The Braided Structures for ω-Smash Coproduct Hopf Algebras

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Abstract

Braided bialgebras were introduced by Larson-Towber by considering a bilinear form with certain properties. Here we study the braided structures of ω-smash coproduct Hopf algebras $B_\omega \bowtie H$ as constructed by Caenepeel, Ion, Militaru and Zhu. Necessary and sufficient conditions for a class of ω-smash coproduct Hopf algebras to be braided Hopf algebras are given in terms of properties of their components. As applications of our results some special cases are discussed and explicit examples are given.

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1 Introduction

As a dual concept of quasitriangular bialgebra, braided bialgebra were introduced by Larson-Towber in [5] as a tool for providing solutions to the quantum Yang-Baxter equations. Since then this notion has been studied extensively. Some investigation related to braided Hopf algebras can be found in [3, 4, 5, 6].

Let B and H be coalgebras over a commutative ring R. Given an R-linear map

$$\omega : B \otimes H \to H \otimes B,$$

the *ω-smash coproduct coalgebra* $B_\omega \ltimes H$ is defined as the R-module $B \otimes H$ with comultiplication

$$\Delta_{B_\omega \ltimes H} = (I_B \otimes \omega \otimes I_H) \circ (\Delta_B \otimes \Delta_H),$$

and counit $\varepsilon_B \otimes \varepsilon_H$, where certain conditions are to be imposed on ω to ensure the required properties of $\Delta_{B_\omega \ltimes H}$ and $\varepsilon_B \otimes \varepsilon_H$ (see [2, Definition 3.1], [1, 2.14]). The usual smash coproduct $B \times H$ (see [7]) and the twisted smash coproduct $B \times_r H$ (see [12]) are special cases of an ω-smash coproduct coalgebra $B_\omega \ltimes H$. If B and H are bialgebras we may consider $B \otimes H$ as algebra with componentwise multiplication, and in [2] necessary and sufficient conditions are given to make $B_\omega \ltimes H$ with this multiplication a bialgebra. Furthermore, if B and H are Hopf algebras the bialgebra $B_\omega \ltimes H$ is also a Hopf algebra which we call the *ω-smash coproduct Hopf algebra* and denote it by $B_\omega \bowtie H$ (also see [2]).

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The aim of this paper is to study braided structures for a class of ω-smash coproduct Hopf algebras $B_\omega \bowtie H$, where the linear map ω obeys the right normal condition, i.e., $\omega(1_B \otimes h) = h \otimes 1_B$.

In Section 2, we recall the notions of an ω-smash coproduct Hopf algebra $B_\omega \bowtie H$ and the braided Hopf algebra (H, σ) (from [1], [2] and [5]) and then give some definitions and basic results needed in the sequel.

In Section 3, we show that if $B_\omega \bowtie H$ is an ω-smash coproduct Hopf algebra with ω a right normal linear map, then $(B_\omega \bowtie H, \sigma)$ is a braided Hopf algebra if and only if σ can be written as (for $a, b \in B$, $h, g \in H$)

$$\sigma(a \otimes h, b \otimes g) = \sum p(a_{(1)}, b_{(1)})\, u(a_{(2)}, g_{(2)})\, v(h_{(2)}, b_{(2)})\, \tau(h_{(1)}, g_{(1)}),$$

where $p : B \otimes B \to R$, $\tau : H \otimes H \to R$, $u : B \otimes H \to R$, and $v : H \otimes B \to R$ are linear maps satisfying certain compatibility conditions.

In Section 4, some special cases are considered, and in Section 5, an explicit example is constructed.

Throughout R will denote a (fixed) commutative ring with unit, and we follow [11] and [1] for the terminology on coalgebras and Hopf algebras. For a coalgebra C and $c \in C$, we write $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$. The antipode of a Hopf algebra H is denoted by S (or S_H).

2 Preliminaries

Let B and H be R-coalgebras and consider a linear map $\omega : B \otimes H \to H \otimes B$. Then a comultiplication is defined on the R-module $B \otimes H$ by

$$\Delta_{B_\omega \ltimes H} = (I_B \otimes \omega \otimes I_H) \circ (\Delta_B \otimes \Delta_H) \tag{2.1}$$

and an R-linear map is given by

$$\varepsilon_{B_\omega \ltimes H} =: \varepsilon_B \otimes \varepsilon_H : B_\omega \ltimes H \to R. \tag{2.2}$$

If the triple $(B \otimes H, \Delta_{B_\omega \ltimes H}, \varepsilon_{B_\omega \ltimes H})$ forms a coalgebra, then it is called a *smash coproduct* of B and H and we denote it by $B_\omega \ltimes H$. This imposes certain conditions on the map ω and to describe these we write for $b \in B$ and $h \in H$,

$$\omega(b \otimes h) = \sum {}^{\omega}h \otimes {}^{\omega}b.$$

Then we get for comultiplication and counit

$$\Delta_{B_\omega \ltimes H}(b \otimes h) = \sum (b_{(1)} \otimes {}^{\omega}h_{(1)}) \otimes ({}^{\omega}b_{(2)} \otimes h_{(2)}), \tag{2.1$'$}$$

$$\varepsilon_{B_\omega \ltimes H}(b \otimes h) = \varepsilon_B(b)\varepsilon_H(h). \tag{2.2$'$}$$

Proposition 2.1 *With the notation above, $B_\omega \ltimes H$ is a smash coproduct if and only if the following conditions hold for $b \in B$ and $h \in H$:*

(1) $(I_H \otimes \varepsilon_B)\, \omega(b \otimes h) = \varepsilon_B(b)h$; *(left conormal condition)*

(2) $(\varepsilon_H \otimes I_B)\, \omega(b \otimes h) = \varepsilon_H(h)b$; *(right conormal condition)*

(3) $\sum ({}^{\omega}h)_{(1)} \otimes ({}^{\omega}h)_{(2)} \otimes {}^{\omega}b = \sum {}^{\omega}h_{(1)} \otimes {}^{\bar{\omega}}h_{(2)} \otimes {}^{\bar{\omega}}({}^{\omega}b)$;

(4) $\sum {}^{\omega}h \otimes ({}^{\omega}b)_{(1)} \otimes ({}^{\omega}b)_{(2)} = \sum {}^{\bar{\omega}}({}^{\omega}h) \otimes {}^{\bar{\omega}}b_{(1)} \otimes {}^{\omega}b_{(2)}$.

Proof. See [2, Section 3] or [1, 2.14]. $\square$

Let B, H be bialgebras and $\omega : B \otimes H \to H \otimes B$ a linear map such that $B_{\omega}\ltimes H$ is a coalgebra. The canonical multiplication on $B \otimes H$ makes $B_{\omega}\ltimes H$ an algebra and it becomes a bialgebra provided it satsifies the compatibility conditions, that is, if $\Delta_{B_{\omega}\ltimes H}$ is a multiplicative map (e.g., [1, 13.1]). In this case we call $B_{\omega}\ltimes H$ an *ω-smash coproduct bialgebra* and denote it by $B_{\omega}\bowtie H$.

Proposition 2.2 *Let B, H be bialgebras and $\omega : B \otimes H \to H \otimes B$ a linear map. Then $B_{\omega}\bowtie H$ is an ω-smash coproduct bialgebra if and only if the conditions (1)-(4) in (2.1) hold and ω is an algebra map.*

Futhermore, if B and H are Hopf algebras with antipodes S_B and S_H, then $B_{\omega}\bowtie H$ is a Hopf algebra with an antipode which is, for $b \in B$ and $h \in H$, given by

$$S_{B_{\omega}\bowtie H}(b \otimes h) = \sum S_B({}^{\omega}b) \otimes S_H({}^{\omega}h).$$

Proof. See [2, Corollary 4.8] for the first part. To show that $S_{B_{\omega}\bowtie H}$ is an antipode of $B_{\omega}\bowtie H$ we compute, for all $b \in B, h \in H$,

$$\begin{array}{rcl}
(S_{B_{\omega}\bowtie H} * I)(b \otimes h) & = & \sum S_{B_{\omega}\bowtie H}(b_{(1)} \otimes^{\omega} h_{(1)})({}^{\omega}b_{(2)} \otimes h_{(2)}) \\
& = & \sum [S_B({}^{\bar{\omega}}b_{(1)}) \otimes S_H({}^{\bar{\omega}}({}^{\omega}h_{(1)}))]({}^{\omega}b_{(2)} \otimes h_{(2)}) \\
& = & \sum S_B({}^{\bar{\omega}}b_{(1)}){}^{\omega}b_{(2)} \otimes S_H({}^{\bar{\omega}}({}^{\omega}h_{(1)}))h_{(2)} \\
& \overset{2.1(4)}{=} & \sum S_B(({}^{\omega}b)_{(1)})({}^{\omega}b)_{(2)} \otimes S_H({}^{\omega}h_{(1)})h_{(2)} \\
& \overset{2.1(1)}{=} & \sum \varepsilon_B(b) \otimes S_H(h_{(1)})h_{(2)} \\
& = & \varepsilon_B(b)\varepsilon_H(h).
\end{array}$$

Similarly, we can verify that $(I * S_{B_{\omega}\bowtie H})(b \otimes h) = \varepsilon_B(b)\varepsilon_H(h)$. This completes the proof. $\square$

Definition 2.3 Let B and H be bialgebras, a linear map $\omega : B \otimes H \to H \otimes B$ is called *right normal*, if ω satifies the *right normal condition*, for all $h \in H$,

$$\omega(1_B \otimes h) = h \otimes 1_B. \tag{$*$}$$

Example 2.4 (1) Let B and H be Hopf algebras, $\omega = \tau_{B,H} : B \otimes H \to H \otimes B$ be the switch map. Then $B_{\omega}\bowtie H = B \otimes H$ is the usual tensor product of Hopf algebras B and H.

(2) Let B, H be Hopf algebras and B a left H-comodule bialgebra with left comodule structure map $\rho : B \to H \otimes B$, $\rho(b) = \sum b^{(1)} \otimes b^{<2>}$ such that $\sum hb^{(1)} \otimes b^{<2>} = \sum b^{(1)}h \otimes b^{<2>}$, for all $b \in B$ and $h \in H$. Let

$$\omega : B \otimes H \to H \otimes B, \quad b \otimes h \mapsto \sum b^{(1)}h \otimes b^{<2>}.$$

Then $B_{\omega}\bowtie H = B \times H$ is the usual smash coproduct Hopf algebra from Molnar [7].

(3) Let B and H be Hopf algebras, B an H-bicomodule coalgebra with left comodule structure map $\rho_l : B \to H \otimes B$, $\rho_l(b) = \sum b^{(1)} \otimes b^{<2>}$ and right comodule structure map $\rho_r : B \to B \otimes H$, $\rho_r(b) = \sum b^{<1>} \otimes b^{(2)}$ such that four additional conditions hold. Define

$$\omega : B \otimes H \to H \otimes B, b \otimes h \mapsto \sum b^{(1)} h S_H(b^{<2>(2)}) \otimes b^{<2><1>}.$$

Then $B_\omega \bowtie H = B \times_r H$ is the twisted smash coproduct Hopf algebra of B and H (see [12, Theorem 2.4] for detail).

(4) Let B and H be Hopf algebras, $R = \sum R^{(1)} \otimes R^{(2)} \in B \otimes H$ an invertible element such that

(i) $\sum \varepsilon_B(R^{(1)}) R^{(2)} = 1_H, \sum R^{(1)} \varepsilon_H(R^{(2)}) = 1_B$;
(ii) $\sum \Delta_B(R^{(1)}) \otimes R^{(2)} = \sum R^{(1)} \otimes r^{(1)} \otimes R^{(2)} r^{(2)}$;
(iii) $\sum R^{(1)} \otimes \Delta_H(R^{(2)}) = \sum R^{(1)} r^{(1)} \otimes r^{(2)} \otimes R^{(2)}$.

Consider the map

$$\omega : B \otimes H \to H \otimes B, \quad b \otimes h \mapsto \sum R^{(2)} h U^{(2)} \otimes R^{(1)} b U^{(1)},$$

where $R^{-1} = U = \sum U^{(1)} \otimes U^{(2)}$. Then $B_\omega \bowtie H$ is an ω-smash coproduct Hopf algebra.

Notice that the linear maps ω in (2.4)(1)-(3) are right normal (Definition 2.3) and the linear map ω in (2.4)(4) is not right normal unless H is commutative. In fact, in the latter case, for all $h \in H$,

$$\omega(1_B \otimes h) = \sum R^{(2)} h U^{(2)} \otimes R^{(1)} U^{(1)} = \sum R^{(2)} U^{(2)} h \otimes R^{(1)} U^{(1)} = h \otimes 1_B.$$

In what follows we will only consider the ω-smash coproduct Hopf algebras $B_\omega \bowtie H$ for which the linear maps ω are right normal. The next lemma gives some properties of $B_\omega \bowtie H$ for this case.

Lemma 2.5 *Let $B_\omega \bowtie H$ be an ω-smash coproduct Hopf algebra with a right normal linear map ω. Then for all $b \in B$ and $h \in H$,*

(1) $\sum {}^\omega 1_H h \otimes {}^\omega b = \sum {}^\omega h \otimes {}^\omega b = \sum h^\omega 1_H \otimes {}^\omega b$;

(2) $\sum {}^{\bar\omega} 1_H {}^\omega 1_H \otimes {}^{\bar\omega} b_{(1)} \otimes {}^\omega b_{(2)} = \sum {}^\omega 1_H \otimes ({}^\omega b)_{(1)} \otimes ({}^\omega b)_{(2)}$
$= \sum {}^\omega 1_H {}^{\bar\omega} 1_H \otimes {}^{\bar\omega} b_{(1)} \otimes {}^\omega b_{(2)}$.

Proof. Since ω is an algebra map, for all $a, b \in B$ and $h, g \in H$,

$$\omega(ab \otimes hg) = \omega(a \otimes h)\omega(b \otimes g), \text{ that is}$$

$$\sum {}^\omega(hg) \otimes {}^\omega(ab) = \sum {}^\omega h {}^{\bar\omega} g \otimes {}^\omega a {}^{\bar\omega} b.$$

Putting $h = 1_H$, and $b = 1_B$ in the equation above and using the right normal condition $(*)$, we obtain $\sum {}^\omega 1_H g \otimes {}^\omega a = \sum {}^\omega g \otimes {}^\omega a$. In a similar manner we can get $\sum {}^\omega h \otimes {}^\omega b = \sum h^\omega 1_H \otimes {}^\omega b$. Thus (1) follows. Putting $h = 1_H$ in (2.1)(4), we get

$$\sum {}^\omega 1_H \otimes ({}^\omega b)_{(1)} \otimes ({}^\omega b)_{(2)} = \sum {}^{\bar\omega}({}^\omega 1_H) \otimes {}^{\bar\omega} b_{(1)} \otimes {}^\omega b_{(2)}.$$

Referring to (1) we now obtain (2). $\square$

Next we recall the definition of a braided Hopf algebra from [5] and give some new definitions.

Definition 2.6 A *braided Hopf algebra* is a pair (H, σ), where H is a Hopf algebra over R and $\sigma : H \otimes H \to R$ is a linear map satisfying, for all $x, y, z \in H$,

(BR1) $\sigma(xy, z) = \sum \sigma(x, z_{(1)})\sigma(y, z_{(2)})$;

(BR2) $\sigma(1_H, x) = \varepsilon(x)$;

(BR3) $\sigma(x, yz) = \sum \sigma(x_{(1)}, z)\sigma(x_{(2)}, y)$;

(BR4) $\sigma(x, 1_H) = \varepsilon(x)$;

(BR5) $\sum \sigma(x_{(1)}, y_{(1)})x_{(2)}y_{(2)} = \sum y_{(1)}x_{(1)}\sigma(x_{(2)}, y_{(2)})$.

As a consequence, we notice that σ is convolution invertible with $\sigma^{-1}(x, y) = \sigma(S_H(x), y)$.

Definition 2.7 Let B, H be Hopf algebras and $u : B \otimes H \to R$ a linear map. (B, H, u) is called a *dual compatible u-Hopf algebra pair* if, for all $a, b \in B$ and $h, g \in H$,

(DC1) $u(ab, h) = \sum u(a, h_{(1)})u(b, h_{(2)})$;

(DC2) $u(1_B, h) = \varepsilon_H(h)$;

(DC3) $u(b, hg) = \sum u(b_{(1)}, h)u(b_{(2)}, g)$;

(DC4) $u(b, 1_H) = \varepsilon_B(b)$.

Clearly, u is convolution invertible with $u^{-1}(b, h) = u(S_B(b), h)$, that is, u is invertible in $\mathrm{Hom}(B \otimes H, R)$ which means, for all $b \in B$, $h \in H$,

$$(u * u^{-1})(b \otimes h) = \sum u(b_{(1)}, h_{(1)})u^{-1}(b_{(2)}, h_{(2)}) = \varepsilon_B(b)\varepsilon_H(h) = (u^{-1} * u)(b \otimes h).$$

Definition 2.8 Let B, H be Hopf algebras with linear map $v : H \otimes B \to R$. Then (H, B, v) is called a *skew dual compatible v−Hopf algebra pair* if, for all $a, b \in B$ and $h, g \in H$,

(SDC1) $v(hg, b) = \sum v(h, b_{(2)})v(g, b_{(1)})$;

(SDC2) $v(1_H, b) = \varepsilon_B(b)$;

(SDC3) $v(h, ab) = \sum v(h_{(1)}, b)v(h_{(2)}, a)$;

(SDC4) $v(h, 1_B) = \varepsilon_H(h)$.

Clearly, v is convolution invertible with $v^{-1}(h, b) = v(S_H^{-1}(h), b)$ provided the antipode S_H of H is bijective.

Definition 2.9 Let B, H be Hopf algebras, $\omega : B \otimes H \to H \otimes B$ a linear map satisfying the conditions in Proposition 2.1, (B, H, u) a dual compatible u−Hopf algebra pair, (H, B, v) a skew dual compatible v−Hopf algebra pair. Given a linear map $p : B \otimes B \to R$, the pair (B, p) is called a (u, v)−*weakly braided Hopf algebra* if, for all $a, b, c \in B$,

(WBR1) $p(ab, c) = \sum p(a_{(1)}, c_{(1)})u(a_{(2)}, {}^{\omega}1_H)p(b, {}^{\omega}c_{(2)})$;

(WBR2) $p(1_B, b) = \varepsilon(b)$;

(WBR3) $p(a, bc) = \sum p(a_{(1)}, c_{(1)})v({}^{\omega}1_H, c_{(2)})p({}^{\omega}a_{(2)}, b)$;

(WBR4) $p(b, 1_B) = \varepsilon(b)$;

(WBR5) $\sum b_{(1)}a_{(1)}p(a_{(2)}, b_{(2)}) = \sum p(a_{(1)}, b_{(1)})\, u(a_{(2)}, {}^{\bar{\omega}}1_H)\, v({}^{\omega}1_H, b_{(3)})\, {}^{\omega}a_{(3)}{}^{\bar{\omega}}b_{(2)}$.

For the linear map $p : B \otimes B \to R$, we have the following

Proposition 2.10 *If, with the above notation, (B, p) is a (u, v)-weakly braided Hopf algebra, then for all $a, b \in B$,*

$$\sum p(a_{(1)}, b_{(1)})u(a_{(2)}, {}^{\omega}1_H)p(S_B(a_{(3)}), {}^{\omega}b_{(2)}) = \varepsilon_B(a)\varepsilon_B(b).$$

Proof. This follows from the equalities, for all $a, b \in B$,

$$\begin{aligned}\sum p(a_{(1)}, b_{(1)})u(a_{(2)}, {}^{\omega}1_H)p(S_B(a_{(3)}), {}^{\omega}b_{(2)}) &\overset{\text{WBR1}}{=} \sum p(a_{(1)}S_B(a_{(2)}), b) \\ &\overset{\text{WBR2}}{=} \varepsilon_B(a)\varepsilon_B(b).\end{aligned}$$

Notice that (u, v)-weakly braided Hopf algebras generalize braided Hopf algebras as seen from the following examples.

Example 2.11 Let (B, σ) be a braided Hopf algebra, H any Hopf algebra with a linear map $\omega : B \otimes H \to H \otimes B$ satisfying the conditions in Proposition 2.1. Take $u = \varepsilon_B \otimes \varepsilon_H$ and $v = \varepsilon_H \otimes \varepsilon_B$. Then it is easy to see that (B, H, u) is a dual compatible u-Hopf algebra pair, (H, B, v) is a skew dual compatible v-Hopf algebra pair and (B, σ) is a (u, v)-weakly braided Hopf algebra.

Example 2.12 Let $H = B$ be a Hopf algebra and $\omega = \tau_B$ the switch map. Then a braided Hopf algebra (B, σ) is just a (u, v)-weakly braided Hopf algebra, where $u = v = \varepsilon_B \otimes \varepsilon_B$ are linear maps as in the Definitions 2.7 and 2.8.

3 The braided structure of $B_\omega \bowtie H$

In this section B and H will be Hopf algebras with linear map $\omega : B \otimes H \to H \otimes B$ which is right normal and such that $B_\omega \bowtie H$ is an ω-smash coproduct Hopf algebra.

Let $(B_\omega \bowtie H, \sigma)$ be a braided Hopf algebra, where $\sigma : (B_\omega \bowtie H) \otimes (B_\omega \bowtie H) \to R$ is a linear form. For all $a, b \in B$ and $h, g \in H$, define

$$\begin{aligned} p : B \otimes B \to R, &\quad p(a, b) = \sigma(a \otimes 1_H, b \otimes 1_H); \\ \tau : H \otimes H \to R, &\quad \tau(h, g) = \sigma(1_B \otimes h, 1_B \otimes g); \\ u : B \otimes H \to R, &\quad u(b, h) = \sigma(b \otimes 1_H, 1_B \otimes h); \\ v : H \otimes B \to R, &\quad v(h, b) = \sigma(1_B \otimes h, b \otimes 1_H). \end{aligned}$$

The following properties are easily derived.

Proposition 3.1 *With the above notation, if σ satisfies the conditions* (BR2) *and* (BR4)*, then, for all $b \in B$ and $h \in H$,*

(1) $p(1_B, b) = \varepsilon(b) = p(b, 1_B)$;

(2) $\tau(1_H, h) = \varepsilon(h) = \tau(h, 1_H)$;

(3) $u(1_B, h) = \varepsilon(h),\ u(b, 1_H) = \varepsilon(b)$;

(4) $v(1_H, b) = \varepsilon(b),\ v(h, 1_B) = \varepsilon(h)$.

Proof. The proof follows by direct calculations. □

Proposition 3.2 *Let $(B_\omega \bowtie H, \sigma)$ be a braided Hopf algebra with σ a bilinear form on $B_\omega \bowtie H$. Then for all $a, b \in B$ and $h, g \in H$,*

(1) $\sum \sigma(1_B \otimes h, b_{(1)} \otimes {}^\omega 1_H)\, {}^\omega b_{(2)} = \sum b_{(1)} \sigma(1_B \otimes h, b_{(2)} \otimes 1_H)$;

(2) $\sum \sigma(b_{(1)} \otimes {}^\omega 1_H, 1_B \otimes h)\, {}^\omega b_{(2)} = \sum b_{(1)} \sigma(b_{(2)} \otimes 1_H, 1_B \otimes h)$;

(3) $\sigma(a \otimes h, b \otimes g) = \sum p(a_{(1)}, b_{(1)})\, u(a_{(2)}, g_{(2)})\, v(h_{(2)}, b_{(2)})\, \tau(h_{(1)}, g_{(1)})$.

Proof. By (BR5), we have

$$\begin{aligned} &\sum \sigma(a_{(1)} \otimes {}^\omega h_{(1)}, b_{(1)} \otimes {}^w g_{(1)})({}^\omega a_{(2)} {}^w b_{(2)} \otimes h_{(2)} g_{(2)}) \\ &\quad = \sum (b_{(1)} a_{(1)} \otimes {}^w g_{(1)} {}^\omega h_{(1)}) \sigma({}^\omega a_{(2)} \otimes h_{(2)}, {}^w b_{(2)} \otimes g_{(2)}). \end{aligned} \tag{3.1}$$

Put $a = 1_B, g = 1_H$ in (3.1) and apply $(I_B \otimes \varepsilon_H)$ to both sides of (3.1), then (1) follows. (2) is seen by putting $b = 1_B, h = 1_H$ and then applying $(1_B \otimes \varepsilon_H)$ to both sides of (3.1).

For (3) we need some more computations. By (BR1) and (BR3), for all $a, b, a', b' \in B$ and $h, g, h', g' \in H$, we have

$$\begin{aligned} \sigma(aa' \otimes hh', bb' \otimes gg') &= \sigma((a \otimes h)(a' \otimes h'), (b \otimes g)(b' \otimes g')) \\ &\overset{\text{BR1}}{=} \sum \sigma(a \otimes h, (b \otimes g)_{(1)}(b' \otimes g')_{(1)})\, \sigma(a' \otimes h', (b \otimes g)_{(2)}(b' \otimes g')_{(2)}) \\ &\overset{\text{BR3}}{=} \sum \sigma((a \otimes h)_{(1)}, (b' \otimes g')_{(1)})\, \sigma((a \otimes h)_{(2)}, (b \otimes g)_{(1)}) \cdot \\ &\qquad \sigma((a' \otimes h')_{(1)}, (b' \otimes g')_{(2)})\, \sigma((a' \otimes h')_{(2)}, (b \otimes g)_{(2)}) \\ &= \sum \sigma(a_{(1)} \otimes {}^\omega h_{(1)}, b'_{(1)} \otimes {}^{\bar\omega} g'_{(1)})\, \sigma({}^\omega a_{(2)} \otimes h_{(2)}, b_{(1)} \otimes {}^w g_{(1)}) \cdot \\ &\qquad \sigma(a'_{(1)} \otimes {}^{\bar w} h'_{(1)}, {}^{\bar\omega} b'_{(2)} \otimes g'_{(2)})\, \sigma({}^{\bar w} a'_{(2)} \otimes h'_{(2)}, {}^w b_{(2)} \otimes g_{(2)}). \end{aligned} \tag{3.2}$$

Putting $a = b' = 1_B, g = h' = 1_H$ in (3.2), we get

$$\begin{aligned} \sigma(a' \otimes h, b \otimes g') &= \sum \sigma(1_B \otimes {}^\omega h_{(1)}, 1_B \otimes {}^{\bar\omega} g'_{(1)})\, \sigma({}^\omega 1_B \otimes h_{(2)}, b_{(1)} \otimes {}^w 1_H) \cdot \\ &\qquad \sigma(a'_{(1)} \otimes {}^{\bar w} 1_H, {}^{\bar\omega} 1_B \otimes g'_{(2)})\, \sigma({}^{\bar w} a'_{(2)} \otimes 1_H, {}^w b_{(2)} \otimes 1_H) \\ &\overset{(*)}{=} \sum \sigma(1_B \otimes h_{(1)}, 1_B \otimes g'_{(1)})\, \sigma(1_B \otimes h_{(2)}, b_{(1)} \otimes {}^w 1_H) \cdot \\ &\qquad \sigma(a'_{(1)} \otimes {}^{\bar w} 1_H, 1_B \otimes g'_{(2)})\, \sigma({}^{\bar w} a'_{(2)} \otimes 1_H, {}^w b_{(2)} \otimes 1_H) \\ &\overset{3.2(1,2)}{=} \sum \sigma(1_B \otimes h_{(1)}, 1_B \otimes g'_{(1)})\, \sigma(1_B \otimes h_{(2)}, b_{(2)} \otimes 1_H) \cdot \\ &\qquad \sigma(a'_{(2)} \otimes 1_H, 1_B \otimes g'_{(2)})\, \sigma(a'_{(1)} \otimes 1_H, b_{(1)} \otimes 1_H) \\ &= \sum p(a'_{(1)}, b_{(1)}) u(a'_{(2)}, g'_{(2)}) v(h_{(2)}, b_{(2)}) \tau(h_{(1)}, g'_{(1)}). \end{aligned}$$

This completes the proof. □

Proposition 3.3 *Let $(B_\omega \bowtie H, \sigma)$ be a braided Hopf algebra. Then σ can be decomposed to*

$$\sigma(a \otimes h, b \otimes g) = \sum p(a_{(1)}, b_{(1)})\, u(a_{(2)}, g_{(2)})\, v(h_{(2)}, b_{(2)})\, \tau(h_{(1)}, g_{(1)})$$

such that p, τ, u, v satisfy for all $a, b \in B$, $h, g \in H$,

(1) $\sum b_{(1)}u(b_{(2)},h)=\sum u(b_{(1)},h_{(2)})\tau({}^{\omega}1_H,h_{(1)}){}^{\omega}b_{(2)}$;
(2) $\sum b_{(1)}v(h,b_{(2)})=\sum v(h_{(2)},b_{(1)})\tau(h_{(1)},{}^{\omega}1_H){}^{\omega}b_{(2)}$;
(3) $\sum u(b,h_{(1)})h_{(2)}=\sum h_{(1)}{}^{\omega}1_H u({}^{\omega}b,h_{(2)})$;
(4) $\sum v(h_{(1)},b)h_{(2)}=\sum v(h_{(2)},{}^{\omega}b){}^{\omega}1_H h_{(1)}$;
(5) $p(a,b)1_H=\sum {}^{w}1_H{}^{\omega}1_H p({}^{\omega}a,{}^{w}b)$;
(6) $\sum p(a,b_{(1)})v(h,b_{(2)})=\sum p(a_{(1)},b_{(1)})u(a_{(2)},{}^{\omega}1_H)v(h,{}^{\omega}b_{(2)})$;
(7) $\sum p(a_{(1)},b)u(a_{(2)},h)=\sum p(a_{(1)},b_{(1)})v({}^{\omega}1_H,b_{(2)})u({}^{\omega}a_{(2)},h)$;
(8) $\sum u(a,h_{(2)})\tau(g,h_{(1)})=\sum u(a,h_{(1)})\tau(g,h_{(2)})$;
(9) $\sum v(h_{(2)},a)\tau(h_{(1)},g)=\sum v(h_{(1)},a)\tau(h_{(2)},g)$.

Proof. Since σ satisfies (BR5) and has the decomposition

$$\sigma(a\otimes h,b\otimes g)=\sum p(a_{(1)},b_{(1)})\,u(a_{(2)},g_{(2)})\,v(h_{(2)},b_{(2)})\,\tau(h_{(1)},g_{(1)}),$$

the equation (3.1) takes the form

$$\begin{aligned}&\sum p(a_{(1)},b_{(1)})\,u(a_{(2)},({}^{\bar\omega}g_{(1)})_{(2)})\,v(({}^{\omega}h_{(1)})_{(2)},b_{(2)})\cdot\\ &\qquad\tau(({}^{\omega}h_{(1)})_{(1)},({}^{\bar\omega}g_{(1)})_{(1)})\cdot({}^{\omega}a_{(3)}{}^{\bar\omega}b_{(3)}\otimes h_{(2)}g_{(2)})\\ &\quad=\sum(b_{(1)}a_{(1)}\otimes{}^{\bar\omega}g_{(1)}{}^{\omega}h_{(1)})\cdot p(({}^{\omega}a_{(2)})_{(1)},({}^{\bar\omega}b_{(2)})_{(1)})\cdot\\ &\qquad\quad u(({}^{\omega}a_{(2)})_{(2)},g_{(3)})\,v(h_{(3)},({}^{\bar\omega}b_{(2)})_{(2)})\,\tau(h_{(2)},g_{(2)}).\end{aligned}\tag{3.3}$$

Putting $b=1_B,h=1_H$ in (3.3) and applying $(I_B\otimes\varepsilon_H)$ to both sides of (3.1) we get (1), and applying $(\varepsilon_B\otimes I_H)$ we get (3). Put $a=1_B,g=1_H$ in (3.3); then applying $(I_B\otimes\varepsilon_H)$ and $(\varepsilon_B\otimes I_H)$ to both sides of (3.3) yields (2) and (4), respectively. Put $h=g=1_H$ in (3.3); then applying $(I_B\otimes\varepsilon_H)$ to both sides yields (5).

By (BR1) we have

$$\sigma((a\otimes h)(b\otimes g),c\otimes l)=\sum\sigma(a\otimes h,c_{(1)}\otimes{}^{\omega}l_{(1)})\sigma(b\otimes g,{}^{\omega}c_{(2)}\otimes l_{(2)}).$$

Using the decomposition of σ, the above equation takes the following form.

$$\begin{aligned}&\sum p(a_{(1)}b_{(1)},c_{(1)})\,u(a_{(2)}b_{(2)},l_{(2)})\,v(h_{(2)}g_{(2)},c_{(2)})\,\tau(h_{(1)}g_{(1)},l_{(1)})\\ &\quad=\sum p(a_{(1)},c_{(1)})u(a_{(2)},({}^{\omega}l_{(1)})_{(2)})\,v(h_{(2)},c_{(2)})\,\tau(h_{(1)},({}^{\omega}l_{(1)})_{(1)})\cdot\\ &\qquad\quad p(b_{(1)},({}^{\omega}c_{(3)})_{(1)})\,u(b_{(2)},l_{(3)})\,v(g_{(2)},({}^{\omega}c_{(3)})_{(2)})\,\tau(g_{(1)},l_{(2)}).\end{aligned}\tag{3.4}$$

Putting $h=l=1_H,b=1_B$ in (3.4) yields (6), and putting $b=c=1_B,h=1_H$ in (3.4) yields (8). By (BR3) we have

$$\sigma(a\otimes h,(b\otimes g)(c\otimes l))=\sum\sigma(a_{(1)}\otimes{}^{\omega}h_{(1)},c\otimes l)\sigma({}^{\omega}a_{(2)}\otimes h_{(2)},b\otimes g).$$

Using the decomposition of σ, the above equation takes the form

$$\begin{aligned}&\sum p(a_{(1)},b_{(1)}c_{(1)})\,u(a_{(2)},g_{(2)}l_{(2)})\,v(h_{(2)},b_{(2)}c_{(2)})\,\tau(h_{(1)},g_{(1)}l_{(1)})\\ &\quad=\sum p(a_{(1)},c_{(1)})\,u(a_{(2)},l_{(2)})\,v(({}^{\omega}h_{(1)})_{(2)},c_{(2)})\,\tau(({}^{\omega}h_{(1)})_{(1)},l_{(1)})\cdot\\ &\qquad\quad p(({}^{\omega}a_{(3)})_{(1)},b_{(1)})\,u(({}^{\omega}a_{(3)})_{(2)},g_{(2)})\,v(h_{(3)},b_{(2)})\,\tau(h_{(2)},g_{(1)}).\end{aligned}\tag{3.5}$$

Putting $h=l=1_H,b=1_B$ in (3.5) yields (7), and putting $a=b=1_B,l=1_H$ in (3.5) yields (9). This completes the proof. $\square$

Proposition 3.4 *Let $(B_\omega \bowtie H, \sigma)$ be a braided Hopf algebra with*

$$\sigma(a \otimes h, b \otimes g) = \sum p(a_{(1)}, b_{(1)}) u(a_{(2)}, g_{(2)}) v(h_{(2)}, b_{(2)}) \tau(h_{(1)}, g_{(1)})$$

for all $a, b \in B$ and $h, g \in H$. Then

(1) *(H, τ) is a braided Hopf algebra;*

(2) *(B, H, u) is a dual compatible u-Hopf algebra pair;*

(3) *(H, B, v) is a skew dual compatible v-Hopf algebra pair;*

(4) *(B, p) is a (u, v)-weakly braided Hopf algebra.*

Proof. (1). Condition (BR2) and (BR4) follow from Proposition 3.1(2). Putting $a = b = c = 1_B$ in (3.4) yields

$$\tau(hg, l) = \sum \tau(h, l_{(1)}) \tau(g, l_{(2)})$$

and so (BR1) holds. Putting $a = b = c = 1_B$ in (3.5) yields

$$\tau(h, gl) = \sum \tau(h_{(2)}, g)\, \tau(h_{(1)}, l)$$

and so (BR3) holds. Putting $a = b = 1_B$ in (3.3) and applying $(\varepsilon_B \otimes I_H)$ to both sides yields

$$\tau(h_{(1)}, g_{(1)}) h_{(2)} g_{(2)} = \sum g_{(1)} h_{(1)} \tau(h_{(2)}, g_{(2)})$$

and (BR5) holds. Thus (H, τ) is a braided Hopf algebra.

(2). Condition (DC2) and (DC4) follow from Proposition 3.1(3). Putting $c = 1_B$, $h = g = 1_H$ in (3.4) yields

$$u(ab, l) = \sum u(a, l_{(1)}) u(b, l_{(2)})$$

and so (DC1) holds. Putting $b = c = 1_B, h = 1_H$ in (3.5) yields

$$\begin{aligned} u(a, gl) &= \sum u(a_{(1)}, l_{(2)})\, \tau({}^{\omega}1_H, l_{(1)})\, u({}^{\omega}a_{(2)}, g) \\ &\overset{3.3(1)}{=} \sum u(a_{(2)}, l)\, u(a_{(1)}, g) \end{aligned}$$

and (DC3) holds. Thus (B, H, u) is a dual compatible u-Hopf algebra pair.

(3). (SDC2) and (SDC4) follow from Proposition 3.1(4). Putting $a = b = 1_B, l = 1_H$ in (3.4) yields

$$\begin{aligned} v(hg, c) &= \sum v(h_{(2)}, c_{(1)})\, \tau(h_{(1)}, {}^{\omega}1_H)\, v(g, {}^{\omega}c_{(2)}) \\ &\overset{3.3(2)}{=} \sum v(h, c_{(2)})\, v(g, c_{(1)}) \end{aligned}$$

and thus (SDC1) holds. Putting $a = 1_B, g = l = 1_H$ in (3.5) yields

$$v(h, bc) = \sum v(h_{(1)}, c)\, v(h_{(2)}, b)$$

and (SDC3) holds. Thus (H, B, v) is a skew dual compatible v-Hopf algebra pair.

(4). (WBR2) and (WBR4) follow from Proposition 3.1(1). Putting $h = g = l = 1_H$ in (3.4) yields

$$p(ab, c) = \sum p(a_{(1)}, c_{(1)})\, u(a_{(2)}, {}^{\omega}1_H)\, p(b, {}^{\omega}c_{(2)})$$

and (WBR1) holds. Put $h = g = l = 1_H$ in (3.5) to obtain

$$p(a, bc) = \sum p(a_{(1)}, c_{(1)})\, v({}^{\omega}1_H, c_{(2)})\, p({}^{\omega}a_{(2)}, b)$$

and (WBR3) holds. Put $h = g = 1_H$ in (3.3) and apply $(I_B \otimes \varepsilon_H)$ to both sides to get

$$\begin{aligned}
&\sum b_{(1)}a_{(1)}\, p(a_{(2)}, b_{(2)})\\
&\quad = \sum p(a_{(1)}, b_{(1)})\, u(a_{(2)}, ({}^{w}1_H)_{(2)})\, v(({}^{\omega}1_H)_{(2)}, b_{(2)})\, \tau(({}^{\omega}1_H)_{(1)}, ({}^{w}1_H)_{(1)})\, {}^{\omega}a_{(3)}\, {}^{w}b_{(3)}\\
&\quad \overset{2.1(3)}{=} \sum p(a_{(1)}, b_{(1)})\, u(a_{(2)}, {}^{w}1_H)\, v(({}^{\omega}1_H)_{(2)}, b_{(2)})\, \tau(({}^{\omega}1_H)_{(1)}, {}^{\bar{w}}1_H)\, {}^{\omega}a_{(3)}\, {}^{w}({}^{\bar{w}}b_{(3)})\\
&\quad \overset{3.3(2)}{=} \sum p(a_{(1)}, b_{(1)})u(a_{(2)}, {}^{w}1_H)\, v({}^{\omega}1_H, b_{(3)})\, {}^{\omega}a_{(3)}\, {}^{w}b_{(2)},
\end{aligned}$$

and so (WBR5) holds. Thus (B, p) is a (u, v)-weakly braided Hopf algebra. □

We now discuss the sufficiency of the conditions in the following theorem.

Theorem 3.5 *Let $B_\omega \bowtie H$ be an ω-smash coproduct Hopf algebra and assume that*
(H, τ) is a braided Hopf algebra,
(B, H, u) is a dual compatible $u-$Hopf algebra pair,
(H, B, v) is a skew dual compatible $v-$Hopf algebra pair, and
(B, p) is a (u, v)-weakly braided Hopf algebra
such that the conditions (1)-(9) in Proposition 3.3 are satisfied.

Then $(B_\omega \bowtie H, \sigma)$ is a braided Hopf algebra where , for all $a, b \in B$, $h, g \in H$,

$$\sigma(a \otimes h, b \otimes g) = \sum p(a_{(1)}, b_{(1)})\, u(a_{(2)}, g_{(2)})\, v(h_{(2)}, b_{(2)})\, \tau(h_{(1)}, g_{(1)}).$$

Proof. It is not difficult to verify that, for all $b \in B, h \in H$,

$$\sigma(1_B \otimes 1_H, b \otimes h) = \sigma(b \otimes h, 1_B \otimes 1_H) = \varepsilon_B(b)\varepsilon_H(h).$$

Thus (BR2) and (BR4) hold and it remains to show that (BR1), (BR3) and (BR5) are satisfied for σ. To prove (BR1), it suffices to show that for all $a, b, c \in B$, $h, g, l \in H$,

$$\sigma((a \otimes h)(b \otimes g), c \otimes l) = \sum \sigma(a \otimes h, c_{(1)} \otimes {}^{\omega}l_{(1)})\sigma(b \otimes g, {}^{\omega}c_{(2)} \otimes l_{(2)}).$$

To this end we compute

$$\begin{aligned}
&\sigma((a \otimes h)(b \otimes g), c \otimes l)\\
&= \sum p(a_{(1)}b_{(1)}, c_{(1)})u(a_{(2)}b_{(2)}, l_{(2)})v(h_{(2)}g_{(2)}, c_{(2)})\tau(h_{(1)}g_{(1)}, l_{(1)})\\
&\overset{\text{BR1,DC1,SDC1}}{=} \sum p(a_{(1)}b_{(1)}, c_{(1)})u(a_{(2)}, l_{(3)})u(b_{(2)}, l_{(4)})v(g_{(2)}, c_{(2)})v(h_{(2)}, c_{(3)})\\
&\qquad \tau(h_{(1)}, l_{(1)})\tau(g_{(1)}, l_{(2)})\\
&\overset{3.3(8)}{=} \sum p(a_{(1)}b_{(1)}, c_{(1)})u(a_{(2)}, l_{(2)})u(b_{(2)}, l_{(4)})v(g_{(2)}, c_{(2)})v(h_{(2)}, c_{(3)})
\end{aligned}$$

$$
\begin{aligned}
& \tau(h_{(1)}, l_{(1)})\tau(g_{(1)}, l_{(3)}) \\
\overset{3.3(6)}{=} \quad & \sum p(a_{(1)}b_{(1)}, c_{(1)})u(a_{(2)}b_{(2)}, {}^{\omega}1_H)u(a_{(3)}, l_{(2)})u(b_{(3)}, l_{(4)}) \\
& v(g_{(2)}, {}^{\omega}c_{(2)})v(h_{(2)}, c_{(3)})\tau(h_{(1)}, l_{(1)})\tau(g_{(1)}, l_{(3)}) \\
\overset{\mathrm{DC}1}{=} \quad & \sum p(a_{(1)}b_{(1)}, c_{(1)})u(a_{(2)}, ({}^{\omega}1_H)_{(1)})u(b_{(2)}, ({}^{\omega}1_H)_{(2)}) \\
& u(a_{(3)}, l_{(2)})u(b_{(3)}, l_{(4)})v(g_{(2)}, {}^{\omega}c_{(2)})v(h_{(2)}, c_{(3)})\tau(h_{(1)}, l_{(1)})\tau(g_{(1)}, l_{(3)}) \\
\overset{\mathrm{WBR}1}{=} \quad & \sum p(a_{(1)}, c_{(1)})u(a_{(2)}, {}^{w}1_H)p(b_{(1)}, {}^{w}c_{(2)})u(a_{(3)}, ({}^{\omega}1_H)_{(1)}) \\
& u(b_{(2)}, ({}^{\omega}1_H)_{(2)})u(a_{(4)}, l_{(2)})u(b_{(3)}, l_{(4)})v(g_{(2)}, {}^{\omega}c_{(3)}) \\
& v(h_{(2)}, c_{(4)})\tau(h_{(1)}, l_{(1)})\tau(g_{(1)}, l_{(3)}) \\
\overset{\mathrm{DC}3}{=} \quad & \sum p(a_{(1)}, c_{(1)})u(a_{(2)}, {}^{w}1_H({}^{\omega}1_H)_{(1)})p(b_{(1)}, {}^{w}c_{(2)}) \\
& u(b_{(2)}, ({}^{\omega}1_H)_{(2)})u(a_{(3)}, l_{(2)})u(b_{(3)}, l_{(4)})v(g_{(2)}, {}^{\omega}c_{(3)}) \\
& v(h_{(2)}, c_{(4)})\tau(h_{(1)}, l_{(1)})\tau(g_{(1)}, l_{(3)}) \\
\overset{2.1(3)}{=} \quad & \sum p(a_{(1)}, c_{(1)})u(a_{(2)}, {}^{w}1_H{}^{\omega}1_H)p(b_{(1)}, {}^{w}c_{(2)}) \\
& u(b_{(2)}, {}^{\bar{\omega}}1_H)u(a_{(3)}, l_{(2)})u(b_{(3)}, l_{(4)})v(g_{(2)}, {}^{\bar{\omega}}({}^{\omega}c_{(3)})) \\
& v(h_{(2)}, c_{(4)})\tau(h_{(1)}, l_{(1)})\tau(g_{(1)}, l_{(3)}) \\
\overset{2.5(2)}{=} \quad & \sum p(a_{(1)}, c_{(1)})u(a_{(2)}, {}^{\omega}1_H)p(b_{(1)}, ({}^{\omega}c_{(2)})_{(1)})u(b_{(2)}, {}^{\bar{\omega}}1_H)u(a_{(3)}, l_{(2)}) \\
& u(b_{(3)}, l_{(4)})v(g_{(2)}, {}^{\bar{\omega}}({}^{\omega}c_{(2)})_{(2)})v(h_{(2)}, c_{(3)})\tau(h_{(1)}, l_{(1)})\tau(g_{(1)}, l_{(3)}) \\
\overset{3.3(6)}{=} \quad & \sum p(a_{(1)}, c_{(1)})u(a_{(2)}, {}^{\omega}1_H)p(b_{(1)}, ({}^{\omega}c_{(2)})_{(1)})u(a_{(3)}, l_{(2)}) \\
& u(b_{(2)}, l_{(4)})v(g_{(2)}, ({}^{\omega}c_{(2)})_{(2)})v(h_{(2)}, c_{(3)})\tau(h_{(1)}, l_{(1)})\tau(g_{(1)}, l_{(3)}) \\
\overset{3.3(2)}{=} \quad & \sum p(a_{(1)}, c_{(1)})u(a_{(2)}, {}^{\omega}1_H)p(b_{(1)}, ({}^{\omega}({}^{w}c_{(3)}))_{(1)})u(a_{(3)}, l_{(2)})u(b_{(2)}, l_{(4)}) \\
& v(g_{(2)}, ({}^{\omega}({}^{w}c_{(3)}))_{(2)})v(h_{(3)}, c_{(2)})\tau(h_{(2)}, {}^{w}1_H)\tau(h_{(1)}, l_{(1)})\tau(g_{(1)}, l_{(3)}) \\
\overset{2.1(3)}{=} \quad & \sum p(a_{(1)}, c_{(1)})u(a_{(2)}, ({}^{\omega}1_H)_{(2)})p(b_{(1)}, ({}^{\omega}c_{(3)})_{(1)})u(a_{(3)}, l_{(2)})u(b_{(2)}, l_{(4)}) \\
& v(g_{(2)}, ({}^{\omega}c_{(3)})_{(2)})v(h_{(3)}, c_{(2)})\tau(h_{(2)}, ({}^{\omega}1_H)_{(1)})\tau(h_{(1)}, l_{(1)})\tau(g_{(1)}, l_{(3)}) \\
\overset{\mathrm{BR}3,\mathrm{DC}3}{=} \quad & \sum p(a_{(1)}, c_{(1)})u(a_{(2)}, ({}^{\omega}1_H)_{(2)}l_{(2)})p(b_{(1)}, ({}^{\omega}c_{(3)})_{(1)})u(b_{(2)}, l_{(4)}) \\
& v(g_{(2)}, ({}^{\omega}c_{(3)})_{(2)})v(h_{(2)}, c_{(2)})\tau(h_{(1)}, ({}^{\omega}1_H)_{(1)}l_{(1)})\tau(g_{(1)}, l_{(3)}) \\
\overset{2.5(1)}{=} \quad & \sum p(a_{(1)}, c_{(1)})u(a_{(2)}, ({}^{\omega}l_{(1)})_{(2)})p(b_{(1)}, ({}^{\omega}c_{(3)})_{(1)})u(b_{(2)}, l_{(3)}) \\
& v(g_{(2)}, ({}^{\omega}c_{(3)})_{(2)})v(h_{(2)}, c_{(2)})\tau(h_{(1)}, ({}^{\omega}l_{(1)})_{(1)})\tau(g_{(1)}, l_{(2)}) \\
= \quad & \sum \sigma(a \otimes h, c_{(1)} \otimes {}^{\omega}l_{(1)})\sigma(b \otimes g, {}^{\omega}c_{(2)} \otimes l_{(2)}).
\end{aligned}
$$

In a similar manner, we can show that (BR3) is satisfied for σ. To prove that (BR5) holds it suffices to show, for all $a, b \in B, h, g \in H$,

$$
\begin{aligned}
& \sum (b_{(1)}a_{(1)} \otimes {}^{w}g_{(1)}{}^{\omega}h_{(1)})\, \sigma({}^{\omega}a_{(2)} \otimes h_{(2)}, {}^{w}b_{(2)} \otimes g_{(2)}) \\
& \qquad = \sum \sigma(a_{(1)} \otimes {}^{\omega}h_{(1)}, b_{(1)} \otimes {}^{w}g_{(1)})\, ({}^{\omega}a_{(2)}{}^{w}b_{(2)} \otimes h_{(2)}g_{(2)}).
\end{aligned}
$$

In fact we have

$$
\begin{aligned}
&\sum(b_{(1)}a_{(1)}\otimes {}^w g_{(1)}{}^\omega h_{(1)})\,\sigma({}^\omega a_{(2)}\otimes h_{(2)}, {}^w b_{(2)}\otimes g_{(2)})\\
&\qquad = \quad \sum(b_{(1)}a_{(1)}\otimes {}^w g_{(1)}{}^\omega h_{(1)})\,p(({}^\omega a_{(2)})_{(1)}, ({}^w b_{(2)})_{(1)})\\
&\qquad\qquad\quad u(({}^\omega a_{(2)})_{(2)}, g_{(3)})\,v(h_{(3)}, ({}^w b_{(2)})_{(2)})\,\tau(h_{(2)}, g_{(2)})\\
&\qquad \overset{2.1(4)}{=} \quad \sum(b_{(1)}a_{(1)}\otimes {}^{\bar w}({}^w g_{(1)})\,{}^{\bar\omega}({}^\omega h_{(1)}))\,p({}^{\bar\omega}a_{(2)}, {}^{\bar w}b_{(2)})\\
&\qquad\qquad\quad u({}^\omega a_{(3)}, g_{(3)})\,v(h_{(3)}, {}^w b_{(3)})\,\tau(h_{(2)}, g_{(2)})\\
&\qquad \overset{2.5(1)}{=} \quad \sum(b_{(1)}a_{(1)}\otimes {}^w g_{(1)}{}^{\bar w}1_H{}^{\bar\omega}1_H{}^\omega h_{(1)})\,p({}^{\bar\omega}a_{(2)}, {}^{\bar w}b_{(2)})\\
&\qquad\qquad\quad u({}^\omega a_{(3)}, g_{(3)})\,v(h_{(3)}, {}^w b_{(3)})\,\tau(h_{(2)}, g_{(2)})\\
&\qquad \overset{3.3(5)}{=} \quad \sum(b_{(1)}a_{(1)}\otimes {}^w g_{(1)}{}^\omega h_{(1)})\,p(a_{(2)}, b_{(2)})\\
&\qquad\qquad\quad u({}^\omega a_{(3)}, g_{(3)})\,v(h_{(3)}, {}^w b_{(3)})\,\tau(h_{(2)}, g_{(2)})\\
&\qquad \overset{2.5(1)}{=} \quad \sum(b_{(1)}a_{(1)}\otimes {}^w 1_H g_{(1)}h_{(1)}{}^\omega 1_H)\,p(a_{(2)}, b_{(2)})\\
&\qquad\qquad\quad u({}^\omega a_{(3)}, g_{(3)})\,v(h_{(3)}, {}^w b_{(3)})\,\tau(h_{(2)}, g_{(2)})\\
&\qquad \overset{\mathrm{BR}5}{=} \quad \sum(b_{(1)}a_{(1)}\otimes {}^w 1_H h_{(2)}g_{(2)}{}^\omega 1_H)\,p(a_{(2)}, b_{(2)})\\
&\qquad\qquad\quad u({}^\omega a_{(3)}, g_{(3)})\,v(h_{(3)}, {}^w b_{(3)})\,\tau(h_{(1)}, g_{(1)})\\
&\qquad \overset{3.3(3,4)}{=} \quad \sum(b_{(1)}a_{(1)}\otimes h_{(3)}g_{(3)})\,p(a_{(2)}, b_{(2)})\\
&\qquad\qquad\quad u(a_{(3)}, g_{(2)})\,v(h_{(2)}, b_{(3)})\,\tau(h_{(1)}, g_{(1)})\\
&\qquad \overset{\mathrm{WBR}5}{=} \quad \sum({}^\omega a_{(3)}{}^w b_{(2)}\otimes h_{(3)}g_{(3)})\,p(a_{(1)}, b_{(1)})\,u(a_{(2)}, {}^w 1_H)\\
&\qquad\qquad\quad v({}^\omega 1_H, b_{(3)})\,u(a_{(4)}, g_{(2)})\,v(h_{(2)}, b_{(4)})\,\tau(h_{(1)}, g_{(1)})\\
&\qquad \overset{\mathrm{SDC}1}{=} \quad \sum({}^\omega a_{(3)}{}^w b_{(2)}\otimes h_{(3)}g_{(3)})\,p(a_{(1)}, b_{(1)})\,u(a_{(2)}, {}^w 1_H)\\
&\qquad\qquad\quad v(h_{(2)}{}^\omega 1_H, b_{(3)})\,u(a_{(4)}, g_{(2)})\,\tau(h_{(1)}, g_{(1)})\\
&\qquad \overset{2.5(1)}{=} \quad \sum({}^\omega a_{(3)}{}^w b_{(2)}\otimes h_{(3)}g_{(3)})\,p(a_{(1)}, b_{(1)})\,u(a_{(2)}, {}^w 1_H)\\
&\qquad\qquad\quad v(h_{(2)}{}^\omega 1_H, b_{(3)})\,u(a_{(4)}, g_{(2)})\,\tau(h_{(1)}, g_{(1)})\\
&\qquad \overset{3.3(1)}{=} \quad \sum({}^\omega({}^{\bar\omega}a_{(4)}){}^w b_{(2)}\otimes h_{(3)}g_{(4)})\,p(a_{(1)}, b_{(1)})\,u(a_{(2)}, {}^w 1_H)\\
&\qquad\qquad\quad v({}^\omega 1_H h_{(2)}, b_{(3)})\,u(a_{(3)}, g_{(3)})\,\tau({}^{\bar\omega}1_H, g_{(2)})\,\tau(h_{(1)}, g_{(1)})\\
&\qquad \overset{\mathrm{BR}1}{=} \quad \sum({}^\omega({}^{\bar\omega}a_{(4)}){}^w b_{(2)}\otimes h_{(3)}g_{(3)})\,p(a_{(1)}, b_{(1)})\,u(a_{(2)}, {}^w 1_H)\\
&\qquad\qquad\quad v({}^\omega 1_H h_{(2)}, b_{(3)})\,u(a_{(3)}, g_{(2)})\,\tau({}^{\bar\omega}1_H h_{(1)}, g_{(1)})\\
&\qquad \overset{2.1(3)}{=} \quad \sum({}^\omega a_{(4)}{}^w b_{(2)}\otimes h_{(3)}g_{(3)})\,p(a_{(1)}, b_{(1)})\,u(a_{(2)}, {}^w 1_H)\\
&\qquad\qquad\quad v(({}^\omega 1_H)_{(2)}h_{(2)}, b_{(3)})\,u(a_{(3)}, g_{(2)})\,\tau(({}^\omega 1_H)_{(1)}h_{(1)}, g_{(1)})\\
&\qquad \overset{3.3(2)}{=} \quad \sum({}^\omega a_{(4)}{}^w({}^{\bar w}b_{(3)})\otimes h_{(4)}g_{(3)})\,p(a_{(1)}, b_{(1)})\,u(a_{(2)}, {}^w 1_H)\\
&\qquad\qquad\quad v(({}^\omega 1_H)_{(3)}h_{(3)}, b_{(2)})\,\tau(({}^\omega 1_H)_{(2)}h_{(2)}, {}^{\bar w}1_H)\,u(a_{(3)}, g_{(2)})\\
&\qquad\qquad\quad \tau(({}^\omega 1_H)_{(1)}h_{(1)}, g_{(1)})
\end{aligned}
$$

$$
\begin{aligned}
\overset{\text{BR3}}{=} \quad & \sum ({}^{\omega}a_{(4)}{}^{w}({}^{\bar{w}}b_{(3)}) \otimes h_{(3)}g_{(3)})\, p(a_{(1)}, b_{(1)})\, u(a_{(2)}, {}^{w}1_H) \\
& \quad v(({}^{\omega}1_H)_{(2)}h_{(2)}, b_{(2)})\, \tau(({}^{\omega}1_H)_{(1)}h_{(1)}, {}^{\bar{w}}1_H g_{(1)})\, u(a_{(3)}, g_{(2)}) \\
\overset{2.1(3)}{=} \quad & \sum ({}^{\omega}a_{(4)}{}^{w}b_{(3)} \otimes h_{(3)}g_{(3)})\, p(a_{(1)}, b_{(1)})\, u(a_{(2)}, ({}^{w}1_H)_{(2)}) \\
& \quad v(({}^{\omega}1_H)_{(2)}h_{(2)}, b_{(2)})\, \tau(({}^{\omega}1_H)_{(1)}h_{(1)}, ({}^{w}1_H)_{(1)}g_{(1)})\, u(a_{(3)}, g_{(2)}) \\
\overset{\text{DC3}}{=} \quad & \sum ({}^{\omega}a_{(3)}{}^{w}b_{(3)} \otimes h_{(3)}g_{(3)})\, p(a_{(1)}, b_{(1)})\, u(a_{(2)}, ({}^{w}1_H)_{(2)}g_{(2)}) \\
& \quad v(({}^{\omega}1_H)_{(2)}h_{(2)}, b_{(2)})\, \tau(({}^{\omega}1_H)_{(1)}h_{(1)}, ({}^{w}1_H)_{(1)}g_{(1)}) \\
\overset{2.5(1)}{=} \quad & \sum ({}^{\omega}a_{(3)}{}^{w}b_{(3)} \otimes h_{(2)}g_{(2)})\, p(a_{(1)}, b_{(1)})\, u(a_{(2)}, ({}^{w}g_{(1)})_{(2)}) \\
& \quad v(({}^{\omega}h_{(1)})_{(2)}, b_{(2)})\, \tau(({}^{\omega}h_{(1)})_{(1)}, ({}^{w}g_{(1)})_{(1)}) \\
= \quad & \sum \sigma(a_{(1)} \otimes {}^{\omega}h_{(1)}, b_{(1)} \otimes {}^{w}g_{(1)})({}^{\omega}a_{(2)}{}^{w}b_{(2)} \otimes h_{(2)}g_{(2)}).
\end{aligned}
$$

Finally, we conclude that $(B_\omega \bowtie H, \sigma)$ is a braided Hopf algebra. $\square$

Combining the Propositions 3.2, 3.3, 3.4 and Theorem 3.5, we obtain the main result of this section.

Theorem 3.6 *Let $B_\omega \bowtie H$ be an ω-smash coproduct Hopf algebra. Then the following are equivalent:*

(a) *$(B_\omega \bowtie H, \sigma)$ is a braided Hopf algebra;*

(b) *for all $a, b \in B$ and $h, g \in H$,*

$$\sigma(a \otimes h, b \otimes g) = \sum p(a_{(1)}, b_{(1)})\, u(a_{(2)}, g_{(2)})\, v(h_{(2)}, b_{(2)})\, \tau(h_{(1)}, g_{(1)}),$$

and (H, τ) is a braided Hopf algebra,
(B, H, u) is a dual compatible u-Hopf algebra pair,
(H, B, v) is a skew dual compatible v-Hopf algebra pair,
(B, p) is a (u, v)-weakly braided Hopf algebra, and
p, τ, u, v satisfy the conditions 3.3(1)-(9).

4 Applications

In this section, we will discuss some applications of Theorem 3.6.

Let B and H be Hopf R-algebras, B a left H-comodule bialgebra, we know from Example 2.4(2) that the usual smash coproduct Hopf algebra $B \times H$ can be viewed as a special case of an ω-smash coproduct Hopf algebra $B_\omega \bowtie H$, where the right normal linear map $\omega : B \otimes H \to H \otimes B$, is given by $\omega(b \otimes h) = \sum b^{(1)}h \otimes b^{<2>}$, for $b \in B$ and $h \in H$. So, we can repeat the Definitions 2.7 - 2.9 in terms of the usual smash coproduct. Especially, Theorem 3.6 takes the following form.

Theorem 4.1 *Let $B \times H$ be a smash coproduct Hopf algebra. Then the following are equivalent:*

(a) *$(B \times H, \sigma)$ is a braided Hopf algebra with σ a bilinear form on $B \times H$;*

(b) *for all* $a, b \in B$ *and* $h, g \in H$,

$$\sigma(a \otimes h, b \otimes g) = \sum p(a_{(1)}, b_{(1)})u(a_{(2)}, g_{(2)})v(h_{(2)}, b_{(2)})\tau(h_{(1)}, g_{(1)}),$$

(H, τ), (B, H, u), (H, B, v) *and* (B, p) *have the properties as in* Theorem 3.6(b), *and for* p, τ, u, v,

(1) $\sum b_{(1)}u(b_{(2)}, h) = \sum u(b_{(1)}, h_{(2)})\tau(b_{(2)}{}^{(1)}, h_{(1)})b_{(2)}{}^{<2>}$;

(2) $\sum b_{(1)}v(h, b_{(2)}) = \sum v(h_{(2)}, b_{(1)})\tau(h_{(1)}, b_{(2)}{}^{(1)})b_{(2)}{}^{<2>}$;

(3) $\sum u(b, h_{(1)})h_{(2)} = \sum h_{(1)}b^{(1)}u(b^{<2>}, h_{(2)})$;

(4) $\sum v(h_{(1)}, b)h_{(2)} = \sum v(h_{(2)}, b^{<2>},)b^{(1)}h_{(1)}$;

(5) $p(a, b)1_H = \sum b^{(1)}a^{(1)}p(a^{<2>}, b^{<2>})$;

(6) $\sum p(a, b_{(1)})v(h, b_{(2)}) = \sum p(a_{(1)}, b_{(1)})u(a_{(2)}, b_{(2)}{}^{(1)})v(h, b_{(2)}{}^{<2>})$;

(7) $\sum p(a_{(1)}, b)u(a_{(2)}, h) = \sum p(a_{(1)}, b_{(1)})v(a_{(2)}{}^{(1)}, b_{(2)})u(a_{(2)}{}^{<2>}, h)$;

(8) $\sum u(a, h_{(2)})\tau(g, h_{(1)}) = \sum u(a, h_{(1)})\tau(g, h_{(2)})$;

(9) $\sum v(h_{(2)}, a)\tau(h_{(1)}, g) = \sum v(h_{(1)}, a)\tau(h_{(2)}, g)$.

Similarly by Example 2.4(3), Theorem 3.6 takes the following form for the twisted smash coproduct.

Theorem 4.2 *Let* $B \times_r H$ *be a twisted smash coproduct Hopf algebra. Then the following are equivalent:*

(a) $(B \times_r H, \sigma)$ *is a braided Hopf algebra with* σ *a bilinear form on* $B \times_r H$;

(b) *for all* $a, b \in B$ *and* $h, g \in H$,

$$\sigma(a \otimes h, b \otimes g) = \sum p(a_{(1)}, b_{(1)})u(a_{(2)}, g_{(2)})v(h_{(2)}, b_{(2)})\tau(h_{(1)}, g_{(1)}),$$

(H, τ), (B, H, u), (H, B, v) *and* (B, p) *have the properties as in* Theorem 3.6(b), *and for* p, τ, u, v,

(1) $\sum b_{(1)}u(b_{(2)}, h) = \sum u(b_{(1)}, h_{(2)})\tau(b_{(2)}{}^{(1)}S_H(b_{(2)}{}^{<2>(2)}), h_{(1)})b_{(2)}{}^{<2><1>}$;

(2) $\sum b_{(1)}v(h, b_{(2)}) = \sum v(h_{(2)}, b_{(1)})\tau(h_{(1)}, b_{(2)}{}^{(1)}S_H(b_{(2)}{}^{<2>(2)}))b_{(2)}{}^{<2><1>}$;

(3) $\sum u(b, h_{(1)})h_{(2)} = \sum h_{(1)}b^{(1)}S_H(b^{<2>(2)})u(b^{<2><1>}, h_{(2)})$;

(4) $\sum v(h_{(1)}, b)h_{(2)} = \sum b^{(1)}S_H(b^{<2>(2)})h_{(1)}v(h_{(2)}, b^{<2><1>})$;

(5) $p(a, b)1_H = \sum b^{(1)}S_H(b^{<2>(2)})a^{(1)}S_H(a^{<2>(2)})p(a^{<2><1>}, b^{<2><1>})$;

(6) $\sum p(a, b_{(1)})v(h, b_{(2)}) = \sum p(a_{(1)}, b_{(1)})u(a_{(2)}, b_{(2)}{}^{(1)}S_H(b_{(2)}{}^{<2>(2)}))$
$v(h, (b_{(2)}{}^{<2><1>})$;

(7) $\sum p(a_{(1)}, b)u(a_{(2)}, h) = \sum p(a_{(1)}, b_{(1)})v(a_{(2)}{}^{(1)}S_H((a_{(2)}{}^{<2>(2)}), b_{(2)})$
$u(a_{(2)}{}^{<2><1>}, h)$;

(8) $\sum u(a, h_{(2)})\tau(g, h_{(1)}) = \sum u(a, h_{(1)})\tau(g, h_{(2)})$;

(9) $\sum v(h_{(2)}, a)\tau(h_{(1)}, g) = \sum v(h_{(1)}, a)\tau(h_{(2)}, g)$.

Any commutative Hopf algebra H has a trivial braided structure $\tau(h,g)=\varepsilon_H(h)\varepsilon_H(g)$, for all $h,g\in H$, and this yields the following.

Theorem 4.3 *Let $B_\omega\bowtie H$ be an ω-smash coproduct Hopf algebra with H a commutative Hopf algebra. Assume that (B,H,u) is a dual compatible u-Hopf algebra pair, (H,B,v) is a skew dual compatible v-Hopf algebra pair, (B,p) is a (u,v)-weakly braided Hopf algebra, and that, for $a,b\in B$ and $h\in H$,*

(1) $\sum b_{(1)}u(b_{(2)},h)=\sum u(b_{(1)},h)b_{(2)}$;

(2) $\sum b_{(1)}v(h,b_{(2)})=\sum v(h,b_{(1)})b_{(2)}$;

(3) $\sum u(b,h_{(1)})h_{(2)}=\sum h_{(1)}{}^{\omega}1_H u({}^{\omega}b,h_{(2)})$;

(4) $\sum v(h_{(1)},b)h_{(2)}=\sum v(h_{(2)},{}^{\omega}b){}^{\omega}1_H h_{(1)}$;

(5) $p(a,b)1_H=\sum {}^{w}1_H{}^{\omega}1_H\, p({}^{\omega}a,{}^{w}b)$;

(6) $\sum p(a,b_{(1)})v(h,b_{(2)})=\sum p(a_{(1)},b_{(1)})u(a_{(2)},{}^{\omega}1_H)v(h,{}^{\omega}b_{(2)})$;

(7) $\sum p(a_{(1)},b)u(a_{(2)},h)=\sum p(a_{(1)},b_{(1)})v({}^{\omega}1_H,b_{(2)})u({}^{\omega}a_{(2)},h)$.

Then $(B_\omega\bowtie H,\sigma)$ is a braided Hopf algebra, with the bilinear form

$$\sigma: B_\omega\bowtie H\otimes B_\omega\bowtie H\to R,\quad \sigma(a\otimes h,b\otimes g)=\sum p(a_{(1)},b_{(1)})u(a_{(2)},g)v(h,b_{(2)}),$$

for all $a,b\in B$ and $h,g\in H$.

Proof. Take $\tau(h,g)=\varepsilon_H(h)\varepsilon_H(g)$, for all $h,g\in H$. We always have that (H,τ) is a braided Hopf algebra and the conditions 3.3(8)-3.3(9) hold. In this case, the conditions 3.3(1) and 3.3(2) change to 4.3(1) and 4.3(2). Thus we have that p,u,v,τ satisfy the all conditions in Theorem 3.5 and we conclude that $(B_\omega\bowtie H,\sigma)$ is a braided Hopf algebra. □

Theorem 4.4 *Let $B_\omega\bowtie H$ be an ω-smash coproduct Hopf algebra, and assume (H,τ), (B,p) to be braided Hopf algebras such that, for $a,b,\in B$ and $h\in H$,*

(1) $\sum b\,\varepsilon_H(h)=\sum \tau({}^{\omega}1_H,h){}^{\omega}b$;

(2) $\sum b\,\varepsilon_H(h)=\sum \tau(h,{}^{\omega}1_H){}^{\omega}b$;

(3) $p(a,b)1_H=\sum {}^{w}1_H{}^{\omega}1_H\, p({}^{\omega}a,{}^{w}b)$.

Then $(B_\omega\bowtie H,\sigma)$ is a braided Hopf algebra, with bilinear form

$$\sigma: B_\omega\bowtie H\otimes B_\omega\bowtie H\to R,\quad \sigma(a\otimes h,b\otimes g)=\sum p(a,b)\tau(h,g),$$

for all $a,b\in B,h,g\in H$.

Proof. By assumption, (H,τ) and (B,p) are braided Hopf algebras. With the linear forms $u=\varepsilon_B\otimes\varepsilon_H$ and $v=\varepsilon_H\otimes\varepsilon_B$, we always have that (B,H,u) is a dual compatible u-Hopf algebra pair, (H,B,v) is a skew dual compatible v-Hopf algebra pair, (B,p) is a (u,v)-weakly braided Hopf algebra, and the conditions 3.3(3)-3.3(4) and 3.3(6)-3.3(9) hold. In this case, the conditions 3.3(1), 3.3(2) and 3.3(5) take the forms 4.4(1), 4.4(2) and 4.4(3). Thus we have that p,u,v,τ satisfy all conditions in Theorem 3.5 and from this we conclude that $(B_\omega\bowtie H,\sigma)$ is a braided Hopf algebra. □

Combining Theorem 4.3 and Theorem 4.4 we get the

Corollary 4.5 *Let $B_\omega \bowtie H$ be an ω-smash coproduct Hopf algebra, H a commutative Hopf algebra, and (B, p) a braided Hopf algebra such that, for $a, b \in B$,*

$$p(a,b)1_H = \sum {}^{\bar{\omega}}1_H {}^{\omega}1_H p({}^{\omega}a, {}^{\bar{\omega}}b).$$

Then $(B_\omega \bowtie H, \sigma)$ is a braided Hopf algebra, with bilinear form

$$\sigma : B_\omega \bowtie H \otimes B_\omega \bowtie H \to R, \quad (a \otimes h) \otimes (b \otimes g) = \sum p(a,b)\varepsilon_H(h)\varepsilon_H(g),$$

for all $a, b \in B, h, g \in H$.

5 Examples

To end our paper we construct explicit examples of ω-smash coproduct Hopf algebras over a ring R. First assume that 2 is invertible in R and let $B = H_4$ be Sweedler's 2-generated Hopf R-algebra (see [8]). This is a free R-module with basis $1, g, x, gx$ and as an algebra it has the generators g and x with relations

$$g^2 = 1, \ x^2 = 0, \ xg = -gx.$$

The coalgebra structure and antipode of H_4 are given by

$$\Delta(g) = g \otimes g, \ \Delta(x) = x \otimes g + 1 \otimes x, \ \Delta(gx) = gx \otimes 1 + g \otimes gx;$$
$$\epsilon(g) = 1, \ \epsilon(x) = 0, \ \epsilon(gx) = 0; \quad S(g) = g, \ S(x) = gx.$$

Let $H = R\mathbb{Z}_2$ be the (group) Hopf algebra [11], where $\mathbb{Z}_2$ is written multiplicatively as $\{1, a\}$.

Lemma 5.1 *Define a linear map $\omega : B \otimes H \to H \otimes B$ by*

$$\begin{array}{rccccccc} \omega : & 1_B \otimes 1_H & \mapsto & 1_H \otimes 1_B, & 1_B \otimes a & \mapsto & a \otimes 1_B \\ & g \otimes 1_H & \mapsto & 1_H \otimes g, & g \otimes a & \mapsto & a \otimes g \\ & x \otimes 1_H & \mapsto & a \otimes x, & x \otimes a & \mapsto & 1_H \otimes x \\ & gx \otimes 1_H & \mapsto & a \otimes gx, & gx \otimes a & \mapsto & 1_H \otimes gx. \end{array}$$

Then $B_\omega \bowtie H$ is an ω-smash coproduct Hopf algebra and ω is right normal.

Proof. A direct calculation shows that ω is well defined, conormal, comultiplicative and an algebra map. By Proposition 2.2, we see that $B_\omega \bowtie H$ is an ω-smash coproduct Hopf algebra. It is obvious that ω is right normal. □

We prepare to compute braidings of $B_\omega \bowtie H$. For this we need

Lemma 5.2 *Let $B = H_4, H = R\mathbb{Z}_2$ and ω the linear map defined above. Then*

(1) *$(R\mathbb{Z}_2, \tau)$ is a braided Hopf algebra, where $\tau : R\mathbb{Z}_2 \otimes R\mathbb{Z}_2 \to R$ is given by*

τ	1	a
1	1	1
a	1	-1

(2) $(H_4, R\mathbb{Z}_2, u)$ *is a dual compatible u-Hopf algebra pair, where* $u : H_4 \otimes R\mathbb{Z}_2 \to R$ *is given by*

u	1	a
1	1	1
g	1	-1
x	0	0
gx	0	0

(3) $(R\mathbb{Z}_2, H_4, v)$ *is a skew dual compatible v-Hopf algebra pair, where* $v : R\mathbb{Z}_2 \otimes H_4 \to R$ *is given by*

v	1	g	x	gx
1	1	1	0	0
a	1	-1	0	0

(4) (H_4, p) *is a* (u, v)*-weakly braided Hopf algebra, where* $p : H_4 \otimes H_4 \to R$ *is given by*

p	1	g	x	gx
1	1	1	0	0
g	1	1	0	0
x	0	0	0	0
gx	0	0	0	0

and p, τ, u, v *satisfy the conditions 3.3(1)-(9) (as in Theorem 3.6).*

Proof. The proof is straightforward and left to the reader. □

Combining Lemma 4.6 and 4.7, we see that all conditions of Theorem 3.6 are satisfied for $B = H_4$ and $H = R\mathbb{Z}_2$. Thus we have

Proposition 5.3 *Let* $B = H_4$, $H = R\mathbb{Z}_2$ *and* ω *the linear map given above. Then* $(B_\omega \bowtie H, \sigma)$ *is a braided Hopf algebra, where*

$$\sigma(a \otimes h, b \otimes g) = \sum p(a_{(1)}, b_{(1)}) u(a_{(2)}, g_{(2)}) v(h_{(2)}, b_{(2)}) \tau(h_{(1)}, g_{(1)})$$

is given by

σ	$1\otimes 1$	$1\otimes a$	$g\otimes 1$	$g\otimes a$	$x\otimes 1$	$x\otimes a$	$gx\otimes 1$	$gx\otimes a$
$1\otimes 1$	1	1	1	1	0	0	0	0
$1\otimes a$	1	1	-1	1	0	0	0	0
$g\otimes 1$	1	-1	1	-1	0	0	0	0
$g\otimes a$	1	1	-1	-1	0	0	0	0
$x\otimes 1$	0	0	0	0	0	0	0	0
$x\otimes a$	0	0	0	0	0	0	0	0
$gx\otimes 1$	0	0	0	0	0	0	0	0
$gx\otimes a$	0	0	0	0	0	0	0	0

To find other braidings recall that if 2 is invertible in R, then H_4 is quasitriangular and selfdual. Using the Hopf algebra isomorphism $H_4 \cong H_4^*$ described in [9] at the end of Section 2 (or specializing [10, Proposition 8]) we can compute all the braided structures of H_4.

Lemma 5.4 *For any $\alpha \in R$, (H_4, p_α) is a braided Hopf algebra, where p_α is given by*

p_α	1	g	x	gx
1	1	1	0	0
g	1	-1	0	0
x	0	0	α	$-\alpha$
gx	0	0	α	α

Moreover, for all $\alpha \in R$, p_α satisfies the condition (5) in Proposition 3.3.

Proof. This is proved by a direct computation. □

Observe that the assumption about invertibility of 2 in R is not necessary for computing the braided structures of H_4. Since $R\mathbb{Z}_2$ is commutative, by applying Corollary 4.5 we obtain a class of new braidings for $H_{4\omega}\bowtie R\mathbb{Z}_2$.

Proposition 5.5 *Let $B = H_4$, $H = R\mathbb{Z}_2$ and ω the linear map given above. Then for all $\alpha \in R$, $(B_\omega \bowtie H, \sigma_\alpha)$ is a braided Hopf algebra, where*

$$\sigma_\alpha(a \otimes h, b \otimes g) = \sum p(a,b)\varepsilon_H(h)\varepsilon_H(g)$$

is given by

σ_α	$1\otimes 1$	$1\otimes a$	$g\otimes 1$	$g\otimes a$	$x\otimes 1$	$x\otimes a$	$gx\otimes 1$	$gx\otimes a$
$1\otimes 1$	1	1	1	1	0	0	0	0
$1\otimes a$	1	1	1	1	0	0	0	0
$g\otimes 1$	1	1	-1	-1	0	0	0	0
$g\otimes a$	1	1	-1	-1	0	0	0	0
$x\otimes 1$	0	0	0	0	α	α	-α	-α
$x\otimes a$	0	0	0	0	α	α	-α	-α
$gx\otimes 1$	0	0	0	0	α	α	α	α
$gx\otimes a$	0	0	0	0	α	α	α	α

By Lemma 4.7(1), $R\mathbb{Z}_2$ has a (unique) non-trivial braided structure τ (see also [10]). If the characteristic of R is 2, then τ satisfies the conditions (1)-(2) in Theorem 4.4. Applying now Theorem 4.4 we can compute a third braiding family $\sigma_\alpha, \alpha \in R$ for $H_{4\omega}\bowtie R\mathbb{Z}_2$, where

$$\sigma_\alpha(a \otimes h, b \otimes g) = \sum p(a,b)\tau(h,g), \qquad \text{for } a, b \in B,\ h, g \in H.$$

But in this case (characteristic 2) τ becomes trivial, so we are in the situation considered above.

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