Coprime Preradicals and Modules

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Abstract

The paper is concerned with the study of coprime elements in the big lattice of preradicals of module categories. In particular we are interested in the module theoretic characterization of this property. Since preradicals are closely related to fully invariant submodules the results are different to those for coprimeness in the lattice of submodules of a given module.

1 Introduction

The definition of primeness of ideals in a ring R is based on the product of ideals. A similar formalism can be developed replacing the product by the intersection of ideals and the resulting theory is concerned with reducibility of rings and factor rings. For the dual notions one may start with some "coproduct" or else the sum of ideals.

The class R-**pr** of preradicals (subfunctors of the identity) on the category of R-modules allows four operations, that is, the meet $\wedge$, the join $\vee$, the product $\cdot$ and the coproduct $:$ (see Section 2). The triple $(R\text{-}\mathbf{pr}, \wedge, \vee)$ behaves like a lattice, except that R-**pr** need not be a set; hence it is called a *big lattice*.

Now primeness can be considered with respect to the product as well as the $\wedge$ and this was done in various papers (e.g., [4], [6], [5]). Since for any preradical τ and $N \in R\text{-}\mathbf{Mod}$, $\tau(N) \subseteq N$ is a invariant submodule, the application of the resulting theory focusses on the structure of fully invariant submodules rather than of (ordinary) submodules.

Dually coprimeness can be studied in R-**pr** based on the coproduct : or on the join $\vee$. The purpose of this paper is to develop such theories. For the sake of generality we do not only consider preradicals on R-**Mod** but on the category $\sigma[M]$, consisting of submodules of M-generated modules, where M is any R-module. We denote the class of preradicals of $\sigma[M]$ by M-**pr** and it is obvious that the operations $\wedge, \vee, \cdot, :$ are also defined on M-**pr**.

On the lattice M-**hpr** of *hereditary preradicals* on $\sigma[M]$, coprimeness with respect to : was investigated in [10] (where the coprime modules are called *duprime*). As we shall see these notions in general differ from those derived in M-**pr** since there are obviously more preradicals than left exact preradicals.

Applied to modules, the coproduct of preradicals induces a "coproduct" of fully invariant submodules of any module N and the notions of coprimeness of modules and fully invarinat submodules. Again there is a difference between these notions and the notion of coprimeness considered in [3] and [2].

In Section 2 basic facts about the big lattice M-**pr** are provided.

In Section 3 coprime preradicals are defined and investigated. In particular the case when the top element $\underline{1}$ is coprime is considered. The results are used to introduce a coproduct for fully invariant submodules for any module and to observe its properties in Section 4. We learn that the condition for $\underline{1}$ to be coprime in M-**pr** in general is stronger then to be coprime in the lattice of left exact preradicals (see Remarks 4.7). Applied to the ring R the first condition forces the ring to be simple (see 3.10), whereas the latter condition requires R to be a left strongly prime ring (see [10, Theorem 3.3]).

In Section 5 we consider coprimeness based on the join $\vee$, a condition which is weaker than coprimeness derived from the coproduct :. The results are related to decompositions of modules into fully invariant submodules.

To place our results within similar investigations let us recall that there are various (big) lattices associated to a category $\sigma[M]$: The big lattices of all preradicals, all idempotent preradicals, all radicals, all idempotent radicals, and the lattices of all hereditary preradicals and all hereditary radicals on $\sigma[M]$. All these lattices have (possibly different) meets and joins, some of them have in addition products and coproducts. Notice that all these lattices except M-**pr** can be characterized by certain classes of modules (pretorsion or pretorsion free classes). Although they are all subclasses of M-**pr**, in general they need not be sublattices, that is, the binary operations may be different. However, the operations $\wedge$, $\cdot$ and : on M-**pr** can be restricted to the lattice of all hereditary preradicals M-**hpr** where $\wedge$ and $\cdot$ coincide. There is a surjective assignment

$$h : M\text{-}\mathbf{pr} \to M\text{-}\mathbf{hpr}, \quad \tau \mapsto h(\tau),$$

defined by putting $h(\tau)(N) = N \cap \tau(\widehat{N})$ for any $N \in \sigma[M]$, where $\widehat{N}$ denotes the M-injective hull of N. This assignment respects arbitrary meets and

$$h(\tau : \rho) \leq (h(\tau) : h(\rho)).$$

From this context it is clear that hereditary preradicals which are coprime in M-**pr** are certainly coprime in M-**hpr**. In particular, a coprime module M is duprime (i.e., $\underline{1}$ is coprime in M-**hpr**).

Investigating coprimeness in a general setting is expected to be of help for studying this notion for coalgebras. This will be done elsewhere.

2 Preliminaries

Let R be an associative ring with unit and R-**Mod** the category of unital left R-modules. For a (fixed) left R-module M, we denote by $\sigma[M]$ the full subcategory of R-**Mod** whose objects are all modules subgenerated by M, and by M-**pr** the big lattice of all preradicals in $\sigma[M]$, that is, the class of all subfunctors of the identity functor of $\sigma[M]$. By $\underline{1}$ and $\underline{0}$ we denote the top and bottom element of this lattice, respectively. For $M = R$, $\sigma[M]$ is equal to R-**Mod** and R-**pr** is the big lattice of preradicals in R-**Mod**.

Recall that a preradical ρ is said to be *hereditary* if for any submodule $K \subset N$, $\rho(K) = K \cap \rho(N)$, and ρ is *cohereditary* (or *right exact*) if for any epimorphism $f : N \to L$, $\rho(L) = f(\rho(N))$.

2.1. Basic preradicals. For $N \in \sigma[M]$ and any fully invariant submodule $K \subset N$, the preradicals α_K^N and ω_K^N are defined by putting, for any $L \in \sigma[M]$,

$$\begin{aligned} \alpha_K^N(L) &= \textstyle\sum\{f(K) \,|\, f : N \to L\}, \\ \omega_K^N(L) &= \textstyle\bigcap\{g^{-1}(K) \,|\, g : L \to N\}. \end{aligned}$$

The following assertions are easy to verify.

Properties. Let $N, L \in \sigma[M]$ and $K \subseteq N$ a fully invariant submodule.

(i) $\alpha_N^N(L)$ is the trace of N in L.

(ii) ω_0^N is the reject of N in L.

(iii) If L is N-injective, then $\alpha^N_K(L) = \alpha^K_K(L)$.

(iv) If L is N-projective, then $\omega^N_K = \omega^{N/K}_0$

(v) If N is projective in $\sigma[M]$, then α^N_K is a cohereditary preradical.

(vi) If N is M-injective, then ω^N_K is a hereditary preradical.

Note that for $N, K \in \sigma[M]$ and $\tau \in M$-**pr**, $\tau(N) = K$ holds if and only if K is a fully invariant submodule of N and

$$\alpha^N_K \le \tau \le \omega^N_K.$$

M-**pr** is an atomic lattice and the atoms are precisely the set of (hereditary) preradicals

$$\{\alpha^{\widehat{S}}_S \mid S \text{ a simple module in } \sigma[M]\},$$

where $\widehat{S}$ denotes the M-injective hull of S.

2.2. Operations on M-pr. There are four binary operations in M-**pr** denoted by $\wedge$, $\vee$, $\cdot$ and $:$ and defined by putting, for $\tau, \rho \in M$-**pr** and $N \in \sigma[M]$,

$$\begin{array}{rcl} (\tau \wedge \rho)(N) & = & \tau(N) \cap \rho(N), \\ (\tau \vee \rho)(N) & = & \tau(N) + \rho(N), \\ (\tau \cdot \rho)(N) & = & \tau(\rho(N)), \\ (\tau : \rho)(N) & & \text{such that } (\tau : \rho)(N)/\tau(N) = \rho(N/\tau(N)). \end{array}$$

The meet $\wedge$ and the join $\vee$ can be defined for classes of preradicals by (X an index class)

$$\begin{array}{rcl} (\bigwedge\{\tau_i \mid i \in X\})(N) & = & \bigcap\{\tau_i(N) \mid i \in X\}, \\ (\bigvee\{\tau_i \mid i \in X\})(N) & = & \sum\{\tau_i(N) \mid i \in X\}. \end{array}$$

Any preradical τ in M-**pr** may be described in terms of the α's or ω's in the following way:

$$\tau = \bigvee\{\alpha^N_{\tau(N)} \mid N \in \sigma[M]\} \quad \text{or} \quad \tau = \bigwedge\{\omega^N_{\tau(N)} \mid N \in \sigma[M]\}.$$

Recall that τ in M-**pr** is said to be *idempotent* if $\tau \cdot \tau = \tau$, and it is a *radical* if $(\tau : \tau) = \tau$.

2.3. Associated preradicals. To any $\tau \in M$-**pr** we assign the preradicals

$$\begin{array}{ll} e(\tau) = \bigwedge\{\rho \in M\text{-}\mathbf{pr} \mid \rho \cdot \tau = \tau\} & = \text{the } \textit{equalizer} \text{ of } \tau, \\ a(\tau) = \bigvee\{\rho \in M\text{-}\mathbf{pr} \mid \rho \cdot \tau = \underline{0}\} & = \text{the } \textit{annihilator} \text{ of } \tau \\ c(\tau) = \bigvee\{\rho \in M\text{-}\mathbf{pr} \mid (\tau : \rho) = \tau\} & = \text{the } \textit{coequalizer} \text{ of } \tau, \\ t(\tau) = \bigwedge\{\rho \in M\text{-}\mathbf{pr} \mid (\tau : \rho) = \underline{1}\} & = \text{the } \textit{totalizer} \text{ of } \tau. \end{array}$$

Pseudo complements have been studied in various lattices and big lattices in ring theory, for example in the lattice of hereditary torsion theories by Golan [7], in the big lattice of Serre subcategories by Raggi and Signoret [8], in the lattice of hereditary pretorsion classes by Raggi, Ríos and Wisbauer [9], in the big lattice of herdeditary and cohereditary classes by Alvarado, Rincon and Ríos [1] and in the big lattice of preradicals over a ring [6].

2.4. Pseudo complements. *Let $\tau \in M$-**pr**.*

*(1) There exists a unique pseudo complement $\tau^{\perp} \in M$-**pr** such that*

  (i) $\tau \wedge \tau^{\perp} = \underline{0}$, and

  *(ii) for any $\rho \in M$-**pr** with $\tau \wedge \rho = \underline{0}$, $\rho \leq \tau^{\perp}$.*

(2) $\tau^{\perp}$ is a left exact radical.

(3) $\tau^{\perp} \leq a(\tau)$ and $\tau^{\perp} \leq t(\tau)$.

(4) For any simple module $S \in \sigma[M]$, $(\alpha_S^{\widehat{S}})^{\perp} = \omega_0^{\widehat{S}}$.

Proof. (1) and (2) can be shown similar to the proof of [6, Theorem 4]. (3) and (4) are easy to verify. □

Recall that maximal elements in (big) lattices are called *coatoms*, and M-**pr** is said to be *coatomic* if for any $\underline{1} \neq \tau \in M$-**pr** there exists a maximal $\rho \in M$-**pr** such that $\tau \leq \rho$. For any ring R with identity the big lattice R-**pr** is coatomic whereas for arbitrary M the big lattice M-**pr** need not be so.

The following observation is obvious.

2.5 Lemma. *Let G be a generator in $\sigma[M]$ and $\tau \in M$-**pr**. Then $\tau = \underline{1}$ if and only if $\tau(G) = G$.*

The next theorem describes when M-**pr** is coatomic.

2.6 Theorem. *For M and a generator G in $\sigma[M]$, the following conditions are equivalent:*

*(a) The big lattice M-**pr** is coatomic;*

(b) every fully invariant proper submodule of G is contained in a maximal fully invariant submodule.

Proof. (a) $\Rightarrow$ (b) Let $K \subset G$ be a fully invariant proper submodule. Then there exists a preradical τ such that $\tau(G) = K$ and hence $\tau \neq \underline{1}$. By (a) there exists a coatom $\rho \in M$-**pr** such that $\tau \leq \rho$, thus $K \subseteq \rho(G)$. We claim that

$\rho(G)$ is maximal fully invariant submodule of G. Since $\rho \neq \underline{1}$ whe know from the preceding Lemma that $\rho(G) \neq G$. Now suppose $\rho(G) \subseteq L \subseteq G$ where L is a fully invariant submodule of G. Assume $L \neq G$; then $\omega_L^G \neq \underline{1}$ and $\rho \leq \omega_L^G$. Since ρ is a coatom this implies $\rho = \omega_L^G$ and so $\rho(G) = L$.

(b) $\Rightarrow$ (a) Let $\tau \in M$-**pr** different from $\underline{1}$, that is, $\tau(G) \neq G$, and choose a maximal fully invariant submodule $L \subset G$ containing $\tau(G)$. Then ω_L^G is a coatom in M-**pr** such that $\rho \leq \omega_L^G$. $\square$

The following example shows that M-**pr** need not be atomic.

2.7 Example. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{p^\infty}$, for some prime p.. Then $G = \bigoplus_{\mathbb{N}} \mathbb{Z}_{p^n}$ is a generator in $\sigma[M]$ without any maximal fully invariant submodules. Hence, by 2.6, M-**pr** is not coatomic.

Notice that $\tau = \alpha_M^M \in M$-**pr** with $\tau(M) = M$ and $\tau(\mathbb{Z}_p) = 0$, thus $\tau \neq \underline{1}$.

Now we characterize some classes of modules by the lattice structure of M-**pr**.

2.8 Theorem. *For M the following conditions are equivalent:*

(a) M is a homogeneous semisimple module;

(b) $\alpha_K^N = \omega_K^N$ for all $0 \neq N \in \sigma[M]$ and fully invariant submodules $K \subseteq N$;

(c) $\alpha_N^N = \omega_N^N$ for all $0 \neq N \in \sigma[M]$.

Proof. (a) $\Rightarrow$ (b) Let M be homogeneous semisimple. Then any nonzero $N \in \sigma[M]$ is homogeneous semisimple and 0 and N are its only fully invariant submodules. For $K = 0$, $\omega_0^N = \underline{0} = \alpha_0^N$, since N is a cogenerator in $\sigma[M]$. For $K = N$, $\alpha_N^N = \underline{1} = \omega_N^N$, since N is a generator in $\sigma[M]$.

(b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (a) Condition (c) implies for any nonzero $N \in \sigma[M]$, $\alpha_N^N = \omega_N^N = \underline{1}$, that is, N is a generator. Thus $\sigma[M]$ has a simple generator and M is homogeneous semisimple. $\square$

2.9 Theorem. *Assume that $\sigma[M]$ has a non-zero M-projective module P. Then the following conditions are equivalent:*

(a) M is a homogeneous semisimple module;

(b) $\alpha_0^N = \omega_0^N$ for all $0 \neq N \in \sigma[M]$;

(c) each $0 \neq N \in \sigma[M]$ is a cogenerator of $\sigma[M]$.

Proof. (a) $\Rightarrow$ (b) is immediate by 2.8.

(b) $\Rightarrow$ (c) This is clear since $\alpha_0^N = \underline{0}$.

(c) $\Rightarrow$ (a) By the given condition it is clear that $\sigma[M]$ has a unique simple module S which has to be M-injective. Since P cogenerates S, there is an inclusion $S \to P$ which splits. Thus S is an M-projective module generating all simple modules in $\sigma[M]$, that is, S is a generator in $\sigma[M]$ (e.g. [12, 18.5]). $\square$

3 Coprime preradicals

3.1 Definition. A nonzero $\tau \in M$-**pr** is called *coprime* if $\tau \leq (\mu : \rho)$, where $\mu, \rho \in M$-**pr**, implies $\tau \leq \mu$ or $\tau \leq \rho$.

The existence of coprimes is guaranteed by the following fact.

3.2 Proposition. *The atoms in M-***pr** *are coprime preradicals.*

Proof. Any atom τ of M-**pr** is of the form $\tau = \alpha_S^{\widehat{S}}$, for a simple module $S \in \sigma[M]$. Now suppose $\tau \leq (\mu : \rho)$ for some $\mu, \rho \in M$-**pr** and $\tau \not\leq \mu$. Then $\mu(\widehat{S}) = 0$ and

$$\rho(\widehat{S}) = \rho(\widehat{S}/\mu(\widehat{S})) = (\mu : \rho)(\widehat{S})/\mu(\widehat{S}) = (\mu : \rho)(\widehat{S}) \leq \tau(\widehat{S}) = S,$$

hence $\tau \leq \rho$. $\square$

As shown in the next theorem to any coprime preradical an idempotent coprime preradical can be associated.

3.3 Theorem. *For any coprime $\tau \in M$-***pr***, its equalizer $e(\tau)$ is an (idempotent) coprime preradical.*

Proof. Suppose $e(\tau) \leq (\mu : \rho)$ for some $\mu, \rho \in M$-**pr**. Then

$$\tau = e(\tau) \cdot \tau \leq (\mu : \rho) \cdot \tau \leq (\mu \cdot \tau : \rho \cdot \tau),$$

and hence $\tau \leq \mu \cdot \tau$ or $\tau \leq \rho \cdot \tau$, implying $\tau = \mu \cdot \tau$ or $\tau = \rho \cdot \tau$, and by definition of the equalizer, $e(\tau) \leq \rho$ or $e(\tau) \leq \rho$. $\square$

3.4 Lemma. *Let K be a fully invariant submodule of $N \in \sigma[M]$.*

(1) $e(\alpha_K^N) = \alpha_K^K$.

(2) If α_K^N is a coprime preradical, then α_K^K is coprime.

Proof. (1) First observe that $\alpha_K^K \alpha_K^N(N) = \alpha_K^K(K) = K$ and therefore $\alpha_K^K \alpha_K^N = \alpha_K^K$.

On the other hand, $e(\alpha_K^N)(K) = e(\alpha_K^N)\alpha_K^N(N) = \alpha_K^N(N) = K$, thus $e(\alpha_K^N) \geq \alpha_K^K$ and so $\alpha_K^K = e(\alpha_K^N)$.

(2) is a consequence of (1). $\square$

Note that $\alpha_S^S = e(\alpha_S^{\widehat{S}})$ and hence the trace of any simple module $S \in \sigma[M]$ is a coprime preradical.

The following observation provides a sufficient condition for a coprime preradical to be a maximal coprime preradical.

3.5 Theorem. *Let $\tau \in M$-**pr** be coprime. If τ is not small in M-**pr**, then τ is a maximal coprime preradical.*

Proof. Let $\eta \in M$-**pr** be coprime with $\tau \leq \eta$ and let $\underline{1} \neq \rho \in M$-**pr** such that $\tau \vee \rho = \underline{1}$. Hence we have $\eta \leq (\tau : \rho)$ and thus $\eta \leq \tau$ or $\eta \leq \rho$. The latter case implies $\tau \leq \eta \leq \rho$, a contradiction. Hence $\eta \leq \tau$ and so $\eta = \tau$. $\square$

3.6 Definition. For $\tau, \rho \in M$-**pr** define the *totalizer of ρ relative to τ* by

$$t_\tau(\rho) = \bigwedge \{\eta \in M\text{-}\mathbf{pr} \mid (\rho : \eta) \geq \tau\}.$$

3.7. Properties of the relative totalizer.

(1) $\tau = \underline{1}$, *then* $t_\tau(\rho) = t(\rho)$.

(2) $\tau \leq \tau'$, *then* $t_\tau(\rho) \leq t_{\tau'}(\rho)$.

(3) $\rho \leq \rho'$, *then* $t_\tau(\rho') \leq t_\tau(\rho)$.

(4) $\tau \geq t_\tau(\rho)$ *and* $(\rho : t_\tau(\rho)) \geq \tau$.

(5) $\rho \geq \tau$ *if and only if* $t_\tau(\rho) = \underline{0}$.

(6) $t_\tau(\underline{0}) = \tau$.

Notice that $t_\tau(\quad)$ may be thought of as an assignment $t_\tau : M$-**pr** $\to M$-**pr**.

3.8 Theorem. *For $\tau \in M$-**pr** the following are equivalent:*

(a) τ *is a coprime preradical;*

(b) *for each* $\eta \in M$-**pr**, $\tau \leq \eta$ *or* $\tau = t_\tau(\eta)$;

(c) $\operatorname{Im} t_\tau = \{\underline{0}, \tau\}$.

Proof. (a) $\Rightarrow$ (b) Let $\eta \in M$-**pr** be such that $\tau \not\leq \eta$ and suppose $\tau \leq (\eta : \rho)$. Then $\tau \leq \rho$, hence $\tau \leq t_\tau(\eta)$ and thus $\tau = t_\tau(\eta)$.

(b) $\Rightarrow$ (c) Let $\eta \in M$-**pr**. If $\tau \leq \eta$ we get $t_\tau(\eta) = \underline{0}$. Now suppose $\tau \not\leq \eta$; then $\tau = t_\tau(\eta)$ and hence $\operatorname{Im} t_\tau = \{\underline{0}, \tau\}$.

(c) $\Rightarrow$ (a) Assume now that $\tau \leq (\mu : \rho)$, hence $t_\tau(\mu) \leq \rho$. Now $\tau = t_\tau(\mu)$ implies $\tau \leq \rho$. On the other hand, $\tau \neq t_\tau(\mu)$ implies $t_\tau(\mu) = \underline{0}$ and hence $\tau \leq \mu$ showing that τ is coprime. $\square$

3.9 Corollary. *For M the following conditions are equivalent:*

(a) $\underline{1}$ is a coprime preradical;

*(b) for each $\underline{1} \neq \tau \in M$-**pr**, $t(\tau) = \underline{1}$;*

(c) $\operatorname{Im} t = \{\underline{0}, \underline{1}\}$.

3.10 Theorem. *Assume that M-**pr** is coatomic. Then the following are equivalent:*

(a) $\underline{1}$ is a coprime preradical;

*(b) M-**pr** has a unique coatom which is a radical;*

(c) each generator $G \in \sigma[M]$ is simple as an $(R, \operatorname{End}(G))$-bimodule.

Proof. (a) $\Rightarrow$ (b) Let ρ, ρ' be two different coatoms in M-**pr**. Then $(\rho : \rho') = \underline{1}$ implying $\rho = \underline{1}$ or $\rho' = \underline{1}$. This is a contradiction and hence there is only one coatom ρ.

If $(\rho : \rho) \neq \rho$, then $(\rho : \rho) = \underline{1}$ contradicting our assumption. Thus $(\rho : \rho) = \rho$, that is, ρ is a radical.

(b) $\Rightarrow$ (c) Let G be a generator in $\sigma[M]$ and ρ the unique coatom in M-**pr**. Thus if $\rho(G) = N$, we have $\rho = \omega_N^G$ and N is the unique maximal fully invariant submodule of G; since ω_N^G is a radical we have $\omega_N^G(G/N) = 0$.

Let $N_1 = \alpha_{G/N}^{G/N}(G)$; if $N_1 \subseteq N$, then for each $f \in \operatorname{Hom}(G/N, G)$, $\operatorname{Im} f \subset N_1 \subseteq N$ and so $G/N \subseteq f^{-1}(N)$. Thus $G/N = \omega_N^G(G/N) = 0$, a contradiction. Therefore $N_1 = G$, that is, $\alpha_{G/N}^{G/N}(G) = G$ and $G \in \sigma[G/N]$.

Since $\alpha_N^G \leq \omega_N^G$, we have $\alpha_N^G(G/N) = 0$, so $\alpha_N^G(K) = 0$ for each $K \in \sigma[G/N]$. In particular $\alpha_N^G(G) = 0$, thus $N = 0$ and so G is simple as an $(R, \operatorname{End}(G))$-bimodule.

(c) $\Rightarrow$ (a) Let G be a generator in $\sigma[M]$ which is simple as an $(R, \operatorname{End}(G))$-bimodule. Then ω_0^G is the the unique coatom of M-**pr** and is a radical.

Consider $\tau, \tau' \in M$-**pr** with $\tau \neq \underline{1} \neq \tau'$. Then $\tau \leq \omega_0^G$ and $\tau' \leq \omega_0^G$ and so $(\tau : \tau') \leq (\omega_0^G : \omega_0^G) = \omega_0^G < \underline{1}$. This implies that $\underline{1}$ is a coprime preradical. $\square$

The hypothesis in 3.10 that M-**pr** is coatomic is necessary as Example 2.7 shows.

4 Coprime submodules and modules

The general properties of preradicals may be expressed by properties of certain submodules and modules. For this we introduce an

4.1. Internal coproduct. For $N \in \sigma[M]$ and fully invariant submodules $L, L' \subseteq N$, define an *internal coproduct* as the fully invariant submodule of N,

$$(L' :_N L) = (\omega^N_{L'} : \omega^N_L)(N),$$

which has the following properties:

(i) $(L' :_N L) = \bigcap\{f^{-1}(L) \mid f \in \mathrm{End}(N),\ L' \subset \mathrm{Ke}\ f\}$.

(ii) $L + L' \subseteq (L :_N L')$.

(iii) If $H \subseteq N$ is a fully invariant submodule with $L, L' \subseteq H$, then

$$(L :_H L') \subseteq (L :_N L').$$

(iv) For $\eta, \rho \in M$-**pr** and any $N \in \sigma[M]$,

$$(\eta : \rho)(N) \subseteq (\eta(N) :_N \rho(N)).$$

4.2 Definition. Let $N \in \sigma[M]$ and $K \subseteq N$ a fully invariant submodule. We say that K *is coprime in* N if for any fully invariant submodules $L, L' \subseteq N$, $K \subseteq (L :_N L')$ implies $K \subseteq L$ or $K \subseteq L'$.

N is called a coprime module if N is coprime in N.

4.3. Remark. Notice that the definition of the coproduct $(L' :_N L)$ in 4.1 only applies to fully invariant submdodules $L', L \subseteq N$ (since it refers to ω's). However, its characterization in 4.1(i) can be used to define a coproduct $L \Box_N L'$ for any submodules $L', L \subseteq N$. This was considered in [3] and applied to define "coprime" modules which differ from those defined in 4.2 (see Remarks 4.7).

The following observation gives us a relation between coprime preradicals and coprime submodules.

4.4 Theorem. *Let $N \in \sigma[M]$ and $0 \neq K \subseteq N$ a fully invariant submodule. Then the following properties are equivalent:*

(a) K is coprime in N;

(b) α_K^N is a coprime preradical.

Proof. (a) $\Rightarrow$ (b) Since $0 \neq K$ also $\alpha_K^N \neq \underline{0}$. Let $\mu, \rho \in M$-**pr** be such that $\alpha_K^N \leq (\mu : \rho)$; then

$$K = \alpha_K^N(N) \subseteq (\mu : \rho)(N) \subseteq (\mu(N) :_N \rho(N)),$$

and K being coprime in N we conclude $K \subseteq \mu(N)$ or $K \subseteq \rho(N)$ and hence $\alpha_K^N \leq \mu$ or $\alpha_K^N \leq \rho$.

(b) $\Rightarrow$ (a) Let $L, L' \subseteq N$ be fully invariant submodules such that $K \subseteq (L :_N L')$ and therefore $K \subseteq (\omega_L^N : \omega_{L'}^N)(N)$, hence $\alpha_K^N \leq (\omega_L^N : \omega_{L'}^N)$ and our condition implies $\alpha_K^N \leq \omega_L^N$ or $\alpha_K^N \leq \omega_{L'}^N$. This means $K \subseteq L$ or $K \subseteq L'$ proving that K is coprime in N. $\square$

4.5 Theorem. *For an R-module N, the following are equivalent:*

(a) N is a coprime module;

(b) α_N^N is a coprime preradical;

(c) for each proper fully invariant submodule $K \subset N$, $\alpha_N^N = \alpha_{N/K}^{N/K}$;

(d) for each proper fully invariant submodule $K \subset N$, N/K generates N;

*(e) for any $\tau, \eta \in M$-**pr**, $N \in \mathbb{T}_{(\tau:\eta)}$ implies $N \in \mathbb{T}_\tau$ or $N \in \mathbb{T}_\eta$, where $\mathbb{T}_\eta = \{X \in \sigma[M] \mid \eta(X) = X\}$.*

Proof. (a) $\Leftrightarrow$ (b) This is an immediate consequence of Theorem 4.4.

(b) $\Rightarrow$ (c) For $K \subset N$ a proper fully invariant submodule,

$$(\alpha_K^N : \alpha_{N/K}^{N/K})(N)/\alpha_K^N(N) = \alpha_{N/K}^{N/K}(N/K) = N/K, \text{ and } (\alpha_K^N : \alpha_{N/K}^{N/K})(N) = N,$$

so $\alpha_N^N \leq (\alpha_K^N : \alpha_{N/K}^{N/K})$ and hence $\alpha_N^N \leq \alpha_{N/K}^{N/K}$ or $\alpha_N^N \leq \alpha_K^N$. Since $K \subset N$ this implies $\alpha_N^N = \alpha_{N/K}^{N/K}$.

(c) $\Rightarrow$ (d) is obvious.

(d) $\Rightarrow$ (a) Let $K, L \subseteq N$ be fully invariant submodules such that $N = (K :_N L) = (\omega_K^N : \omega_L^N)(N)$, and therefore $(\omega_K^N : \omega_L^N)(N)/\omega_K^N(N) = \omega_L^N(N/K) = N/K$. Hence $\alpha_{N/K}^{N/K} \leq \omega_L^N$.

If $N/K = 0$ we obtain $N = K$, while $N/K \neq 0$ implies $\alpha_{N/K}^{N/K}(N) = N$, hence $\omega_L^N(N) = N$ and so $N = L$.

(a) $\Rightarrow$ (e) Let $N \in \mathbb{T}_{(\tau:\eta)}$, that is, $N = (\tau : \eta)(N)$ and hence $(\omega_{\tau(N)}^N : \omega_{\eta(N)}^N)(N) = N$ which means $(\tau(N) :_N \eta(N)) = N$. By hypothesis, $N = \tau(N)$ or $N = \eta(N)$, in other words $N \in \mathbb{T}_\tau$ or $N \in \mathbb{T}_\eta$.

(e) $\Rightarrow$ (a) Let $N \in \sigma[M]$ satisfy condition (e). Consider fully invariant submodules $K, L \subseteq N$ such that $N = (K :_N L)$. So $N = (\omega_K^N : \omega_L^N)(N)$, hence $N \in \mathbb{T}_{(\omega_K^N : \omega_L^N)}$ and hence (by (e)) $N \in \mathbb{T}_{\omega_K^N}$ or $N \in \mathbb{T}_{\omega_L^N}$ which means $N = K$ or $N = L$. $\square$

4.6 Corollary. *(1) Let M be a module with no non-trivial fully invariant submodules. Then M is a coprime module.*

(2) Let M be projective in $\sigma[M]$. Then M is a coprime module if and only if it has no non-trivial fully invariant submodules.

(3) The ring R is coprime if and only if R is a simple ring.

Proof. (1) This is obvious.

(2) One direction follows by (1).

Assume M to be coprime and $K \subseteq M$ a fully invariant submodule. Then M/K generates M which - by [11, Lemma 2.8] - is not possible unless $K = 0$.

(3) This is a special case of (2). $\square$

Notice that $\mathbb{Z}_{p^\infty}$ is a coprime module which has many fully invariant submodules.

4.7 Remarks. (1) In [10, Theorem 3.1], *duprime modules M* are defined as modules, for which the identity is coprime in the lattice of hereditary preradicals on $\sigma[M]$. For modules M which are projective in $\sigma[M]$, this is equivalent to M being strongly prime, that is, each nonzero submodule of M subgenerates M (see [10, Theorem 3.3]). It follows from Corollary 4.6 that this condition is different from M being coprime in M-**pr**. For example, $\mathbb{Z}$ is a duprime module but $\mathbb{Z}$ is not coprime in our sense (see 4.2).

(2) Let M be self-injective. Then, as observed in 2.1, ω_K^M is a hereditary preradical for any fully invariant submodule $K \subseteq M$. Hence $\underline{1}$ coprime in M-**pr** is equivalent to $\underline{1}$ being coprime in M-**hpr**, that is, M is coprime if and only if it is duprime.

(2) The coprimeness derived from the comultiplication of submodules in [3] also differs from coprimeness defined in 4.2. To illustrate this consider the rationals $\mathbb{Q}$ as $\mathbb{Z}$-module: $\mathbb{Q}$ is duprime (see [10]) and coprime (as in 4.2) but is not coprime in the sense of [3].

(3) In general, M coprime as in [3] $\Rightarrow$ M coprime (from 4.2) $\Rightarrow$ M duprime. Hence, since $\mathbb{Z}_{p^\infty}$ is coprime in the sense of [3] it is also coprime and duprime.

We consider one more example to show the difference between the various notions of coprimeness.

4.8 Example. Consider a (nonassociative) ring A with unit as module over its multiplication algebra $M(A)$, which is the subring of $\mathrm{End}_{\mathbb{Z}}(A)$ generated by left and right multiplication with elements from A (see [13, p. 6]). Then the subcategory $\sigma[_{M(A)}A]$ of $M(A)$-**Mod** reflects ring properties of A. In particular, a prime ring A is duprime if and only if its central closure is a simple ring (see [10, Example 4.14]). However, A is coprime (as in 4.2) if and only if A is a simple ring: To see this, recall that (by 4.5) A coprime implies that for any proper ideal $I \subset A$, A is generated by A/I as an $M(A)$-module. Since the image of any $M(A)$-morphism $A/I \to A$ is annihilated by I, this can only happen if $I = 0$. Thus A is a simple ring.

Notice that all $M(A)$-submodules of A are in fact fully invariant submodules (since $\mathrm{End}_{M(A)}(A)$ is just the center of A). Hence our coprimeness condition coincides with that from [3].

4.9 Proposition. *For a module N, let $K \subseteq H \subseteq N$ be submodules such that K is fully invariant in H and H fully invariant in N.*

(1) If K is coprime in N then K is coprime in H.

(2) If K is coprime in N then K is a coprime module.

(3) If α_K^N is a coprime preradical then so is α_K^H.

Proof. (1) Let $L, L' \subseteq H$ be fully invariant submodules such that $K \subseteq (L :_N L')$, that is, $K \subseteq (\omega_L^H : \omega_{L'}^H)(H)$, hence

$$K + L/L \subseteq (\omega_L^H : \omega_{L'}^H)(H)/L = (\omega_L^H : \omega_{L'}^H)(H)/\omega_L^H(H) = \omega_{L'}^H(H/\omega_L^H).$$

On the other hand, $(L :_N L')/L = (\omega_L^N : \omega_{L'}^N)(N)/L = \omega_{L'}^N(N/L)$. Noting that for any morphism $f : N/L \to N$, $f(H/L) \subseteq H$, we obtain

$$K + L/L \subseteq (f|_{H/L})^{-1}(L') \subseteq f^{-1}(L').$$

Hence $K \subseteq (L :_H L')$ and by hypothesis $K \subseteq L$ or $K \subseteq L'$, showing that K is a coprime submodule of H.

(2) and (3) are consequences of (1). □

Observe that while item (2) above can be obtained from Lemma 3.4(2) and Theorem 4.4, item (3) provides a generalization of Lemma 3.4(2).

Now we come to a partial converse of Proposition 4.9.

4.10 Theorem. *Let $N \subseteq Q$ be a fully invariant submodule of a self-injective module Q. Then N is a coprime module if and only if N is coprime in Q.*

Proof. One implication is shown in Proposition 4.9.

Let $K, L \subseteq Q$ be fully invariant submodules with $N \subseteq (K :_Q L)$, that is,

$$N/(K \cap N) \simeq (N + K)/K \subseteq \omega^Q_L(Q/K) = (\omega^Q_K : \omega^Q_L)(Q)/K = (K :_Q L)/K.$$

Since $K \cap N$ and $L \cap N$ are fully invariant submodules of N,

$$\begin{aligned}(K \cap N :_N L \cap N)/(K \cap N) &= (\omega^N_{K\cap N} : \omega^N_{L\cap N})(N)/(K \cap N) \\ &= \omega^N_{L\cap N}(N/(K \cap N)) \subseteq N/(K \cap N).\end{aligned}$$

Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & N/(K \cap N) \arrow[r] \arrow[d, dotted, "f"'] & Q/K \arrow[d, dotted, "g"] \\ 0 \arrow[r] & N \arrow[r] & Q\,. \end{tikzcd}$$

Given any f, there is a g making the diagram commutative (by self-injectivity of Q). On the other hand, any g yields an f by restriction. Hence

$$\begin{aligned}N/(K \cap N) &= \omega^Q_{L\cap N}(Q/K) \cap (N/(K \cap N)) \\ &= [\bigcap\{g^{-1}(L) \,|\, g : Q/K \to Q\}] \cap (N/(K \cap N)) \\ &= \bigcap\{f^{-1}(L \cap N) \,|\, f : N/(K \cap N) \to N\} \subset N/(K \cap N),\end{aligned}$$

and therefore $\omega^N_{L\cap N}(N/(K \cap N)) = N/(K \cap N)$, that is, $N = (K \cap N :_N L \cap N)$. Since the module N is coprime this implies $N = K \cap N$ or $N = L \cap N$ which means $N \subseteq K$ or $N \subseteq L$, showing that N is a coprime submodule in Q $\square$

4.11. Remark. Let G be a generator in $\sigma[M]$. Then for any coatom $\tau \in M$-**pr**, $\tau(G)$ is a maximal fuly invariant submodule of G and $\tau = \omega^G_{\tau(G)}$. Therefore there is a bijection between the coatoms of M-**pr** and the maximal fully invariant submodules of G. Hence the class of all coatoms in M-**pr** is a set.

4.12 Theorem. *Assume that the set of all coatoms in M-**pr** is not empty and let $\tau \in M$-**pr** be coprime.*

*(1) Either there exists a unique coatom $\rho \in M$-**pr** such that $\tau \not\leq c(\rho)$, or, for each coatom $\rho \in M$-**pr**, $\tau \leq c(\rho)$.*

(2) *Assume there exists a self-projective generator in $\sigma[M]$. Then $\tau \not\leq c(\rho)$ for a coatom $\rho \in M$-**pr** if and only if $\tau = \alpha_{G/N}^{G/N}$, where $N \subset G$ is a maximal fully invariant submodule.*

Proof. (1) For any coatoms $\rho_1 \neq \rho_2$, $(c(\rho_1) : c(\rho_2)) = \underline{1}$ and hence $\tau \leq (c(\rho_1) : c(\rho_2))$ implying $\tau \leq c(\rho_1)$ or $\tau \leq c(\rho_2)$.

(2) Let $\tau \not\leq c(\rho)$ for a coatom $\rho \in M$-**pr** and G a generator in $\sigma[M]$. Then $N = \rho(G) \subset G$ is a maximal fully invariant submodule. Since $\rho = \omega_N^G$, $\tau \not\leq \omega_0^{G/N}$ and so $\tau(G/N) = G/N$ which means $\alpha_{G/N}^{G/N} \leq \tau$. Notice that

$$(\omega_0^{G/N} : \alpha_{G/N}^{G/N})(G)/\omega_0^{G/N}(G) = \alpha_{G/N}^{G/N}(G/N) = G/N.$$

So we get $(\omega_0^{G/N} : \alpha_{G/N}^{G/N}) = \underline{1}$. Hence $\tau \leq (\omega_0^{G/N} : \alpha_{G/N}^{G/N})$ and since $\tau \not\leq \omega_0^{G/N}$ this implies $\tau \leq \alpha_{G/N}^{G/N}$, that is, $\tau = \alpha_{G/N}^{G/N}$.

Conversely, assume that $\tau = \alpha_{G/N}^{G/N}$ where N is a maximal fully invariant submodule of a generator G in $\sigma[M]$. Then $\tau = \alpha_{G/N}^{G/N} \not\leq \omega_0^{G/N}$ where ω_N^G is the coatom in M-**pr**. $\square$

4.13 Corollary. *For each maximal fully invariant submodule N of a self-projective generator G in $\sigma[M]$, the preradical $\alpha_{G/N}^{G/N}$ is a maximal coprime preradical in M-**pr**.*

Proof. Let $\tau \in M$-**pr** be coprime with $\alpha_{G/N}^{G/N} \leq \tau$. Since G is self-projective we have that $\alpha_N^G(G/N) = 0$ and so $\tau \not\leq \alpha_N^G$. Since

$$(\alpha_N^G : \alpha_{G/N}^{G/N})(G)/N = \alpha_{G/N}^{G/N}(G/N) = G/N,$$

this implies $(\alpha_N^G : \alpha_{G/N}^{G/N}) = \underline{1}$, therefore $\tau \leq (\alpha_N^G : \alpha_{G/N}^{G/N})$ and so $\tau \leq \alpha_{G/N}^{G/N}$ which means $\tau = \alpha_{G/N}^{G/N}$. $\square$

4.14 Theorem. *Let $N \in \sigma[M]$ be such that for any fully invariant submodules $K, L \subseteq N$, $(\omega_K^N : \omega_L^N) = \omega_{K:_N L}^N$. Then for each coprime preradical $\tau \in M$-**pr** with $\tau(N) \neq 0$, $\tau(N)$ is a coprime submodule in N.*

Proof. Let $K, L \subseteq N$ be fully invariant submodules with $\tau(N) \leq (K :_N L)$. Then $\tau \leq \omega_{(K:_N L)}^N = (\omega_K^N : \omega_L^N)$ and thus $\tau \leq \omega_K^N$ or $\tau \leq \omega_L^N$. Therefore $\tau(N) \subseteq K$ or $\tau(N) \subseteq L$, showing that $\tau(N)$ is a coprime submodule in N. $\square$

4.15 Lemma. *Let $\mathcal{C}$ be a subclass of coprime preradicals of M-**pr** which is linearly ordered. Then $\bigvee_{\tau \in \mathcal{C}} \tau$ is a coprime preradical.*

Proof. Let $\rho = \bigvee_{\tau\in\mathcal{C}}$ and suppose that $\rho \leq (\mu : \eta)$ for $\mu, \eta \in M$-**pr**. Assume there exists $\tau \in \mathcal{C}$ such that for each $\nu \in \mathcal{C}$ with $\tau \leq \nu$ we have $\nu \leq \mu$, then $\rho \leq \mu$.

On the other hand, assume that for each $\tau \in \mathcal{C}$ there exists $\nu \in \mathcal{C}$ with $\tau \leq \nu$ and $\nu \not\leq \mu$. In this case $\nu \leq \eta$ and so, for every $\nu \in \mathcal{C}$ with $\tau \leq \nu$ we get $\nu \leq \eta$ and therefore $\rho \leq \eta$. Thus ρ is a coprime element in M-**pr**. $\square$

4.16. Remark. Referring to Zorn's Lemma for classes, Lemma 4.15 implies that for any coprime preradical $\tau \in M$-**pr** there exists a maximal coprime $\rho \in M$-**pr** such that $\tau \leq \rho$.

4.17 Theorem. *For M the following conditions are equivalent:*

*(a) Each element in M-**pr** is coprime;*

*(b) M-**pr** is linearly ordered and each element of M-**pr** is a radical;*

*(c) M-**pr** is linearly ordered and for each $N \in \sigma[M]$ the fully invariant submodules are coprime in N;*

*(d) for each $\tau \in M$-**pr**, the subclass $\{\alpha^N_{\tau N} \mid N \in \sigma[M]\}$ of M-**pr** is linearly ordered, and, for each $N \in \sigma[M]$, the nonzero fully invariant submodules are coprime in N.*

Proof. (a) $\Rightarrow$ (b) Consider $\tau \neq \rho$ in M-**pr**. Then $\tau \vee \rho \leq (\tau : \rho)$ and hence $\tau \vee \rho \leq \tau$ or $\tau \vee \rho \leq \rho$, thus $\rho \leq \tau$ or $\tau \leq \rho$. Thus M-**pr** is linearly ordered. Now, for each $\tau \in M$-**pr**, $(\tau : \tau) \leq (\tau : \tau)$, hence $(\tau : \tau) \leq \tau$ which means that τ is a radical.

(b) $\Rightarrow$ (a) Take $\tau, \rho, \eta \in M$-**pr** and assume $\tau \leq (\rho : \eta)$. Without loss of generality suppose $\eta \leq \rho$. Then $\tau \not\leq \eta$ implies $\eta < \tau$ and so $(\rho : \eta) \leq (\eta : \eta) = \eta < \tau$, a contradiction.

(a) $\Rightarrow$ (c) Let K be a nonzero fully invariant submodule of $N \in \sigma[M]$. Then α^N_K is a coprime preradical and, by 4.4, K is coprime in N.

(c) $\Rightarrow$ (d) is obvious.

(d) $\Rightarrow$ (a) Since $\tau = \bigvee\{\alpha^N_{\tau N} \mid N \in \sigma[M]\}$ it follows by Lemma 4.15 that τ is coprime. $\square$

4.18 Theorem. *Let M be such that each element in M-**pr** is coprime. Then:*

(1) For each $N \in \sigma[M]$ the lattice of fully invariant submodules of N is linearly ordered.

(2) The category $\sigma[M]$ has a unique simple module (up to isomorphism).

(3) *Each nonzero $N \in \sigma[M]$ has maximal submodules.*

(4) *For any nonzero $N, N' \in \sigma[M]$, $\mathrm{Hom}(N, N') \neq 0$ or $\mathrm{Hom}(N', N) \neq 0$.*

(5) *If M-**pr** is coatomic, then any generator G of $\sigma[M]$ is simple as $(R, \mathrm{End}(G))$-bimodule.*

Proof. (1) is clear.

(2) Let S, S' be simple modules in $\sigma[M]$. Then $\alpha_S^S \leq \alpha_{S'}^{S'}$ implies $\alpha_S^S = \alpha_{S'}^{S'}$ and hence $S \simeq S'$.

(3) Since M-**pr** is linearly ordered and for any simple module $S \in \sigma[M]$, $\omega_0^S(S) = 0$ and $\alpha_S^S(S) = S$, we must have $\omega_0^S < \alpha_S^S$. So $\mathbb{T}_{\omega_0^S}$ is a proper pretorsion subclass of $\mathbb{T}_{\alpha_S^S}$ and hence $\mathbb{T}_{\omega_0^S} = 0$ which means that, for each $N \in \sigma[M]$, $\omega_0^S(N) \neq N$ and so N has a maximal submodule.

(4) Assume there exist two modules $N, N' \in \sigma[M]$ such that $\mathrm{Hom}(N, N') = 0$ and $\mathrm{Hom}(N', N) = 0$. Then $0 \oplus N'$ and $N \oplus 0$ are fully invariant submodules of $N \oplus N'$ such that $(0 \oplus N' :_{N \oplus N'} N \oplus 0) = N \oplus N'$. This means that $N \oplus N'$ is not a coprime module, a contradiction.

(5) Assume M-**pr** to be coatomic and let G be a generator in $\sigma[M]$. Then, by Theorem 2.6, G has maximal fully invariant submodules. Now it follows by (1) that there is a unique maximal fully invariant submodule in G. Applying Theorem 3.10, we get G is simple as $(R, \mathrm{End}(G))$-bimodule. $\square$

4.19. Remark. If M-**pr** is linearly ordered then in particular the lattice M-**hpr** of left exact preradicals is linearly ordered, a condition which was investigated in [11, Theorem 2.5].

5 $\vee$-coprime preradicals and modules

The definition of coprime preradicals was referring to the coproduct $(\tau : \rho)$ of two preradicals $\tau, \rho \in M$-**pr**. Similar definitions make sense when this is replaced by the sum $\tau \vee \rho$ of preradicals.

5.1 Definition. A preradical $\tau \in M$-**pr** is called

$\vee$-coprime	if for any $\mu, \rho \in M$-**pr**, $\tau \leq \mu \vee \rho$ implies $\tau \leq \mu$ or $\tau \leq \rho$,
coirreducible	if $\tau = \mu \vee \rho$ implies $\tau = \mu$ or $\tau = \rho$.

We collect basic properties of these notions.

5.2 Theorem. *Let $\tau \in M$-**pr**.*

(1) τ coprime $\Rightarrow \tau$ $\vee$-coprime $\Rightarrow \tau$ coirreducible.

(2) τ idempotent and coirreducible $\Rightarrow \tau$ $\vee$-coprime.

*(3) If M-**pr** is distributive, then τ coirreducible $\Rightarrow \tau$ $\vee$-coprime.*

Proof. (1) Let τ be coprime and assume $\tau \leq \mu \vee \rho$. Then $\tau \leq (\mu : \rho)$ and hence $\tau \leq \mu$ or $\tau \leq \rho$, that is, τ is $\vee$-coprime.

Let τ be $\vee$-coprime and assume $\tau = \mu \vee \rho$. Then $\tau \leq \mu$ or $\tau \leq \rho$ which means $\tau = \mu$ or $\tau = \rho$ proving that τ is coirreducible.

(2) Let τ be idempotent and coirreducible and suppose $\tau \leq \mu \vee \rho$. Then

$$\tau = \tau^2 \leq (\mu \vee \rho) \cdot \tau = \mu \cdot \tau \vee \rho \cdot \tau \leq \tau,$$

hence $\mu \cdot \tau \vee \rho \cdot \tau = \tau$ and $\mu \cdot \tau = \tau$ or $\rho \cdot \tau = \tau$. This implies $\tau \leq \mu$ or $\tau \leq \rho$ and thus τ is $\vee$-coprime.

(3) Assume $\tau \leq \mu \vee \rho$. Then $\tau = (\mu \vee \rho) \wedge \tau = (\mu \wedge \tau) \vee (\rho \wedge \tau)$. Therefore $\tau = \mu \wedge \tau$ or $\tau = \rho \wedge \tau$, thus $\tau \leq \mu$ or $\tau \leq \rho$ showing that τ is $\vee$-coprime. $\square$

5.3 Theorem. *Let $\tau \in M$-**pr** be $\vee$-coprime but not small in M-**pr**. Then τ is a maximal $\vee$-coprime element in M-**pr**.*

Proof. Let $\rho \in M$-**pr** be $\vee$-coprime such that $\tau \leq \rho$ and chose $\underline{1} \neq \eta \in M$-**pr** with $\tau \vee \eta = \underline{1}$, so $\rho \leq \tau \vee \eta$, implying $\rho \leq \tau$ or $\rho \leq \eta$. The latter implies $\tau \leq \rho \leq \eta$ and $\underline{1} = \tau \vee \eta = \eta$, a contradiction. Thus $\tau = \rho$ showing the maximality of τ. $\square$

5.4 Theorem. *Let $\tau \in M$-**pr**.*

(1) τ coirreducible implies $e(\tau)$ coirreducible.

(2) τ $\vee$-coprime implies $e(\tau)$ $\vee$-coprime.

Proof. (1) Suppose $e(\tau) = \eta \vee \rho$ for $\eta, \rho \in M$-**pr**. Then

$$\tau = e(\tau) \cdot \tau = (\eta \vee \rho) \cdot \tau = \eta \cdot \tau \vee \rho \cdot \tau,$$

and therefore $\tau = \eta \cdot \tau$ or $\tau = \rho \cdot \tau$ which implies $e(\tau) = \eta$ or $e(\tau) = \rho$. Thus $e(\tau)$ is coirreducible.

(2) The proof is similar to the proof of (1). $\square$

5.5 Definition. Let K, L, L' be fully invariant submodules of an R-module N. We say that K *is $\vee$-coprime in* N if $K \subset L + L'$ implies $K \subset L$ or $K \subset L'$, and K *is bi-hollow in* N if $K = L + L'$ implies $K = L$ or $K = L'$.

Furthermore, N is called *bi-hollow* if it is bi-hollow as a submodule of itself.

Clearly, if N has no non-trivial fully invariant submodules contained in K, then K is trivially bi-hollow in N.

5.6. Remark. Let $f : P \to N$ be an epimorphism with P self-projective and $\mathrm{Ke}\, f$ small in P. If N is bi-hollow then P is bi-hollow.

5.7 Theorem. *Let $K \subseteq N$ be a fully invariant submodule.*

(1) The following conditions are equivalent:

(a) K is $\vee$-coprime in N;

(b) α_K^N is a $\vee$-coprime preradical.

(2) The following are equivalent:

(a) K is bi-hollow in N;

(b) α_K^N is a coirreducible preradical.

Proof. (1) (a) $\Rightarrow$ (b) Assume that $\alpha_K^N \leq \eta \vee \rho$ for $\eta, \rho \in M$-**pr**. Then $K = \alpha_K^N(N) \subseteq \eta(N) + \rho(N)$, hence $N \leq \eta(N)$ or $N \leq \rho(N)$, therefore $\alpha_K^N \leq \rho$ or $\alpha_K^N \leq \eta$, proving that α_K^N is $\vee$-coprime.

(b) $\Rightarrow$ (a) Suppose that $K \subseteq L + L'$ with fully invariant submodules $L, L' \subseteq N$. Then $\alpha_K^N \leq \alpha_L^N \vee \alpha_{L'}^N$, hence $\alpha_K^N \leq \alpha_L^N$ or $\alpha_K^N \leq \alpha_{L'}^N$, that is $K \subseteq L$ or $K \leq L'$. Thus N is $\vee$-coprime in N.

(2) The proof is similar to the proof of (1). $\square$

5.8 Corollary. *For an R-module N the following are equivalent:*

(a) N is bi-hollow;

(b) α_N^N is a coirreducible preradical.

5.9 Remarks. The notions of $\vee$-coprime and bi-hollow coincide if $K = N$, and, by Theorem 5.2(2), α_N^N is coirreducible if and only if it is $\vee$-coprime.

5.10 Corollary. *Let $f\colon P \to N$ be an epimorphism of modules with $Ke(f)$ small in P and P self-projective. If α_N^N is coirreducible then so is α_P^P.*

Proof. This follows from Theorem 5.7 and Remark 5.6. $\square$

5.11 Proposition. *Let $K \subset H \subset N$ be submodules with K fully invariant in H and H fully invariant N. Then:*

(1) K bi-hollow ($\vee$-coprime) in N implies K bi-hollow ($\vee$-coprime) in H.

(2) K bi-hollow in N implies K bi-hollow.

(3) α^N_K coirreducible ($\vee$-coprime) implies α^H_K coirreducible ($\vee$-coprime).

(4) α^N_K coirreducible implies α^K_K coirreducible.

Proof. (1) Assume $K \subset N$ to be $\vee$-coprime in N. Let $K \subseteq L + L'$ with L, L' fully invariant in H and hence in N. So we have $K \subseteq L$ or $K \subseteq L'$ and thus K is $\vee$-coprime in H. Similar arguments apply to bi-hollow submodules.

(2) This is an immediate consequence of (1).

(3) This is a consequence of (1) and Theorem 5.7.

(4) This is clear by (2) and Corollary 5.8. □

5.12 Theorem. *Let N be a fully invariant submodule of a self-injective module Q. Then N is bi-hollow if and only if N is bi-hollow in Q.*

Proof. The if part is immediate by Proposition 5.11(2).

Now suppose that N is bi-hollow and let $L, K \subset Q$ be fully invariant submodules such that $N = K + L$. Since Q is self-injective, K, L are also fully invariant in N and $N = K$ or $N = L$. Thus N is bi-hollow in Q. □

5.13 Theorem. *Assume that M-**pr** is coatomic and let G be a generator in $\sigma[M]$. The following conditions are equivalent:*

(a) G has a unique maximal fully invariant submodule N;

*(b) $\underline{1} \in M$-**pr** is coirreducible.*

Proof. (a) ⇒ (b) Assume $\underline{1} = \mu \vee \rho$ for $\mu, \rho \in M$-**pr**, hence $G = \mu(G) + \rho(G)$. Now, if $\mu \neq \underline{1}$, then $\mu(G) \neq G$ and so $\mu(G) \subseteq N$. Since N is the unique maximal fully invariant submodule of G, $\rho(G) \subseteq N$ is not possible and hence $\rho(G) = G$ and $\rho = \underline{1}$.

(b) ⇒ (a) Suppose that $\underline{1}$ is coirreducible and let $\rho \neq \rho'$ be coatoms in M-**pr**. Then $\rho \vee \rho' = \underline{1}$, a contradiction. Therefore there is a unique coatom ρ in M-**pr** and $N = \rho(G)$ is the unique maximal fully invariant submodule of G. □

Coirreducible preradicals need not be coprime; for this consider any ring R with a unique nonzero maximal (two-sided) ideal I. Then $\underline{1}$ is coirreducible but not coprime.

5.14 Theorem. *Let M be such that M-**pr** is coatomic and $\sigma[M]$ has a self-projective generator G. Then for a $\vee$-coprime $\tau \in M$-**pr**, either* *(i) there exists a unique maximal fully invariant submodule $N \subset G$ such that $\tau \not\leq \omega_0^{G/N}$, or*

(ii) for each maximal fully invariant submodule $N \subset G$, $\tau \leq \omega_0^{G/N}$.

Proof. For distinct maximal fully invariant submodule $N, N' \subset G$, $\omega_0^{G/N} \vee \omega^{G/N'} = \underline{1} \geq \tau$, and therefore $\tau \leq \omega_0^{G/N}$ or $\tau \leq \omega_0^{G/N'}$. Hence $\tau \not\leq \omega_0^{G/N}$ implies $\tau \leq \omega_0^{G/N'}$ for all fully invariant submodule $N' \subset G$ distinct from N. □

Notice that with the hypothesis of the preceding theorem, for any maximal fully invariant submodule $N \subset G$, $\alpha_{G/N}^{G/N} \not\leq \omega_0^{G/N}$, hence $\alpha_{G/N}^{G/N}$ is a $\vee$-coprime preradical satisfying condition (i) in Theorem 5.14. On the other hand we observe:

5.15 Theorem. *Let G be a generator in $\sigma[M]$ with a maximal fully invariant submodule $N \subset G$. Assume there exists a projective cover $p : P \to G/N$ in $\sigma[M]$. Then α_P^P is a maximal $\vee$-coprime element of M-**pr**.*

Proof. By 5.10 (and 5.9) α_P^P is $\vee$-coprime and $\alpha_{G/N}^{G/N} \leq \alpha_P^P$. Suppose $\alpha_P^P \leq \tau$ for some $\vee$-coprime $\tau \in M$-**pr**. First observe that $\tau \not\leq \omega_0^{G/N}$. By self-projectivity of P, there exists a nonzero $g : P \to G$, yielding a commutative diagram

$$\begin{tikzcd} & P \arrow[dl, "g"'] \arrow[d, "p"] & \\ G \arrow[r] & G/N \arrow[r] & 0. \end{tikzcd}$$

Hence $\alpha_P^P(G) \not\subseteq N$, and so $(\alpha_P^P \vee \omega_0^{G/N})(G) = \alpha_P^P(G) + N = G$, that is, $\alpha_P^P \vee \omega_0^{G/N} = \underline{1}$. Since τ is $\vee$-coprime this implies $\tau \leq \alpha_P^P$, thus $\tau = \alpha_P^P$. □

The following example shows that even though $\alpha_{G/N}^{G/N}$ is a maximal coprime preradical, it need not be maximal as $\vee$-coprime preradical.

5.16 Example. Let

$$R = \begin{pmatrix} \mathbb{Q} & 0 \\ \mathbb{R} & \mathbb{R} \end{pmatrix}, \quad P = \begin{pmatrix} \mathbb{Q} & 0 \\ \mathbb{R} & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 0 & 0 \\ \mathbb{R} & \mathbb{R} \end{pmatrix},$$

and $S = R/I$. Then the natural morphism $P \to S$ is a projective cover and clearly $\alpha_S^S < \alpha_P^P$. Hence α_P^P is $\vee$-coprime but cannot be coprime.

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