes_R S)$, the statements follow from 15.8 and 15.11. □

References: Anderson-Fuller [1], DeMeyer-Ingraham [10], Faith [12], Pierce [33], Wisbauer [40, 276].

16 Modules and rings of fractions

1.Faithfully flat modules. 2.Properties of the ring of fractions. 3.Properties of the module of fractions. 4.Algebras of fractions. 5.Modules over algebras of fractions. 6.Special algebras of fractions.

In this section we study the forming of quotient modules and quotient algebras with respect to multiplicative subsets of R. Since this is closely related to tensoring with quotient rings of R, we recall some properties of tensor functors for the category of R-modules.

Recall that an R-module U is *faithfully flat* if U_R is flat and, for any R-module N, $U \otimes_R N = 0$ implies $N = 0$. The following is shown in [40, 12.17]:

16.1 Faithfully flat modules.

For an R-module U, the following assertions are equivalent:

(a) U is faithfully flat;

(b) U is flat and $U \otimes_R R/I \neq 0$, for every proper (maximal) ideal $I \subset R$;

(c) U is flat and $UI \neq U$, for every proper (maximal) ideal $I \subset R$;

(d) the functor $U \otimes_R - : R\text{-Mod} \to \mathbb{Z}\text{-Mod}$

(i) is exact and reflects zero morphisms, or

(ii) preserves and reflects exact sequences.

An important class of flat R-modules is obtained by forming the ring of fractions with respect to multiplicative subsets $S \subset R$ containing 1:

On the set $R \times S$ an equivalence relation is given by

$$(a, s) \sim (b, t) \text{ if } (at - bs)s' = 0, \text{ for some } s' \in S.$$

For the equivalence classes of $(a, s) \in R \times S$, denoted by $[a, s]$, we define addition and multiplication by

$$[a, s] + [b, t] = [at + bs, st], \quad [a, s][b, t] = [ab, st].$$

This yields a ring denoted by RS^{-1}, called the *ring of fractions of R* with respect to S. There is a ring morphism

$$R \to RS^{-1}, \ r \mapsto [r, 1], \text{ with kernel } \{r \in R \mid rs = 0, \text{ for some } s \in S\}.$$

16.2 Properties of the ring of fractions. *We use the notation above.*

(1) RS^{-1} is a flat R-module.

(2) Every ideal of RS^{-1} is of the form IRS^{-1}, for some ideal $I \subset R$.

(3) IRS^{-1} *is a prime ideal in* RS^{-1} *if and only if* I *is a prime ideal in* R *with* $I \cap S = \emptyset$.

(4) If R *is artinian or noetherian, then* RS^{-1} *has the same property.*

(5) For any ideal $I \subset R$ *and* $\pi : R \to R/I$, $RS^{-1}/IRS^{-1} \simeq (R/I)\pi(S)^{-1}$.

Proof. (1) It is to show that $I \otimes RS^{-1} \to IRS^{-1}, a \otimes [1, s] \mapsto [a, s]$, is injective for any ideal $I \subset R$ (e.g., [40, 12.16]). This is a special case of an isomorphism which will be obtained in 16.3(1).

(2) For any ideal $L \subset RS^{-1}$, consider $I = \{a \in R \,|\, [a, 1] \in L\}$.
Then obviously $IRS^{-1} \subset L$. For $[a, s] \in L$, $[1, s][a, 1] \in IRS^{-1}$ and so $IRS^{-1} = L$.

(3) For a prime ideal $P \subset RS^{-1}$, $p = \{a \in R \,|\, [a, 1] \in P\}$ is a prime ideal in R and $P = pRS^{-1}$. Since P contains no invertible elements of RS^{-1}, $p \cap S = \emptyset$.

Now let p be a prime ideal of R with $p \cap S = \emptyset$ and $[a, s][b, t] \in pRS^{-1}$. Then $[ab, 1][1, st][st, 1] \in pRS^{-1}$ and $ab \in p$, implying $a \in p$ or $b \in p$. Hence $[a, s]$ or $[b, t] \in pRS^{-1}$ and pRS^{-1} is a prime ideal.

(4) This is an obvious consequence of (2).

(5) The isomorphism is given by

$$RS^{-1}/IRS^{-1} \to (R/I)\pi(S)^{-1},\ [a, s] + IRS^{-1} \mapsto [a + I, s + I]. \qquad \square$$

For any R-module M and a multiplicative subset $S \subset R$ containing 1, we define a *module of fractions* MS^{-1} in the following way:
On the set $M \times S$ consider the equivalence relation

$$(m, s) \sim (n, t) \text{ if } (mt - ns)s' = 0, \text{ for some } s' \in S.$$

On the set of equivalence classes, denoted by $[m, s]$, addition and scalar multiplication are defined by

$$[m, s] + [n, t] = [mt + ns, st], \quad [a, s][m, t] = [am, st].$$

This makes MS^{-1} an RS^{-1}-module, and there is a canonical R-linear map

$$M \to MS^{-1},\ m \mapsto [m, 1],\ \text{with kernel } \{m \in M \mid ms = 0 \text{ for some } s \in S\}.$$

16.3 Properties of the module of fractions.

Let M, N *be* R*-modules. We use the above notation.*

(1) $M \otimes_R RS^{-1} \simeq MS^{-1}$.

(2) If M *is a flat* R*-module, then* MS^{-1} *is a flat* RS^{-1}*-module.*

(3) *Every RS^{-1}-submodule of MS^{-1} is of the form KS^{-1}, for some R-submodule $K \subset M$.*

(4) *If M is a finitely generated R-module, then MS^{-1} is a finitely generated RS^{-1}-module.*

(5) *If M is a noetherian (artinian) R-module, then MS^{-1} is a noetherian (artinian) RS^{-1}-module.*

(6) *A finite subset $\{m_1, \ldots, m_k\} \subset M$ is in the kernel of $M \to MS^{-1}$ if and only if there exists $s \in S$ with $sm_i = 0$, for $i = 1, \ldots, k$.*

(7) $MS^{-1} \otimes_R N \simeq MS^{-1} \otimes_{RS^{-1}} NS^{-1} \simeq (M \otimes_R N)S^{-1}$.

(8) *For an RS^{-1}-module L, the map $L \to LS^{-1}$, $l \mapsto [l, 1]$ is an isomorphism and $Hom_R(M, L) \simeq Hom_{RS^{-1}}(MS^{-1}, L)$.*

(9) *For RS^{-1}-modules L, L', we have $L \otimes_R L' \simeq L \otimes_{RS^{-1}} L'$ and $Hom_R(L, L') \simeq Hom_{RS^{-1}}(L, L')$.*

(10) *For any submodule $N \subset M$, $MS^{-1}/NS^{-1} \simeq (M/N)S^{-1}$.*

Proof. (1) The map $M \times RS^{-1} \to MS^{-1}, (m, [r, s]) \mapsto [rm, s]$, is R-bilinear and hence yields an R-linear map $M \otimes_R RS^{-1} \to MS^{-1}$, which is in fact RS^{-1}-linear.

Consider the assignment $MS^{-1} \to M \otimes_R RS^{-1}, [m, s] \mapsto m \otimes [1, s]$.
If $[m, s] = [m', s']$, then $ts'm = tsm'$ for some $t \in S$ and

$$m \otimes [1, s] = m \otimes [ts', tss'] = ts'm \otimes [1, tss'] = tsm' \otimes [1, tss'] = m' \otimes [1, s'].$$

Hence our assignment defines in fact an inverse to the above mapping.

(2) Since RS^{-1} is a flat R-module by 16.2, this follows from (1).

(3) For any RS^{-1}-submodule $L \subset MS^{-1}$, consider

$$K = \{m \in M \mid [m, 1] \in L\} \subset M.$$

K is an R-submodule and obviously $KS^{-1} \subset L$. For $[l, s] \in L$ we observe $[l, s] = [1, s][l, 1] \in KS^{-1}$ and hence $KS^{-1} = L$.

(4) and (5) are immediate consequences of (3).

(6) Under the given conditions, there exist $s_i \in S$ with $s_i m_i = 0$, and $s := s_1 \cdots s_k$ has the property demanded.

(7) This follows from (1) and basic properties of the tensor product.

(8) Assume $[l, 1] = 0$. Then there exists $s \in S$ with $sl = 0$ which implies $l = [1, s][s, 1]l = 0$. Hence the map is injective.

Consider $[l, s] \in LS^{-1}$. The image in LS^{-1} of $[1, s]l \in L$ is in fact $[l, s]$ and the map is surjective.

Referring to (1), the second isomorphism is obtained from 15.6.

(9) These isomorphisms are derived from (7) and (8).

(10) Since $-\otimes_R RS^{-1}$ is an exact functor this follows from (1). □

Applied to R-algebras, the above techniques again yield algebras and we want to consider some basic relations between an algebra and its algebra of fractions. Recall that an algebra is said to be *reduced* if it contains no non-zero nilpotent elements.

16.4 Algebras of fractions.

Assume A is an R-algebra and $S \subset R$ is a multiplicative subset. Then:

(1) $AS^{-1} \simeq A \otimes_R RS^{-1}$ is an RS^{-1}-algebra.

(2) Every (left) ideal of AS^{-1} is of the form KS^{-1}, for some (left) ideal $K \subset A$.

(3) The canonical map $A \to AS^{-1}$, $a \mapsto [a, 1]$, is an R-algebra morphism.

(4) If A is commutative, associative, alternative, power-associative or a Jordan algebra, then AS^{-1} is of the same type.

(5) Assume A is a power-associative algebra. Then:
(i) For any nil (left) ideal $N \subset A$, NS^{-1} is a nil (left) ideal in AS^{-1}.
(ii) Every nilpotent element $[a, s] \in AS^{-1}$ is of the form $[a, s] = [ta, ts]$, for some $t \in S$ and $ta \in A$ nilpotent.
(iii) If A is reduced then AS^{-1} is reduced.

Proof. (1) AS^{-1} is a scalar extension of A by RS^{-1} (see 15.4).

(2) This can be seen with the proof of (3) in 16.3.

(3),(4) and (5)(i) are obvious.

(5)(ii) Assume for $[a, s] \in AS^{-1}$ and $n \in \mathbb{N}$, we have $[a, s]^n = [a^n, s^n] = 0$. Then $ta^n = 0$, for some $t \in S$, and $(ta)^n = 0$. Obviously $[a, s] = [ta, ts]$.

(5)(iii) This is immediate from (ii). □

If A is an associative R-algebra the above constructions can also be extended to unital A-modules and with the proofs of 16.3 we obtain:

16.5 Modules over algebras of fractions.

Let A be an associative R-algebra with unit, S a multiplicative subset of R, and M a unital left A-module. Considering M as an R-module we form the module of fractions MS^{-1}. Then:

(1) $MS^{-1} \simeq M \otimes_R RS^{-1} \simeq M \otimes_A AS^{-1}$, and for any right A-module N,
$N \otimes_A MS^{-1} \simeq NS^{-1} \otimes_{AS^{-1}} MS^{-1}$.

(2) *If M is a flat A-module, then MS^{-1} is a flat AS^{-1}-module.*

(3) *Every AS^{-1}-submodule of MS^{-1} is of the form KS^{-1}, for some A-submodule $K \subset M$.*

(4) *For any submodules $U, V \subset M$, $(U \cap V)S^{-1} = US^{-1} \cap VS^{-1}$.*

(5) *If M is a finitely generated A-module, then MS^{-1} is a finitely generated AS^{-1}-module.*

(6) *If M is a noetherian (artinian) A-module, then MS^{-1} is a noetherian (artinian) AS^{-1}-module.*

(7) *If U is a direct summand in M, then US^{-1} is a direct summand in MS^{-1}.*

(8) *If every (finitely generated) A-submodule of M is a direct summand, then every (finitely generated) AS^{-1}-submodule of MS^{-1} is a direct summand.*

As an application of the above observations we state:

16.6 Special algebras of fractions. *Let A be an R-algebra.*

(1) *If A is alternative and (strongly) regular, then AS^{-1} is (strongly) regular.*

(2) *If every (finitely generated) left ideal of A is a direct summand, then every (finitely generated) left ideal of AS^{-1} is a direct summand.*

(3) *If every (finitely generated) ideal of A is a direct summand, then every (finitely generated) ideal of AS^{-1} is a direct summand.*

Proof. (1) Assume A is regular. For any $[a, s] \in AS^{-1}$, there exists $b \in A$ with $a = aba$, and we have $[a, s][bs, 1][a, s] = [a, s]$, i.e., AS^{-1} is regular.

If A is reduced, then AS^{-1} is reduced by 16.4.

(2) Left ideals of A are $L(A)$-submodules of A, where $L(A)$ denotes the left multiplication algebra of A (see 2.1). Similar to the construction in 15.11, we obtain a surjective ring morphism $L(A) \otimes_R RS^{-1} \to L(AS^{-1})$ and we see that left ideals of AS^{-1} are exactly its $L(A) \otimes_R RS^{-1}$-submodules.

Now the assertion follows from 16.5(7).

(3) This is obtained by the same proof as (2). □

References: Bourbaki [7], Matsumura [25].

Chapter 5

Local-global techniques

In studying R-algebras A, the properties of the ring R may be of importance. In some instances it is of advantage to change to other base rings with "better" properties by scalar extensions and then try to win insight about the algebra A over R. For example, one can try to transfer problems to algebras over local rings. If they are solvable in this special case one can try to extend the solution to the general case. These procedures are called *local-global techniques.* This reduction process is based on the rings of quotients introduced in the preceding chapter.

17 Localization at prime ideals

1.Localization at prime ideals. 2.Morphisms and localization. 3.Reduction to fields. 4.Endomorphisms of finitely generated modules. 5.R as a direct summand. 6.Central idempotents and nilpotent elements. 7.Localization over regular rings. 8.Tensor product over regular rings. 9.Associative regular algebras. 10.Associative strongly regular algebras. 11.Exercises.

As special types of multiplicative subsets of R we can take $S = R \setminus p$ for any prime ideal $p \subset R$. Then the ring of fractions RS^{-1} is denoted by R_p and is called the *localization of R at p.*

We denote by $\mathcal{P}$ the set of all prime ideals of R (the *prime spectrum*), and by $\mathcal{M}$ the set of all maximal ideals of R (the *maximal spectrum*).

17.1 Localization at prime ideals. *We use the above notation.*

(1) *For every $p \in \mathcal{P}$, R_p is a local ring with maximal ideal pR_p, and $R_p/pR_p \simeq Q(R/p)$, the quotient field of R/p.*
The prime ideals in R_p correspond to the prime ideals of R contained in p.

(2) *For every $m \in \mathcal{M}$, R_m is a local ring with maximal ideal mR_m and $R_m/mR_m \simeq R/m$.*

(3) *$F_{\mathcal{P}} = \bigoplus_{p\in\mathcal{P}} R_p$ and $F_{\mathcal{M}} = \bigoplus_{m\in\mathcal{M}} R_m$ are faithfully flat R-modules.*

(4) *For any R-module M, we have a canonical monomorphism*

$$M \to \prod\nolimits_{\mathcal{M}} M_m,\ k \mapsto ([k,1]_m).$$

Proof. (1) and (2) follow from 16.2.

(3) The R_p are flat by 16.2 and direct sums of flat modules are again flat ([40, 36.1]). By (2), for every maximal ideal $m \subset R$, $mR_m \neq R_m$. Hence $mF_{\mathcal{M}} \neq F_{\mathcal{M}}$ and $F_{\mathcal{M}}$ is faithfully flat by 16.1.

The same argument applies to $F_{\mathcal{P}}$.

(4) For the kernel K of this map, we have $K \otimes_R R_m = 0$ for all $m \in \mathcal{M}$, hence $K = 0$ by (3). □

From 17.1(3) we immediately deduce the behaviour of morphisms:

17.2 Morphisms and localization.
Let $f : M \to N$ be a morphism of R-modules.

(1) f is monic if and only if, for every $m \in \mathcal{M}$, $f_m : M \otimes R_m \to N \otimes R_m$ is monic.

(2) f is epic if and only if, for every $m \in \mathcal{M}$, f_m is epic.

(3) f is an isomorphism if and only if, for every $m \in \mathcal{M}$, f_m is an isomorphism.

In some cases the test of a module to be zero can also be achieved by tensoring with certain factor rings:

17.3 Reduction to fields. *Let M and N be R-modules.*

(1) Assume R is a local ring with maximal ideal m and
M is finitely generated, or m is a t-nilpotent ideal.
Then $M \otimes_R R/m = 0$ implies $M = 0$.

(2) Assume M is finitely generated, or mR_m is t-nilpotent for every $m \in \mathcal{M}$.
(i) Then $M \otimes_R R/m = 0$ for all $m \in \mathcal{M}$ implies $M = 0$.
(ii) For $f : N \to M$ the following are equivalent:
(a) f is surjective;
(b) for every $m \in \mathcal{M}$, $f \otimes id : N \otimes_R R/m \to M \otimes_R R/m$ is surjective.

Proof. (1) Assume $M \otimes R/m = 0$, i.e., $M = mM$. Since R is local, m is the Jacobson radical of R. If M is finitely generated, the Nakayama Lemma ([40, 21.13]) implies $M = 0$. If m is a t-nilpotent ideal, then $mM = M$ is only possible for $M = 0$ (e.g., [40, 43.5]).

(2)(i) Assume $M \otimes R/m = 0$ for all $m \in \mathcal{M}$. By $R/m \simeq R_m/mR_m$ (see 17.1), $M \otimes_R R_m/mR_m = 0$. Since R_m is a local ring with maximal ideal mR_m, we know from (1) that $M \otimes_R R_m = 0$ for all $m \in \mathcal{M}$. By 17.1(3), this implies $M = 0$.

(ii) (a) $\Rightarrow$ (b) is obvious.

$(b) \Rightarrow (a)$ Put $L = M/Nf$. The functor $- \otimes_R R/m$ yields the exact sequence

$$N \otimes_R R/m \xrightarrow{f \otimes id} M \otimes_R R/m \longrightarrow L \otimes_R R/m \longrightarrow 0.$$

Since all $f \otimes id$ are surjective, we obtain $L \otimes R/m = 0$ for all $m \in \mathcal{M}$ and $L = 0$ by (2). Hence f is surjective. □

The next proposition collects characterizations of automorphisms of finitely generated modules:

17.4 Endomorphisms of finitely generated modules.

Let M be a finitely generated R-module. Then:

(1) If I is an ideal in R with $IM{=}M$, then $(1 - r)M = 0$ for some $r \in I$.

(2) Every surjective endomorphism of M is an isomorphism.

(3) For $f \in End_R(M)$, the following are equivalent:

(a) f is an isomorphism;
(b) f is an epimorphism;
(c) f is a pure monomorphism;
(d) for every $m \in \mathcal{M}$, $f \otimes id : M \otimes R/mR \to M \otimes R/mR$ is surjective;
(e) for every $m \in \mathcal{M}$, $f \otimes id : M \otimes R/mR \to M \otimes R/mR$ is injective;
(f) for every $m \in \mathcal{M}$, $f \otimes id : M \otimes R_m \to M \otimes R_m$ is surjective;
(g) for every $m \in \mathcal{M}$, $f \otimes id : M \otimes R_m \to M \otimes R_m$ is a pure monomorphism.

Proof. (1) This is a well-known property of modules over commutative rings (e.g., [40, 18.9]).

(2) Consider some $f \in End_R(M)$ with $Mf = M$, and denote by $R[f]$ the R-subalgebra of $End_R(M)$ generated by f and id_M. M is a faithful module over the commutative ring $R[f]$, and for the ideal $< f > \subset R[f]$ generated by f, we have $< f > M = M$. By (1), $(id_M - gf)M = 0$, for some $g \in R[f]$, and hence $gf = id_M$, i.e., f is an isomorphism.

(3) Obviously, (a) implies all the other assertions.

$(b) \Rightarrow (a)$ is shown in (2). $(d) \Rightarrow (b)$ follows from 17.3.

$(c) \Rightarrow (e) \Rightarrow (g)$ Pure monomorphisms remain (pure) monomorphisms under tensor functors (e.g. [40, 34.5]).

$(e) \Rightarrow (d)$ This is well-known for the finite dimensional R/mR-vector space $M \otimes R/mR$.

$(f) \Rightarrow (b)$ follows from 17.2.

$(g) \Rightarrow (e)$ Apply the functor $- \otimes_{R_m} R_m/mR_m$ and $R_m/mR_m \simeq R/m$. □

17.5 R as a direct summand.

Let M be a finitely generated projective R-module. Assume M contains R as a submodule. Then R is a direct summand of M.

Proof. The inclusion $i : R \to M$ splits if and only if the map

$$Hom(i, R) : Hom_R(M, R) \to Hom_R(R, R) \simeq R$$

is surjective. Tensoring with R/m, $m \in \mathcal{M}$, we obtain (using 15.8)

$$Hom_R(M, R) \otimes_R R/m \simeq Hom_{R/m}(M \otimes_R R/m, R/m) \to R/m.$$

Since M is faithful, $M \otimes_R R/m \neq 0$ (e.g., [40, 18.9]) and hence this map is surjective for all $m \in \mathcal{M}$. By 17.3, this implies that $Hom(i, R)$ is surjective and i splits. □

If A is an R-algebra, $A_p \simeq A \otimes_R R_p$ is an R_p-algebra, for every $p \in \mathcal{P}$ (see 16.4). We list some properties of these scalar extensions:

17.6 Central idempotents and nilpotent elements.

Let A be an R-algebra.

(1) Assume for every $m \in \mathcal{M}$, all idempotents in A_m are central. Then all idempotents in A are central.

(2) If A is power-associative, then the following are equivalent:
(a) A is reduced;
(b) for every $m \in \mathcal{M}$, A_m is reduced.

Proof. (1) The canonical map $A \to \prod_{\mathcal{M}} A_m$, $a \mapsto ([a, 1]_m)$, is an injective algebra morphism (see 17.1). For every idempotent $f \in A$, f_m is an idempotent in A_m and hence is central by our assumption. So the image of f under the above map is in the centre of $\prod_{\mathcal{M}} A_m$. This implies that f is in the centre of A.

(2) $(a) \Rightarrow (b)$ This follows from 16.4(5).

$(b) \Rightarrow (a)$ Assume in the proof of (1) that $f \in A$ is nilpotent. Then $f_m = 0$ for all $m \in \mathcal{M}$ (because of (b)) and hence $f = 0$. □

In general, the map $R \to R_m$ need not be surjective and hence it may be difficult to transfer properties from R to R_m. However, over regular rings the situation is different:

17.7 Localization over regular rings.

Let M be an R-module, $m \subset R$ a maximal ideal and $J = Jac(R)$.

(1) Assume R is regular. Then:

(i) The canonical map $R \to R_m$ is surjective with kernel m and

$$R_m \simeq R/m \simeq \varinjlim \{R/eR \mid e^2 = e \in m\}.$$

(ii) The kernel of the canonical map $M \to M_m$ is mM and

$$M_m \simeq M/mM \simeq M \otimes_R R_m \simeq \varinjlim \{M/eM \mid e^2 = e \in m\}.$$

The map $M \to M_m$, $a \mapsto a + mM$, is a direct limit of splitting R-module epimorphisms and hence is pure in $\sigma[M]$.

(2) Assume R/J is regular. Then $(M/JM)_m \simeq M/mM$.

Proof. (1)(i) Consider $[a,s] \in R_m$, $a \in R$, $s \in R \setminus m$. Then for some $t \in R$, $s(1-ts) = 0$, and $s(a - tsa) = 0$ implies $[a,s] = [at,1]$, i.e., the map is surjective.

Therefore R_m is regular and so its Jacobson radical mR_m is zero. By 17.1, this means $R_m \simeq R/m$. m is maximal and contained in the kernel of $R \to R_m$. Hence the two ideals coincide.

Since R is regular, $m \simeq \varinjlim\{Re \mid e^2 = e \in m\}$. This implies the other isomorphisms with direct limits.

(ii) By (i), we have $M_m \simeq M \otimes R_m \simeq M \otimes R/m \simeq M/mM$. Hence mM is contained in the kernel of $M \to M_m$.

Assume $a \in M$ is in this kernel. Then $sa = 0$ for some $s \in R \setminus m$. Choose $t \in R$ with $s(1-ts) = 0$. Then $1 - ts \in m$ and $a = (1-ts)a \in mM$.

(2) M/JM is an R/J-module and (since $J \subset m$) we may identify $R \setminus m$ and $(R/J) \setminus (m/J)$. Putting $\overline{m} = m/J$ we have by (1),

$$(M/JM)_{\overline{m}} \simeq (M/JM)/\overline{m}(M/JM) \simeq M/mM.$$

$\square$

17.8 Tensor product over regular rings.

Let R be a regular ring and A an associative unital R-algebra. Assume M and M' are left A-modules with $M' \in \sigma[M]$ and M finitely presented in $\sigma[M]$. Then the map from 15.7,

$$\nu_M : Hom_A(M, M') \otimes_R R_m \to Hom_A(M, M' \otimes_R R_m),$$

is an isomorphism for every $m \in \mathcal{M}$. In particular,

$$End_A(M) \otimes_R R_m \simeq End_{A_m}(M \otimes_R R_m).$$

Proof. Taking into account that the canonical map $M \to M_m$ is a pure epimorphism in $\sigma[M]$ (see 17.7), the assertion is shown with the same proof as the corresponding result for Pierce stalks in 18.5. □

As a first local-global result we characterize (von Neumann) regular R-algebras by their localizations at maximal ideals of R:

17.9 Associative regular algebras.

Let A be an associative R-algebra with unit.

(1) *The following statements are equivalent:*
   (a) *A is regular;*
   (b) *for every $m \in \mathcal{M}$, A_m is a regular algebra.*

(2) *If A is a central algebra, then the following are equivalent:*
   (a) *A is regular;*
   (b) *R is regular and for every $m \in \mathcal{M}$, A/mA is regular;*
   (c) *for every $m \in \mathcal{M}$, A_m is regular.*

(3) *R is regular if and only if R_m is a field, for every $m \in \mathcal{M}$.*

Proof. (1) $(a) \Rightarrow (b)$ This follows from 16.6.

$(b) \Rightarrow (a)$ Assume all $A_m \simeq A \otimes R_m$ to be regular. Then all $A \otimes R_m$-modules are flat and hence, by 17.1, all A-modules are flat, i.e., A is regular (see 7.4).

(2) $(a) \Rightarrow (b)$ R is the centre of the regular ring A and hence is regular (e.g., [40, 3.16]). Obviously, all factor rings A/mA are regular.

$(b) \Rightarrow (c)$ Since R is regular, $A_m \simeq A/mA$ by 17.7.

$(c) \Rightarrow (a)$ This is shown in (1).

(3) Observing 17.7 this is a special case of (2). □

Strongly regular algebras are regular algebras without nilpotent elements or with all idempotents central. They can also be described by their algebras of fractions. Further characterizations will be obtained in 18.4.

17.10 Associative strongly regular algebras.

Let A be an associative R-algebra with unit.

(1) *Assume for every $m \in \mathcal{M}$, A_m is a division algebra. Then A is a strongly regular algebra.*

(2) *If A is a central R-algebra, then the following are equivalent:*
   (a) *A is strongly regular;*
   (b) *R is regular and A/mA is a division algebra, for every $m \in \mathcal{M}$;*
   (c) *for every $m \in \mathcal{M}$, A_m is a division algebra.*

Proof. (1) If all A_m are division algebras, then A is regular by 17.9, and is reduced by 17.6. Hence A is strongly regular.

(2) $(a) \Rightarrow (b)$ As centre of the regular algebra A, R is regular. Assume $K \subset A$ is a finitely generated left ideal such that $K + mA/mA$ is a non-zero ideal in A/mA. Then $K = eA$ for some idempotent $e \in R$. Since $K \not\subset mA$ we have $e \notin m$. This implies $K + mA \supset (Re + m)A = A$ and A/mA has no non-trivial left ideal, i.e., it is a division algebra.

$(b) \Rightarrow (c)$ This follows from 17.7.

$(c) \Rightarrow (a)$ is shown in (1). □

17.11 Exercises.

(1) Let A be an associative central R-algebra with unit. Prove:

(i) A is fully idempotent if and only if A_m is fully idempotent for every $m \in \mathcal{M}$.
(ii) A is a left V-ring (every simple left A-module is injective, [40, 23.5]) if and only if A_m is a left V-ring for every $m \in \mathcal{M}$ ([55, Theorem 6]).

(2) Suppose that in R every prime ideal is maximal, and let $m \subset R$ be a maximal ideal. Prove that $R_m \simeq R/I$, where the ideal $I \subset R$ is generated by the idempotents in m ([253]).

References: Armendariz-Fisher-Steinberg [55], Bourbaki [7], Matsumura [25], Burkholder [99], v.Oystaeyen-v.Geel [218], Szeto [252, 253], Villamayor-Zelinsky [260], Wisbauer [276].

18 Pierce stalks of modules and rings

1.Properties of Pierce stalks. 2.Lifting of idempotents. 3.Central idempotents and nilpotent elements. 4.Pierce stalks of regular algebras. 5.Tensor product with Pierce stalks. 6.Direct summands in stalks. 7.Self-projective refinable modules. 8.Lifting of special summands. 9.Indecomposable stalks. 10.Modules with local Pierce stalks. 11.Bimodules with local Pierce stalks. 12.Modules with simple Pierce stalks. 13.Bimodules with simple Pierce stalks. 14.Modules with local and perfect Pierce stalks. 15.Commutative locally perfect rings. 16.Pierce stalks and polyform modules. 17.Exercises.

As we have seen in the preceding section, localizing at prime ideals yields local rings. Now we study a localization which leads to rings without non-trivial idempotents.

Let $B(R)$ denote the set of all idempotents in R. With the operations for $e, f \in B(R)$,

$$e \uplus f = e(1-f) + (1-e)f \quad \text{and} \quad e \cdot f = ef,$$

$B(R)$ forms a *Boolean* ring (every element is idempotent).

Denote by $\mathcal{X}$ the set of all maximal ideals in $B(R)$. Since maximal ideals are prime, for every $x \in \mathcal{X}$, the set $S = B(R) \setminus x$ is closed under multiplication in $B(R)$ and R, i.e., it is a multiplicative subset of R.

Therefore we may construct the rings of fractions $R_x := RS^{-1}$, and for an R-module M, the module of fractions $M_x := MS^{-1}$. These are called the *Pierce stalks* of R and M respectively.

Since $x \in \mathcal{X}$ is prime, $e \in B(R) \setminus x$ if and only if $1 - e \in x$ and hence

$$Ke\,(R \to R_x) = \{r \in R \mid r(1-e) = 0, \text{ for some } e \in x\}.$$

From this we see that the image of $B(R)\setminus x$ under the canonical projection $R \to R/xR$ consists only of the unit element. Since obviously $xR_x = 0$ we obtain by 16.2(5),

$$R_x \simeq R_x/xR_x \simeq R/xR.$$

18.1 Properties of Pierce stalks.

With the above notation we have for any $x \in \mathcal{X}$ and any R-module M:

(1) $xR = \varinjlim \{eR \mid e \in x\}$, $\quad R_x \simeq R/xR \simeq \varinjlim \{R/eR \mid e \in x\}$ *and*

$$M_x \simeq M \otimes_R R_x \simeq \varinjlim \{M/eM \mid e \in x\} \simeq M/xM.$$

The canonical map $M \to M_x$, $m \mapsto m + xM$, is a direct limit of splitting A-module epimorphisms and a pure epimorphism in $\sigma[M]$.

(2) R_x has no non-trivial idempotents.

(3) $F_{\mathcal{X}} = \bigoplus_{x\in\mathcal{X}} R_x$ is a faithfully flat R-module.

(4) The canonical map $M \to \prod_{\mathcal{X}} M_x$, $m \mapsto (m + xM)_{x\in\mathcal{X}}$, is monic.

Proof. (1) For any $e, f \in x$, $e \uplus f \uplus e\cdot f = e+f-ef \in x$ and hence $eR, fR \subset (e \uplus f)R$. Therefore the set $xR = \{eR \mid e \in x\}$ is directed with respect to inclusion and so $xR = \varinjlim\{eR \mid e \in x\}$.

The other assertions are derived from the direct limit of the sequences

$$0 \longrightarrow eR \longrightarrow R \longrightarrow R/eR \longrightarrow 0,$$

and isomorphisms from 16.3.

(2) Let $[a, 1]$ be an idempotent in R_x, $a \in R$. Then $[a^2 - a, 1] = 0$ and hence there is an $e \in x$ with $(a^2 - a)(1 - e) = 0$. Then $c = a(1 - e)$ is an idempotent in R with $[a, 1] = [c, 1]$. Now $c \in x$ implies $[a, 1] = 0$ and $(1 - c) \in x$ implies $[a, 1] = 1$.

(3) R_x is flat by 16.2 and hence $F_{\mathcal{X}}$ is also a flat R-module.

For any R-module N we have to show that $N \otimes R_x \simeq N_x = 0$, for every $x \in \mathcal{X}$, implies $N = 0$. It suffices to show this for cyclic modules $N = Ra$. Put $\mathcal{X} = \{x_\lambda\}_\Lambda$. Then $a(1 - e_\lambda) = 0$, for suitable idempotents $e_\lambda \in x_\lambda \in \mathcal{X}$.

Consider the ideal in $B(R)$ generated by all $1-e_\lambda$. Since this ideal is not contained in any maximal ideal, it has to be equal to $B(R)$, and hence we can write

$$1 = c_1(1 - e_1) \uplus \cdots \uplus c_k(1 - e_k)\,, \text{ where } c_1, \ldots, c_k \in B(R).$$

This also yields a linear combination in R

$$1 = r_1(1 - e_1) + \cdots + r_k(1 - e_k)\,, \quad \text{where } r_1, \ldots, r_k \in R,$$

implying $a = a1 = ar_1(1 - e_1) + \cdots + ar_k(1 - e_k) = 0$.

(4) The kernel of this map is $\bigcap_{\mathcal{X}} xM$ which is zero by (3). □

Let us state some observations about Pierce stalks.

18.2 Lifting of idempotents.

Let A be an R-algebra with unit and $x \in \mathcal{X}$.

(1) Idempotents lift under the canonical map $A \to A_x$.

(2) Assume A is finitely generated as R-module. Then central idempotents lift to central idempotents under $A \to A_x$.

Proof. (1) Consider some $u \in A$ such that u_x is idempotent. Then $(u^2 - u)_x = 0$ and hence $(u^2 - u)e = 0$, for some $e \in B(R) \setminus x$ (see 16.3). Then $ue \in A$ is idempotent and $(ue)_x = u_x e_x = u_x$.

(2) Let A be generated by $a_1, \ldots, a_k$ as R-module. Consider any $u \in A$ such that u_x is a central idempotent. Then, for all $i, j \leq k$,

$$(u^2 - u)_x = 0, \qquad [a_i(a_j u) - (a_i a_j)u]_x = 0,$$
$$(a_i u - u a_i)_x = 0, \quad [u(a_i a_j) - (u a_i)a_j]_x = 0.$$

By 16.3, we have for some $e \in B(R) \setminus x$,

$$(u^2 - u)e = 0, \qquad [a_i(a_j u) - (a_i a_j)u]e = 0,$$
$$(a_i u - u a_i)e = 0, \quad [u(a_i a_j) - (u a_i)a_j]e = 0.$$

Then $ue \in A$ is idempotent with associators $(a_i, a_j, ue) = (ue, a_i, a_j) = 0$ and commutators $[ue, a_i] = 0$. Since every element in A is an R-linear combination of the a_i's, we conclude that ue is in fact a central idempotent (see 1.11). □

18.3 Central idempotents and nilpotent elements.

Let A be an R-algebra.

(1) If A is a central R-algebra with unit, the following are equivalent:

(a) Every idempotent in A is central;

(b) for every $x \in X$, A_x has no non-trivial idempotents.

(2) If A is power-associative, the following are equivalent:

(a) A is reduced;

(b) for every $x \in \mathcal{X}$, A_x is reduced.

Proof. (1) $(a) \Rightarrow (b)$ Assume for $u \in A$ and $x \in \mathcal{X}$ that the element $u_x \in A_x$ is idempotent. Then $(u^2 - u)_x = 0$ and hence $(u^2 - u)e = 0$ for some $e \in B(R) \setminus x$ (see 18.2). Since $(ue)^2 = ue$ we have $ue \in B(R)$ by assumption. From the relation $u_x = u_x e_x = (ue)_x$ we conclude $u_x = 1_x$ if $ue \notin x$ or $u_x = 0_x$ otherwise (see 18.1(2)).

$(b) \Rightarrow (a)$ For any idempotent $f \in A$, f_x is an idempotent and hence equal to 0 or 1 in A_x. Hence the image of f under the canonical embedding $A \to \prod_{\mathcal{X}} A_x$ (see 18.1) is in the centre of $\prod_{\mathcal{X}} A_x$ and therefore f is in the centre of A.

(2) This is obtained with the same proof as 17.6(2). □

In 17.9 we investigated localization of regular algebras at maximal ideals of R. Now we ask for properties of Pierce stalks of regular algebras:

18.4 Pierce stalks of regular algebras.

Let A be an associative R-algebra with unit.

(1) The following are equivalent:

(a) A is regular;

(b) for every $x \in \mathcal{X}$, A_x is regular.

(2) Assume for every $x \in \mathcal{X}$, A_x is a division algebra. Then A is strongly regular.

(3) If A is a central R-algebra, then the following are equivalent:
(a) A is strongly regular;
(b) for every $x \in \mathcal{X}$, A_x is a division algebra.

(4) R is regular if and only if R_x is a field, for every $x \in \mathcal{X}$.

Proof. (1) This is obtained with the proof of 17.9(1).

(2) Assume all A_x are division algebras. Then A is regular by (1) and is reduced by 18.3, i.e., A is strongly regular.

(3) $(a) \Rightarrow (b)$ Assume A is strongly regular. Then all idempotents in A are central. Since idempotents lift under $A \to A_x$ by 18.2, we see that all idempotents of A_x belong to R_x and hence are 0 or 1 (see 18.1). As a regular algebra without non-trivial idempotents, A_x is a division algebra.

$(b) \Rightarrow (a)$ is shown in (2).

(4) is a special case of (3). □

For Pierce stalks we get the isomorphisms considered in 15.7 also for modules M which are finitely presented in $\sigma[M]$ (see 7.1, 7.2):

18.5 Tensor product with Pierce stalks.

Let A be an associative unital R-algebra, M and M' left A-modules with $M' \in \sigma[M]$ and M finitely presented in $\sigma[M]$. Then the map from 15.7,

$$\nu_M : Hom_A(M, M') \otimes_R R_x \to Hom_A(M, M' \otimes_R R_x),$$

is an isomorphism for every $x \in \mathcal{X}$. In particular,

$$End_A(M) \otimes_R R_x \simeq End_{A_x}(M \otimes_R R_x).$$

Proof. With the surjection $R \to R_x$ we get from the proof of 15.7(2) the diagram

$$\begin{array}{ccccc} Hom_A(M, M') \otimes_R R & \to & Hom_A(M, M') \otimes_R R_x & \to & 0 \\ \downarrow\simeq & & \downarrow\nu_M & & \\ Hom_A(M, M' \otimes_R R) & \to & Hom_A(M, M' \otimes_R R_x) & \to & 0 \end{array} .$$

Since $M' \to M'_x$ is a direct limit of splitting module epimorphisms by 18.1, it is pure in $\sigma[M]$ (see 7.2) and hence the functor $Hom_A(M, -)$ is exact with respect to it, i.e., the lower row is exact and hence ν_M is surjective. By 15.7, ν_M is also injective.

For the ring isomorphism apply 15.8. □

18.6 Direct summands in stalks.

Let M be a finitely generated self-projective left module over an associative R-algebra A. Assume for some $x \in \mathcal{X}$ and some finitely generated submodules $K, L \subset M$, we have $M_x = K_x \oplus L_x$.

Then there exists an idempotent $e \in B(R) \setminus x$, such that $eM = eK \oplus eL$.

Proof. With the canonical maps $M \to M_x \to K_x$, consider the diagram

$$\begin{tikzcd} & & M \arrow[dl, "id"'] \arrow[d] \\ K \arrow[r] & M \arrow[r] & K_x \end{tikzcd} ,$$

which can be extended commutatively by some $\alpha : M \to K$ with the properties

$$[K(id - \alpha)]_x = 0, \quad [M(\alpha - \alpha^2)]_x = 0.$$

Replacing K by L in the above diagram we can find $\beta : M \to L$ with

$$[L(id - \beta)]_x = 0, \quad [M(\beta - \beta^2)]_x = 0,$$

and we observe

$$[M(id - (\alpha + \beta))]_x = 0, \quad [M\alpha\beta]_x = 0 = [M\beta\alpha]_x \ .$$

Since all these submodules are finitely generated, there exists $e \in B(R) \setminus x$ with

$$\begin{aligned} eK(id - \alpha) = 0, &\quad eM(\alpha - \alpha^2) = 0, \\ eL(id - \beta) = 0, &\quad eM(\beta - \beta^2) = 0, \\ eM(id - (\alpha + \beta)) = 0, &\quad eM\alpha\beta = 0 = eM\beta\alpha \ . \end{aligned}$$

From these identities we derive $eM\alpha = eK$, $eM\beta = eL$, $eM\alpha \cap eM\beta = 0$ and finally

$$eM = eM\alpha \oplus eM\beta = eK \oplus eL.$$

□

For self-projective modules we have nice characterizations of refinable modules.

18.7 Self-projective refinable modules.

Let A be an associative R-algebra. For a finitely generated self-projective A-module M, the following are equivalent:

(a) M is (strongly) refinable;

(b) $M/Rad(M)$ is refinable and direct summands lift modulo $Rad(M)$;

(c) for every $x \in \mathcal{X}$, M_x is (strongly) refinable.

(d) $End_A(M)$ *is left refinable;*

(e) for every $x \in \mathcal{X}$, $End_{A_x}(M_x)$ *is (strongly) refinable.*

Proof. $(a) \Rightarrow (b)$ Since $Rad(M)$ is fully invariant this is clear by 8.4.

$(b) \Rightarrow (a)$ We show that direct summands lift modulo every submodule of M. For a submodule $K \subset M$, consider submodules $K \subset U, V \subset M$ for which $M/K = U/K \oplus V/K$. Denote by L, X the submodules $K \subset L \subset U$ and $K \subset X \subset V$ with $L/K = Rad(U/K)$ and $X/K = Rad(V/K)$. We have the commutative diagram of canonical module morphisms

$$\begin{array}{ccc} M & \longrightarrow & U/K \oplus V/K \\ \downarrow & & \downarrow \\ M/Rad(M) & \longrightarrow & U/L \oplus V/X \end{array} .$$

Since $M/Rad(M)$ is refinable there is a direct summand V^o of $M/Rad(M)$ which is mapped onto V/X. By assumption, V^o can be lifted to a direct summand V' of M.

Commutativity of the above diagram implies $V' + L + X = V + L$. Since X/K is superfluous in $(V + L)/K$, we conclude $V' + L = V + L$ and $U + V' = U + V = M$. By M-projectivity of M and M/V', the canonical map $U \to M \to M/V'$ splits and hence there exists a direct summand $U' \subset U$ of M with $U' \oplus V' = M$. Now we have

$$M/K = (U' + K)/K \oplus (V' + K)/K = U/K \oplus V/K,$$

and hence the direct summand $U' \subset M$ is mapped onto U/K.

$(a) \Rightarrow (c)$ Since all xM are fully invariant this is clear by 8.4. Observe that the M_x are self-projective, hence for them refinable is equivalent to strongly refinable.

$(c) \Rightarrow (a)$ Let $M = K + L$ with K, L finitely generated and put $\mathcal{X} = \{x_\lambda\}_\Lambda$. For any $x_\lambda \in \mathcal{X}$, $M_{x_\lambda} = K_{x_\lambda} + L_{x_\lambda}$ is strongly refinable and hence we can find finitely generated submodules $K^\lambda \subset K$ and $L^\lambda \subset L$ with $M_{x_\lambda} = K^\lambda_{x_\lambda} \oplus L^\lambda_{x_\lambda}$.

By 18.6, there exist $e_\lambda \in B(R) \setminus x_\lambda$ with $e_\lambda M = e_\lambda K^\lambda \oplus e_\lambda L^\lambda$.

The ideal in $B(R)$ generated by all the e_λ is not contained in any maximal ideal and hence is equal to $B(R)$. Therefore we have $1 = r_1 e_1 \uplus \ldots \uplus r_k e_k$ with the $e_i \in \{e_\lambda\}_\Lambda$ and $r_i \in B(R)$. By an orthogonalization procedure, we obtain from this orthogonal idempotents $e_i' \in e_i B(R)$ with

$$\begin{array}{c} 1 = e_1' \uplus \cdots \uplus e_k' = e_1' + \cdots + e_k' \ , \quad \text{and} \\ M = (e_1' + \cdots + e_k')M = e_1' K^1 \oplus e_1' L^1 \oplus \cdots \oplus e_k' K^k \oplus e_k' L^k \ . \end{array}$$

Putting $K' = e_1' K^1 \oplus \cdots \oplus e_k' K^k \subset K$ and $L' = e_1' L^1 \oplus \cdots \oplus e_k' L^k \subset L$ we now have $M = K' \oplus L'$.

$(a) \Rightarrow (d)$ Recall that, for a finitely generated self-projective module M and any left ideal $I \subset End_A(M)$, we have $I = Hom_A(M, MI)$ (see [40, 18.4]).

Assume $End_A(M) = I + J$ with left ideals I, J. Then $M = MI + MJ$ and there is an idempotent $e \in I$ with $M = Me + MJ$. Then $End_A(M) = End_A(M)e + J$ and hence $End_A(M)$ is left refinable.

$(d) \Rightarrow (a)$ Assume $M = K + L$ for any submodules $K, L \subset M$. Then, by projectivity, $End_A(M) = Hom_A(M, K + L) = Hom_A(M, K) + Hom_A(M, L)$ (see [40, 18.4]) and there exists an idempotent $e \in Hom_A(M, K)$ with

$$End_A(M) = End_A(M)e + Hom_A(M, L).$$

Applying these morphisms to M, we obtain $M = Me + L$ with $Me \subset K$, i.e., M is refinable.

$(c) \Leftrightarrow (e)$ Since $End_{A_x}(M_x) \simeq End_A(M)_x$ by 18.5 and M_x is self-projective, this equivalence follows from $(a) \Leftrightarrow (d)$ proved above. □

Examples for refinable modules are finitely generated self-projective A-modules M which are semiperfect or f-semiperfect in $\sigma[M]$ (see 8.12, 8.13).

In the proof of $(b) \Rightarrow (a)$ in 18.7, self-projectivity of M was only needed to conclude that $M = U + V'$ with a direct summand $V' \subset M$ implies the existence of a direct summand $U' \subset U$ of M with $U' \oplus V' = M$. Without projectivity the same conlusion also holds for any fully invariant submodule $U \subset M$: If $V' = Me$ for some idempotent $e \in End_A(M)$ we can choose $U' = U(1 - e) = M(1 - e) \subset U$.

Again the same argument applies for any submodule $U \subset M$ if the idempotent $e \in End_A(M)$ can be considered as element of R. Hence we have:

18.8 Lifting of special summands.

Let M be a finitely generated module over an associative R-algebra A, such that $M/Rad(M)$ is refinable and direct summands lift modulo $Rad(M)$.

Assume $M/K = U/K \oplus V/K$ with submodules $K \subset U, V \subset M$. If

(i) U is a fully invariant submodule, or

(ii) idempotents in $End_A(M)$ are central and are induced by idempotents in R,

then U/K can be lifted to a direct summand of M.

Condition (ii) above occurs in the next cases where we consider Pierce stalks with special properties.

18.9 Indecomposable stalks.

Let M be a finitely generated left module over an associative R-algebra A, and assume that R is canonically isomorphic to the centre of $End_A(M)$. Consider the statements

(i) for every $x \in \mathcal{X}$, M_x is an indecomposable A-module;

(ii) every idempotent in $End_A(M)$ is central.

Then (i) $\Rightarrow$ (ii) always holds.

If M is finitely presented in $\sigma[M]$, then also (ii) $\Rightarrow$ (i).

Proof. (i) $\Rightarrow$ (ii) If M_x is indecomposable then $End_A(M_x)$ has no non-trivial idempotents. By 15.8, the ring $End(M)_x$ is a subring of $End_{A_x}(M_x)$ and hence also has no non-trivial idempotents. Now we see from 18.3 that idempotents in $End(M)$ are central.

(ii) $\Rightarrow$ (i) If M is finitely presented in $\sigma[M]$, $End_A(M)_x \simeq End_{A_x}(M_x)$ by 18.5 and has no non-trivial idempotents by 18.3, i.e., M_x is an indecomposable module. $\square$

18.10 Modules with local Pierce stalks.

Let A be an associative unital R-algebra, M a finitely generated A-module and $B = End(_AM)$. Assume that R is canonically isomorphic to the centre of B. Then the following assertions are equivalent:

(a) *For every $x \in \mathcal{X}$, M_x is a local A-module;*

(b) *M is a refinable A-module and idempotents in $End_A(M)$ are central;*

(c) *$M/Rad(M)$ is refinable, direct summands lift modulo $Rad(M)$ and idempotents in $End_A(M)$ are central.*

Proof. (a) $\Rightarrow$ (b) Since local modules are indecomposable, we know from 18.9 that idempotents in $End_A(M)$ are central. Assume $M = U + V$ and hence $M_x = U_x + V_x$ for any $x \in \mathcal{X}$. Since every M_x is local, $M_x = U_x$ or $M_x = V_x$, i.e.,

$$0 = M_x/U_x \simeq (M/U)_x \quad \text{or} \quad 0 = M_x/V_x \simeq (M/V)_x.$$

For all $x_\lambda \in \mathcal{X}$ with $(M/U)_{x_\lambda} = 0$, we find $e_\lambda \in B(R) \setminus x_\lambda$ with $e_\lambda(M/U) = 0$, and for $x_\mu \in \mathcal{X}$ with $(M/V)_{x_\mu} = 0$ we have $f_\mu \in B(R) \setminus x_\mu$ with $f_\mu(M/V) = 0$.

Denote by E the ideal in $B(R)$ generated by all e_λ and by F the ideal generated by all f_μ. Then the ideal $E \uplus F$ is not contained in any maximal ideal and hence $E \uplus F = B(R)$ and $1 = \tilde{e} \uplus \tilde{f}$, for some $\tilde{e} \in E$ and $\tilde{f} \in F$. By construction, $\tilde{e}(M/U) = 0$ and hence $\tilde{e}M \subset U$. Also, $\tilde{e}M = \tilde{e}U$ and $\tilde{f}M = \tilde{f}V \subset V$. This implies

$$M = (\tilde{e} \uplus \tilde{f})(U + V) = \tilde{e}U + \tilde{f}U + V = \tilde{e}M + V,$$

and hence $\tilde{e}M$ is the direct summand we were looking for.

(b) $\Rightarrow$ (a) Consider $x \in \mathcal{X}$ and $M = K + L$. Then $M_x = K_x + L_x$. Since M is refinable, there exists an idempotent $e \in End_A(M)$ with $Me \subset K$ and $Me + L = M$. By our assumptions we have $e \in B(R)$. Now $e \in x$ implies $L_x = M_x$ whereas $e \notin x$ means $K_x \supset (eM)_x = M_x$. Hence every proper submodule of M_x is superfluous.

(b) $\Leftrightarrow$ (c) This is a consequence of 8.4 and 18.8. $\square$

Considering an A-module $_AM$ with $B = End_A(M)$ as a left $A \otimes_R B^o$-module we have a commutative endomorphism algebra (see 4.2) and obtain as a corollary:

18.11 Bimodules with local Pierce stalks.

Let A be an associative unital R-algebra, M an A-module and $B = End_A(M)$. Assume the canonical map $R \to End_{A\otimes B^o}(M)$ is an isomorphism and M is finitely generated as $A \otimes_R B^o$-module. Then the following are equivalent:

(a) *For every $x \in \mathcal{X}$, M_x is a local $A \otimes B^o$-module;*

(b) *M is a refinable $A \otimes B^o$-module;*

(c) *$M/Rad({}_{A\otimes B^o}M)$ is refinable and direct summands lift modulo $Rad({}_{A\otimes B^o}M)$.*

Next we consider an even more special situation.

18.12 Modules with simple Pierce stalks.

Let A be an associative unital R-algebra, M a finitely generated A-module and $B = End_A(M)$. Assume that R is canonically isomorphic to the centre of B. Then the following assertions are equivalent:

(a) *For every $x \in X$, M_x is a simple A-module;*

(b) *every finitely generated (cyclic) A-submodule of M is a direct summand and idempotents in $End_A(M)$ are central.*

If M is finitely presented in $\sigma[M]$, then $(a),(b)$ are equivalent to:

(c) *M is a self-generator and, for any $x \in X$, $End_A(M)_x$ is a division algebra.*

Proof. $(a) \Rightarrow (b)$ By 18.9, idempotents in $End_A(M)$ are central.

Consider a finitely generated A-submodule $U \subset M$. Since every M_x is simple, $U_x = M_x$ and hence $0 = M_x/U_x \simeq (M/U)_x$, or $U_x = 0$. For all $x_\lambda \in \mathcal{X}$ with $(M/U)_{x_\lambda} = 0$ we find $e_\lambda \in B(R) \setminus x_\lambda$ with $e_\lambda(M/U) = 0$, and for $x_\mu \in \mathcal{X}$ with $U_{x_\mu} = 0$ we have $f_\mu \in x_\mu$ with $f_\mu U = 0$.

Denote by E the ideal in $B(R)$ generated by all e_λ, and by F the ideal generated by all f_μ. Then $E \uplus F = B(R)$ and $1 = \tilde{e} \uplus \tilde{f}$ for some $\tilde{e} \in E$ and $\tilde{f} \in F$.

For any $g \in E \cap F$, we observe $gU = 0$ and $g(M/U) = 0$, implying $gM = 0$ and $g = 0$ (M is a faithful R-module). Therefore $E \cap F = 0$ and $1 = \tilde{e} \uplus \tilde{f} = \tilde{e} + \tilde{f}$. By construction, $\tilde{e}(M/U) = 0$, hence $\tilde{e}M \subset U$ and $(1-\tilde{e})U = 0$, implying $U = \tilde{e}U \subset \tilde{e}M$.

From this we see $U = \tilde{e}M$ and U is a direct summand of M.

$(b) \Rightarrow (a)$ Any cyclic A-submodule $\bar{U} \neq 0$ of M_x has the form $\bar{U} = (U + xM)/xM$, for some cyclic A-submodule $U \subset M$. By (b), $U = eM$ for a central idempotent $e \in B$ and $\bar{U} = (eM + xM)/xM$. Since $e \notin x$ we conclude $eM + xM = M$ and $\bar{U} = M_x$, i.e., M_x is a simple A-module.

Now assume M is finitely presented in $\sigma[M]$, hence $End_A(M)_x \simeq End_{A_x}(M_x)$ (18.5).

$(a) \Rightarrow (c)$ By (b), M generates any finitely generated submodule and therefore all its submodules. If M_x is a simple module, $End_{A_x}(M_x)$ is a division algebra by Schur's Lemma.

$(c) \Rightarrow (a)$ Assume M is a self-generator. Since xM is fully invariant, M_x is also a self-generator (see 5.2). If $End_{A_x}(M_x)$ is a division algebra, every endomorphism of M_x is invertible and hence M_x is simple. □

Similar to 18.11 we obtain as a corollary for bimodules:

18.13 Bimodules with simple Pierce stalks.
Let A be an associative unital R-algebra, M an A-module and $B = End(_AM)$. Assume the canonical map $R \to End_{A\otimes B^o}(M)$ is an isomorphism and M is finitely generated as an $A \otimes_R B^o$-module. Then the following are equivalent:

(a) M_x is a simple $A \otimes_R B^o$-module, for every $x \in X$;

(b) every finitely generated $A \otimes_R B^o$-submodule of M is a direct summand.

If M is perfect in $\sigma[M]$ (see 8.14), for any fully invariant submodule $U \subset M$, M/U is perfect in $\sigma[M/U]$ and hence M_x is perfect in $\sigma[M_x]$, for every $x \in \mathcal{X}$. However, this property is not enough to make M perfect in $\sigma[M]$. We only obtain that $M/Rad(M)$ is regular in $\sigma[M/Rad(M)]$.

18.14 Modules with local and perfect Pierce stalks.
Let A be an associative unital R-algebra, M a finitely generated self-projective A-module and $B = End(_AM)$. Assume R is canonically isomorphic to the centre of B. Then the following are equivalent:

(a) For every $x \in \mathcal{X}$, M_x is local and perfect in $\sigma[M_x]$;

(b) For every $x \in \mathcal{X}$, $B_x = End_A(M)_x$ is a local and left perfect ring;

(c) $\bar{B} = B/Jac(B)$ is (strongly) regular, $Jac(B)$ is right t-nilpotent and idempotents in B are central.

Assume M is a self-generator. Then (a)-(c) are equivalent to:

(d) $\bar{M} = M/Rad(M)$ is regular in $\sigma[\bar{M}]$, $Rad(M)^{(\mathbb{N})} \ll M^{(\mathbb{N})}$ and idempotents in $End_A(M)$ are central.

Proof. $(a) \Leftrightarrow (b)$ A finitely generated, self-projective module M is perfect in $\sigma[M]$ if and only if $End_A(M)$ is left perfect (see [40, 43.8]).

Since $B_x = End_A(M)_x \simeq End_{A_x}(M_x)$ (by 18.5) the assertion is clear.

$(b) \Rightarrow (c)$ By 18.3, idempotents in B are central.

Assume $Jac(B^{(\mathbb{N})}) + L = B^{(\mathbb{N})}$. Since $[Rad(B)]_x \subset Rad(B_x)$ we have, by perfectness of B_x,

$$[Rad(B^{(\mathbb{N})})]_x \subset Rad(B_x^{(\mathbb{N})}) \ll M_x^{(\mathbb{N})}.$$

This implies $L_x = B_x^{(\mathbb{N})}$, for every $x \in \mathcal{X}$, and hence $L = B^{(\mathbb{N})}$. We conclude $Rad(B^{(\mathbb{N})}) \ll B^{(\mathbb{N})}$ and $Rad(B)$ is right t-nilpotent by [40, 43.4].

Put $\mathcal{X} = \{x_\lambda\}_\Lambda$. Since all B_x are local, there are unique maximal left ideals $x_\lambda B \subset K_\lambda \subset B$, with $Rad(B_{x_\lambda}) = K_\lambda / x_\lambda B$. On the other hand, for every maximal left ideal $K \subset B$, the set $\{e \in B(R) \,|\, e(B/K) = 0\}$ is a maximal ideal in $B(R)$ and we conclude that $Rad(B) = \bigcap_\Lambda K_\lambda$.

Now we claim that $\bar{B}$ has no nilpotent elements. For this we have to show that all nilpotent elements in B belong to $Rad(B)$: Assume $f^n = 0$, for some $f \in B$, $n \in \mathbb{N}$. Then $f^n_{x_\lambda} = 0$, for every $x_\lambda \in \mathcal{X}$, and hence $Bf \subset \bigcap_\Lambda K_\lambda = Rad(B)$, implying $f \in Rad(B)$.

Next we show that $Jac(\bar{B}_x) = 0$, for every $x \in \mathcal{X}$. Assume for $f \in \bar{B}$ we have $f_x \in Jac(\bar{B}_x)$. Then $f_x^n = 0$, for some some $n \in \mathbb{N}$, and hence $ef^n = (ef)^n = 0$ for a suitable $e \in B(R) \setminus x$. However, $\bar{B}$ has no nilpotent elements and therefore $ef = 0$, implying $f_x = 0$.

From what we have seen so far we conclude that $\bar{B}_x$ is a division ring for every $x \in \mathcal{X}$, and by 18.4, $\bar{B}$ is a regular ring.

$(c) \Rightarrow (b)$ Since $\bar{B}$ is regular, $(B/Rad(B))_x$ is also regular and hence $(Rad(B))_x = Rad(B_x)$ is right t-nilpotent, for every $x \in \mathcal{X}$.

Therefore idempotents lift modulo $Rad(B_x)$. They also lift modulo xB (see 18.2). Since idempotents in B are central, $\bar{B}_x$ is a regular algebra without non-trivial idempotents (see 18.3), i.e., it is a division algebra.

$(c) \Leftrightarrow (d)$ Since M is finitely generated and self-projective we know that $Jac(B) = Hom_A(M, Rad(M))$ and $End_A(\bar{M}) \simeq B/Jac(B)$ (see [40, 22.2]).

By our assumptions, $\bar{M}$ is also a self-generator and hence $\bar{M}$ is regular in $\sigma[\bar{M}]$ if and only if $\bar{B}$ is a regular ring.

It follows from [40, 43.4] that $Rad(M^{(\mathbb{N})}) \ll M^{(\mathbb{N})}$ if and only if $Jac(B)$ is right t-nilpotent. □

As a special case we describe commutative rings with perfect Pierce stalks which are called *locally perfect.*

18.15 Commutative locally perfect rings.

For the ring R with Jacobson radical J, the following are equivalent:

(a) *For every $x \in \mathcal{X}$, R_x is a perfect ring;*

(b) *$\bar{R} = R/J$ is regular and J is t-nilpotent;*

(c) *for every $m \in \mathcal{M}$, R_m is a perfect ring.*

Proof. $(a) \Rightarrow (b)$ Assume all R_x are perfect. Then $Jac(R_x)$ is a nil ideal and hence we obtain from 16.4 that $Jac(R_x) = JR_x$. R_x has no non-trivial idempotents (see

18.1) and idempotents lift modulo $Jac(R_x)$. Hence $(R/J)_x \simeq R_x/JR_x$ is a semisimple ring without non-trivial idempotents, i.e., it is a field, and by 18.4, R/J is regular.

As in the proof of $(b) \Rightarrow (c)$ in 18.14 we see that J is t-nilpotent.

$(b) \Rightarrow (a)$ is a special case of 18.14.

$(b) \Rightarrow (c)$ If R/J is regular, $(R/J)_m \simeq R_m/JR_m$ is a field and hence $JR_m = mR_m$, for every $m \in \mathcal{M}$. Now t-nilpotence of J implies t-nilpotence of $Jac(R_m) = mR_m$ and hence R_m is perfect.

$(c) \Rightarrow (b)$ Assume all R_m are perfect. Then $Jac(R_m) = mR_m$ is a nil ideal and hence we obtain $JR_m = mR_m$ by 16.4. This implies that $(R/J)_m \simeq R_x/JR_x \simeq R/m$ is a field and hence R/J is regular by 17.9. $\square$

We close this section with a view on the Pierce stalks of the idempotent closure of polyform modules.

18.16 Pierce stalks and polyform modules.

Let M be a polyform A-module, $T = End_R(\widehat{M})$, and B the Boolean ring of central idempotents of T and $\widetilde{M} = MB$ the idempotent closure of M (see 11.15).

For a maximal ideal x of B, denote by

$$\varphi_x : \widetilde{M} \to \widetilde{M}_x, \quad m \mapsto m + \widetilde{M}x,$$

the canonical map of $\widetilde{M}$ to the Pierce stalk $\widetilde{M}_x$. Then:

(1) $Ke\,\varphi_x = \{m \in \widetilde{M} \mid \varepsilon(m) \in x\}$ and $(M)\varphi_x = \widetilde{M}_x$.

(2) If M is a PSP-module, then $\widetilde{M}_x$ is strongly prime.

Proof. (1) Clearly $(m)\varphi_x = 0$ if and only if $me = 0$ for some $e \notin x$. According to 11.12, $e \in (1 - \varepsilon(m))C$. Hence $1 - \varepsilon(m) \notin x$ and $\varepsilon(m) \in x$.

Conversely, assume $\varepsilon(m) \in x$. Then $1 - \varepsilon(m) \notin x$. Since $m(1 - \varepsilon(m)) = 0$ we obtain $(m)\varphi_x = 0$.

Consider $m \in \widetilde{M}$. By 11.15, $m = \sum_{i=1}^n m_i e_i$, where $m_i \in M$ and $e_i \in B$ are pairwise orthogonal idempotents. If all these idempotents belong to x, then $0 = (m)\varphi_x \in (M)\varphi_x$. Assume $e_1 \notin x$. Then $1 - e_1, e_2, \ldots, e_n \in x$ and $m_1 - \sum_{i=1}^n m_i e_i \in \widetilde{M}x$. So also in this case, $(m)\varphi_x = (m_1)\varphi_x \in (M)\varphi_x$. Hence $(M)\varphi_x = \widetilde{M}_x$.

(2) Let M be a PSP-module and $0 \neq y \in \widetilde{M}_x$. Clearly $\widetilde{M}_x \subset \widehat{M}_x$ and $y = (z)\varphi_x$ for some $z \in M$. By 14.11, $AzT = \widehat{M}\varepsilon(z)$. Since $y = (z)\varphi_x \neq 0$, $\varepsilon(z) \notin x$ and hence

$$AyT_x = (AzT)\varphi_x = (\widehat{M}\varepsilon(m))\varphi_x = \widehat{M}_x.$$

Now it follows from 13.3(h) that $\widetilde{M}_x$ is strongly prime. $\square$

18.17 Exercises.

Let A denote an associative central R-algebra with unit.

(1) Show that a cyclic A-module M has local Pierce stalks if and only if, for any $m, k \in M$ and $Am = M$, there exists an idempotent $e \in R$ with $A(k - em) = M$.

(2) Show that the following are equivalent ([98, 102]):
(a) A has local Pierce stalks as left A-module;
(b) every element in A is the sum of a unit and a central idempotent;
(c) every cyclic left A-module has local Pierce stalks;
(d) ${}_AA$ is refinable and every idempotent in A is central.

(3) Assume A has local Pierce stalks as left A-module. Prove that $A/Jac(A)$ is biregular if and only if it is strongly regular ([98]).

(4) Assume A is finitely generated as an R-module and idempotents lift from $A_x/Jac(A)_x$ to A_x, for every $x \in \mathcal{X}$. Prove that idempotents lift from $A/Jac(A)$ to A ([179, Proposition 2]).

(5) Prove that for the ring R, the following are equivalent ([125]):
(a) R is locally perfect;
(b) every non-zero R-module has a maximal submodule
(c) every non-zero R_x-module has a maximal submodule, for every $x \in \mathcal{X}$.

References: Armendariz-Fisher-Steinberg [55], Bourbaki [7], Burgess-Raphael [95], Burgess-Stephenson [96, 97, 98], Burkholder [99, 100, 102], Dauns-Hofmann [113], Faith [125], Hamsher [148], Hofmann [160], Kambara-Oshiro [172], Kirkman [179], Koifman [181], Lesieur [186], Matsumura [25], Nicholson [212], Oystaeyen-Geel [218], Pierce [32], Renault [226], Stock [248], Szeto [252], Villamayor-Zelinsky [260], Wisbauer [276].

19 Projectives and generators

1.N-projective modules. 2.Self-projective modules. 3.Projective modules and $-\otimes R/m$. 4.Projective modules. 5.Faithfully flat algebras. 6.Generators. 7.Self-generators. 8.Self-progenerators. 9.Progenerators in A-Mod. 10.Tensorproduct of self-projective modules. 11.Exercises.

We continue to study properties of modules over associative R-algebras by properties of Pierce stalks or localizations at prime ideals.

Throughout this section A and B will be *associative unital* R-algebras.

19.1 N-projective modules.

Let M and N be A-modules with M finitely generated.

(1) If M is N-projective, then $M \otimes_R B$ is $N \otimes_R B$-projective as $A \otimes_R B$-module and

$$Hom_A(M, N) \otimes_R B \simeq Hom_{A \otimes_R B}(M \otimes_R B, N \otimes_R B).$$

(2) The following assertions are equivalent:

(a) M is N-projective;

(b) for every $x \in \mathcal{X}$, M_x is N_x-projective and

$$Hom_A(M, N) \otimes_R R_x \simeq Hom_{A_x}(M_x, N_x);$$

(c) for every $m \in \mathcal{M}$, M_m is N_m-projective and

$$Hom_A(M, N) \otimes_R R_m \simeq Hom_{A_m}(M_m, N_m).$$

If R is a locally perfect ring, or ${}_AM$ is finitely generated and ${}_RN$ is noetherian, then (a)-(c) are also equivalent to:

(d) for every $m \in \mathcal{M}$, M/mM is N/mN-projective and

$$Hom_A(M, N) \otimes_R R/m \simeq Hom_{A/mA}(M/mM, N/mN).$$

Proof. (1) Assume M to be N-projective. Then it is also $N \otimes_R B$-projective as an A-module (see [40, 18.2]). From the isomorphism in 15.6,

$$Hom_{A \otimes_R B}(M \otimes_R B, N \otimes_R B) \simeq Hom_A(M, N \otimes_R B),$$

we conclude that $M \otimes_R B$ is $N \otimes_R B$-projective as an $A \otimes_R B$-module.

The isomorphism stated is taken from 15.8.

(2) $(a) \Rightarrow (b), (c), (d)$ is a consequence of (1).

$(b) \Rightarrow (a)$ Let K be a submodule of N. In an obvious way we construct the commutative exact diagram with vertical monomorphisms obtained from 15.8,

$$\begin{array}{ccccccc}
0 \to & Hom_A(M,K)\otimes_R R_x & \to & Hom_A(M,N)\otimes_R R_x & \to & Hom_A(M,N/K)\otimes_R R_x & \\
& \downarrow & & \downarrow\simeq & & \downarrow & \\
0 \to & Hom_{A_x}(M_x,K_x) & \to & Hom_{A_x}(M_x,N_x) & \to & Hom_{A_x}(M_x,(N/K)_x) & \to 0.
\end{array}$$

By assumption the central vertical map is an isomorphism. Therefore the other vertical maps are also isomorphisms and the right morphism in the upper row is epic. Since $\bigoplus_{\mathcal{X}} R_x$ is faithfully flat, this implies that

$$Hom_A(M,N) \to Hom_A(M,N/K) \to 0$$

is an exact sequence, i.e., M is N-projective.

$(c) \Rightarrow (a)$ is shown with the same proof.

$(d) \Rightarrow (a)$ Let us first assume that R is a local ring with t-nilpotent maximal ideal m. In view of 17.3, we can follow the proof of $(b) \Rightarrow (a)$ replacing R_x by R/m to show that M is N-projective.

Assume (d) for locally perfect R. Then R_m is local and perfect, the radical of R_m is equal to mR_m and is t-nilpotent. By the canonical isomorphism $R_m/mR_m \simeq R/m$, we see from above that $M\otimes R_m$ is $N\otimes R_m$-projective. From 15.8 we have a monomorphism

$$(*) \qquad Hom_A(M,N)\otimes_R R_m \to Hom_{A_m}(M_m,N_m).$$

Tensoring with R_m/mR_m we get

$$\begin{array}{rcl}
Hom_A(M,N)\otimes_R R/m & \to & Hom_{A_m}(M_m,N_m)\otimes_{R_m} R_m/mR_m \\
& & \simeq Hom_{A/mA}(M/mM,N/mN),
\end{array}$$

with an isomorphism given by 15.8. Since the composition of the two maps is an isomorphism by assumption and mR_m is t-nilpotent, we conclude from 17.3 that $(*)$ is an isomorphism, for every $m \in \mathcal{M}$. Now apply (c) to see that M is N-projective.

Now assume (d) for ${}_AM$ finitely generated and ${}_RN$ noetherian. There exists an exact sequence $A^n \to M \to 0$, for some $n \in I\!N$. From this we deduce

$$Hom_A(M,N) \subset Hom_A(A^n,N) \simeq N^n.$$

Since N^n is R-noetherian, $Hom_A(M,N)$ is a finitely generated R-module and obviously $Hom_A(M,N/K)$ is a finitely generated R-module for every submodule $K \subset M$. Referring to 17.3 we again follow the arguments for $(b) \Rightarrow (a)$ to get the desired result. □

19.2 Self-projective modules.

For a finitely generated A-module M, the following are equivalent:

(a) *M is self-projective;*

(b) *M is finitely presented in $\sigma[M]$, and for every $x \in \mathcal{X}$, M_x is self-projective;*

(c) *for every $x \in \mathcal{X}$, M_x is self-projective and*

$$End_A(M) \otimes_R R_x \simeq End_{A_x}(M_x);$$

(d) *for every $m \in \mathcal{M}$, M_m is self-projective and*

$$End_A(M) \otimes_R R_m \simeq End_{A_m}(M_m).$$

Proof. $(a) \Leftrightarrow (c) \Leftrightarrow (d)$ follows from 19.1.

$(c) \Rightarrow (b)$ is clear and $(b) \Rightarrow (c)$ follows from 18.5. □

19.3 Projective modules and $-\otimes R/m$.

Put $J = Jac(R)$. Consider the exact sequence of A-modules

$$(*) \qquad 0 \longrightarrow K \xrightarrow{\alpha} P \xrightarrow{\beta} M \longrightarrow 0\,,$$

with P projective as an A-module and finitely generated as R-module. Then the following assertions are equivalent:

(a) *$(*)$ splits in A-Mod;*

(b) *$(*)$ splits in R-Mod and $\quad 0 \to K/JK \xrightarrow{\alpha'} P/JP \xrightarrow{\beta'} M/JM \to 0$ splits in A/JA-Mod;*

(c) *$(*)$ splits in R-Mod and $\quad 0 \to K/mK \xrightarrow{\alpha'} P/mP \xrightarrow{\beta'} M/mM \to 0$ splits in A/mA-Mod, for every $m \in \mathcal{M}$.*

Proof. $(a) \Rightarrow (b)$ and $(a) \Rightarrow (c)$ are obvious.

$(b) \Rightarrow (a)$ We construct the exact diagram with canonical projections,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{\alpha} & P & \xrightarrow{\beta} & M & \longrightarrow & 0 \\
 & & \downarrow p_K & & \downarrow p_P & & \downarrow p_M & & \\
0 & \longrightarrow & K/JK & \xrightarrow{\alpha'} & P/JP & \xrightarrow{\beta'} & M/JM & \longrightarrow & 0\,.
\end{array}$$

Since $(*)$ splits as an R-sequence, the lower sequence is exact. By assumption, there exists $\gamma : M \to P/JP$ with $\gamma\beta' = p_M$. By the Homotopy Lemma we can find some $\delta : P \to K/JK$ with $\alpha\delta = p_K$.

P being A-projective there exists $h : P \to K$ with $hp_k = \delta$ and $\alpha h p_K = \alpha\delta = p_K$. Since $JK \ll K$ this implies that αh is a surjective endomorphism of the finitely generated R-module K. Hence αh is an isomorphism by 17.4, and the upper sequence splits in *A-Mod*.

$(c) \Rightarrow (a)$ Since $(*)$ splits in *R-Mod*, K is finitely generated as R-module and hence as an A-module. This implies that M is finitely presented in *A-Mod* and by 15.8,

$$Hom_A(M,A) \otimes_R R_m \simeq Hom_{A_m}(M_m, A_m).$$

According to 19.1, $(*)$ splits (i.e., M is projective) if and only if $(*)$ splits after tensoring with R_m (i.e., M_m is A_m-projective).

Recall that in the local ring R_m, $Jac\, R_m = mR_m$. By the canonical isomorphism $R_m/mR_m \simeq R/m$, we deduce from $(b) \Rightarrow (a)$ that $(*)$ splits after tensoring with R_m and the assertion is shown. $\square$

The above results yields several descriptions of projectives in *A-Mod*:

19.4 Projective modules.

Let M be a finitely generated A-module. The following are equivalent:

(a) *M is A-projective (projective in A-Mod);*

(b) *M is finitely presented in A-Mod, and for every $x \in \mathcal{X}$, M_x is A_x-projective;*

(c) *M is finitely presented in A-Mod, and for every $m \in \mathcal{M}$, M_m is A_m-projective;*

If A is finitely generated as an R-module, then (a)-(c) are equivalent to:

(d) *There exists an (A,R)-exact sequence $P \to M \to 0$, where P is A-projective and for every $m \in \mathcal{M}$, M/mM is A/mA-projective.*

Proof. $(a) \Leftrightarrow (b) \Leftrightarrow (c)$ follow from 19.1 and the fact that the isomorphisms necessary hold for finitely presented A-modules (see 15.8).

$(a) \Rightarrow (d)$ is obvious.

$(d) \Rightarrow (a)$ Since A and M are finitely generated as R-modules, we may assume that P is also a finitely generated R-module. Now apply 19.3. $\square$

In the above cases we had to check a whole family of scalar extensions to get results over the ring R. For associative algebras which are faithfully flat R-modules, it is enough to consider only one extension:

19.5 Faithfully flat algebras.

Let M, N be A-modules. If B is faithfully flat as R-module, then:

(1) *M is a finitely generated A-module if and only if $M \otimes_R B$ is a finitely generated $A \otimes_R B$-module.*

(2) *M is a finitely presented A-module if and only if $M \otimes_R B$ is a finitely presented $A \otimes_R B$-module.*

(3) Assume ${}_AM$ is finitely generated. Then M is N-projective if and only if $M \otimes_R B$ is $N \otimes_R B$-projective and

$$Hom_A(M, N) \otimes_R B \simeq Hom_{A\otimes_R B}(M \otimes_R B, N \otimes_R B).$$

(4) M is finitely generated and A-projective if and only if $M \otimes_R B$ is finitely generated and $A \otimes_R B$-projective.

Proof. (1) If M is finitely generated there exists an epimorphism $A^n \to M$ yielding an epimorphism $(A \otimes_R B)^n \to M \otimes_R B$. Hence $M \otimes_R B$ is finitely generated as $A \otimes_R B$-module.

Assume $M \otimes_R B$ is finitely generated with generating set $m_1 \otimes 1, \ldots, m_k \otimes 1$. Denote by M' the A-submodule of M generated by $m_1, \ldots, m_k$. Tensoring the inclusion $M' \to M$ with $- \otimes_R B$ we get an epimorphism $M' \otimes_R B \to M \otimes_R B$. Since B is faithfully flat this means $M' = M$ and ${}_AM$ is finitely generated.

(2) M is a finitely presented A-module if and only if there exists an exact sequence $0 \to K \to A^n \to M \to 0$, where K is finitely generated. Tensoring with $- \otimes_R B$ and applying (1), the resulting sequence shows that $M \otimes_R B$ is a finitely presented $A \otimes_R B$-module.

The converse conclusion is obtained in a similar way.

(3) This is shown with the proof of 19.1.

(4) If M is finitely generated and A-projective, the epimorphism in the proof of (1) splits and hence $M \otimes_R B$ is $A \otimes_R B$-projective.

Assume $M \otimes_R B$ to be finitely generated and $A \otimes_R B$-projective. Then by (2), M is a finitely presented A-module and by 15.8,

$$Hom_A(M, A) \otimes_R B \simeq Hom_{A\otimes_R B}(M \otimes_R B, A \otimes_R B).$$

Now we learn from (3) that M is A-projective. $\square$

Finally we investigate the behaviour of generators under localizations (see Section 5 for definitions). We start with a general observation:

19.6 Generators.

Let M be an A-module. If M is a generator in $\sigma[M]$, then $M \otimes_R B$ is a generator in $\sigma[M \otimes_R B]$ as $A \otimes_R B$-module.

Proof. It is easily verified that $M \otimes_R B$ belongs to $\sigma[M]$ as an A-module. Then also any $A \otimes_R B$-submodule L of $(M \otimes_R B)^{(\mathbb{N})}$ belongs to $\sigma[M]$ and hence is M-generated as an A-module. Applying an isomorphism from 15.6, we obtain the commutative exact diagram

$$\begin{array}{ccccc} M \otimes_R Hom_A(M,L) & \longrightarrow & L & \longrightarrow & 0 \\ \downarrow\simeq & & \| & & \\ (M \otimes_R B) \otimes_B Hom_{A\otimes_R B}(M \otimes_R B, L) & \longrightarrow & L & \longrightarrow & 0 \end{array} .$$

Hence $M \otimes_R B$ generates the submodules of $(M \otimes_R B)^{(\mathbb{N})}$ and therefore it is a generator in $\sigma[M \otimes_R B]$ as $A \otimes_R B$-module. □

19.7 Self-generators.

Let M be an A-module which is finitely presented in $\sigma[M]$. Then the following are equivalent:

(a) M is a self-generator;

(b) for every $x \in \mathcal{X}$, M_x is a self-generator as A_x-module.

Assume R is regular. Then (a),(b) are also equivalent to:

(c) For every $m \in \mathcal{M}$, M_m is a self-generator as A_m-module.

Proof. $(a) \Rightarrow (b),(c)$ For $x \in \mathcal{X}$ and $m \in \mathcal{M}$, xM and mM are fully invariant submodules. Since $M_x \simeq M/xM$ and, for regular R, $M_m \simeq M/mM$, the assertion follows from 5.2.

$(b) \Rightarrow (a)$ M is a self-generator if the map $\mu : M \otimes_R Hom_A(M,K) \to K$ is surjective, for every (cyclic) submodule $K \subset M$. Tensoring with R_x we obtain the commutative diagram

$$\begin{array}{ccccc} M \otimes_R Hom_A(M,K) \otimes_R R_x & \xrightarrow{\mu\otimes id_{R_x}} & K \otimes_R R_x & & \\ \downarrow & & \downarrow\simeq & & \\ M_x \otimes_{R_x} Hom_{A_x}(M_x, K_x) & \longrightarrow & K_x & \longrightarrow & 0 \end{array} .$$

By 18.5 and 15.6, the first vertical map is an isomorphism. By assumption, the lower sequence is exact. Therefore $\mu \otimes id_{R_x}$ is surjective, for every $x \in \mathcal{X}$. Hence μ is surjective by 18.1.

$(c) \Rightarrow (a)$ This can be shown with the above proof. The necessary isomorphisms are obtained from 17.8 and 15.6. □

Applying our knowledge of projective modules we obtain the following local-global characterization of self-progenerators (see 5.10):

19.8 Self-progenerators.

For a finitely generated A-module M, the following are equivalent:

(a) M is a self-progenerator;

(b) M is finitely presented in $\sigma[M]$, and for every $x \in \mathcal{X}$, M_x is a self-progenerator;

(c) for every $x \in \mathcal{X}$, M_x is a self-progenerator and

$$End_A(M) \otimes_R R_x \simeq End_{A_x}(M_x);$$

(d) for every $m \in \mathcal{M}$, M_m is a self-progenerator and

$$End_A(M) \otimes_R R_m \simeq End_{A_m}(M_m).$$

Assume R is locally perfect. Then (a)-(d) are also equivalent to:

(e) for every $m \in \mathcal{M}$, M/mM is a self-progenerator and

$$End_A(M) \otimes_R R/m \simeq End_{A/mA}(M/mM).$$

Proof. $(a) \Rightarrow (b)$ Since M is finitely generated and M-projective, it is finitely presented in $\sigma[M]$. By 19.2, M_x is a self-projective A_x-module. By 19.6, M_x is a self-generator as an A_x-module.

$(b) \Rightarrow (c), (d)$ The isomorphisms follow from 18.5.

$(c) \Rightarrow (a)$ By 19.2, under the given conditions M is self-projective. The proof of $(b) \Rightarrow (a)$ in 19.7 can be used to show that M is a self-generator.

$(d) \Rightarrow (a)$ Use the same argument as for $(c) \Rightarrow (a)$.

$(a) \Rightarrow (e)$ is obtained from 19.1 and 19.6.

$(e) \Rightarrow (a)$ By 19.2, M is self-projective. Applying 17.3(3), we again use the proof of $(b) \Rightarrow (a)$ in 19.7 to conclude that M is a self-generator. □

If M is a faithful A-module which is finitely generated as module over $End_A(M)$, then A is isomorphic to a submodule of a finite direct sum of copies of M (M is cofaithful) and hence $\sigma[M] = A$-*Mod* (4.4). From the preceding results we obtain:

19.9 Progenerators in A-*Mod*.

For a finitely generated A-module M and $B = End_A(M)$, the following are equivalent:

(a) M is a progenerator in A-Mod;

(b) ${}_AM$ is a faithful self-progenerator and M_B is finitely generated;

(c) M is finitely presented in A-Mod and, for every $x \in \mathcal{X}$, M_x is a progenerator in A_x-Mod;

(d) M is finitely presented in A-Mod and for every $m \in \mathcal{M}$, M_m is a progenerator in A_m-Mod.

If A is finitely generated as an R-module, then (a)-(d) are equivalent to:

(e) *For every $m \in \mathcal{M}$, M/mM is a progenerator in A/mA-Mod and* $End_A(M) \otimes_R R/m \simeq End_{A/mA}(M/mM)$.

Proof. $(a) \Rightarrow (b)$ Any generator in A-*Mod* is finitely generated (and projective) over its endomorphism ring (see 5.5).

$(b) \Rightarrow (a)$ As pointed out above, under the given conditions $\sigma[M] = A$-*Mod*.

$(a) \Rightarrow (c), (d), (e)$ By 19.8, M_x and M_m are self-progenerators. Since M_B is finitely generated we deduce $\sigma[M_x] = A_x$-*Mod* and $\sigma[M_m] = A_m$-*Mod*, for all $x \in X$ and all $m \in \mathcal{M}$.

$(c) \Rightarrow (a)$ Since ${}_AM$ is finitely presented in A-*Mod*, for every A-module K, $yHom_A(M, K) \otimes R_x \simeq Hom_A(M, K \otimes R_x)$ (see 18.5). By 19.8, M is a progenerator in $\sigma[M]$. Following the proof $(b) \Rightarrow (a)$ of 19.7, we see that M generates every A-module K and hence $\sigma[M] = A$-*Mod*.

$(d) \Rightarrow (a)$ is shown with the same proof.

$(e) \Rightarrow (a)$ If the canonical map $M \otimes Hom_A(M, K) \to K$ is surjective for every finitely generated A-module K, then M is a generator in A-*Mod*.

Since A is a finitely generated A-module, K is also a finitely generated R-module. Observing 17.3 we apply the proof of $(b) \Rightarrow (a)$ in 19.7 to obtain the assertion. □

The type of modules considered above is closed under tensor products:

19.10 Tensorproduct of self-projective modules.
Let M be an A-module and N a B-module, both finitely generated.

(1) *If M and N are self-projective, then $M \otimes_R N$ is self-projective as an $A \otimes_R B$-module.*

(2) *If M and N are self-progenerators, then $M \otimes_R N$ is a self-progenerator as an $A \otimes_R B$-module.*

(3) *If M is A-projective and N is B-projective, then $M \otimes_R N$ is $A \otimes_R B$-projective.*

(4) *If M is a progenerator in A-Mod and N is a progenerator in B-Mod, then $M \otimes_R N$ is a progenerator in $A \otimes_R B$-Mod.*

In all these cases,

$$End_{A \otimes_R B}(M \otimes_R N) \simeq End_A(M) \otimes_R End_B(N).$$

Proof. (1) For N there is an R-epimorphism $R^{(\Lambda)} \to N$. Tensoring with M yields the A-epimorphism $M \otimes_R R^{(\Lambda)} \to M \otimes_R N$. This implies $\sigma[M \otimes_R N] \subset \sigma[M]$ and similarly $\sigma[M \otimes_R N] \subset \sigma[N]$.

Consider an exact $A \otimes_R B$-sequence

$$(*) \qquad M \otimes_R N \to X \to 0.$$

Since M is projective in $\sigma[M]$, we obtain the exact B-sequence

$$Hom_A(M, M \otimes_R N) \to Hom_A(M, X) \to 0.$$

By 15.7, $Hom_A(M, M \otimes_R N) \simeq Hom_A(M, M) \otimes_R N$ and we notice that the above sequence lies in fact in $\sigma[N]$. Hence $Hom_B(N, -)$ yields the upper sequence exact in the commutative diagram

$$\begin{array}{ccccc} Hom_B(N, Hom_A(M, M \otimes_R N)) & \to & Hom_B(N, Hom_A(M, X)) & \to & 0 \\ \downarrow\simeq & & \downarrow\simeq & & \\ Hom_{A\otimes_R B}(M \otimes_R N, M \otimes_R N)) & \to & Hom_{A\otimes_R B}(M \otimes_R N, X) & \to & 0 \end{array},$$

with the vertical isomorphisms given in 15.6. So $Hom_{A\otimes_R B}(M \otimes_R N, -)$ is exact with respect to $(*)$, i.e., $M \otimes_R N$ is self-projective.

(2) From the above proof we notice that

$$Hom_{A\otimes_R B}(M \otimes_R N, -) \simeq Hom_B(N, Hom_A(M, -))$$

is a functorial isomorphism on $\sigma[M \otimes_R N]$. Since both functors on the right side are exact and faithful, the same is true for $Hom_{A\otimes_R B}(M \otimes_R N, -)$ and hence $M \otimes_R N$ is a generator in $\sigma[M \otimes_R N]$.

(3),(4) Repeat the proof of (1),(2), replacing $(*)$ by exact sequences

$$A \otimes_R B \to X \to 0.$$

The isomorphism for the endomorphism rings follows from 15.9. $\square$

19.11 Exercises.

Let A be an associative unital R-algebra and M a finitely presented module in A-Mod. Prove:

(1) For an A-module N, the following are equivalent:
(a) N is M-generated;
(b) for every $m \in \mathcal{M}$, N_m is M_m-generated;
(c) for every $x \in \mathcal{X}$, N_x is M_x-generated.

(2) The following are equivalent:
(a) M is a generator in A-Mod;
(b) for every $x \in \mathcal{X}$, M_x is a generator in A_x-Mod;
(c) for every $m \in \mathcal{M}$, M_m is a generator in A_m-Mod.

(3) For an exact sequence $P \to M \to 0$ in A-Mod, the following are equivalent:
(a) $P \to M \to 0$ splits in A-Mod;
(b) for every $x \in \mathcal{X}$, $P_x \to M_x \to 0$ splits in A_x-Mod;
(c) for every $m \in \mathcal{M}$, $P_m \to M_m \to 0$ splits in A_m-Mod.

(In Szeto [254, Theorem 3.1] the first part is erroneously stated for M finitely generated instead of finitely presented; see also Endo [122, Propositions 2.4, 2.5]).

References: Armendariz-Fisher-Steinberg [55], Burgess [94], Burkholder [99, 100, 102], Byun [103], Dauns-Hofmann [113], Endo [122], Magid [189], Matsumura [25], Kambara-Oshiro [172], Pierce [32], Szeto [254], Villamayor-Zelinsky [260], Wisbauer [276].

20 Relative semisimple modules

1.Properties of (M,R)-projectives. 2.Characterization of (M,R)-injectives. 3.(A,R)-projectives and (A,R)-injectives. 4.(M,R)-injectives in $\sigma[M]$. 5.(A,R)-semisimple and (A,R)-regular modules. 6.Properties of (A,R)-semisimple modules. 7.Finitely generated (A,R)-semisimple modules. 8.Properties of (A,R)-regular modules. 9.Left (A,R)-semisimple algebras. 10.Left (A,R)-semisimple algebras. Properties. 11.Left (A,R)-regular algebras. 12.Left (A,R)-regular algebras. Properties. 13.Exercises.

Throughout this section A is an *associative unital* R-algebra. (Left) A-modules are R-modules and exact sequences in A-*Mod* are also exact sequences in R-*Mod*.

Definitions. A sequence in A-*Mod*

$$(*) \quad 0 \longrightarrow K \xrightarrow{f} L \xrightarrow{g} N \longrightarrow 0$$

is said to be (A,R)-*exact* if it is exact in A-*Mod* and it splits in R-*Mod*. In this case f is called an *R-split A-monomorphism* and g is an *R-split A-epimorphism*.

Let M,P,Q be A-modules. P is called (M,R)-*projective* if $Hom_A(P,-)$ is exact with respect to all (A,R)-exact sequences in $\sigma[M]$.

Q is called (M,R)-*injective* if $Hom_A(-,Q)$ is exact with respect to all (A,R)-exact sequences in $\sigma[M]$.

By standard arguments it is easily seen that direct sums and direct summands of (M,R)-projective modules are again (M,R)-projective. Similarly, direct products and direct summands of (M,R)-injective modules are (M,R)-injective.

Obviously, any projective module in $\sigma[M]$ is (M,R)-projective and every M-injective module is (M,R)-injective.

If R is a semisimple ring, then for modules in $\sigma[M]$, (M,R)-projective and (M,R)-injective are synonymous to projective and injective in $\sigma[M]$.

20.1 Properties of (M,R)-projectives. *Let M,P be A-modules.*

(1) *If P is (M,R)-projective, then $P\otimes_R T$ is $(M\otimes_R T,T)$-projective for any scalar R-algebra T.*

(2) *For $P\in\sigma[M]$, the following are equivalent:*

   (a) *P is (M,R)-projective;*

   (b) *every (A,R)-exact sequence $L\to P\to 0$ in $\sigma[M]$ splits.*

   If P is finitely presented in $\sigma[M]$, then (a) is equivalent to:

   (c) *for every $x\in\mathcal{X}$, P_x is (M_x,R_x)-projective.*

   If P is finitely presented in A-Mod, then (a) is equivalent to:

   (d) *for every $m\in\mathcal{M}$, P_m is (M_m,R_m)-projective.*

Proof. (1) By 15.6, there is a functorial isomorphism on $\sigma[M \otimes_R T]$,

$$Hom_{A\otimes T}(P \otimes_R T, -) \simeq Hom_A(P, -).$$

(2) $(a) \Rightarrow (b)$ is obvious and $(a) \Rightarrow (c), (d)$ is clear by (1).

$(b) \Rightarrow (a)$ Consider an (A, R)-exact sequence $(*)$ in $\sigma[M]$ and a morphism $P \to N$. Forming a pullback we have the commutative exact diagram

$$\begin{array}{ccccccccc} 0 & \to & K & \to & Q & \to & P & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & K & \to & L & \to & N & \to & 0 \end{array} .$$

Since the lower sequence is R-splitting, the same holds for the upper sequence, which in fact splits as an A-sequence by (b). Hence P is (M, R)-projective.

$(c) \Rightarrow (a)$ From an (A, R)-exact sequence $(*)$ in $\sigma[M]$ and isomorphisms provided by 18.5, we have the commutative diagram

$$\begin{array}{cccc} Hom_A(P, L) \otimes_R R_x & \to & Hom_A(P, N) \otimes_R R_x & \\ \downarrow\simeq & & \downarrow\simeq & \\ Hom_{A_x}(P_x, L_x) & \to & Hom_{A_x}(P_x, N_x) & \to 0. \end{array}$$

By assumption, the lower row is exact and hence the upper morphism is epic, for all $x \in \mathcal{X}$. By 18.1, $Hom_A(P, L) \to Hom_A(P, N)$ is epic and P is (M, R)-projective.

$(d) \Rightarrow (a)$ Apply the same proof as above with isomorphisms from 15.8. □

Dual to part of 20.1 we have the

20.2 Characterization of (M, R)-injectives.

For an A-module M and $Q \in \sigma[M]$, the following are equivalent:

(a) Q is (M, R)-injective;

(b) every (A, R)-exact sequence $0 \to Q \to L$ in $\sigma[M]$ splits.

There need not be (M, R)-projectives in $\sigma[M]$ but there are enough (M, R)-injectives. To see this we first look at the situation in A-*Mod*.

20.3 (A, R)-projectives and (A, R)-injectives.

(1) For any R-module X, $A \otimes_R X$ is (A, R)-projective.

(2) An A-module P is (A, R)-projective if and only if the map

$$A \otimes_R P \to P, \ a \otimes p \mapsto ap,$$

splits in A-Mod.

(3) For any R-module Y, $Hom_R(A, Y)$ is (A, R)-injective.

(4) An A-module Q is (A, R)-injective if and only if the map

$$Q \to Hom_R(A, Q),\ q \mapsto [a \mapsto aq],$$

splits in A-Mod.

Proof. (1) By 15.6, there is a functorial isomorphism on R-*Mod*,

$$Hom_A(A \otimes_R X, -) \simeq Hom_R(X, -).$$

Since the functor on the right is exact on (A, R)-exact sequences, the same is true for the functor on the left.

(2) The A-epimorphism $A \otimes_R P \to P$ is R-split by $m \mapsto 1 \otimes m$. Hence it is A-split if and only if P is (A, R)-projective.

(3) By Hom-tensor relations ([40, 12.12]), there are functorial isomorphisms on A-*Mod*,

$$Hom_R(-, Y) \simeq Hom_R(A \otimes_A -, Y) \simeq Hom_A(-, Hom_R(A, Y)).$$

The functor on the left is exact on (A, R)-exact sequences and so the same holds for the functor on the right.

(4) The A-monomorphism $Q \to Hom_R(A, Q)$ is R-split by $f \mapsto (1)f$. So it is A-split if and only if Q is (A, R)-injective. □

With the above we are now able to show:

20.4 (M, R)-injectives in $\sigma[M]$.

Let M be an A-module. Every module in $\sigma[M]$ is an R-split submodule of an (M, R)-injective module in $\sigma[M]$.

Proof. By 20.3, every $N \in \sigma[M]$ is an R-split submodule of the (A, R)-injective module $Hom_R(A, N)$.

Choose any generator G in $\sigma[M]$ and form the trace in A-*Mod*,

$$\widetilde{N} = Tr(G, Hom_R(A, N)).$$

Then N is an R-split submodule of $\widetilde{N} \in \sigma[M]$ and it is easily verified that $\widetilde{N}$ is (M, R)-injective. □

Definitions. An A-module M is said to be (A, R)-*semisimple* if every (A, R)-exact sequence in $\sigma[M]$ splits. M is called (A, R)-*regular* if every (A, R)-exact sequence is pure in $\sigma[M]$ (see 7.2).

Obviously, over a semisimple ring R, every (A, R)-semisimple module is a semisimple A-module and every (A, R)-regular module M is regular in $\sigma[M]$. In general we have the following characterizations:

20.5 (A, R)**-semisimple and** (A, R)**-regular modules.**

(1) For an A-module M, the following are equivalent:
(a) M is (A, R)-semisimple;
(b) every A-module (in $\sigma[M]$) is (M, R)-projective;
(c) every A-module (in $\sigma[M]$) is (M, R)-injective.

(2) For an A-module M, the following are equivalent:
(a) M is (A, R)-regular;
(b) every pure projective module in $\sigma[M]$ is (M, R)-projective;
(c) every finitely presented module in $\sigma[M]$ is (M, R)-projective.

Proof. (1) The assertions follow easily from the definitions.

(2) $(a) \Rightarrow (b)$ Given (a), every pure projective module in $\sigma[M]$ is projective relative to (A, R)-exact sequences.

$(b) \Rightarrow (c)$ Finitely presented modules are pure projective in $\sigma[M]$.

$(c) \Rightarrow (a)$ Given (c), every (A, R)-exact sequence is pure in $\sigma[M]$. □

We consider some properties of these modules.

20.6 Properties of (A, R)**-semisimple modules.**

Assume M is an (A, R)-semisimple A-module. Then:

(1) Every module in $\sigma[M]$ is (A, R)-semisimple.

(2) For every ideal $I \subset R$, M/IM is $(A/IA, R/I)$-semisimple.

(3) For every multiplicative subset $S \subset R$, MS^{-1} is (AS^{-1}, RS^{-1})-semisimple.

(4) If R is an integral domain with quotient field Q, then $M \otimes_R Q$ is a semisimple $A \otimes_R Q$-module.

(5) If R is hereditary and M is R-projective, then M is hereditary in $\sigma[M]$.

Proof. (1) is easy to see and (2) is a consequence of (1).

(3) Every $N \in \sigma[MS^{-1}]$ is (M, R)-projective and hence $N \otimes_R RS^{-1} \simeq N$ is (MS^{-1}, RS^{-1})-projective by by 20.1.

Therefore MS^{-1} is (AS^{-1}, RS^{-1})-semisimple by 20.5.

(4) By (3), $M \otimes_R Q$ is $(A \otimes_R Q, Q)$-semisimple and hence semisimple as an $A \otimes_R Q$-module.

(5) Since every submodule of M is projective as an R-module, it is also projective in $\sigma[M]$. □

20.7 Finitely generated (A, R)-semisimple modules.

Let M be an (A, R)-semisimple A-module.

(1) *If R is a perfect ring, or M is finitely generated as R-module and R is a semilocal ring, then $Rad({}_AM) = Jac(R)M$.*

(2) *If M is finitely generated and self-projective as A-module, and R is locally perfect, then M is a progenerator in $\sigma[M]$.*

(3) *If M is finitely generated as R-module and self-projective as A-module, then M is a progenerator in $A/An(M)$-Mod.*

Proof. (1) Denote $J = Jac(R)$. By 6.17, $JM \subset Rad({}_AM)$.

Since $M \otimes_R R/J \simeq M/JM$ is a semisimple A-module, $Rad({}_AM) \subset JM$.

(2) For every $m \in \mathcal{M}$, M/mM is a semisimple A-module and hence a self-progenerator. So the assertion follows from 19.8.

(3) Under the given conditions, M is finitely generated over its endomorphism ring and hence $\sigma[M] = A/An(M)$-*Mod* (4.4). Now the statement follows from (2). □

20.8 Properties of (A, R)-regular modules.

Assume M is an (A, R)-regular A-module and there is a generating set of finitely presented modules in $\sigma[M]$. Then:

(1) *For every multiplicative subset $S \subset R$, MS^{-1} is (AS^{-1}, RS^{-1})-regular.*

(2) *If R is an integral domain with quotient field Q, then $M \otimes_R Q$ is regular in $\sigma[M \otimes_R Q]$.*

Proof. Since there are enough finitely presented objects in $\sigma[M]$, pure exact sequences are direct limits of splitting sequences in $\sigma[M]$ (see 7.2).

(1) Consider an RS^{-1}-splitting sequence in $\sigma[MS^{-1}]$,

$$(*) \quad 0 \to K \to L \to N \to 0\,.$$

This is also an R-splitting sequence in $\sigma[M]$ and hence - by assumption - a direct limit of splitting sequences in $\sigma[M]$,

$$0 \to K_\lambda \to L_\lambda \to N_\lambda \to 0.$$

Tensoring with RS^{-1} we obtain a family of splitting sequences in $\sigma[MS^{-1}]$ whose direct limit yields $(*)$. Hence $(*)$ is pure in $\sigma[MS^{-1}]$.

(2) By (1), $M \otimes_R Q$ is $(A \otimes_R Q, Q)$-regular and hence regular in $\sigma[M \otimes_R Q]$. □

Definitions. An R-algebra A is called *left (A, R)-regular* if A is (A, R)-regular as left A-module. A is said to be *left (A, R)-semisimple* if A is (A, R)-semisimple as left A-module.

These classes of algebras have nice properties.

20.9 Left (A, R)-semisimple algebras. Characterizations.
For the algebra A the following are equivalent:

(a) *A is left (A, R)-semisimple;*

(b) *every left A-module is (A, R)-projective;*

(c) *for any A-module N, $A \otimes_R N \to N$, $a \otimes n \mapsto an$, splits in A-Mod;*

(d) *every left A-module is (A, R)-injective;*

(e) *for any A-module N, $N \to Hom_R(A, N)$, $n \mapsto [a \mapsto an]$, splits in A-Mod.*

Proof. $(a) \Leftrightarrow (b) \Leftrightarrow (d)$ is clear by 20.5.

$(b) \Leftrightarrow (c)$ and $(d) \Leftrightarrow (e)$ follow from 20.3.

$(a) \Rightarrow (d)$ and $(a) \Rightarrow (e)$ follow from 20.8. □

20.10 Left (A, R)-semisimple algebras. Properties.
Let A be a left (A, R)-semisimple algebra and $M \in A$-Mod.

(1) *For every ideal $I \subset R$, A/IA is $(A/IA, R/I)$-semisimple.*

(2) *If M is a flat R-module, then M is a flat A-module.*

(3) *If M is a projective R-module, then M is a projective A-module.*

(4) *If M is an injective R-module, then M is an injective A-module.*

(5) *If R is a regular ring, then A is a regular ring.*

(6) *If R is a hereditary ring and A is a projective R-module, then A is a left hereditary ring.*

(7) *If R is a locally perfect ring, or A is finitely generated as R-module and R is a semilocal ring, then $Jac(A) = Jac(R)A$.*

Proof. (1) and (6) follow from 20.6. (3) and (4) are obvious.

(7) is a consequence of 20.7.

(2) and (5) will be shown in 20.12 in a more general situation. □

20.11 Left (A, R)-regular algebras. Characterizations.

For the algebra A, the following are equivalent:

(a) *A is left (A, R)-regular;*

(b) *every pure projective (finitely presented) module in A-Mod is (A, R)-projective;*

(c) *for any finitely presented A-module N, $A \otimes_R N \to N$, $a \otimes n \mapsto an$, splits in A-Mod;*

(d) *every pure injective module in A-Mod is (A,R)-injective;*

(e) *for every $x \in \mathcal{X}$, A_x is left (A_x, R_x)-regular;*

(f) *for every $m \in \mathcal{M}$, A_m is left (A_m, R_m)-regular.*

Proof. $(a) \Leftrightarrow (b) \Rightarrow (d)$ is clear by 20.5.

$(b) \Leftrightarrow (c)$ follows from 20.3.

$(d) \Leftrightarrow (a)$ If every pure injective module is injective with respect to a short exact sequence, then this sequence is pure (see 7.2).

$(a) \Rightarrow (e)$ and $(a) \Rightarrow (f)$ follow from 20.8.

$(e) \Rightarrow (b)$ For any finitely presented A-module P and $x \in \mathcal{X}$, P_x is a finitely presented A_x-module and hence is (A_x, R_x)-projective by (d). As shown in 20.1, this implies that P is (A,R)-projective and hence A is left (A,R)-regular.

$(f) \Rightarrow (b)$ This is seen with the same proof. □

Clearly, a regular algebra A is left (and right) (A,R)-regular for any R. For regular rings R the converse is also true:

20.12 Left (A,R)-regular algebras. Properties.

Assume the R-algebra A is left (A,R)-regular. Then:

(1) *A left A-module which is flat as an R-module is flat as an A-module.*

(2) *If R is a regular ring, then A is a regular ring.*

Proof. (1) Let M be an A-module which is flat as R-module. Then M is a direct limit of finitely generated projective R-modules P_λ, $\lambda \in \Lambda$. Obviously all $A \otimes_R P_\lambda$ are projective A-modules and hence

$$A \otimes_R M \simeq \varinjlim (A \otimes_R P_\lambda)$$

is a flat A-module. Since $A \otimes_R M \to M$ splits in A-*Mod*, M is also a flat A-module.

(2) Assume R is regular. Then every R-module is flat and by (1), every A-module is flat in A-*Mod*. Hence A is regular. □

Remarks. The investigations in this section are part of *relative homological algebra*. The basic idea is that the class of short exact sequences as central notion is replaced by a more restricted class and homological algebra is developped *relative* to this class. For this, notions like *relative projective*, *relative injective*, *relative projective resolutions*, and so on, are used. This can be subsumed under general *purity* (see Exercise 5 below). A presentation of these ideas for arbitrary categories can be found in Schubert [38].

20.13 Exercises.

Let A be an associative R-algebra with unit.

(1) Schanuel's Lemma *(compare [40, 50.2]).*

For an A-module M, consider (A,R)-exact sequences in $\sigma[M]$,

$0 \to K \to P \to N \to 0$ *and* $0 \to L \to Q \to N \to 0,$

where P and Q both are (M,R)-projective. Prove $K \oplus Q \simeq L \oplus P$.

(2) Show that the following statements are equivalent:
(a) Every (A,R)-projective A-module is A-projective;
(b) every short exact sequence in A-Mod is (A,R)-exact;
(c) every left ideal A is an R-direct summand;
(d) every (A,R)-injective A-module is A-injective.

(3) Assume A is a finitely presented R-module and R is regular. Show that every finitely presented (A,R)-projective A-module is projective.

(4) Let A be left (A,R)-semisimple and projective as R-module. Prove:
(i) If R is a perfect ring, then A is a left perfect ring.
(ii) If R is a QF (quasi Frobenius) ring, then A is a QF ring.

(5) Consider the class of A-modules

$$\mathcal{P} = \{A \otimes_R X \mid X \in R\text{-}Mod\}.$$

Show:
(i) An exact sequence is (A,R)-exact if and only if it is $\mathcal{P}$-pure
(i.e., every $P \in \mathcal{P}$ is projective with respect to it).
(ii) A is left (A,R)-semisimple if and only if every $\mathcal{P}$-pure exact sequence splits
(i.e., A is $\mathcal{P}$-pure semisimple).

(6) Assume M is an (A,R)-semisimple self-projective A-module, and A and M are finitely generated as R-modules.

Show that $End_A(M)$ is left $(End_A(M),R)$-semisimple.

(7) For any subring $B \subset A$, the definition of (A,B)-exact sequences and (A,B)-projective modules (as given at the beginning of this section) makes sense. Many of the results obtained hold in fact in this more general situation.

Assume $C \subset B \subset A$ are subrings with the same unit element. Show for an A-module M:
(i) If M is (A,B)-projective and (B,C)-projective, then M is (A,C)-projective.
(ii) If M is (A,C)-projective, then M is (A,B)-projective.

References: Cunningham [112], Hattori [153], Higman [157], Hirata-Sugano [158], Hochschild [159], McMahon-Mewborn [196], Mishina-Skornjakov [26], Schubert [38], Sugano [250, 251], Wisbauer [271].

Chapter 6

Radicals of algebras

21 Radicals defined by some classes of algebras

1.Simple factor algebras. 2.Radicals defined by simple algebras. 3.Radicals of finite dimensional algebras. 4.Stalks with zero radical. 5.Proposition. 6.Brown-McCoy radical. 7.Structure of module finite algebras. 8.Prime and semiprime. 9.The prime radical. 10.Weakly local algebras. 11.Uniserial algebras. 12.The weakly local radical. 13.Exercises.

The classical Wedderburn Structure Theorem for a finite dimensional associative algebra A over a field states that the factor algebra A/N by the nil radical N is a finite direct product of simple algebras. These simple algebras are matrix rings over division rings.

There were many attempts to extend this type of theorem to more general situations. For associative rings A the Jacobson radical $Jac\, A$ can be defined as the sum of all quasi-regular (left) ideals and $A/Jac\, A$ is a subdirect product of primitive rings. Again, a certain ideal is determined by properties of its elements and the factor ring by this ideal allows certain statements about its structure.

In general, looking for a *radical* X in any algebra A there are two points to be watched. On the one hand one has to care for the structure of A/X, and on the other hand one would like to have an internal characterization of the ideal X.

Let us start with the first problem.

Definitions. An algebra without any proper non-zero ideals is called *quasi-simple.* A quasi-simple algebra A with $A^2 \neq 0$ is called *simple.*

An ideal $K \subset A$ is said to be *modular* if there exists $e \in A$ with

$$e \notin K,\ a - ae \in K,\ a - ea \in K \text{ for every } a \in A.$$

Obviously, the image of such an element e under $A \to A/K$ is the unit element in A/K. In a unital algebra every proper ideal is modular (with $e = 1$).

From the above definitions we obtain by 1.8:

21.1 Simple factor algebras.

Consider an ideal K in an algebra A.

(1) *The factor algebra A/K is quasi-simple if and only if K is a maximal $M(A)$-submodule (maximal ideal) of A.*

(2) *A/K is a simple algebra if and only if K is a maximal ideal with $A^2 \not\subset K$.*

(3) *A/K is a simple ring with unit if and only if K is a maximal modular ideal.*

The representation of algebras as subdirect products of factor algebras described in 1.10 yields our next results. As for associative rings, the *descending chain condition (dcc)* on ideals is of importance:

21.2 Radicals defined by simple algebras.

Let A be an R-algebra.

(1) *The radical of A as an $M(A)$-module is*

$$Rad(A) = \bigcap \{K \subset A \mid K \text{ a maximal ideal }\},$$

and $Rad(A/Rad(A)) = 0$.
$Rad(A) = 0$ if and only if A is a subdirect product of quasi-simple algebras.
If A has dcc on ideals, then $A/Rad(A)$ is a finite product of quasi-simple algebras.

(2) *The* Albert radical *of A is defined as*

$$Alb(A) = \bigcap \{K \subset A \mid K \text{ a maximal ideal with } A^2 \not\subset K\},$$

and $Alb(A/Alb(A)) = 0$.
$Alb(A) = 0$ if and only if A is a subdirect product of simple algebras.
If A has dcc on ideals K with $A^2 \not\subset K$, then $A/Alb(A)$ is a finite product of simple algebras.

(3) *The* Brown-McCoy radical *of A is defined as*

$$BMc(A) = \bigcap \{K \subset A \mid K \text{ a maximal modular ideal }\},$$

and $BMc(A/BMc(A)) = 0$. $BMc(A) = 0$ if and only if A is a subdirect product of simple algebras with units.
If A has dcc on modular ideals, then $A/BMc(A)$ is a finite product of simple algebras with units.

(4) *$Jac(M(A))A \subset Rad(A) \subset Alb(A) \subset BMc(A)$.*
If $A^2 = A$, then $Rad(A) = Alb(A)$.
If A has a unit, then $Rad(A) = Alb(A) = BMc(A)$.

(5) *Assume A is a finitely generated R-module, or R is a perfect ring. Then $Jac(R)A \subset Rad(A)$.*

Proof. As mentioned above, most of the statements are immediately derived from 1.10. For the second part of (3) it is to check that, for any modular ideals U, V with $U + V = A$, $U \cap V$ is also a modular ideal:

Assume $e \in A$ with $e \notin U$, $a - ae \in U$, $a - ea \in U$ for $a \in A$. It is easily verified that we may assume $e \in V$. Choosing similarly an $f \notin V$ and $f \in U$ the element $e + f$ shows that $U \cap V$ is a modular ideal.

(4) These relations follow from the definitions and 6.16.

(5) This is a special case of 6.17. □

For finite dimensional algebras we obtain as a corollary:

21.3 Radicals of finite dimensional algebras.
Let A be a finite dimensional algebra over a field K. Then:

(1) *$Rad(A) = 0$ if and only if A is a direct product of quasi-simple algebras.*

(2) *$Alb(A) = 0$ if and only if A is a direct product of simple algebras.*

(3) *$BMc(A) = 0$ if and only if A is a direct product of simple algebras with unit.*

Next we show that the above radicals of A are zero if all Pierce stalks have zero radical. Recall that we denote by $\mathcal{X}$ the set of all maximal ideals in the Boolean ring of idempotents of R (see 18.1).

21.4 Stalks with zero radical. *Let A be an R-algebra.*

(1) *If $Rad_{M(A_x)}(A_x) = 0$ for every $x \in \mathcal{X}$, then $Rad(A) = 0$.*

(2) *If $Alb(A_x) = 0$ for every $x \in \mathcal{X}$, then $Alb(A) = 0$.*

(3) *If $BMc(A_x) = 0$ for every $x \in \mathcal{X}$, then $BMc(A) = 0$.*

Proof. (1) A is a subdirect product of the A_x by 18.1. Assume all $Rad_{M(A_x)}(A_x) = 0$. Then the A_x are subdirect products of algebras without non-trivial ideals and the same is true for A, i.e., $Rad(A) = 0$.

The same argument yields (2) and (3). □

21.5 Proposition. *Let A be an R-algebra. The Brown-McCoy radical and the Albert radical of the R-algebra A are equal to the corresponding radical of the $\mathbb{Z}$-algebra (ring) A.*

Proof. We have to show that maximal $\mathbb{Z}$-ideals $I \subset A$ with $A^2 \not\subset I$ are also R-ideals.

Assume I is a maximal $\mathbb{Z}$-ideal in A. Then RI is an R- and a $\mathbb{Z}$-ideal containing I and hence $RI = I$ or $RI = A$. However, the last equality would imply $A^2 \subset RIA \subset IRA \subset IA \subset I$, contradicting the choice of I. □

By construction the radicals defined above yield satisfying structure theorems for algebras with zero radical. However, so far we have no internal characterizations for these radicals. For the Brown-McCoy radical we can get such a characterazition similar to the Jacobson radical of associative algebras (see 6.16).

For an element a in any algebra A we define $S(A)$ as the ideal of A generated by the elements $ax - x + ya - y$ for all $x, y \in A$, i.e.

$$S(a) := < \{ax - x + ya - y \mid x, y \in A\} > \subset A.$$

An element $a \in A$ is called *S-quasi-regular* if $a \in S(a)$.
An ideal $I \subset A$ is said to be *S-quasi-regular* if every element of I is S-quasi-regular.

21.6 Brown-McCoy radical. Internal characterization.

(1) For any algebra A the Brown-McCoy radical $BMc(A)$ is equal to the largest S-quasi-regular ideal.

(2) $BMc(A)$ contains no central idempotents.

Proof. (1) Assume $a \in A$ is not S-quasi-regular, i.e. $a \notin S(a)$. Then the set of ideals $I \subset A$ with $S(a) \subset I$ and $a \notin I$ is non-empty and inductive (by inclusion). By Zorn's Lemma it contains a maximal element, say J. We show that this is a maximal ideal in A.

Consider $b \in A \setminus J$ and the ideal $< b >$ generated by b. Then, by maximality of J, $a \in < b > +J$ and hence

$$a + ax - x \in < b > +J, \quad \text{for all} \quad x \in A.$$

This implies $< b > +J = A$. Since $S(a) \subset J$, J is modular and hence J is a maximal modular ideal not containing a, i.e. $a \notin BMc(A)$.

Now let Q be an S-quasi-regular ideal and K a maximal modular ideal in A. Assume $Q \not\subset K$. Then $K + Q = A$. By assumption, there exists $e \in A \setminus K$ with $b - be \in K$ and $b - eb \in K$ for every $b \in A$. Let us write $e = k + q$ with $k \in K$, $q \in Q$. Then

$$b - be = b - bq - bk \in K, \text{ and hence } b - bq \in K.$$

Similarly we obtain $b - qb \in K$. Since q is S-quasi-regular we conclude

$$q \in S(q) \subset K \quad \text{ and } \quad e = k + q \in K,$$

contradicting the choice of e. Hence $Q \subset BMc(A)$.

(2) Assume $e \in BMc(A)$ is a central idempotent. Then $e = ex - x$ for some $x \in A$ implying $e = e^2 = ex - ex = 0$. $\square$

If the R-algebra A is finitely generated as R-module, then $M(A) \subset A^k$ for some $k \in I\!N$ (see 2.12). Hence the structure of $M(A)$ is related to the structure of A:

21.7 Structure of module finite algebras.

Assume the R-algebra A is finitely generated as R-module.

(1) The following statements are equivalent:
   (a) A is a finite direct product of quasi-simple algebras;
   (b) $M(A)$ is left semisimple.

(2) If R is artinian, then the following are equivalent:
   (a) A is a finite direct product of quasi-simple algebras;
   (b) $Rad(A) = 0$;
   (c) $M(A)$ is left semisimple;
   (d) $Jac(M(A)) = 0$.

(3) If R is artinian, then:
   (i) $A/Rad(A)$ is a finite direct product of quasi-simple algebras.
   (ii) $M(A)/Jac(M(A))$ is a left semisimple algebra.
   (iii) $Rad(A) Jac(M(A)) \cdot A$.

Proof. (1) $(a) \Rightarrow (b)$ Since $M(A) \subset A^k$ and A is a semisimple $M(A)$-module, $M(A)$ is left semisimple.

$(b) \Rightarrow (a)$ If $M(A)$ is left semisimple, every left $M(A)$-module is semisimple.

(2) Since R is artinian, A and $M(A)$ have dcc on (left) ideals. Hence $(a) \Leftrightarrow (b)$ by 21.2 and $(c) \Leftrightarrow (d)$ by the structure theory of associative left artinian rings.

$(a) \Leftrightarrow (c)$ is shown in (1).

(3) (i) and (ii) are obvious by (2).

We always have $Jac(M(A)) \cdot A \subset Rad(A)$. By (i), $A/(Jac(M(A)) \cdot A)$ is a semisimple $M(A)$-module and hence $Jac(M(A)) \cdot A \supset Rad(A)$. $\square$

Definitions. Let A be an R-algebra. A proper ideal $P \subset A$ is called a *prime ideal* if, for any two ideals $U, V \subset A$, $UV \subset P$ implies $U \subset P$ or $V \subset P$.

P is a *semiprime ideal* if $U^2 \subset P$ implies $U \subset P$, for every ideal $U \subset A$.

A is called a *prime algebra* if 0 is a prime ideal. It is called a *semiprime algebra* if the ideal 0 is semiprime, i.e., if A has no non-zero ideals whose square is zero.

21.8 Prime and semiprime. *Let A be an R-algebra.*

(1) *Every maximal ideal $K \subset A$ with $A^2 \not\subset K$ is a prime ideal.*

(2) *A is a semiprime algebra if and only if, for any ideals $U, V \subset A$, $UV = 0$ implies $U \cap V = 0$.*

(3) *If A is (semi-) prime, then the centroid $C(A)$ is also a (semi-) prime algebra.*

Proof. (1) Consider ideals $U, V \subset A$ with $UV \subset K$. Assume $U \not\subset K$ and $V \not\subset K$. Then $K + U = K + V = A$ and

$$A^2 = (K + U)(K + V) \subset K,$$

a contradiction.

(2) Assume A is semiprime and $UV = 0$. Then $(U \cap V)^2 \subset UV = 0$ and hence $U \cap V = 0$. If the second property is given and $U^2 = 0$, then $U = U \cap U = 0$, i.e., A is semiprime.

(3) Let A be prime. Consider two ideals $I, J \subset C(A)$ with $IJ = 0$. Then $(AI)(AJ) = A^2IJ = 0$ and hence $AI = 0$ or $AJ = 0$. This implies $I = 0$ or $J = 0$ showing that $C(A)$ is prime.

Putting $I = J$ we obtain our assertion for semiprime algebras. □

Obviously, P is a (semi) prime ideal if and only if A/P is a (semi) prime algebra. It is easy to verify that A is (semi) prime as R-algebra if and only if it is (semi) prime as a $\mathbb{Z}$-algebra (ring).

21.9 The prime radical.

For an R-algebra A the prime radical *is defined as*

$$Pri(A) := \bigcap \{P \subset A \mid P \textit{ is a prime ideal }\}.$$

It has the following properties:

(1) *$Pri(A)$ is a semiprime ideal and $Pri(A/Pri(A)) = 0$.*

(2) *$Pri(A) = 0$ if and only if A is a subdirect product of prime algebras.*

(3) *$Pri(A) \subset Alb(A)$.*

Proof. (1) and (2) are immediate consequences of 1.10.

(3) This follows from the fact that any simple algebra is prime. □

A word of caution is in order. An associative algebras A with unit is semiprime if and only if $Pri(A) = 0$ (e.g., [40, 3.13]). In general, $Pri(A) = 0$ still implies that A is semiprime. However, the converse conclusion need no longer be true.

Definitions. Let A be an R-algebra. An ideal $L \subset A$ is called *weakly local* if it is contained in a unique maximal ideal $K \subset A$ with $A^2 \not\subset K$.

A is called a *weakly local algebra* if 0 is a weakly local ideal.

Of course, every simple algebra is weakly local. The ring R is weakly local if and only if it is local in the classical sense. In general we have the following characterizations:

21.10 Weakly local algebras.

For an R-algebra A the following are equivalent:

(a) *A is a weakly local algebra;*

(b) *A is a local $M(A)$-module and $Rad(A) = Alb(A)$;*

(c) *$Rad(A)$ is a maximal ideal and $A^2 \not\subset Rad(A)$;*

(d) *$A/Rad(A)$ is a simple algebra;*

(e) *A is finitely generated as $M(A)$-module, and every factor algebra B of A is indecomposable and $B^2 \neq 0$.*

Proof. $(a) \Rightarrow (b)$ Assume K is the only maximal ideal in A and $A^2 \not\subset K$. Then $K = Alb(A) = Rad(A)$.

$(b) \Rightarrow (c)$ $Alb(A)$ is a maximal ideal and hence $A^2 \not\subset Rad(A)$.

$(c) \Leftrightarrow (d)$ This is clear from the definitions.

$(c) \Rightarrow (e)$ As a local $M(A)$-module, A is finitely generated and every factor algebra is indecomposable (see 8.2). The kernel of every surjective algebra morphism $A \to B$ is contained in $Rad(A)$ and $A^2 \not\subset Rad(A)$ implies $B^2 \neq 0$.

$(e) \Rightarrow (a)$ Consider two maximal ideals $K_1, K_2 \subset A$. Since $(A/K_i)^2 \neq 0$, $A^2 \not\subset K_i$. Assume $K_1 \not\subset K_2$. Then $A/(K_1 \cap K_2) \simeq A/K_1 \oplus A/K_2$ is a non-trivial decomposition of a factor algebra, a contradiction. □

Recall that a module M is called *uniform* if every submodule is essential in it, and M is *uniserial* if its submodules are linearly ordered by inclusion.

21.11 Uniserial algebras.

For an R-algebra A the following statements are equivalent:

(a) *The ideals in A are linearly ordered by inclusion;*

(b) *every factor algebra B of A is a uniform $M(B)$-module.*

Under these conditions, an algebra A with unit is weakly local.

Proof. $(a) \Rightarrow (b)$ Factor modules of uniserial modules are uniserial, hence uniform.

$(b) \Rightarrow (a)$ Every factor algebra B of A has simple or zero socle as an $M(B)$- and $M(A)$-module and hence is uniserial by 8.15.

If A has a unit, $Rad(A)$ is a unique maximal ideal. $\square$

Based on the notions considered above we introduce another radical:

21.12 The weakly local radical.
For an R-algebra A the weakly local radical *is defined as*

$$Loc(A) := \bigcap \{L \subset A \mid L \text{ is a weakly local ideal }\}.$$

It has the following properties:

(1) $Loc(A/Loc(A)) = 0$ and $Loc(A) \subset Alb(A)$.

(2) $Loc(A) = 0$ if and only if A is a subdirect product of weakly local algebras.

(3) If $A^2 = A$ and A is finitely generated as $M(A)$-module and has dcc on ideals, then $A/Loc(A)$ is a finite product of weakly local algebras.

Proof. (1), (2) follow immediately from the definitions and 1.10.

(3) We may assume $Loc(A) = 0$. Choose a minimal set of weakly local ideals L_i with $L_1 \cap \cdots \cap L_k = 0$ and denote by K_i the unique maximal ideal of A with $L_i \subset K_i$.

Suppose $K_i = K_j$ for some $i \neq j$. Then $L_i \cap L_j$ is a weakly local ideal: Assume $L_i \cap L_j \subset K$ for some maximal ideal $K \neq K_i$. Then $L_iL_j \subset K$ and hence $L_i \subset K$ or $L_j \subset K$, since K is a prime ideal. Hence L_i or L_j is contained in two maximal ideals, a contradiction.

Therefore, by our minimality assumption, we have $K_i \neq K_j$ for $i \neq j$ and $L_i + L_j$ is not contained in any maximal ideal of A. Hence $L_i + L_j = A$ and the assertion follows from the Chinese Remainder Theorem 1.10. $\square$

21.13 Exercises.

(1) Let A be an associative algebra. Show for matrix rings ($n \in I\!N$):

$$BMc(A^{(n,n)}) = (BMc(A))^{(n,n)}.$$

(2) Let A be an associative semiperfect ring with unit. Assume $\{e_1, \ldots, e_n\}$ to be a complete set of primitive central idempotents modulo $Jac(A)$. Prove that ([178])

$$Loc(A) = \sum_{i \neq j} Ae_iAe_jA.$$

(3) Let $(A, +, \cdot\,)$ be a simple associative ring. Prove ([247]):

(i) Assume $S \subset A$ is a subring such that $S^2 \neq 0$ and $SAS = S$. Then S has a unique maximal ideal

$$\{t \in S \mid StS = 0\}.$$

(ii) Let $a \in A$ such that $a^2 \neq 0$. Then the ring aAa has a unique maximal ideal

$$\{s \in aAa \mid asa = 0\}.$$

(iii) Let $c \in A$ such that $AcA \neq 0$. Define a new product on $(A, +)$ by

$$* : A \times A \to A, \quad (a, b) \mapsto acb.$$

Then the ring $(A, +, *)$ has a unique maximal ideal $\{a \in A \mid cac = 0\}$.

(4) Let A be an associative ring with unit which has a unique maximal ideal M. Prove ([240, Theorem 3]):

A is either a local ring or every maximal right ideal $I \neq M$ is an idempotent non-two-sided ideal.

Furthermore, $(IJ)^2 = IJ$ for any maximal right ideals I, J different from M.

References: Albert [43], Behrens [68, 69], Brown-McCoy [91], Gray [17], Jenner [167], Kerner-Thode [178], Pritchard [223], Satyanarayana-Deshpande [240], Schafer [37], Stewart-Watters [247], Zwier [289].

22 Solvable and nil ideals

1.Solvable ideals. 2.The solvable radical. 3.Nilpotent ideals. 4.Nil ideals. 5.The nil radical. 6.Exercises.

Now we consider some radicals defined by internal properties. In order to transfer nilpotency of ideals to non-associative algebras the following notion is useful:

For a subalgebra B of an R-algebra A the *derived series of subalgebras* is formed by

$$B^{(1)} := B, \quad B^{(2)} := B^2, \quad B^{(n+1)} := B^{(n)}B^{(n)}, \quad \text{for } n \in I\!N.$$

Observe that, for an ideal B in a non-associative algebra A, in general $B^{(n)}$ need not be an ideal in A. It is an ideal in associative and in alternative algebras. However, $A^{(2)}$ is an ideal in any algebra A.

A subalgebra B of an algebra A is called *solvable* if $B^{(n)} = 0$ for some $n \in I\!N$.

Of course, in associative algebras a subalgebra is solvable if and only if it is nilpotent. The usefulness of this notion is based on technical properties:

22.1 Solvable ideals. *Let A be an R-algebra.*

(1) Assume B is an ideal in A. Then A is solvable if and only if B and A/B are solvable.

(2) If B and C are solvable ideals in A, then $B + C$ is also a solvable ideal.

Proof. (1) Subalgebras and factor algebras of solvable algebras are again solvable.

Assume B and A/B are solvable and $p : A \to A/B$ is the canonical homomorphism. Since $p(A^{(n)}) = (p(A))^{(n)} = 0$ for some $n \in I\!N$, we have $A^{(n)} \subset B$. Choosing $t \in I\!N$ such that $B^{(t)} = 0$ yields

$$A^{(n+t)} = (A^{(n)})^{(t)} \subset B^{(t)} = 0 \ ,$$

i.e., A is solvable.

(2) By isomorphism theorems (see 1.8), $(B + C)/C \simeq B/(B \cap C)$ and hence is solvable. Now (1) implies that $B + C$ is also solvable. $\square$

The sum of all solvable ideals in A is called the *solvable radical* and we denote it by $Sol(A)$. In general $Sol(A)$ need not be a solvable ideal and it may happen that $Sol(A/Sol(A)) \neq 0$. In case A has the ascending chain condition on ideals, $Sol(A)$ is a finite sum of solvable ideals and hence is solvable by 22.1.

We collect some properties of this radical:

22.2 The solvable radical.

Let A be an R-algebra with solvable radical $Sol(A)$. Then:

(1) *Every finitely generated ideal B with $B \subset Sol(A)$ is solvable.*

(2) *Assume D is an R-algebra with $Sol(D) = 0$. Then every surjective algebra morphism $f : A \to D$ factorizes over $A/Sol(A)$.*

(3) *$Sol(A) \subset Alb(A)$.*

(4) *If $Sol(A)$ is solvable, then $Sol(A/Sol(A)) = 0$.*

(5) *Assume for every ideal $B \subset A$, B^2 is also an ideal. Then $Sol(A) \subset Pri(A)$ and A is semiprime if and only if $Sol(A) = 0$.*

Proof. (1) Every finitely generated ideal $B \subset Sol(A)$ is contained in a finite sum of solvable ideals and hence is solvable by 22.1.

(2) If B is a solvable ideal in A then $f(B)$ is a solvable ideal in D and hence is zero. Therefore $B \subset Ke\, f$ and $Sol(A) \subset Ke\, f$ and the factorization follows by 1.7.

(3) If A/K is a simple factor algebra of A, then $Sol(A/K) = 0$ and $Sol(A) \subset K$ by (2). This implies $Sol(A) \subset Alb(A)$.

(4) Let $p : A \to A/Sol(A)$ denote the canonical morphism. Assume for the ideal $B \subset A$ that $p(B)$ is solvable. Then $(B + Sol(A))/Sol(A) \simeq p(B)$ and hence is solvable. Now $B + Sol(A)$ is solvable by 22.1, i.e., $B \subset Sol(A)$ and $p(B) = 0$.

(5) Under the given condition, for every prime ideal $P \subset A$, the algebra A/P contains no solvable ideal. Hence $Sol(A) \subset Pri(A)$.

$Sol(A) = 0$ if and only if A has no non-zero ideals B with $B^2 = 0$. □

Another extension of nilpotency of ideals to non-associative algebras is based on the following notion:

A subalgebra B of an R-algebra A is said to be *nilpotent (of degree t)* if, for some $t \in I\!N$, any product of t elements $b_1, \ldots, b_t \in B$ is zero with every possible bracketing. Nilpotent ideals can be characterized in the multiplication algebra.

For any subset $B \subset A$ we denote by $M_B(A)$ the subalgebra of $M(A)$, generated by left and right multiplication with elements from B, i.e.

$$M_B(A) := < \{R_b, L_b \mid b \in B\} > \subset M(A).$$

22.3 Nilpotent ideals.

An ideal B of an R-algebra A is nilpotent if and only if $M_B(A)$ is a nilpotent subalgebra of $M(A)$.

Proof. Assume $B \subset A$ is nilpotent of degree $k \in I\!N$. Then any product with more than k elements from B is zero.

A product of k elements T_i from $M_B(A)$, $T = T_1 \cdots T_k$, is a sum of expressions containing a product of at least k left or right multiplications S_i with elements from B. Since B is an ideal, for any $x \in A$, we have $S_1 x \in B$ and $S_k \cdots S_1 x = 0$. Hence $T = 0$ and $M_B(A)$ is nilpotent (of degree k).

Now assume B is a subalgebra of A and $M_B(A)$ is nilpotent of degree $t \in I\!N$. We first show that any product of at least 2^n ($n \in I\!N$) elements from B can be written as

$$S_n \cdots S_1 b \quad \text{with } \ b \in B \text{ and } \ S_i \in M_B(A).$$

This is obtained by induction on n. For $n = 1$ we observe that every product of two elements $b, b_1 \in B$ can be written as

$$bb_1 = R_{b_1} b = S_1 b \quad \text{with } S_1 := R_{b_1} \in M_B(A).$$

Similarly, in every product of at least 2^{n+1} elements from B, there is a last left or right multiplication $S_{n+1} \in M_B(A)$ to be performed. By induction hypothesis, the remaining factor is of the form $S_n \cdots S_1 b$ with $b \in B$, $S_i \in M_B(A)$.

Since $M_B(A)$ is nilpotent of degree t, B is nilpotent of degree 2^t. □

Remark. If B is a nilpotent ideal in A and A/B is also nilpotent, then in general A need not be nilpotent. Therefore this notion of nilpotency is not suitable for the definiton of a radical for arbitrary algebras.

Obviously, the sum of all nilpotent ideals in A is contained in the solvable radical $Sol(A)$ of A. Let us now consider a radical which contains $Sol(A)$.

An element $a \in A$ is called *nilpotent (of degree $n \in I\!N$)* if an n-fold product of a is zero for some suitable bracketing.

An ideal in A is said to be a *nil ideal* if every element in it is nilpotent. A is called a *nil algebra* if every element in A is nilpotent.

For an n-fold product of $a \in A$ there are finitely many possibilities of bracketing which we enumerate with an index and we denote the ν-th possibility by $P^n_\nu(a)$. With this notation, $a \in A$ is nilpotent if and only if $P^n_\nu(a) = 0$ for some pair $n, \nu \in I\!N$.

Similarly to 22.1 we have:

22.4 Nil ideals. *Let A be an R-algebra.*

(1) *Assume B is an ideal in A. Then A is a nil algebra if and only if B is a nil ideal and A/B is a nil algebra.*

(2) *The sum of nil ideals in A is again a nil ideal.*

Proof. (1) We have to show that $a \in A$ is nilpotent if B and A/B are nil. Let $p : A \to A/B$ denote the canonical morphism. Then there exist $n, \nu \in I\!N$ such that

$$P_\nu^n(p(a)) = p(P_\nu^n(a)) = 0, \text{ i.e., } P_\nu^n(a) \in B.$$

Since B is nil, there exist $m, \mu \in I\!N$ with $P_\mu^m(P_\nu^n(a)) = 0$. Hence we have an nk-fold product of a which is zero.

(2) Applying an isomorphism theorem we conclude from (1) that every finite sum of nil ideals is nil (see proof of 22.1(2)).

Every element in an infinite sum of nil ideals is contained in a finite partial sum and hence is nilpotent by the above argument. Therefore any sum of nil ideals is a nil ideal. $\square$

The sum of all nil ideals in an R-algebra A is called the *nil radical* of A and is denoted by $Nil(A)$. It is easy to see that the nil radical of an R-algebra A is equal to the nil radical of A as $\mathbb{Z}$-algebra (ring). It has the following properties:

22.5 The nil radical. *Let A be an R-algebra.*

(1) For surjective algebra morphisms $f : A \to B$, $f(Nil(A)) \subset Nil(B)$.

(2) $Nil(A/Nil(A)) = 0$.

(3) $Sol(A) \subset Nil(A) \subset BMc(A)$.

(4) $Nil(R)A \subset Nil(A)$.

Proof. (1) This is clear since $f(Nil(A))$ is a nil ideal in B.

(2) is a consequence of 22.4.

(3) In every simple algebra with unit the nil radical is zero. Hence by (1), $Nil(A)$ is contained in every maximal modular ideal of A, i.e., $Nil(A) \subset BMc(A)$.

Since every solvable ideal is nil, $Sol(A) \subset Nil(A)$.

(4) is obvious. $\square$

In general, $Nil(A)$ need not be contained in $Alb(A)$. For example, in any simple Lie algebra A, $Alb(A) = 0$ and $Nil(A) = A$.

For finite dimensional associative algebras A over fields, the properties of an ideal to be *nil*, *nilpotent* or *solvable* are equivalent and hence $Sol(A) = Nil(A)$.

This is also true for finite dimensional alternative and Jordan algebras (e.g., Schafer [37, Theorem 3.2 and 4.3]). In these algebras also $Nil(A) = BMc(A)$ holds (see [37, 3.12 and 4.6]).

22.6 Exercises.

(1) Let A be an associative algebra. An ideal $I \subset A$ is called semi-nilpotent *if each subring generated by a finite subset of I is nilpotent. The Levitzki radical of A is defined by*

$$Lev(A) := \sum \{I \subset A \mid I \text{ a semi-nilpotent ideal}\}.$$

Prove that $Lev(A)$ is a semi-nilpotent ideal and $Lev(A/Lev(A)) = 0$.

(2) Let A be a Lie algebra. The Frattini subalgebra $\Phi(A)$ of A is defined as the intersection of all maximal subalgebras of A.

An element $x \in A$ is said to be non-generating *if the following holds: If $x \in S$ for some generating set $S \subset A$, then $S \setminus \{x\}$ is also a generating set. Prove:*

(i) $\Phi(A)$ is the set of all non-generating elements of A.

(ii) If A is nilpotent, then $A^2 \subset \Phi(A)$.

(iii) Assume A has finite length as an R-module and let $K \subset L \subset A$ be ideals such that $K \subset \Phi(A)$ and L/K is nilpotent. Then L is a nilpotent Lie algebra.

(3) Let A be a solvable Lie R-algebra which has finite length as an R-module. Prove that the Frattini subgroup $\Phi(A)$ is a nilpotent ideal in A.

(4) Let A be a Lie algebra with an ideal $K \subset A$ such that A/K^2 is nilpotent. Prove that A is also nilpotent.

(5) Prove that the following are equivalent for a Lie algebra A:

(a) For some $l \in I\!N$, $A^{(l-1)} \neq 0$ and $A^{(l)} = 0$ (A is solvable);

(b) for some $l \in I\!N$, there exists a series of ideals (or subalgebras)

$$A = I_0 \supset I_1 \supset \cdots \supset I_l = 0,$$

with $I_i^2 \subset I_{i+1}$ for $i = 0, \ldots, l-1$.

(6) Let A be a Lie R-algebra which has finite length as an R-module. Prove:

(i) The sum of all nilpotent ideals is a nilpotent ideal in A.

(ii) The sum of all solvable ideals is a solvable ideal in A.

References: Albert [43], Bakhturin [3], Behrens [68, 69], Brown-McCoy [91], Gray [17], Jenner [167], Schafer [37].

Chapter 7

Modules for algebras

Bimodules over arbitrary R-algebras A are defined as R-modules M with two R-linear maps

$$A \otimes_R M \to M, \quad M \otimes_R A \to M.$$

With obvious notation, for these modules associator and commutator are defined for $a, b \in A$, $m \in M$ (compare 1.11),

$$(a, b, m) = (ab)m - a(bm), \quad [a, m] = am - ma,$$

and similarly (a, m, b) and (m, a, b).

For bimodules over associative algebras additional conditions are imposed:

$$(a, b, m) = (a, m, b) = (m, a, b) = 0, \quad \text{for all } a, b \in A,\ m \in M.$$

Following a suggestion in Eilenberg [120], bimodules over algebras with identities can be defined by imposing corresponding identities in terms of the module multiplication. For example, bimodules over alternative algebras A should satisfy

$$(a, a, m) = (a, m, a) = (m, a, a) = 0, \quad \text{for all } a \in A,\ m \in M,$$

and over Jordan algebras A, for all $a, b \in A$, $m \in M$ (see [19, p. 95]),

$$am = ma,\ a(a^2m) = a^2(am) \ \text{ and } (a^2, b, m) = 2(a, b, am).$$

The category of alternative (Jordan) bimodules over an alternative (Jordan) algebra A is equivalent to the full module category over the corresponding associative *universal enveloping algebra* $U(A)$ (see [19]). Hence this setting provides a connection between A and a full module category over an associative ring. Such a connection turned out to be so useful for associative algebras A (with A-*Mod*). However, in general, the algebra $U(A)$ is not closely related to the algebra A. For example, A need not be a faithful module over $U(A)$. Therefore it may be very difficult to conclude from properties of A to properties of $U(A)$ (e.g., [166]). Moreover, $U(A)$ has to be defined for every variety of algebras independently and the definitions are quite cumbersome. For these reasons, by the approach outlined above it is not so easy to apply the rich module theory over associative rings to non-associative algebras. Results in this direction can be found in Jacobson [166, 19], Schafer [241], Zhevlakov [287], Slinko-Shestakov [243] and Zhevlakov-Slinko-Shestakov-Shirshov [41].

In the next section we suggest a different approach to make associative module theory accessible to non-associative algebras without referring to identities.

23 The category $\sigma[A]$

1.Properties of $\sigma[A]$. 2.The centre of an $M(A)$-module. 3.Algebras with units. 4.A finitely presented in $M(A)$-Mod. 5.Proposition. 6.A as an A^e-module. 7.Direct projectivity. 8.Weak product of quasi-simple algebras. 9.Corollary. 10.Finite product of simple algebras. 11.Modules for quasi-simple algebras. 12.Properties of weak products of quasi-simple algebras.

To apply module theory to arbitrary algebras A, it makes sense not to focus on the relationship between A and the category of all (non-associative) bimodules over A but to a suitable subcategory of it. This category should be small enough to be closely related to the algebra A and should be large enough to provide effective techniques from category theory. Both requirements are met by the category $\sigma[A]$, a subcategory of $M(A)$-*Mod* (and the bimodules over A), which is nothing but a special case of the category $\sigma[M]$ defined for any module M over an associative algebra. We repeat some definitions and properties in this new context.

Let A be an R-algebra and $M(A)$ its multiplication algebra (see 2.1). We consider A as a left module over the associative algebra $M(A)$. The $M(A)$-submodules are the (two-sided) ideals of A and $C(A) := End({}_{M(A)}A)$ is the *centroid* of A (see 2.6).

An $M(A)$-module is called *A-generated* if it is a homomorphic image of a direct sum $A^{(\Lambda)}$ in $M(A)$-*Mod*, for some index set Λ.

For two modules $M, N \in \sigma[A]$ the *trace of M in N* is defined as

$$Tr(M,N) := \sum \{Im\, f \mid f \in Hom_{M(A)}(M,N)\} = M Hom_{M(A)}(M,N).$$

N is M-generated if and only if $N = Tr(M,N) = M Hom_{M(A)}(M,N)$.

By $\sigma[A]$, or $\sigma[{}_{M(A)}A]$, we denote the full subcategory of $M(A)$-*Mod* whose objects are submodules of A-generated module.

$\sigma[A]$ is a Grothendieck category and with the above notation we have from 4.4:

23.1 Properties of $\sigma[A]$. *Let A be an R-algebra.*

(1) Morphisms in $\sigma[A]$ have kernels and cokernels.

(2) For objects $\{N_\lambda\}_\Lambda$ in $\sigma[A]$, the direct sum $\bigoplus_\Lambda N_\lambda$ in $M(A)$-Mod belongs to $\sigma[A]$ and is the coproduct in $\sigma[A]$.

(3) The finitely generated (cyclic) submodules of $A^{(\mathbb{N})}$ form a set of generators in $\sigma[A]$. The direct sum of these modules is a generator in $\sigma[A]$.

(4) Objects in $\sigma[A]$ are finitely (co-) generated in $\sigma[A]$ if and only if they are finitely (co-) generated in $M(A)$-Mod.

(5) For objects $\{N_\lambda\}_\Lambda$ in $\sigma[A]$, let $\prod_\Lambda N_\lambda$ denote the product in $M(A)$-Mod (cartesian product). Then $Tr(\sigma[A], \prod_\Lambda N_\lambda)$ is the product of $\{N_\lambda\}_\Lambda$ in $\sigma[A]$.

(6) *Every simple module in* $\sigma[A]$ *is a factor module of an ideal of* A.

(7) *If* A *is a finitely generated* $C(A)$*-module, then* $\sigma[A] = M(A)$*-Mod.*

Although the category $\sigma[A]$ has a generator, it need not have a finitely generated or a projective generator. This is the main difference to full module categories. If $\sigma[A]$ has a finitely generated projective generator, then it is equivalent to a full module category (see 5.10).

Every left $M(A)$-module may be regarded as left module over the left multiplication algebra $L(A)$ and the right multiplication algebra $R(A)$.

Recall that for any $a \in A$, we denote by L_a and R_a the left and the right multiplication by a in A (see 2.1). Putting $L_a m = am$ and $R_a m = ma$ we see that every $M(A)$-module is also a bimodule over A.

23.2 The centre of an $M(A)$-module.

In any $M(A)$-module M, we define the *centre* $Z_A(M)$ *of* M as

$$\{m \in M \mid L_a m = R_a m,\ L_a L_b m = L_{ab} m,\ R_b R_a m = R_{ab} m,\ \text{for all } a, b \in A\}.$$

Using associator and commutator we have

$$Z_A(M) = \{m \in M \mid [a, m] = 0,\ (a, b, m) = (m, a, b) = 0, \quad \text{for all } a, b, \in A\}.$$

Obviously, $Z_A(A) = Z(A)$, the centre of the algebra A (see 1.11) and $Z_A(M)$ is a $Z(A)$-submodule of M.

Moreover, for any ideal $I \subset A$, $Z_A(I) = I \cap Z(A)$.

It is easy to verify that over algebras with units the centre of modules is closely connected to morphisms:

23.3 Algebras with units.

Let A *be an* R*-algebra with unit* e_A *and* $M \in M(A)$*-Mod. Then the map*

$$Hom_{M(A)}(A, M) \to Z_A(M), \quad \gamma \mapsto (e_A)\gamma,$$

is a $Z(A)$*-module isomorphism.*

M *is* A*-generated as an* $M(A)$*-module if and only if*

$$M = AHom_{M(A)}(A, M) = AZ_A(M).$$

An ideal $I \subset A$ *is* A*-generated if and only if*

$$I = AHom_{M(A)}(A, I) = A(I \cap Z(A)).$$

23.4 A finitely presented in $M(A)$-*Mod*.

Let A be an R-algebra. Assume $M(A)$ is finitely generated as an R-module. If A has a unit or A is finitely generated and projective as R-module, then A is finitely presented in $M(A)$-Mod.

Proof. Let A have a unit. The map $M(A) \to A$, $\lambda \mapsto \lambda(1)$, splits as an R-morphism and hence its kernel is a finitely generated R-module. Then it is also a finitely generated $M(A)$-module and A is a finitely presented $M(A)$-module.

Now assume A is a finitely generated and projective R-module. Then there exists an $M(A)$-epimorphism $M(A)^k \to A$, for some $k \in I\!N$, which splits as an R-morphism. Hence its kernel is finitely generated as an R-module and as an $M(A)$-module. So A is a finitely presented $M(A)$-module. $\square$

In general, we do not know if for any module finite R-algebra A, $M(A)$ is also finitely generated as an R-module. However we conclude from 3.7 and 3.13:

23.5 Proposition. *Let A be an alternative or a Jordan algebra which is a finitely generated R-module. Then $M(A)$ is finitely generated as an R-module.*

A is called an *affine R-algebra* if it is finitely generated as an R-algebra. Obviously, any unital R-algebra which is a finitely generated R-module is an affine R-algebra.

For associative algebras A we have a special situation. Recall that such an A is a left module over $A^e = A \otimes_R A^o$ (by $(a \otimes b) \cdot x = axb$).

23.6 A as an A^e-module.

Let A be an associative R-algebra with unit.

(1) $\mu : A^e \to A$, $a \otimes b \mapsto ab$, is a surjective A^e-module morphism.

(2) $Ke\,\mu$ is generated as an A^e-submodule by $\{a \otimes 1 - 1 \otimes a \,|\, a \in A\}$.

(3) If A is an affine R-algebra, then A is finitely presented in A^e-Mod.

Proof. (1) This is easily verified.

(2) For $\sum a_i \otimes b_i \in Ke\,\mu$, $\mu(\sum a_i \otimes b_i) = \sum a_i b_i = 0$ and hence

$$\sum a_i \otimes b_i = \sum (a_i \otimes 1)(1 \otimes b_i - b_i \otimes 1).$$

(3) Let A be generated as an R-algebra by $a_1, \ldots, a_k \in A$. We want to show that $Ke\,\mu$ is generated as an A^e-module by $\{a_i \otimes 1 - 1 \otimes a_i \,|\, i \leq k\}$.

By assumption we know that every element in A is an R-linear combination of finite products of the elements $a_1, \ldots, a_k$. Assume $a \in A$ is a product of $n+1$ such elements. Without restriction $a = a_1 b$, with b a product of n of these elements, and

$$a_1 b \otimes 1 - 1 \otimes a_1 b = (a_1 \otimes 1)(b \otimes 1 - 1 \otimes b) + (1 \otimes b)(a_1 \otimes 1 - 1 \otimes a_1).$$

By induction we see that any $a \otimes 1 - 1 \otimes a$ belongs to the A^e-submodule generated by $\{a_i \otimes 1 - 1 \otimes a_i \,|\, i \leq k\}$. $\square$

An important effect of a unity in an associative algebra is that it makes the algebra projective as a left module. In general, a unit still implies a weak form of projectivity for the bimodule structure.

23.7 Direct projectivity.

Let A be a unital algebra.

(1) *Let $e \in A$ be a central idempotent and $U \subset A$ an ideal. Then every $M(A)$-epimorphism $h : U \to Ae$ splits.*

(2) *A is a direct projective $M(A)$-module.*

(3) *Let $h : A \to A$ be an $M(A)$-epimorphism. Then $(Z(A))h = Z(A)$ and h is an isomorphism.*

(4) *Let $e \in A$ be a central idempotent and assume Ae is a local $M(A)$-module. Then $eZ(A)$ is a local ring.*

(5) *Assume A has a unique maximal ideal. Then $Z(A)$ is a local ring.*

Proof. (1) For a central idempotent $e \in A$ and $U \subset A$, consider an $M(A)$-epimorphism $h : U \to Ae$. Choose $x \in U$ with $(x)h = e$. Then for any $a \in A$,

$$a(ex) = e(ax) = (x(ax))h = x(ae) = (ex)a.$$

Similarly for any $a, b \in A$,

$$(ab)(ex) = (a(b(x)h))x = (a(bx))hx = (a(bx))e = a(b(ex)).$$

So $[a, ex] = 0$ and $(a, b, ex) = 0 = (ex, a, b)$, for all $a, b, \in A$, i.e., $ex \in Z(A) \cap U$. Hence the map

$$k : Ae \to U,\ ae \mapsto aex,$$

is an $M(A)$-homomorphism with $kh = id_{Ae}$.

(2) Any direct summand of A is of the type Ae, for some central idempotent $e \in A$. By (1), any epimorphism $A \to Ae$ splits, i.e., A is a direct projective $M(A)$-module.

(3) For an epimorphism $h : A \to A$, obviously $(Z(A))h \subset Z(A)$. By (1), there exists $k : A \to A$ with $kh = id_A$. Since $End_{M(A)}(A) \simeq Z(A)$ is commutative, also $hk = id_A$ and so h and k are isomorphisms. Hence

$$Z(A) = (Z(A))kh \subset (Z(A))h \subset Z(A).$$

(4) Under the given conditions, Ae has a unique maximal $M(A)$-submodule. Since Ae is direct projective (by (2)), $End_{M(A)}(Ae) \simeq eZ(A)$ is a local ring by 8.3.

(5) is a special case of (4). □

Assume A has a decomposition as a direct sum of $M(A)$-submodules (ideals), $A = \bigoplus_\Lambda I_\lambda$. Then it is easily checked that $I_\lambda I_\mu = 0$ if $\lambda \neq \mu$. Hence the product in A can be formed by components and A can be considered as the *weak product* of the algebras I_λ (see 1.4).

Transferring the characterization of semisimple modules in 6.9, we obtain characterizations of special classes of algebras which demonstrate the usefulness of our concepts. The assertions about the radicals are taken from 21.2.

Recall that for module finite R-algebras A, $\sigma[A] = M(A)$-*Mod*.

23.8 Weak product of quasi-simple algebras.

For an R-algebra A the following statements are equivalent:

(a) A is a weak product of quasi-simple algebras;

(b) every ideal in A is a direct summand;

(c) every module in $\sigma[A]$ is semisimple;

(d) every module in $\sigma[A]$ is A-projective;

(e) every module in $\sigma[A]$ is A-injective;

(f) every short exact sequence in $\sigma[A]$ splits;

(g) every simple module in $\sigma[A]$ is A-projective;

(h) every cyclic module in $\sigma[A]$ is A-injective.

If A is a finitely generated $M(A)$-module, then (a)-(h) are equivalent to:

(i) $Rad_{M(A)}(A) = 0$ and A is finitely cogenerated as $M(A)$-module.

If A is a finitely generated R-module, then (a)-(i) are equivalent to:

(j) $M(A)$ is a left semisimple algebra.

23.9 Corollary. *Let A be a weak product of quasi-simple algebras. Then:*

(1) The centroid $C(A)$ is a regular ring and A is a projective generator in $\sigma[A]$.

(2) If A is a finitely generated $M(A)$-module, then $C(A)$ is a left semisimple ring.

In the above situation we may have $A^2 = 0$. In the next cases this trivial situation is excluded. Recall that the centroid of algebras A with $A^2 = A$ is commutative (see 2.7).

23.10 Finite product of simple algebras.

Let A be an R-algebra which is finitely generated as $M(A)$-module.

(1) The following statements are equivalent:

(a) A is a finite product of simple algebras;

(b) $A^2 = A$ and every module in $\sigma[A]$ is semisimple;

(c) $Alb(A) = 0$ and A is finitely cogenerated as $M(A)$-module.

Under theses conditons, $C(A)$ is a finite product of fields.

(2) The following are also equivalent:

(a) A is a finite product of simple algebras with units;

(b) A has a unit and every module in $\sigma[A]$ is semisimple;

(c) $BMc(A) = 0$ and A is finitely cogenerated as $M(A)$-module.

Under theses conditons, $C(A) \simeq Z(A)$ is a finite product of fields.

Of course, all the module theoretic formulations in 23.8 also apply to 23.10.

It is known from linear algebra that a ring R is a field if and only if every R-module is free. Applying the above results this can be extended to arbitrary algebras:

23.11 Modules for quasi-simple algebras.

For any algebra A the following are equivalent:

(a) A is quasi-simple;

(b) every module in $\sigma[A]$ is isomorphic to $A^{(\Lambda)}$, for some set Λ.

Proof. $(a) \Rightarrow (b)$ Assume A is quasi-simple. Then A is a generator in $\sigma[A]$ and every module $N \in \sigma[A]$ is a sum of submodules isomorphic to A. Hence N is a direct sum of such modules (e.g. [40, 20.1]), i.e., $N \simeq A^{(\Lambda)}$ for some Λ.

$(b) \Rightarrow (a)$ Consider any simple module $E \in \sigma[A]$. By assumption, $E \simeq A^{(\Lambda)}$ and hence $A \simeq E$ is a simple $M(A)$-module, i.e., A is a quasi-simple algebra (see 21.1). □

The algebras considered in 23.8 and 23.10 have all good properties we may expect in module theory. In fact it is easy to show (compare [40, 20.4]):

23.12 Properties of weak products of quasi-simple algebras.

Assume the algebra A is a weak product of quasi-simple algebras. Then:

(1) $\sigma[A]$ has a semisimple cogenerator;

(2) every (finitely generated) ideal of A is projective in $\sigma[A]$;

(3) every finitely generated ideal of A is a direct summand;

(4) every simple module in $\sigma[A]$ is A-injective;

(5) for all $N \in \sigma[A]$, $Rad_{M(A)}(A) = 0$;

(6) every (finitely) A-generated module has a projective cover in $\sigma[A]$;

(7) A is a projective generator in $\sigma[A]$;

(8) A is an injective cogenerator in $\sigma[A]$.

None of these properties is sufficient to assure that A is a semisimple $M(A)$-module. They characterize various interesting larger classes of algebras.

Remarks. (1) Let A be an alternative, Jordan, Lie or Malcev algebra and denote by $U(A)$ the corresponding universal enveloping algebra. By the properties of $U(A)$, there exists an algebra morphism $U(A) \to End_R(A)$ whose image is just the multiplication algebra $M(A)$. Hence every $M(A)$-module - in particular every module in $\sigma[A]$ - is an $U(A)$-module, i.e., it is an alternative, Jordan, Lie or Malcev (see [121]) module, respectively.

(2) In case A is an associative and commutative algebra with unit, say $A = R$, we have $M(A) = A$ and $\sigma[A] = A$-*Mod*. Hence in general results about $\sigma[A]$ extend the module theory over associative commutative rings.

References. Elduque [121], Jacobson [166], Wisbauer [270, 267].

24 Generator properties of A

1.Algebras with large centroid. 2.Algebras with unit and large centre. 3.Self-generator algebras. 4.Finitely presented self-generators. 5.Algebras satisfying (G.4). 6.A as a generator in $\sigma[A]$. 7.Properties of generator algebras. 8.A generating $M(A)$-Mod. 9.Example. 10.Tensor product of Azumaya algebras.

In general, A is far from being a generator in $\sigma[A]$. In this section we investigate the following conditions on an algebra A related to this property:

$(G.1)$ For every non-zero ideal $U \subset A$, $U \cap Z(A) \neq 0$.

$(G.2)$ For every non-zero ideal $U \subset A$, $Hom_{M(A)}(A, U) \neq 0$.

$(G.3)$ Every ideal in A is A-generated.

$(G.4)$ For every morphism $f : A^k \to A^n$ in $\sigma[A]$, $k, n, \in \mathbb{N}$, $Ke\, f$ is A-generated.

$(G.5)$ A is a generator in $\sigma[A]$.

$(G.6)$ A is a generator in $M(A)$-*Mod*.

We always have $(G.6) \Rightarrow (G.5) \Rightarrow (G.4)$ and $(G.5) \Rightarrow (G.3) \Rightarrow (G.2)$.

If A has no absolute zero-divisors, then $(G.1) \Rightarrow (G.2)$. In algebras A with unit, we may assume $U \cap Z(A) = Hom_{M(A)}(A, U)$ for every ideal $U \subset A$, and hence $(G.1) \Leftrightarrow (G.2)$.

Definitions. We say an algebra A has *large centre* if A satisfies $(G.1)$, and A has *large centroid* if it satisfies $(G.2)$.

24.1 Algebras with large centroid.

Let A be an algebra with large centroid. Then:

(1) For any ideal $U \subset A$, $Tr(A, U)$ is an essential $M(A)$-submodule of U.

(2) A is quasi-simple if and only if $C(A)$ is a division algebra.

(3) A is a finite product of quasi-simple algebras if and only if $C(A)$ is left semisimple.

Proof. (1) For every non-zero submodule $K \subset U$, there exists a non-zero $\alpha \in Hom(A, K)$ and $0 \neq A\alpha \subset K \cap Tr(A, U)$.

(2) If A is quasi-simple, $C(A)$ is a divison algebra by Schur's Lemma.

Assume $C(A)$ is a division algebra and consider any ideal $U \subset A$. Then there exists a non-zero $\alpha \in Hom(A, U) \subset C(A)$. Since α is invertible, $A = A\alpha \subset U$ and hence $U = A$.

(3) If A is a finite product of quasi-simple algebras, then $C(A)$ is left semisimple by 23.8.

Now assume $C(A)$ is left semisimple and consider any ideal $U \subset A$. By (1), $Tr(A, U) = AHom(A, U)$ is an essential submodule of U. $Hom(A, U)$ is a left ideal in $C(A)$ and hence is generated by some idempotent $e \in C(A)$ and $Tr(A, U) = Ae$. Therefore Ae is essential and a direct summand in U, i.e., $U = Ae$ is a direct summand in A and A is a direct sum of simple $M(A)$-modules (see 23.8).

Since there are only finite families of orthogonal idempotents in $C(A)$, A is a finite product of quasi-simple algebras. □

For algebras with unit the above statements have the following form:

24.2 Algebras with unit and large centre.

Assume A is an algebra with unit and large centre. Then:

(1) For every ideal $U \subset A$, $A(U \cap Z(A))$ is an essential $M(A)$-submodule of U.

(2) A is simple if and only if $Z(A)$ is a field.

(3) A is a finite product of simple algebras if and only if $Z(A)$ is a finite direct product of fields.

(4) A is (semi-) prime if and only if $Z(A)$ is (semi-) prime.

Proof. (4) Obviously, for a (semi-) prime A, $Z(A)$ is also (semi-) prime.

Let $U, V \subset A$ be ideals with $UV = 0$. Then $(U \cap Z(A))(V \cap Z(A)) = 0$. If $Z(A)$ is prime this implies $U \cap Z(A) = 0$ or $V \cap Z(A) = 0$ and hence $U = 0$ or $V = 0$.

Putting $U = V$ we obtain the assertion for semiprime algebras. □

Algebras A with property $(G.3)$ are *self-generators* as $M(A)$-modules; we call them *self-generator algebras*. Besides the properties shown in 24.1 and 24.2 we observe:

24.3 Self-generator algebras.

Assume the algebra A is a self-generator as an $M(A)$-module. Then:

(1) A generates every simple module in $\sigma[A]$.

(2) For every ideal $U \subset A$ with $U\,C(A) \subset U$, A/U is a self-generator algebra.

(3) For every $a \in A$ and $f \in End(A_{C(A)})$, there exists $\mu \in M(A)$ with $f(a) = \mu a$.

(4) If $C(A)$ is a regular algebra, then every ideal in A, which is finitely generated as an $M(A)$-module, is a direct summand.

(5) If A has a unit, then for every ideal $U \subset A$, $U = A(U \cap Z(A))$, and every factor algebra of A is a self-generator algebra.

(6) If A has a unit, then $Nil(Z(A))A = Nil(A)$.

Proof. (1) By 23.1, every simple module in $\sigma[A]$ is a factor module of an ideal in A. Since ideals in A are A-generated, all simple modules in $\sigma[A]$ are A-generated.

(2) Under the given conditions, U is a fully invariant submodule of A and the assertion follows from 5.2.

(3) This is also shown in 5.2.

(4) Consider a finitely generated ideal $U \subset A$. Then there is a module epimorphism $f: A^k \to U$, $k \in I\!N$, which may be regarded as an element of the regular ring $End_{M(A)}(A^k)$. Therefore $(A^k)f = U$ is a direct summand in A (see 7.6).

(5) The first assertion follows from 23.3. Every ideal in A is also a $Z(A)$-module and hence the second assertion is obtained from (2).

(6) As already observed in 22.5, $Nil(Z(A))A \subset Nil(A)$ always holds. By (4), $Nil(A) = (Nil(A) \cap Z(A))A$. Since $Nil(A) \cap Z(A) \subset Nil(Z(A))$ the statement is proved. $\square$

Under a certain finiteness condition, 19.7 yields a local-global characterization for self-generator algebras. Again $\mathcal{X}$ denotes the set of maximal ideals in the Boolean ring of idempotents of R.

24.4 Finitely presented self-generators.

Assume A is an R-algebra such that A is finitely presented in $\sigma[A]$. Then the following are equivalent:

(a) A is a self-generator algebra;

(b) for every $x \in \mathcal{X}$, A_x is a self-generator algebra.

Next we ask for algebras satisfying $(G.4)$. Applying [40, 15.9], we obtain:

24.5 Algebras satisfying (G.4).

For any algebra A with centroid $C(A)$, the following are equivalent:

(a) For every morphism $f: A^k \to A^n$, $k, n \in I\!N$, $Ke\,f$ is A-generated;

(b) for every morphism $f: A^k \to A$, $k \in I\!N$, $Ke\,f$ is A-generated;

(c) A is flat as a right $C(A)$-module.

Since all modules over regular rings are flat, every algebra with regular centroid satisfies $(G.4)$ but need not satisfy $(G.1)$ or $(G.2)$.

Algebras with $(G.5)$ we call *generator algebras*. By 5.3 and 23.3 they can be described in the following way:

24.6 A as a generator in $\sigma[A]$.

For an R-algebra A with centroid $C(A)$, the following are equivalent:

(a) *A is a generator algebra (generator in $\sigma[A]$);*

(b) *the functor $Hom_{M(A)}(A, -) : \sigma[A] \to C(A)$-Mod is faithful;*

(c) *A generates every (cyclic) submodule of $A^{(\mathbb{N})}$;*

(d) *$A^{(\mathbb{N})}$ is a self-generator as an $M(A)$-module;*

(e) *for every (cyclic) submodule $U \subset A^{(\mathbb{N})}$, $U = AHom_{M(A)}(A, U)$.*

If A has a unit the following is also equivalent to (a)-(e):

(f) *For every $M \in \sigma[A]$, $M = AZ_A(M)$.*

In addition to the above these algebras have many nice properties:

24.7 Properties of generator algebras.

Assume A is a generator algebra. Then:

(1) *A is flat as a $C(A)$-module.*

(2) *For any Λ, $M(A)$-submodules of $A^{(\Lambda)}$ are $End(A_{C(A)})$-submodules.*

(3) *Every ideal of A is an $End(A_{C(A)})$-submodule.*

(4) *$M(A)$ is a dense subalgebra of $End(A_{C(A)})$, i.e., for any $a_1, \ldots, a_k \in A$ and $f \in End(A_{C(A)})$, there exists $\mu \in M(A)$ with $f(a_i) = \mu a_i$ for $i = 1, \ldots, k$.*

(5) *$\sigma[{}_{M(A)}A] = \sigma[{}_{End(A_{C(A)})}A]$.*

Proof. (1) follows from 24.5. (2),(3) and (4) are special cases of 5.3.

(5) $\sigma[{}_{M(A)}A]$ consists of modules of the form V/U with some $M(A)$-submodules $U \subset V \subset A^{(\Lambda)}$. By (2), these are also $End(A_{C(A)})$-submodules and hence belong to $\sigma[{}_{End(A_{C(A)})}A]$. $\square$

Finally we turn to algebras satisfying $(G.6)$:

24.8 A generating $M(A)$-*Mod*.

For an R-algebra A, the following are equivalent:

(a) *A is a generator in $M(A)$-Mod;*

(b) *$Hom_{M(A)}(A, -) : M(A)$-Mod $\to C(A)$-Mod is a faithful functor;*

(c) *A is a generator in $\sigma[A]$ and is finitely generated as a $C(A)$-module;*

(d) *$M(A) \simeq End(A_{C(A)})$ and A is finitely generated and projective as a $C(A)$-module.*

If A is central (i.e., $R = C(A)$), then (a)-(d) are also equivalent to:

(e) A is a finitely generated, projective generator in $M(A)$-Mod;

(f) $Hom_{M(A)}(A, -) : M(A)\text{-}Mod \to R\text{-}Mod$ is a category equivalence.

Proof. $(a) \Leftrightarrow (b)$ For this apply 5.3.

$(a) \Leftrightarrow (d)$ See the characterization of generators in full module categories in 5.5.

$(a) \Leftrightarrow (c)$ If A is a finitely generated $C(A)$-module, $\sigma[A] = M(A)$-*Mod* (23.1).

$(d) \Rightarrow (e)$ Since R is commutative this follows from 5.5(2).

$(e) \Rightarrow (a)$ is trivial.

$(e) \Leftrightarrow (f)$ is a consequence of 5.10. □

Definiton. A central R-algebra A which is a generator in $M(A)$-*Mod* is called an *Azumaya algebra over R.*

24.9 Example. Assume A is a quasi-semisimple algebra which is finitely generated as $C(A)$-module and $C(A)$ is commutative. Then A is an Azumaya algebra over $C(A)$.

In particular, every quasi-semisimple algebra with $A^2 = A$ is an Azumaya algebra over $C(A)$.

Proof. As a semisimple $M(A)$-module, A is a generator in $\sigma[A]$ and, by 23.1, $\sigma[A] = M(A)$-*Mod.* $A^2 = A$ implies that $C(A)$ is commutative (see 2.7) and the assertion follows from 24.8. □

Besides the characterization for Azumaya algebras considered in 24.8 we will find additional descriptions when studying projectivity properties in the next sections. One of the important properties is immediately obtained from 19.10:

24.10 Tensor product of Azumaya algebras.

Assume A and B are unital Azumaya algebras over R. Then $A \otimes_R B$ is also an Azumaya algebra over R.

References. Delale [115], Wisbauer [267].

25 Projectivity properties of A

1.Semi-projective algebras. 2.Intrinsically projective algebras. 3.Properties of intrinsically projective algebras. 4.Intrinsically projective unital algebras. 5.Self-projective algebras. 6.A projective in $\sigma[A]$. 7.Self-projective unital algebras. 8.Radical of self-projective algebras. 9.A projective in $M(A)$-Mod. 10.Properties of projective algebras. 11.Tensor product of projective algebras. 12.Scalar extensions of projective algebras. 13.Exercises.

We are interested in the following projectivity properties of an algebra A as an $M(A)$-module:

$(P.1)$ A is intrinsically projective as an $M(A)$-module (see 5.7), i.e., every diagram of $M(A)$-modules with exact row,

$$\begin{array}{ccccc} & & A & & \\ & & \downarrow & & \\ A^n & \xrightarrow{g} & U & \longrightarrow & 0\,, \end{array}$$

with $n \in I\!N$ and $U \subset A$, can be extended commutatively by some morphism $A \to A^n$. A is semi-projective if the above condition holds for $n = 1$.

$(P.2)$ A is self-projective as an $M(A)$-module, i.e., every diagram of $M(A)$-modules with exact row,

$$\begin{array}{ccccc} & & A & & \\ & & \downarrow & & \\ A & \xrightarrow{g} & X & \longrightarrow & 0\,, \end{array}$$

can be extended commutatively by some morphism $A \to A$.

$(P.3)$ A is projective in $\sigma[A]$.

$(P.4)$ A is projective in $M(A)$-*Mod*.

The implications $(P.4) \Rightarrow (P.3) \Rightarrow (P.2)$ are trivial. Since A-projective implies A^n-projective we also have $(P.2) \Rightarrow (P.1)$.

For associative unital A we may also ask when A is projective in $A \otimes_R A^o$-*Mod*. Such algebras are called *separable R-algebras* (see 28.2). Obviously they satisfy $(P.4)$.

For algebras satisfying identities, one might also consider the case that A is projective over the corresponding universal enveloping algebra. This condition is stronger than $(P.4)$. For example, it is satisfied by *separable* alternative or Jordan algebras over R (see Remarks 29.10).

We are going to characterize algebras with the properties given above.

25.1 Semi-projective algebras.

For an algebra A with centroid $C(A)$, the following are equivalent:

(a) A is semi-projective (i.e., satisfies $(P.1)$ for $n = 1$);

(b) for any $f \in C(A)$, $C(A)f = Hom_{M(A)}(A, Af)$;

(c) the map $J \mapsto AJ$ from cyclic left ideals in $C(A)$ to ideals of A is injective.

If A has a unit, then (a)-(c) are equivalent to:

(d) For every $c \in Z(A)$, $Z(A)c = Ac \cap Z(A)$.

Proof. Apply the proof of 5.7 for $n = 1$. □

25.2 Intrinsically projective algebras.

For an algebra A with centroid $C(A)$, the following are equivalent:

(a) A is intrinsically projective (i.e., satisfies $(P.1)$);

(b) for any finitely generated left ideal $J \subset C(A)$, $J = Hom_{M(A)}(A, AJ)$.

If A is a finitely generated $M(A)$-module, then $(a), (b)$ are equivalent to:

(c) For every left ideal $J \subset C(A)$, $J = Hom_{M(A)}(A, AJ)$;

(d) the map $J \mapsto AJ$ from left ideals in $C(A)$ to ideals of A is injective.

If A has a unit, then $(a) - (c)$ are equivalent to:

(e) For every ideal $I \subset Z(A)$, $I = AI \cap Z(A)$.

Proof. The assertions are special cases of 5.7. □

25.3 Properties of intrinsically projective algebras.

Assume A is intrinsically projective and finitely generated as an $M(A)$-module by $a_1, \ldots, a_k \in A$. Then (see 2.11)

$$C(A) \simeq (a_1, \ldots, a_k)C(A) \subset A^k.$$

(1) For every left ideal $J \subset C(A)$,

$$(a_1, \ldots, a_k)C(A) \cap (A^k)J = (a_1, \ldots, a_k)J.$$

(2) For every proper left ideal $J \subset C(A)$, $A \neq AJ$.

(3) If $A_{C(A)}$ is flat, then $(a_1, \ldots, a_k)C(A)$ is a pure submodule of $A^k_{C(A)}$.

(4) If $A_{C(A)}$ is a projective $C(A)$-module, then $C(A)$ is isomorphic to a direct summand of $A^k_{C(A)}$ (and hence $A_{C(A)}$ is a generator in Mod-$C(A)$).

Proof. This is a particular case of the situation described in 5.8. □

For unital algebras the above proposition yields:

25.4 Intrinsically projective unital algebras.

Assume A is a intrinsically projective unital algebra with centre $Z(A)$.

(1) *If $A_{Z(A)}$ is flat, then $Z(A)$ is a pure submodule of $A_{Z(A)}$.*

(2) *If $A_{Z(A)}$ is a projective $Z(A)$-module, then $Z(A)$ is a direct summand of $A_{Z(A)}$.*

Proof. Put $k = 1$ and $a_1 = 1$ in 25.3. □

Property (P.2) defines a class of algebras of particular interest:

25.5 Self-projective algebras.

(1) *For an algebra A, the following are equivalent:*

(a) *A is self-projective (as an $M(A)$-module);*

(b) *A is A^n-projective for all $n \in I\!N$.*

(2) *Let A be a self-projective algebra, $U \subset A$ an ideal with $U\,C(A) \subset U$, and $p : A \to A/U$ the canonical map. Then:*

(i) *$Hom_{M(A)}(A, p) : C(A) \to C(A/U)$ is surjective.*

(ii) *A/U is self-projective (as $M(A/U)$-module).*

(iii) *For every ideal $J \subset C(A)$, A/AJ is self-projective.*

(iv) *For every scalar R-algebra S, $A \otimes_R S$ is a self-projective algebra.*

Proof. (1) This is a basic property of projectivity (see [40, 18.2]).

(2) (i) By A-projectivity of A, the map

$$Hom_{M(A)}(A, p) : Hom_{M(A)}(A, A) \to Hom_{M(A)}(A, A/U)$$

is surjective. Since U is fully invariant we identify $Hom_{M(A)}(A, A/U) = C(A/U)$.

(ii) The factor of the self-projective $M(A)$-module A by a fully invariant submodule U is again a self-projective $M(A)$-module (see [40, 18.2]). Then A/U is also self-projective as $M(A/U)$-module (see 2.3).

(iii) For any ideal $J \subset C(A)$, AJ is fully invariant in A.

(iv) This follows from 19.1. □

Before turning to unital algebras we consider general algebras with (P.3).

25.6 A projective in $\sigma[A]$.

For an algebra A, the following are equivalent:

(a) *A is projective in $\sigma[A]$;*

(b) *A is $A^{(\Lambda)}$-projective for any set Λ;*

(c) the functor $Hom_{M(A)}(A, -) : \sigma[A] \to C(A)$*-Mod is exact.*

If A is a finitely generated $M(A)$*-module, (a)-(c) are equivalent to:*

(d) A is self-projective;

(e) A is finitely presented in $\sigma[A]$*, and for every* $x \in \mathcal{X}$*,* A_x *is self-projective;*

(f) for every $x \in \mathcal{X}$*,* A_x *is self-projective with centroid* $C(A)_x$*;*

(g) for every $m \in \mathcal{M}$*,* A_m *is self-projective with centroid* $C(A)_m$*.*

If A is projective in $\sigma[A]$*, then for any left ideal* $J \subset C(A)$*,*

$$J = Hom_{M(A)}(A, AJ).$$

Proof. The first equivalences are obtained from [40, 18.3 and 18.4]. The rest is an application of 19.2. □

Now we investigate the properties regarded above for unital algebras.

25.7 Self-projective unital algebras.

For a unital algebra A, the following are equivalent:

(a) A is self-projective (as an $M(A)$*-module);*

(b) A is projective in $\sigma[A]$*;*

(c) for any surjective algebra morphism $f : A \to B$*,* $Z(A)f = Z(B)$*, i.e., for every ideal* $U \subset A$*,*

$$Z(A/U) = Z(A)/(U \cap Z(A));$$

(d) every factor algebra of A is self-projective;

(e) for any A-generated $M(A)$*-module M and epimorphism* $g : M \to N$*,* $Z_A(M)g = Z_A(N)$*;*

(f) for any $M \in \sigma[A]$*,* $a_1, \ldots, a_k \in A$ *and* $m_1, \ldots, m_k \in Z_A(M)$*: if* $\sum_{i=1}^k a_i m_i \in Z_A(M)$*, then there exist* $c_1, \ldots, c_k \in Z(A)$ *with*

$$\sum_{i=1}^k a_i m_i = \sum_{i=1}^k c_i m_i\,;$$

(g) for any $a \in A$*,*

$$(a + < \{[a,x], (a,x,y), (y,x,a) \mid x, y \in A\} >) \cap Z(A) \neq \emptyset,$$

where $< X >$ *denotes the ideal generated by* $X \subset A$*.*

Proof. $(a) \Leftrightarrow (b)$ is shown in 25.6.

$(b) \Rightarrow (c)$ In a unital algebra all ideals are fully invariant and $C(A) \simeq Z(A)$. So the assertion follows from 25.5(2).

$(c) \Rightarrow (a)$ Consider the diagram with exact row,

$$\begin{array}{ccccc} & & A & & \\ & & \downarrow_f & & \\ A & \xrightarrow{g} & X & \longrightarrow & 0\,. \end{array}$$

Then $(1)f \in Z(B)$ and hence there exists some $e \in Z(A)$ with $(e)g = 1$. Now the morphism $A \to A$ defined by $1 \mapsto e$, extends the diagram as desired.

$(b) \Leftrightarrow (e)$ Using the isomorphism $Z_A(M) \simeq Hom_{M(A)}(A, M)$, this is shown similarly to $(a) \Leftrightarrow (c)$.

$(b) \Rightarrow (f)$ With the elements given in (f) we have the diagram

$$\begin{array}{ccc} & & A \\ & & \downarrow_g \\ A^k & \xrightarrow{f} & \sum Am_i \end{array} ,$$

with $(b_1, \ldots, b_k)f := \sum b_i m_i$ for $b_i \in A$ and $(1)g := \sum_{i=1}^k a_i m_i$.

By (b), there exists a morphism $h = (h_1, \ldots, h_k) : A \to A^k$ with $hf = g$. For this we have $(1)h_i =: c_i \in Z(A)$ and

$$(1)g = \sum_{i=1}^k a_i m_i = (1)hf = \sum_{i=1}^k c_i m_i\,.$$

$(f) \Rightarrow (c)$ is obvious by putting $M = A$.

$(c) \Leftrightarrow (g)$ For $a \in A$ and $f : A \to B$ we have $(a)f \in Z(Af)$ if and only if

$$I_a := < \{[x, a], (a, x, y), (y, x, a) \mid x, y \in A\} > \subset Ke\, f.$$

If $(a + I_a) \cap Z(A) \neq \emptyset$, then there exists an element in $Z(A)$ which is mapped to $(a)f$ and so (c) holds true.

Now assume (c). For the canonical map $g : A \to A/I_a$, $(a)g \in Z(Ag)$, and there exists a preimage $b \in Z(A)$ of $(a)g$, which means $b \in (a + I_a) \cap Z(A)$. □

For the radical of the centroid of a self-projective ring we obtain:

25.8 Radical of self-projective algebras.

Let A be finitely generated and self-projective as an $M(A)$-module. Then:

(1) $Jac\,(C(A)) = Hom_{M(A)}(A, Rad\,A)$.

(2) $C(A/Rad\,A) \simeq C(A)/Jac\,(C(A))$.

(3) If A has a unit, then

$$Jac\,(Z(A)) = Z(A) \cap BMc(A) \quad \text{and} \quad Jac\,(Z(A))A \subset BMc(A).$$

(4) If A has a unit and is a self-generator, then $BMc(A) = Jac\,(Z(A)) \cdot A$.

Proof. (1) and (2) are known for endomorphism rings of finitely generated self-projective modules (see [40, 22.2]).

(3) For a ring A with unit, $Rad(A) = BMc(A)$. From (1) we obtain

$$Jac\,(Z(A)) = Hom_{M(A)}(A, BMc(A)) = Z(A) \cap BMc(A).$$

(4) Since A is a self-generator, $I = (I \cap Z(A)) \cdot A$ for any ideal $I \subset A$ (see 24.3). Hence the assertion follows from (3). □

In the next proposition we state properties equivalent to (P.4) which are immediately obtained from general module theory:

25.9 A projective in $M(A)$-*Mod*.

For any algebra A, the following assertions are equivalent:

(a) A is projective in $M(A)$-Mod;

(b) A is a direct summand of $M(A)^{(\Lambda)}$, for some set Λ;

(c) the functor $Hom_{M(A)}(A, -) : M(A)\text{-}Mod \to C(A)\text{-}Mod$ is exact.

If A has a unit e the next assertions are also equivalent to (a):

(d) The surjective map $M(A) \to A$, $\rho \mapsto \rho \cdot e$, splits in $M(A)$-Mod;

(e) the map $Hom_{M(A)}(A, M(A)) \to Z(A)$, $f \mapsto (e)f \cdot e$, is surjective.

From this we derive as a corollary:

25.10 Properties of projective algebras.

Assume the algebra A is finitely generated and projective in $M(A)$-Mod. Then:

(1) $C(A)$ is a direct summand of $A^k_{C(A)}$, for some $k \in I\!N$.

(2) $A_{C(A)}$ is a generator for the right $C(A)$-modules.

(3) If A has a unit, then $Z(A)$ is a direct summand in $A_{Z(A)}$.

Proof. (1) There exists a splitting sequence $M(A)^k \to A \to 0$ of $M(A)$-modules. Applying $Hom_{M(A)}(-, A)$ we get the splitting sequence $0 \to C(A) \to A^k$.

(2) This property is equivalent to (1).

(3) In case A has a unit we can chose $k = 1$ in (1). □

A remarkable property of the class of projective R-algebras is that it is closed under tensor products:

25.11 Tensor product of projective algebras.

Assume A and B are unital R-algebras such that A and B are projective as $M(A)$-, resp. $M(B)$-modules.

Then $A \otimes_R B$ is projective as an $M(A \otimes_R B)$-module and

$$Z(A) \otimes_R Z(B) \simeq Z(A \otimes_R B).$$

Proof. Since the algebras considered are unital there is a surjective map $M(A) \otimes_R M(B) \to M(A \otimes_R B)$ (cf. 15.10) and the assertions follow from 19.10. $\square$

Since every scalar R-algebra S is projective over $M(S) = S$, we can specialize the above results to scalar extensions:

25.12 Scalar extensions of projective algebras.

Let A be a unital R-algebra which is projective as an $M(A)$-module. Then:

(1) For any scalar R-algebra S, $A \otimes_R S$ is projective as $M(A \otimes_R S)$-module with centre $Z(A) \otimes_R S$.

(2) For every ideal $I \subset Z(A)$, A/AI is a projective $M(A/AI)$-module with centre $Z(A)/I$.

25.13 Exercises.

(1) Let A be a unital associative self-projective algebra. Show that for any $n \in \mathbb{N}$, the matrix algebra $A^{(n,n)}$ is also self-projective (as bimodule).

(2) Let A be a unital associative left self-injective regular algebra. Prove that A is self-projective as an $M(A)$-module ([258, Theorem 4]).

References. Tyukavkin [258], Wisbauer [266, 267].

26 Ideal algebras and Azumaya rings

1.Ideal algebras with units. 2.Ideal algebras. 3.Properties of ideal algebras. 4.Azumaya rings. 5.Azumaya rings with unit. 6.Properties of Azumaya rings. 7.Correspondence of submodules. 8.Local-global characterizations of Azumaya rings. 9.Ideal algebras and Azumaya rings. 10.Ideal algebras over locally perfect rings. 11.Example. 12.Exercises.

In this section we investigate algebras A which are projective generators in $\sigma[A]$. To begin with we consider a slightly more general class of algebras:

Definition. An R-algebra A is said to be an *ideal algebra* if the map $I \mapsto AI$ defines a bijection between the ideals in R and the two-sided ideals of A.

For unital algebras we have:

26.1 Ideal algebras with units.

For an R-algebra A with unit, the following are equivalent:

(a) A is an ideal algebra;

(b) (i) for any ideal $U \subset A$, $U = (U \cap R)A$, and
(ii) for any (finitely generated) ideal $I \subset R$, $I = R \cap AI$;

(c) (i) for any ideal $U \subset A$, $U = (U \cap R)A$, and
(ii) A is a faithfully flat R-module;

(d) for every $m \in \mathcal{M}$, A_m is an ideal R_m-algebra;

(e) for every $x \in \mathcal{X}$, A_x is an ideal R_x-algebra.

Proof. $(a) \Leftrightarrow (b)$ This is clear since the inverse map of $I \to AI$ is given by $U \to U \cap R$, for ideals $I \subset R$, $U \subset A$.

$(b) \Leftrightarrow (c)$ follows from 5.9.

$(c) \Rightarrow (d),(e)$ It follows from 5.9 that in fact AS^{-1} is an ideal RS^{-1}-algebra for every multiplicative $S \subset R$.

$(d),(e) \Rightarrow (c)$ For any ideal $U \subset A$, $A(U \cap R) \subset U$. Localizing with respect to $m \in \mathcal{M}$ we obtain by 16.5 and (d),

$$A(U \cap R)_m = A_m(U \cap R)_m = A_m(U_m \cap R_m) = U_m.$$

From this we conclude $A(U \cap R) = U$.

Since all A_m are faithfully flat R_m-modules, A is a faithfully flat R-module.

The same proof applies for $x \in \mathcal{X}$. □

By definition, a central R-algebra A is an ideal algebra if and only if it is an ideal module over $M(A)$. We transfer the characterization of these modules in 5.9 to algebras.

Notice that for a central ideal R-algebra A and $m \in \mathcal{M}$, A_m is an ideal R_m-algebras (see above) but need not be a central R_m-algebra.

26.2 Ideal algebras.

Assume A is a central R-algebra and a finitely generated $M(A)$-module. Then the following are equivalent:

(a) A is an ideal algebra;

(b) (i) A is a self-generator and
(ii) intrinsically projective as an $M(A)$-module;

(c) (i) A is a self-generator as an $M(A)$-module, and
(ii) faithfully flat as an R-module.

Proof. The assertions are derived from the description of ideal modules in 5.9 and of intrinsically projective modules in 25.2. □

The following properties are immediate consequences of 26.2.

26.3 Properties of ideal algebras.

Assume A is a central ideal algebra. Then:

(1) Every maximal ideal $U \subset A$ is of the form $U = Am$, for some maximal ideal $m \subset R$.

(2) A is a quasi-simple algebra if and only if R is a field.

(3) For every maximal ideal $m \subset R$, A/mA is a quasi-simple algebra.

By slightly strengthening the projectivity or the generator condition we arrive at the following class of algebras:

Definition. A central R-algebra A is called *Azumaya ring (over R)* if A is finitely generated, self-projective and self-generator as an $M(A)$-module.

Notice that by definition Azumaya rings have commutative centroids. Obviously, quasi-simple rings (and finite products of quasi-simple rings) A with commutative centroids are Azumaya rings (over $C(A)$).

Again applying results from module theory we obtain:

26.4 Azumaya rings.

Assume A is a central R-algebra and is finitely generated as an $M(A)$-module. Then the following are equivalent:

(a) *A is an Azumaya ring;*

(b) *A is a projective generator in* $\sigma[A]$*;*

(c) *A is a generator in* $\sigma[A]$ *and*
  (α) $AI \neq A$ *for every (maximal) ideal* $I \subset R$*, or*
  (β) *A is faithfully flat as an R-module;*

(d) *(i)* $AI \neq A$ *for every (maximal) left ideal* $I \subset R$*, and*
  (ii) for any finitely A-generated $M(A)$*-module U, the canonical map* $A \otimes_R Hom_{M(A)}(A, U) \to U$ *is injective;*

(e) *there are canonical isomorphisms*
  (i) for every R-module Y, $Y \simeq Hom_{M(A)}(A, A \otimes_R Y)$ *and*
  (ii) for every $X \in \sigma[A]$, $A \otimes_R Hom_{M(A)}(A, X) \simeq X$*;*

(f) $Hom_{M(A)}(A, -) : \sigma[A] \to R\text{-}Mod$ *is an equivalence of categories.*

Proof. $(a) \Leftrightarrow (b)$ A finitely generated self-projetive module A is projective in $\sigma[A]$ (see 5.1). A projective self-generator A is a generator in $\sigma[A]$ (see [40, 18.5]).

If A is a generator in $\sigma[A]$, A is a flat R-module (see 5.3) and hence in (c), (α) and (β) are equivalent (see [40, 12.17]).

The remaining equivalences are special cases of 5.10. □

For unital Azumaya rings A, $R = C(A)$ is the centre of A and we have a nice internal characterization:

26.5 Azumaya rings with unit.

For a central R-algebra A with unit, the following are equivalent:

(a) *A is an Azumaya ring;*

(b) *for every ideal* $U \subset A$*,*

$$U = (U \cap R)A \quad \text{and} \quad Z(A/U) = R/U \cap R.$$

Proof. We take $Z(A) = R$. For unital algebras A, the first condition in (b) is equivalent for A to be a self-generator (see 24.3) and the second condition is equivalent for A to be self-projective (see 25.7). □

26.6 Properties of Azumaya rings.

Assume A is an Azumaya ring over R. Then:

(1) *A is an ideal algebra.*

(2) $M(A)$ *is dense in* $End_R(A)$*.*

(3) *For every ideal* $U \subset A$*,* A/U *is an Azumaya ring (over* $C(A/U)$*).*

(4) For every maximal ideal $m \subset R$, A/Am is a central quasi-simple R/m-algebra.

(5) If $A^2 = A$, then, for every maximal ideal $m \subset R$, A/Am is a central simple R/m-algebra.

(6) If A has a unit, then for any unital R-algebra B, $Z(A \otimes_R B) \simeq Z(B)$.

Proof. (1) is obvious by 26.2, (2) is shown in 24.7.

(3) By 19.7, A/U is a self-generator and by 25.5, A/U is self-projective.

(4),(5) Notice that $Am \cap R = m$ by 26.2. Hence by (3), A/Am is an Azumaya ring with centroid R/m (see 25.5).

If $A^2 = A$, then $(A/mA)^2 \neq 0$ and A/mA is simple by 26.2.

(6) Applying the isomorphism $B \simeq Hom_{M(A)}(A, A \otimes_R B)$ (see 26.4) the assertion is derived from 15.12. □

26.7 Correspondence of submodules.

Assume A is an Azumaya R-algebra and B any R-algebra.

(1) There is an isomorphism of $M(B)$-modules,

$$B \to Hom_{M(A)}(A, A \otimes_R B),\ b \mapsto [a \mapsto a \otimes b].$$

(2) There is a bijection between the $M(A) \otimes_R M(B)$-submodules $U \subset A \otimes_R B$ and the left ideals $I \subset B$, given by

$$U \mapsto Hom_{M(A)}(A, U), \quad I \mapsto A \otimes_R I.$$

(3) Assume $A^2 = A$ and B has a unit. Then (identifying B with $1 \otimes B$) the above bijection is between the ideals $U \subset A \otimes_R B$ and ideals $I \subset B$, given by

$$U \mapsto U \cap B \text{ and } I \mapsto A \otimes_R I.$$

Proof. (1) Obviously the isomorphism in 26.4(e) is in fact an $M(B)$-morphism.

(2) By (1), we may identify $Hom_{M(A)}(A, U)$ with an ideal in B. Regarding $M(A) \otimes_R M(B)$-submodules of $A \otimes_R B$ as $M(A)$- and $M(B)$-submodules, the correspondence is provided by the canonical isomorphisms in 26.4(e).

Notice that in general $M(A) \otimes_R M(B)$-submodules need not coincide with the $M(A \otimes_R B)$-submodules (ideals) of $A \otimes_R B$.

(3) Consider an ideal $U \subset A \otimes_R B$. Since B has a unit, there is an algebra homomorphism $M(A) \to M(A \otimes_R B)$ and hence U is an $M(A)$-submodule.

From the canonical isomorphisms in 26.4(e), we obtain an $M(A)$-isomorphism $U \simeq A \otimes_R Hom_{M(A)}(A, U)$. If we show that $Hom_{M(A)}(A, U)$ is an $M(B)$-submodule the correspondence wanted follows from (2).

For any $a \in A$ and $b \in B$, $L_a \otimes L_b \in M(A \otimes_R B)$ and hence

$$L_a \otimes L_b(A \otimes_R Hom_{M(A)}(A,U)) = L_a A \otimes_R L_b Hom_{M(A)}(A,U) \subset U.$$

Now $A^2 = A$ implies

$$A \otimes_R L_b Hom_{M(A)}(A,U) = A \otimes_R Hom_{M(A)}(A, L_b U) \subset U.$$

Since A generates $L_b U$ as an $M(A)$-module we deduce $L_b U \subset U$. A similar argument shows $R_b U \subset U$, for all $b \in B$, and so U is an $M(B)$-module. □

Remarks. The correspondence considered in 26.7 was originally proved for associative central simple algebra (e.g., Jacobson [19, p. 109]). It was observed in Stewart [246, Corollary 2.2] that it also holds for non-associative algebras. For associative Azumaya rings with units the result is due to Azumaya [60, Proposition 2.9]. Similar relations for *strictly simple m-ary algebras* over fields are given in Röhrl [229, § 5].

To formulate some characterizations of Azumaya rings recall that we denote by $\mathcal{M}$ the set of all maximal ideals of R (see 17.1) and by $\mathcal{X}$ the set of all maximal ideals in the ring of idempotents of R (see 18.1).

From 19.8 we obtain:

26.8 Local-global characterizations of Azumaya rings.

Let A be a central R-algebra which is finitely generated as an $M(A)$-module. Then the following are equivalent:

(a) A is an Azumaya ring;

(b) A is finitely presented in $\sigma[A]$ and, for every $x \in \mathcal{X}$, A_x is an Azumaya ring;

(c) for every $x \in \mathcal{X}$, A_x is an Azumaya ring with centroid R_x;

(d) for every $m \in \mathcal{M}$, A_m is an Azumaya ring with centroid R_m.

Assume R is locally perfect. Then (a)-(d) are also equivalent to:

(e) For every $m \in \mathcal{M}$, A/mA is a central quasi-simple R/m-algebra.

Our next results show that ideal algebras are very close to Azumaya rings.

26.9 Ideal algebras and Azumaya rings.

(1) For a central R-algebra A the following are equivalent:

(a) A is an Azumaya ring;

(b) A is an ideal algebra and is a self-projective $M(A)$-module;

(c) A is an ideal algebra and is a generator in $\sigma[A]$.

(2) *Let A be an ideal algebra over R.*
If A is a projective R-module, then A is a generator in Mod-R and is an Azumaya ring. In particular, if R is perfect then A is an Azumaya ring.

Proof. (1) $(a) \Leftrightarrow (b)$ is clear from 26.2. $(a) \Rightarrow (c)$ is obvious.

$(c) \Rightarrow (a)$ A is a finitely generated generator in $\sigma[A]$ which is faithfully flat over its endomorphism ring. Hence A is projective in $\sigma[A]$ by 5.10.

(2) The first assertion follows from 5.9.
If R is perfect, the flat R-module A is projective. □

26.10 Ideal algebras over locally perfect rings.

Assume A is a central R-algebra and R is locally perfect. Then the following are equivalent:

(a) *A is an Azumaya ring;*

(b) *A is finitely presented in $\sigma[A]$ and A is an ideal algebra;*

(c) *A is finitely presented in $\sigma[A]$ and, for every $x \in \mathcal{X}$, A_x is an ideal algebra;*

(d) *for every $x \in \mathcal{X}$, A_x is an ideal algebra with centroid R_x;*

(e) *for every $m \in \mathcal{M}$, A_m is an ideal algebra with centroid R_m.*

Proof. $(a) \Rightarrow (b)$ Follows from 26.6 and the fact that Azumaya algebras are finitely generated and projective in $\sigma[A]$.

$(b) \Rightarrow (c)$ Apply 5.9(2).

Recalling 26.9(2), the remaining assertions are derived from 26.8. □

26.11 Example.

Consider the central R-algebra $B = End_R(R^{(\mathbb{N})})$, the ideal

$$I = \{f \in B \mid Im\, f \subset R^k \text{ for some } k \in \mathbb{N}\},$$

and put $A = B/I$. Then (cf. [99]):

(1) A is an R-central algebra.

(2) If R is a field, A is a central simple R-algebra.

(3) For any $m \in \mathcal{M}$, A/mA is a central simple R/m-algebra.

(4) A is an Azumaya ring if and only if R is artinian.

(5) Let R be a non-artinian ring with the maximal ideals finitely generated. Then for every $m \in \mathcal{M}$, A/mA is a central simple R/m-algebra but A is not an Azumaya ring.

(6) If R is a field, then I is the only ideal in B and B is a self-projective $M(B)$-module but is not an Azumaya ring.

Remarks. The investigation of (unital associative) ideal algebras was initiated in Ranga Rao [224].

The study of associative unital Azumaya rings A by properties of bimodules related to A already occurs in Artin [57]. His incorrect Proposition 2.3 was the first attempt to characterize these rings. The investigation was continued in Delale [115]. For associative rings, 26.5 corresponds to [115, Théorème 5.1]. Delale calls these rings *anneaux affines sur le centre* or *algèbres affines*. In Azumaya [60], the same rings are studied under the name *separable rings*, and in Burkholder [99] they are called *Azumaya rings*.

Here we use the name *Azumaya rings* also more generally for non-associative rings of this type. They were already considered in [267].

The investigations in Burkholder [101] are concerned with the question when products of associative Azumaya rings or simple rings are Azumaya rings. In particular it is pointed out that an infinite product of isomorphic copies of a simple ring need not be an Azumaya ring whereas any product of division rings is an Azumaya ring.

26.12 Exercises.

(1) Let A be a central R-algebra which is finitely generated as an $M(A)$-module. Assume R is a semihereditary ring. Prove:

(i) If A is a self-generator, then A is a semi-projective $M(A)$-module.

(ii) If A is flat R-module, then A is intrinsically projective as $M(A)$-module.

(iii) If A is a generator in $\sigma[A]$, then A is an Azumaya ring.

Hint: 5.6.

(2) Assume A is an associative unital Azumaya ring over R. Show that every module in $\sigma[A]$ is (A,R)-projective as a left A-module.

References: Artin [57], Azumaya [60], Burkholder [99, 100, 101], Delale [115], Jacobson [19], Pierce [32], Ranga Rao [224], Röhrl [229], Stewart [246], Wisbauer [267].

27 Cogenerator and injectivity properties of A

1.Algebras satisfying (C.1). 2.Self-cogenerator algebras. 3.Algebras satisfying (C.3). 4.Semi-injective algebras. 5.Intrinsically injective algebras. 6.Self-injective algebras. 7.Cogenerator algebras. 8.Cogenerator algebras with unit. 9.Azumaya rings as cogenerators. 10.Azumaya algebras as cogenerators. 11.Algebras with Morita dualities. 12.Quasi-Frobenius algebras. 13.Projective Quasi-Frobenius algebras. 14.Exercises.

Dual to generator properties we consider cogenerator conditons on an algebra A.

$(C.1)$ For every proper ideal $U \subset A$, $Hom_{M(A)}(A/U, A) \neq 0$.

$(C.2)$ For every ideal $U \subset A$, A/U is cogenerated by A.

$(C.3)$ For every $f : A^k \to A^n$ in $\sigma[A]$, $k, n, \in I\!N$, $Coke\, f$ is cogenerated by A.

$(C.4)$ A is a weak cogenerator in $\sigma[A]$.

$(C.5)$ A is a cogenerator in $\sigma[A]$.

We always have $(C.5) \Rightarrow (C.4)$, $(C.5) \Rightarrow (C.2) \Rightarrow (C.1)$ and $(C.5) \Rightarrow (C.3)$. If A is a finitely generated $M(A)$-module, then $(C.4) \Rightarrow (C.3)$.

27.1 Algebras satisfying $(C.1)$.

Let A be an algebra.

(1) Assume A satisfies $(C.1)$. Then A is
quasi-simple if and only if $C(A)$ is a division algebra;
a weak product of quasi-simple algebras if and only if $C(A)$ is regular;
a finite product of quasi-simple algebras if and only if $C(A)$ is left semisimple.

(2) If A is finitely generated as an $M(A)$-module, the following are equivalent:
(a) A satisfies $(C.1)$;
(b) for every maximal ideal $I \subset A$, $An_{C(A)}(I) \neq 0$.

Proof. (1) If A is quasi-simple, $C(A)$ is a divison algebra. If A is a weak (finite) product of quasi-simple algebras, then $C(A)$ is regular (left semisimple) by 23.9.

For any ideal $U \subset A$, consider a non-zero $f : A/U \to A$ and the projection $p : A \to A/U$. Then $pf \in C(A)$.

If $C(A)$ is a division algebra, then pf is invertible and $U \subset Ke\, pf = 0$.

Assume $C(A)$ is regular. Let $U \trianglelefteq A$. Without restriction we may assume that f is injective. Then $Ke\, pf = U$ is a direct summand (see 7.6), hence $U = A$. This implies that A is a semisimple $M(A)$-module.

If $C(A)$ is left semisimple, there are only finite families of orthogonal idempotents in $C(A)$ and A is a finite product of quasi-simple algebras.

(2) This is obvious by the fact that every proper ideal of A is contained in a maximal ideal of A. □

An algebra A which satisfies $(C.2)$ is called a *self-cogenerator algebra.* From Module Theory and 27.1 we obtain:

27.2 Self-cogenerator algebras.

Let A be a self-cogenerator algebra. Then:

(1) A cogenerates every simple module in $\sigma[A]$.

(2) For every $a \in A$ and $f \in End(A_{C(A)})$, there exists $\mu \in M(A)$ with $f(a) = \mu a$.

(3) If $C(A)$ is regular, then A is a weak product of quasi-simple algebras.

For algebras satisfying $(C.3)$ we transfer from [40, 15.9]:

27.3 Algebras satisfying $(C.3)$.

For any algebra A with centroid $C(A)$, the following are equivalent:

(a) For every morphism $f : A^k \to A^n$, $k, n \in I\!N$, Coke f is cogenerated by A;

(b) for every morphism $f : A \to A^n$, $n \in I\!N$, Coke f is cogenerated by A;

(c) A is FP-injective as a right $C(A)$-module.

Algebras with $(C.5)$ we call *cogenerator algebras.* As we will see, such algebras are self-injective provided they have commutative centroid. Hence we shall characterize them after some remarks about injectivity properties of A.

Consider the following properties of A as $M(A)$-module:

$(I.1)$ A is intrinsically injective as an $M(A)$-module (see 5.7), i.e., every diagram of $M(A)$-modules with exact row,

$$\begin{array}{ccccc} 0 & \longrightarrow & U & \longrightarrow & A^n \\ & & \downarrow & & \\ & & A & & , \end{array}$$

with $n \in I\!N$ and U a factor module of A, can be extended commutatively by some $A^n \to A$. A is called *semi-injective* if the above condition holds for $n = 1$.

$(I.2)$ A is weakly A-injective.

$(I.3)$ A is self-injective (i.e., injective in $\sigma[A]$).

The implication $(I.3) \Rightarrow (I.2)$ is obvious. If A is finitely generated as an $M(A)$-module, $(I.2) \Rightarrow (I.1)$ (then U in $(I.1)$ is finitely generated).

We are going to characterize algebras with the properties given above. From 6.12 we obtain:

27.4 Semi-injective algebras.

For a central R-algebra A, the following are equivalent:

(a) *A is semi-injective;*

(b) *$fR = Hom_{M(A)}(A/Ke\, f, A)$, for every $f \in R$ $(= C(A))$;*

(c) *the map $I \mapsto Ke\, I$ from cyclic right ideals in R to ideal of A is injective.*

If A has a unit, then (a)-(c) are equivalent to:

(d) *For every $c \in R$, $cR = An_R An_A(cR)$.*

Next we consider algebras with $(I.1)$:

27.5 Intrinsically injective algebras.

For a central R-algebra A, the following are equivalent:

(a) *A is intrinsically injective;*

(b) *$I = Hom_{M(A)}(A/Ke\, I, A)$ for every finitely generated right ideal $I \subset R$;*

(c) *the map $I \mapsto Ke\, I$ from finitely generated right ideals in R to ideals of A is injective.*

If A has a unit, (a)-(c) are equivalent to:

(d) *For every finitely generated ideal $I \in R$, $I = An_R An_A(I)$.*

Algebras with $(C.3)$ are called *self-injective algebras*. They will be of great importance in localization theory. Here we list some elementary facts.

27.6 Self-injective algebras.

For a central R-algebra A, the following are equivalent:

(a) *A is a self-injective algebra;*

(b) *A is injective in $\sigma[A]$;*

(c) *the functor $Hom_{M(A)}(-, A) : \sigma[A] \to Mod\text{-}R$ is exact.*

If A has a unit, (a)-(c) are equivalent to:

(d) *For every ideal $U \subset A$ and $f \in Hom_{M(A)}(U, A)$, there exists $a \in R$ such that $(u)f = ua$ for all $u \in U$.*

Proof. All these equivalences are clear from Module Theory. The characterization in (d) corresponds to *Baer's Lemma* for left modules over associative unital rings. □

Remark. For an associative commutative algebra A with unit, the following are equivalent:

(a) A is a cogenerator in A-Mod;

(b) A is an injective cogenerator in A-Mod;

(c) A is finitely cogenerated and injective in A-Mod.

For non-commutative A, (b) and (c) are still equivalent (e.g., [40, 48.12]) but (a) and (b) are not.

Replacing A-*Mod* by $\sigma[A]$ we will see that the above equivalences remain true for any algebra with unit. More generally we obtain as an application of 6.7:

27.7 Cogenerator algebras.

For a central R-algebra A, the following are equivalent:

(a) A is a cogenerator in $\sigma[A]$;

(b) A is self-injective and self-cogenerator as $M(A)$-module;

(c) $A = \bigoplus_\Lambda \widehat{E}_\lambda$, where the E_λ form a (minimal) representing set of the simple modules in $\sigma[A]$ and $\widehat{E}_\lambda$ is the injective hull of E_λ in $\sigma[A]$;

(d) $A = \bigoplus_\Lambda A_\lambda$, where the A_λ are R-algebras which are indecomposable self-injective self-cogenerators as bimodules.

If A has a unit the decomposition above is finite and we get:

27.8 Cogenerator algebras with unit.

For a central R-algebra A with unit, the following are equivalent:

(a) A is a cogenerator in $\sigma[A]$;

(b) $A = \widehat{E}_1 \oplus \cdots \oplus \widehat{E}_k$, where the E_i are (up to isomorphism) all the simple modules in $\sigma[A]$, and $\widehat{E}_i$ is the injective hull of E_i in $\sigma[A]$;

(c) $A = A_1 \oplus \cdots \oplus A_k$, where the algebras A_i are indecomposable self-injective self-cogenerators as bimodules;

(d) R is a semiperfect ring and for every maximal ideal $m \subset R$, $A_m = A \otimes_R R_m$ is a self-injective self-cogenerator algebra.

Under these conditions, A_R cogenerates all simple R-modules and A_R is FP-injective.

Proof. From 27.7 we get the equivalence of (a) to (c).

$(b) \Rightarrow (d)$ R is the endomorphism ring of a finitely cogenerated self-injective module and hence semiperfect. By [40, 47.10], A_R cogenerates all simple R-modules. Hence $A_m \neq 0$ for every maximal ideal $m \subset R$, and $A_i \simeq A \otimes R_{m_i}$, for maximal ideals $m_i \subset R$, by 6.7.

$(d) \Rightarrow (c)$ Since R is semiperfect we have $R = R_{m_1} \oplus \cdots \oplus R_{m_k}$ for maximal ideals $m_i \subset R$. Therefore $A = (A \otimes_R R_{m_1}) \oplus \cdots \oplus (A \otimes_R R_{m_k})$, where the $A_i = A \otimes_R R_{m_i}$ are self-injective self-cogenerators by assumption.

Since A is a cogenerator in $\sigma[A]$, the R-module A_R is FP-injective (by 27.3). $\square$

Remark. For a local-global characterization of A being a cogenerator in $\sigma[A]$ (as in (d)) the condition on R to be semiperfect is necessary. For example, for a commutative regular ring R, the local rings R_m are in fact fields for every maximal ideal $m \subset R$ and R need not be a cogenerator in R-*Mod*.

Recall that a self-projective self-generator central R-algebra A is an *Azumaya ring*, and it is an *Azumaya algebra*, provided it is a finitely generated module over R. From 6.8 we derive:

27.9 Azumaya rings as cogenerators.
For a self-projective central R-algebra A with unit, the following are equivalent:

(a) A is a cogenerator in $\sigma[A]$;

(b) A is self-injective, self-generator and finitely cogenerated;

(c) every module which cogenerates A is a generator in $\sigma[A]$;

(d) A is an Azumaya ring and R is a PF-ring.

For module finite algebras the above theorem specializes to:

27.10 Azumaya algebras as cogenerators.
For a central R-algebra A with unit, the following are equivalent:

(a) A is a self-projective cogenerator in $\sigma[A]$ and A_R is finitely generated;

(b) A is a self-projective cogenerator in $\sigma[A]$ and $\sigma[A] = M(A)$-Mod;

(c) $M(A)$ is left injective and finitely cogenerated (= left PF-ring);

(d) A is an Azumaya algebra and R is a PF-ring.

Proof. $(a) \Rightarrow (b)$ For A_R finitely generated, $\sigma[A] = M(A)$-*Mod*.

$(b) \Rightarrow (a)$ Since $M(A) \in \sigma[A]$ we have an exact sequence

$$0 \to M(A) \to A^k, \quad k \in \mathbb{N}.$$

Exactness of $Hom_{M(A)}(-, A)$ yields the exact sequence $R^k \to A_R \to 0$. Hence A_R is a finitely generated R-module.

$(b) \Rightarrow (c)$ According to 27.9 and 27.7, A is an injective generator in $M(A)$-*Mod* and hence $M(A)$ is a direct summand of A^k for some $k \in \mathbb{N}$.

$(c) \Rightarrow (d)$ Since $M(A)$ is cogenerated by A, condition (c) implies that $M(A)$ is a direct summand of a finite direct sum of copies of A. Hence A is a generator in $\sigma[A] = M(A)$-*Mod*, i.e., an Azumaya algebra (24.8). A being a cogenerator in $\sigma[A]$ we know from 27.9 that A is self-injective and finitely cogenerated and R is a PF-ring.

$(d) \Rightarrow (b)$ By 24.8, $Hom(A, -) : \sigma[A] \to R$-*Mod* is an equivalence. Since $R = End_{M(A)}(A)$ is a PF-ring the module ${}_{M(A)}A$ is also a self-cogenerator. $\square$

In view of 27.7, the description of Morita dualities in [40, 47.12] has the form:

27.11 Algebras with Morita dualities.
For a central R-algebra A, the following are equivalent:

(a) *A is linearly compact as an $M(A)$-module and a cogenerator in $\sigma[A]$;*

(b) *A is a cogenerator in $\sigma[A]$ and A_R is injective in R-Mod;*

(c) *$Hom_{M(A)}(-, A)$ defines a duality between the submodules of finitely A-generated $M(A)$-modules and the submodules of finitely generated R-modules;*

(d) *A_R is an injective cogenerator in R-Mod with essential socle, R is linearly compact, and $M(A)$ is a dense subring of $End(A_R)$;*

(e) *all factor modules of ${}_{M(A)}A$ and R_R are A-reflexive.*

Combined with finiteness conditions our results yield the description of a two-sided version of Quasi-Frobenius rings. For this we transfer from [40, 48.14.II]:

27.12 Quasi-Frobenius algebras.
For a central R-algebra A, the following statements are equivalent:

(a) *A is a noetherian injective generator in $\sigma[A]$;*

(b) *A is an artinian projective cogenerator in $\sigma[A]$;*

(c) *A is a noetherian projective cogenerator in $\sigma[A]$;*

(d) *A is an injective generator in $\sigma[A]$ and R is artinian.*

One interesting aspect of the above characterizations is the interplay between projectivity and injectivity properties. In case we already know that A is projective we obtain from [40, 48.14]:

27.13 Projective Quasi-Frobenius algebras.
For a self-projective central R-algebra A with unit, the following are equivalent:

(a) *A is a noetherian cogenerator in $\sigma[A]$;*

(b) *A is noetherian, self-injective and self-generator as $M(A)$-module;*

(c) *A is a cogenerator in $\sigma[A]$ and R is artinian;*

(d) *A is artinian and noetherian and the A-injective hulls of simple modules are projective in $\sigma[A]$;*

(e) *every injective module is projective in $\sigma[A]$;*

(f) *A is an Azumaya ring and every projective module is injective in $\sigma[A]$;*

(g) $A^{(\mathbb{N})}$ *is an injective cogenerator in* $\sigma[A]$.

Remark: It follows from [162, Proposition 3] that for self-injective A the ascending (descending) chain condition on *essential* ideals already implies that A is noetherian (artinian) as bimodule.

The algebra A in 27.12 and 27.13 has finite length as an $M(A)$-module and hence $Hom_{M(A)}(-, A)$ defines a duality between the finitely generated modules in $\sigma[A]$ and the finitely generated modules in R-*Mod* (see [40, 47.13]).

27.14 Exercises.

(1) Let A *be an algebra which is coherent (or noetherian) in* $\sigma[A]$. *Assume that for any finitely generated ideal* $U \subset A$, *every* $f \in Hom_{M(A)}(U, A)$ *extends to* A.

Prove that A *is weakly* A*-injective.*

(2) Let A *be central* R*-algebra with unit. Prove:*

(i) If A *is weakly* A*-injective as an* $M(A)$*-module, then*

(α) *for any finitely generated ideals* $U, V \subset A$,

$$An_R(U \cap V) = An_R(U) + An_R(V);$$

(β) *for any finitely generated ideal* $I \subset R$, $I = An_R An_A(I)$.

(ii) Assume A *is a self-generator and coherent (noetherian) in* $\sigma[A]$. *If* A *satisfies* (α) *and* (β) *in (i), then* A *is weakly* A*-injective.*

(3) Let A *be an intrinsically injective central* R*-algebra. Prove that* R *is noetherian if and only if* A *satisfies dcc on annihilators of finitely generated ideals of* R.

References. Menini-Orsatti [197], Hauger-Zimmermann [154], Wisbauer [277].

Chapter 8

Separable and biregular algebras

28 Associative separable algebras

1.Associative Azumaya algebras. 2.Associative separable algebras. 3.Tensor product of separable algebras. 4.Transitivity of separability. 5.Properties. 6.Separable and relative semisimple algebras. 7.Separable and Azumaya algebras. 8.Characterizations of separable algebras. 9.Local-global characterization. 10.Simple extensions. 11.Separable field extensions. 12.Separable algebras over fields. 13.Examples of separable algebras. 14.Exercises.

In this section A will always denote an associative R-algebra with unit.

A is a left module over $A^e = A \otimes_R A^o$ and there is a surjective A^e-module morphism (see 23.6)

$$\mu : A^e \to A, \; a \otimes b \mapsto ab.$$

Recall that an Azumaya algebra (over R) is a central R-algebra A which is a generator in $M(A)$-*Mod* (see 24.8).

Assume A is an Azumaya algebra. It is obvious from 24.8 that the opposite algebra A^o is also an Azumaya algebra over R and by 24.10, A^e is an Azumaya algebra over R. We can use this to prove:

28.1 Associative Azumaya algebras.

For a central R-algebra A the following are equivalent:

(a) A is an Azumaya algebra;

(b) $A^e \simeq End_R(A)$ and A is a finitely generated, projective R-module;

(c) A is a generator in A^e-Mod;

(d) A is a projective generator in A^e-Mod;

(e) A is projective in A^e-Mod;

(f) A is a finitely generated R-module and one of the following holds:
(i) for every $x \in \mathcal{X}$, A_x is an Azumaya algebra;
(ii) for every $m \in \mathcal{M}$, A_m is an Azumaya algebra;
(iii) for every $m \in \mathcal{M}$, A/mA is an Azumaya algebra;
(iv) A^e is an ideal algebra.

Proof. $(a) \Rightarrow (b)$ The properties of A as an R-module follow from 24.8. It remains to show that the map

$$\varphi : A^e \to M(A) \simeq End_R(A), \quad a \otimes b \mapsto L_a R_b,$$

is injective. $Ke\,\varphi$ is an ideal in A^e. Since by the preceding remark A^e is an Azumaya algebra, we have $Ke\,\varphi = I \cdot A^e$ for some ideal $I \in R$ (see 24.3). This however implies $I \cdot A^e \cdot A = IA = 0$, hence $I = 0$ and $Ke\,\varphi = 0$.

$(b) \Leftrightarrow (c)$ is another application of 5.5.

$(c) \Leftrightarrow (d)$ can be seen with the same proof as $(d) \Rightarrow (e)$ in 24.8.

$(d) \Rightarrow (e)$ is trivial.

$(e) \Rightarrow (d)$ will be shown in 28.7.

$(a) \Leftrightarrow (f), (i), (ii)$ follows from the characterization of Azumaya rings 26.8 since module finite Azumaya rings are Azumaya algebras.

$(a) \Leftrightarrow (f), (iii)$ can be derived from 19.3.

$(a) \Rightarrow (f), (iv)$ By 19.10, the tensor product of Azumaya algebras is again an Azumaya algebra and in particular an ideal algebra.

$(f), (iv) \Rightarrow (iii)$ Assume A^e is an ideal algebra. Then for every $m \in \mathcal{M}$, A^e/mA^e is a semisimple algebra by 26.3 and hence A/mA is a separable R/m-algebra. $\square$

Definition. The (associative unital) R-algebra A is called *separable over* R, or R*-separable*, if A is projective as an A^e-module.

Notice that this definition is only for unital algebras. A notion of separability for associative algebras without units is considered in Taylor [257].

It follows from the surjective homomorphism of R-algebras (see 2.5),

$$A^e \to M(A), \quad a \otimes b \mapsto L_a R_b,$$

that any separable R-algebra is projective in $M(A)$-*Mod*.

For the characterization of these algebras we have:

28.2 Associative separable algebras.

For the R-algebra A, the following are equivalent:

(a) *A is a separable R-algebra;*

(b) *the map $\mu : A^e \to A$ splits (in A^e-Mod);*

(c) *the functor $Hom_{A^e}(A, -) : A^e\text{-}Mod \to Z(A)\text{-}Mod$ is exact;*

(d) *the map $\mu' : Hom_{A^e}(A, A^e) \to End_{A^e}(A)$, $f \mapsto f\mu$, is surjective;*

(e) *there exists* $e \in A^e$ *with* $\mu(e) = 1$ *and* $(a \otimes 1)e = (1 \otimes a)e$ *for all* $a \in A$.
Such an e *is called a* separability idempotent *for* A.

Proof. The equivalence of (a), (b) and (c) follows from general module theory.

$(b) \Rightarrow (d)$ Apply $Hom_{A^e}(A, -)$ to the sequence in (b).

$(d) \Rightarrow (a)$ Since μ' is surjective there exists an $f : A \to A^e$ with $f\mu = id_A$. Hence the sequence in (b) splits.

$(b) \Rightarrow (e)$ The splitting of μ is established by an A^e-morphism $\varepsilon : A \to A^e$ with $\varepsilon\mu = id_A$. Then the element $e := (1)\varepsilon$ has the desired properties.

On the other hand, assume some $e \in A^e$ has the given properties. Then the map $\varepsilon : A \to A^e$, $a \mapsto (a \otimes 1) \cdot e$, defines an A^e-morphism with $\varepsilon\mu = id_A$. □

Next we observe that the class of associative separable algebras is closed under tensor products:

28.3 Tensor product of separable algebras.

Let S *and* T *denote two scalar* R*-algebras.*

(1) *Assume* A *and* B *are separable algebras over* S *and* T*, respectively. Then* $A \otimes_R B$ *is a separable algebra over* $S \otimes_R T$ *and*

$$Z(A \otimes_R B) \simeq Z(A) \otimes_R Z(B).$$

(2) *Assume* A *is a separable* R*-algebra. Then* $A \otimes_R T$ *is a separable* T*-algebra and*

$$Z(A \otimes_R T) \simeq Z(A) \otimes_R T.$$

Proof. (1) Apply 19.10. (2) is a special case of (1) (for $B = T$). □

28.4 Transitivity of separability.

Let S *be a scalar* R*-algebra and* A *an* S*-algebra. Then* A *is also an* R*-algebra and:*

(1) *If* A *is separable over* R*, then* A *is separable over* S*.*

(2) *If* A *is* S*-separable and* S *is* R*-separable, then* A *is* R*-separable.*

(3) *Let* A *be separable over* R *and assume* A *is finitely generated, projective and faithful as an* S*-module. Then* S *is separable over* R*.*

Proof. (1) This follows from the surjective map $A \otimes_R A^o \to A \otimes_S A^o$.

(2) Under the given conditions, for every $A \otimes_R A^o$-module M we have an isomorphism

$$Hom_{A \otimes_R A^o}(A, M) \simeq Hom_{A \otimes_S A^o}(A, Hom_{S \otimes_R S}(S, M)),$$

both sides being isomorphic to the centre of M (see 23.2).

If A and S are separable as demanded, the two Hom-functors on the right side are exact. Hence $Hom_{A\otimes_R A^o}(A,-)$ is also exact and A is R-separable.

(3) By assumption, A is a direct summand of $A \otimes_R A^o$. Since A is S-projective, $A \otimes_R A^o$ is $S \otimes_R S$-projective and so is A.

The map $S \to A$, $s \mapsto s \cdot 1$, is injective and it follows from 17.5 that S is a direct summand of A. Hence S is also $S \otimes_R S$-projective, i.e., S is R-separable. □

Although by definition separability is a two-sided property, our next results show that it also has a strong influence on the one-sided structure of A:

28.5 Properties. *Let A be a separable R-algebra. Then:*

(1) A is left (A,R)-semisimple.

(2) For any maximal ideal $I \subset A$, $I = mA$ for some maximal ideal $m \subset Z(A)$.

(3) If A is projective as an R-module, then A is finitely generated as an R-module.

Proof. (1) By 15.6, there is an isomorphism, functorial in M, $N \in A$-*Mod*,

$$Hom_{A^e}(A, Hom_R(M,N)) \to Hom_A(M,N).$$

By assumption, $Hom_{A^e}(A,-)$ is an exact functor. Since $Hom_R(M,-)$ is exact on R-splitting sequences, we conclude that $Hom_A(M,-)$ is exact on (A,R)-exact sequences. So every A-module M is (A,R)-projective and A is left (A,R)-semisimple.

(2) Consider the ideal $m = I \cap Z(A)$ in $Z(A)$. Since A is a separable $Z(A)$-algebra (see 28.4), A/I is a simple algebra with centre $Z(A)/m$ (see 25.7). This implies that $Z(A)/m$ is a field.

Since A/mA is a central separable $Z(A)/m$-algebra (see 28.3), it is left semisimple (by (3)) with simple centre, i.e., A/mA is a simple algebra. Obviously $mA \subset I$ and maximality of mA implies $I = mA$.

(3) Assume A is projective as an R-module. Then A^o is projective as an R-module and there are

$$\{f_\lambda : A^o \to R\}_\Lambda \text{ and } \{g_\lambda \in A^o\}_\Lambda \text{ with } b = \textstyle\sum_\Lambda (b)f_\lambda g_\lambda \text{ for all } b \in A^o,$$

where only a finite number of the $(b)f_\lambda$'s are not zero (dual basis). In obvious notation we have morphisms

$$\begin{array}{ccccc} A^o & \xrightarrow{f} & R^{(\Lambda)} & \xrightarrow{g} & A^o\,, \\ b & \mapsto & ((b)f_\lambda)_\Lambda & \mapsto & \sum_\Lambda (b)f_\lambda g_\lambda\,, \end{array}$$

with $fg = id_{A^o}$. Tensoring with A we obtain the maps

$$\begin{array}{ccccc} A \otimes_R A^o & \stackrel{id\otimes f}{\longrightarrow} & A \otimes_R R^{(\Lambda)} & \stackrel{id\otimes g}{\longrightarrow} & A \otimes A^o \,, \\ a \otimes b & \mapsto & (a \otimes (b)f_\lambda)_\Lambda & \mapsto & \sum a \otimes (b)f_\lambda g_\lambda \,, \end{array}$$

and $(id \otimes f)(id \otimes g) = id_{A\otimes_R A^o}$. Let $e = \sum x_i \otimes y_i$ denote a separability idempotent of A (see 28.2). For any $a \in A$ we have, with the above maps,

$$(\alpha) \quad (a \otimes 1)e = \sum_{i\leq k} ax_i \otimes \sum_\Lambda (y_i)f_\lambda g_\lambda,$$

$$(\beta) \quad (1 \otimes a)e = \sum_{i\leq k} x_i \otimes \sum_\Lambda (y_i a)f_\lambda g_\lambda.$$

By the choice of $e \in A^e$, (α) and (β) describe the same element. From (α) we see that the subset $\Lambda' \subset \Lambda$, for which the f_λ occuring are nonzero, is finite. Applying the map μ to the expression in (β), we obtain

$$a = \mu((1 \otimes a)e) = \sum_{i\leq k}\sum_{\Lambda'} x_i((y_i a)f_\lambda)g_\lambda = \sum_{i\leq k}\sum_{\Lambda'} (y_i a)f_\lambda \cdot x_i g_\lambda \,.$$

Hence the finite set $\{x_i g_\lambda \,|\, i \leq k,\ \lambda \in \Lambda'\}$ generates A^o (and so A) as an R-module. □

There is another important connection between separable and relative semisimple algebras:

28.6 Separable and relative semisimple algebras.

Let S be a scalar R-algebra. Assume A is a separable R-algebra and B is a left (B, S)-semisimple S-algebra.

Then $A \otimes_R B$ is a left $(A \otimes_R B, S)$-semisimple algebra.

Proof. We have to show that every left $A\otimes_R B$-module M is $(A\otimes_R B, S)$-projective, i.e., the following canonical map splits as an $A \otimes_R B$-morphism,

$$(A \otimes_R B) \otimes_S M \to M, \quad (a \otimes b) \otimes m \mapsto (a \otimes b)m.$$

By assumption, the canonical map $B \otimes_S M \to M$ is split by some B-homomorphism $\varepsilon : M \to B \otimes_S M$.

Choose a separability idempotent of A, $e = \sum x_i \otimes y_i \in A^e$ (see 28.2). It is easily verified that the map $A \otimes_R M \to M$ is split by

$$\delta : M \longrightarrow A \otimes_R M, \quad m \mapsto \sum x_i \otimes y_i m.$$

By the way, this yields another proof for 28.5(1). Now the composition

$$M \stackrel{\delta}{\longrightarrow} A \otimes_R M \stackrel{id\otimes\varepsilon}{\longrightarrow} (A \otimes_R B) \otimes_S M, \quad m \mapsto \sum x_i \otimes \varepsilon(y_i m),$$

is the $A \otimes_R B$-morphism wanted. □

Now we show that associative central separable algebras are generators in A^e-*Mod*, i.e., are Azumaya algebras. This completes the proof of 28.1.

28.7 Separable and Azumaya algebras.

For a central R-algebra A the following are equivalent:

(a) A is a separable R-algebra;

(b) $A \otimes_R A^o$ is a separable R-algebra;

(c) $A \otimes_R A^o$ is left $(A \otimes_R A^o, R)$-semisimple;

(d) A is finitely generated as R-module and $A \otimes_R A^o$ is left $(A \otimes_R A^o, R)$-regular;

(e) $M(A)$ is a separable R-algebra;

(f) A is finitely generated as R-module and $M(A)$ is left $(M(A), R)$-semisimple;

(g) A is an Azumaya algebra.

Proof. $(a) \Rightarrow (b)$ is shown in 28.3; $(b) \Rightarrow (c)$ follows from 28.5.

$(c) \Rightarrow (a)$ is obvious since the canonical map $A \otimes_R A^o \to A$ is R-split.

$(c) \Rightarrow (d)$ is clear by $(b) \Rightarrow (g)$ (below).

$(d) \Rightarrow (a)$ Since A is a finitely generated R-module, A is finitely presented in A^e-*Mod* and the assertion is clear by 20.11.

$(b) \Rightarrow (g)$ Since A is A^e-projective, the trace ideal of A, $Tr(A, A^e) \subset A^e$, has the property $Tr(A, A^e) \cdot A = A$ (e.g., [40, 18.7]).

Assume $Tr(A, A^e) \neq A^e$. Then the trace ideal is contained in a maximal ideal of A^e which is of the form $m \cdot A^e$, for some maximal ideal $m \subset R$, i.e.,

$$Tr(A, A^e) \subset m \cdot A^e \ \text{ and hence } \ m \cdot A = m \cdot A^e \cdot A = A.$$

This is not possible by 25.2, (e). Hence $Tr(A, A^e) = A^e$ and A is a generator in A^e-*Mod* (and also in $M(A)$-*Mod*).

In particular, A is finitely generated and projective as an R-module.

$(g) \Rightarrow (e)$ Since A is finitely generated and projective as an R-module, $M(A) \simeq End_R(A)$ is R-separable (see examples 28.13).

$(e) \Rightarrow (g)$ Assume $M(A)$ is R-separable. Since the map $M(A) \to A$ splits as R-morphism, A is $M(A)$-projective by 28.5.

Since R is also the centre of $M(A)$, we conclude from $(a) \Rightarrow (g)$ that $M(A)$ and A are finitely generated and projective R-modules. So for any maximal ideal $m \in R$, $M(A) \neq mM(A)$ and $A \neq mA$. By this the left semisimple algebras A/mA are generators for the left modules over $M(A/mA) \simeq M(A) \otimes_R R/m$.

By 19.9, this implies that A is a generator in $M(A)$-*Mod*.

$(e) \Rightarrow (f)$ is clear by $(e) \Rightarrow (g)$ and 28.5.

$(f) \Rightarrow (a)$ Assume $M(A)$ is left $(M(A), R)$-semisimple. Since the canonical map $M(A) \to A$ is R-split, A is a projective $M(A)$-module and the assertion follows from 20.7. □

The following theorem shows that the study of separable algebras can be split in two parts: (central) Azumaya algebras and commutative separable algebras.

28.8 Characterizations of separable algebras.
For the R-algebra A with unit and centre $Z(A)$, the following are equivalent:

(a) A is a separable R-algebra;

(b) $A \otimes_R A^o$ is a separable R-algebra;

(c) $A \otimes_R A^o$ is left $(A \otimes_R A^o, R)$-semisimple;

(d) $M(A)$ is a separable R-algebra;

(e) A is an Azumaya algebra over $Z(A)$ and $Z(A)$ is separable over R.

If A is finitely generated as an R-module, then (a)-(e) are equivalent to:

(f) $A \otimes_R A^o$ is left $(A \otimes_R A^o, R)$-regular;

(g) $M(A)$ is left $(M(A), R)$-semisimple and $Z(A)$ is separable over R.

Proof. All the statements are readily obtained from 28.4 and (the proof of) 28.7. The implication $(a) \Rightarrow (c)$ also follows from 28.6. □

If A is finitely generated as an R-algebra or as an R-module we also have nice local-global characterizations for A to be separable:

28.9 Local-global characterization of separable algebras.
Let A be finitely generated as an R-algebra. The following are equivalent:

(a) A is a separable R-algebra;

(b) for every $x \in \mathcal{X}$, A_x is a separable R_x-algebra;

(c) for every $m \in \mathcal{M}$, A_m is a separable R_m-algebra.

(d) $A \otimes_R S$ is a separable S-algebra for some scalar R-algebra S which is faithfully flat as R-module.

If A is a finitely generated R-module, then (a)-(d) are equivalent to:

(e) for every $m \in \mathcal{M}$, A/mA is a separable R/m-algebra.

Proof. $(a) \Leftrightarrow (b) \Leftrightarrow (c)$ Since A is finitely generated as an R-algebra, A is a finitely presented A^e-module by 23.6. Now apply 19.4.

$(a) \Leftrightarrow (d)$ follows from 19.5(4).

$(a) \Rightarrow (e)$ is a consequence of 28.3.

$(e) \Rightarrow (a)$ Assume A is a finitely generated R-module. The map $\mu : A^e \to A$ splits as an R-morphism and A^e is finitely generated as an R-module. Hence the statement can be derived from 19.4(d). □

There is also a notion of *separability* in field theory and we want to establish the relationship with our considerations.

Recall that a non-constant polynomial in $K[X]$ is called *separable* if its irreducible factors have no multiple roots. A field extension $L : K$ is called *separable* if for every $a \in L$, the minimal polynomial of a over K is a separable polynomial.

28.10 Simple extensions.

Let $L : K$ be a simple algebraic field extension, i.e., $L = K(a)$ with $a \in L$ algebraic over K, and denote by $f \in K[X]$ the minimal polynomial of a. For any field extension $Q : K$ assume

$$f = p_1^{n_1} \cdots p_k^{n_k} \in Q[X],$$

with the p_i distinct irreducible polynomials in $Q[X]$ and $n_i \in \mathbb{N}$. Then

$$L \otimes_K Q \simeq \prod_{i=1}^{k} Q[X]/p_i^{n_i} Q[X].$$

Proof. From $L \simeq K[X]/fK[X]$ we obtain by applying $Q \otimes_K -$,

$$Q \otimes_K L \simeq Q \otimes_K K[X]/fK[X] \simeq Q[X]/fQ[X] \simeq \prod_{i=1}^{k} Q[X]/p_i^{n_i} Q[X],$$

with the last isomorphism given by the Chinese Remainder Theorem. □

We see from 28.5 that separable algebras over a field are finite dimensional. Hence we restrict the proof of the next observation to finite field extensions (the first three conditions are also equivalent in the infinite case).

28.11 Separable field extensions.

For a finite dimensional field extension $L : K$, the following are equivalent:

(a) $L : K$ is a separable field extension;

(b) for any (finite) field extension $Q : K$, Nil$(L \otimes_K Q) = 0$;

(c) Nil$(L \otimes_K L) = 0$;

(d) *L is separable as a K-algebra.*

Proof. $(a) \Rightarrow (b)$ Since any finite separable field extension is simple we may assume $L = K(a)$ for some $a \in L$. For the separable minimal polynomial $f \in K[X]$ of a, we have $f = p_1 \cdots p_k \in Q[X]$, with the p_i distinct irreducible polynomials in $Q[X]$. By 28.10, we have the isomorphism

$$L \otimes_K Q \simeq \prod_{i=1}^{k} Q[X]/p_i Q[X],$$

where the $Q[X]/p_i Q[X]$ are fields and hence $L \otimes_K Q$ contains no nilpotent elements.

$(b) \Rightarrow (c)$ is trivial.

$(c) \Rightarrow (d)$ It is well known that finite dimensional (commutative) algebras with zero nil radical are semisimple. Over a semisimple algebra every module is projective. In particular, L is $L \otimes_K L$-projective, i.e., L is a separable K-algebra.

$(d) \Rightarrow (a)$ Assume $L : K$ is not a separable field extension. Then K has characteristic $p \neq 0$.

Let $f \in K[X]$ be the minimal polynomial of some $a \in L$ which is not separable over K and denote by Q the splitting field of f. Then there exists a polynomial

$$h(X) = \sum_{i=1}^{r} b_i X^i \quad \text{with} \quad b_i \in Q, \ b_r \neq 0, \quad \text{and } f(X) = h(X)^p.$$

Obviously r is smaller than the degree of f and hence the elements $1, a, \ldots, a^r$ in L are linearly independent over K. Then also the elements $1 \otimes 1, a \otimes 1, \ldots, a^r \otimes 1$ in $L \otimes_K Q$ are linearly independent over Q. In particular, $0 \neq c := \sum_{j=1}^{r} a^j \otimes b_j$ but

$$c^p = (\sum_{j=1}^{r} a^j \otimes b_j)^p = \sum_{j=1}^{r} a^{jp} \otimes b_j^p = \sum_{j=1}^{r} (b_j a^j)^p \otimes 1 = h(a)^p \otimes 1 = 0.$$

So there is a non-zero nilpotent element in $L \otimes_K Q$. This contradicts the fact that the separable Q-algebra $L \otimes_K Q$ is semisimple by 28.5. □

We are now able to characterize separable algebras over fields in the following way:

28.12 Separable algebras over fields.

For a finite dimensional algebra A over a field K, the following are equivalent:

(a) *A is a separable K-algebra;*

(b) *for any (finite) field extension $L : K$, $Nil(A \otimes_K L) = 0$;*

(c) *for any (finite) field extension $L : K$, $A \otimes_K L$ is left semisimple;*

(d) *$Nil(A \otimes_K A^o) = 0$;*

(e) $Nil(A \otimes_K Z(A)) = 0$;

(f) A *is left semisimple and* $Z(A)$ *is a separable* K*-algebra;*

(g) $A \otimes_K Z(A)$ *is a separable* $Z(A)$*-algebra.*

Proof. $(a) \Rightarrow (b)$ For any field extension $L : K$, $A \otimes_K L$ is a separable L-algebra and hence is left semisimple by 28.5.

$(b) \Leftrightarrow (c)$ A finite dimensional associative algebra is left semisimple if and only if it has zero nil (or Jacobson) radical.

$(a) \Leftrightarrow (d)$ If A is K-separable, then $A \otimes_K A^o$ is also K-separable and hence has zero radical (by 28.5).

On the other hand, if $Nil(A \otimes_K A^o) = 0$, then $A \otimes_K A^o$ is left semisimple and every $A \otimes_K A^o$-module is projective.

$(d) \Leftrightarrow (e)$ is obvious.

$(e) \Leftrightarrow (f)$ From (e) it is easily seen that $Nil(A)$ and $Nil(Z(A) \otimes_K Z(A))$ are zero. By the implication $(a) \Leftrightarrow (d)$ we conclude that $Z(A)$ is K-separable. The same conclusion could be derived from 28.4.

$(f) \Rightarrow (a)$ Since A is left semisimple, it is also semisimple as $A \otimes_{Z(A)} A^o$-module. Hence it is a projective generator in $\sigma[A]$ which is equal to $M(A)$-*Mod* since A is finitely generated as $Z(A)$-module (see 23.1). So A is an Azumaya algebra (see 28.1).

Now the assertion follows from 28.4.

$(g) \Leftrightarrow (a)$ This is a special case of 28.9. □

We finally list some examples:

28.13 Examples of separable algebras.

As before R denotes an associative commutative ring with unit.

(1) For any multiplicative subset $S \subset R$, RS^{-1} is a separable R-algebra.
In particular, the field of rational numbers $\mathbb{Q}$ is a separable $\mathbb{Z}$-algebra.

(2) For every $n \in \mathbb{N}$, the matrix ring $R^{(n,n)}$ is a separable R-algebra.

(3) For any finitely generated projective R-module M, $End_R(M)$ is a separable R-algebra.

(4) For any finite group G of order n, the group algebra $R[G]$ is R-separable if n is invertible in R.

Proof. (1) By 16.3, the map $RS^{-1} \otimes_R RS^{-1} \to RS^{-1}$ is an isomorphism.

(2) Let e_{ij} denote the matrix in $R^{(n,n)}$ having 1 in the (i,j)-position and 0 elsewhere. Then

$$e = \sum_{i=1}^{n} e_{i1} \otimes e_{1i}$$

is a separability idempotent for $R^{(n,n)}$ (see 28.2). This is a special case of (3).

(3) Assume M is a finitely generated projective R-module. Then by 15.8,

$$End_R(M) \otimes_R R/m \simeq End_{R/m}(M/mM)$$

for every $m \in \mathcal{M}$. On the right we have the endomorphism ring of the finite dimensional R/m-vector space M/mM which is a simple algebra with centre R/m. Hence $End_R(M)$ is (central) R-separable by 28.9.

Another way to see that $End_R(M)$ is R-separable is to refer to the fact that M is a direct summand of a free module R^n, and hence $End_R(M) = fR^{(n,n)}f$, for some idempotent $f \in R^{(n,n)}$.

(4) A separability idempotent for $R[G]$ is of the form

$$e = \frac{1}{n} \sum_{g \in G} g \otimes g^{-1} \in R[G] \otimes_R R[G]^o.$$

$\square$

Remark. For the problem which (semi) group rings are Azumaya see Cheng [106], DeMeyer-Hardy [116], DeMeyer-Janusz [117], Okninski [214], Szeto-Wong [256].

28.14 Exercises.

Let A denote an associative unital R-algebra.

(1) Assume A is finitely generated and projective as an R-module. Show that the following are equivalent:
(a) A is R-separable;
(b) for any R-scalar algebra S, $A \otimes_R S$ is left $(A \otimes_R S, S)$-semisimple;
(c) for any R-scalar algebra S, $Jac(A \otimes_R S) = A \otimes_R Jac\, S$;
(d) for any R-scalar algebra S, $Nil(A \otimes_R S) = A \otimes_R Nil\, S$;
(e) for any R-scalar algebra S, $BMc(A \otimes_R S) = A \otimes_R Jac\, S$.

(2) Let A be R-separable with separability idempotent $e \in A^e$. Show that $A^e = A^e e \oplus A^e(1-e)$, with $A^e(1-e) = Ke\,\mu$ and $A^e e \simeq A$ as left A^e-modules.

(3) Assume A to be finitely generated as an R-module and that for every associative semiprime R-algebra B, $A \otimes_R B$ is semiprime. Prove that A is R-separable ([239]).

(4) Let A be R-central. Prove that the following are equivalent ([86, 88, 119]):
(a) A is R-separable;
(b) there is some $e \in A^e$ such that $e1 = 1$ and $eA \subset R$;
(c) $A^e \otimes_R (A^e)^o \simeq End_R(A^e)$.

(5) Let R be a Prüfer ring (a semihereditary integral domain, [40, 40.4]). Assume that A is a separable R-algebra which is finitely generated and torsion free as an R-module. Prove that A is a left and right semihereditary ring ([264, 2.1]).

References: Auslander-Goldman [58], Braun [86, 87, 88], Cheng [106], DeMeyer-Hardy [116], DeMeyer-Ingraham [10], DeMeyer-Janusz [117], Dicks [119], Hattori [153], Higman [157], Hirata-Sugano [158], Knus-Ojanguren [23], Magid [189], McMahon-Mewborn [196], Okninski [214], Pierce [33], Ranga Rao [224], Saito [239], Sugano [250, 251], Szeto-Wong [256], Taylor [257], Wehlen [264], Wisbauer [266, 271].

29 Non-associative separable algebras

1.Azumaya and quasi-separable algebras. 2.Azumaya and separable algebras. 3.Azumaya algebras with unit. 4.Local-global characterizations of Azumaya algebras. 5.Quasi-separable algebras. 6.Separable algebras. 7.Quasi-separable algebras over fields. 8.Separable algebras over fields. 9.Examples of non-associative separable algebras. 10.Remarks. 11.Exercises.

As shown in 28.8, an associative unital algebra A is R-separable if and only if its multiplication algebra $M(A)$ is R-separable. Associative separable algebras are assumed to have a unit. Studying non-associative algebras this is too strong a condition, for example, it would exclude all Lie algebras from our considerations. These observations motivate the following definitions (introduced in Müller [208]):

Definitions. An R-algebra A is called *quasi-separable* over R if its multiplication algebra $M(A)$ is an R-separable (associative) algebra. A is called *separable* over R if $M(A)$ is R-separable and $M^*(A) = M(A)$ (see 2.1).

As mentioned before, associative unital algebras are separable in the above sense if and only if they are separable in the sense of 28.2.

Again separable and Azumaya algebras are closely related. Our next result also relates Azumaya (quasi-separable) algebras to quasi-simple central algebras. Recall that $\mathcal{M}$ denotes the set of all maximal ideals of R.

29.1 Azumaya and quasi-separable algebras.

Assume A is a central R-algebra. Then the following are equivalent:

(a) A is an Azumaya algebra;

(b) A is an Azumaya ring and is a finitely generated R-module.

(c) A is a finitely generated, projective R-module and $M(A)$ is a separable R-algebra;

(d) A is a finitely generated, projective R-module and $M(A)$ is left $(M(A), R)$-semisimple;

(e) A is a finitely generated, projective R-module and $M(A) \simeq End_R(A)$;

(f) A is a finitely generated, projective R-module and for every $m \in \mathcal{M}$, $A \otimes_R R/m$ is a quasi-simple central R/m-algebra.

Proof. $(a) \Leftrightarrow (b)$ is trivial since $\sigma[A] = M(A)$-Mod (see 23.1).

$(a) \Leftrightarrow (c) \Leftrightarrow (e)$ follows from 24.8. $(a) \Rightarrow (f)$ is clear by 26.6.

$(c) \Rightarrow (d)$ holds by 28.5 and $(d) \Rightarrow (a)$ by 20.7.

$(f) \Rightarrow (e)$ The inclusion $M(A) \to End_R(A)$ yields the commutative diagram

$$\begin{array}{ccc} M(A) \otimes_R R/m & \longrightarrow & End_R(A) \otimes_R R/m \\ \downarrow & & \downarrow \\ M(A \otimes_R R/m) & \longrightarrow & End_{R/m}(A \otimes_R R/m) \end{array} ,$$

with the first vertical map surjective and the second bijective by 15.11.

If $A \otimes_R R/m$ is a quasi-simple central R/m-algebra, then the lower map is bijective. Hence the upper map is surjective. Since $End_R(A)$ is a finitely generated R-module, $M(A) = End_R(A)$ by 17.3. □

29.2 Azumaya and separable algebras.

For a central R-algebra A, the following are equivalent:

(a) *A is an Azumaya algebra and $A^2 = A$;*

(b) *A is a finitely generated, projective R-module, $M^*(A) = M(A)$ and $M(A)$ is a separable R-algebra;*

(c) *A is a finitely generated, projective R-module, $M^*(A) = M(A)$ and $M(A)$ is left $(M(A), R)$-semisimple;*

(d) *A is finitely generated and projective as an R-module, and for every $m \in \mathcal{M}$, $A \otimes_R R/m$ is a simple central R/m-algebra.*

In particular, any R-algebra A, which is a finite product of simple algebras and is finitely generated as R-module, is separable over its centroid.

Proof. $(a) \Leftrightarrow (b) \Leftrightarrow (c)$ By 29.1, it only remains to show that for Azumaya algebras, $A^2 = A$ is equivalent to $M^*(A) = M(A)$.

Obviously, $M^*(A) = M(A)$ implies $A = M(A)A = M^*(A)A = A^2$.

Assume $A^2 = A$ and that $M^*(A)$ is a proper ideal in $M(A) \simeq End({}_RA)$. Then $M^*(A)$ is contained in a maximal ideal of $M(A)$ which is of the form $mM(A)$ for some maximal ideal $m \subset R$ (see 28.5). This implies

$$A = A^2 = M^*(A)A \subset mM(A)A = mA,$$

which is not possible (e.g., [40, 18.9]). So $M^*(A) = M(A)$.

$(a) \Rightarrow (d)$ $A^2 = A$ obviously implies $(A/mA)^2 \neq 0$ and hence the assertion follows from 29.1.

$(d) \Rightarrow (b)$ For simple algebras A/mA, $M^*(A/mA) = End_{R/m}(A/mA)$. Now follow the proof $(f) \Rightarrow (e)$ in 29.1 with $M(A)$ replaced by $M^*(A)$.

Finally, assume A is a finite product of simple algebras and is finitely generated as an R-module. Then $\sigma[A] = A\text{-}Mod$ and $M(A)$ is left semisimple (see 23.8). □

For unital algebras the above considerations simplify to:

29.3 Azumaya algebras with unit.

For a central R-algebra A with unit, the following are equivalent:

(a) *A is an Azumaya algebra;*

(b) *$M(A)$ is a separable R-algebra;*

(c) *A is a finitely generated R-module and $M(A)$ is left $(M(A), R)$-semisimple;*

(d) (i) *A is a finitely generated, projective R-module and*
(ii) *for every $m \in \mathcal{M}$, $A \otimes_R R/m$ is a simple algebra with centre R/m.*

Proof. $(a) \Leftrightarrow (d) \Rightarrow (b)$ and $(d) \Rightarrow (c)$ are clear by 29.2 and 28.5.

$(c) \Rightarrow (a)$ follows from 20.7.

$(b) \Rightarrow (a)$ By 29.2, it remains to show that A is a finitely generated and projective R-module.

$M(A)$ is a central separable R-algebra (see 2.10) and hence $M(A)$ is finitely generated and projective as an R-module (see 28.8).

Since the map $M(A) \to A$, $\lambda \mapsto \lambda(1)$, splits as an R-morphism, it splits in fact as an $M(A)$-morphism by 28.5. So A is a direct summand of $M(A)$ and hence A is also finitely generated and projective as an R-module. □

If we assume $M(A)$ to be finitely generated as R-module we obtain:

29.4 Local-global characterizations of Azumaya algebras.

Assume A is a central R-algebra with unit and $M(A)$ is finitely generated as an R-module. Then the following assertions are equivalent:

(a) *A is an Azumaya algebra;*

(b) *A is an Azumaya ring;*

(c) *for every $x \in \mathcal{X}$, A_x is an Azumaya algebra;*

(d) *for every $m \in \mathcal{M}$, A_m is an Azumaya algebra;*

(e) *for every $m \in \mathcal{M}$, A/mA is a central simple R/m-algebra.*

Proof. By 23.4, A is a finitely presented $M(A)$-module. Hence the assertions follow from 19.9. □

Now we turn to not necessarily central algebras. Nevertheless we usually want the centroid to be commutative. Here we have to distinguish between the multiplication algebras of A as an R-algebra and an $C(A)$-algebra (see 2.10).

29.5 Quasi-separable algebras.

For an R-algebra A with commutative centroid $C(A)$, the following are equivalent:

(a) A is a quasi-separable R-algebra and $C(A) \subset M({}_RA)$;

(b) A is an Azumaya algebra, $C(A) \subset M({}_RA)$ and $C(A)$ is R-separable;

(c) A is a generator in $M({}_RA)$-Mod and $C(A)$ is R-separable.

Proof. $(a) \Rightarrow (b)$ $C(A) \subset M({}_RA)$ implies that $C(A)$ is the centre of $M({}_RA)$ (see 2.10).

So $M({}_RA)$ is a central separable $C(A)$-algebra and $C(A)$ is separable over R by 28.4. Now we see from 29.1 that A is an Azumaya algebra over $C(A)$.

$(b) \Rightarrow (c)$ A being an Azumaya algebra, A is a generator in $M({}_{C(A)}A)$-*Mod*. Under the given conditions $M({}_RA) = M({}_{C(A)}A)$.

$(c) \Rightarrow (a)$ A being a generator in $M({}_RA)$-*Mod*, $M({}_RA) \simeq End({}_{C(A)}A)$ and A is a finitely generated and projective $C(A)$-module. This implies $C(A) \subset M({}_RA)$, $M({}_RA) = M({}_{C(A)}A)$ and $C(A)$ is the centre of $M({}_RA)$ (see 2.10).

Hence $M({}_RA)$ is a central separable $C(A)$-algebra. Since $C(A)$ is separable over R, $M({}_RA)$ is R-separable by 28.4. □

Similar to the above we characterize separable algebras. Here stronger conditions imply that the centroid is commutative:

29.6 Separable algebras.

For an R-algebra A with centroid $C(A)$, the following are equivalent:

(a) A is a separable R-algebra;

(b) $A^2 = A$, A is an Azumaya algebra, $C(A) \subset M({}_RA)$ and $C(A)$ is R-separable;

(c) $A^2 = A$, A is a generator in $M({}_RA)$-Mod and $C(A)$ is R-separable.

Proof. $(a) \Rightarrow (b)$ Assume A is a separable R-algebra, i.e., $M({}_RA)$ is separable and $M^*(A) = M({}_RA)$. This implies $A^2 = A$ and $C(A)$ is the centre of $M({}_RA)$ (see 2.10).

Now the assertion follows from 29.5.

$(b) \Rightarrow (a)$ For an Azumaya algebra A with $A^2 = A$, $M^*(A) = M({}_{C(A)}A)$ by 29.2. Since $M({}_RA) = M({}_{C(A)}A)$ under the given conditions, the claim follows from 29.5.

$(b) \Leftrightarrow (c)$ follows immediately from 29.5. □

29.7 Quasi-separable algebras over fields.

Assume A is a finite dimensional algebra A over a field K with commutative centroid. Then the following are equivalent:

(a) A is a quasi-separable K-algebra;

(b) *for any (finite) field extension $L : K$, $A \otimes_K L$ is a finite product of quasi-simple algebras;*

(c) *for any (finite) field extension $L : K$, $Rad(A \otimes_K L) = 0$;*

(d) *A is a finite product of quasi-simple algebras and $C(A)$ is a separable K-algebra.*

Proof. $(a) \Rightarrow (b)$ By definition, $M(A)$ is a finite dimensional, separable K-algebra. For any field extension $L : K$, $M(A) \otimes_K L \simeq M(A \otimes_K L)$ is a separable L-algebra and hence is left semisimple by 28.5.

So $A \otimes_K L$ is a semisimple $M(A \otimes_K L)$-module and hence it is a finite direct product of quasi-simple algebras.

$(c) \Leftrightarrow (c)$ is obvious by the definition of the radical (see 21.2).

$(b) \Leftrightarrow (d)$ For $K = L$ we see that A is a product of quasi-simple algebras. For any field extension $L : K$, the centroid of $A \otimes_K L$ ($\simeq C(A) \otimes_K L$) is a left semisimple algebra and hence $C(A)$ is separable over K by 28.12.

$(d) \Rightarrow (a)$ Since A is a product of quasi-simple algebras, A is an Azumaya algebra (over $C(A)$). Hence $M(A)$ is a separable $C(A)$-algebra. So $M(A)$ is a separable R-algebra by 28.4. □

With nearly the same arguments we obtain characterizations of separable algebras over fields. Condition (b) was originally used to define this class of algebras (e.g., [19]):

29.8 Separable algebras over fields.

Assume A is a finite dimensional algebra over a field K and $M^*(A) = M(A)$. Then the following are equivalent:

(a) *A is a separable K-algebra;*

(b) *for any (finite) field extension $L : K$, $A \otimes_K L$ is a (finite) product of simple algebras;*

(c) *for any (finite) field extension $L : K$, $Alb(A \otimes_K L) = 0$;*

(d) *A is a (finite) product of simple algebras and $C(A)$ is a separable K-algebra;*

(e) *$A \otimes_K C(A)$ is a separable $C(A)$-algebra.*

Proof. Notice that $M^*(A) = M(A)$ implies $A^2 = A$ and $C(A)$ is the centre of $M(A)$ (see 2.10). Now the equivalences from (a) to (d) follow from 29.7.

$(a) \Rightarrow (e)$ By definition, $M(A)$ is K-separabel, and hence
$M(A) \otimes_K C(A) \simeq M(A \otimes_K C(A))$ is $C(A)$-separabel.

$(e) \Rightarrow (a)$ $C(A)$ being the centre of $M(A)$, the assertion follows from 28.12 and the above isomorphism. □

From Müller [208] we have the following

29.9 Examples of non-associative separable algebras.

(1) For some $n \in I\!N$, let R be a ring with $2, 3, n$ invertible. Denote by $gl(n, R)$ the (n, n)-matrices $R^{(n,n)}$ with the Lie product $[a, b] = ab - ba$, and by $sl(n, R)$ the matrices in $R^{(n,n)}$ with trace zero. Then

$$gl(n, R) = Z(gl(n, R)) \oplus sl(n, R),$$

and $sl(n, R)$ is an Azumaya Lie algebra over R.

(2) Assume A is an associative Azumaya algebra with unit over a ring R with $\frac{1}{2} \in R$. Then A with the Jordan product $a \times b = \frac{1}{2}(ab + ba)$ (see 3.10) is an Azumaya Jordan algebra.

The decomposition in (1) follows from linear algebra. For every maximal ideal $m \subset R$, $sl(A) \otimes_R R/m$ is a central simple R/m-algebra.

In (2), for every maximal ideal $m \subset R$, A/mA is a central simple associative R/m-algebra. By a result of Herstein ([18, Theorem 1.1]), A is also central simple as a Jordan algebra.

29.10 Remarks. Condition (b) in 29.8 was originally used to define (non-associative) separable algebras over fields (cf. Jacobson [19]).

There are various definitions for non-associative separable algebras over rings in the literature (e.g., [82, 83, 267, 271]). In Bix [82, 83] separability for *alternative* and *Jordan* R-algebras A is defined through the separability of the associative *unital multiplication envelope* $U(A)$. For rings R with 2 invertible it was shown in [267, Satz 4.1] that these algebras are characterized by the separability of $M(A)$ if A is finitely generated as R-module. Hence, for all the definitions mentioned, the *central separable* algebras A are exactly the *Azumaya algebras* (compare [271, Satz 2.1]).

The equivalence of (a) and (d) in 29.2 is proved, e.g., in Bix [82, 83] and [267, Satz 4.1] for alternative and Jordan algebras, and also in Zhelyabin [286, Lemma 3.1] for central alternative algebras.

In Zhelyabin [286] examples of alternative and Jordan algebras A are given which are projective over their universal enveloping algebra but are not separable, in contrast to the associative case, where projectivity over A^e is sufficient for separability.

A unital alternative algebra A is R-separable if and only if A is a direct product of an associative separable algebra and Cayley-Dickson (octonion) algebras whose centres are separable R-algebras (cf. Bix [83, Theorem 4.5]).

29.11 Exercises.

(1) Let A be a unital separable R-algebra which is finitely generated and projective as R-module. Show that for any R-scalar algebra S,

$$BMc(A \otimes_R S) = A \otimes_R Jac\, S.$$

(2) Prove that a central R-algebra A is Azumaya if and only if A is a finitely generated projective R-module and for every $m \in \mathcal{M}$, $M(A \otimes_R R_m)$ is a finite matrix ring over R_m.

(3) Assume A is an Azumaya algebra over R. Show that every algebra endomorphism of A is an automorphism.

References: Auslander-Goldman [58], Bix [82, 83], Hirata-Sugano [158], Müller [208], Sugano [250, 251], Zhelyabin [286], Wisbauer [271, 266, 270].

30 Biregular algebras

1.Biregular algebras. 2.Properties of biregular algebras. 3.Algebras with regular centroid. 4.Characterization of biregular algebras. 5.A regular in $\sigma[A]$. 6.Biregular Azumaya rings. 7.Biregular Azumaya algebras. 8.Module finite biregular algebras with units. 9.Biregular algebras with units. 10.Co-semisimple algebras. 11.Self-projective co-semisimple algebras. 12.Module finite multiplication algebras. 13.Examples. 14.Exercises.

Definition. An algebra A is called *biregular* if every principal ideal of A is a direct summand.

Obviously, A is biregular if and only if every ideal $I \subset A$ generated by a single element is of the form $I = Ae$, for an idempotent $e \in C(A)$.

First examples of biregular algebras are quasi-simple and weak products of quasi-simple algebras (see 23.8). On the other hand, every biregular algebra which is locally noetherian as an $M(A)$-module is a weak product of quasi-simple algebras.

An algebra A with unit is biregular if and only if every principal ideal is generated by an idempotent of the centre of A. A commutative associative ring is biregular if and only if it is (von Neumann) regular.

Remark. For associative rings biregularity was defined in Arens-Kaplansky [51] by the property that every principal ideal is generated by a central idempotent. Since in general $C(A) \not\subset A$, this condition is stronger than our definition. Of course, for rings with identity the two properties coincide.

Algebras who do not allow non-trivial idempotents may nevertheless be biregular in our sense (e.g., semisimple Lie algebras).

For biregular algebras with commutative centroid we immediately obtain from 18.12 a characterization by their Pierce stalks. Again $\mathcal{X}$ denotes the set of maximal ideals in the ring of idempotents of R:

30.1 Biregular algebras.

Assume A is a central R-algebra which is finitely generated as an $M(A)$-module. Then the following are equivalent:

(a) A is biregular;

(b) every finitely generated ideal is a direct summand in A;

(c) for every $x \in \mathcal{X}$, A_x is a quasi-simple algebra.

For biregular algebras with $A^2 = A$ we notice:

30.2 Properties of biregular algebras.

Assume A is a central biregular R-algebra and $A^2 = A$. Then:

(1) A is an ideal algebra and every finitely generated ideal is principal.

(2) Every ideal in A is idempotent and R is a regular ring.

(3) Every factor algebra of A is biregular.

(4) Every ideal in A is an intersection of maximal ideals.

(5) Every prime ideal is a maximal ideal in A.

(6) For every $x \in \mathcal{X}$, A_x is a simple algebra.

Proof. (1) By definition, every finitely generated ideal $U \subset A$ is A-generated and so A is a self-generator.

Assume Ae and Af are principal ideals of A, with idempotents $e, f \in C(A)$. Then $Ae+Af = A(e+f-ef)$ is a principal ideal and (by induction) every finitely generated ideal is principal.

(2) Every principal ideal in A is of the form Ae, for some idempotent $e \in R$, and $(Ae)^2 = (A^2)e^2 = Ae$. This implies that every ideal of A is idempotent. By 2.7, R is a (von Neumann) regular ring.

(3) The factor algebras are of the form $A/AI \simeq A \otimes_R R/I$ and the canonical map $A \to A/AI$ preserves direct summands.

(4) Obviously a biregular algebra A has no superfluous $M(A)$-submodules. So $Rad(A) = 0$, i.e., 0 is the intersection of maximal ideals in A.

By (3), every factor algebra of A is biregular and so every ideal in A is the intersection of the maximal ideals containing it.

(5) Assume A is a prime biregular ring. Any finitely generated ideal is of the form Ae for some idempotent $e \in R$ and $Ae \cdot A(1-e) = 0$. This implies $Ae = 0$ or $A(1-e) = 0$ and A must be a simple algebra.

(6) This is clear by 30.1 since $(A_x)^2 \neq 0$. □

Before characterizing idempotent biregular algebras we have a look at algebras with regular centroid.

30.3 Algebras with regular centroid.

Assume A is a central R-algebra, finitely generated as $M(A)$-module and R is regular. Then:

(1) For every ideal $I \subset R$, $I = Hom_{M(A)}(A, AI)$.

(2) For each $m \in \mathcal{M}$, there exists a maximal ideal $M \subset A$ with $m = Hom_{M(A)}(A, M)$.

If A has a unit, the above properties can be written in the following form:

(3) For every ideal $I \subset R$, $AI \cap R = I$.

(4) For each $m \in \mathcal{M}$, there exists a maximal ideal $M \subset A$ with $M \cap R = m$.

Proof. (1) This is a special case of the more general fact that finitely generated modules with regular endomorphism rings are intrinsically projective (see 5.6).

(2) We see from (1) that $Am \neq A$ for $m \in \mathcal{M}$. Hence $Am \subset M$ for some maximal ideal $M \subset A$. By the maximality of m, $m = Hom_{M(A)}(A, M)$.

Obviously, in unital algebras, (1) corresponds to (3) and (2) to (4). □

Remark. For associative rings, (2) and (4) were proved in Nauwelaerts-Oystaeyen [209]. In fact their proof also applies in the non-associative case.

30.4 Characterization of biregular algebras.
Assume A is a central R-algebra, is finitely generated in $M(A)$-Mod and $A^2 = A$. Then the following are equivalent:

(a) A is biregular;

(b) R is a regular ring and A is an ideal algebra;

(c) R is regular and A is a self-generator as an $M(A)$-module;

(d) for every $x \in \mathcal{X}$, A_x is a simple algebra;

(e) R is regular and for every $m \in \mathcal{M}$, A_m is a simple algebra;

(f) R is regular and for every maximal ideal $M \subset A$, $M = AHom_{M(A)}(A, M)$ (i.e., M is A-generated).

Proof. $(a) \Rightarrow (b) \Rightarrow (c)$ are clear by 30.2. $(b) \Rightarrow (f)$ is trivial.

$(c) \Rightarrow (a)$ This is shown in 24.3.

$(d) \Leftrightarrow (e)$ Over a regular ring R, localizing with respect to $x \in \mathcal{X}$ corresponds to localizing with respect to $m \in \mathcal{M}$.

$(f) \Rightarrow (b)$ For any $m \in \mathcal{M}$, there is a maximal ideal $M \subset A$ with $m = Hom_{M(A)}(A, M)$ (by 30.3) and hence $Am = AHom_{M(A)}(A, M) = M$. This implies that $A_m \simeq A/Am \simeq A/M$ is a simple algebra. □

Recall that a short exact sequence $(*)\quad 0 \to K \to L \to N \to 0$ is *pure in* $\sigma[A]$, if the functor $Hom_A(P, -)$ is exact with respect to $(*)$ for every finitely presented module P in $\sigma[A]$ (see 7.2).

A module $L \in \sigma[A]$ is *regular* in $\sigma[A]$ if every exact sequence of type $(*)$ is pure in $\sigma[A]$ (see 7.2). Regular modules generalize semisimple modules. They have a number of interesting properties (see 7.3, or [40, 37.3, 37.4]). In particular we have for the $M(A)$-module A:

30.5 A regular in $\sigma[A]$.

For an algebra A the following are equivalent:

(a) *A is regular in $\sigma[A]$;*

(b) *every short exact sequence in $\sigma[A]$ is pure;*

(c) *every module in $\sigma[A]$ is flat (or absolutely pure) in $\sigma[A]$;*

(d) *every finitely presented module is projective in $\sigma[A]$.*

If A is finitely presented in $\sigma[A]$, then (a)-(d) are equivalent to:

(e) *Every module in $\sigma[A]$ is weakly A-injective;*

(f) *A is a biregular algebra.*

The algebras described in the second part of the theorem above have some more interesting characterizations if they are idempotent:

30.6 Biregular Azumaya rings.

Assume A is a central R-algebra with $A^2 = A$ and is finitely generated as an $M(A)$-module. Then the following conditions are equivalent:

(a) *A is finitely presented and regular in $\sigma[A]$;*

(b) *A is a biregular ring and is finitely presented in $\sigma[A]$;*

(c) *A is a biregular ring and is self-projective as $M(A)$-module;*

(d) *A is a biregular ring and is a generator in $\sigma[A]$;*

(e) *A is an Azumaya ring and R is regular;*

(f) *A is a generator in $\sigma[A]$ and R is regular;*

(g) *A is finitely presented in $\sigma[A]$ and for every $x \in \mathcal{X}$, A_x is a simple algebra;*

(h) *R is regular and for every $m \in \mathcal{M}$, $A \otimes R/m$ is a central simple R/m-algebra.*

Proof. $(a) \Leftrightarrow (b) \Leftrightarrow (c)$ follows from 30.5.

$(c) \Leftrightarrow (d) \Leftrightarrow (e)$ are special cases of 26.9(1). $(e) \Rightarrow (f)$ is clear.

$(f) \Rightarrow (e)$ By 30.4, A is an Azumaya ring. So A is a generator in $\sigma[A]$ which is faithfully flat over its endomorphism ring R. By 5.10, A is projective in $\sigma[A]$.

$(b) \Leftrightarrow (g)$ is shown in 30.2.

$(e) \Leftrightarrow (h)$ is an application of 26.8. □

As seen in 29.1, Azumaya rings which are finitely generated as modules over their centroid are Azumaya algebras. Recalling that for such algebras $\sigma[A] = M(A)$-*Mod*, we obtain:

30.7 Biregular Azumaya algebras.

Assume A is a central R-algebra, is finitely presented in $M(A)$-Mod and $A^2 = A$. Then the following conditions are equivalent:

(a) *A is a biregular algebra and finitely generated as R-module;*

(b) *A is a biregular algebra and a generator in $M(A)$-Mod;*

(c) *A is an Azumaya algebra and R is a regular ring;*

(d) *$M(A)$ is a (von Neumann) regular ring;*

(e) *for every $x \in \mathcal{X}$, A_x is simple and finite dimensional over its centre.*

Proof. $(a) \Leftrightarrow (b)$ is clear since $\sigma[A] = M(A)$-*Mod*. $(a) \Leftrightarrow (c)$ follows from 30.6.

$(a) \Leftrightarrow (d)$ Assume A is regular in $\sigma[A] = M(A)$-*Mod*. Then every finitely presented module in $M(A)$-*Mod* is projective and hence $M(A)$ is a (von Neumann) regular algebra. On the other hand, if $M(A)$ is regular, then A is regular as an $M(A)$-module and hence it is biregular.

$(a) \Leftrightarrow (e)$ is an application of 19.9. □

There are various ways to ensure that A is finitely presented in $M(A)$-*Mod* (see 23.4). We consider two particular cases.

30.8 Module finite biregular algebras with units.

Assume A is a central alternative or Jordan R-algebra with unit. Then the following conditions are equivalent:

(a) *A is a biregular algebra and finitely generated as R-module;*

(b) *A is a biregular algebra and a generator in $M(A)$-Mod;*

(c) *A is an Azumaya algebra and R is a regular ring;*

(d) *A is a finitely generated R-module and $M(A)$ is a regular ring.*

Proof. By 23.4, 3.7 and 3.13, any of the conditions implies that A is finitely presented in $M(A)$-*Mod* and hence the assertions follow from 30.7. □

30.9 Biregular algebras with units.

Assume A is a central R-algebra with unit and $M(A)$ is finitely generated as R-module. Then the following are equivalent:

(a) *A is a biregular algebra;*

(b) *A is a biregular algebra and a generator in $\sigma[A]$-Mod;*

(c) *A is an Azumaya algebra (ring) and R is regular;*

(d) *$M(A)$ is a regular ring.*

Proof. Since $M(A)$ is finitely generated as R-module, A is finitely presented in $M(A)$-*Mod* by 23.4. Hence the statements follow from 30.7. □

It is well-known that the ring R is regular if and only if every simple R-module is injective, i.e., if R is a co-semisimple ring. This equivalence is no longer true for non-commutative regular rings and left modules. What can be said about the relationship between biregular rings and A-injectivity of simple $M(A)$-modules? We will see that at least one of the above implications holds in the general case.

We call an algebra A *co-semisimple* if it is a *co-semisimple* $M(A)$-module, i.e., if every simple module in $\sigma[A]$ is A-injective. From 6.18 we have:

30.10 Co-semisimple algebras.

For an R-algebra A, the following are equivalent:

(a) *A is a co-semisimple $M(A)$-module;*

(b) *every finitely cogenerated module in $\sigma[A]$ is A-injective;*

(c) *for every ideal $U \subset A$, $Rad(A/U) = 0$;*

(d) *every proper ideal in A is the intersection of maximal ideals.*

By the observations in 30.2, it is clear that biregular algebras A with $A^2 = A$ are co-semisimple $M(A)$-modules.

30.11 Self-projective co-semisimple algebras.

Let A be a central R-algebra. Assume A is co-semisimple and self-projective as an $M(A)$-module. Then:

(1) *A is a generator in $\sigma[A]$.*

(2) *If A is finitely generated as an $M(A)$-module, then A is a biregular Azumaya ring and R is regular.*

(3) *If A is finitely generated as R-module, then A is an Azumaya algebra (and R is regular).*

Proof. (1) Since simple modules in $\sigma[A]$ are injective, they are A-generated. A being self-projective, it is a generator in $\sigma[A]$ (see [40], 18.5).

(2) A is a progenerator in $\sigma[A]$ and $Hom_{M(A)}(A,-) : \sigma[A] \to R\text{-}Mod$ is an equivalence of categories. This implies that R is a co-semisimple ring and hence is regular. By this equivalence, it is easy to see that A is a biregular ring.

(3) This follows immediately from (2), since $\sigma[A] = M(A)$-*Mod*. □

For module finite multiplication algebras several notions coincide:

30.12 Module finite multiplication algebras.

Let A be a central R-algebra with unit. Assume $M(A)$ is a finitely generated R-module. Then the following are equivalent:

(a) A is a co-semisimple $M(A)$-module;

(b) $M(A)$ is left co-semisimple (left V-ring);

(c) A is biregular;

(d) $M(A)$ is a regular ring;

(e) A is an Azumaya algebra and R is regular.

Proof. $(a) \Leftrightarrow (b)$ is clear since $\sigma[A] = M(A)$-*Mod*.

$(c) \Leftrightarrow (d) \Leftrightarrow (e)$ This follows from $\sigma[A] = M(A)$-*Mod* and the fact that A is finitely presented in $M(A)$-*Mod* (see 30.7).

$(c) \Rightarrow (a)$ is shown in 30.2.

$(a) \Rightarrow (e)$ We know from $(a) \Leftrightarrow (b)$ that $M(A)$ is a fully idempotent algebra and hence its centre R is regular (e.g., [40, 3.16]). Hence for every $m \in \mathcal{M}$, $A_m \simeq A/mA$ is a finite dimensional central R/m-algebra with $BMc(A) = 0$, i.e., A/mA is a finite product of simple algebras (see 21.3). Therefore all A/mA are Azumaya algebras and so is A by 29.4. □

30.13 Examples.

Consider the ring $R^{(2,2)}$ of 2 by 2 matrices over R and some subring $S \subset R^{(2,2)}$. Denote by $A_{R,S}$ the ring of sequences of elements in $R^{(2,2)}$ which are eventually constant with values in S, i.e.,

$$A_{R,S} = \{(a_1, \dots, a_k, s, s, \dots) \mid k \in \mathbb{N},\ a_i \in R^{(2,2)},\ s \in S\}.$$

(1) Assume R is a field and $S = \{\begin{pmatrix} r & 0 \\ 0 & t \end{pmatrix}, r, t \in R\}$.
Then $A_{R,S}$ is regular but not biregular (cf. [173]).

(2) Let R be a field and $S = \{\begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix}, r \in R\}$.
Then for every prime ideal $P \subset A_{R,S}$, $A_{R,S}/P$ is regular but $A_{R,S}$ is not regular (cf. [138]).
For $Z = Z(A_{R,S})$, $A_{R,S}/ZA_{R,S}$ is a nil ring and $ZA_{R,S}$ is regular and biregular (cf. [173]).

(3) Assume $R = \mathbb{R}$ (the reals) and $S = \{\begin{pmatrix} r & t \\ -t & r \end{pmatrix}, r, t \in \mathbb{R}\}$.

Then $A_{R,S}$ is biregular (all Pierce stalks are simple) but is not an Azumaya ring (cf. [99, 100]).

Remarks. A is co-semisimple if for every ideal $U \subset A$, $Rad(A/U) = 0$. Varying this condition one may ask when all $Alb(A/U) = 0$ or $BMc(A/U) = 0$. Associative algebras satisfying the last condition were first considered in Andrunakievic [46] under the name *N-semisimple*. For algebras with units all these properties coincide.

30.14 Exercises.

(1) Let A be a central R-algebra with unit. For an ideal $I \subset A$ define the annihilator

$$I^a = \{a \in A \mid aI = Ia = 0\}.$$

Show: (i) If I is A-generated or A is associative, then I^a is an ideal.

(ii) If A is biregular, then for any finitely generated ideal $I \subset A$, $I^{aa} = I$.

(iii) For any family of ideals $I_\lambda \subset A$, $(\sum_\Lambda I_\lambda)^a = \bigcap_\Lambda I^a_\lambda$.

(iv) Assume A is associative or a self-generator (as $M(A)$-module) and $I^{aa} = I$ for all ideals $I \subset A$. Then

(α) For every maximal ideal $M \subset A$, M^a is a minimal ideal.

(β) If $BMc(A) = 0$, A is a finite product of simple algebras.

(v) Assume A is biregular and for every maximal ideal $M \subset A$, $M^a \neq 0$. Then A is a finite product of simple algebras ([46]).

(2) Let A be an associative regular ring with unit which is left self-injective. Prove that the following are equivalent ([228, Proposition 1.6]):

(a) A is biregular;

(b) every prime ideal in A is maximal.

(3) Let A be an associative biregular algebra. Prove that for any $n \in \mathbb{N}$, the matrix ring $A^{(n,n)}$ is biregular.

(4) Let A be a semiprime alternative central R-algebra with unit. Assume A is a finitely generated R-module and R is regular. Prove that A is an Azumaya algebra.

(5) Let A be an associative central R-algebra with unit which is finitely generated as an R-algebra. Assume R is locally perfect and $Z(A/AJ) = R/J$ for $J = Jac(R)$. Prove that A is an Azumaya ring if and only if A/AJ is biregular.

(6) Let A be an associative algebra with unit. A is called π-regular *if for every $a \in A$ there exist $x \in A$ and $n \in I\!N$ such that $a^n x a^n = a^n$.*

Prove that a π-regular and reduced A is biregular.

(7) Let R be a regular ring and assume that A is an associative unital R-algebra which is finitely generated as an R-module. Prove that the following are equivalent ([52, Theorem 2]):
(a) A is semiprime;
(b) A is regular;
(c) A is biregular;
(d) A is semiprime and separable over its centre.

(8) Let R be a self-injective regular ring. Let A be an associative unital semiprime R-algebra which is finitely generated, non-singular and faithful as an R-module. Prove that A is left and right self-injective and an Azumaya algebra over its centre ([146, Proposition 3.3]).

(9) Prove that a biregular associative PI-ring is regular ([54]).

References: Andrunakievic [46], Arens-Kaplansky [51], Armendariz [52], Armendariz-Fisher [54], Burkholder [99, 100, 101], Dauns-Hofmann [113], Goursaud-Pascaud-Valette [146], Gray [17], Michler-Villamayor [199], Nauwelaerts-Oystaeyen [209], Renault [228], Wehlen [263], Wisbauer [268, 269, 276, 270].

31 Algebras with local Pierce stalks

1.Refinable algebras. 2.Idempotent refinable algebras. 3.Finitely lifting algebras. 4.Lifting algebras. 5.Supplemented algebras. 6.Self-projective f-semiperfect algebras. 7.Self-projective semiperfect algebras. 8.Perfect algebras. 9.Module finite perfect algebras. 10.Semiperfect associative algebras. 11.Structure of semiperfect algebras. 12.Structure of left perfect algebras. 13.Structure of left and right perfect algebras. 14.Self-projective associative left semiperfect algebras. 15.Exercises.

Biregular rings were characterized by quasi-simple Pierce stalks. In 8.4 we investigated more generally modules whose Pierce stalks are local modules. We now reconsider these properties for algebras as bimodules.

31.1 Refinable algebras.

Assume A is a central R-algebra and a finitely generated $M(A)$-module. Then the following are equivalent:

(a) *A is refinable as an $M(A)$-module;*

(b) *A is strongly refinable as an $M(A)$-module;*

(c) *every factor algebra of A is refinable over its multiplication algebra;*

(d) *direct summands (decompositions) lift modulo every ideal of A;*

(e) *for every $x \in \mathcal{X}$, A_x is a local $M(A)$-module;*

(f) *$A/Rad(A)$ is refinable and direct summands lift modulo $Rad(A)$.*

Proof. $(a) \Leftrightarrow (b) \Leftrightarrow (c)$ Since ideals are fully invariant submodules of A, the assertions follow from 8.5.

$(a) \Leftrightarrow (d)$ is clear by 8.4.

$(a) \Leftrightarrow (e) \Leftrightarrow (f)$ is an application of 18.10. □

Recall that an algebra A with $A^2 = A$ is said to be *weakly local* if it has a unique maximal ideal (see 21.10).

31.2 Idempotent refinable algebras.

Assume A is a central R-algebra, is finitely generated in $M(A)$-Mod and $A^2 = A$. Then the following are equivalent:

(a) *A is refinable as an $M(A)$-module;*

(b) *For every $x \in \mathcal{X}$, A_x is a weakly local algebra;*

(c) *for every $x \in \mathcal{X}$, xA is a weakly local ideal;*

(d) *$A/Alb(A)$ is refinable and direct summands lift modulo $Alb(A)$.*

Proof. $(a) \Leftrightarrow (b)$ is obvious by $A^2 = A$.

$(b) \Leftrightarrow (c)$ Since $A_x \simeq A/xA$ (see 18.1), the assertion is clear by the definitions.

$(a) \Leftrightarrow (d)$ For an idempotent algebra A, $Rad(A) = Alb(A)$ (see 21.2). □

A somewhat stronger condition is considered in the next proposition.

31.3 Finitely lifting algebras.
Assume A is a central R-algebra and a finitely generated $M(A)$-module. Then the following are equivalent:

(a) A is a finitely lifting $M(A)$-module;

(b) A is f-supplemented and π-projective;

(c) for any cyclic (finitely generated) ideal $U \subset A$, there is a decomposition $A = X \oplus Y$, with $X \subset U$ and $Y \cap U \ll Y$;

(d) for any finitely generated ideals $U, V \subset A$ with $U + V = A$, there exists an idempotent $e \in R$, with $Ae \subset U$ and $U(id - e) \ll A(1 - e)$ (and $A(id - e) \subset V$);

(e) as an $M(A)$-module, A is amply f-supplemented and supplements are direct summands;

(f) $A/Rad(A)$ is biregular and direct summands lift modulo $Rad(A)$.

If A has a unit and satifies the above conditions, then R is an f-semiperfect ring and $Jac(R) = R \cap BMc(A)$.

Proof. $(a) \Leftrightarrow (b)$ follows from 8.10(2). $(a) \Rightarrow (f)$ is clear by 8.8.

(a), (c), (d) and (e) are equivalent by 8.8.

$(f) \Rightarrow (a)$ By 31.1, A is strongly refinable and by 8.8, A is lifting.

Since algebras with units are direct projective (by 23.7) and $C(A) = Z(A) = R$, the assertion about the centre follows from 8.3. □

Remark. As shown in Nicholson [212, Theorem 2.1], a module has the exchange property if and only if its endomorphism ring is (left) refinable. Since f-semiperfect rings are refinable, we know from the preceding propositon that a finitely lifting algebra A with unit has the exchange property as an $M(A)$-module.

Similar to 31.3 we find equivalent conditions for A to be lifting:

31.4 Lifting algebras.
Assume A is a central R-algebra and a finitely generated $M(A)$-module. Then the following are equivalent:

(a) A is a lifting $M(A)$-module;

(b) *A is supplemented and π-projective;*

(c) *for every ideal $U \subset A$, there is a decomposition $A = X \oplus Y$, with $X \subset U$ and $Y \cap U \ll Y$;*

(d) *for any ideals $U, V \subset A$ with $U + V = A$, there is an idempotent $e \in R$, with $Ae \subset U$ and $U(id - e) \ll A(id - e)$ (and $A(id - e) \subset V$);*

(e) *for every ideal $U \subset A$, there is an idempotent $e \in R$, with $Ae \subset U$ and $U(id - e) \ll A(id - e)$;*

(f) *as an $M(A)$-module, A is amply supplemented and supplements are direct summands in A;*

(g) *$A/Rad(A)$ is a finite product of quasi-simple algebras and direct summands lift modulo $Rad(A)$;*

(h) *A is a finite product of algebras with unique maximal ideals.*

If A has a unit and the above properties, then R is a semiperfect ring and $Jac(R) = R \cap BMc(A)$.

Proof. $(a) \Rightarrow (b)$ follows from 8.9.

The equivalence of (a), (c), (d) and (e) follows from 8.9 and 8.10(2).

$(a) \Leftrightarrow (f)$ See the corresponding proof in 31.3

$(a) \Rightarrow (g)$ This can be deduced from the decomposition theorem 8.11. It can also be easily seen from (f) since in fact decompositions lift modulo $Rad(A)$ (see 31.1).

$(g) \Rightarrow (f)$ Assume $A = A_1 \oplus \ldots \oplus A_n$ with each of the ideals A_i having unique maximal ideals as algebras. Then

$$A/Rad(A) = A_1/Rad(A_1) \oplus \ldots \oplus A_n/Rad(A_n)$$

is a finite product of simple algebras. Every summand in $A/Rad(A)$ is a sum of the simple components and so can be lifted to a direct summand of A.

Unital algebras are direct projective and hence the assertion about $Jac(R)$ follows from 8.3. □

The characterization of supplemented modules in 8.6 yields:

31.5 Supplemented algebras.
Assume the algebra A is finitely generated as an $M(A)$-module.

(1) *The following are equivalent:*

  (a) *A is supplemented as an $M(A)$-module;*

  (b) *every maximal ideal in A has a supplement in A;*

(c) A is an irredundant (finite) sum of local $M(A)$-submodules.

(2) If A is supplemented, $A/Rad(A)$ is a finite product of quasi-simple algebras.

(3) If A is f-supplemented, then $A/Rad(A)$ is biregular.

For self-projective (f-) supplemented algebras we get from 8.13:

31.6 Self-projective f-semiperfect algebras.
Assume the R-algebra A is finitely generated and self-projective as an $M(A)$-module. Then the following are equivalent:

(a) A is f-semiperfect in $\sigma[A]$;

(b) A is f-supplemented as an $M(A)$-module;

(c) $A/Rad(A)$ is biregular and decompositions lift modulo $Rad(A)$.

The algebras considered above are in particular refinable and finitely lifting. Without the finiteness restriction we have from 8.12:

31.7 Self-projective semiperfect algebras.
Assume the central R-algebra A is finitely generated and self-projective as an $M(A)$-module. Then the following are equivalent:

(a) A is semiperfect in $\sigma[A]$;

(b) A is supplemented as an $M(A)$-module;

(c) every finitely A-generated module has a projective cover in $\sigma[A]$;

(d) $A/Rad(A)$ is a finite product of simple algebras and decompositions lift modulo $Rad(A)$;

(e) every simple factor module of A has a projective cover in $\sigma[A]$;

(f) A is a finite direct product of algebras with unique maximal ideals;

(g) R is a semiperfect ring.

Finally we transfer the charcterization of perfect modules from 8.14:

31.8 Perfect algebras.
Assume the central R-algebra A is finitely generated and self-projective as an $M(A)$-module. Then the following are equivalent:

(a) A is perfect in $\sigma[A]$;

(b) every (indecomposable) A-generated flat module is projective in $\sigma[A]$;

(c) $A^{(\mathbb{N})}$ is semiperfect in $\sigma[A]$;

(d) $A/Rad(A)$ *is semisimple and* $Rad(A^{(\mathbb{N})}) \ll A^{(\mathbb{N})}$;

(e) R *is a perfect ring.*

From the above we see that, for example, an Azumaya ring over R is (semi-) perfect in $\sigma[A]$ if and only if R is a (semi-) perfect ring.

In the following situation perfectness of A and $M(A)$ are related:

31.9 Module finite perfect algebras.

Let A *be a self-projective central* R*-algebra and assume that* A *and* $M(A)$ *are finitely generated as* R*-modules.*

Then A *is perfect in* $\sigma[A]$ *if and only if* $M(A)$ *is a left (or right) perfect ring.*

Proof. Under the conditions given for A, R is a perfect ring. As a module finite algebra with perfect centre, $M(A)$ is a left perfect ring (Exercise (4)).

Now assume $M(A)$ is a left perfect ring. Then the self-projective $M(A)$-module $A^{(\mathbb{N})}$ is supplemented and hence A is perfect in $\sigma[A]$. □

The next observation was already made in Miyashita [201, Proposition 4.4].

31.10 Semiperfect associative algebras.

Let A *be an associative algebra with unit. If* A *is a left semiperfect ring, then* A *is supplemented as an* $M(A)$*-module.*

Proof. Consider an ideal $U \subset A$ with left complement L and right complement K. Then $K \subset AK = UK + LK$ and $A = U + LK$. For any ideal $V \subset A$ with $A = U + V$, $KA = KU + KV$ and so $A = U + KV$. By minimality of K, $K = KV \subset V$. Similarly we have $L = VL \subset V$ and hence $LK \subset V$, showing that LK is an $M(A)$-complement of U in A. □

With this knowledge we can prove a structure theorem for left semiperfect algebras which is partly due to Szeto [255, Theorem 2.1]. Notice that for such algebras $A/Jac(A)$ is left semisimple and hence a product of simple algebras, implying $Jac(A) = BMc(A) = Rad_{M(A)}(A)$.

31.11 Structure of semiperfect algebras.

Let A *be an associative central* R*-algebra* A *with unit. Then the following are equivalent:*

(a) A *is left semiperfect and* π*-projective as an* $M(A)$*-module;*

(b) A *is left semiperfect and refinable as an* $M(A)$*-module;*

(c) A *is left semiperfect and central idempotents lift modulo* $Jac(A)$*;*

(d) A is a finite product of finite matrix rings over local rings.

Proof. $(a) \Leftrightarrow (b) \Leftrightarrow (c)$ In view of 31.10, all these equivalences are clear by 31.4.

$(a) \Rightarrow (d)$ We know from 31.10 that A is a supplemented $M(A)$-module and hence by 31.4, A is a finite product of algebras with unique maximal ideals.

So assume A itself has a unique maximal ideal. Being left semiperfect, $A = K_1 \oplus \cdots \oplus K_r$ with local projective left A-modules K_i. The trace $Tr(K, A)$ is a two-sided ideal which is not superfluous and hence $Tr(K, A) = A$. From this it is obvious that $K_1 \simeq K_i$ for $i = 2, \ldots, r$.

So $A \simeq End_A(K_1^r) \simeq S_1^{(r,r)}$, with the local ring $S_1 = End_A(K_1)$.

$(d) \Rightarrow (a)$ It is obvious that an algebra with this structure is left (and right) semiperfect. Moreover, all these matrix rings have a unique maximal ideal and hence the assertion follows from 31.4. □

For left perfect associative algebras we have essentially the same characterization with a stronger condition on the rings for the matrices. Since left semiperfect is in fact equivalent to right semiperfect, the above theorem is left-right symmetric. However, left perfect is not equivalent to right perfect.

31.12 Structure of left perfect algebras.

Let A be an associative central R-algebra A with unit. Then the following are equivalent:

(a) A is left perfect and π-projective as an $M(A)$-module;

(b) A is left perfect and refinable as an $M(A)$-module;

(c) A is left perfect and central idempotents lift modulo $Jac(A)$;

(d) A is a finite product of finite matrix rings over left perfect local rings.

Remarks. An associative ring S with unit is called a *left Steinitz ring* if any linearly independent subset of a free left module F can be extended to a basis of F. As shown in Lenzing [185] and Brodskii [90], left Steinitz rings are exactly the left perfect local rings. Hence the rings considered in 31.12 correspond to the *central perfect rings* as defined in Neggers-Allen [210]. The question which group rings share these properties is discussed in [210].

Left perfect local rings need not be right perfect (cf. Example 2 in [90]). Hence left and right perfect algebras for which central idempotents lift modulo the Jacobson radical form a proper subclass of the algebras considered in 31.12. They were characterized in Courter [110] by the property that every module is *rationally complete*. Since matrix rings are right perfect if and only if the base ring is, the following statements (contained in the Main Theorem of [110]) follow immediately from 31.12.

31.13 Structure of left and right perfect algebras.

Let A be an associative central R-algebra A with unit. Then the following are equivalent:

(a) A is a finite product of left and right perfect algebras with unique maximal ideals;

(b) A is left and right perfect and central idempotents lift modulo $Jac(A)$;

(c) A is a finite product of finite matrix rings over left and right perfect local rings.

For semiperfect associative algebras which are self-projective as $M(A)$-modules a structure theorem was proved in Miyashita [201, Theorem 4.6]. It is a consequence of our next theorem which can easily be deduced from the above results:

31.14 Self-projective associative left semiperfect algebras.

Let A be an associative central R-algebra A with unit. Then the following are equivalent:

(a) A is left semiperfect (perfect) and self-projective as an $M(A)$module;

(b) A is a finite product of finite matrix rings over (left perfect) local rings which are self-projective as bimodules.

Proof. $(a) \Rightarrow (b)$ Because of 31.11 and 31.12 we have only to argue about self-projectivity. At the end of the proof of 31.11 we had the isomorphism $A \simeq End_A(K_1^r) \simeq S_1^{(r,r)}$, with $S_1 = End_A(K_1)$. Since A is self-projective as a bimodule and ideals in A are determined by ideals in S_1 we see that S_1 is also self-projective as a bimodule (see 25.7).

$(b) \Rightarrow (a)$ Similar to the above argument we see that under the given conditions, A is a product of algebras which are self-projective as bimodules. Now it is easily checked that A is also self-projective as a bimodule.

The other assertions follow immediately from 31.11, respectively 31.12. $\square$

31.15 Exercises.

(1) A module M is called Σ-direct projective *if any direct sum of copies of M is direct projective (see [11, 11.2]).*

Let A be a unital algebra. Assume A is a finite product of weakly local algebras. Prove:

(i) A is Σ-direct projective as an $M(A)$-module.
(ii) $Z(A)$ is a semiperfect ring.
(iii) If A is a generator in $\sigma[A]$ then A is an Azumaya ring.

(2) Let A be an associative unital algebra. Assume factor rings of A have no proper (central) idempotents. Show that A has a unique maximal ideal.

(3) Let A be an associative unital algebra with unique maximal ideal $M \subset A$. Prove:

- *(i) If every right ideal of A has a non-zero annihilator then A is a local ring.*
- *(ii) A is a local ring or any maximal right ideal is an idempotent non-two-sided ideal.*

(4) Let A be an associative unital central R-algebra which is finitely generated as an R-module. Prove that the following are equivalent ([272, Satz 4.5]):

- *(a) A is a left (or right) perfect ring+*
- *(b) $M(A)$ is a left (or right) perfect ring;*
- *(c) R is a perfect ring.*

References. Brodskii [90], Courter [110], Lenzing [185], Miyashita [201], Neggers-Allen [210], Nicholson [212], Satyanarayana-Deshpande [240], Szeto [255], Wisbauer [272].

Chapter 9

Localization of algebras

In this chapter we apply our knowledge about localization in categories of type $\sigma[M]$ to algebras considered as bimodules. It turns out that in this setting many properties known from the commutative case, which do not transfer to left modules over non-commutative associative rings, are preserved. Moreover, associativity is not needed for many basic constructions.

Consider three different methods to extend the integers $\mathbb{Z}$ to the rationals $\mathbb{Q}$:

(I) Define an equivalence relation on the set $\mathbb{Z} \times \{\mathbb{Z} \setminus 0\}$ by

$$(a, b) \sim (c, d) \Leftrightarrow ad = bc.$$

On the set of equivalence classes $\mathbb{Z} \times \{\mathbb{Z} \setminus 0\}/\sim$, addition and multiplication is introduced by

$$\begin{array}{rcl} (a_1, b_1) + (a_2, b_2) & = & (a_1b_2 + a_2b_1, b_1b_2), \\ (a_1, b_1) \cdot (a_2, b_2) & = & (a_1a_2, b_1b_2). \end{array}$$

These definitions are independent of the choice of representatives and the construction yields (a model of) the rationals $\mathbb{Q}$.

(II) $\mathbb{Q}$ can be identified with the injective hull of $\mathbb{Z}$.

(III) $\mathbb{Q}$ can be represented as a direct limit of $\mathbb{Z}$-modules,

$$\mathbb{Q} \simeq \varinjlim \{Hom_{\mathbb{Z}}(n\mathbb{Z}, \mathbb{Z}) \,|\, n \in \mathbb{N}\}.$$

These constructions are readily extended to any prime commutative ring and they yield isomorphic quotient rings.

For non-commutative associative prime rings A the situation is different. In general, a relation defined as in (I) would no longer be an equivalence relation. Special conditions (*Ore conditions*) are necessary to copy this construction. Trying (II) we realize that the injective hull of A in A-*Mod* need not have a ring structure - unless A is left non-singular. Similar problems arise if we try to imitate (III).

Applying torsion theory we find that (II) and (III) still are of interest for associative rings A (and yield isomorphic results) if we replace *injective hull* in (II) by the *quotient module* $Q_{\mathcal{T}}(A)$ for a suitable hereditary torsion class $\mathcal{T}$ in A-*Mod*, and replace in (III) the ideals by $\mathcal{T}$-dense left ideals in A (see 9.20). This follows from the fundamental formula

$$\varinjlim \{Hom_A(U, A/Q_{\mathcal{T}}(A)) \,|\, U \in \mathcal{L}(A, \mathcal{T})\} \simeq Hom_A(A, Q_{\mathcal{T}}(A)),$$

where $\mathcal{L}(A, \mathcal{T})$ is the set of $\mathcal{T}$-dense left ideals, derived in 9.17. From there we also have the isomorphisms

$$Q_{\mathcal{T}}(A) \simeq Hom_A(A, Q_{\mathcal{T}}(A)) \simeq Hom_A(Q_{\mathcal{T}}(A), Q_{\mathcal{T}}(A)),$$

which allow to transfer the ring structure of the endomorphism ring to the quotient module $Q_{\mathcal{T}}(A)$. Since endomorphism rings are associative it is clear that this kind of constructing quotient rings only apply to associative A. Notice that here we get a quotient ring by module theoretic construction and not by inverting certain elements.

The above considerations have a two-sided counterpart. For a torsion theory $\mathcal{T}$ in $\sigma[A]$, the fundamental formula reads as

$$\varinjlim \{Hom_{M(A)}(U, A/Q_{\mathcal{T}}(A)) \,|\, U \in \mathcal{L}(A, \mathcal{T})\} \simeq Hom_{M(A)}(A, Q_{\mathcal{T}}(A)),$$

with $\mathcal{L}(A, \mathcal{T})$ the set of $\mathcal{T}$-dense (two-sided) ideals. We still have the isomorphism

$$Hom_{M(A)}(A, Q_{\mathcal{T}}(A)) \simeq End_{M(A)}(Q_{\mathcal{T}}(A)).$$

Now suppose that $Q_{\mathcal{T}}(A)$ is A-generated as $M(A)$-module. Then we get

$$Q_{\mathcal{T}}(A) = A Hom_{M(A)}(A, Q_{\mathcal{T}}(A)) \simeq A/\mathcal{T}(A)\, End_{M(A)}(Q_{\mathcal{T}}(A)),$$

which means that $Q_{\mathcal{T}}(A)$ is written as a kind of product of the ring $A/\mathcal{T}(A)$ and the ring $End_{M(A)}(Q_{\mathcal{T}}(A))$. It is tempting to introduce a multiplication on $Q_{\mathcal{T}}(A)$ by putting

$$(as) \cdot (bt) := (ab)st, \qquad \text{for } a, b \in A,\ s, t \in End_{M(A)}(Q_{\mathcal{T}}(A)),$$

and linear extension to all elements of $Q_{\mathcal{T}}(A)$. Of course, this assignement has to be independent of the choice of representatives of an element of type as. This is the case if $End_{M(A)}(Q_{\mathcal{T}}(A))$ is a commutative ring. Observe that along this way associativity of A is no longer of importance and $Q_{\mathcal{T}}(A)$ will certainly be as non-associative as $A/\mathcal{T}(A)$.

We will study such situations in the next sections. For prime and semiprime rings A we will consider the torsion class in $\sigma[A]$ determined by the injective hull of A in $\sigma[A]$. More general, we will also use the torsion class determined by the injective hull of A/I in $\sigma[A]$ for any (semi-) prime ideal $I \subset A$.

Another approach to find quotient rings for associative algebras A from bimodule properties is to study torsion theories in the category $A \otimes_R A^o$-*Mod* and then restrict the constructions to A-generated modules. We refer to [259, 93, 8] for details.

There is yet another possibility to generalize construction (II) from $\mathbb{Z}$ to an arbitrary central R-algebra A which is non-singular as R-module by defining a multiplication on the R-injective hull of A (*almost classical localization*). More details are given in Exercise 32.12(12). For regular R this construction can be related to the *orthogonal completion* of A (see [73, 4, 22]).

32 The central closure of semiprime rings

1.Extended centroid of a semiprime algebra. 2.Central closure of semiprime algebras. 3.Properties of the central closure. 4.Self-injective semiprime algebras. 5.Idempotent closure of semiprime algebras. 6.Semiprime algebras with non-zero socle. 7.Algebras with large centroids. 8.Large centres and finite uniform dimension. 9.Semiprime rings as subgenerators. 10.Corollary. 11.Central orders in Azumaya algebras. 12.Exercises.

The following observation is very important for our further investigations. It shows that - generalizing the situation for commutative associative rings - any semiprime algebra A is non-A-singular (polyform) as a bimodule. This allows us to apply our results concerning this type of modules to semiprime algebras.

Recall that $\widehat{A}$ denotes the A-injective hull of A (in $\sigma[A]$), $C(A) = End_{M(A)}(A)$ is the centroid of A.

32.1 Extended centroid of a semiprime algebra.

Let A be any semiprime algebra and $T := End_{M(A)}(\widehat{A})$. Then:

(1) A is a polyform $M(A)$-module.

(2) $C(A) \subset T$ and T is a commutative, regular and self-injective ring.

(3) $\widehat{A}$ is the quotient module of A for the Lambek torsion theory in $\sigma[A]$ and

$$T \simeq \varinjlim \{Hom_{M(A)}(U, A) \,|\, U \trianglelefteq_{M(A)} A\} \simeq Hom_{M(A)}(A, \widehat{A}).$$

(4) T is semisimple if and only if A has finite uniform dimension as $M(A)$-module.

T is called the *extended centroid* of A.

Proof. (1) According to 10.8, we have to show that for any essential ideals $U \subset V$ of A, $Hom_{M(A)}(V/U, A) = 0$.

Take any $f \in Hom_{M(A)}(V/U, A)$. Then $(V/U)f$ is an ideal in A and obviously $\{U \cap (V/U)f\}^2 = 0$. Now A semiprime implies $U \cap (V/U)f = 0$ and hence $(V/U)f = 0$ (since $U \trianglelefteq A$), i.e., $f = 0$.

(2) By (1) and 11.2, $C(A) \subset T$ and T is regular and self-injective ring.

To prove commutativity consider any $f, g \in T$. Then $U := Af^{-1} \cap Ag^{-1} \cap A$ is an essential ideal in A. For any ideal $K \subset A$ we observe $U^2 \cap K \supset (U \cap K)^2 \neq 0$ and hence the ideal generated by U^2 is also essential in A. Moreover, for any $a, b \in U$,

$$(ab)fg = (a(bf))g = (ag)(bf) = (ab)gf.$$

This implies $U^2(fg - gf) = 0$ and $fg - gf$ annihilates the ideal generated by U^2, i.e., $fg = gf$ (by (1)).

(3) Apply 9.13 and 9.17. (4) holds by 11.2. □

32.2 Central closure of semiprime algebras.

Let A be any semiprime algebra with A-injective hull $\widehat{A}$ and $T := End_{M(A)}(\widehat{A})$. Then:

(1) $\widehat{A} = AHom_{M(A)}(A, \widehat{A}) = AT$, and a ring structure is defined on $\widehat{A}$ by

$$(as)\cdot(bt) := (ab)st, \quad \textit{for } a,b \in A,\ s,t \in T,$$

and linear extension.

(2) A is a subring of $\widehat{A}$.

(3) $\widehat{A}$ is a semiprime ring with centroid $C(\widehat{A}) = T$ (hence a T-algebra).

(4) $\widehat{A}$ is self-injective as an $M(\widehat{A})$-module.

(5) If A is a prime ring, then $\widehat{A}$ is also a prime ring.

$\widehat{A}$ is called the *central closure* of A.

Proof. (1) Every injective module in $\sigma[A]$ is A-generated, in particular

$$\widehat{A} = AHom_{M(A)}(A, \widehat{A}).$$

By 32.1, A is polyform (non-A-singular) and hence (by 9.17)

$$Hom_{M(A)}(A, \widehat{A}) = End_{M(A)}(\widehat{A}).$$

To show that the definition makes sense, we verify that for $a_i, b \in A,\ s_i, t \in T$,

$$\textstyle\sum a_i s_i = 0 \text{ implies } (\sum a_i s_i)(bt) = 0 \text{ and } (bt)(\sum a_i s_i) = 0.$$

By definition, $\quad (\sum a_i s_i)(bt) := \sum (a_i b) s_i t = (\sum a_i s_i) tb = 0$, and

by commutativity of T, $\quad (bt)(\sum a_i s_i) := \sum (b a_i) t s_i = b(\sum a_i s_i) t = 0$.

(2) Obviously, $A \simeq A \cdot id_{\widehat{A}} \subset AT$.

(3),(4) For any ideal $U \subset \widehat{A}$, $U \cap A$ is an ideal in A and $U^2 = 0$ implies $(U \cap A)^2 = 0$. Hence $U \cap A = 0$ and $U = 0$, i.e., $\widehat{A}$ is semiprime.

By the definitions, we may assume $T, M(A) \subset End_T(\widehat{A})$ and $M(\widehat{A}) = M(A)T$. This implies

$$T = End_{M(A)}(\widehat{A}) = End_{M(\widehat{A})}(\widehat{A}) = C(\widehat{A}).$$

Moreover, from the middle equality we conclude that the self-injective $M(A)$-module $\widehat{A}$ is also self-injective as an $M(\widehat{A})$-module. $\square$

Recalling results about polyform modules from 11.11, 11.12 and 11.13, we have:

32.3 Properties of the central closure.

Let A be a semiprime algebra, $\widehat{A}$ its central closure and $T = End_{M(\widehat{A})}(\widehat{A})$.

(1) For any $m \in \widehat{A}$, $An_T(m)$ is generated by an idempotent in T and $\widehat{A}$ is a non-singular right T-module.

(2) Every essential ideal of $\widehat{A}$ is essential as an $M(A)$-submodule.

(3) For every submodule (subset) $K \subset \widehat{A}$, there exists an idempotent $\varepsilon(K) \in T$, such that $An_T(K) = (1 - \varepsilon(K))T$.

(4) For $M(A)$-submodules $K \trianglelefteq L \subset \widehat{A}$, $\varepsilon(K) = \varepsilon(L)$.

(5) Every finitely generated T-submodule of $\widehat{A}$ is T-injective.

(6) If $\widehat{A}$ is a finitely generated $M(\widehat{A})$-module, then $\widehat{A}$ is a generator in T-Mod.

(7) For $q_1, \ldots, q_n \in \widehat{A}$ assume $q_1 \notin \sum_{i=2}^n q_i T$. Then there exists $\mu \in M(A)$ such that $\mu q_1 \neq 0$ and $\mu q_i = 0$ for $i = 2, \ldots, n$.

The central closure of a semiprime algebra is self-injective (as bimodule). This class of rings obviously contains all rings which are direct sums of simple algebras. We list some of their properties from 11.13 and 32.4.

32.4 Self-injective semiprime algebras.

Let A be a self-injective semiprime central R-algebra.

(1) For $m_1, \ldots, m_n \in A$, the following are equivalent:

(a) $m_1 R \cap \sum_{i=2}^n m_i R = 0$.

(b) $An_{M(A)}(m_2, \ldots, m_n) m_1 \trianglelefteq A m_1$.

(2) For any ideal $N \subset A$ and $m \in A$, $(\mathcal{T}_N(A) : m)_{M(A)} = An_{M(A)}(m\varepsilon(N))$.

(3) For $m_1, \ldots, m_n, m \in A$ and $U = An_{M(A)}(m_2, \ldots, m_n)$ are equivalent:

(a) There exists $h \in M(A)$ with $hm_1 = m\,\varepsilon(Um_1)$ and $hm_i = 0$, for $i = 2, 3, \ldots, n$;

(b) there exist $r_1, \ldots, r_k \in M(A)$ such that for $s_1, \ldots, s_n \in M(A)$ the relations

$$\sum_{l=1}^n s_l r_j m_l = 0\,, \text{ for } j = 1, \ldots, k, \quad \text{imply } s_1 m \in \mathcal{T}_{Um_1}(A).$$

Applying the construction of the idempotent closure of a module (in 11.15) to an algebra, we obtain a subalgebra of the central closure:

32.5 Idempotent closure of semiprime algebras.

Let A be a semiprime R-algebra, $T = End_{M(A)}(\widehat{A})$, B the Boolean ring of idempotents of T. The idempotent closure of A as an $M(A)$-module, $\widetilde{A} = AB$, is an R-algebra and:

(1) *For any $a \in \widetilde{A}$, there exist $a_1, \ldots, a_k \in A$ and pairwise orthogonal $e_1, \ldots, e_k \in B$, such that*
(i) $a = \sum_{i=1}^k a_i e_i$,
(ii) $e_i = \varepsilon(a_i)e_i$, for $i = 1, \ldots, k$, and
(iii) $\varepsilon(a) = \sum_{i=1}^k e_i$.

(2) *For every prime ideal $K \subset \widetilde{A}$, $P = K \cap A$ is a prime ideal in A and*

$$\widetilde{A}/K = (A + K)/K \simeq A/P.$$

The set $x = \{e \in B | \widetilde{A}e \subset K\}$ is a maximal ideal in B and $K = PB + \widetilde{A}x$.

(3) *For any prime ideal $P \subset A$, there exists a prime ideal $K \subset \widetilde{A}$ with $K \cap A = P$.*

Proof. (1) This follows immediately from 11.15.

(2) Consider two ideals $I, J \subset A$ with $IJ \subset P$. Then IB and JB are ideals in $\widetilde{A}$ and $(IB)(JB) \subset K$. Hence $IB \subset K$ or $JB \subset K$. Assume $IB \subset K$. Then $I \subset IB \subset K \cap A \subset P$. So P is a prime ideal in A.

Consider $a \in A$ and $e \in B$. Since $(\widetilde{A}e)[\widetilde{A}(1-e)] = 0$, $\widetilde{A}e \subset K$ or $\widetilde{A}(1-e) \subset K$. Hence $ae \in K$ or $a(1-e) \in K$, which means $ae + K = K$ or $ae + K = a + K$.

Therefore for any $d \in \widetilde{A}$, there exists $r \in A$, with $d + K = r + K$. This implies $\widetilde{A}/K = (A + K)/K \simeq A/P$.

An easy argument shows that x is a maximal ideal in the Boolean ring B.

Put $U = PB + \widetilde{A}x$. Clearly $U \subset K$. For any $a \in K$, choose $a_1, \ldots, a_k \in A$ and orthogonal idempotents $e_1, \ldots, e_k \in B$ satisfying the conditions (ii) and (iii) of (1). In case all $e_1, \ldots, e_k \in x$, then $a \in \widetilde{A}x \subset U$.

Assume, without restriction, $e_1 \notin x$. Since $e_i e_1 = 0$ for $i \neq 1$, we have $e_2, \ldots, e_k \in K$. Also $1 - e_1 \in x$. Therefore

$$a_1 = a + a_1(1 - e_1) \in K + \widetilde{A}x = K$$

and $a_1 \in P$. Consequently, $a_1 e_1 \in PB$ and

$$a = a_1 e_1 + \sum_{i=2}^k a_i e_i \in PB + \widetilde{A}x.$$

So in any case $a \in U$, implying $K = U$.

(3) Put $S = A \setminus P$. Obviously, for any $a, b \in S$,

$$\emptyset \neq (M(A)a)(M(A)b) \cap S \subset (M(\widehat{A})a)(M(\widehat{A})b) \cap S.$$

Using this relationship we can show (as in the associative case) that an ideal $K \subset \widetilde{A}$, which is maximal with respect to $K \cap S = \emptyset$, is a prime ideal.

Clearly $I = K \cap A \subset P$. As shown above, $\widetilde{A}/K = (A + K)/K \simeq A/I$. Obviously, P/I is a prime ideal in A/I. Hence there exists a prime ideal $J \supset K$ of $\widetilde{A}$, for which $J/K = (P + K)/K$. This means $J = P + K$,

$$J \cap A = P + K \cap A = P + I = P,$$

and $J \cap S = \emptyset$. By the choice of K we conclude $K = J$. □

32.6 Semiprime algebras with non-zero socle.

Let A be a semiprime algebra and $T := End_{M(A)}(\widehat{A})$.

(1) For any minimal ideal $B \subset A$, $BT \subset B$.

(2) $Soc_{M(A)}A = Soc_{M(\widehat{A})}\widehat{A}$.

(3) If $Soc_{M(A)}A \trianglelefteq A$, then T is a product of fields.

Proof. (1) For any $t \in T$, $(A)t^{-1}$ is an essential $M(A)$-submodule in A and hence contains the minimal ideal B. Therefore B and $(B)t$ are minimal ideals in A implying $B = (B)t$ or $B \cap (B)t = 0$. The last equality means $0 = B(B)t = (B^2)t$ and from this we see that t is zero on the ideal generated by B^2, which is in fact B. So $BT \subset B$.

(2) is an immediate consequence of (1).

(3) Assume $Soc_{M(A)}A \trianglelefteq A$ and $Soc_{M(A)}A = \bigoplus_\Lambda B_\lambda$, where all B_λ are minimal ideals in A. Then

$$T = Hom_{M(A)}(Soc_{M(A)}A, \widehat{A}) \simeq End_{M(A)}(Soc_{M(A)}A) \simeq \prod_\Lambda End_{M(A)}(B_\lambda),$$

where each $End_{M(A)}(B_\lambda)$ is a field (by Schur's Lemma). □

It is obvious that an associative commutative ring is semiprime if and only if it is non-singular. More generally we observe:

32.7 Algebras with large centroids.

Let A be an algebra with large centroid $C(A)$.

(1) The following are equivalent:
  (a) A is a semiprime algebra;
  (b) every essential ideal is rational and A has no absolute zero-divisors;
  (c) A is non-A-singular (as bimodule) and A has no absolute zero-divisors.

(2) If A is semiprime, then $End_{M(A)}(\widehat{A})$ is the maximal ring of quotients of the centroid $End_{M(A)}(A)$.

Proof. (1) $(a) \Rightarrow (b)$ This follows from 32.1. $(b) \Leftrightarrow (c)$ Apply 11.1.

$(b) \Rightarrow (a)$ Let $U \subset A$ be an ideal with $U^2 = 0$. Then there exists an ideal $K \subset A$, such that $U \oplus K$ is essential, hence rational in A, and $(U \oplus K)U = 0$.

For any $f \in Hom_{M(A)}(A, U)$,

$$(U \oplus K)fA = (U \oplus K) \cdot (A)f = 0.$$

Since there are no absolut zero-divisors, $(U \oplus K)f = 0$. $U \oplus K$ being rational in A we conclude $f = 0$ and $Hom_{M(A)}(A, U) = 0$. The large centroid now implies $U = 0$, showing that A is semiprime.

(2) This follows from 11.5. □

There are quite different conditions which make a semiprime ring to have large centre. Of course, this is the case for any Azumaya ring or any self-generator algebra. It is long known that associative semiprime PI-rings have large centres and it was shown in Slater [242] that the same is true for purely alternative prime algebras A provided $3A \neq 0$. In an attempt to subsume this similarities between associative PI-algebras and special non-associative algebras, L.H. Rowen studied polynomial identities of non-associative algebras. Among other things he proved that semiprime Jordan algebras with normal polynomial identity and also *centrally admissible* algebras have large centres ([235, Theorem 2.5], [236, Theorem 3.3]).

For algebras with finite uniform dimension we get the following two-sided version of Goldie's Theorem 11.7:

32.8 Large centres and finite uniform dimension.
Let A be a semiprime algebra with unit, $C(A)$ the centre of A and $T := End_{M(A)}(\widehat{A})$. The following conditions are equivalent:

(a) A has finite uniform dimension as $M(A)$-module and
  (i) A has large centre, or
  (ii) for every ideal $V \trianglelefteq A$, there is a monomorphism $A \to V$;

(b) T is the classical quotient ring of $C(A)$ and
  (i) for every ideal $V \trianglelefteq A$, $VT = \widehat{A}$, or
  (ii) $\widehat{A}$ is a direct sum of simple algebras.

Proof. $(a.i) \Rightarrow (a.ii)$ Apply 11.6.

$(a.ii) \Rightarrow (b.i)$ By 11.5, T is the classical quotient ring of $C(A)$.

Since A has finite uniform dimension, every monomorphism $A \to V$ has essential image and hence is invertible in T (by 9.18). This implies $VT = \widehat{A}$

$(b.i) \Rightarrow (b.ii)$ $VT = \widehat{A}$ for all $V \trianglelefteq A$ implies that $\widehat{A}$ has no proper essential ideals and hence is a finite direct sum of simple $M(\widehat{A})$-submodules.

$(b.ii) \Rightarrow (a.i)$ Obviously $\widehat{A}$ and A have finite uniform dimension.

For any non-zero ideal $U \subset A$, UT is an ideal - and hence a direct summand - in $\widehat{A}$. This implies $UT \cap T \neq 0$.

Consider any non-zero $q := u_1t_1 + \cdots + u_kt_k \in UT \cap T$. Then there exists a non-zero-divisor $s \in C(A)$ such that $t_is \in C(A)$ for all $i \leq k$ and

$$0 \neq qs = u_1t_1s + \cdots + u_kt_ks \in U \cap T.$$

$\square$

Adapting 11.8 we are able to characterize semiprime algebras A which are subgenerators in $M(A)$-*Mod*. In particular, this includes R-algebras which are finitely generated as R-modules.

32.9 Semiprime rings as subgenerators.

Let A be a semiprime algebra, $T = End_{M(A)}(\widehat{A})$ and suppose $M(A) \in \sigma[A]$. Then:

(1) $M(A)$ is left non-singular and $Q_{\mathcal{S}}(M(A)) = Q_{\max}(M(A))$.

(2) $\widehat{A}$ is a generator in $Q_{\max}(M(A))$-Mod.

(3) $\widehat{A}_T$ is finitely generated and T-projective, and $Q_{\max}(M(A)) \simeq End_T(\widehat{A})$.

(4) If A has finite uniform dimension, then T and $Q_{\max}(M(A))$ are left semisimple.

As a consequence we observe (see 10.7):

32.10 Corollary. *For a semiprime algebra A the following are equivalent:*

(a) A is a subgenerator in $M(A)$-Mod;

(b) $\widehat{A}$ is finitely generated as module over $T = End_{M(A)}(A)$.

In this case $\widehat{A}$ is a (bi)regular Azumaya algebra over T (see 30.14 (7)).

It was noticed before that any R-algebra A, which is finitely generated as an R-module, is a subgenerator in $M(A)$-*Mod*. This question is of great interest in the theory of PI-algebras (see comments after 35.14).

In general, for a semiprime algebra A, $M(\widehat{A})$ need not coincide with $Q_{\max}(M(A))$. If A is a subgenerator in $M(A)$-*Mod*, we have

$$M(A) \subset M(\widehat{A}) \subset End_T(\widehat{A}) \simeq Q_{\max}(M(A)).$$

If, moreover, $\widehat{A}$ is a direct sum of simple algebras, $M(\widehat{A})$ is left semisimple, hence self-injective, and we conclude

$$M(\widehat{A}) = End_T(\widehat{A}) \simeq Q_{\max}(M(A)).$$

Then by 29.1, $\widehat{A}$ is an Azumaya T-algebra and we have:

32.11 Central orders in Azumaya algebras.

For a semiprime central R-algebra with unit, the following are equivalent:

(a) A has large centre, finite uniform dimension, and is a subgenerator in $M(A)$-Mod;

(b) $T = End_{M(A)}(\widehat{A})$ is the classical quotient ring of R and
(i) $M(\widehat{A})$ is left semisimple and finitely generated as T-module, or
(ii) $\widehat{A}$ is an Azumaya algebra over T.

Remarks. The extended centroid and central closure where first introduced in Martindale [191] for associative prime rings. With an elegant construction Amitsur expanded these notions to associative semiprime rings in [45]. Further generalizations to non-associative prime and semiprime rings occured in Erickson-Martindale-Osborn [124] and Baxter-Martindale [62]. In Rowen [234, Theorem 2] it was implicitely observed that the central closure of an associative semiprime (PI-) algebra A is A-injective as an $M(A)$-module. The interpretation and construction of the central closure of any semiprime ring as self-injective envelope of A as an $M(A)$-module was given in [273].

Notice that in the associative case the extended closure is obtained as a subalgebra of (larger) left rings of quotients (cf. Exercise (9)).

Extended centroids of skew polynomial rings are studied in Rosen-Rosen [231]. For an investigation of the extended centroid of normed and C^*-algebras we refer to Cabrera-Rodriguez [104] and Ara [50].

32.12 Exercises.

(1) Let A be any semiprime algebra. Prove that A is a cogenerator in $\sigma[A]$ if and only if A is a weak product of simple algebras.

(2) Let A be an associative algebra. A left ideal $I \subset A$ is called completely semiprime *if for any $a \in A$, $a^2 \in I$ implies $a \in I$. Recall that the circle composition of $a, b \in A$ is defined by $a \circ b := a + b - ab$. Prove ([48]):*

(i) Let $I \subset A$ be a completely semiprime left ideal. Then for $a \in A$, $b \in I$,
$a - ba \in I$ implies that $a \in I$, and
$b \circ a \in I$ implies that $a \circ b \in I$.

(ii) Suppose that $I \subset A$ is a completely semiprime, maximal left ideal. Then I is a two-sided ideal and the factor ring A/I is a division ring.

(3) Prove that for an alternative algebra A, the following are equivalent ([156]):
(a) A is a subdirect product of rings without zero-divisors;
(b) A is reduced.

(4) Prove that for an associative algebra A, the following are equivalent ([47]):
(a) A is a subdirect product of division rings;
(b) every ideal $I \subset A$, when considered as a ring, contains an ideal $U \subset I$, such that I/U is a strongly regular ring.

(5) Let A be an associative algebra with unit and $\{e_\lambda\}_\Lambda$ a set of idempotents in A. Prove ([80, Theorem 4]):
(i) If $I \subset A$ is a completely semiprime ideal, then $I+\sum_\Lambda e_\lambda A$ is a completely semiprime ideal.
(ii) If Λ is finite and $I \subset A$ is a semiprime ideal such that $(1-e_\lambda)Ae_\lambda \subset I$ for all $\lambda \in \Lambda$, then $I + \sum_\Lambda e_\lambda A$ is a semiprime ideal.

(6) Let A be an associative algebra with unit. Prove ([262]):
(i) The following are equivalent for a left ideal $I \subset A$:
(a) For any left ideals $K, L \supset I$, $KL \subset I$ implies $K = I$ or $L = I$;
(b) for any left ideals K, L, $(I + K)(I + L) \subset I$ implies $K \subset I$ or $L \subset I$;
(c) for $a, b \in A$, $(a + I)A(b + I) \subset I$ implies $a \in I$ or $b \in I$.
A proper left ideal $I \subset A$ satisfying this conditions is called weakly prime.
(ii) For a left ideal $I \subset A$ which is not two-sided, the following are equivalent:
(a) I is weakly prime;
(b) for every left ideal $L \subset A$, $IL \subset A$ implies $L \subset I$;
(c) for every $b \in A$, $IAb \subset I$ implies $b \in I$.
(iii) Suppose that every proper left ideal in A is weakly prime. Then A is a simple algebra.

(7) Let A be an associative regular central unital R-algebra. Let $T = End_{M(A)}(\widehat{A})$ denote the extended centroid. Prove that the following are equivalent ([49, 2.2]):
(a) $T = Q_{\max}(R)$;
(b) any non-zero ideal of A which has no proper essential extension in A contains a non-zero central element, and for every idempotent $e \in A$, $eC(A) \trianglelefteq C(eAe)$ (as $C(A)$-module).

(8) Let A, B be associative semiprime K-algebras, K a field, and let T and S denote their extended centroids. Prove ([195]):
The tensor product $A \otimes_K B$ is semiprime if and only if $T \otimes_K S$ is semiprime.

(9) Martindale ring of quotients.
Let A be an associative semiprime algebra. Denote by $\mathcal{F}$ the filter of all ideals of A with zero left (=right) annihilator. Then

$$Q_0(A) := \varinjlim \{Hom_A(I, {}_A A) \mid I \in \mathcal{F}\}$$

is a ring extension of A, called the Martindale left ring of quotients. *Prove ([45, 4]):*
(i) $Q_0(A)$ and $Q_{\max}(A)$ have the same centre which is isomorphic $T := End_{M(A)}(\widehat{A})$ and we have canonical inclusions

$$A \subset \widehat{A} \subset Q_0(A) \subset Q_{\max}(A).$$

(ii) $Q_0(A)$ is injective as a T-modules.

(10) Let A be an associative PI-algebra with central closure $\widehat{A}$ and extended centroid T. Prove that $\widehat{A} = Q_{\max}(A)$ if and only if $\widehat{A}$ is finitely generated as T-module ([70, Lemma 5]).

(11) Semiprime algebras with many uniform ideals.
Let A be an R-algebra with extended centroid T. Prove that the following are equivalent:
(a) Every ideal in A contains a uniform ideal;
(b) T is a product of fields.

(12) Almost classical localization *([73]).*
Let A be a central R-algebra which is non-singular as an R-module.
(i) Prove that R is semiprime and $\varinjlim \{Hom_R(I, A) \mid I \trianglelefteq R\} \simeq Hom_R(R, \widetilde{A}) \simeq \widetilde{A}$, where $\widetilde{A}$ denotes the R-injective hull of A.
(ii) For essential ideals $I, J \subset R$ and $f \in Hom_R(I, A)$, $g \in Hom_R(J, A)$, put

$$f \cdot g : IJ \to A,\ rs \mapsto f(r)g(s), \quad r \in I,\, s \in J.$$

Show that this map is properly defined and its (unique) extension to a map $R \to \widetilde{A}$ defines a product on $\widetilde{A}$.
(iii) Prove that with this product $\widetilde{A}$ is an R-algebra which contains A as a subalgebra and has the (regular) quotient ring $\widehat{R}$ (injective hull) as centre.
(iv) Assume A is (semi) prime. Show that $\widetilde{A}$ is (semi) prime.
(iv) Assume A to be an alternative regular algebra without nilpotent elements. Show that $\widetilde{A}$ is also alternative regular without nilpotent elements ([73, 145]).

References. Amitsur [45], Baxter-Martindale [62, 63], Beidar [71], Beidar-Martindale-Mikhalev [4], Beidar-Mikhalev [73], Beidar-Wisbauer [76, 75], Cabrera-Rodriguez [104], Cobalea-Fernandez [108], González [145], Martindale [191], Matczuk [195], Rosen-Rosen [231], Passman [220], Rowen [234, 235, 236], Slater [242], Wisbauer [273].

33 Closure operations in $\sigma[A]$

1.A-singular modules for semiprime rings. 2.A-singularity over left-nonsingular rings. 3.Singular closure in $\sigma[A]$. 4.Relations with the central closure. 5.$M(A)$-submodules of $A^{(\Lambda)}$. 6.Correspondence of closed submodules of $A^{(\Lambda)}$.

Recall that a module $N \in \sigma[A]$ is said to be *singular in* $\sigma[A]$, or A-*singular*, if $N \simeq L/K$, for some $L \in \sigma[A]$ and $K \trianglelefteq L$.

We denote by $\mathcal{S}$ the pretorsion class of all A-singular modules in $\sigma[A]$, and for any $X \in \sigma[A]$ we write $\mathcal{S}(X)$ for the largest A-singular submodule of X.

For any semiprime ring A, $\mathcal{S}(A) = 0$ (see 32.1) and $\mathcal{S}$ is a torsion class, i.e., it is closed under extensions in $\sigma[A]$ (see 10.2).

Over an associative ring A with unit, a left module is singular if and only if the annihilator of each of its elements is an essential left ideal. This characterization is based on the fact that A is projective as a left A-module and hence any homomorphism from A to a singular left A-module has essential kernel.

In general, A is not projective as a bimodule (even if it is associative) and hence we do not have a corresponding characterization of A-singular modules in $\sigma[A]$. However, any homomorphism from a non-A-singular module to an A-singular module has essential kernel. So we may expect a similar characterization of A-singularity in case A is non-A-singular, in particular if A is semiprime. This is what we show now.

33.1 A-singular modules for semiprime rings.
Let A be a semiprime ring. Then for any $N \in \sigma[A]$, the following are equivalent:

(a) N is A-singular;

(b) for every cyclic $M(A)$-submodule $X \subset N$, there exists an essential ideal $I \trianglelefteq A$ such that $IX = 0$ (or $XI = 0$).

If A is associative, (b) is equivalent to:

(c) For every $x \in N$, there exists an essential ideal $I \trianglelefteq A$ with $Ix = 0$ (or $xI = 0$).

Proof. $(a) \Rightarrow (b)$ Let $X \subset N$ be generated by one element. By 10.3, X is contained in a finitely A-generated, A-singular module $\widetilde{X}$. For this we have an epimorphism $f : A^n \to \widetilde{X}$, for some $n \in \mathbb{N}$. Since A is non-A-singular, for each inclusion $\varepsilon_i : A \to A^n$, $i = 1, \ldots, n$, we have $Ke\, \varepsilon_i f \trianglelefteq A$. Then $I := \bigcap_{i=1}^n Ke\, \varepsilon_i f$ is an essential ideal in A and

$$I\widetilde{X} = I(A^n)f \subset (I^n)f = 0.$$

Similarly we get $\widetilde{X}I = 0$. From this the assertion follows.

$(b) \Rightarrow (a)$ Let $X \subset N$ be a cyclic $M(A)$-submodule and let $H \trianglelefteq A$ be such that $HX = 0$. Take $\widetilde{X}$ to be a finitely A-generated essential extension of X. Then we have the exact diagram

$$\begin{array}{ccccccc} & & & A^n & & & \\ & & & \downarrow f & & & \\ 0 \longrightarrow & X & \longrightarrow & \widetilde{X} & \xrightarrow{p} & \widetilde{X}/X & \longrightarrow 0\,, \\ & & & \downarrow & & & \\ & & & 0 & & & \end{array}$$

and for every inclusion $\varepsilon_i : A \to A^n$, $i = 1, \ldots, n$,

$$I_i := (X)(\varepsilon_i f)^{-1} \trianglelefteq A, \text{ and } I := \bigcap_{i=1}^{n} I_i \trianglelefteq A.$$

For this we have

$$<HI> \widetilde{X} = <HI> (A^n)f \subset H(I^n)f \subset HX = 0\,,$$

where $<HI>$ denotes the ideal generated by HI. This shows that $< HI >^n \subset Ke\,f$. Also, $< HI >$ is an essential ideal in A since for every non-zero ideal $L \subset A$, $0 \neq (I \cap H \cap L)^2 \subset IH \cap L$. Hence $\widetilde{X}$ and X are A-singular.

$(b) \Leftrightarrow (c)$ For an associative algebra A, the ideal generated by $x \in N$ has the form $(\mathbb{Z} + A)x(\mathbb{Z} + A)$. Hence for any ideal $I \subset A$, $Ix = 0$ implies

$$I(\mathbb{Z} + A)x(\mathbb{Z} + A) = Ix(\mathbb{Z} + A) = 0.$$

$\square$

For an associative left non-singular ring A with unit, we have a nice one sided characterization of the A-singular (bi)modules in $\sigma[A]$. Denoting the singular submodule of any left module X by $\mathcal{S}_l(X)$ we obtain:

33.2 A-singularity over left-nonsingular rings.

For an associative semiprime ring A with unit, the following are equivalent:

(a) $\mathcal{S}_l(A) = 0$, i.e., A is left non-singular;

(b) for every module $X \in \sigma[A]$, $\mathcal{S}(X) = \mathcal{S}_l(X)$.

Proof. $(a) \Rightarrow (b)$ By Proposition 33.1, the elements of $\mathcal{S}(X)$ are annihilated by essential ideals. Since A is semiprime, any essential ideal $I \subset A$ is also essential as left ideal. So $\mathcal{S}(X) \subset \mathcal{S}_l(X)$.

Denote $N := \mathcal{S}_l(X)$. This is obviously an A-bimodule and hence it is contained in an A-generated essential extension $\widetilde{N} \in \sigma[A]$. Consider the exact sequence of A-bimodules

$$0 \to N \to \widetilde{N} \to \widetilde{N}/N \to 0.$$

By construction, $\widetilde{N}/N$ is A-singular and hence left singular by the above argument.

Considering this sequence in A-*Mod*, we have $\widetilde{N}$ as an extension of the left singular A-modules N and $\widetilde{N}/N$. Since $\mathcal{S}_l(A) = 0$ we know that the class of singular left modules is closed under extensions and hence $\widetilde{N}$ is left singular.

By construction, there is a bimodule epimorphism $f : A^{(\Lambda)} \to \widetilde{N}$. For every inclusion $\varepsilon_\lambda : A \to A^{(\Lambda)}$, $Ke\,\varepsilon_\lambda f$ is an ideal, which is essential as a left ideal, since $\widetilde{N}$ is left singular. Then it is certainly essential as an ideal showing that $\widetilde{N}$ is A-singular.

$(b) \Rightarrow (a)$ Since $A \in \sigma[A]$, we have $\mathcal{S}_l(A) = \mathcal{S}(A) = 0$. □

We are now going to outline the transfer of the closure operations for modules in $\sigma[M]$ to the category $\sigma[A]$, where A is any ring with multiplication algebra $M(A)$.

Recall that for modules $K \subset N$ in $\sigma[A]$ we have two types of 'closures' of K in N: the maximal essential extension of K in N (which is unique if N is non-A-singular), and the A-singular closure of K in N defined by $[K]_N/K := \mathcal{S}(N/K)$.

Using the characterization of A-singular modules given in 33.1 we obtain:

33.3 Singular closure in $\sigma[A]$.

Let A be a semiprime ring and $K \subset N$ in $\sigma[A]$. Then

$$[K]_N = \{x \in N \mid Ix \subset K \text{ (or } xI \subset K) \text{ for some essential ideal } I \trianglelefteq A\}.$$

Over a semiprime ring A, for any $N \in \sigma[A]$, the A-singular closed submodules are in one-to-one correspondence with the closed submodules of $N/\mathcal{S}(N)$ (by 12.4). Hence in what follows we will focus on non-A-singular modules.

Applying 12.7 to A, and recalling that the central closure $\widehat{A}$ is just the A-injective hull of A we have:

33.4 Relations with the central closure.

Let A be a semiprime ring with central closure $\widehat{A}$ and $T := End_{M(A)}(\widehat{A})$.

(1) There exists bijections between

(i) the closed ideals in A,

(ii) the ideals which are direct summands in $\widehat{A}$,

(iii) the ideals which are direct summands in T.

(2) (i) For any essential ideal $I \trianglelefteq T$, $\widehat{A}I \subset \widehat{A}$ is an essential ideal.

(ii) For every $\widehat{A}$-generated essential ideal $V \subset \widehat{A}$, $Hom_{M(\widehat{A})}(\widehat{A}, V) \trianglelefteq T$.

Proof. (1) is immediately clear by 12.7(1).

(2) It is easy to see that the $M(A)$-submodule $\widehat{A}I$ is in fact an ideal in $\widehat{A}$. Moreover, since T is commutative, $Hom_{M(A)}(\widehat{A}, V)$ is a (two-sided) ideal in T and $Tr(\widehat{A}, V)$ (in $M(A)$-*Mod*) is an ideal in $\widehat{A}$. With this remark the assertions follow from 12.7(2). □

Applying the correspondence theorem for submodules of $M^{(\Lambda)}$ and $\widehat{M}^{(\Lambda)}$ to $A^{(\Lambda)}$ and $\widehat{A}^{(\Lambda)}$ a new phenomenon occurs (similar to 33.4): In $\widehat{A}^{(\Lambda)}$ we have $M(A)$-submodules and $M(\widehat{A})$-submodules. It is a nice aspect of the theory that closed $M(A)$-submodules and closed $M(\widehat{A})$-submodules coincide. We are going to prove this fact, which is similar to the observation (made in 12.11) that for associative rings A with unit the closed left A-submodules of $Q(A)^{(\Lambda)}$ are precisely the same as the closed left $Q(A)$-submodules.

33.5 $M(A)$**-submodules of** $A^{(\Lambda)}$**.**

Let A be a semiprime ring with central closure $\widehat{A}$ and $T := End_{M(A)}(\widehat{A})$. Then:

(1) $Hom_{M(A)}(\widehat{A}, \widehat{A}^{(\Lambda)}) = Hom_{M(\widehat{A})}(\widehat{A}, \widehat{A}^{(\Lambda)})$.

(2) Closed $M(A)$-submodules of $\widehat{A}^{(\Lambda)}$ are closed $M(\widehat{A})$-submodules, and conversely.

(3) Closed $M(A)$-submodules of $\widehat{A}^{(\Lambda)}$ are $\widehat{A}$-generated $M(\widehat{A})$-submodules.

(4) If A is a finitely generated $M(A)$-module, then for every T-submodule $X \subset Hom_{M(\widehat{A})}(\widehat{A}, \widehat{A}^{(\Lambda)})$,

$$Hom_{M(\widehat{A})}(\widehat{A}, \widehat{A}X) = X.$$

Proof. (1) Recall that $T = End_{M(A)}(\widehat{A}) = End_{M(\widehat{A})}(\widehat{A})$. Now the assertion follows from the fact that every $f \in Hom_{M(A)}(\widehat{A}, \widehat{A}^{(\Lambda)})$ is determined by the $f\pi_\lambda \in T$, where $\pi_\lambda : \widehat{A}^{(\Lambda)} \to \widehat{A}$ denote the canonical projections.

(2) Since $M(\widehat{A}) = M(A)T$ we have to show that every closed $M(A)$-submodule $U \subset \widehat{A}^{(\Lambda)}$ is a T-submodule:

Let $t \in T$ and $I \trianglelefteq A$ such that $It \subset A$. Then $Ut \cdot I = U(It) \subset U$. Since U is closed this implies $Ut \subset U$ (by 33.3).

Clearly every $M(A)$-closed submodule is $M(\widehat{A})$-closed.

Suppose that $V \subset \widehat{A}^{(\Lambda)}$ is a closed $M(\widehat{A})$-submodule. Let $u \in \widehat{A}^{(\Lambda)}$ be such that $uI \subset V$ for some $I \trianglelefteq A$. Then obviously $uIT \subset V$, where IT is an essential ideal in $\widehat{A}$. Since V is $M(\widehat{A})$-closed this implies $u \in V$ showing that V is $M(A)$-closed.

(3) By 12.8, a closed submodule $U \subset \widehat{A}^{(\Lambda)}$ is $\widehat{A}$-generated as $M(A)$-module. Now it is obvious by (1) that $\widehat{A}$ generates U as $M(\widehat{A})$-module.

(4) Applying (1) this follows from 12.9. $\square$

We are now prepared to present the following

33.6 Correspondence of closed submodules of $A^{(\Lambda)}$.

Let A be a semiprime ring which is finitely generated as $M(A)$-module. We put $T := End_{(M(A)}(\widehat{A})$ and identify $T^{(\Lambda)} = Hom_{M(\widehat{A})}(\widehat{A}, \widehat{A}^{(\Lambda)})$. Then there are bijections between

(i) the closed $M(A)$-submodules of $A^{(\Lambda)}$,

(ii) the closed $M(\widehat{A})$-submodules of $\widehat{A}^{(\Lambda)}$,

(iii) the closed T-submodules of $T^{(\Lambda)}$.

References. Ferrero [129, 130, 131], Ferrero-Wisbauer [132].

34 Strongly and properly semiprime algebras

1.Lemma. 2.Associative PSP algebras. 3.Strongly semiprime rings. 4.Properly semiprime rings. 5.Pierce stalks and PSP algebras. 6.Fully idempotent rings. 7.Biregular and PSP rings. 8.Central closure as Azumaya ring. 9.Example. 10.Exercises.

Definition. An algebra A is said to be *properly semiprime (PSP)* if it is PSP as an $M(A)$-module. A is called *strongly semiprime (SSP)* if it is SSP as an $M(A)$-module.
A is called *strongly prime* if it is a strongly prime $M(A)$-module.

Obviously, every strongly prime algebra is SSP, and every SSP algebra is PSP. Strongly prime algebras will be studied in the next section.

We need a slight non-degeneracy condition to make PSP algebras to be semiprime.

34.1 Lemma. *Let A be an algebra such that no non-zero ideal of A annihilates A.*

(1) If A is a PSP algebra then A is semiprime.

(2) If A is a strongly prime algebra then A is prime.

Proof. (1) Assume $K \subset A$ is a non-zero finitely generated ideal with $K^2 = 0$. Consider $U = \{L_a \mid a \in K\}$. By assumption, $A/\mathcal{T}_K(A) \in \sigma[K]$. Now $UK = 0$ implies $U(A/\mathcal{T}_K(A)) = 0$ and $KA \subset \mathcal{T}_K(A)$. So $KA \subset \mathcal{T}_K(A) \cap K = 0$, contradicting the given condition.

(2) Consider non-zero ideals $I, J \subset A$ with $IJ = 0$. Put $U = \{L_a \mid a \in I\}$. By assumption, $A \in \sigma[J]$ and hence $IA = UA = 0$, a contradiction. □

In general, a strongly prime ring need not be prime. For this consider the cyclic group $\mathbb{Z}_p$ of prime order p with the trivial multiplication $ab = 0$, for all $a, b \in \mathbb{Z}_p$. This is a strongly prime (simple) ring which is not prime.

For associative A we have the notions of PSP and left PSP algebra. We observe the following relationship between these properties:

34.2 Associative PSP algebras.
Let A be an associative semiprime algebra with unit.

(1) In $\sigma[A]$, $\mathcal{T}_U(A) = An_A(U)$ for any ideal $U \subset A$.

(2) If A is a PSP ring, then A is left PSP.

(3) If A is an SSP ring, then A is left SSP.

(4) If A is a strongly prime ring, then A is left strongly prime.

Proof. (1) $\mathcal{T}_U(A)$ is an ideal in A and $\mathcal{T}_U(A) \cap U = 0$, hence $\mathcal{T}_U(A) \subset An_A(U)$.

Consider $f \in Hom_{M(A)}(An_A(U), I_A(U))$ and put $K = (U)f^{-1}$. Then

$$(Kf)^2 \subset U(Kf) = (UK)f \subset (UAn_A(U))f = 0.$$

Since A is semiprime, $Kf = 0$ and so $Im\, f \cap U = 0$, implying $f = 0$. Hence $\mathcal{T}_U(A) \supset An_A(U)$ (compare 14.18).

(2) Consider $a, b \in A$ and assume without restriction $A = L(A) \subset M(A)$. Since A is PSP, there exist $x_1, \ldots, x_k \in M(A)$ such that

$$An_{M(A)}(x_1 a, \ldots, x_k a)b \subset \mathcal{T}_{M(A)a}(A) = An_A(M(A)a) \subset An_M(Aa).$$

In particular,

$$An_A(x_1 a, \ldots, x_k a)b \subset An_M(Aa).$$

Now assume $x_i = \sum_{j=1}^{m_i} L_{y_{ij}} R_{z_{ij}}$. Then

$$An_A(\{y_{ij}a \,|\, 1 \leq j \leq m_i, 1 \leq i \leq k\})b \subset An_M(Aa).$$

By 14.18, $\mathcal{T}_{Aa}(A) = An_A(Aa)$. Hence by 14.6, the above relation implies that A is left PSP.

(3) and (4) are shown in a similar way. □

34.3 Strongly semiprime rings.

Let A be a ring which is not annihilated by any non-zero ideal and $T = End_{M(A)}(\widehat{A})$. Then the following conditions are equivalent:

(a) A is an SSP ring;

(b) A is semiprime and for every essential ideal $U \subset A$, $A \in \sigma[U]$;

(c) for every ideal $I \subset A$, $\widehat{A} = IT \oplus \mathcal{T}_I(\widehat{A})$.

(d) A is semiprime and the central closure $\widehat{A}$ is a direct sum of simple ideals.

If A is associative, then (a)-(d) are equivalent to:

(e) A is semiprime and for every ideal $I \subset A$, $A/An_A(I) \in \sigma[I]$;

Proof. By 34.1, A being SSP implies that A is semiprime.

$(a) \Leftrightarrow (b)$ Since A is semiprime, A is a polyform $M(A)$-module by 32.2. Hence the assertion follows from 14.10.

$(a) \Leftrightarrow (c)$ This is also obtained from 14.10.

$(a) \Leftrightarrow (d)$ The central closure $\widehat{A}$ is a direct sum of simple ideals if and only if it is semisimple as an $(M(A), T)$-bimodule. Now apply 14.4.

$(a) \Leftrightarrow (e)$ By 34.1, R is a semiprime ring. Hence by 34.2, $\mathcal{T}_U(A) = An_A(U)$. Now the equivalence is clear by 14.3. □

34.4 Properly semiprime rings.

Let A be a semiprime ring, $T = End_{M(A)}(\widehat{A})$, B the Boolean ring of idempotents of T and $\widetilde{A} = AB$ the idempotent closure of A. Then the following are equivalent:

(a) A is a PSP ring;

(b) for every $a \in A$, $M(A)aT = \widehat{A}\varepsilon(a)$;

(c) for every finitely generated ideal $K \subset A$, $KT = \widehat{A}\varepsilon(K)$;

(d) $\widetilde{A}$ is a PSP ring.

Under the given conditions, for every $a \in A$, $\mathcal{T}_a(\widehat{A}) = \widehat{A}(id - \varepsilon(a))$.

Proof. Since semiprime rings are polyform as bimodules, these equivalences essentially are obtained from 14.11.

We only have to show that (d), i.e., $\widetilde{A}$ is PSP as an $M(\widetilde{A})$-module, is equivalent to $\widetilde{A}$ being PSP as an $M(A)$-module. This follows readily from $M(\widetilde{A}) = M(A)B$. □

34.5 Pierce stalks and PSP algebras.

Let A be a semiprime PSP algebra with extended centroid T. Denote by B the Boolean ring of idempotents of T, and by $\mathcal{X}$ the set of maximal ideals in B. Consider the idempotent closure $\widetilde{A} = AB$ of A and the canonical projections $\varphi_x : \widetilde{A} \to \widetilde{A}_x$, for $x \in \mathcal{X}$.

Assume that for any ideal $K \subset A$, $K \subset M(A)(KA)$. Then:

(1) For every $x \in \mathcal{X}$, $\widetilde{A}_x = \widetilde{A}/\widetilde{A}x$ is a prime and strongly prime ring.

(2) For every minimal prime ideal $P \subset A$ there exists an $x \in \mathcal{X}$ such that $\widetilde{A}_x \simeq A/P$ and $P = A \cap \widetilde{A}x$.

Proof. (1) By 34.4, $\widetilde{A}$ is a PSP algebra. It follows from 18.16 that $\widetilde{A}_x$ is a strongly prime $M(\widetilde{A})$-module. Since $M(\widetilde{A}_x) = M(\widetilde{A})/An_{M(\widetilde{A})}\widetilde{A}_x$, $\widetilde{A}_x$ is a strongly prime $M(\widetilde{A}_x)$-module, i.e., $\widetilde{A}_x$ is a strongly prime algebra.

Consider any non-zero ideal $I \subset \widetilde{A}_x$. Since $(A)\varphi_x = \widetilde{A}_x$ we have $I \subset M(\widetilde{A}_x)(I\widetilde{A}_x)$. In particular, $I\widetilde{A}_x \neq 0$. By 34.1, this implies that $\widetilde{A}_x$ is a prime ring.

(2) Consider a minimal prime ideal $P \subset A$. By 32.5, there exists a prime ideal $K \subset \widetilde{A}$ with $K \cap A = P$. Moreover,

$$x := \{e \in B \,|\, e\widetilde{A} \subset K\} \in \mathcal{X},\ K = PB + x\widetilde{A} \text{ and } \widetilde{A}/K \simeq A/P\,.$$

From above it follows that $\widetilde{A}x$ is a prime ideal in $\widetilde{A}$. Now by 32.5, $\widetilde{A}x \cap A$ is a prime ideal in A. But $\widetilde{A}x \cap A \subset K \cap A = P$ and P is a minimal prime ideal. Therefore $\widetilde{A}x \cap A = P$ and $K = PB + \widetilde{A}x = \widetilde{A}x$. This shows that $\widetilde{A}_x = \widetilde{A}/K \simeq A/P$ is a prime and strongly prime ring. □

In [77, Theorem 3.5] a result similar to 34.5 is stated for associative left PSP algebras with a weak symmetry condition. Moreover, an example of an associative semiprime PI-algebra is constructed which is left PSP, coincides with its central closure but is not biregular.

An algebra A is said to be *fully idempotent* if, for any ideal $I \subset A$, $I = M(A)I^2$. Clearly such rings are semiprime. We show a module property of them:

34.6 Fully idempotent rings.
Let A be a fully idempotent ring. Then A is pseudo regular as $M(A)$-module.

Proof. Put $S = End_{M(A)}(A)$ and $T = End_{M(A)}(\widehat{A})$. For any subset $U \subset A$, define $L_o(U) = \{L_u \mid u \in U\}$. Consider $a \in A$ and $I = M(A)a$. By assumption,

$$I = M(A)I^2 = M(A)L_o(I)I,$$

and hence $a \in M(A)L_o(I)M(A)a$. Therefore there exists $\alpha \in M(A)L_o(I)M(A)$ with $\alpha a = a$. Obviously, $\alpha \in M(A)$ and $\alpha A \subset I \subset M(A)aT$. So A is a pseudo regular $M(A)$-module (see 14.21). $\square$

The next result reveils the connection between biregular and PSP rings. For simplicity we only consider rings with units, although our methods apply to the general case.

34.7 Biregular and PSP rings.
Let A be a semiprime ring with unit, $T = End_{M(A)}(\widehat{A})$, B the Boolean ring of idempotents of T and $\widetilde{A} = AB$ the idempotent closure of A. Then the following are equivalent:

(a) A is biregular;

(b) A is a fully idempotent PSP ring;

(c) for every $a \in A$, $M(A)a = A\varepsilon(a)$;

(d) $\widetilde{A}$ is biregular;

(e) A is PSP and every prime ideal in A is maximal;

Proof. $(a) \Rightarrow (b)$ Assume A is biregular. Then A is fully idempotent.

For any $a \in A$, $M(A)a = Ae$ for some idempotent $e \in C$. e extends to a unique idempotent $\bar{e} \in T$ and $M(A)aT = AT\bar{e} = \widehat{A}\bar{e}$. As easily checked, $\bar{e} = \varepsilon(a)$. So A is PSP by 34.4.

$(b) \Rightarrow (a)$ By 34.6, fully idempotent rings are pseudo regular $M(A)$-modules. Hence the assertion follows from 14.23.

$(b) \Leftrightarrow (c)$ Since A is polyform, these assertions follow from 14.23.

$(b) \Rightarrow (d)$ Since A is polyform, $\widetilde{A}$ is PSP by 14.11. We show that $\widetilde{A}$ is fully idempotent. Any element in $\widetilde{A}$ can be written as $a = \sum_{i=1}^{k} a_i e_i$, for some $a_i \in A$ and pairwise orthogonal $e_i \in B$. Put $K = M(\widetilde{A})a$ and $K_i = M(A)a_i$. Clearly $K = \sum_{i=1}^{k} K_i e_i B$. Since A is fully idempotent, $K_i = M(A)K_i^2$. Hence

$$M(\widetilde{A})K^2 = \sum_{i=1}^{k} M(A)K_i^2 e_i B = \sum_{i=1}^{k} K_i e_i B = K,$$

and $\widetilde{A}$ is fully idempotent. So $\widetilde{A}$ is biregular by $(a) \Leftrightarrow (b)$.

$(d) \Rightarrow (a)$ Consider $a \in A$. By assumption, $M(\widetilde{A})a = \widetilde{A}\varepsilon(a)$. Obviously, $M(\widetilde{A}) = M(A)B$ and $M(\widetilde{A})a = M(A)aB$. Put $c = \varepsilon(a)$. Clearly $\varepsilon(c) = c$ and $c \in M(A)aB$. By 32.5, there exist $a_1, \ldots, a_k \in M(A)a$ and pairwise orthogonal $e_1, \ldots, e_k \in B$, such that

$$c = \sum_{i=1}^{k} a_i e_i \text{ and } c = \varepsilon(c) = \sum_{i=1}^{k} e_i.$$

Therefore $ce_i = a_i e_i$, $ce_i = e_i$ and $(1 - a_i)e_i = 0$, for all $i = 1, \ldots, k$. Put

$$b = (...(((1 - a_1)(1 - a_2))(1 - a_3)...)(1 - a_k).$$

Clearly $be_i = 0$ for all $i = 1, \ldots, k$. Hence $bc = 0$. Since $a_i \in M(A)a$, $b = 1 + d$ where $d \in M(A)a$. Further since $ac = a\varepsilon(a) = a$ and $d \in M(A)a$, $dc = d$. So

$$0 = bc = (1 + d)c = c + dc = c + d \text{ and } \varepsilon(a) = c = -d \in M(A)a.$$

Hence $M(A)a \subset A\varepsilon(a) \subset M(A)a$ and $M(A)a = A\varepsilon(a)$.

$(a) \Rightarrow (e)$ We have shown above that A is a PSP ring. Since A is biregular, every prime ideal is maximal.

$(e) \Rightarrow (a)$ By 34.4, $\widetilde{A}$ is a PSP algebra and by 34.5, $\widetilde{A}_x$ is a prime ring, for every maximal ideal x in the Boolean ring of central idempotents of $\widetilde{A}$. According to 32.5, $\widetilde{A}_x \simeq A/(A \cap \widetilde{A}x)$. Hence $A \cap \widetilde{A}x$ is a prime ideal in A and - by hypothesis - it is in fact maximal. So $\widetilde{A}_x$ is a simple ring. Now it follows from 30.4 that $\widetilde{A}$ is a biregular ring. $\square$

Combining our results 14.13, 14.14 and 26.4 we have:

34.8 Central closure as Azumaya ring.

Let A be a semiprime ring with unit, $T = End_{M(A)}(\widehat{A})$ and $\widehat{A}$ the central closure of A. Then the following are equivalent:

(a) $\widehat{A}$ is an Azumaya ring;

(b) $\widehat{A}$ is a PSP ring and the module ${}_{M(\widehat{A})}\widehat{A}$ is finitely presented in $\sigma[\widehat{A}]$;

(c) *$\widehat{A}$ is a biregular ring and the module ${}_{M(\widehat{A})}\widehat{A}$ is projective in $\sigma[\widehat{A}]$;*

(d) *the module ${}_{M(\widehat{A})}\widehat{A}$ is a generator in $\sigma[\widehat{A}]$;*

(e) *$M(\widehat{A})$ is a dense subring of $End_T(\widehat{A})$;*

(f) *for any $m_1, \ldots, m_n, m \in A$ there exist $r_1, \ldots, r_k \in M(A)$ such that for $s_1, \ldots, s_n \in M(A)$ the relations*

$$\sum_{l=1}^{n} s_l r_j m_l = 0 \quad \text{for } j = 1, \ldots, k,$$

imply $s_1 m \in \mathcal{T}_{Um_1}(A)$ for $U = An_{M(A)}(m_2, \ldots, m_n)$;

(g) *A is a PSP ring and for any $m_1, \ldots, m_n \in A$ there exist $r_1, \ldots, r_k \in M(A)$ such that for $s_1, \ldots, s_n \in M(A)$ the relations*

$$\sum_{l=1}^{n} s_l r_j m_l = 0 \quad \text{for } j = 1, \ldots, k,$$

imply $s_1 m_1 \in \mathcal{T}_{Um_1}(A)$ for $U = An_{M(A)}(m_2, \ldots, m_n)$.

34.9 Example. There exists a biregular (PI-) algebra A with unit, whose central closure is not biregular. So A is PSP as an $M(A)$-module but $\widehat{A}$ is not PSP.

Proof. Let F be field of non-zero characteristic p, $K = F(X)$ the field of the rational functions and $G = F(X^p)$ a subfield of K. Consider the embedding

$$\alpha : K \to M_p(G) := End_G(K), \quad a \mapsto L_a.$$

Denote by $Q = M_p(G)^{\mathbb{N}}$ and $I = M_p(G)^{(\mathbb{N})}$. We have an embedding

$$\psi : K \to M_p(G)^{\mathbb{N}} \subset Q, \quad k \mapsto \{\alpha(k)\}_{\mathbb{N}}.$$

Clearly I is an ideal in Q. Put $A = I + \psi(K)$. Obviously, A is a biregular ring with unit, whose maximal right ring of quotients is Q, and the centre T of Q is the extended centroid of A.

Hence $R = AT$ is the central closure of A. I is an ideal in R and R/I is a commutative ring. Let $y \in Q$ denote an element all whose coefficients are equal to X and put $z = \psi(X)$. Then $y^p = z^p$. Therefore $y - z + I$ is a non-zero nilpotent element in the commutative ring R/I. Hence R cannot be biregular. □

Remark. Similar to Goldie's theorem for associative rings, the following was proved by Zelmanov (with different techniques, see [281, Theorem 1]): Let A be a (semi-) prime Jordan algebra satisfying the annihilator chain condition and not containing an infinite direct sum of inner ideals. Then A is an order in a (semi-) simple Jordan algebra with dcc for inner ideals.

34.10 Exercises.

(1) Call an algebra A a left PSP algebra *if A is PSP a an $L(A)$-module, and call A* left strongly prime *if it is strongly prime as an $L(A)$-module. Prove ([76, 9.2]):*
(i) Assume A is left PSP and no non-zero left ideal of A annihilates the right nucleus $N_r(A)$. Then A is a semiprime ring.
(ii) Assume A is left strongly prime and no ideal of A annihilates A. Then A is a prime ring.

(2) Let A be an associative algebra with unit.

(i) Prove that the following assertions are equivalent ([77, 4.4]):
(a) A is regular and biregular;
(b) A is regular and left PSP;
(c) A is left PSP and all prime factor algebras are regular;
(d) the idempotent closure $\widetilde{A}$ is regular and biregular.

(ii) Assume that A is reduced. Prove that A is regular if and only if every prime factor algebra is regular ([77, 4.5]).

(3) Let A be an associative algebra with unit. A is called u-regular (unit-regular) *if, for every $r \in A$, there exists an invertible $s \in A$ with $r = rsr$.*

Prove ([77, 4.6, 4.7]):

(i) Let A be biregular and $\mathcal{X}$ the set of maximal ideals of the Boolean ring of central idempotents of A. Assume that A_x is u-regular, for all $x \in \mathcal{X}$. Then A is a u-regular algebra.

(ii) The following assertions are equivalent:
(a) A is u-regular and biregular;
(b) A is u-regular and left PSP;
(c) A is a left PSP ring whose prime factor rings are u-regular;
(d) the idempotent closure $\widetilde{A}$ is u-regular and biregular.

(4) Let A be an associative algebra with unit.

A is said to be of bounded index (of nilpotency) *if there exists $n \in I\!N$ such that $a^n = 0$ for each nilpotent $a \in A$. The least such $n \in I\!N$ is called the* index of A.

Prove ([151, 53]):

(i) If A is semiprime and $n \in I\!N$, the following are equivalent:
(a) A is of bounded index n;
(b) for every subset $X \subset A$, $An_A(X^n) = An_A(X^{n+1})$;
(c) for every $x \in A$, $An_A(x^n) = An_A(x^{n+1})$;
(d) for each $x \in A$ and each ideal $I \subset A$, $An_A(x^n I) = An_A(x^{n+1} I)$.

(ii) Suppose A has index $n \in I\!N$, and denote by N the sum of all nilpotent ideals of A. Then for any nil subring $X \subset A$, $X^n \subset N$.

(5) Let A be an associative semiprime algebra of index $n \in \mathbb{N}$. Prove ([151, 53]):
(i) Every nil subring of A is nilpotent.
(ii) A has no non-zero nilpotent left (or right) ideal.
(iii) A is left (and right) non-singular.
(iv) If A is prime then A is left strongly prime.
(v) If A is left strongly semiprime, then A has acc on left (and right) annihilators.

(6) Let A be an associative semiprime algebra of index $n \in \mathbb{N}$.

(i) Prove that the following are equivalent ([74, Theorem 4]):
(a) Every left ideal in A is idempotent;
(b) every right ideal in A is idempotent;
(c) for each $a \in A$, there exists an idempotent $e \in M(A)a$ such that $ae = a$;
(d) for any ideal $L \subset A$ and any $a_1, \ldots, a_m \in A$ and $b_1, \ldots, b_n \in A$, there exists an idempotent $e \in L$ such that $a_i e = ea_i$ and $b_j e = b_j = eb_j$, for all $i \leq m$, $j \leq n$.

(ii) Suppose that for every minimal prime ideal $P \subset A$, A/P is a regular ring. Show that for each $a \in A$ there exists $x \in A$ such that $a^n x a^n = a^n$ ([74, Lemma 7]).

(7) Let A be an associative central R-algebra with unit which is an algebraic extension *of R (each element satisfies a non-zero polynomial with coefficient in R).*

Prove ([56]):

If A is semiprime and has acc on left (or right) annihilators, then A has large centre and the central closure $\widehat{A}$ is a left semisimple algebra.

References. Armendariz-Hajarnavis [56], Beidar-Wisbauer [75, 76, 77], Hannah [151], Wisbauer [273], Zelmanov [281].

35 Prime and strongly prime rings

1.Lemma. 2.Extended centroid of prime algebras. 3.Tensor products with central closures. 4.Multiplication ideal of central closures. 5.Prime algebras with non-zero socle. 6.Strongly prime algebras. 7.Characterization of strongly prime rings. 8.More characterizations of left strongly prime rings. 9.Proposition. 10.Prime algebra with large centroids. 11.Prime algebras with large centres. 12.Strongly prime subgenerators. 13.Alternative subgenerators. 14.Central orders in central simple algebras. 15.Subgenerator with large centroid. 16.Corollary. 17.Centroid of strongly affine algebras. 18.Strongly affine ideals and multiplication algebra. 19.The strongly affine dimension of strongly affine algebras. 20.Exercises.

Applying the constructions in 32.1 and 32.2 we obtain:

35.1 Lemma. *Assume A is a prime algebra. Then:*

(1) The centroid of A is a prime ring and the extended centroid $End_{M(A)}(\widehat{A})$ is a field.

(2) The central closure $\widehat{A}$ of A is a prime algebra.

Proof. (1) A prime ring A is a uniform $M(A)$-module and the assertion follows from 11.1 and 32.1(2).

(2) For any ideals $U, V \subset \widehat{A}$, $U \cap A$ and $V \cap A$ are ideals in A, and $UV = 0$ implies $(U \cap A)(V \cap A) = 0$. Hence $U = 0$ or $V = 0$, i.e., $\widehat{A}$ is prime. $\square$

In general, the extended centroid of a prime algebra need not be an algebraic extension of the centroid (see Exercise 35.20(4)). However, under certain conditions it may coincide with the quotient field of the centroid (see 35.10).

If A is a prime algebra, every ideal generated by $b \in A$ is essential, and hence any $h \in End_{M(A)}(\widehat{A})$ is uniquely determined by its restriction to $M(A)b$, i.e., by the image of b. Since $(A)h \cap A \neq 0$, we may choose $b \in A$ such that $a := (b)h \in A$. Hence we can identify the elements of the extended centroid with certain pairs $(a, b) \in A \times A$.

35.2 Extended centroid of prime algebras.

For any prime algebra A, consider the set

$$\mathcal{C}' := \{(a, b) \in A \times (A \setminus 0) \mid a = 0 \text{ or } An_{M(A)}(a) = An_{M(A)}(b)\},$$

with an equivalence relation

$$(a, b) \sim (a', b') \text{ if there exist } \mu, \nu \in M(A) \text{ such that } \mu b = \nu b' \neq 0 \text{ and } \mu a = \nu a'.$$

On the set of equivalence classes $\mathcal{C} := \mathcal{C}'/\sim$ we define operations

$$\begin{array}{rcl}[a,b]+[c,d] & := & [\beta a+\delta c, \beta b], \text{ with } \beta b = \delta d \neq 0,\ \beta,\delta \in M(A);\\ [a,b]\cdot[c,d] & := & \begin{cases}[\gamma c, \alpha b], & \text{with } \alpha a = \gamma d \neq 0, \text{ if } a \neq 0,\ \alpha,\gamma \in M(A),\\ [0,b], & \text{if } a = 0.\end{cases}\end{array}$$

It is routine work to check that with this operations $\mathcal{C}$ is a ring, in fact a field as becomes evident from the observation

Property (1). *$\mathcal{C}$ is isomorphic to the extended centroid $T := End_{M(A)}(\widehat{A})$.*

Proof. For $h \in T$ and non-zero $b, b' \in A$, $((b)h, b)$ and $((b')h, b')$ are equivalent elements in $\mathcal{C}'$. Hence we have a map

$$\varphi : T \to \mathcal{C}, \quad h \mapsto [(b)h, b].$$

On the other hand, every $(a,b) \in \mathcal{C}'$ defines an $M(A)$-homomorphism

$$h' : M(A)b \to M(A)a, \quad \mu b \mapsto \mu a,$$

which extends uniquely to $h : \widehat{A} \to \widehat{A}$. Obviously, any pair of elements equivalent to (a,b) yield the same $h : \widehat{A} \to \widehat{A}$.

So φ is a bijection and it is easy to verify that it is a ring isomorphism. $\square$

Property (2). *If A is an associative prime algebra, then*

(i) $\mathcal{C}' = \{(a,b) \in A \times A \mid axb = bxa,$ *for all* $x \in A\}$;

(ii) the equivalence relation $\sim$ on $\mathcal{C}'$ is given by

$$(a,b) \sim (a',b') \text{ if } axb' = bxa', \text{ for all } x \in A;$$

(iii) the operations on $\mathcal{C}$ are given by

$$\begin{array}{rcll}[a,b]+[c,d] & = & [axd+bxc, bxd], & \\ [a,b]\cdot[c,d] & = & [axc, bxd], & \text{with } bxd \neq 0,\ x \in A.\end{array}$$

Proof. (*i*) For $(a,b) \in \mathcal{C}'$ and $a \neq 0$,

$$L_{ax} - R_{xa} \in An_{M(A)}(a) = An_{M(A)}(b), \text{ i.e., } axb = bxa, \text{ for all } x \in A.$$

Let $(a,b) \in A \times A$ with $axb = bxa$, for all $x \in A$. For $\mu = \sum L_{x_i} R_{y_i} \in An_{M(A)}(b)$,

$$(\mu a)xb = \sum (x_i a y_i)xb = \sum (x_i b y_i)xa = 0, \text{ for all } x \in A,$$

and hence $\mu a = 0$ (by primeness of A). Applying the same argument for $a \neq 0$ we have $An_{M(A)}(a) = An_{M(A)}(b)$.

(*ii*) Let $(a,b),(a',b') \in \mathcal{C}'$ be such that $axb' = b'xa$, for all $x \in A$. Choose $x \in A$ such that $bxb' \neq 0$ and put $\mu = R_{xb'}$, $\nu = L_{bx}$. Then $\mu b = bxb' = \nu b'$, and

$$\mu a = R_{xb'}a = axb' = b'xa = L_{bx}a' = \nu a'.$$

This shows $(a,b) \sim (a',b')$.

(*iii*) In the definition of the operations on $\mathcal{C}$ put

$$\beta = R_{xd},\ \delta = L_{bx},\ \alpha = R_{xd} \text{ and } \gamma = L_{ax}, \text{ for suitable } x \in A. \qquad \square$$

There are nice properties concerning tensor products with central closures which we are going to display now.

35.3 Tensor products with central closures.
Let A be a prime algebra, $T = End_{M(A)}(\widehat{A})$ and B any T-algebra.

(1) If B has no absolute zero-divisors, then any non-zero ideal $I \subset \widehat{A} \otimes_T B$ contains a non-zero element $a \otimes b$, where $a \in \widehat{A}$ and $b \in B$.

(2) If B is (semi-) prime then $\widehat{A} \otimes_T B$ is (semi-) prime.

(3) If B is prime then $\widehat{A} \otimes_T \widehat{B}$ is self-injective as bimodule with centroid $End_{M(B)}(\widehat{B})$.

Proof. (1) Any non-zero $x \in I$ is of the form $\sum_{i=1}^k a_i \otimes b_i$, and we may assume $a_1 \notin \sum_{i=2}^k a_i T$. By 32.3, we can choose $\mu \in M(A)$ such that $\mu a_1 \neq 0$ and $\mu a_i = 0$, for $i = 2, \ldots, k$. Moreover, pick $c \in B$ with $cb_1 \neq 0$. Then

$$I \ni (\mu \otimes L_c)(\sum_{i=1}^k a_i \otimes b_i) = \mu a_1 \otimes cb_1 \neq 0.$$

(2) Consider non-zero ideals $I, J \subset \widehat{A} \otimes_T B$ with $IJ = 0$. Applying (1), choose non-zero $a \otimes b \in I$ and $c \otimes d \in J$. Then

$$(M(\widehat{A})a)(M(\widehat{A})c) \otimes (M(B)b)(M(B)d) \subset M(\widehat{A} \otimes B)(a \otimes b) \cdot M(\widehat{A} \otimes B)(c \otimes d) \subset IJ = 0.$$

Since $(M(\widehat{A})a)(M(\widehat{A})c) \neq 0$ we conclude $(M(B)b)(M(B)d) = 0$, a contradiction to B being prime.

The same argument applies for B semiprime.

(3) Let $I \subset \widehat{A} \otimes_T \widehat{B}$ be any ideal and $f : I \to \widehat{A} \otimes_T \widehat{B}$ a non-zero $M(\widehat{A} \otimes_T \widehat{B})$-homomorphism. $Im f \subset \widehat{A} \otimes_T \widehat{B}$ is a non-zero ideal and hence contains a non-zero element $a \otimes b$, $a \in \widehat{A}$, $b \in \widehat{B}$. The set

$$U := \{u \in I \,|\, (u)f \in M(\widehat{A} \otimes_T \widehat{B})(a \otimes b)\},$$

is an ideal in $\widehat{A} \otimes_T \widehat{B}$ and there is a non-zero element $x \otimes y$, $x \in \widehat{A}$, $y \in \widehat{B}$ (by (1)). Without restriction assume $U = M(\widehat{A} \otimes_T \widehat{B})(x \otimes y)$. We have

$$(x \otimes y)f = \mu a \otimes \nu b, \text{ for some } \mu \in M(\widehat{A}) \text{ and } \nu \in M(\widehat{B}).$$

Since $\widehat{A} \otimes_T \widehat{B}$ is prime (by (2)), f is a monomorphism. So it is obvious that x and μa have the same annihilator in $M(\widehat{A})$, and y and νb have the same annihilator in $M(\widehat{B})$. Hence (by 35.2) there are elements $g \in T$ and $h \in End_{M(B)}(\widehat{B})$ with $(x)g = \mu a$ and $(y)h = \nu b$. Tensoring the two maps we have

$$(x \otimes y)g \otimes h = \mu a \otimes \nu b = (x \otimes y)f.$$

So $f|_U$ can be extended to an isomorphism of $\widehat{A} \otimes_T B$, which also extends f. This means that $\widehat{A} \otimes_T \widehat{B}$ is self-injective as a bimodule.

The construction shows that every endomorphism of $\widehat{A} \otimes_T \widehat{B}$ is of the form

$$g \otimes h \in T \otimes_T End_{M(B)}(\widehat{B}) \simeq End_{M(B)}(\widehat{B}). \qquad \square$$

In particular we obtain for associative algebras:

35.4 Multiplication ideal of central closures.

Let A be an associative prime algebra and $T = End_{M(A)}(\widehat{A})$. Then the canonical map (2.5)

$$\widehat{A} \otimes_T \widehat{A}^o \to M^*(\widehat{A}),\ a \otimes b \mapsto L_a R_b,$$

is an isomorphism.

Proof. Since the map is always surjective it remains to show that its kernel is zero. If this is not the case it contains a non-zero element $a \otimes b$ (by 35.3). This means $aAb = 0$, a contradiction to A being prime. $\square$

As an application of 32.6 we notice:

35.5 Prime algebras with non-zero socle.

Let A be a prime algebra with $T = End_{M(A)}(\widehat{A})$ and $B = Soc_{M(A)}A \neq 0$. Then:

(1) B is equal to $Soc_{M(\widehat{A})}\widehat{A}$.

(2) $T = End_{M(A)}(B)$.

(3) The image of the canonical map $M(A) \to End_T(B)$ is dense.

(4) If A is associative, then B is a simple ring.

(5) If $\widehat{A}$ is simple, then $B = A = \widehat{A}$.

(6) If B contains a central element then $B = A$ (hence A is simple).

Proof. (1) and (2) follow from 32.6.

(3) Apply the Density Theorem (5.4) to the simple $M(A)$-module B.

(4) Let I be an ideal in the ring B. Then BIB is an ideal in A and hence $I \supset BIB = B$.

(4),(5) and (6) are consequences of (1). □

We are now going to explore which conditions on A make $\widehat{A}$ a simple algebra without any additional properties. Recall that an algebra A is said to be *strongly prime* if A is strongly prime as an $M(A)$-module.

35.6 Strongly prime algebras.

For any algebra A, the following conditions are equivalent:

(a) A is a strongly prime algebra;

(b) $\widehat{A}$ is generated (as an $M(A)$-module) by any non-zero ideal of A;

(c) A is a prime ring and A is contained in any ideal of the central closure $\widehat{A}$;

(d) A is a prime ring and the central closure $\widehat{A}$ is a simple ring;

(e) for $a \in A$, $0 \neq b \in A$ there are $\mu_1, \ldots, \mu_k \in M(A)$ such that

$$\bigcap_{i=1}^{k} An_{M(A)}(\mu_i(b)) \subset An_{M(A)}(a).$$

If A has a unit these conditions are equivalent to:

(f) For $0 \neq b \in A$ there are $\mu_1, \ldots, \mu_k \in M(A)$ such that

$$\bigcap_{i=1}^{k} An_{M(A)}(\mu_i(b)) \subset An_{M(A)}(1).$$

Proof. Using the fact that the central closure $\widehat{A}$ is a simple ring if and only if $\widehat{A}$ has no fully invariant $M(A)$-submodules, the equivalence of the first five conditions is obtained by 13.3.

$(e) \Rightarrow (f)$ is trivial.

$(f) \Rightarrow (a)$ The condition implies that A is subgenerated by the ideal $M(A)b$. This assures (a). □

Recall that for any $X \in \sigma[A]$, we write $\mathcal{S}(X)$ for the largest A-singular submodule of X. Since a semiprime ring A is non-A-singular (see 32.1) we obtain from 13.5 the following

35.7 Characterization of strongly prime rings.

For a semiprime ring A, the following properties are equivalent:

(a) *A is a strongly prime ring (in $\sigma[A]$);*

(b) *any module in $\sigma[A]$ is A-singular or a subgenerator in $\sigma[A]$;*

(c) *every non-A-singular module in $\sigma[A]$ is an absolute subgenerator in $\sigma[A]$;*

(d) *for every non-zero $N \in \sigma[A]$,*

$$\mathcal{S}(N) = \bigcap \{K \subset N \mid N/K \text{ is an absolute subgenerator in } \sigma[A]\};$$

(e) *there exists an absolute subgenerator in $\sigma[A]$.*

An associative ring A is an object of $\sigma[A]$ and of A-*Mod*. Accordingly A can be strongly prime in each of these categories. Left strongly prime rings A (i.e., A strongly prime in A-*Mod*) were characterized in 13.6. Combining this with 35.7 we arrive at a description of left strongly prime associative rings by properties of two-sided modules.

35.8 More characterizations of left strongly prime rings.

For an associative semiprime ring A with unit, the following are equivalent:

(a) *A is a left strongly prime ring;*

(b) *any module $N \in \sigma[A]$ is A-singular (in $\sigma[A]$) or is a subgenerator in A-Mod;*

(c) *every non-A-singular module in $\sigma[A]$ is an absolute subgenerator in A-Mod;*

(d) *for every non-zero $N \in \sigma[A]$,*

$$\mathcal{S}(N) = \bigcap \{K \subset N \mid \begin{array}{l} K \text{ is a sub-bimodule and} \\ N/K \text{ is an absolute subgenerator in } A\text{-}Mod\}; \end{array}$$

(e) *there exists a module in $\sigma[A]$ which is an absolute subgenerator in A-Mod.*

Proof. $(a) \Rightarrow (b)$ Let A be left strongly prime. Then A is left non-singular and, by 33.2, modules which are not A-singular in $\sigma[A]$ are not singular as left A-modules. Hence, by 13.6, they are subgenerators in A-*Mod*.

$(b) \Rightarrow (c)$ By the same argument this follows from 13.6.

$(c) \Rightarrow (d)$ By 35.7, $\mathcal{S}(N)$ is the intersection of those $K \subset N$, for which N/K is an absolute subgenerator in $\sigma[A]$. Then N/K is non-A-singular and hence an absolute subgenerator in A-*Mod*.

$(d) \Rightarrow (e) \Rightarrow (a)$ are obvious since $\mathcal{S}(A) = 0$. $\square$

We observe that any algebra A, which has no absolute zero-divisors, satifies condition $(*)$ of 13.1 as an $M(A)$-module:

For any ideal $U \subset A$ and $0 \neq u \in U$, the left multiplication $L_u : A \to U, a \mapsto ua$, has the property

$$0 \neq L_u \in M(A) \ \text{ and } L_u(A/U) = 0.$$

Since prime algebras have no absolute zero-divisors we obtain from 13.1:

35.9 Proposition. *For any ring A, the following properties are equivalent:*

(a) ${}_{M(A)}A$ is a prime module;

(b) $M(A)$ is a prime ring.

${}_{M(A)}A$ being a prime module evidently implies that A is a prime ring. The converse conclusion only holds in special cases.

Recall that a module is *compressible* if it can be embedded in every non-zero submodule.

35.10 Prime algebra with large centroids.

For any algebra A, $C(A) = End_{M(A)}(A)$ and $T = End_{M(A)}(\widehat{A})$, the following are equivalent:

(a) A is (finitely) cogenerated (as an $M(A)$-module) by any of its non-zero ideals;

(b) A is a compressible $M(A)$-module;

(c) A has large centroid and one of the following holds:

(i) A is a prime algebra;

(ii) $M(A)$ is a prime algebra;

(iii) A is a strongly prime algebra;

(iv) $\widehat{A}$ has no fully invariant $M(A)$-submodules;

(v) every non-zero ideal is rational in A (as an $M(A)$-module).

Under these conditions, $\widehat{A}$ is a simple algebra and T is the quotient field of $C(A)$.

Proof. $(a) \Rightarrow (b)$ Assume A is cogenerated by any non-zero ideal. Then A is a prime $M(A)$-module, A is a prime algebra and $C(A)$ is a commutative prime ring. For an ideal I of A, any non-zero $\alpha \in Hom_{M(A)}(A, I) \subset c(A)$ is a monomorphism, i.e., $A \simeq A\alpha \subset I$.

$(c.i) \Rightarrow (c.iii)$ T being a field, for any non-zero ideal I of A,

$$I \cdot T \supset A \cdot Hom_{M(A)}(A, I) \cdot T = A \cdot T = \widehat{A}.$$

This shows that A is a strongly prime $M(A)$-module.

From this it is easy to see that any condition in (c) characterizes A as a strongly prime algebra (see 13.3, 13.1).

$(c) \Rightarrow (b) \Rightarrow (a)$ are obvious (since T is a field).

The assertion about T follows from 11.5. □

A prime algebra has no absolute zero-divisors and so large centre implies large centroid. Hence we have as a special case of 35.10:

35.11 Prime algebras with large centres.

Let A be any prime algebra with centre $Z(A)$ and assume the central closure $\widehat{A}$ has a unit. Then the following are equivalent:

(a) A has large centre;

(b) T is isomorphic to the quotient field of $Z(A)$.

In this case, $\widehat{A}$ is simple, A is strongly prime and $M(A)$ is a prime algebra.
If A has a unit, (a) and (b) are equivalent to:

(c) A is a compressible $M(A)$-module;

(d) for any $0 \neq b \in A$, there exists $\mu \in M(A)$ such that

$$An_{M(A)}(\mu(b)) = An_{M(A)}(1).$$

Proof. $(a) \Rightarrow (b)$ Assume A has large centre. Then for every non-zero $h \in T$ there exist non-zero $a, b \in Z(A)$ such that $(b)h = a$. The elements of T are uniquely determined by suitable pairs in $Z(A) \times Z(A)$ (see 35.2). All central elements have the same annihilator in $M(A)$, and in $Z(A) \times Z(A)$, $(a, b) \sim (a', b')$ if and only if $ab' = b'a$. This shows that T is just the quotient field of $Z(A)$.

$(b) \Rightarrow (a)$ Denote by e the unit of the central closure $\widehat{A}$. For every ideal $U \subset A$, $UT = \widehat{A}$ and hence we have

$$e = u_1 t_1 + \cdots u_k t_k, \text{ for } u_i \in U, t_i \in T.$$

Since T is the quotient field of $Z(A)$, there exists a non-zero $d \in Z(A)$ such that $t_i d \in Z(A)$, for $i = 1, \ldots, k$. By this,

$$d = ed = u_1 t_1 d + \cdots u_k t_k d \in U \cap Z(A).$$

Every non-zero ideal of $\widehat{A}$ contains a non-zero central element which is invertible in $\widehat{A}$. Hence $\widehat{A}$ is a simple algebra.

Now assume A has a unit.

$(a) \Leftrightarrow (c)$ follows from 35.10.

$(c) \Leftrightarrow (d)$ For any monomorphism $\alpha : A \to M(A) \cdot b$ and $(1)\alpha = \mu b$, $\mu \in M(A)$, we have $An_{M(A)}(1) = An_{M(A)}(\mu b)$.

On the other hand, this condition implies $\mu b \in Z(A)$ and assures the existence of the monomorphism wanted. $\square$

For A a strongly prime subgenerator in $M(A)$-*Mod* we have:

35.12 Strongly prime subgenerators.

Let A be an algebra and $T = End_{M(A)}(\widehat{A})$. The following are equivalent:

(a) A is a strongly prime $M(A)$-module and $M(A) \in \sigma[A]$;

(b) $M(A)$ is subgenerated by every non-zero ideal of A;

(c) $M(A)$ is finitely cogenerated by every non-zero ideal of A;

(d) A is a prime algebra and $\widehat{A}$ is a finite dimensional, simple T-algebra.

Proof. $(a) \Leftrightarrow (b)$ Every non-zero ideal subgenerates A and A subgenerates $M(A)$.

$(b) \Leftrightarrow (c)$ This follows from the projectivity of $M(A)$.

$(a) \Leftrightarrow (d)$ By 35.6, $\widehat{A}$ simple is equivalent to A being strongly prime (as $M(A)$-module). By 13.1, $M(A) \in \sigma[A]$ is equivalent to $\widehat{A}$ being finite dimensional over T. $\square$

Using the structure theory of certain classes of algebras we obtain, for example:

35.13 Alternative subgenerators.

For an associative or alternative algebra A with unit satisfying $M(A) \in \sigma[A]$, the following properties are equivalent:

(a) A is a prime algebra;

(b) A is a strongly prime algebra;

(c) $M(A)$ is a prime algebra.

In this case, $\widehat{A}$ is a finite dimensional algebra over $T = End_{M(A)}(\widehat{A})$.

Proof. If A is a prime ring and $M(A) \in \sigma[A]$, $\widehat{A}$ is a finite dimensional prime associative or alternative algebra, hence it is simple. By 35.6, A is a strongly prime $M(A)$-module.

The other implications are evident. $\square$

The above theorem can be extended to appropriate wider classes of rings. It is well-known, for example, that associative prime PI-rings satisfy the conditions of the theorem. They also have large centroids (as in 35.10, 35.11). In fact we have:

35.14 Central orders in central simple algebras.

Let A be prime R-algebra with centre $Z(A)$ and assume $\widehat{A}$ has a unit. Put $T = End_{M(A)}(\widehat{A})$. Then the following are equivalent:

(a) A has large centre and is a subgenerator in $M(A)$-Mod;

(b) T is the quotient field of $Z(A)$ and $\widehat{A}$ is a finite dimensional, central simple T-algebra.

If A is associative, $(a), (b)$ are equivalent to:

(c) A is a PI-algebra.

Proof. $(a) \Leftrightarrow (b)$ is a special case of 32.11 and 35.12.

$(b) \Leftrightarrow (c)$ This is essentially Posner's Theorem (e.g., [36, Theorem 1.7.9]). □

The equivalence of (a) and (c) also holds for some non-associative algebras. For example, for a prime alternative algebra A which is not associative, the central closure $\widehat{A}$ is a Cayley-Dickson algebra which has dimension 8 over its centre, and $[X, Y]^4$ is a central polynomial for A ([36, C.33]).

It is shown in [235, Theorem 1.5] that for every (non-associative) prime algebra satisfying a *normal polynomial identity*, the central closure has finite dimension over its centroid. As mentioned before (remarks after 32.7), prime Jordan algebras with such identities have large centres.

In 35.11 and 35.14 we needed a unit in $\widehat{A}$ to describe orders in central simple algebras. This is not essential as we will see in our next result.

35.15 Subgenerator with large centroid.

Let A be a prime algebra with large centroid $C(A)$, $T := End_{M(A)}(A)$, and assume $M(A) \in \sigma[A]$. Then:

(1) $\widehat{A}$ is a central simple T-algebra of finite dimension, say $n \in \mathbb{N}$, and

$$M^*(\widehat{A}) = M(\widehat{A}) = End_T(\widehat{A}) \simeq T^{(n,n)}.$$

(2) T is the quotient field of $C(A)$ and of the centre of $M^*(A)$.

(3) $M^*(A)$ and $M(A)$ are central orders in $M(\widehat{A})$.

Proof. $\widehat{A}$ is a generator in $M(\widehat{A})$-*Mod* and $Q_{\max}(M(A) \simeq M(\widehat{A}) = End_T(\widehat{A})$ (see 32.9). By 35.10, T is the quotient field of $C(A)$ which is isomorphic to the centre of $M(A)$ (2.10).

The $C(A)$-algebra $M^*(A)$ is an ideal in $M(A)$ and hence $M^*(\widehat{A}) = M^*(A)T$ is an ideal in the simple algebra $M(\widehat{A})$, i.e., $M^*(\widehat{A}) = M^*(A)T = M(\widehat{A})$.

By 35.14 (applied to $M(A)$), $M^*(A)$ has non-trivial centre which is an ideal in $C(A)$ and hence has the same quotient field T. □

Based on the facts used in the proof above we can state:

35.16 Corollary. *For an algebra A, the following are equivalent:*

(a) A is prime with large centre and $M(A) \in \sigma[A]$;

(b) A is strongly prime, $M(A) \in \sigma[A]$ and T is the quotient field of $C(A)$;

(c) for the quotient field S of $C(A)$, AS ($\subset \widehat{A}$) is a central simple, finite dimensional S-algebra

Proof. $(a) \Rightarrow (b)$ is shown in 35.15. $(b) \Rightarrow (c)$ is obvious.

$(c) \Rightarrow (b)$ The simple algebra AS is self-injective and essential as $M(AS)$-submodule in $\widehat{A}$, hence $AS = \widehat{A}$. By 32.10, $M(A) \in \sigma[A]$.

$(b) \Rightarrow (a)$ Under the given conditions, $\widehat{A}$ is simple and finite dimensioanl over its centroid. It remains to show that A has large centroid.

As observed in the proof of 35.15, $M^*(A)$ is a prime (associative) PI-algebra. Now, for every non-zero ideal $I \subset A$, $An_{M^*(A)}(A/I)$ is a non-zero ideal in $M^*(A)$ and hence contains a non-zero central element γ (e.g., [36, 1.6.27]), yielding a non-zero $M(A)$-homomorphism

$$A \to I, \quad a \mapsto \gamma a.$$

□

Recall that A is an *affine R-algebra* if it is finitely generated as R-algebra. An affine algebra A is said to be *strongly affine* if it has large centroid and $M(A) \in \sigma[A]$.

Without loss of generality we will assume that any strongly affine prime R-algebra is faithful as R-module which implies that R is a domain.

It follows from Posner's theorem that an associative prime algebra is strongly affine if and only if it is an affine PI-algebra.

35.17 Centroid of strongly affine algebras.

Let A be a strongly affine prime R-algebra, K the quotient field of R and $T := End_{M(A)}(\widehat{A})$. Then:

(1) T is a finitely generated field extension of K.

(2) If A is simple, T is an algebraic extension of K and R.

Proof. (1) If A is a strongly affine prime R-algebra, then $AK \subset \widehat{A}$ is a strongly affine prime K-algebra and we may assume $AK = A$.

Let B be a T-basis for $\widehat{A}$, and $a_1, \ldots, a_n$ a generating set for the K-algebra A. Denote by S the subfield of T generated over K by the coefficients of all a_i and the structural constants of the T-algebra $\widehat{A}$ with respect to the basis B. Then

$$\widetilde{A} := \sum\nolimits_{b \in B} Sb$$

is an S-subalgebra of $\widehat{A}$ containing all a_i, so $A \subset \widetilde{A}$. For $b \in B$, there exists a non-zero $\alpha \in c(A)$ such that $b\alpha \in A \subset \widetilde{A}$. Since B is a basis for $\widetilde{A}$, this implies $\alpha \in S$. Now, for any $\gamma \in c(A)$ we have $(b)\alpha\gamma \in A$, hence $\alpha\gamma \in S$ and $\gamma \in S$. Thus, $C(A) \subset S$ and $S = T$, i.e., T is a finitely generated field extension of K.

(2) Since A is simple we have $\widehat{A} = A$ and $C(A) = T$. Now we denote by S not the subfield, but the K-sub*algebra* of T generated by the same elements as above. We again have $A \subset \sum_{b\in B} Sb$ and hence

$$\textstyle\sum_{b\in B} Sb = \sum_{b\in B} Tb, \text{ implying } S = T.$$

Hence the field T is a finitely generated algebra over K. It is well-known from field theory that this implies that T is algebraic over K (hence over R). □

Definition. An ideal P in an algebra A is called a *strongly affine ideal* if A/P is a strongly affine algebra.

In associative, alternative or Jordan strongly affine rings, any prime ideal is strongly affine. This need not be the case for arbitrary strongly affine algebras (as is shown by an example in [221]).

35.18 Strongly affine ideals and multiplication algebra.

Let P be a strongly affine prime ideal in an algebra A. Then:

(1) $U := An_{M^*(A)}(A/P)$ is a prime ideal in $M^*(A)$.

(2) For an ideal $Q \subset A$ properly containig P, $An_{M^*(A)}(A/P) \subset An_{M^*(A)}(A/Q)$ and this is a proper inclusion.

Proof. (1) $An_{M^*(A)}(A/P)$ is the kernel of the algebra morphism $M^*(A) \to M^*(A/P)$, induced by the projection $A \to A/P$ (see 2.3). By 35.15, $M^*(A/P)$ is a prime algebra which means that $An_{M^*(A)}(A/P)$ is a prime ideal.

(2) $An_{M^*(A)}(A/P) \subset An_{M^*(A)}(A/Q)$ is obvious.

Since P is prime we have $AQ \not\subset P$ and hence the above inclusion is proper. □

The above result can be used to get information about the length of chains of strongly affine prime ideals in any strongly affine prime algebra.

Definition. By the *strongly affine Krull dimension* of a strongly affine prime R-algebra A, denoted by $sa.dim\,A$, we mean the length of a maximal chain of strongly affine prime ideals in A.

For an associative strongly affine (PI) algebra B, every prime ideal is strongly affine and $sa.dim\,B = K.dim\,B$ is the *Krull dimension* of B, i.e., the length of a maximal chain of prime ideals in B.

For any commutative domains $R \subset S$, we denote by $tr.deg_R S$ the *transcendence degree* of S over R (see [36, 1.10]). Recall that for the quotient fields $\overline{R} \supset R$ and $\overline{S} \supset S$, $tr.deg_R S = tr.deg_{\overline{R}} \overline{S}$.

Combining 35.15 and 35.17 with an inequality given in [36, p. 84], we have

$$a.dim\, A \leq K.dim\, M^*(A) \leq tr.deg\,_R centre(M^*(A)) = tr.deg\,_R T < \infty.$$

It is shown in [221, Theorem 1.1] that these are in fact equalities:

35.19 The strongly affine dimension of strongly affine algebras.
For any strongly affine prime R-algebra A,

$$a.dim\, A = K.dim\, M^*(A) = tr.deg\,_R T.$$

Proof. Let K denote the quotient field of R. We have to show that for any $t < tr.deg\,_R T$, there exists a chain of strongly affine prime ideals $0 \subset P_1 \subset \cdots \subset P_t \subset A$. We will proceed by induction on t. For $t = 0$ there is nothing prove.

Assume $t \geq 1$.

In the algebra $Z := centre(M^*(A))$ we choose $c_1, \ldots, c_t$ algebraically independent over K, and put $S := K[c_1, \ldots, c_{t-1}] \setminus 0$. Consider the set

$$\mathcal{M} := \{I \subset A \,|\, I \text{ ideal and } sA \not\subset I \text{ for every } s \in S\}.$$

Then $0 \in \mathcal{M}$, $\mathcal{M}$ is inductively ordered (because A is finitely generated), and, by Zorn's Lemma, it contains a maximal element P. We will show that P is a non-zero strongly affine prime ideal in A.

Suppose $P = 0$. Then for any non-zero ideal $I \subset A$, there exists an $s \in S$ such that $sA \subset I$. This implies that the algebra AS^{-1} is a simple KS^{-1}-algebra. In this case, by 35.17, T is an algebraic extension of KS^{-1}, a contradiction, so $P \neq 0$.

Now consider ideals $I, J \subset A$, both containing P, and suppose $IJ \subset P$. Then there exist $s_1, s_2 \in S$ for which $s_1 A \subset I$ and $s_2 A \subset J$, so that $s_1 s_2 A^2 \subset IJ \subset P$. This implies

$$s_1 s_2 M^*(A) A \subset P,\ s_1 s_2 Z A \subset P, \text{ and finally } s_1 s_2 S A \subset P,$$

contradicting the choice of P. Thus P is a prime ideal in A.

Denote by $\overline{}$ the canonical maps $A \to A/P$ and $M^*(A) \to M^*(A/P)$, and put $\overline{A} := A/P$. It is easy to see that for any non-zero ideal $\overline{I} \subset \overline{A}$, there exists an $s \in S$ such that $\overline{s}\overline{A} \subset \overline{I}$. Hence $\overline{A}$ has large centroid and its central closure is a simple algebra whose dimension over its centroid is smaller than $dim_T \widehat{A}$, hence is finite. So $\overline{A}$ is a strongly affine prime algebra.

Since the elements $\overline{c}_1, \ldots, \overline{c}_{t-1}$ are algebraically independent over K, we have $tr.deg\,_K Z(M^*(\overline{A})) \geq t - 1$. Hence by the induction hypothesis, there exists a chain of strongly affine prime ideals $\overline{0} \subset \overline{P}_1 \subset \cdots \subset \overline{P}_{t-1} \subset \overline{A}$ from which we can pass to the chain

$$0 \subset P \subset P_1 \subset \cdots \subset P_{t-1} \subset A.$$

$\square$

Remarks. If A is an affine K-algebra and $dim_K \widehat{A} = n$, then we can choose a maximal chain of strongly affine prime ideals $0 \subset P_1 \subset \cdots \subset P_t \subset A$ of length $t = tr.deg\,_K T$ such that for each $i = 1, \ldots, t$, the central closure of A/P_i has dimension n over its centroid (see [221, Remark]).

For the construction of free (alternative, Jordan) algebras we refer to [221]. Central orders in non-associative quaternion algebras are studied in [184].

Of course, algebras with certain identities may allow a more precise structure theory. For example, prime (non-commutative) Jordan algebras are described in [108, 128]. The central closure of an alternative non-associative prime algebra is a Cayley-Dickson algebra (e.g., [41, Chap. 9]). Similarly it is shown in [134] that the central closure of any prime Malcev algebra which is not a Lie algebra is a 7-dimensional central simple algebra over the extended centroid.

35.20 Exercises.

(1) Let A and B be associative prime K-algebras, K a field. Let T and S denote their extended centroids and $\widehat{A}$, $\widehat{B}$ the central closures. Prove ([213]):

(i) If $\gamma(a \otimes b) = 0$ for $a \in \widehat{A}$, $b \in \widehat{B}$ and $\gamma \in T \otimes_K S$, then either $\gamma = 0$ or $a \otimes b = 0$.

(ii) The following are equivalent:

(a) $T \otimes_K S$ is a field;

(b) each non-zero ideal of $A \otimes B$ contains a non-zero element $a \otimes b$, $a \in A$, $b \in B$.

(iii) If A and B are simple K-algebras such that $T \otimes_K S$ is a field, then $A \otimes_K B$ is a simple K-algebra.

(2) Let K be a field and P the ideal in the polynomial ring $K[X, Y]$, generated by $XY - YX - X$. Prove that $K[X, Y]/P$ is a central, prime, self-injective (as bimodule) K-algebra which is not simple ([115, 9.7.2]).

(3) An ring A is said to be fully prime *if every ideal in A is prime. Prove for an associative ring A:*

(i) A is fully prime if and only if the ideals of A are idempotent and are linearly ordered by inclusion.

(ii) Let A be a fully prime ring. Then

– $Z(A)$ is a field or $Z(A) = 0$;

– for each $n \in I\!N$, the matrix ring $A^{(n,n)}$ is fully prime;

– if A satifies a polynomial identity, then A is a finite dimensional central simple algebra.

(4) (i) Consider the subsets of $I\!N \times I\!N \times Z\!\!Z$ ([114, § 7]),

$$\begin{aligned} E &:= \{(t, n, g) \in Z\!\!Z^3 \mid 0 \le n \le t\}, \\ F &:= \{(t, n, g) \in Z\!\!Z^3 \mid 0 \le n \le t,\ n(n+1) \le 2g \le n(2t - n + 1)\}. \end{aligned}$$

Define a product on E by

$$(t,n,g)\cdot(t',n',g') := (t+t', n+n', g+g'+tn').$$

Prove:

(α) *With this product E is a monoid with submonoid F.*

(β) *The lexicographic order on E, also denoted by $\leq$, is compatible with this product; in fact, for any $e, e', e'' \in E$,*

$$e' \leq e'' \Leftrightarrow e\cdot e' \leq e\cdot e'' \Leftrightarrow e'\cdot e \leq e''\cdot e\,.$$

(ii) Let K be any field. Denote by $A := K[F]$ and $B := K[E]$ the monoid algebras of F and E respectively. Obviously both algebras are integral domains (check the 'leading' coefficients) and we may assume $A \subset B$. Prove:

(α) *$(t,n,g) \in E$ commutes with the elements $(1,0,0)$, $(1,1,1) \in A$ if and only if $t = n = 0$.*

Conclude $Z(A) \simeq K$ and $Z(B) \simeq K[\mathbb{Z}]$ (i.e., $Z(B)$ is not algebraic over $Z(A)$).

(β) *For monomials $(t,n,g) \in B$,*

$$(t,n,g) = (t,n,\frac{n(n+1)}{2})\cdot(0,0,g-\frac{n(n+1)}{2}) \in A\cdot Z(B).$$

Conclude that B is A-generated as an $M(A)$-module.

(γ) *For any $(t',n',g') \in B$ there exists $t \in \mathbb{N}$ - only dependent on t' - such that*

$$(t,1,1)\cdot(t',n',g')\cdot(t,1,t) \in A,$$

where $(t,1,1)$, $(t,1,t) \in A$.

Conclude that $A \trianglelefteq B$ as an $M(A)$-module.

(δ) *B is contained in the central closure of A and the extended centroid of A is not algebraic over the centre of A.*

References. Blair-Tsutsui [84], Delale [114, 115], Erickson-Martindale-Osborne [124], Ferrero [129], Ferrero-Wisbauer [132], Lee-Waterhouse [184], Martindale [191], Nicholson-Watters [213], Polikarpov-Shestakov [221], Wisbauer [273, 274].

36 Localization at semiprime ideals

1.$\mathcal{T}_I$-dense ideals. 2.Quotient modules with respect to an ideal. 3.Quotient ring with respect to semiprime ideals.

We know from general torsion theory that for an arbitrary R-algebra A, any injective module in $\sigma[A]$ determines a hereditary torsion class. Here we will sketch the torsion theory derived from the injective hull of a factor module A/I, where I is a semiprime ideal of A. It will turn out that the quotient module of A with respect to such a torsion class has an algebra structure. Basic for this construction is the fact that in this setting the square of a dense ideal in A is again dense.

Let $I \subset A$ be an ideal in the algebra A. Consider the torsion class defined by the A-injective hull $\widehat{A/I}$ of A/I,

$$\mathcal{T}_I := \{X \in \sigma[A] \mid Hom_{M(A)}(X, \widehat{A/I}) = 0\}.$$

In general, A need not be $\mathcal{T}_I$-torsionfree, i.e., $\mathcal{T}_I(A) \neq 0$. By construction, A/I is $\mathcal{T}_I$-torsionfree and hence $\mathcal{T}_I(A) \subset I$.

Recall that $I \subset A$ is a *(semi) prime ideal* if A/I is a (semi) prime algebra.

36.1 $\mathcal{T}_I$-dense ideals. *Let $I, U \subset A$ be ideals.*

(1) The following assertions are equivalent:

(a) U is $\mathcal{T}_I$-dense in A, i.e., $Hom_{M(A)}(A/U, \widehat{A/I}) = 0$;

(b) for any $x \in A \setminus I$ and $y \in A$, there exists $\mu \in M(A)$ such that $\mu x \notin I$ and $\mu y \in U$.

(2) If I is semiprime then the following are equivalent:

(a) U is $\mathcal{T}_I$-dense in A;

(b) $U + I$ is $\mathcal{T}_I$-dense in A;

(c) $(U + I)/I \trianglelefteq A/I$;

(d) for any ideal $V \subset A$, $UV \subset I$ implies $V \subset I$.

Moreover, if U is $\mathcal{T}_I$-dense, then the ideal generated by U^2 is $\mathcal{T}_I$-dense in A.

(3) If I is prime then the following are equivalent:

(a) U is $\mathcal{T}_I$-dense in A;

(b) $U \not\subset I$.

In case U, V are $\mathcal{T}_I$-dense ideals, the ideal generated by UV is $\mathcal{T}_I$-dense in A.

Proof. (1) $(a) \Rightarrow (b)$ Consider a non-zero $f : A/U \to \widehat{A/I}$. Since $(A/U)f \cap A/I \neq 0$ we can find $x \in A \setminus I$ und $y \in A \setminus U$ satisfying $(y + U)f = x + I$. Then for any $\mu \in M(A)$, $\mu y \in U$ implies $\mu x \in I$.

$(b) \Rightarrow (a)$ Suppose (b) does not hold. Then there exist $x \in A \setminus I$ and $y \in A$ such that for $\mu \in M(A)$, $\mu y \in U$ implies $\mu x \in I$. From this we get a non-zero homomorphism

$$f : M(A)y + U \;/\; U \to M(A)x + I \;/\; I \subset A/I,\ \mu y + U \mapsto \mu x + I,$$

implying that $A/U \notin \mathcal{T}_I$.

(2) $(a) \Rightarrow (b)$ is obvious.

$(b) \Leftrightarrow (c)$ From $A/(U + I) \simeq (A/I)/(U + I/I)$ we conclude that $(U + I)$ is $\mathcal{T}_I$-dense in A if and only if $(U + I)/I$ is dense in A/I with respect to the torsion theory determined by the injective hull of A/I in $\sigma[A/I]$. By 32.1, this is equivalent to $(U + I)/I \trianglelefteq A/I$.

$(c) \Rightarrow (d)$ Assume $UV = 0$ for some ideal $V \subset A$. Then

$$(U + I/I)(V + I/I) = (UV + I)/I = 0.$$

However, an essential ideal in the semiprime ring A/I cannot be annihilated by a non-zero ideal. Hence $(UV + I)/I = 0$ and $UV \subset I$.

$(d) \Rightarrow (c)$ Assume $(U + I/I) \cap V/I = 0$ for some $I \subset V \subset A$. Then

$$(U + I/I)(V/I) = (UV + I)/I = 0$$

and $UV \subset I$. By our assumption, this implies $V \subset I$.

$(c) \Rightarrow (a)$ Assume $U \subset A$ is not $\mathcal{T}_I$-dense. Then there exist $U \subset V \subset A$ and a non-zero $f : V/U \to A/I$. Now $(U + I)V/U$ is in the kernel of f and therefore $(U + I/I)(V/U)f = 0$. Since A/I is semiprime this implies $f = 0$.

To prove the remaining assertion, let $U \subset A$ be a $\mathcal{T}_I$-dense ideal. Assume the ideal generated by U^2 (we also denote it by U^2) is not $\mathcal{T}_I$-dense. Then there exists a non-zero $M(A)$-morphism $f : W/U^2 \to A/I$, with $U^2 \subset W \subset U$. Obviously, $(U + I)(W/U^2)f = 0$ and hence

$$\{(U + I)/I \cap (W/U^2)f\}^2 = 0 \text{ in } A/I.$$

Since A/I is semiprime, the expression within the brackets has to be zero. Moreover, U being $\mathcal{T}_I$-dense in A implies that $(U + I)/I$ is $\mathcal{T}_I$-dense in A/I, whis is $\mathcal{T}_I$-torsionfree by definition. Hence by 9.6, $(U + I)/I \trianglelefteq A/I$ and we conclude $(W/U^2)f = 0$, a contradiction.

(3) If $U \subset A$ is not $\mathcal{T}_I$-dense there is a non-zero $g \in Hom_{M(A)}(W/U, A/I)$ with $U \subset W \subset A$. Now $(U + I/I)(W/U)g = 0$ implies $U + I/I = 0$, i.e., $U \subset I$.

If $U \subset I$, $A/U \to A/I$ is non-zero and hence U is not $\mathcal{T}_I$-dense in A.

From this characterization it is clear that the product of $\mathcal{T}_I$-dense ideals generates a $\mathcal{T}_I$-dense ideal in A. □

According to our definition after 9.14, for any $N \in \sigma[A]$, we have the quotient module with respect to $\mathcal{T}_I$,

$$Q_{\mathcal{T}_I}(N) := E_{\mathcal{T}_I}(N/\mathcal{T}_I(N)).$$

For our purpose we need a slight variation of this notion.

36.2 Quotient modules with respect to an ideal.
Let $I \subset A$ be any ideal. With the above notation put

$$(1) \qquad Q_I(N) := Tr(A, E_{\mathcal{T}_I}(N/\mathcal{T}_I(N)),$$

and call this the *quotient module of* N with respect to I.

For any A-generated module N, $N/\mathcal{T}_I(N) \subset Q_I(N)$ is a $\mathcal{T}_I$-dense submodule.

For any A-generated module L, $Hom_{M(A)}(L, Q_I(N)) = Hom_{M(A)}(L, E_{\mathcal{T}_I}(N/\mathcal{T}_I(N))$ and hence we obtain from 9.17,

$$(2) \quad Hom_{M(A)}(L, Q_I(N)) = Hom_{M(A)}(Q_I(L), Q_I(N)).$$

In particular, for $L = A$ we have

$$(3) \quad Q_I(N) = AHom_{M(A)}(A, Q_I(N)) = AHom_{M(A)}(Q_I(A), Q_I(N)),$$

and for $L = A = N$ we have

$$(4) \qquad Q_I(A) = AHom_{M(A)}(A, Q_I(A)) = AEnd_{M(A)}(Q_I(A)).$$

The relations given above are true for any ideal $I \subset A$. Again we have special properties for semiprime ideals I. The preceding results enable us to prove:

36.3 Quotient ring with respect to semiprime ideals.
Let $I \subset A$ be a semiprime ideal. Then:

(1) $T := End_{M(A)}(Q_I(A))$ is a commutative ring.

(2) $Q_I(A) = AT$ has a ring structure defined by (linear extension of)

$$(as) \cdot (bt) := (ab)st, \qquad for\ a, b \in A,\ s, t \in T.$$

$Q_I(A)$ is called the quotient ring with respect to I.

(3) *The canonical map* $A \to Q_I(A)$ *is a ring homomorphism with kernel* $\mathcal{T}_I(A)$.

(4) T *is the centroid of* $Q_I(A)$.

(5) *For any* A*-generated* $N \in \sigma[A]$, $Q_I(N)$ *is a* $Q_I(A)$*-generated* $Q_I(A)$*-bimodule.*

Proof. (1) Consider any $f, g \in T$ and put $\bar{A} = A/\mathcal{T}_I(A)$. Then $U := \bar{A}f^{-1} \cap \bar{A}g^{-1} \cap \bar{A}$ is a $\mathcal{T}_I$-dense ideal in $\bar{A}$. Moreover, for any $a, b \in U$,

$$(ab)fg = (a(bf))g = (ag)(bf) = (ab)gf.$$

This implies $U^2(fg - gf) = 0$ and $fg - gf$ annihilates the $\mathcal{T}_I$-dense ideal generated by U^2 (see 36.1). Hence $fg = gf$.

(2) $Q_I(A) = AT$ was pointed out in 36.2. The proof for the correctness of the definition is the same as in 32.2(1).

(3) and (4) are clear by the construction.

(5) By 36.2, $Q_I(N) = A Hom_{M(A)}(Q_I(A), Q_I(N))$. Since $Hom_{M(A)}(Q_I(A), Q_I(N))$ is a (left) T-module we have

$$Q_I(N) = Q_I(A) Hom_{M(Q_I(A))}(Q_I(A), Q_I(N)).$$

$\square$

If A is a semiprime algebra we may apply the above construction to $I = 0$. Then essential ideals are $\mathcal{T}_0$-dense and hence $E_{\mathcal{T}_0}(A) = \widehat{A}$ is A-generated and $\widehat{A} = Q_0(A)$ is the central closure of A as considered in 32.2.

References. Beachy [65], Beachy-Blair [66], Delale [115], Lambek [183], Wisbauer [278].

Chapter 10

Group actions on algebras

In this chapter we will apply the module theory developed in the previous parts to study the action of a group G on any algebra A. For this we will construct the skew group algebra $A'G$ and consider A as a module over the skew group algebras $L(A)'G$ and $M(A)'G$.

37 Skew group algebras

1.Group action on A. 2.Group action on $M^*(A)$. 3.Group action on $A \otimes_R A^o$. 4.Skew group algebra. 5.Module structures of A and $A'G$. 6.G-multiplication algebra. 7.Augmentation map. 8.Finiteness conditions. 9.G-invariant and fixed subsets. 10.Trace map.

As before R will denote an associative commutative ring with unit and A will be an R-algebra. $M(A)$ (resp. $M^*(A)$) denotes the multiplication algebra (ideal) of A, and $L(A)$ (resp. $L^*(A)$) stands for the left multiplication algebra (ideal) of A. Recall that we write module homomorphisms on the side opposite to the scalars.

Let G be a (multiplicative) group with unit e. If G is finite the order of G is denoted by $|G|$.

37.1 Group action on A.
We say that G *acts on* A if there is a group homomorphism

$$i : G \to Aut_R(A),$$

where $Aut_R(A)$ is the group of all R-algebra automorphisms of A.

We let G act from the left and write the action of $g \in G$ on $a \in A$ as

$${}^g a := i(g)(a).$$

In case $Ke\, i = \{e\}$ the action is called *faithful* and if $i(g) = id_A$ for every $g \in G$, the action is called *trivial*.

Throughout this section G will be a group acting on the R-algebra A.

37.2 Group action on $M^*(A)$.

As noticed in 2.3, every R-algebra automorphism $h : A \to A$ induces an R-algebra automorphism

$$h_m : M^*(A) \to M^*(A),\ L_a \mapsto L_{{}^h a},\ R_a \mapsto R_{{}^h a},$$

and hence the action of any group G on A yields an action of G on $M^*(A)$,

$$i_m : G \to Aut_R(M^*(A)),\ g \mapsto i(g)_m.$$

By definition, for every $\rho \in M^*(A)$, $a \in A$ and $g \in G$,

$$({}^g\rho)({}^g a) = {}^g(\rho a).$$

We will often make use of these facts without further reference.

From this it is obvious that an action of G on A also induces an action of G on the left multiplication ideal $L^*(A)$.

37.3 Group action on $A \otimes_R A^o$.

Every R-algebra automorphism $h : A \to A$ induces an R-algebra automorphism

$$h_e := h \otimes h : A \otimes_R A^o \to A \otimes_R A^o,$$

and an action $i : G \to Aut_R(A)$ yields an action

$$i_e : G \to Aut_R(A \otimes_R A^o), \quad g \mapsto i(g)_e.$$

If A is an associative algebra with unit, A is a left module over $A \otimes_R A^o$ and by definition, for every $a \otimes b \in A \otimes_R A^o$, $c \in A$ and $g \in G$,

$${}^g(a \otimes b)({}^g c) = {}^g(acb).$$

Moreover, the action of G on A also induces an action on the (extended) centroid. This will be outlined in 41.1 and 42.3.

37.4 Skew group algebra.

The *skew group algebra* $A'G$ is defined as the direct sum $A^{(G)}$, in which every element can be written uniquely as

$$x = \sum_{g \in G} a_g g \text{ with only finitely many nonzero } a_g \in A,$$

with multiplication given by

$$(ag) \cdot (bh) = a({}^g b)gh, \text{ for } a, b \in A \text{ and } g, h \in G.$$

Putting $r(ag) = (ra)g$, for $r \in R$, $A'G$ becomes an R-algebra which has A as a subalgebra by the embedding

$$A \to A'G, \; a \mapsto a \cdot e.$$

In case A has a unit 1, then $1 \cdot e$ is the unit of $A'G$ and

$$G \to A'G, \; g \mapsto 1 \cdot g,$$

is a semigroup monomorphism (since ${}^g1 = 1$ for any $g \in G$) and we may consider G as a subset of the right nucleus of $A'G$.

As easily seen, if A is associative, then $A'G$ is also an associative algebra. Other identities (e.g., alternativity) need not transfer from A to $A'G$.

If the action of G on A is trivial, $A'G$ is the *group algebra* $A[G]$, with the multiplication

$$(ag) \cdot (bh) = ab\,gh, \text{ for } a, b \in A \text{ and } g, h \in G.$$

Remark. By the ($M(A)$-) isomorphism $A^{(G)} \simeq A \otimes_R R[G]$, we may identify $A'G$ and $A \otimes_R R[G]$ with the multiplication

$$(a \otimes g) \cdot (b \otimes h) = a({}^gb) \otimes gh, \text{ for } a, b \in A \text{ and } g, h \in G.$$

On the other hand, $A \otimes_R R[G]$ with componentwise multiplication,

$$(a \otimes g) \cdot (b \otimes h) = ab \otimes gh, \text{ for } a, b \in A \text{ and } g, h \in G,$$

yields the group algebra $A[G]$.

It is clear that A and $A'G$ both are $R[G]$-modules.

37.5 Module structures of A and $A'G$.

By the fact that the action of G on A induces an action on $M^*(A)$ we may form the skew group algebra $M^*(A)'G$. The $M^*(A)$-module structure of A transfers to the direct sum $A^{(G)}$ and this yields an $M^*(A)'G$-module structure on $A'G$ and A by

$$\begin{array}{lcll} M^*(A)'G \times A'G & \to & A'G, & (\rho g, bh) \mapsto \rho({}^gb)gh, \\ M^*(A)'G \times A & \to & A, & (\rho g, b) \quad \mapsto \rho({}^gb), \end{array}$$

where the operations are linearly extended to finite sums.

The same mappings also yield an $L^*(A)'G$-modules structure on $A'G$ and A.

In particular, if A is an associative algebra with unit, we can identify $L^*(A)$ with A, and A is a left $A'G$-module with

$$A'G \times A \to A, \; (ag, c) \mapsto a({}^gc).$$

Moreover, G acts on $A \otimes_R A^o$ (see 37.3) and we have the module action

$$(A \otimes_R A^o)'G \times A \to A, \; ((a \otimes b)g, c) \mapsto a({}^gc)b.$$

37.6 G-multiplication algebra.

The R-subalgebra of $End(_RA)$ generated by all left and right multiplications and $i(G)$ is called the *G-multiplication algebra* $M_G(A)$ of A, i.e.,

$$M_G(A) = < \{L_a, R_a \mid a \in A\} \cup \{i(G)\} > \quad \subset End(_RA).$$

This is an associative R-algebra with unit and there is an algebra morphism

$$M^*(A)'G \to M_G(A), \ \mu g \mapsto \mu\, i(g),$$

whose kernel is just the annihilator of A in $M^*(A)'G$. This map is surjective in case $M^*(A) = M(A)$.

37.7 Augmentation map.

The *(left) augmentation map*

$$\alpha : A'G \to A, \ \sum_{g\in G} a_g g \mapsto \sum_{g\in G} a_g,$$

is an $M^*(A)'G$- and $L^*(A)'G$-epimorphism and $Ke\,\alpha$ is a left ideal in $A'G$.

Let A have a unit. For a family $\{g_\lambda\}_\Lambda$ of elements in G, the following are equivalent:

(a) The group G is generated by $\{g_\lambda\}_\Lambda$;

(b) $Ke\,\alpha = \sum_{\lambda\in\Lambda} A'G \cdot (g_\lambda - e)$.

Proof. Denote $L := \sum_{\lambda\in\Lambda} A'G \cdot (g_\lambda - e)$. Clearly $L \subset Ke\,\alpha$.

$(a) \Rightarrow (b)$ Suppose $\sum_{g\in G} a_g g \in Ke\,\alpha$, i.e., $\sum_{g\in G} a_g = 0$. Then

$$\sum_{g\in G} a_g g = \sum_{g\in G} a_g (g - e).$$

Consider the set $G_o := \{h \in G \mid h - e \in L\} \subset G$. For any $h \in G_o$ and $\lambda \in \Lambda$,

$$\begin{array}{rcll} hg_\lambda - e &=& (h-e) + h(g_\lambda - e) & \in L, \\ hg_\lambda^{-1} - e &=& (h-e) - hg_\lambda^{-1}(g_\lambda - e) & \in L. \end{array}$$

Because $e \in G_o$, this implies $G = G_oG = G_o$ and $L = Ke\,\alpha$.

$(b) \Rightarrow (a)$ For any subgroup $H \subset G$, define

$$I_H := \{\sum_{g\in G} a_g g \mid \sum_{h\in H} a_{gh} = 0, \text{ for every } g \in G\} \subset A'G.$$

Clearly for any $h \in H$, $h - e \in I_H$. Moreover, I_H is a left ideal in $A'G$. Obviously it is closed under addition and left multiplication by elements from A. It remains to show that it is closed under left multiplication by elements of G.

Let $\sum\limits_{g\in G} a_g g \in I_H$ and $\sigma \in G$. Then

$$\sigma \cdot (\sum_{g\in G} a_g g) = \sum_{g\in G} ({}^{\sigma}a_g)\sigma g = \sum_{g\in G} ({}^{\sigma}a_{\sigma^{-1}g})g,$$

and for each $g \in G$,

$$\sum_{h\in H} {}^{\sigma}a_{\sigma^{-1}gh} = {}^{\sigma}(\sum_{h\in H} a_{\sigma^{-1}gh}) = 0.$$

This proves that I_H is a left ideal in $A'G$.

Let $\Gamma \subset G$ be a set of representatives for the left cosets G/H. Then $\sum\limits_{g\in G} a_g g \in I_H$ implies

$$\sum_{g\in G} a_g = \sum_{\gamma\in\Gamma}(\sum_{h\in H} a_{\gamma h}) = 0,$$

and hence $I_H \subset I_G = Ke\,\alpha$.

Now let H denote the subgroup generated by $\{g_\lambda\}_\Lambda$. Since all $g_\lambda - e \in I_H$, we have by (b), $I_H = Ke\,\alpha = L$. For $x \in G \setminus H$, $x - e \in Ke\,\alpha$ but $x - e \notin I_H$, a contradiction. Thus $G = H$ is generated by $\{g_\lambda\}_\Lambda$. □

37.8 Finiteness conditions.

Let G act on an R-algebra A with unit.

(1) If the augmentation map $\alpha : A'G \to A$ splits as $R[G]$-morphism, then G is a finite group.

(2) $Ke\,\alpha$ is finitely generated as an $M(A)'G$- (or $L(A)'G$-) module if and only if the group G is finitely generated.

Proof. (1) Let $\beta : A \to A'G$ be an $R[G]$-morphism with $\beta\alpha = id_A$. Then

$$(1)\beta = \sum_{i=1}^{n} a_i g_i, \text{ for } 0 \neq a_i \in A, g_i \in G.$$

For every $h \in G$,

$$(1)\beta = ({}^{h}1)\beta = h \cdot (\sum_{i=1}^{n} a_i g_i) = \sum_{i=1}^{n} ({}^{h}a_i) h g_i.$$

Comparing coefficients we obtain ${}^{h}a_i = a_{hg_i}$, for $i = 1, \dots, n$. Without restriction, we may assume $g_1 = e$ and then ${}^{h}a_1 = a_h$. For $h \in G \setminus \{g_1, \dots, g_n\}$ this implies ${}^{h}a_1 = a_h = 0$ and hence $a_1 = 0$, a contradiction. So G has to be a finite group.

(2) follows immediately from 37.7. □

37.9 G-invariant and fixed subsets.
A subset $K \subset A$ is called *G-invariant* if for every $g \in G$,

$$^{g}K := \{^{g}a \mid a \in K\} \subset K.$$

The G-invariant elements of a subset $K \subset A$ are called the *fixed points* of K,

$$K^G := \{a \in K \mid {}^{g}a = a \text{ for all } g \in G\}.$$

In particular, the fixed points of A,

$$A^G := \{a \in A \mid {}^{g}a = a \text{ for all } g \in G\},$$

form an R-subalgebra called the *fixed ring (algebra)* of A.

37.10 Trace map.
If G is a finite group acting on A, then there is an R-linear map, called the *trace map*,

$$tr_G : A \to A^G,\ a \mapsto \sum_{g \in G} {}^{g}a.$$

For any G-invariant subgroup $I \subset A$, it induces a map $tr_G : I \to I^G$.
If $a \in A^G$, then $tr_G(a) = |G| \cdot a$ and hence

$$|G| \cdot A^G \subset tr_G(A) \quad \text{and} \quad |G| \cdot I^G \subset tr_G(I).$$

For associative algebras A, tr_G is a left and right A^G-morphism.

References. Alfaro-Ara-del Rio [44], García-del Río [142].

38 Associative skew group algebras

1.Properties of ${}_{A'G}A$. 2.Proposition. 3.The trace of A in $A'G$. 4.Maschke's theorem for skew group algebras. 5.Theorem.

In this section G will be any group, A will always be an *associative* R-algebra with unit and $i : G \to Aut_R(A)$ a group action.

In this situation A is a left $A'G$-module (see 37.5) and we want to investigate this module structure. We begin with some elementary facts which are easily verified.

38.1 Properties of ${}_{A'G}A$.

(1) The $A'G$-submodules of A are the G-invariant left ideals of A.

(2) A is a cyclic $A'G$-module (since $\alpha : A'G \to A$ is surjective).

(3) A is a finitely presented $A'G$-module if and only if the group G is finitely generated.

(4) (i) If ${}_{A'G}A$ is a faithful module then the action of G is faithful.
(ii) If the action of G is trivial, then ${}_{A'G}A$ is faithful if and only if $G = \{e\}$.

(5) The Jacobson radical $Jac(A)$ is G-invariant.

Proof. (3) A is a finitely presented $A'G$-module if and only if the kernel of the augmentation map $A'G \to A$ is finitely generated as an $A'G$-module. So the assertion follows from 37.8.

The other statements are easily verified. $\square$

Let ${}_{A'G}M$ be any left $A'G$-module. The set of all *G-invariant elements of M* is denoted by

$$M^G := \{m \in M \mid gm = m, \text{ for every } g \in G\}.$$

For $M = A$ we obtain the fixed ring A^G.

Every $A'G$-module is an A-module (by the inclusion $A \to A'G$) and M^G is an A^G-submodule of M, since for any $a \in A^G$, $m \in M^G$ and $g \in G$,

$$g(am) = g(ae)m = (gae)m = ({}^g a)gm = am.$$

In 23.3 the importance of the centre of $M(A)$-modules was pointed out. Replacing $M(A)$ by $A'G$, the same arguments show the significance of the fixed elements of $A'G$-modules. In particular, we have that $End_{A'G}(A)$ is a subring of A.

38.2 Proposition. *Let M be an $A'G$-module.*

(1) The (evaluation) map

$$\Phi_M : Hom_{A'G}(A, M) \to M^G,\ f \mapsto (1)f,$$

is an isomorphism of left A^G-modules.

(2) $\Phi_A : End_{A'G}(A) \to A^G$ is an algebra isomorphism.

(3) M is A-generated as an $A'G$-module if and only if

$$M = AHom_{A'G}(A, M) = AM^G.$$

(4) An $A'G$-submodule $I \subset A$ is A-generated if and only if $I = AI^G$.

(5) An $A'G$-submodule $I \subset A$ is fully invariant if and only if $I \cdot A^G = I$.
In particular, every G-invariant two-sided ideal in A is a fully invariant $A'G$-submodule of A.

Applying the above observations to the left $A'G$-module $A'G$ for a finite group G we obtain:

38.3 The trace of A in $A'G$.
Let G be a finite group acting on A.

(1) There is an A^G-isomorphism

$$\Phi_{A'G} : Hom_{A'G}(A, A'G) \to (\sum_{g\in G} g) \cdot A,\ f \mapsto (1)f.$$

(2) Moreover, we may identify

$$(\sum_{g\in G} g) \cdot A = (\sum_{g\in G} g)A'G = \{\sum_{g\in G} ({}^g a)g \mid a \in A\}.$$

(3) For the trace of A in $A'G$, we have

$$Tr_{A'G}(A, A'G) = A \cdot (\sum_{g\in G} g) \cdot A = A'G(\sum_{g\in G} g)A'G.$$

Proof. (1) Consider $f \in Hom_{A'G}(A, A'G)$ and $(1)f = \sum_{g\in G} a_g g$. Then for any $h \in G$,

$$(1)f = h \cdot (1)f = \sum_{g\in G} ({}^h a_g)hg.$$

Hence ${}^h a_e = a_h$ and

$$(1)f = \sum_{g\in G} ({}^g a_e)g = (\sum_{g\in G} g) \cdot a_e \in (\sum_{g\in G} g) \cdot A.$$

It is easy to check that each element $(\sum_{g\in G} g)\cdot b$, $b\in A$, determines an $A'G$-morphism

$$A \to A'G,\ a \mapsto a\cdot(\sum_{g\in G} g)\cdot b.$$

(2) For each $bh \in A'G$,

$$(\sum_{g\in G} g)\cdot bh = \sum_{g\in G}({}^{g}b)gh = \sum_{g\in G}({}^{gh^{-1}}b)g = (\sum_{g\in G} g)\cdot({}^{h^{-1}}b) \in (\sum_{g\in G} g)\cdot A.$$

This proves the first equality in (2).

Similar arguments yield the second equality.

(3) The first equality follows directly from 38.2. For the second equality apply (2) and $A'G(\sum_{g\in G} g) = A\cdot(\sum_{g\in G} g)$. □

The following classical result concerning finite groups is of great importance.

38.4 Maschke's Theorem for skew group algebras.
Let $A'G$ be a skew group algebra with G a finite group such that $|G|^{-1}\in A$.

(1) Let $W, U \in A'G$-Mod, and let $W = U\oplus V$ be a direct sum of A-modules. Then U is a direct summand in W as an $A'G$-module.

(2) Any exact sequence $0\to K\to L\to M\to 0$ in $A'G$-Mod, which splits in A-Mod, is also splitting in $A'G$-Mod.

(3) If A is left semisimple then $A'G$ is left semisimple.

Proof. ([28, 0.1]) (1) Consider $W, U \in A'G$-*Mod* such that $W = U\oplus V$ is a direct sum of A-left modules, and π_U the projection of W onto U in A-*Mod*. Defining

$$p: W\to U,\ w\mapsto |G|^{-1}\cdot\sum_{g\in G} g^{-1}(g\cdot w)\pi_U,$$

one can easily verify that p is an $A'G$-homomorphism. Now we show that $W = Ke\,p\oplus U$. For any $u\in U$ and $g\in G$, we have $(g\cdot u)\pi_U = g\cdot u$ and therefore

$$(u)p = |G|^{-1}\sum_{g\in G} g^{-1}(g\cdot u)\pi_U = |G|^{-1}\sum_{g\in G} g^{-1}gu = |G|^{-1}\sum_{g\in G} u = u,$$

which means that $p^2 = p \in End_{A'G}(W)$. Hence $W = Im\,p\oplus Ke\,p = U\oplus Ke\,p$ as $A'G$-modules.

(2) and (3) are easy consequences of (1). □

If the group G is finite we say that a left $A'G$-module M is *without (additive) $|G|$-torsion* if $|G|\cdot m = 0$ implies $m = 0$ for every $m\in M$.

This condition allows a generalized form of Maschke's Theorem.

38.5 Theorem. *Let G be a finite group. Let $U \subset M$ be $A'G$-modules without additive $|G|$-torsion. Then $U \trianglelefteq_A M$ if and only if $U \trianglelefteq_{A'G} M$.*

Proof. ([31, p. 31]) $U \trianglelefteq_A M$ obviously implies $U \trianglelefteq_{A'G} M$. Suppose ${}_{A'G}U \trianglelefteq_{A'G} M$.

Case 1: First assume that $M = U \oplus V$ is a direct sum of A-modules. We claim that $V = 0$. For every $g \in G$, we have $M = g \cdot U \oplus g \cdot V = U \oplus gV$ in A-*Mod.* From the last decomposition we obtain the projection $\pi_g : M \to U$ in A-*Mod.* Then $\pi : M \to U$, $m \mapsto \sum_{g \in G} (m)\pi_g$ becomes an $A'G$-homomorphism, because for every $h \in G$ and $m \in M$, we get from $m = u + g \cdot v$ that $hm = hu + hgv$, with $hu \in U$. Then $(hm)\pi_{hg} = hu = h(m)\pi_g$. It follows that

$$(hm)\pi = \sum_{g \in G} (hm)\pi_g = \sum_{g \in G} (hm)\pi_{hg} = \sum_{g \in G} h(m)\pi_g = h \cdot (m)\pi.$$

Of course $(u)\pi = |G| \cdot u$, for every $u \in U$. Since U was assumed to be $|G|$-torsionfree we obtain $Ke\,\pi \cap U = 0$. This implies $Ke\,\pi = 0$ because of ${}_{A'G}U \trianglelefteq_{A'G} M$.

For an arbitrary $v \in V$, we have $(v)\pi \in U$ and $|G| \cdot v - (v)\pi \in Ke\,\pi = 0$. Therefore $|G| \cdot v = (v)\pi \in U \cap V = 0$ and consequently $v = 0$.

Case 2: Choose an A-submodule V of M which is maximal with respect to the condition $U \cap V = 0$. Then as A-modules we have $U \oplus V \trianglelefteq M$. Let $W := \bigcap_{g \in G} g \cdot (U \oplus V)$. Since G is finite, W is an essential A-submodule of M and, in addition, W is also an $A'G$-submodule of M. Now $U \subset W \subset U \oplus V$ and hence

$$W = (U \oplus V) \cap W = U \oplus (V \cap W).$$

From case 1 we get $W = U$ an $V \cap W = 0$. Then V must be zero (since $W \trianglelefteq_A M$). So $U = U \oplus V \trianglelefteq_A M$. □

References: Auslander-Reiten-Smalø [59], García-del Río [142], Montgomery [28], Passman [31].

39 Generator and projectivity properties of ${}_{A'G}A$

1.Generator properties. 2.Algebras with large fixed rings. 3.Theorem. 4.Corollary. 5.${}_{A'G}A$ as a self-generator. 6.A^G noetherian or artinian. 7.A as a flat module over A^G. 8.${}_{A'G}A$ as a generator in $\sigma[{}_{A'G}A]$. 9.Density theorem. 10.${}_{A'G}A$ as a generator in $A'G$-Mod. 11.Projectivity properties. 12.${}_{A'G}A$ intrinsically projective. 13.Lemma. 14.Purity and intrinsical projectivity. 15.${}_{A'G}A$ noetherian or artinian. 16.${}_{A'G}A$ self-projective. 17. ${}_{A'G}A$ projective in $A'G$-Mod. 18.Corollary. 19.The radical of ${}_{A'G}A$. 20.A as a right $A'G$-module. 21.$A'G$-projectivity of A. 22.$A_{A'G}$ self-projective. 23.Exercises.

We make the same assumptions as in the preceding section. G is be any group, A an *associative* R-algebra with unit and $i : G \to Aut_R(A)$ a group action.

Here we consider various generator and projectivity conditions of A as an $A'G$-module. Similar to the situation in § 24 we begin with

39.1 Generator properties.

(G^l.1) For every nonzero G-invariant left ideal $I \subset A$, $I^G \neq 0$.

(G^l.2) Every G-invariant left ideal of A is ${}_{A'G}A$-generated (${}_{A'G}A$ is a self-generator.)

(G^l.3) For every $A'G$-morphism $f : A^k \to A^n$, $k, n \in I\!N$, $Ke\, f$ is A-generated.

(G^l.4) ${}_{A'G}A$ is a generator in $\sigma[{}_{A'G}A]$.

(G^l.5) ${}_{A'G}A$ is a generator in $A'G$-*Mod.*

Obviously, (G^l.5) $\Rightarrow$ (G^l.4) $\Rightarrow$ (G^l.3) and (G^l.4) $\Rightarrow$ (G^l.2) $\Rightarrow$ (G^l.1) always hold.

Property (G^l.1) is the question for non-zero fixed elements in non-zero left ideals. The situation for bimodules suggests the following

Definition. We say an algebra A has a *large fixed ring* if $I^G \neq 0$, for every G-invariant left ideal $0 \neq I \subset A$.

Adapting the arguments in the proof of 24.2 we obtain:

39.2 Algebras with large fixed rings.

Assume A is an algebra with unit and large fixed ring. Then:

(1) For any G-invariant left ideal $U \subset A$, $AU^G \trianglelefteq U$ as an $A'G$-submodule.

(2) A is a simple $A'G$-module if and only if A^G is a skew field.

(3) A is a semisimple $A'G$-module if and only if A^G is a left semisimple ring.

For semiprime algebras A without G-torsion we recall results by Bergman and Isaac (from 1973) concerning algebras with large fixed algebras (see [28]):

39.3 Theorem. *Let G be finite and A without additive $|G|$-torsion (without unit).*

(1) If $tr_G(A)$ is nilpotent then A is nilpotent.

(2) If A is a semiprime ring then A^G is also semiprime.

(3) If I is a G-invariant left ideal in A then either I is nilpotent or $tr_G(I) \neq 0$.

Since for any G-invariant subgroup $I \subset A$, $tr_G(I) \subset I^G$ we obtain the

39.4 Corollary. *Let G be a finite group and A a semiprime algebra without additive $|G|$-torsion. Then A has a large fixed ring.*

To characterize rings satisfying condition $(G^l.2)$, we consider the following two mappings:

Let $\mathcal{L}_G(A)$ denote the set of G-invariant left ideals in A, and $\mathcal{L}(A^G)$ the left ideals in the fixed ring A^G. Then we have

$$\begin{array}{rl} \delta_A: & \mathcal{L}(A^G) \to \mathcal{L}_G(A), \quad I \mapsto A \cdot I \qquad \text{and} \\ Hom_{A'G}(A,-): & \mathcal{L}_G(A) \to \mathcal{L}(A^G), \quad U \mapsto Hom_{A'G}(A,U) \simeq U^G (= U \cap A^G). \end{array}$$

Both mappings are order preserving and we obtain the following characterization of $(G^l.2)$ in view of 38.2.

39.5 ${}_{A'G}A$ as a self-generator.

The following assertions are equivalent:

(a) A is a self-generator as a left $A'G$-module;

(b) for every G-invariant left ideal $U \subset A$, $U = A \cdot Hom_{A'G}(A,U)$ $(= A \cdot U^G)$;

(c) the map $Hom_{A'G}(A,-) : \mathcal{L}_G(A) \to \mathcal{L}(A^G)$ is injective;

(d) the map $\delta_A : \mathcal{L}(A^G) \to \mathcal{L}_G(A)$ is surjective.

If the conditions above hold, chain conditions transfer from left ideals in A^G to submodules of ${}_{A'G}A$:

39.6 A^G noetherian or artinian.

Let ${}_{A'G}A$ be a self-generator. Then

(1) If A^G is a left noetherian ring, then ${}_{A'G}A$ is a noetherian module.

(2) If A^G is a left artinian ring, then ${}_{A'G}A$ is an artinian module.

Proof. Let $U_1 \subset U_2 \subset \ldots$ be an ascending chain of G-invariant left ideals in A. Then $Hom_{A'G}(A,U_1) \subset Hom_{A'G}(A,U_2) \subset \ldots$ is an ascending chain of left ideals in A^G, which becomes stationary if A^G is left noetherian. Since $U_k = A \cdot Hom_{A'G}(A,U_k)$, for every $k \in I\!N$ (by 39.5), the chain $U_1 \subset U_2 \subset \ldots$ becomes eventually stationary.

The same argument applies for descending chains. □

By [40, 15.9], group actions satisfying (G^l.3) have the following characterization:

39.7 A as a flat module over A^G.
The following are equivalent:

(a) *For every $A'G$-morphism $f : A^k \to A^n$, $k, n \in I\!N$, $Ke\,f$ is A-generated;*

(b) *for every $A'G$-morphism $f : A^k \to A$, $k \in I\!N$, $Ke\,f$ is A-generated;*

(c) *A is flat as a right A^G-module.*

As a direct consequence from the equivalences above we note that (G^l.3) holds whenever A^G is a regular ring.

For a more explicit study of the property described in 39.7 we refer to [142]. For algebras A which are projective over A^G see [171].

Combining 5.2 with the observations in 38.2 yields the following characterizations of condition (G^l.4):

39.8 $_{A'G}A$ as a generator in $\sigma[_{A'G}A]$.
The following assertions are equivalent:

(a) *$_{A'G}A$ is a generator in $\sigma[_{A'G}A]$;*

(b) *for every $_{A'G}A$-submodule $U \subset A^k$, $k \in I\!N$, $U = A \cdot Hom_{A'G}(A, U) = A \cdot U^G$;*

(c) *the fix point functor $(-)^G := Hom_{A'G}(A, -) : \sigma[_{A'G}A] \to A^G$-Mod is faithful;*

(d) *for every module $X \in \sigma[_{A'G}A]$, $X = A \cdot X^G$.*

For algebras characterized above we have in particular the

39.9 Density theorem.
Let $_{A'G}A$ be a generator in $\sigma[_{A'G}A]$. Then

(1) *$A'G$ acts dense in $End(A_{A^G})$, i.e., for any $a_1, \dots, a_k \in A$ and $f \in End(A_{A^G})$, there exists an $x \in A'G$ with $f(a_i) = x \cdot a_i$, for $i = 1, \dots, k$;*

(2) *$\sigma[_{A'G}A] = \sigma[_{End(A_{A^G})}A]$.*

Condition (G^l.5) obviously implies that A is a faithful $A'G$-module and hence the action of G on A has to be faithful. The characterization of generators in full module categories yields part of our next proposition.

39.10 $_{A'G}A$ as a generator in $A'G$-*Mod*.
The following conditions are equivalent for a finite group G:

(a) *$_{A'G}A$ is a generator in $A'G$-Mod;*

(b) *A is a finitely generated, projective right A^G-module, and $A'G \simeq End_{A^G}(A_{A^G})$;*

(c) $Tr_{A'G}(A, A'G) = AHom_{A'G}(A, A'G) = A'G$;

(d) $A'G = A \cdot (\sum\limits_{g\in G} g) \cdot A$;

(e) *there exist* $x_1 \ldots, x_n, y_1, \ldots, y_n \in A$, *such that*

$$\sum_{i=1}^{n} x_i(\sum_{g\in G} g)y_i = 1 \cdot e \in A'G.$$

Proof. $(a) \Leftrightarrow (b) \Leftrightarrow (c)$ is an application of 5.5.

$(c) \Leftrightarrow (d) \Leftrightarrow (e)$ follow from 38.3. □

Remark. Condition (e) of 39.10 (in different notation) was used to define *Galois extensions of rings* by Y. Miyashita in [202] in an attempt to generalize the results from Chase-Harrison-Rosenberg [105] from commutative to non-commutative rings. For a commutative algebra A, the conditions of 39.10 are equivalent to ${}_{A'G}A$ being a *projective* generator in $A'G$-*Mod*.

Now we turn to the investigation of projectivity conditions on A as an $A'G$-module. We will focus on the following three

39.11 Projectivity properties.

$(P^l.1)$ ${}_{A'G}A$ is intrinsically projective (see 5.7).

$(P^l.2)$ ${}_{A'G}A$ is self-projective.

$(P^l.3)$ ${}_{A'G}A$ is projective in $A'G$-*Mod*.

Since ${}_{A'G}A$ is always a cyclic module (A with unit), $(P^l.2)$ is equivalent to ${}_{A'G}A$ being projective in $\sigma[{}_{A'G}A]$. It is clear that $(P^l.3) \Rightarrow (P^l.2) \Rightarrow (P^l.1)$. We will give more characterizations of these properties.

Let us begin with a special case of 5.7.

39.12 ${}_{A'G}A$ intrinsically projective.

The following assertions are equivalent:

(a) ${}_{A'G}A$ *is intrinsically projective;*

(b) *for every left ideal* $I \subset A^G$, $I = Hom_{A'G}(A, A \cdot I)$;

(c) *the map* $\delta_A : \mathcal{L}(A^G) \to \mathcal{L}_G(A)$ *is injective;*

(d) *for every left ideal* $I \subset A^G$, $I = (A \cdot I) \cap A^G$.

${}_{A'G}A$ being intrinsically projective is closely related to the condition that A^G is a pure submodule of A_{A^G}. This is based on the following more general observation.

39.13 Lemma. *Let $S \subset A$ be a subring of A with the same unit, and assume S_S to be a pure submodule of A_S. Then, for every left ideal $I \subset S$, $I = (A \cdot I) \cap S$.*

Proof. Let $0 \to S \to A \to N \to 0$ be a pure exact sequence in *Mod-S*. Then for every left S-module X,

$$0 \to S \otimes_S X \to A \otimes_S X \to N \otimes_S X \to 0$$

is exact (see [40, 34.5]). For any left ideal $I \subset S$, put $X := S/I$. Now the assertion follows from [40, 34.9]. □

Taking $S = A^G$, the equality $I = (A \cdot I) \cap A^G$, for every left ideal $I \subset A^G$, is a characterization for ${}_{A'G}A$ to be intrinsically projective and we obtain as a corollary of 39.12 and 39.13 (see also [40, 36.6]):

39.14 Purity and intrinsical projectivity.

(1) If A^G is a pure submodule of A_{A^G}, then ${}_{A'G}A$ is intrinsically projective.

(2) If A_{A^G} is a flat A^G-module, the following statements are equivalent:

(a) ${}_{A'G}A$ is intrinsically projective;

(b) A^G is a pure submodule of A_{A^G}.

Projectivity conditions allow to transfer properties of the module to the endomorphism ring. The implications here are converse to 39.6.

39.15 ${}_{A'G}A$ noetherian or artinian.

Let ${}_{A'G}A$ be intrinsically projective.

(1) If ${}_{A'G}A$ is a noetherian module, then A^G is a left noetherian ring.

(2) If ${}_{A'G}A$ is an artinian module, then A^G is a left artinian ring.

The next proposition is a characterization of condition (P^l.2). Notice that here ${}_{A'G}A$ is finitely generated (since A has a unit).

39.16 ${}_{A'G}A$ self-projective.

The following statements are equivalent:

(a) ${}_{A'G}A$ is self-projective (projective in $\sigma[{}_{A'G}A]$);

(b) $Hom_{A'G}(A, -) : \sigma[{}_{A'G}A] \to A^G$-Mod is exact;

(c) for every G-invariant left ideal $I \subset A$, the sequence

$$0 \to I^G \to A^G \to (A/I)^G \to 0$$

is exact in A^G-Mod;

(d) *for every G-invariant left ideal $I \subset A$ and every $a \in A$ with $a + I \in (A/I)^G$, there exists $b \in A^G$ such that $a - b \in I$;*

(e) *for every $a \in A$, $(a + \sum_{g,h \in G} A({}^g a - {}^h a)) \cap A^G \neq \emptyset$.*

Proof. $(a) \Leftrightarrow (b)$ is clear from module theory (5.1).

$(b) \Leftrightarrow (c)$ follows from the (functorial) isomorphism of 38.2.

$(a) \Rightarrow (d)$ Let $a + I \in (A/I)^G$ for an arbitrary G-invariant left ideal $I \subset A$. Consider the $A'G$-homomorphism

$$f : A \to A/I, \quad x \mapsto x(a + I).$$

With the projection $p : A \to A/I$, we have the diagram (in $\sigma[{}_{A'G}A]$)

$$\begin{array}{ccccc} & & A & & \\ & & \downarrow f & & \\ A & \xrightarrow{p} & A/I & \to & 0\,, \end{array}$$

which can be extended commutatively by an $\alpha \in End_{A'G}(A)$ if ${}_{A'G}A$ is self-projective. Then $b := (1)\alpha \in A^G$ satisfies $a - b \in I$.

$(d) \Rightarrow (a)$ Reverse the arguments in $(a) \Rightarrow (d)$.

$(d) \Leftrightarrow (e)$ Notice that $a + I \in (A/I)^G$ if and only if $a - {}^g a \in I$, for every $g \in G$, and hence $\sum_{g,h \in G} A({}^g a - {}^h a) \subset I$. So $(e) \Rightarrow (d)$ is obvious.

The converse conclusion follows by the fact that $\sum_{g,h \in G} A({}^g a - {}^h a)$ is a G-invariant left ideal. $\square$

In particular, statement (d) gives a useful criterion to check self-projectivity of ${}_{A'G}A$. This was observed independently in [265] and [141]. García and del Río call this property *G-lifting*. An example of a ring A and a group G, such that ${}_{A'G}A$ is intrinsically projective but not self-projective, will be given in 43.1.

Now we look at algebras satisfying $(P^l.3)$. This implies in particular, that the augmentation map $\alpha : A'G \to A$ splits as $A'G$-morphism and hence, by 37.8, G has to be a finite group. Hence without loss of generality we restrict the next proposition to this case.

39.17 ${}_{A'G}A$ projective in $A'G$-*Mod*.

For a finite group G acting on A, the following are equivalent:

(a) *${}_{A'G}A$ is projective in $A'G$-Mod;*

(b) *the augmentation $\alpha : A'G \to A$ splits in $A'G$-Mod;*

(c) there exists $a \in A$ *such that* $\sum\limits_{g\in G} {}^g a = 1$*;*

(d) the trace map $tr_G : A \to A^G$ *is surjective.*

Proof. $(a) \Leftrightarrow (b)$ and $(c) \Leftrightarrow (d)$ are obvious.

$(b) \Rightarrow (c)$ Let $\beta : A \to A'G$ be an $A'G$-morphisms with $\beta\alpha = id_A$. We conclude from 38.3 that $(1)\beta = \sum\limits_{g\in G} ({}^g a)g$, for some $a \in A$, and hence

$$1 = (1)\beta\alpha = \sum_{g\in G} {}^g a.$$

$(c) \Rightarrow (b)$ Let $a \in A$ such that $\sum_{g\in G} {}^g a = 1$. Then we see (from 38.3) that

$$\beta : A \to A'G,\ x \mapsto x \cdot \sum_{g\in G} ({}^g a)g,$$

defines an $A'G$-morphism satisfying $(1)\beta\alpha = 1$, i.e., $\beta\alpha = id_A$. □

As a special case we observe:

39.18 Corollary. *Let* G *be a finite group and assume* $|G|^{-1} \in A$*. Then* ${}_{A'G}A$ *is projective in* $A'G$*-Mod.*

Proof. If $|G|^{-1} \in A$ then $\sum\limits_{g\in G} (|G|^{-1})^g = 1$ and ${}_{A'G}A$ is projective by 39.17. □

Under projectivity conditions the radical of a module and the radical of the endomorphism ring are related. In our situation this yields:

39.19 The radical of ${}_{A'G}A$**.**

(1) If ${}_{A'G}A$ *is self-projective then*

$$Jac(A^G) = Hom_{A'G}(A, Rad({}_{A'G}A)) = Rad({}_{A'G}A) \cap A^G.$$

(2) If ${}_{A'G}A$ *is self-projective and* $Rad({}_{A'G}A)$ *is generated by* ${}_{A'G}A$*, then*

$$Rad({}_{A'G}A) = A \cdot Jac(A^G).$$

(3) If ${}_{A'G}A$ *is a projective* $A'G$*-module, then*

$$Jac(A^G) = Jac(A) \cap A^G.$$

Proof. (1) The first equality is a special case of [40, 22.2]; the second one follows from 38.2.

(2) By (1), $Jac(A^G) = Hom_{A'G}(A, Rad({}_{A'G}A))$. Then by assumption,

$$A \cdot Jac(A^G) = A \cdot Hom_{A'G}(A, Rad({}_{A'G}A)) = Rad({}_{A'G}A).$$

(3) In view of (1) it remains to show that $Rad({}_{A'G}A) = Jac(A)$.

Since ${}_{A'G}A$ is projective, $Rad({}_{A'G}A) = Jac(A'G)A$. Under the given conditions it follows from [30, Theorem 16.3] that $Jac(A'G)A = (Jac(A)A'G)A = Jac(A)$. □

In particular, $|G|^{-1} \in A$ implies projectivity of ${}_{A'G}A$ (by 39.18) and hence $Jac(A^G) = Jac(A) \cap A^G$. This was first observed by S. Montgomery (see [28, 1.14]).

39.20 ***A*** **as a right** ***A'G*****-module.**

Recall that A is an associative ring with unit. An action of G on A induces an action on A^o, and the map

$$\varphi : A'G \to (A^o)'G,\ ag \mapsto ({}^{g^{-1}}a)g^{-1},$$

is an anti-isomorphism of algebras. Thus we may identify the algebras $(A'G)^o$ and $(A^o)'G$. Doing this the left $(A^o)'G$-structure on A^o corresponds to the right $A'G$-structure on A given by ([59, p. 7])

$$A \times A'G \to A,\ (b, ag) \mapsto {}^{g^{-1}}(ba).$$

It was observed in [59, Proposition 1.7] that $A'G$-projectivity is a left-right symmetric property of A. With arguments similar to the proof of 39.17 we obtain:

39.21 ***A'G*****-projectivity of** ***A*****.**

For G acting on A the following are equivalent:

(a) *A is a projective left $A'G$-module;*

(b) *the right augmentation map $A'G \to A$, $\sum_{g\in G} a_g g \mapsto \sum_{g\in G} {}^{g^{-1}}(a_g)$ splits in Mod-$A'G$;*

(c) *A is a projective right $A'G$-module.*

It is easily verified that for any right $A'G$-module N, $Hom_{A'G}(A, N) \simeq N^G$ as right A^G-modules (compare 38.2) and right self-projectivity over $A'G$ has similar characterizations as left self-projectivity in 39.16 (see [142, Proposition 10]).

39.22 $\boldsymbol{A_{A'G}}$ **self-projective.**

The following statements are equivalent:

(a) *$A_{A'G}$ is self-projective (as right $A'G$-module);*

(b) $Hom_{A'G}(A,-) : \sigma[A_{A'G}] \to Mod\text{-}A^G$ *is exact;*

(c) *for every G-invariant right ideal $I \subset A$ and every $a \in A$ with $a + I \in (A/I)^G$, there exists $b \in A^G$ such that $a - b \in I$;*

By 39.21, A is projective in $A'G$-*Mod* if and only if A is projective in *Mod*-$A'G$. Notice that a similar symmetry does not hold for self-projectivity of A as $A'G$-module. This is shown by Example 43.2.

39.23 Exercises.

Let G be a group acting on the associative algebra A with unit.

(1) Prove that A is a self-projective $A'G$-module if and only if ([142, Proposition 10])

$$\text{for every } a \in A,\ (a + A^G) \cap \sum_{g,h\in G} A({}^g a - {}^h a) \neq \emptyset.$$

(2) Let A be a projective left $A'G$-module. Prove ([59, Corollary 1.12]): For any left $A'G$-module M, $M^G = (\sum_{g\in G} g) \cdot M$.

(3) Assume that $A'G$ is a regular ring. Prove ([44, Theorem 2.4]):
(i) For every subgroup $H \subset G$, $A'H$ is regular.
(ii) G is locally finite.
(iii) For every finite subgroup $H \subset G$, $1 \in tr_H(A)$ and R^H is regular.

(4) A is called a Frobenius extension *of A^G if A is a finitely generated projective A^G-module and $A \simeq Hom_{A^G}(A, A^G)$ as an (A, A^G)-bimodule.*

Assume G is finite and $A'G$ is a biregular ring. Prove that A is a Frobenius extension of A^G ([142, Proposition 17]).

For the next problems recall that A may also be considered as right $A'G$-module (see 39.20) and hence $A \otimes_{A^G} A$ is an $(A'G, A'G)$-bimodule.

(5) Let A be a generator in $A'G$-Mod and assume G to be finite.

Prove ([59, Proposition 1.6]):
(i) The map $A \otimes_{A^G} A \to A'G$, $x \otimes y \mapsto x \cdot (\sum_{g\in G} g) \cdot y$ is an $(A'G, A'G)$-isomorphism.
(ii) The multiplication map $A \otimes_{A^G} A \to A$ is a split (A, A)-epimorphism.

(6) Let A be (self-) injective as a right A-module. Prove ([123, Theorem 2.2]):
(i) If ${}_{A'G}A$ is self-projective, then $A_{A'G}$ is self-injective.
(ii) If G is finite and $A_{A'G}$ is self-injective, then ${}_{A'G}A$ is self-projective.

References. Alfaro-Ara-del Rio [44], Auslander-Reiten-Smalø [59], García-del Río [141, 142], Cohen [109], Fisher-Osterburg [137], Jøndrup [171], Kharchenko [22], Montgomery [28], Miyashita [202], Park [219], Passman [30, 31], Wilke [265].

40 ${}_{A'G}A$ as an ideal module and progenerator

1.${}_{A'G}A$ as an A^G-ideal module. 2.${}_{A'G}A$ as a self-progenerator. 3.A^G-ideal modules and self-progenerators. 4.Corollary. 5.Corollary. 6.${}_{A'G}A$ as a generator in $A'G$-Mod and projectivity. 7.Outer automorphisms and simple rings. 8.Outer automorphisms and projectivity. 9.Example. 10.Group actions on factor algebras. 11.Progenerators and factor rings. 12.Proposition. 13.Action of subgroups. 14.Remark. 15.A as a module over the group ring. 16.Exercises.

Again we will assume throughout this section that G is any group, A is an *associative* R-algebra with unit and $i : G \to Aut_R(A)$ a group action.

Now we combine the projectivity and generating properties studied in the previous section. In particular we consider the following conditions on the $A'G$-module A:

($PG.1$) ${}_{A'G}A$ is an intrinsically projective self-generator.

($PG.2$) ${}_{A'G}A$ is a self-projective self-generator.

($PG.3$) ${}_{A'G}A$ is a projective generator in $A'G$-*Mod*.

Condition ($PG.1$) defines *ideal modules*. Since ${}_{A'G}A$ is cyclic, ($PG.2$) makes A a self-progenerator. If ($PG.3$) is satisfied then G is finite and the pair (A, G) is called *pregalois* (in [59]) or A is called a *G-Galois extension* (in [109]).

Obviously, $(PG.3) \Rightarrow (PG.2) \Rightarrow (PG.1)$.

Ideal modules were described in 5.9. Applying this to our situation we obtain:

40.1 ${}_{A'G}A$ as an A^G-ideal module.

The following statements are equivalent for G acting on A:

(a) *${}_{A'G}A$ is an A^G-ideal module ($Hom_{A'G}(A, -)$ yields a bijection between G-invariant left ideals in A and left ideals in A^G);*

(b) *${}_{A'G}A$ is intrinsically projective and a self-generator;*

(c) *every G-invariant left ideal in A can be written as $A \cdot I$, for some left ideal $I \subset A^G$, and A_{A^G} is faithfully flat.*

Examples of the algebras just described will be given later on.

In case ${}_{A'G}A$ satisfies ($PG.2$), it is a finitely generated projective generator in $\sigma[{}_{A'G}A]$ (self-progenerator) and we have the following characterizations:

40.2 ${}_{A'G}A$ as a self-progenerator.

The following statements are equivalent:

(a) *${}_{A'G}A$ is a self-progenerator;*

(b) $Hom_{A'G}(A, -) : \sigma[_{A'G}A] \to A^G$*-Mod is an equivalence of categories (with inverse* $A \otimes_{A^G} -$*);*

(c) *for every* G*-invariant left ideal* $I \subset A$,

$$I = A \cdot I^G \quad \text{and} \quad (A/I)^G \simeq A^G/I^G.$$

Similar to the close relationship between ideal algebras and Azumaya rings we have connections between A being an ideal algebras or a progenerator over $A'G$.

40.3 A^G-ideal modules and self-progenerators.

The following assertions are equivalent for $_{A'G}A$.

(a) $_{A'G}A$ *is a self-progenerator;*

(b) $_{A'G}A$ *is an* A^G*-ideal module and self-projective;*

(c) $_{A'G}A$ *is intrinsically projective and a generator in* $\sigma[_{A'G}A]$*;*

(d) $_{A'G}A$ *is a generator in* $\sigma[_{A'G}A]$ *and* A^G *is a pure submodule of* A_{A^G}.

Proof. The equivalence of (a)-(c) follows from 5.10, and 39.14 implies $(d) \Leftrightarrow (c)$. $\square$

For generators this yields nice characterizations for self-projectivity.

40.4 Corollary. *Let* $_{A'G}A$ *be a generator in* $\sigma[_{A'G}A]$*. Then the following statements are equivalent:*

(a) $_{A'G}A$ *is self-projective;*

(b) $_{A'G}A$ *is intrinsically projective;*

(c) A^G *is a pure submodule of* A_{A^G}.

As in 26.9 we have special cases when A^G-ideal modules are self-progenerators.

40.5 Corollary. *Let* $_{A'G}A$ *be an* A^G*-ideal module.*

(1) *If* A_{A^G} *is projective then* A^G *is a direct summand in* A_{A^G}.
Moreover, A *is a generator in Mod-*A^G *and* $_{A'G}A$ *is a self-progenerator.*

(2) *If the fixed ring* A^G *is right perfect then* $_{A'G}A$ *is a self-progenerator.*

Next we collect facts which make a generator A in $A'G$-*Mod* projective.

40.6 $_{A'G}A$ as a generator in $A'G$-*Mod* and projectivity.

Let $_{A'G}A$ *be a generator in* $A'G$*-Mod. Then the following are equivalent:*

(a) $_{A'G}A$ *is projective in* $A'G$*-Mod;*

(b) $_{A'G}A$ *is self-projective;*

(c) ${}_{A'G}A$ *is intrinsically projective;*

(d) A^G *is a pure submodule of* A_{A^G}*;*

(e) A^G *is a direct summand in* A_{A^G}*;*

(f) A_{A^G} *is a generator in Mod-*A^G.

Proof. $(a) \Rightarrow (b) \Rightarrow (c)$ and $(e) \Rightarrow (f)$ are obvious. $(c) \Rightarrow (d)$ is shown in 40.4.

$(d) \Rightarrow (e)$ Since ${}_{A'G}A$ is a generator in $A'G$-*Mod*, A_{A^G} is projective and by (d), ${}_{A'G}A$ is an A^G-ideal module. By 40.5, A^G is a direct summand in A_{A^G}.

$(f) \Rightarrow (a)$ If A_{A^G} is a generator in Mod-A^G then A is projective as left module over the ring $End(A_{A^G}) \simeq A'G$. □

If A is commutative and ${}_{A'G}A$ is a (projective) generator in $A'G$-*Mod*, A is called a *Galois extension* of A^G. Such extensions were first investigated by Auslander and Goldman [58, Appendix], and later on by Chase, Harrison and Rosenberg [105]. In Miyashita[202] some of these results were extended to non-commutative rings.

An automorphism g of A is called *inner* if there exists an invertible $u \in A$, such that

$$^{g}x = u^{-1}xu, \text{ for all } x \in A.$$

A group G of automorphisms of A is called *inner* if every $g \in G$ is inner, and G is *outer* if id_A is the only inner automorphism in G. We recall the following facts about outer automorphism groups (see [28, Theorem 2.3, 2.4]):

40.7 Outer automorphisms and simple rings.

Let G be an outer automorphism group of a simple ring A with unit. Then:

(1) $A'G$ *is a simple ring.*

(2) A *is a generator in* $A'G$-*Mod.*

Combined with 40.6 this yields an extended version of [28, Theorem 2.5]:

40.8 Outer automorphisms and projectivity.

Let A be a simple ring and G a finite group of outer automorphisms of A. Then the following statements are equivalent:

(a) ${}_{A'G}A$ *is intrinsically projective;*

(b) ${}_{A'G}A$ *is self-projective;*

(c) ${}_{A'G}A$ *is projective in* $A'G$-*Mod;*

(d) ${}_{A'G}A$ *is a projective generator in* $A'G$-*Mod;*

(e) A_{A^G} *is a generator in Mod-*A^G*;*

(f) A^G *is a simple ring;*

(g) there exists some $a \in A$ *with* $tr_G(a) = 1$*.*

It is not surprising that a self-progenerator ${}_{A'G}A$ need not be a generator in $A'G$*-Mod*.

40.9 Example. Let K be a field and G be an infinite subgroup of $Aut(K)$. Obviously, ${}_{K'G}K$ is a simple module and hence a self-progenerator. However, ${}_{K'G}K$ is neither projective in $K'G$*-Mod* (because $|G| = \infty$, see 39.18) nor a generator in $K'G$*-Mod* (K_{K^G} is not finitely generated, see 39.10).

40.10 Group action on factor algebras.
Let G act on A and consider a two-sided G-invariant ideal $I \subset A$. Then the group G acts on A/I by the group homomorphism $\tau : G \to Aut_R(A/I)$ with

$$\tau(g) : A/I \to A/I, \quad a + I \mapsto {}^g a + I\,.$$

Self-progenerator and self-projective transfer to factor algebras.

40.11 Progenerators and factor rings.
Let G *act on* A *and let* $I \subset A$ *be a two-sided* G*-invariant ideal. Denote* $B := A/I$*.*

(1) If ${}_{A'G}A$ *is self-projective then* ${}_{B'G}B$ *is self-projective.*

(2) If ${}_{A'G}A$ *is a self-generator then* ${}_{B'G}B$ *is a self-generator.*

(3) If ${}_{A'G}A$ *is projective in* $A'G$*-Mod then* ${}_{B'G}B$ *is projective in* $B'G$*-Mod.*

(4) If ${}_{A'G}A$ *is a generator in* $A'G$*-Mod then* ${}_{B'G}B$ *is a generator in* $B'G$*-Mod.*

Proof. (1) As a two-sided G-invariant ideal, I is a fully invariant submodule of ${}_{A'G}A$. Hence ${}_{A'G}B$ is self-projective ([40, 18.2]) and clearly ${}_{B'G}B$ is also self-projective.

(2) Every $B'G$-submodule of B is of the form X/I for a suitable G-invariant left ideal with $I \subset X \subset A$. Then X/I is an $A'G$-submodule of B and, by assumption, there exists an $A'G$-epimorphism $f : A^{(\Lambda)} \to X/I$. Since $I \subset_{A'G} A$ is fully invariant, we conclude $I^{(\Lambda)} \subset Ke\,f$, and we have

$$\begin{tikzcd} A^{(\Lambda)} \arrow[rr, "f"] \arrow[dr] & & X/I \arrow[r] & 0 \\ & A^{(\Lambda)}/I^{(\Lambda)} \arrow[ur, "\alpha"'] & & \end{tikzcd} \quad \text{in } \sigma[{}_{A'G}A],$$

with some $A'G$-homomorphism α, which is in fact a $B'G$-epimorphism.

Since $A^{(\Lambda)}/I^{(\Lambda)} \simeq B^{(\Lambda)}$ this means that ${}_{B'G}B$ is a self-generator.

(3) and (4) are easily derived from the criterions 39.17 and 39.10. They are also proved in [59, Proposition 3.2]. □

The following fact is shown in [59, Theorem 3.3].

40.12 Proposition. *Let G act on A and denote $B := A/Jac(A)$. Then:*

(1) If ${}_{B'G}B$ is projective in $B'G$-Mod, then ${}_{A'G}A$ is projective in $A'G$-Mod.

(2) If ${}_{B'G}B$ is a generator in $B'G$-Mod, then ${}_{A'G}A$ is a generator in $A'G$-Mod.

We will show by Example 43.3 that the assertions in 40.12 are no longer true for projectivity replaced by self-projectivity.

It was observed in [59] that the progenerator property of A in $A'G$-*Mod* is inherited by subgroups and factor groups.

40.13 Action of subgroups.
Let G act on the algebra A and consider any subgroup $H \subset G$.

(1) If A is a projective $A'G$-module, then A is also a projective $A'H$-module.

(2) If A is a generator in $A'G$-Mod, then A is a generator in $A'H$-Mod.

Proof. For $H \subset G$ write G as a disjoint union $G = \dot{\bigcup}_{\gamma\in\Gamma} H\gamma$, $\Gamma \subset G$. Then we have an ($M(A)'H$-) isomorphism

$$\varphi : (A'H)^{(\Gamma)} \to A'G, \quad (\sum_{h\in H} a_h^{(\gamma)} h)_{\gamma\in\Gamma} \mapsto \sum_{\gamma\in\Gamma}\sum_{h\in H} a_h^{(\gamma)} h\gamma.$$

In particular, $A'G$ is a projective (free) $A'H$-module.

(1) If A is a projective $A'G$-module, then A is a direct summand of ${}_{A'G}A'G$ and hence it is a projective $A'H$-module.

(2) The map φ shows that $A'H$ is generated by $A'G$ as an $A'H$-module. If A generates $A'G$ (as $A'G$-module) it also generates $A'H$ (as $A'H$-module). □

40.14 Remark. Let A be a progenerator in $A'G$-*Mod* and $N \subset G$ any normal subgroup. It is shown in [59, Theorem 2.8] that under this conditions ${}_{A^N{}'G/N}A^N$ is a projective generator in $(A^N)'(G/N)$-*Mod*.

We will show in example 43.2 that a similar result does not hold for self-progenerators, and that the assertions of 40.13 do not transfer to self-progenerators.

We close this section by considering the case that G acts trivially on A, i.e., we consider A as a module over the (ordinary) group ring $A[G]$.

40.15 *A* as a module over the group ring.
Let G be a group acting trivially on A and consider A as an $A[G]$-module.

(1) ${}_{A[G]}A$ is a self-progenerator.

(2) ${}_{A[G]}A$ *is a generator in* $A[G]$*-Mod if and only if* $G = \{e\}$ *(and* $A[G] = A$*).*

(3) ${}_{A[G]}A$ *is projective in* $A[G]$*-Mod if and only if* G *is finite and* $|G|^{-1} \in A$*.*

Proof. (1) By assumption, every left ideal $I \subset A$ is G-invariant and $I^G = I$. So ${}_{A[G]}A$ is a self-progenerator by 40.2.

(2) If G consists of at least two elements the trivial action is not faithful and therefore A is not a faithful $A[G]$-module.

(3) By 37.8, G has to be finite. So for every $a \in A$, we obtain

$$tr_G(a) = \sum_{g \in G} {}^g a = |G| \cdot a.$$

Hence there exists $a \in A$, such that $tr_G(a) = 1$ if and only if $|G|^{-1} \in A$ (39.18). □

40.16 Exercises.

(1) Let G *be a finitely generated group acting on an associative* R*-algebra* A *with unit. By 40.10,* G *acts on* A_x*, for every* $x \in \mathcal{X}$*. Prove:*

(i) The following are equivalent:

(a) A *is self-projective as an* $A'G$*-module;*

(b) A_x *is self-projective as an* A'_xG*-module, for each* $x \in \mathcal{X}$*.*

(ii) The following are equivalent:

(a) A *is self-generator as an* $A'G$*-module;*

(b) A_x *is self-generator as an* A'_xG*-module, for each* $x \in \mathcal{X}$*.*

(iii) The following are equivalent:

(a) A *is a progenerator in* $A'G$*-Mod;*

(b) A_x *is a progenerator in* A'_xG*-Mod, for each* $x \in \mathcal{X}$*.*

(2) Let A *be an associative prime Azumaya* R*-algebra of rank* n^2 *(dimension of the central closure over its centre). Prove that the factor group* $Aut_R(A)/Inn(A)$ *is abelian of torsion* n*, where* $Inn(A)$ *denotes the inner automorphisms of* A *([107, Theorem 2]).*

References. Auslander-Reiten-Smalø [59], Kharchenko [22], Cohen [109], Wilke [265].

41 A as an $M(A)'G$-module

1.Group action on the centroid. 2.G-centre and G-centroid. 3.Properties of ${}_{M(A)'G}A$. 4.The G-centre of $M(A)'G$-modules. 5.Proposition. 6.${}_{M(A)'G}A$ self-projective. 7.Algebras with large fixed centre. 8.${}_{M(A)'G}A$ as a $Z(A)^G$-ideal module. 9.${}_{M(A)'G}A$ as a self-progenerator. 10.${}_{M(A)'G}A$ as a generator in $M(A)'G$-Mod. 11.Action of subgroups.

In the next sections we will consider not necessarily associative algebras A. To study A as an $M(A)'G$-module we will combine our investigations on A as an $M(A)$-module and on groups acting on A.

41.1 Group action on the centroid.

For $\gamma \in End_{M(A)}(A)$ and any R-automorphism $h : A \to A$, there is an $M(A)$-homomorphism

$$^h\gamma : A \xrightarrow{h^{-1}} A \xrightarrow{\gamma} A \xrightarrow{h} A,$$

and this induces an algebra automorphism

$$h_c : C(A) \to C(A), \quad \gamma \mapsto {}^h\gamma.$$

Hence the action $i : G \to Aut_R(A)$ yields an action of G on $C(A)$,

$$i_C : G \to Aut_R(C(A)),\ g \mapsto i(g)_C.$$

Since the centre $Z(A)$ of A is invariant under automorphisms, the action of G on A also yields an action

$$i_Z : G \to Aut_R(Z(A)),\ g \mapsto i(g)|_{Z(A)}.$$

For an algebra A with unit 1, we have the commutative diagram

$$\begin{array}{ccccccc} C(A) & \xrightarrow{h_c} & C(A) & \qquad & \gamma & \to & {}^h\gamma \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Z(A) & \xrightarrow{h|_{Z(A)}} & Z(A) & & (1)\gamma & \to & (1)\gamma h\,. \end{array}$$

41.2 G-centre and G-centroid.

Under the actions considered above, the G-invariant elements $Z(A)^G$ are called the *G-centre of A*, and $C(A)^G$ is called the *G-centroid of A*. We consider G as a subset of $Aut_R(A)$ ($\subset End_R(A)$).

(1) $C(A)^G \simeq End_{M_G(A)}(A) = \{\gamma \in C(A) \mid g \circ \gamma = \gamma \circ g$ *for each* $g \in G\}$.

(2) There exists an R-algebra morphism

$$\nu : Z(A)^G \to C(A)^G, \quad a \mapsto L_a.$$

(3) *If A is a faithful $Z(A)^G$-module then ν is injective.*

(4) *$Z(A)^G$ is a right $C(A)^G$-module.*

(5) *If A has a unit then ν is an isomorphism.*

Proof. The arguments are similar to those of 2.8. □

Although our techniques apply to algebras without units, the existence of a unit does simplify the presentation of some results. Hence we assume A to *have a unit* in the rest of this section.

41.3 Properties of $_{M(A)'G}A$.
Let G be a group acting on a ring A with unit.

(1) *A is a cyclic $M(A)'G$-module and the $M(A)'G$-submodules of A are the G-invariant ideals of A.*

(2) *If $_{M(A)'G}A$ is a faithful module, then the action of G on A is faithful.*

The role of the centre of $M(A)$-modules is now taken by

41.4 The G-centre of $M(A)'G$-modules.
*Let G be a group acting on a ring A with unit, and let X be any left $M(A)'G$-module. The G-*centre of X *is defined as*

$$Z_A^G(X) := \{x \in Z_A(X) \,|\, g \cdot x = x, \text{ for every } g \in G\}.$$

For $X = A$ we obtain $Z_A^G(A) = Z(A)^G$ and $Z_A^G(X)$ is a $Z(A)^G$-submodule of X, since

$$g(ax) = (gae)x = ({}^g a)gx = ax, \text{ for any } x \in Z_A^G(X),\ a \in A^G,\ g \in G.$$

Similar to the observations in 23.3 and 38.2 we obtain:

41.5 Proposition. *Let G act on an R-algebra A with unit and let $X \in M(A)'G$-Mod.*

(1) *The (evaluation) map*

$$\Phi_X : Hom_{M(A)'G}(A, X) \to Z_A^G(X),\ f \mapsto (1)f,$$

is an isomorphism of left $Z(A)^G$-modules.

(2) *$\Phi_A : End_{M(A)'G}(A) \to Z(A)^G$ is an algebra isomorphism.*

(3) *X is A-generated as an $M(A)'G$-module if and only if*

$$X = AHom_{M(A)'G}(A, X) = AZ_A^G(X).$$

(4) Any G-invariant ideal $I \subset A$ is A-generated as $M(A)'G$-module if and only if $I = A(I \cap Z(A)^G)$.

(5) Any G-invariant ideal $I \subset A$ is a fully invariant $M(A)'G$-submodule.

It is more or less obvious how to transfer the observations about generating and projectivity conditions from the sections 24, 39, and 40 to A as an $M(A)'G$-module. For example, self-projectivity has the following characterization.

41.6 $_{M(A)'G}A$ self-projective.

The following statements are equivalent for a ring A with unit.

(a) $_{M(A)'G}A$ is self-projective (projective in $\sigma[_{M(A)'G}A]$);

(b) $Hom_{M(A)'G}(A, -) : \sigma[_{M(A)'G}A] \to Z(A)^G$-Mod is exact;

(c) for every G-invariant ideal $I \subset A$, the sequence

$$0 \to Z_A^G(I) \to Z(A)^G \to Z_A^G(A/I) \to 0$$

is exact in $Z(A)^G$-Mod;

(d) for every G-invariant ideal $I \subset A$ and every $a \in A$ with $a + I \in Z(A/I)^G$, there exists $b \in Z(A)^G$ such that $a - b \in I$;

(e) for every $a \in A$, $(a + I_a) \cap Z(A)^G \neq \emptyset$,
where I_a denotes the ideal in A generated by the subset

$$\{({}^g a)x - x({}^h a), (({}^g a)x)y - ({}^h a)(xy), (yx)({}^g a) - y(x({}^h a)) \mid x, y, \in A,\ g, h \in G\}.$$

In case A is associative, $I_a = \sum_{g,h \in G} A(({}^g a)x - x({}^h a))A$.

Proof. The proof follows the same lines as 25.7 and 39.16.

(e) is based on the fact that for any $a \in A$ with $a + I \in Z(A/I)^G$, the ideal described is contained in I. □

Definition. We say an algebra A has a *large fixed centre* if for every non-zero G-invariant ideal $I \subset A$, $I \cap Z(A)^G \neq 0$.

The arguments from 24.2 yield in this situation:

41.7 Algebras with large fixed centre.

Assume A is an algebra with unit and large fixed centre. Then:

(1) For any G-invariant ideal $U \subset A$, $A(U \cap Z(A)^G)$ is an essential $M(A)'G$-submodule of U.

(2) A is a simple $M(A)'G$-module if and only if $Z(A)^G$ is a field.

(3) A is a semisimple $M(A)'G$-module if and only if $Z(A)^G$ is a semisimple ring.

For example, an algebra A has a large fixed centre if A has a large centre and $Z(A)$ has a large fixed ring. In particular, any semiprime associative PI-algebra has a large fixed centre provided G is finite and A has no additive $|G|$-torsion (see 39.3).

Combining projectivity and generating properties we consider the following conditions on the $M(A)'G$-module A.

- A is an intrinsically projective self-generator $M(A)'G$-module.
- A is a self-projective self-generator $M(A)'G$-module.
- A is a projective generator in $M(A)'G$-*Mod*.

From the description of ideal modules in 5.9 we have:

41.8 $_{M(A)'G}A$ as a $Z(A)^G$-ideal module.

The following statements are equivalent for G acting on an algebra A with unit.

(a) $_{M(A)'G}A$ is a $Z(A)^G$-ideal module (there is a bijection between G-invariant ideals in A and ideals in $Z(A)^G$);

(b) $_{M(A)'G}A$ is intrinsically projective and a self-generator;

(c) every G-invariant ideal in A can be written as $A \cdot I$, for some ideal $I \subset Z(A)^G$, and A is faithfully flat as a $Z(A)^G$-module.

Next we characterize A as a projective generator in $\sigma[_{M(A)'G}A]$.

41.9 $_{M(A)'G}A$ as a self-progenerator.

The following are equivalent for a group G acting on an algebra A with unit:

(a) $_{M(A)'G}A$ is a self-progenerator;

(b) $Hom_{M(A)'G}(A, -) : \sigma[_{M(A)'G}A] \to Z(A)^G$-Mod is an equivalence of categories (with inverse $A \otimes_{Z(A)^G} -$);

(c) for every G-invariant ideal $I \subset A$,

$$I = A \cdot Z_A^G(I) \quad \text{and} \quad Z(A/I)^G \simeq Z(A)^G / Z_A^G(I).$$

Finally we describe A as a progenerator in $M(A)'G$-*Mod*. Since $End_{M(A)'G}(A)$ is commutative we know that A being a generator in $M(A)'G$-*Mod* already implies that A is projective in $M(A)'G$-*Mod* (see 5.5). Moreover, we know from 37.8 that G has to be a finite group in this case.

41.10 $_{M(A)'G}A$ as a generator in $M(A)'G$-*Mod*.

For a finite group G acting on an algebra A with unit, the following are equivalent:

(a) *A is a generator in $M(A)'G$-Mod;*

(b) *A is a projective generator in $M(A)'G$-Mod;*

(c) *A is a finitely generated, projective right $Z(A)^G$-module, and $M(A)'G \simeq End_{Z(A)^G}(A)$;*

(d) *$Tr_{M(A)'G}(A, M(A)'G) = A Hom_{M(A)'G}(A, M(A)'G) = M(A)'G$;*

(e) *$Hom_{M(A)'G}(A, -) : M(A)'G\text{-}Mod \to Z(A)^G\text{-}Mod$ is an equivalence of categories (with inverse $A \otimes_{Z(A)^G} -$);*

Remark. For an associative and commutative algebra A, the situation above yields *Galois extensions* in the sense of Chase-Harrison-Rosenberg ([105, Theorem 1.3]).

Similar to 40.13 we observe for subgroups:

41.11 Action of subgroups.

Let G be a group acting on an algebra A with unit, and let $H \subset G$ be a subgroup.

(1) *If A is a projective $M(A)'G$-module, then A is also a projective $M(A)'H$-module.*

(2) *If A is a generator in $M(A)'G$-Mod, then A is a generator in $M(A)'H$-Mod.*

Proof. (1) In the proof of 40.13, the map $\varphi : (A'H)^{(\Gamma)} \to A'G$ is in fact an $M(A)'H$-isomorphism.

By assumption, $_{M(A)'G}A$ is a direct summand of $M(A)'G$. Moreover, by 40.13, $M(A)'G$ is a direct sum of copies of $M(A)'H$. Hence $A'H$ is a projective $M(A)'H$-module.

(2) Let A be a generator in $M(A)'G$-*Mod*. As a direct summand, $M(A)'H$ is generated by $M(A)'G$ as an $M(A)'H$-module and so it is A-generated. □

42 The central closure of G-semiprime rings

1.Extended centroid of a G-semiprime algebra. 2.Central G-closure of G-semiprime algebras. 3.Group action on the central closure. 4.Fixed elements of the extended centroid. 5.Remarks. 6.X-outer automorphisms and strongly prime rings.

In this section A will denote a non-associative algebra with unit, and G any group acting on A.

A is said to be *G-semiprime* if it does not contain any non-zero nilpotent G-invariant ideals, and A is called *G-prime* if for any G-invariant ideals $I, J \in A$, $IJ = 0$ implies $I = 0$ or $J = 0$. Obviously, if A is (semi)prime then it is G-(semi)prime.

The module theory used to construct extended centroid and central closure for a semiprime algebra allows similar constructions for G-semiprime algebras. Recalling that G-invariant ideals are just the $M_G(A)$-submodules of A we obtain from 32.1:

42.1 Extended centroid of a G-semiprime algebra.
Let A be any G-semiprime algebra. Let $Q_G(A)$ denote the self-injective hull of A as $M_G(A)$-module and $T_G := End_{M_G(A)}(Q_G(A))$. Then:

(1) A is a polyform $M_G(A)$-module.

(2) $C(A)^G \subset T_G$ and T_G is a commutative, regular and self-injective ring.

(3) $Q_G(A)$ is the quotient module of A for the Lambek torsion theory in $\sigma[_{M_G(A)}A]$ and

$$T_G \simeq \varinjlim \{Hom_{M_G(A)}(U, A) \mid U \trianglelefteq_{M_G(A)} A\} \simeq Hom_{M_G(A)}(A, Q_G(A)).$$

(4) T_G is semisimple if and only if A has finite uniform dimension as an $M_G(A)$-module.

T_G is called the *extended G-centroid* of A.

Proof. Transfer the proof of 32.1. □

Based on the above observations we can also construct the

42.2 Central G-closure of G-semiprime algebras.
Let A be any G-semiprime algebra and $T_G := End_{M_G(A)}(Q_G(A))$. Then:

(1) $Q_G(A) = AHom_{M_G(A)}(A, Q_G(A)) = AT_G$, and a ring structure is defined on $Q_G(A)$ by (linear extension of)

$$(as) \cdot (bt) := (ab)st, \quad \text{for } a, b \in A,\ s, t \in T_G.$$

(2) A is a subring of $Q_G(A)$.

(3) $Q_G(A)$ is a G-semiprime ring with centroid T_G (hence a T_G-algebra).

(4) $Q_G(A)$ is self-injective as an $M(Q_G(A))$-module.

(5) If A is a G-prime ring then $Q_G(A)$ is also a G-prime ring.

We call $Q_G(A)$ the *central G-closure* of A.

Proof. Apply the proof of 32.2. □

Of course, most of the results about the central closure of semiprime algebras find their interpretation for G-semiprime algebras (e.g., 32.3, 32.4, 32.5, 32.6, 32.7).

Since any semiprime algebra is G-semiprime, it has a central closure as well as a central G-closure. To compare these two constructions it is helpful to observe that automorphisms of semiprime algebras can be extended to their central closure:

42.3 Group action on the central closure.
Let $\gamma \in End_{M(A)}(\widehat{A})$ and denote $I := A\gamma^{-1}$. For any R-automorphism $h : A \to A$, ${}^hI$ is an essential ideal in A and there is an $M(A)$-homomorphism

$$ {}^hI \xrightarrow{h^{-1}} I \xrightarrow{\gamma} A \xrightarrow{h} A, $$

which can be uniquely extended to an $M(A)$-homomorphism ${}^h\gamma : \widehat{A} \to \widehat{A}$. Hence h yields an algebra automorphism

$$ h_c : End_{M(A)}(\widehat{A}) \to End_{M(A)}(\widehat{A}), \quad \gamma \mapsto {}^h\gamma, $$

and an action $i : G \to Aut(A)$ yields an action on the extended centroid,

$$ i_C : G \to Aut_R(End_{M(A)}(\widehat{A})), \quad g \mapsto i(g)_C. $$

Recalling that $\widehat{A} = AEnd_{M(A)}(\widehat{A})$, an automorphism $h : A \to A$ now yields an automorphism of the central closure,

$$ \widetilde{h} : AEnd_{M(A)}(\widehat{A}) \to AEnd_{M(A)}(\widehat{A}), \quad a\alpha \mapsto ({}^ha)({}^h\alpha), $$

and we have the group action

$$ i_C : G \to Aut_R(\widehat{A}), \quad g \mapsto \widetilde{i(g)}. $$

In particular, we may consider $\widehat{A}$ as an $M(A)'G$-module. However, $\widehat{A}$ need not be A-generated as an $M(A)'G$-module.

By the action defined above, the fixed elements of $End_{M(A)}(\widehat{A})$ are precisely those $M(A)$-endomorphisms which commute with the elements of G. Hence we have:

42.4 Fixed elements of the extended centroid.

Let G be a group acting on a semiprime algebra A. Then

$$[End_{M(A)}(\widehat{A})]^G = End_{M(A)'G}(\widehat{A}) = \{\gamma \in End_{M(A)}(\widehat{A}) \,|\, g \circ \gamma = \gamma \circ g \textit{ for each } g \in G\}.$$

42.5 Remarks. Let A be an associative semiprime algebra with Martindale left ring of quotients $Q_o(A)$. It was shown (by Kharchenko) that any automorphism of A can be uniquely extended to an automorphism of $Q_o(A)$ (which contains the central closure $\widehat{A}$, see [28, p. 42]). Following Kharchenko, define for any automorphism $g : A \to A$,

$$\Phi_g = \{x \in Q_o(A) \,|\, x({}^g a) = ax, \text{ for all } a \in A\}.$$

g is said to be *X-inner* if $\Phi_g \neq 0$.

An automorphism group G of A is called *X-outer* if the identity is the only X-inner element of G. Clearly, in this case G is an outer automorphism group for $\widehat{A}$.

The results on outer automorphisms of simple rings in 40.7 yield:

42.6 X-outer automorphisms and strongly prime rings.

Let G be an X-outer automorphism group of a strongly prime ring A with unit. Then:

(1) $\widehat{A}'G$ is a simple ring.

(2) $\widehat{A}$ is a generator in $\widehat{A}'G$-Mod.

Proof. Since A is strongly prime, $\widehat{A}$ is a simple ring (see 35.6). By the above remarks, G is outer on $\widehat{A}$ and the assertions follow from 40.7. □

References. Goursaud-Pascaud-Valette [146], Kharchenko [22], Montgomery [28].

43 Examples for group actions

In this section we collect several explicit examples for the situations considered in the previous sections.

43.1 Example. *A ring A and a group G such that ${}_{A'G}A$ is intrinsically projective but not self-projective.*

The ring extension $\mathbb{Z} \subset \mathbb{Z}[i]$. Consider the ring $\mathbb{Z}[i]$ of the Gaussian integers and the group $G := \{id, \varphi\}$, with φ the canonical complex conjugation. Then obviously $\mathbb{Z}[i]^G = \mathbb{Z}$. Clearly $\mathbb{Z}$ is a direct summand and hence a pure submodule of $\mathbb{Z}[i]$. So ${}_{\mathbb{Z}[i]'G}\mathbb{Z}$ is intrinsically projective by 39.14(1).

To see that ${}_{\mathbb{Z}[i]'G}\mathbb{Z}$ is not self-projective, consider the G-invariant ideal $I := \mathbb{Z}[i]\cdot 2$. For $x = i$, we have $x - {}^g x \in I$, for every $g \in G$, hence $x + I \in (\mathbb{Z}[i]/I)^G$.

However, for every $y \in \mathbb{Z}$, we have $x - y = -y + i \notin I$. Therefore ${}_{\mathbb{Z}[i]'G}\mathbb{Z}[i]$ is not self-projective. $\square$

Next we show that self-projectivity over $A'G$ is not a left-right symmetric property (see remarks after 39.22) and that the statements of 40.13 need not hold for self-progenerators.

43.2 Example. *An abelian group G acting on a ring A, such that ${}_{A'G}A$ is a self-progenerator but $A_{A'G}$ is not self-projective.*

Moreover, we have a (normal) subgroup $H \subset G$, such that neither ${}_{A'H}A$ nor ${}_{A^{H\prime}G/H}A^H$ are self-progenerators.

Let k be a field of characteristic $p \neq 0$ and $A = k^{(3,3)}$ be the ring of all 3×3-matrices over k. Let G be the subgroup of $Aut(A)$ generated by the two inner automorphisms of A acting by conjugation with the matrices

$$U = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the order of G is p^2 (and infinite if char k=0). For an arbitrary matrix

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix},$$

we obtain

$$UXU^{-1} = \begin{pmatrix} x_{11} + x_{21} & x_{12} - x_{11} - x_{21} + x_{22} & x_{13} + x_{23} \\ x_{21} & x_{22} - x_{21} & x_{23} \\ x_{31} & x_{32} - x_{31} & x_{33} \end{pmatrix} \qquad \text{(I)}$$

and

$$VXV^{-1} = \begin{pmatrix} x_{11}+x_{31} & x_{12}+x_{32} & x_{13}-x_{11}-x_{31}+x_{33} \\ x_{21} & x_{22} & x_{23}-x_{21} \\ x_{31} & x_{32} & x_{33}-x_{31} \end{pmatrix} \qquad \text{(II)}$$

If $X \in A^G$ then $UXU^{-1} = VXV^{-1} = X$ and hence the fixed ring is

$$A^G = \{\begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \mid a, b, c \in k\}.$$

(i) ${}_{A'G}A$ **is a self-progenerator.**

To show this we have to determine the G-invariant left ideals of A. Let $I \neq 0$ be a G-invariant left ideal of A. If there exists an element $X \in I$ such that $x_{i1} \neq 0$ for at least one index $i \in \{1,2,3\}$, then

$$X_i := \begin{pmatrix} x_{i1} & x_{i2} & x_{i3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in I.$$

Since I is G-invariant,

$$UX_iU^{-1} = \begin{pmatrix} x_{i1} & x_{i2}-x_{i1} & x_{i3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in I, \quad VX_iV^{-1} = \begin{pmatrix} x_{i1} & x_{i2} & x_{i3}-x_{i1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in I.$$

Hence

$$X_i - UX_iU^{-1} = \begin{pmatrix} 0 & x_{i1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in I, \quad X_i - VX_iV^{-1} = \begin{pmatrix} 0 & 0 & x_{i1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in I.$$

By multiplication with suitable matrices in A from the left, it is easy to see that

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in I$, i.e., $I = A$. So for $I \neq R$ we have $I = \begin{pmatrix} 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$.

If there is a matrix $X \in I$ of rank two, we obtain

$$I = I_1 := \begin{pmatrix} 0 & k & k \\ 0 & k & k \\ 0 & k & k \end{pmatrix},$$

and if $\operatorname{rank}(X) \leq 1$ for every $X \in I$, we can find $a, b \in k$ such that

$$I = I_{a,b} := \{\begin{pmatrix} 0 & xa & xb \\ 0 & ya & yb \\ 0 & za & zb \end{pmatrix} \mid x, y, z \in k\}.$$

Now we know what the G-invariant left ideals in A look like and we can prove that ${}_{A'G}A$ is a self-progenerator. For self-projectivity of ${}_{A'G}A$ we have to show, for every G-invariant left ideal $I \subset A$:

If $X - {}^gX \in I$, for every $g \in G$, there exists $Y \in A^G$ such that $X - Y \in I$.

This condition is obviously true for the trivial left ideals of A. If $X - {}^gX \in I_1$, then $x_{21} = x_{31} = 0$ and we can choose

$$Y = \begin{pmatrix} x_{11} & 0 & 0 \\ 0 & x_{11} & 0 \\ 0 & 0 & x_{11} \end{pmatrix} \in A^G.$$

Now let $a, b \in k$ and $a \neq 0$ or $b \neq 0$. For an element $X \in A$ with $X - {}^gX \in I_{a,b}$, for every $g \in G$, we derive from $X - UXU^{-1} \in I_{a,b}$ and (I),

$$x_{21} = x_{31} = 0,\ x_{22} - x_{11} = xa \text{ and } x_{23} = xb, \text{ for some } x \in k.$$

Similarly, by (II) and $X - VXV^{-1} \in I_{a,b}$, we obtain

$$x_{21} = x_{31} = 0,\ x_{32} = ya \text{ and } x_{33} - x_{11} = yb, \text{ for some } y \in k.$$

Choosing

$$Y = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{11} & 0 \\ 0 & 0 & x_{11} \end{pmatrix} \in A^G, \text{ we have } X - Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & xa & xb \\ 0 & ya & yb \end{pmatrix} \in I_{a,b}.$$

Hence ${}_{A'G}A$ is self-projective and it is easy to see that $A \cdot I^G = I$, for every G-invariant left ideal $I \subset A$. So ${}_{A'G}A$ is a self-generator.

(ii) $A_{A'G}$ is not self-projective.

To prove this consider the G-invariant right ideal (see (I), (II))

$$J := \begin{pmatrix} k & k & k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \subset A.$$

$$\text{For } X = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad UXU^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad VXV^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

which implies $X - {}^gX \in J$, for each $g \in G$. However, for every $Y \in A^G$,

$$X - Y = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} = \begin{pmatrix} 1-a & 1-b & -c \\ 0 & 1-a & 1 \\ 0 & 0 & -a \end{pmatrix} \notin J.$$

This proves that $A_{A'G}$ is not self-projective.

(iii) ${}_{A'H}A$ is not self-projective.
Let H denote the subgroup of G generated by $X \mapsto UXU^{-1}$. Then

$$A^H = \{ \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & d & e \end{pmatrix} \mid a,b,c,d,e \in k\}.$$

We show that ${}_{A'H}A$ is not self-projective. Of course, I_1 is also an H-invariant left ideal. For a matrix $X \in A$ with $X - {}^hX \in I_1$, for every $h \in H$, we have only one condition, namely $x_{21} = 0$. In particular, x_{31} may be non-zero. Then $X - Y$ has a non-zero first entry in the third row and hence $X - Y \notin I_1$. So ${}_{A'H}A$ is not self-projective (see 39.16).

(iv) ${}_{A^H{}'G/H}A^H$ is not self-projective.
Because of $UV = VU$, G is an abelian group and hence H is a normal subgroup of G. We denote the elements of G/H by $\overline{g}$ with $g \in G$. There is no difficulty to see that $(A^H)^{G/H} = A^G$. For the G/H-invariant left ideal

$$I := \begin{pmatrix} 0 & k & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \subset A^H \quad \text{and } X := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad VXV^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

and therefore $X - {}^{\overline{g}}X \in I$, for every $\overline{g} \in G/H$. However, for every

$$Y = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ 0 & y_{11} & 0 \\ 0 & 0 & y_{11} \end{pmatrix} \in A^G, \text{ we have } X - Y = \begin{pmatrix} -y_{11} & 1 - y_{12} & -y_{13} \\ 0 & -y_{11} & 0 \\ 0 & 0 & -y_{11} \end{pmatrix} \notin I,$$

which means that ${}_{A^H{}'G/H}A^H$ is not self-projective. □

Now we show that 40.11 need not hold for self-progenerators.

43.3 Example. *A group G acting on A, such that ${}_{A/Jac(A)'G}A/Jac(A)$ is a self-progenerator but ${}_{A'G}A$ is not.*

Consider the ring A of all 2×2 matrices over the ring $\mathbb{Z}_4$, and the automorphism group $G = \{id_A, \varphi\}$ given by

$$\varphi(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

${}_{A'G}A$ is not self-projective:
For the G-invariant left ideal $I = \{\begin{pmatrix} 2a & 0 \\ 2b & 0 \end{pmatrix} \mid a,b \in \mathbb{Z}_4\}$ and $X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in A$,

$$X - {}^{\varphi}X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \in I.$$

Hence $X+I={}^gX+I$, for every $g\in G$, i.e., $X+I\in(A/I)^G$.

It is easily verified that $A^G=\{\begin{pmatrix} a & 2b\\ 2c & d\end{pmatrix}\mid a,b,c,d\in\mathbb{Z}_4\}$, and it is impossible to find $Y\in A^G$ with $Y-X\in I$. So ${}_{A'G}A$ is not self-projective.

It is straightforward to show

$$Jac(A)=\{\begin{pmatrix} 2a & 2b\\ 2c & 2d\end{pmatrix}\mid a,b,c,d\in\mathbb{Z}_4\},$$

and the induced action of G on $A/Jac(A)$ is trivial. Hence ${}_{A/Jac(A)'G}A/Jac(A)$ is a self-progenerator (by 40.15). $\square$

The next two examples show that projectivity of ${}_{A'G}A$ in $A'G$-*Mod* does not depend on the invertibility of $|G|$ in A.

43.4 Examples. *A finite group G acting on an algebra A, $|G|^{-1}\notin A$, and ${}_{A'G}A$ a projective generator in $A'G$-Mod.*

(1) ([59]) Let A be the ring of 2×2-matrices over the field $\mathbb{Z}_2$, and let G be the automorphism group $G=\{\mathrm{id},\varphi\}$ with

$$\varphi\left(\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right):=\begin{pmatrix} 1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix} 1 & 0\\ 1 & 1\end{pmatrix}=\begin{pmatrix} a+b & b\\ a+b+c+d & b+d\end{pmatrix}.$$

${}_{A'G}A$ is projective in $A'G$-*Mod*, since

$$\sum_{g\in G}{}^g\begin{pmatrix} 1 & 1\\ 0 & 0\end{pmatrix}=\begin{pmatrix} 1 & 1\\ 0 & 0\end{pmatrix}+\varphi\left(\begin{pmatrix} 1 & 1\\ 0 & 0\end{pmatrix}\right)=\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix},$$

and ${}_{A'G}A$ is a generator in $A'G$-*Mod*, since for

$$x_1=y_1=\begin{pmatrix} 1 & 1\\ 0 & 0\end{pmatrix}\text{ and } x_2=y_2=\begin{pmatrix} 0 & 1\\ 0 & 1\end{pmatrix}$$

we have

$$x_1(\sum_{g\in G}g)y_1+x_2(\sum_{g\in G}g)y_2=1\cdot\mathrm{id}.$$

(2) Let $A:=\mathbb{Z}_2[x,y\mid xy+yx=1]$ be the first Weyl algebra over $\mathbb{Z}_2$ and $G=\{id,\varphi\}$ with $\varphi(x):=y$ and $\varphi(y):=x$. ${}_{A'G}A$ is a projective generator in $A'G$-*Mod*, since

$$\sum_{g\in G}{}^g(xy)=xy+\varphi(xy)=xy+yx=1,\ \text{and}$$

$$xy(\mathrm{id}+\varphi)1+1(\mathrm{id}+\varphi)yx=1\cdot\mathrm{id}.$$

$\square$

For more examples for A being a progenerator in $A'G$-*Mod* see [59].

The following example (suggested by S. Montgomery) was used in [109] to show that ${}_{A'G}A$ need not be a faithful $A'G$-module, even if G acts faithfully on A. In fact it shows the following:

43.5 Example. *A finite group G acting faithfully on an algebra A, with $|G|^{-1} \in A$ (hence A is projective in $A'G$-Mod) and ${}_{A'G}A$ is a self-progenerator but not a generator in $A'G$-Mod.*

Let $A := \mathbb{Q}^{(3,3)}$, the ring of 3×3-matrices over the rationals $\mathbb{Q}$. Consider the elements in A,

$$x := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad y := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad xy = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = yx.$$

Denote by φ_x, φ_y and φ_{xy} the inner automorphisms of A, defined by conjugation with x, y and xy, respectively. These yield a group G of order 4,

$$G = \{\mathrm{id}_A, \varphi_x, \varphi_y, \varphi_{xy}\}.$$

The order of G is invertible in A and A is left semisimple. Hence (by 38.4) $A'G$ is also left semisimple. In particular, ${}_{A'G}A$ is a semisimple module and hence a self-progenerator. However, ${}_{A'G}A$ is not a generator in $A'G$-*Mod* since ${}_{A'G}A$ is not a faithful $A'G$-module: From $x + y - xy - 1 = 0$ we have for each $a \in A$,

$$(x\varphi_x + y\varphi_y - xy\varphi_{xy} - \mathrm{id}) \cdot a = x^2ax + y^2ay - (xy)^2axy - a = a(x + y - xy - 1) = 0.$$

Therefore $0 \neq x\varphi_x + y\varphi_y - xy(\varphi_{xy}) - \mathrm{id} \in An_{A'G}(A)$. $\square$

43.6 Example. *A finite group G acting on a commutative algebra A with $|G|^{-1} \notin A$, ${}_{A'G}A$ is semisimple (hence a self-progenerator) but neither projective nor a generator in $A'G$-Mod.*

Consider the ring $A := (\mathbb{Z}_2)^3$ with componentwise addition and multiplication, and the group $G = \{\mathrm{id}, \varphi\}$ with the automorphism

$$\varphi : A \to A, (a_1, a_2, a_3) \mapsto (a_3, a_2, a_1).$$

For any $(a_1, a_2, a_3) \in A$,

$$\sum_{g \in G} {}^g(a_1, a_2, a_3) = (a_1, a_2, a_3) + \varphi((a_1, a_2, a_3)) = (a_1 + a_3, 0, a_1 + a_3) \neq (1, 1, 1),$$

Hence ${}_{A'G}A$ is not projective in $A'G$-*Mod*. Moreover, ${}_{A'G}A$ is not a generator in $A'G$-*Mod* since it is not a faithful $A'G$-module: For all $(a_1, a_2, a_3) \in A$,

$$((0,1,0)\varphi - (0,1,0)\mathrm{id}) \cdot (a_1, a_2, a_3) = (0, a_2, 0) - (0, a_2, 0) = (0,0,0),$$

i.e., $0 \neq (0,1,0)\varphi - (0,1,0)\mathrm{id} \in An_{A'G}(A)$.

${}_{A'G}A$ is a semisimple module since we have a decompostion

$${}_{A'G}A = A'G \cdot (1,0,0) \oplus A'G \cdot (0,1,0),$$

where $A'G \cdot (1,0,0) = \{(a_1, 0, a_3) \in A\}$ and $A'G \cdot (0,1,0) = \{(0, a_2, 0) \in A\}$ are simple $A'G$-modules.

Notice that A can also be considered as 3×3-diagonal matrices over $\mathbb{Z}_2$ and the automorphism φ can be described by conjugation with the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \notin A.$$

□

The preceding example can be modified such that ${}_{A'G}A$ is no longer semisimple but still is a self-progenerator.

43.7 Example. *A finite group G acting on a commutative algebra A, with $|G|^{-1} \notin A$, ${}_{A'G}A$ is a self-progenerator but neither semisimple, nor projective nor a generator in $A'G$-Mod.*

Let $A := (\mathbb{Z}_4)^3$ be endowed with componentwise addition and multiplication, and let $G = \{id, \varphi\}$ be the same group as in 43.6. It is obvious from 43.6 that ${}_{A'G}A$ is not faithful and hence is not a generator in $A'G$-*Mod*. Moreover, ${}_{A'G}A$ is not projective in $A'G$-*Mod* since for each $(a_1, a_2, a_3) \in A$,

$$\sum_{g \in G} {}^{g}(a_1, a_2, a_3) = (a_1, a_2, a_3) + \varphi((a_1, a_2, a_3)) = (a_1 + a_3, 2a_2, a_1 + a_3) \neq (1,1,1).$$

To show that ${}_{A'G}A$ is a self-progenerator we have to find all G-invariant ideals in A. Obviously they are of the form $I = (C, D, C)$ with suitable ideals $C, D \subset \mathbb{Z}_4$, i.e., $C, D \in \{\{0\}, 2\mathbb{Z}_4, \mathbb{Z}_4\}$. In fact, the G-invariant ideals in A are

$$\begin{array}{lll} I_0 := (0,0,0) & I_3 := (2\mathbb{Z}_4, 0, 2\mathbb{Z}_4) & I_6 := (2\mathbb{Z}_4, \mathbb{Z}_4, 2\mathbb{Z}_4) \\ I_1 := (0, 2\mathbb{Z}_4, 0) & I_4 := (\mathbb{Z}_4, 0, \mathbb{Z}_4) & I_7 := (\mathbb{Z}_4, 2\mathbb{Z}_4, \mathbb{Z}_4) \\ I_2 := (0, \mathbb{Z}_4, 0) & I_5 := (2\mathbb{Z}_4, 2\mathbb{Z}_4, 2\mathbb{Z}_4) & I_8 := A\,. \end{array}$$

It is easy to see that I_1 and I_3 are small submodules of A and hence ${}_{A'G}A$ is not semisimple.

To show that ${}_{A'G}A$ is self-projective we have to check (by 39.16) that for any G-invariant ideal $I = (C, D, C)$ and $x \in A$ with $x + I \in (A/I)^G$, there exists $y \in A^G$ with $x - y \in I$.

Let $x = (x_1, x_2, x_3) \in A$. Then $x + I \in (A/I)^G$ if and only if

$$(x_1, x_2, x_3) - {}^{\varphi}(x_1, x_2, x_3) = (x_1 - x_3, 0, x_3 - x_1) \in I.$$

In particular, $x_1 - x_3 \in C$. Obviously, $A^G = \{(a_1, a_2, a_3) \in A \mid a_1 = a_3\}$.

Putting $y := (x_3, x_2, x_3) \in A^G$ we have

$$x - y = (x_1 - x_3, 0, 0) \in (C, D, C) = I.$$

Hence ${}_{A'G}A$ is self-projective. ${}_{A'G}A$ is a self-generator if

$$I = A \cdot Hom_{A'G}(A, I) = A \cdot I^G \text{ for all } G\text{-invariant ideals } I \subset A.$$

The inclusion $A \cdot I^G \subset I$ is trivial. Let $(x_1, x_2, x_3) \in I = (C, D, C)$ mit $C = \mathbb{Z}_4 \cdot c$ and $D = \mathbb{Z}_4 \cdot d$, for suitable $c, d \in \mathbb{Z}_4$. There exist $r_1, r_2, r_3 \in \mathbb{Z}_4$ with

$$(x_1, x_2, x_3) = (r_1 c, r_2 d, r_3 c) = (r_1, r_2, r_3) \cdot (c, d, c) \in A \cdot I^G.$$

This finishes our proof. $\square$

References: Auslander-Reiten-Smalø [59], García-del Río [141, 142], Montgomery [28], Wilke [265].

Bibliography

Monographs

[1] Anderson, F., Fuller, K., *Rings and Categories of Modules*, Springer GTM (1974)

[2] Andrunakievic, V.A., Rjabuhin, V.A., *Radicals of Algebras and Structure Theory* (Russian), NAUKA, Moscow (1979)

[3] Bakhturin, J.A., *Identities in Lie Algebras* (Russian), NAUKA, Moscow (1985)

[4] Beidar, K., Martindale, W.S., Mikhalev, A.V., *Rings with Generalized Polynomial Identities*, Marcel Dekker, New York (1995)

[5] Bican, L., Kepka, T., Nemec, P., *Rings, modules and preradicals*, LN Pure Appl. Math. 75, Marcel Dekker, New York (1982)

[6] Bourbaki, N., *Algèbre, Chapitres 1 à 3*, Hermann, Paris (1970)

[7] ——, *Algèbre commutative, Chapitres 1 à 9*, Hermann, Paris (1961-83)

[8] Bueso, J.L., Jara, P., Verschoren, A., *Compatibility, Stability, and Sheaves*, Marcel Dekker, New York (1994)

[9] Chatters, A..W., Hajarnavis, C.R., *Rings with chain conditions*, Pitman, London (1980)

[10] DeMeyer, F., Ingraham, E. *Separable Algebras over Commutative Rings*, Springer LNM 181 (1971)

[11] Dung, N.V., Huynh, D.V., Smith, P.F., Wisbauer, R., *Extending Modules*, Pitman Research Notes 313, Longman (1994)

[12] Faith, C., *Algebra: Rings Modules and Categories I*, Springer, Grundlehren 190 (1973)

[13] ——, *Algebra II, Ring Theory*, Springer, Grundlehren 191 (1976)

[14] Golan, J.S., *Localization of noncommutative rings*, Marcel Dekker, New York (1975)

[15] Goodearl, K.R., *Von Neumann Regular Rings*, Pitman, London (1979)

[16] Goodearl, K.R., Handelman, D., Lawrence, J., *Strongly prime and completely torsionfree rings*, Carlton Math. Series 109, Ottawa (1974)

[17] Gray, M., *A Radical Approach to Algebra*, Addison-Wesley, Reading (1970)

[18] Herstein, I.N., *Topics in Ring Theory*, Math. Lecture Notes, Chicago (1965)

[19] Jacobson, N., *Structure and Representations of Jordan Algebras*, Amer. Math. Soc. Coll. Publ. 39, Providence (1968)

[20] Jacobson, N., *Structure Theory of Quadratic Jordan Algebras*, University of Arkansas Lecture Notes (1981)

[21] Kasch, F., *Ringe und Moduln*, Teubner, Stuttgart (1977)

[22] Kharchenko, V.K., *Automorphisms and Derivations of Associative Rings*, Kluwer Acad. Publ., Dordrecht (1991)

[23] Knus, M.A, Ojanguren, M., *Théorie de la Descente et Algèbre d'Azumaya,* Springer LNM 389 (1974)

[24] Lambek, J., *Lectures on Rings and Modules*, Blaisdell Publ., Waltham e.a. (1966)

[25] Matsumura, H., *Commutative ring theory*, Cambridge University Press (1986)

[26] Mishina, A.P., Skornjakov, L.A., *Abelian Groups and Modules*, AMS Translations 107, Amer. Math. Soc., Providence (1976)

[27] Mohamed, S.H., Müller, B., *Continuous and Discrete Modules*, Lond. Math. Soc. LN 147, Cambridge University Press (1990)

[28] Montgomery, S., *Fixed rings of finite automorphism groups of associative rings*, Springer LNM 818 (1980)

[29] Myung, H.Ch., *Malcev-admissible algebras*, Birkhäuser, Boston e.a. (1986)

[30] Passman, D.S., *Infinite Group Rings*, Marcel Dekker (1971)

[31] ——, *Group Rings, Crossed products, and Galois Theory*, Regional Conf. Ser. Math. 64, Amer. Math. Soc. (1986)

[32] Pierce, R.S., *Modules over commutative regular rings*, Mem. Amer. Math. Soc. 70 (1967)

[33] ——, *Associative Algebras*, Springer, Graduate Texts in Math. 88 (1982)

[34] Postnikov, M.M., *Lie groups and algebras* (Russian), Nauka, Moscow (1982)

[35] Renault, G., *Algèbre non commutative*, Gauthier-Villars, Paris (1975)

[36] Rowen, L.H., *Polynomial identities in Ring Theory*, Academic Press, New York-London (1980)

[37] Schafer, R.D., *An introduction to nonassociative algebras*, Academic Press, New York-London (1966)

[38] Schubert, H., *Homologische Algebra*, Skriptum, Univ. Düsseldorf (1979)

[39] Stenström, B., *Rings of Quotients*, Springer, Berlin (1975)

[40] Wisbauer, R., *Foundations of Module and Ring Theory*, Gordon and Breach, Reading, Paris (1991)

[41] Zhevlakov, K.A., Slinko, A.M., Shestakov, I.P., Shirshov, A.I., *Rings that are nearly associative*, Nauka, Moskau (1978), Translation: Academic Press, New York (1982)

Articles

[42] Albert, A.A., *Non-associative algebras*, Annals of Math. 43, 685-723 (1942)

[43] ——, *The radical of a non-associative algebra*, Bull. Amer. Math. Soc. 48, 891-897 (1942)

[44] Alfaro, R., Ara, P., del Rio, A., *Regular skew group rings*, J. Austral. Math. Soc. 58, 167-182 (1995)

[45] Amitsur, S.A., *On rings of quotients*, Symp. Mathematica 8, 149-164 (1972)

[46] Andrunakievic, V.A., *Biregular rings* (russian), Mat. Sbornik 39, 447-464 (1956)

[47] Andrunakievic, A.A., Andrunakievic, V.A., *On subdirect products of division rings*, Dokl. Akad. Nauk SSSr 269, 311-319 (1983) (Transl. in Soviet Math. Dokl. 27 (1983))

[48] ——, *Completely semiprime and abelian regular ideals of a ring*, Mat. Zametki 39, 311-319 (1986)

[49] Ara, P., *Centers of maximal quotient rings*, Arch. Math. 50, 342-347 (1988)

[50] ——, *Extended centroid of C^*-algebras*, Arch. Math. 54, 358-364 (1990)

[51] Arens, R., Kaplansky, I., *Topologigal representations of algebras*, Trans. Amer. Math. Soc. 63, 457-481 (1949)

[52] Armendariz, E.P., *On semiprime P.I.-algebras over commutative regular rings*, Pac. J. Math. 66, 23-28 (1976)

[53] ——, *On semiprime rings of bounded index*, Proc. Amer. Math. Soc. 85, 146-148 (1982)

[54] Armendariz, E.P., Fisher, J.W., *Regular PI-rings*, Proc. Amer. Math. Soc. 39, 247-251 (1973)

[55] Armendariz, E.P., Fisher, J.W., Steinberg, S.A., *Central localizations of regular rings*, Proc. Amer. Math. Soc. 46, 315-321 (1974)

[56] Armendariz, E.P., Hajarnavis, C.R., *A note on semiprime rings algebraic over their centres*, Comm. Algebra 17, 1627-1631 (1989)

[57] Artin, M., *On Azumaya Algebras and Finite Dimensional Representations of Rings*, J. Algebra 11, 532-563 (1969)

[58] Auslander, M., Goldman, O., *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc. 97, 367-409 (1960)

[59] Auslander, M., Reiten, I., Smalø, S., *Galois actions on rings and finite Galois coverings*, Math. Scand. 65, 5-32 (1989)

[60] Azumaya, G., *Separable Rings*, J. Algebra 63, 1-14 (1980)

[61] Baues, O., *Flache Strukturen auf $\mathcal{GL}(n)$ und zugehörige linkssymmetrische Algebren*, Dissertation, Universität Düsseldorf (1995)

[62] Baxter, W.E., Martindale, W.S., *Central closure of semiprime non-associative rings*, Comm. Algebra 7, 1103-1132 (1979)

[63] ——, *The extended centroid in semiprime rings with involution*, Comm. Algebra 13, 945-985 (1985)

[64] Beachy, J.A., *Some aspects of noncommutative localization*, Noncommutative Ring Theory, Kent State 1975, Springer LNM 545.

[65] ——, *On localization at an ideal*, Canad. Math. Bull. 22, 397-401 (1979)

[66] Beachy, J.A., Blair, W.D., *Localization at semiprime ideals*, J. Algebra 38, 309-314 (1974)

[67] ——, *Rings whose faithful left ideals are cofaithful*, Pac. J. Math. 58, 1-13 (1975)

[68] Behrens, E.A., *Nichtassoziative Ringe*, Math. Ann. 127, 441-452 (1954)

[69] ——, *Zur additiven Idealtheorie in nichtassoziativen Ringen*, Math. Zeit. 64, 169-182 (1956)

[70] Beidar, K.I., *Rings with generalized identities I*, Vestnik Mosk. Univ. Ser. Mat.-Mex. 2, 19-26 (1977)

[71] ——, *Quotient rings of semiprime rings*, Vestnik Mosk. Univ. Ser. Mat.-Mex. 5, 36 - 43 (1978)

[72] ——, *Classical rings of quotients of PI-rings*, Usp. Matem. Nauk 33(6), 197 - 198 (1978)

[73] Beidar K.I., Mikhalev A.V., *Structure of non-degenerated alternative algebras*, Trudi Sem. I.G. Petrov. 12, 59-74 (1987) (Russian)

[74] ——, *Semiprime rings with bounded index of nilpotent elements*, Trudi Sem. I.G. Petrov. 13, 237 - 249 (1988) (Russian)

[75] Beidar K.I., Wisbauer R., *Strongly semiprime modules and rings*, Usp. Mat. Nauk 48, 161-162 (1993)

[76] ——, *Strongly and properly semiprime rings and modules*, Ring Theory, ed. by Jain-Rizvi, World Scientific, Singapore, 58-95 (1993)

[77] ——, *Properly semiprime self-pp-modules*, Comm. Algebra. (1995)

[78] Berning, J., *Beziehungen zwischen links-linearen Toplogien und Modulkategorien*, Dissertation, Universität Düsseldorf (1994)

[79] Bican, L., Jambor, P., Kepka, T., Nemec, P., *Prime and coprime modules*, Fund. Math. 57, 33-45 (1980)

[80] Birkenmeier, G.F., *Idempotents and completely semiprime ideals*, Comm. Algebra 11, 567-580 (1983)

[81] Birkenmeier, G.F., Kim, J.Y., Park, J.K., *A connection between weak regularity and the simplicity of prime factor rings*, Proc. Amer. Math. Soc. (1994)

[82] Bix, R., *Separable Jordan Algebras over Commutative Rings*, J. Algebra 57, 111-143 (1979)

[83] Bix, R., *Separable Alternative Algebras over Commutative Rings*, J. Algebra 92, 81-103 (1985)

[84] Blair, W.D., Tsutsui, H., *Fully prime rings*, Comm. Algebra 22, 5389-5400 (1994)

[85] Boyle, A.K., *Injectives containing no quasi-injective submodules*, Comm. Algebra 4, 775-785 (1976)

[86] Braun, A., *On Artin's Theorem and Azumaya algebras*, J. Algebra 77, 323-332 (1982)

[87] ——, *On a question of G. Bergman and L. Small on Azumaya algebras*, J. Algebra 89, 24-45 (1984)

[88] ——, *On the defining axioms of Azumaya algebras*, J. Algebra 109, 166-171 (1987)

[89] Brodskii, G.M., *Steinitz modules*, Mat. Issled. Kishinev 2(24), 14-28 (1972)

[90] ——, *Annihilator conditions in endomorphism rings of modules*, Mat. Zametki 16, 933-942 (1974)

[91] Brown, B., McCoy, N.H., *Prime ideals in nonassociative rings*, Trans. Amer. Math. Soc. 89, 245-255 (1958)

[92] Bruck, R.H., Kleinfeld, E., *The structure of alternative division rings*, Proc. Amer. Math. Soc. 2, 878-890 (1951)

[93] Bueso, J.L., Jara, P., Merino, L., *A structure sheaf on noetherian rings*, Comm. Algebra 22, 3511-3530 (1994)

[94] Burgess, W.D., *On nonsingular right FPF rings*, Comm. Algebra 12(14), 1729-1750 (1984)

[95] Burgess, W.D., Byrd, K.A., Raphael, R., *Self-injective simple Pierce sheaves*, Arch. Math. 42, 354-361 (1984)

[96] Burgess, W.D., Stephenson, W., *Pierce sheaves of non-commutative rings*, Comm. Algebra 4(1), 51-75 (1976)

[97] ——, *An analogue of the Pierce sheaf for non-commutative rings*, Comm. Algebra 6(9), 863-886 (1978)

[98] ——, *Rings all of whose Pierce stalks are local*, Canad. Math. Bull. 22, 159-164 (1979)

[99] Burkholder, D.G., *Azumaya rings with locally perfect centers*, J. Algebra 103, 606-618 (1986)

[100] ——, *Azumaya rings, Pierce stalks and central ideal algebras*, Comm. Algebra 17, 103-113 (1989)

[101] ——, *Products of Azumaya rings and Kochen's map*, Comm. Algebra 17, 115-134 (1989)

[102] ——, *Modules with local Pierce stalks*, J. Algebra 145, 339-348 (1992)

[103] Byun, H., *Globalization of local epimorphisms*, Comm. Algebra 17(2), 325-330 (1989)

[104] Cabrera Garcia, M., Rodriguez Palacios, A., *Extended centroid and central closure of semiprime normed algebras*, Comm. Algebra 18, 2293-2326 (1990)

[105] Chase, S.U., Harrison, D.K., Rosenberg, A., *Galois theory and Galois cohomology of commutative rings*, Mem. Amer. Math. Soc. 52, 1-20 (1965)

[106] Cheng, C.C., *Separable semigroup algebras*, J. Pure Appl. Alg. 33, 151-158 (1984)

[107] Chi, D.T.M., *Automorphisms of prime Azumaya algebras*, Comm. Alg. 12, 1207-1212 (1984)

[108] Cobalea Vico, A., Fernández López, A., *Prime noncommutative Jordan algebras with nonzero socle and central closure*, Algebras, Groups and Geometries 5, 129-136 (1988)

[109] Cohen, M., *A Morita context related to finite automorphism groups of rings*, Pac. J. Math. 98, 37-54 (1982)

[110] Courter, R., *Finite direct sums of complete matrix rings over perfect completely primary rings*, Canad. J. Math. 21, 430-446 (1969)

[111] ——, *Fully idempotent rings have regular centroids*, Proc. Amer. Math. Soc. 43, 293-296 (1974)

[112] Cunningham, R.S, *Strongly separable pairings of rings*, Trans. Amer. Math. Soc. 148, 399-416 (1970)

[113] Dauns, J., Hofmann, K.H., *The representation of biregular rings by sheaves,* Math. Z. 91, 103-123 (1966)

[114] Delale, J.P., *Localisation et premiers à gauche*, Sem. d'algebre non commutative, Publ. Math. d'Orsay, Conf. 6 et 8 (1972)

[115] ——, *Sur le spectre d'un anneau non commutatif,* Thèse, Université Paris Sud, Centre d'Orsay (1974)

[116] DeMeyer, F., Hardy, D., *Semigroup rings which are separable algebras*, Ring Theory, Antwerp 1985, Springer LNM 1197, 51-59 (1986)

[117] DeMeyer, F., Janusz, G.J., *Group rings which are Azumaya algebras*, Trans. Amer. Math. Soc. 279, 389-395 (1983)

[118] Desale, G., Nicholson, W.K., *Endoprimitive rings*, J. Algebra 70, 548-560 (1981)

[119] Dicks, W., *On a characterization of Azumaya algebras*, Publ. Matemàtiques 32, 165-166 (1988)

[120] Eilenberg, S., *Extension of general algebras*, Ann. Pol. Soc. Math. 21, 125-134 (1948)

[121] Elduque, A., *On Malcev modules*, Comm. Algebra 18, 1551-1561 (1990)

[122] Endo, S., *Completely faithful modules and quasi-Frobenius algebras*, J. Math. Soc. Japan 19, 437-456 (1967)

[123] Enochs, E.E., García, J.J., del Río, A., *When does R Gorenstein imply R^G Gorenstein?*, preprint (1995)

[124] Erickson, T.S., Martindale, W.S., Osborn, J.M., *Prime nonassociative algebras*, Pac. J. Math. 60, 49-63 (1975)

[125] Faith, C., *Locally perfect commutative rings are those whose modules have maximal submodules*, Comm. Algebra 23, 4885-4886 (1995)

[126] Farrand, S.M., *The Multiplication Algebra of a Non-Associative Algebra*, Dissertation, UC San Diego (1981)

[127] Farrand, S.M, Finston, D.R., *On the multiplication ideal of a nonassociative algebra*, Arch. Math. 48, 298-302 (1987)

[128] Fernández López, A., *Prime nondegenerate Jordan algebras with nonzero socle and the symmetric Martindale algebra of quotients*, Coll. Math. 39, 249-256 (1988)

[129] Ferrero, M., *Centred Bimodules over Prime Rings: Closed Submodules and Applications to Ring Extensions*, J. Algebra 172, 470-505 (1995)

[130] ——, *Closed Submodules of Centred Bimodules over Prime Rings, and Applications*, Resenhas IME-USP 2, 143-160 (1995)

[131] ——, *Closed Submodules of Centred Bimodules over Semiprime Rings, and Applications to Ring Extensions*, preprint

[132] Ferrero, M., Wisbauer, R., *Closure operations in modules categories*, Algebra Colloq. (1996)

[133] Filippov, V.T., *Central simple Mal'tsev algebras*, Algebra i Logika 15, 235-242 (1976)

[134] ——, *Prime Mal'tsev algebras*, Matem. Zametki 31, 669-678 (1982)

[135] Finston, D.R., *On Multiplication Algebras*, Trans. Amer. Math. Soc. 293, 807-818 (1986)

[136] Fisher, J.W., *Structure of semiprime PI-rings*, Proc. Amer. Math. Soc. 39, 465-467 (1973)

[137] Fisher, J.W., Osterburg, J., *Finite group actions on noncommutative rings: A survey since 1970*, B.R. McDonald (Ed.), Ring theory and algebra III, Marcel Decker LNPAM 55, 357-393 (1980)

[138] Fisher, J.W., Snider, R.L., *On the von Neumann regularity of rings with regular prime factor rings*, Pac. J. Math. 54, 135-144 (1974)

[139] Fuller, K.R., *Density and Equivalence*, J. Algebra 29, 528-550 (1974)

[140] Gabriel, P., *Des catégories abéliennes*, Bull. Soc. Math. France 90, 323-448 (1962)

[141] García, J.J., del Río, A., *Actions of groups in fully bounded noetherian rings*, to appear

[142] ——, *On flatness and projectivity of a ring as module over a fixed subring*, Math. Scand., to appear

[143] Goel, V.K, Jain, S.K., *π-injective modules and rings whose cyclics are π-injective*, Comm. Algebra 6, 59-73 (1978)

[144] Goldman, O., *Rings and modules of quotients*, J. Algebra 13, 10-47 (1969)

[145] González, S., *The complete ring of quotients of a reduced alternative ring. Orthogonal completion*, Comm. Algebra 17, 2017-2034 (1989)

[146] Goursaud, J.M., Pascaud, J.L., Valette, P., *Sur les travaux de V.K. Kharchenko*, Sém. d'Algèbre Dubreil-Malliavin, Springer LNM 924, 322-355 (1982)

[147] Grabmeier, J., Wisbauer, R., *Computations in algebras of finite rank*, Computational Algebra, Marcel Dekker, LNPAM 151, 131-165 (1994)

[148] Hamsher, R., *Commutative rings over which every module has a maximal submodule*, Proc. Amer. Math. Soc. 18, 1133-1137 (1967)

[149] Handelman, D., *Strongly semiprime rings*, Pac. J. Math. 60, 115-122 (1975)

[150] Handelman, D., Lawrence, J., *Strongly prime rings*, Trans. Amer. Math. Soc. 211, 209-223 (1975)

[151] Hannah, J., *Quotient rings of semiprime rings with bounded index*, Glasgow Math. J. 23, 53-64 (1982)

[152] Harada, M., *Perfect Categories I*, Osaka J. Math. 10, 329-341 (1973)

[153] Hattori, A., *Semisimple algebras over a commutative ring*, J. Math. Soc. Japan 15, 404-419 (1963)

[154] Hauger, G., Zimmermann, W., *Quasi-Frobenius-Moduln*, Arch. Math. 24, 379-386 (1975)

[155] Helmstetter, J., *Algèbres symétriques à gauche*, C.R. Acad. Sc. Paris 272, 1088-1091 (1971)

[156] Hentzel, I.R., *Alternative rings without nilpotent elements*, Proc. Amer. Math. Soc. 42, 373-376 (1974)

[157] Higman, D.G., *Relative Cohomology*, Canad. J. Math. 9, 19-34 (1957)

[158] Hirato, K., Sugano, K., *On semisimple extensions and separable extensions over non commutative rings*, J. Math. Soc. Japan 18, 360-373 (1966)

[159] Hochschild, G., *Relative homologigal algebra*, Trans. Amer. Math. Soc. 82, 246-269 (1956)

[160] Hofmann, K.H., *Representations of algebras by continuous sections*, Bull. Amer. Math. Soc. 78, 291-373 (1972)

[161] Hutchinson, J.J., Leu H.M., *Rings of quotients of R and eRe*, Chinese J. Math. 4, 25-35 (1976)

[162] Huynh, D.V., Dung, N.V., Wisbauer, R., *Quasi-injective modules with acc or dcc on essential submodules*, Arch. Math. 53, 252-255 (1989)

[163] Ion, I.D., *Rings with large centers*, Bull. Math. Soc. Sci. Roumanie 32, 305-310 (1988)

[164] Izawa, T., *Maximal quotient rings of endomorphism rings of $E(R_R)$-torsionfree generators*, Can. J. Math. 33, 585-605 (1981)

[165] Jacobson, N., *A note on non-associative algebras*, Duke Math. J. 3, 544-548 (1937)

[166] ——, *Structure of alternative and Jordan bimodules*, Osaka Math. J. 6, 1-71 (1954)

[167] Jenner, W.E., *The radical of a non-associative ring*, Proc. Amer. Math. Soc. 1, 348-351 (1950)

[168] Johnson, A.E., *Representations of prime rings*, Trans. Amer. Math. Soc. 74, 351-357 (1953)

[169] Johnson, R.E, Wong, E.T., *Quasi-injective modules and irreducible rings*, J. London Math. Soc. 36, 260-268 (1961)

[170] Jøndrup S., *Rings of quotients of some semiprime P.I. rings*, Comm. Algebra 7, 279-286 (1979)

[171] ——, *When is the ring a projective module over the fixed point ring*, Comm. Algebra 16, 1971-1972 (1988)

[172] Kambara, H., Oshiro, K., *On P-exchange rings*, Osaka J. Math. 25, 833-842 (1988)

[173] Kaplansky, I., *Topological representation of algebras*, Trans. Amer. Math. Soc. 68, 62-75 (1950)

[174] ——, *Algebraic and analytic aspects of operator algebras*, Regional Conf. Ser. Math. 1, AMS, Providence, R.I. (1970)

[175] Katayama, H., *A study of rings with trivial preradical ideals*, Tsukuba J. Math. 5, 67-83 (1981)

[176] ——, *On rings for which various types of nonzero preradicals are cofaithful*, Hokkaido Math. J. 12, 49-63 (1983)

[177] Kaup, W., *Jordan algebras and holomorphy*, Functional Analysis, Holomorphy, and Approximation Theory, ed. by S. Machado, Springer LNM 843 (1981)

[178] Kerner, O., Thode, Th., *Primary Rings and Related Radical*, Arch. Math. 27, 372-377 (1976)

[179] Kirkman, E.E., *Inertial coefficient rings and the idempotent lifting property*, Proc. Amer. Math. Soc. 61, 217-222 (1976)

[180] Kleinfeld, E., *Simple alternative rings*, Ann. Math. 58, 544-547 (1953)

[181] Koifman, L.A., *Rings over which every module has a maximal submodule*, Math. Notes Acad. Sci. USSR 7, 359-367 (1970)

[182] Kutami, M., Oshiro, K., *Strongly semiprime rings and nonsingular quasi-injective modules*, Osaka J. Math. 17, 41-50 (1980)

[183] Lambek, J., *Noncommutative localization*, Bull. Amer. Math. Soc. 79, 857-872 (1973)

[184] Lee, H.K., Waterhouse, W.C., *Maximal orders in nonassociative quaternion algebras*, J. Algebra 146, 441-453 (1992)

[185] Lenzing, H., *A homological characterization of Steinitz rings*, Proc. Amer. Math. Soc. 29, 269-271 (1971)

[186] Lesieur, L., *Sur quelques classes d'anneaux non commutatifs*, Comm. Algebra 14, 1481-1488 (1986)

[187] Leu, H.M., *The ring of quotients of a module endomorphism ring*, Tamkang J. Math. 7, 77-86 (1976)

[188] Levitzki, J., *On automorphisms of certain rings*, Annals of Math. 36, 984-992 (1935)

[189] Magid, A.R., *Pierce's representation and separable algebras*, Ill. J. Math. 15, 114-121 (1971)

[190] Markov, V.T., *B-rings with a polynomial identity*, Trud. Sem. Im. Petrov. 7, 232-238 (1981)

[191] Martindale W.S. III, *Prime rings satisfying a generalized polynomial identity*, J. Algebra 12, 576-584 (1969)

[192] ——, *On semiprime PI-rings*, Proc. Amer. Math. Soc. 40, 365-369 (1973)

[193] ——, *Fixed rings of automorphisms and the Jacobson radical*, J. London Math. Soc. 17, 42-46 (1978)

[194] Masaike, K., *On quotient rings and torsionless modules*, Sci. Rep. Tokyo Kyoiku D., Sec. A11, 26-30 (1971)

[195] Matczuk, J., *Central closure of semiprime tensor products*, Comm. Algebra 10, 263-278 (1982)

[196] McMahon, E., Mewborn, A.C., *Separable extensions of noncommutative rings*, Hokkaido Math. J. 13, 74-88 (1984)

[197] Menini, C., Orsatti, A., *Good dualities and strongly quasi-injective modules*, Ann. Mat. Pura Appl. 77, 187-230 (1981)

[198] ——, *Representable equivalences between categories of modules and applications*, Rend. Sem. Mat. Univ. Padova. 82, 203-231 (1989)

[199] Michler, G.O., Villamayor, O.E, *On rings whose simple modules are injective*, J. Algebra 25, 185-201 (1973)

[200] Miyashita, Y., *On quasi-injective modules*, J. Fac. Sci. Hokkaido Univ. 18, 158-187 (1965)

[201] ——, *Quasi-projective modules, perfect modules, and a theorem for modular lattices*, J. Fac. Sci. Hokkaido Univ. 19, 86-110 (1966)

[202] ——, *Finite outer Galois theory of non-commutative rings*, J. Fac. Sci. Hokkaido Univ. 19, 114-134 (1966)

[203] Mohamed, S., Singh, S., *Generalizations of decomposition theorems known over perfect rings*, J. Austral. Math. Soc. 24, 496-510 (1977)

[204] Montgomery, S., *Group actions on rings: some classical problems*, F. van Oystaeyen (Ed.), Methods in Ring Theory, Reidel Publ., 327-346 (1984)

[205] Morrison, D.R., *Bi-regular rings and the ideal lattice isomorphism*, Proc. Amer. Math. Soc. 6, 46-49 (1955)

[206] Moufang, R., *Alternativkörper und der Satz vom vollständigen Vierseit* (D_9), Abhdlg. Math. Sem. Univ. Hamburg 9, 207-222 (1933)

[207] ——, *Zur Struktur von Alternativkörpern*, Math. Ann. 110, 416-430 (1934)

[208] Müller, N., *Nicht assoziative Algebren über Ringen*, Abhdlg. Math. Sem. Univ. Hamburg 40, 115-131 (1974)

[209] Nauwelaerts, E., V. Oystaeyen, F., *Zariski extensions and biregular rings*, Israel J. Math. 37, 315-326 (1980)

[210] Neggers, J., Allen, P.J., *Central perfect rings*, J. Korean Math. Soc. 17, 53-80 (1980)

[211] Nicholson, K., *Semiregular modules and rings*, Canad. J. Math. 28, 1105-1120 (1976)

[212] ——, *Lifting idempotents and exchange rings*, Trans. Amer. Math. Soc. 229, 269-278 (1977)

[213] Nicholson, K., Watters, J.F., *Tensor products and extended centroids*, Proc. Amer. Math. Soc. 88, 215-217 (1983)

[214] Okninski, J., Van Oystaeyen, F., *Cancellative semigroup rings which are Azumaya algebras*, J. Algebra 117, 290-296 (1988)

[215] Osborn, J.M., *Modules over nonassociative rings*, Comm. Algebra 6, 1297-1358 (1978)

[216] Oshiro, K., *Semi-perfect modules and quasi-semiperfect modules*, Osaka J. Math. 20, 337-372 (1983)

[217] ——, *Lifting modules, extending modules and their applications to QF-rings*, Hokkaido Math. J. 13, 310-338 (1984)

[218] Van Oystaeyen, F., Van Geel, J., *Local-global results for regular rings*, Comm. Algebra 4, 811-821 (1976)

[219] Park, J.K., *Artinian skew group rings*, Proc. Amer. Math. Soc. 75, 1-7 (1979)

[220] Passman, D.S., *Computing the symmetric ring of quotients*, J. Algebra 105, 207-235 (1987)

[221] Polikarpov, S.V., Shestakov, I.P., *Nonassociative affine algebras*, Algebra i Logika 29, 709-723 (1990)

[222] Pritchard, F.L., *Ideals in the multiplication algebra of a non-associative K-algebra*, Comm. Algebra 21, 4541-4559 (1993)

[223] ——, *The radical and the multiplication algebra of a non-associative algebra*, Algebras, Groups and Geom. 11, 329-345 (1994)

[224] Ranga Rao, M.L., *Azumaya, semisimple and ideal algebras*, Bull. Amer. Math. Soc. 78, 588-592 (1972)

[225] Renault, G., *Etude des sous-modules complements dans un module*, Bull. Soc. Math. France, Mémoire 9 (1967)

[226] ——, *Anneaux réduits non commutatifs*, J. Math. Pure Appl. 46, 203-214 (1967)

[227] ——, *Anneaux réguliers auto-injectifs à droite*, Bull. Soc. Math. France 101, 237-254 (1973)

[228] ——, *Anneaux biréguliers auto-injectifs à droite*, J. Algebra 36, 77-84 (1975)

[229] Röhrl, H., *The algebra of polynomial functions with coefficients in a nonassociative algebra*, Sem. Ber. FU Hagen 6, 137-172 (1980)

[230] Roitman, M., *A Characterization of QI-Rings*, Math. Publ. Series 76, Univ. of Haifa (1983)

[231] Rosen, J.D., Rosen, M.P., *Extended centroids of skew polynomial rings*, Canad. Math. Bull. 28, 67-76 (1985)

[232] Rowen L.H., *Some results on the center of a ring with polynomial identities*, Bull. Amer. Math. Soc. 79, 219-223 (1973)

[233] ——, *Rings with central polynomials*, J. Algebra 31, 187-222 (1974)

[234] ——, *Maximal quotients of semiprime PI-algebras*, Trans. Amer. Math. Soc. 196, 127-135 (1974)

[235] ——, *Nonassociative rings satisfying normal polynomial identities*, J. Algebra 49, 104-111 (1977)

[236] ——, *Polynomial identities of nonassociative rings, I*, J. Algebra 49, 104-111 (1977)

[237] Rubin, R.A., *Absolutely torsionfree rings*, Pacific J. Math. 46, 503-514 (1973)

[238] Sagle, A.A., *Malcev algebras*, Trans. Amer. Math. Soc. 101, 426-458 (1961)

[239] Saito, R., *A supplement to 'A note on algebras and semiprime rings'*, J. Coll. Dairying 11, 513-514 (1986)

[240] Satyanarayana, M., Deshpande, M.G., *Rings with unique maximal ideals*, Math. Nachr. 87, 213-219 (1979)

[241] Schafer, R.D., *Representations of alternative algebras*, Trans. Amer. Math. Soc. 72, 1-17 (1952)

[242] Slater, M., *Prime alternative rings I*, J. Algebra 15, 229-243 (1970)

[243] Slinko, A.M., Shestakov, I.P., *Right representations of algebras*, Algebra and Logic 16, 98-100 (1974)

[244] Smiley, M.F., *The radical of an alternative ring*, Ann. of Math. 49, 702-709 (1948)

[245] Smith, P.F., *Modules for which every submodule has a unique closure*, Ring Theory, ed. by Jain-Rizvi, World Scientific, Singapore, 302-313 (1993)

[246] Stewart, I., *Tensorial extensions of central simple algebras*, J. Algebra 25, 1-14 (1973)

[247] Stewart, P.N., Watters, J.F., *Morita contexts and rings with a unique maximal ideal*, Math. Chronicle 16, 85-87 (1987)

[248] Stock, J., *On rings whose projective modules have the exchange property*, J. Algebra 103, 437-453 (1986)

[249] Storrer, H.H., *On Goldman's primary decomposition*, Lectures on Rings and Modules (Tulane Univ.), Springer LNM 246 (1972)

[250] Sugano, K., *Note on semisimple extensions and separable extensions*, Osaka J. Math. 4, 265-270 (1967)

[251] ——, *Note on seperability of endomorphism rings*, J. Fac. Sci. Hokkaido Univ., Ser I, 21, 196-208 (1971)

[252] Szeto, G., *A Characterization of Azumaya Algebras*, J. Pure Appl. Algebra 9, 65-71 (1976)

[253] ——, *The localization of zero-dimensional rings*, Bull. Soc. Math. Belgique 28, 193-196 (1976)

[254] ——, *On an exact sequence of modules*, Arch. Math. 29, 67-71 (1977)

[255] ——, *The structure of semiperfect rings*, Comm. Algebra 5(3), 219-229 (1977)

[256] Szeto, G., Wong, Y.F., *A general Maschke Theorem*, Nanta Mathematica 12, 1-4 (1977).

[257] Taylor, J.L., *A bigger Brauer group*, Pac. J. Math. 103, 163-203 (1982)

[258] Tyukavkin, D.V., *Centers of regular self-injective rings*, J. Algebra 172, 830-835 (1995)

[259] Verschoren, A., *Some ideas in noncommutative Algebraic Geometry*, Thesis, University of Antwerp (1978)

[260] Villamayor, O.E., Zelinsky, D., *Galois theory with infinitely many idempotents*, Nagoya Math. J. 35, 83-98 (1969)

[261] Viola-Prioli, J., *On absolutely torsion-free rings*, Pacif. J. Math. 56, 275-283 (1975)

[262] van der Walt, A., *Weakly prime one-sided ideals*, J. Austral. Math. Soc. 38, 84-91 (1985)

[263] Wehlen, J.A., *Algebras over absolutely flat commutative rings*, Trans. Amer. Math.Soc. 196, 149-160 (1974)

[264] ——, *Separable algebras over Prufer domains*, Rocky Mountain J. Math. 5, 145-155 (1975)

[265] Wilke, B., *Zur Modulstruktur von Ringen über Schiefgruppenringen*, Algebra-Berichte 73, München (1994)

[266] Wisbauer, R., *Radikale von separablen Algebren über Ringen*, Math. Z. 139, 9-13 (1974)

[267] ——, *Zentrale Bimoduln und separable Algebren*, Arch. Math. 30, 129-137 (1978)

[268] ——, *Nonassociative left regular and biregular rings*, J. Pure Appl. Algebra 10, 215-226 (1977)

[269] ——, *Co-semisimple modules and nonassociative V-rings*, Comm. Algebra 5, 1193-1209 (1977)

[270] ——, *Nichtassoziative Ringe und Modultheorie*, Habilitationsschrift, Universität Düsseldorf (1977)

[271] ——, *Separable Moduln von Algebren über Ringen*, J. Reine Angew. Math 303/304, 221-230 (1978)

[272] ——, *σ-semiperfekte und σ-perfekte Moduln*, Math. Z. 162, 131-138 (1978)

[273] ——, *Localization of modules and the central closure of rings*, Comm. Algebra 9, 1455-1493 (1981)

[274] ——, *On Prime Modules and Rings*, Comm. Algebra 11, 2249-2265 (1983)

[275] ——, *Restricted M-injective modules and FP-injective endomorphism rings*, Comm. Algebra 14, 255-276 (1986)

[276] ——, *Local-global results for modules over algebras and Azumaya rings*, J. Algebra 135, 440-455 (1990)

[277] ——, *Injectivity and cogenerator properties of a ring as bimodule*, Comm. Algebra 18, 1925-1933 (1990)

[278] ——, *Modules and structure presheaves for arbitrary rings*, Adv. Math. (China) 20, 15-23 (1990)

[279] ——, *On module classes closed under extensions*, Conf. on Rings and Radicals, Shijiazhuang 1994, Pitman RN (1996)

[280] Yue Chi Ming, R., *On V-rings and unit-regular rings*, Rend. Sem. Mat. Univ. Padova 64, 127-140 (1981)

[281] Zelmanov, E.F., *Goldie's theorems for Jordan algebras*, Sib. Mat. J. 28, 44-52 (1987)

[282] Zelmanowitz, J.M., *Endomorphism rings of torsionless modules*, J. Algebra 5, 325-341 (1967)

[283] ——, *Semiprime rings with maximum condition*, J. Algebra 25, 554-574 (1973)

[284] ——, *Representation of rings with faithful polyform modules*, Comm. Algebra 14, 1141-1169 (1986)

[285] ——, *Weakly semisimple modules and density theory*, Comm. Algebra, to appear

[286] Zhelyabin, V.N., *Definition of separability in alternative and Jordan algebras*, Algebra i Logika 24, 392-407 (1985)

[287] Zhevlakov, K.A., *Radical and representations of alternative rings*, Algebra i Logika 11, 162-173 (1972)

[288] Zöschinger, H., Komplemente als direkte Summanden, Arch. Math. 25, 241-253 (1974)

[289] Zwier, P.J., *Prime ideals in a large class of nonassociative rings*, Trans. Amer. Math. Soc. 158, 257-271 (1971)

Index