Decompositions of Modules and Comodules

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Abstract

It is well-known that any semiperfect A ring has a decomposition as a direct sum (product) of indecomposable subrings $A = A_1 \oplus \cdots \oplus A_n$ such that the A_i-Mod are indecomposable module categories. Similarly any coalgebra C over a field can be written as a direct sum of indecomposable subcoalgebras $C = \bigoplus_I C_i$ such that the categories of C_i-comodules are indecomposable. In this paper a decomposition theorem for closed subcategories of a module category is proved which implies both results mentioned above as special cases. Moreover it extends the decomposition of coalgebras over fields to coalgebras over noetherian (QF) rings.

1 Introduction

The close connection between module categories and comodule categories was investigated in [12] and it turned out that there are parts of module theory over algebras which provide a perfect setting for the theory of comodules. In a similar spirit the present paper is devoted to decomposition theorems for closed subcategories of a module category which subsume decomposition properties of algebras as well as of coalgebras.

Let A be an associative algebra over a commutative ring R. For an A-module M we denote by $\sigma[M]$ the category of those A-modules which are submodules of M-generated modules. This is the smallest

Grothendieck subcategory of A-Mod containing M. The inner properties of $\sigma[M]$ are dependent on the module properties of M and there is a well established theory dealing with this relationship.

We define a σ-decomposition

$$\sigma[M] = \bigoplus_{\Lambda} \sigma[N_\lambda],$$

for a family $\{N_\lambda\}_\Lambda$ of modules, meaning that for every module $L \in \sigma[M]$, $L = \bigoplus_\Lambda L_\lambda$,where $L_\lambda \in \sigma[N_\lambda]$. We call $\sigma[M]$ *σ-indecomposable* if no such non-trivial decomposition exists.

The $\sigma[N_\lambda]$ are closely related to fully invariant submodules of a projective generator (if there exists one) and - under certain finiteness conditions - to the fully invariant submodules of an injective cogenerator. Consequently an indecomposable decomposition of $\sigma[M]$ can be obtained provided there is a semiperfect projective generator or an injective cogenerator of locally finite length in $\sigma[M]$.

Such decompositions of $\sigma[M]$ were investigated in Vanaja [10] and related constructions are considered in García-Jara-Merino [4], Năstăsescu-Torrecillas [8] and Green [5].

Let C be a coalgebra over a commutative ring R. Then the dual C^* is an R-algebra and C is a left and right module over C^*. The link to the module theory mentioned above is the basic observation that the category of right C-comodules is subgenerated by C. Moreover, if ${}_RC$ is projective, this category is the same as $\sigma[{}_{C^*}C]$. This is the key to apply module theory to comodules and our decomposition theorem for $\sigma[M]$ yields decompositions of coalgebras and their comodule categories over noetherian (QF) rings. For coalgebras over fields such results were obtained in Kaplansky [6], Montgomery [7], Shudo-Miyamoto [9].

2 Decompositions of module categories

Throughout R will denote an associative commutative ring with unit, A an associative R-algebra with unit, and A-Mod the category of unital left A-modules.

We write $\sigma[M]$ for the full subcategory of A-Mod whose objects are submodules of M-generated modules. $N \in \sigma[M]$ is called a *subgenerator* if $\sigma[M] = \sigma[N]$.

2.1 The trace functor. For any $N, M \in A$-Mod the *trace of M in N* is defined as

$$\mathrm{Tr}(M, N) := \sum \{\mathrm{Im}\, f \mid f \in \mathrm{Hom}_A(M, N)\},$$

and we denote the *trace of $\sigma[M]$ in N* by

$$\mathcal{T}^M(N) := \mathrm{Tr}(\sigma[M], N) = \sum \{\mathrm{Im}\, f \mid f \in \mathrm{Hom}_A(K, N),\ K \in \sigma[M]\}.$$

If N is M-injective, or if M is a generator in $\sigma[M]$, then $\mathcal{T}^M(N) = \mathrm{Tr}(M, N)$.

A full subcategory $\mathcal{C}$ of A-Mod is called *closed* if it is closed under direct sums, factor modules and submodules (hence it is a Grothendieck category). It is straightforward to see that any closed subcategory is of type $\sigma[N]$, for some N in A-Mod.

The next result shows the correspondence between the closed subcategories of $\sigma[M]$ and fully invariant submodules of an injective cogenerator of $\sigma[M]$, provided M has locally finite length.

2.2 Correspondence relations. *Let M be an A-module which is locally of finite length and Q an injective cogenerator in $\sigma[M]$.*

(1) For every $N \in \sigma[M]$, $\sigma[N] = \sigma[\mathrm{Tr}(N, Q)]$.

(2) The map $\sigma[N] \mapsto \mathrm{Tr}(N, Q)$ yields a bijective correspondence between the closed subcategories of $\sigma[M]$ and the fully invariant submodules of Q.

(3) $\sigma[N]$ is closed under essential extensions (injective hulls) in $\sigma[M]$ if and only if $\mathrm{Tr}(N, Q)$ is an A-direct summand of Q.

(4) $N \in \sigma[M]$ is semisimple if and only if $\mathrm{Tr}(N, Q) \subset \mathrm{Soc}({}_AQ)$.

Proof. Notice that by our finiteness condition every cogenerator in $\sigma[M]$ is a subgenerator in $\sigma[M]$. Moreover by the injectivity of Q, $\mathrm{Tr}(\sigma[N], Q) = \mathrm{Tr}(N, Q)$.

(1) $\mathrm{Tr}(N, Q)$ is a fully invariant submodule which by definition belongs to $\sigma[N]$. Consider the N-injective hull $\widehat{N}$ of N (in $\sigma[N]$). This is a direct sum of N-injective hulls $\widehat{E}$ of simple modules $E \in \sigma[N]$. Since Q is a cogenerator we have (up to isomorphism) $\widehat{E} \subset Q$ and so $\widehat{E} \subset \mathrm{Tr}(N, Q)$. This implies $\widehat{N} \in \sigma[\mathrm{Tr}(N, Q)]$ and so $\sigma[N] = \sigma[\mathrm{Tr}(N, Q)]$.

(2) and (4) are immediate consequences of (1).

(3) If $\sigma[N]$ is closed under essential extensions in $\sigma[M]$ then clearly $\mathrm{Tr}(N, Q)$ is an A-direct sumand in Q (and hence is injective in $\sigma[M]$).

Now assume $\mathrm{Tr}(N, Q)$ to be an A-direct sumand in Q and let L be any injective object in $\sigma[N]$. Then L is a direct sum of N-injective hulls $\widehat{E}$ of simple modules $E \in \sigma[N]$. Clearly the $\widehat{E}$'s are (isomorphic to) direct summands of $\mathrm{Tr}(N, Q)$ and hence of Q, i.e., they are M-injective and so L is M-injective, too. $\square$

2.3 Sum and decomposition of closed subcategories. For any $K, L \in \sigma[M]$ we write $\sigma[K] \cap \sigma[L] = 0$, provided $\sigma[K]$ and $\sigma[L]$ have no non-zero object in common. Given a family $\{N_\lambda\}_\Lambda$ of modules in $\sigma[M]$, we define

$$\sum\nolimits_\Lambda \sigma[N_\lambda] := \sigma[\bigoplus\nolimits_\Lambda N_\lambda].$$

This is the smallest closed subcategory of $\sigma[M]$ containing all the N_λ's. Moreover we write

$$\sigma[M] = \bigoplus_\Lambda \sigma[N_\lambda],$$

provided for every module $L \in \sigma[M]$, $L = \bigoplus_\Lambda \mathcal{T}^{N_\lambda}(L)$ (internal direct sum). We call this a *σ-decomposition of* $\sigma[M]$, and we say $\sigma[M]$ is *σ-indecomposable* if no such non-trivial decomposition exists.

In view of the fact that every closed subcategory of A-Mod is of type $\sigma[N]$, for some A-module N, the above definition describes the decomposition of any closed subcategory into closed subcategories.

2.4 σ-decomposition of modules. *For a decomposition $M = \bigoplus_\Lambda M_\lambda$, the following are equivalent:*

(a) for any distinct $\lambda, \mu \in \Lambda$, M_λ and M_μ have no non-zero isomorphic subfactors;

(b) for any distinct $\lambda, \mu \in \Lambda$, $\mathrm{Hom}_A(K_\lambda, K_\mu) = 0$, where K_λ, K_μ are subfactors of M_λ, M_μ, respectively;

(c) for any distinct $\lambda, \mu \in \Lambda$, $\sigma[M_\lambda] \cap \sigma[M_\mu] = 0$;

(d) for any $\mu \in \Lambda$, $\sigma[M_\mu] \cap \sigma[\bigoplus_{\lambda \neq \mu} M_\lambda] = 0$;

(e) for any $L \in \sigma[M]$, $L = \bigoplus_\Lambda \mathcal{T}^{M_\lambda}(L)$.

*If these conditions hold we call $M = \bigoplus_\Lambda M_\lambda$ a σ-*decomposition *of M and in this case*

$$\sigma[M] = \bigoplus\nolimits_\Lambda \sigma[M_\lambda].$$

Proof. $(a) \Rightarrow (b)$ and $(e) \Rightarrow (a)$ are obvious.

$(b) \Rightarrow (c)$ This follows from the plain fact that for any A-module N, each non-zero module in $\sigma[N]$ contains a non-zero subfactor of N.

$(c) \Rightarrow (d)$ Any non-zero module in $\sigma[\bigoplus_{\lambda\neq\mu} M_\lambda]$ contains a non-zero subfactor of M_ν, for some $\nu \in \Lambda \setminus \{\mu\}$. This implies (d).

$(d) \Rightarrow (e)$ It is easy (see [10]) to verify that $L = \bigoplus_\Lambda \mathcal{T}^{M_\lambda}(L)$. □

2.5 Corollary. *Let $\sigma[M] = \bigoplus_\Lambda \sigma[N_\lambda]$ be a σ-decomposition of $\sigma[M]$. Then*

(1) each $\sigma[N_\lambda]$ is closed under essential extensions in $\sigma[M]$;

(2) any $L \in \sigma[N_\lambda]$ is M-injective if and only if it is N_λ-injective;

(3) $M = \bigoplus_\Lambda \mathcal{T}^{N_\lambda}(M)$ is a σ-decomposition of M.

Proof. (1) For any $L \in \sigma[N_\lambda]$, consider an essential extension $L \trianglelefteq K$ in $\sigma[M]$. Then $\mathcal{T}^{N_\lambda}(K) \trianglelefteq K$ and is a direct summand. So $K = \mathcal{T}^{N_\lambda}(K) \in \sigma[N_\lambda]$.

(2) Any $L \in \sigma[N_\lambda]$ is M-injective if it has no non-trivial essential extensions in $\sigma[M]$. Assume L to be N_λ-injective. Then L has no non-trivial essential extensions in $\sigma[N_\lambda]$ and by (1), it has no non-trivial essential extensions in $\sigma[M]$.

(3) Put $M_\lambda := \mathcal{T}^{N_\lambda}(M)$. By definition, $M_\lambda \in \sigma[N_\lambda]$ and it remains to show that $N_\lambda \in \sigma[M_\lambda]$. Let $\widehat{N_\lambda}$ denote the M-injective hull of N_λ. $\widehat{N_\lambda}$ is M-generated, and by (1), $\widehat{N_\lambda} \in \sigma[N_\lambda]$. This implies that $\widehat{N_\lambda}$ is M_λ-generated and so $N_\lambda \in \sigma[M_\lambda]$. □

It is obvious that any σ-decomposition of M is also a fully invariant decomposition. The reverse implication holds in special cases:

2.6 Corollary. *Assume M to be a projective generator or an injective cogenerator in $\sigma[M]$. Then any fully invariant decomposition of M is a σ-decomposition.*

Proof. Let $M = \bigoplus_\Lambda M_\lambda$ be a fully invariant decomposition, i.e., $\text{Hom}_A(M_\lambda, M_\mu) = 0$, for $\lambda \neq \mu$.

Assume M to be a projective generator in $\sigma[M]$. Then clearly every submodule of M_λ is generated by M_λ. Since the M_λ's are projective in $\sigma[M]$, any non-zero (iso)morphism between (sub)factors of M_λ and M_μ yields a non-zero morphism between M_λ and M_μ. So our assertion follows from 2.4.

Now suppose that M is an injective cogenerator in $\sigma[M]$. Then every subfactor of M_λ must be cogenerated by M_λ. From this it follows that for $\lambda \neq \mu$, there are no non-zero maps between subfactors of M_λ and M_μ and so 2.4 applies. □

Remark. 2.4 and 2.6 are shown in Vanaja [10, Proposition 2.2] and related constructions are considered in García-Jara-Merino [4, Section 3] and [3, Theorem 5.2], Năstăsescu-Torrecillas [8, Lemma 5.4] and Green [5, 1.6c].

Following García-Jara-Merino [3], we call a module M *σ-indecomposable* if M has no non-trivial σ-decomposition.

2.7 Corollary. *The following are equivalent:*

(a) $\sigma[M]$ is σ-indecomposable;

(b) M is σ-indecomposable;

(c) any subgenerator in $\sigma[M]$ is σ-indecomposable;

(d) an injective cogenerator which is a subgenerator in $\sigma[M]$, has no fully invariant decomposition.

If there exists a projective generator in $\sigma[M]$ then (a)-(d) are equivalent to:

(e) projective generators in $\sigma[M]$ have no fully invariant decompositions.

It is straightforward to see that a σ-decomposition of the ring A is of the form

$$A = Ae_1 \oplus \cdots \oplus Ae_k, \text{ for central idempotents } e_i \in A,$$

and so A-Mod is σ-indecomposable if and only if A has no non-trivial central idempotent.

By the structure theorem for cogenerators with commutative endomorphism rings (see [11, 48.16]) we have:

2.8 σ-decomposition when $\mathrm{End}_A(M)$ commutative. *Let M be a cogenerator in $\sigma[M]$ with $\mathrm{End}_A(M)$ commutative. Then $M \simeq \bigoplus_\Lambda \widehat{E}_\lambda$, where $\{E_\lambda\}_\Lambda$ is a minimal representing set of simple modules in $\sigma[M]$. This is a σ-decomposition of M and*

$$\sigma[M] = \bigoplus\nolimits_\Lambda \sigma[\widehat{E}_\lambda],$$

where each $\sigma[\widehat{E}_\lambda]$ is indecomposable and contains only one simple module.

A special case of the situation described above is the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z} = \bigoplus_{p\,prime} \mathbb{Z}_{p^\infty}$ and the decomposition of the category of torsion abelian groups as a direct sum of the categories of p-groups,

$$\sigma[\mathbb{Q}/\mathbb{Z}] = \bigoplus\nolimits_{p\,prime} \sigma[\mathbb{Z}_{p^\infty}].$$

Notice that although $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator in $\mathbb{Z}$-Mod with a non-trivial σ-decomposition, $\mathbb{Z}$-Mod is σ-indecomposable. This is possible since $\mathbb{Q}/\mathbb{Z}$ is not a subgenerator in $\mathbb{Z}$-Mod.

In general it is not so easy to get σ-decompositions of modules. We need some technical observations to deal with modules whose endomorphism rings are not commutative.

2.9 Relations on families of modules. Consider any family of A-modules $\{M_\lambda\}_\Lambda$ in $\sigma[M]$. Define a relation $\sim$ on $\{M_\lambda\}_\Lambda$ by putting

$$M_\lambda \sim M_\mu \quad \text{if there exist non-zero morphisms } M_\lambda \to M_\mu \text{ or } M_\mu \to M_\lambda.$$

Clearly $\sim$ is symmetric and reflexive and we denote by $\approx$ the smallest equivalence relation on $\{M_\lambda\}_\Lambda$ determined by $\sim$, i.e.,

$$M_\lambda \approx M_\mu \quad \text{if there exist } \lambda_1, \ldots, \lambda_k \in \Lambda, \text{ such that } M_\lambda = M_{\lambda_1} \sim \cdots \sim M_{\lambda_k} = M_\mu\,.$$

Then $\{M_\lambda\}_\Lambda$ is the disjoint union of the equivalence classes $\{[M_\omega]\}_\Omega$, $\Lambda_\omega \subset \Lambda$.

Assume each $M_\lambda \simeq \widehat{E}_\lambda$, the M-injective hull of some simple module $E_\lambda \in \sigma[M]$. Then

$$\widehat{E}_\lambda \sim \widehat{E}_\mu \textit{ if and only if } \mathrm{Ext}_M(E_\lambda, E_\mu) \neq 0 \textit{ or } \mathrm{Ext}_M(E_\mu, E_\lambda) \neq 0,$$

where Ext_M *denotes the extensions in* $\sigma[M]$*.*

Proof. For any non-zero morphism $\widehat{E}_\lambda \to \widehat{E}_\mu$, there exists a submodule $E_\lambda \subset L \subset \widehat{E}_\lambda$ with a non-splitting exact sequence

$$0 \to E_\lambda \to L \to E_\mu \to 0.$$

Assume such a sequence is given. From this it is easy to construct a non-zero morphism $f : \widehat{E}_\lambda \to \widehat{E}_\mu$. □

A decomposition $M = \bigoplus_\Lambda M_\lambda$ is said to *complement direct summands* if, for every direct summand K of M, there exists a subset $\Lambda' \subset \Lambda$ such that $M = K \oplus (\bigoplus_{\Lambda'} M_\lambda)$ (cf. [1, § 12]). We observe that such decompositions yield fully invariant indecomposable decompositions.

2.10 Lemma. *Let* $M = \bigoplus_\Lambda M_\lambda$ *be a decomposition which complements direct summands, where all* M_λ *are indecomposable. Then* M *has a decomposition* $M = \bigoplus_A N_\alpha$*, where each* $N_\alpha \subset M$ *is a fully invariant submodule and does not decompose non-trivially into fully invariant submodules.*

Proof. Consider the equivalence relation $\approx$ on $\{M_\lambda\}_\Lambda$ (see 2.9) with the equivalence classes $\{[M_\omega]\}_\Omega$ and $\Lambda = \dot{\bigcup}_\Omega \Lambda_\omega$. Then $N_\omega := \bigoplus_{\Lambda_\omega} M_\lambda$ is a fully invariant submodule of M, for each $\omega \in \Omega$, and

$$M = \bigoplus\nolimits_\Omega \left(\bigoplus\nolimits_{\Lambda_\omega} M_\lambda\right) = \bigoplus\nolimits_\Omega N_\omega.$$

Assume $N_\omega = K \oplus L$ for fully invariant $K, L \subset N_\omega$. Since the defining decomposition of N_ω complements direct summands we may assume that Λ_ω is the disjoint union of subsets X, Y such that

$$N_\omega = \left(\bigoplus\nolimits_X M_\lambda\right) \oplus \left(\bigoplus\nolimits_Y M_\lambda\right).$$

By construction, for any $x \in X$, $y \in Y$, we have $M_x \approx M_y$ and it is easy to see that this implies the existence of non-zero morphisms $K \to L$ or $L \to K$, contradicting our assumption. So N_ω does not decompose into fully invariant submodules. □

The following are standard examples of module decompositions which complement direct summands.

2.11 Proposition. *Let $M = \bigoplus_\Lambda M_\lambda$, where each $\mathrm{End}_A(M_\lambda)$ is local.*

(1) If M is M-injective the decomposition complements direct summands.

(2) If M is projective in $\sigma[M]$ and $\mathrm{Rad}(M) \ll M$, then the decomposition complements direct summands.

Proof. For the first assertion we refer to [2, 8.13].

The second condition characterizes M as semiperfect in $\sigma[M]$ (see [11, 42.5]) and the assertion follows from [2, 8.12]. $\square$

2.12 σ-decomposition for locally noetherian modules. *Let M be a locally noetherian A-module. Then M has a σ-decomposition $M = \bigoplus_\Lambda M_\lambda$ and*

$$\sigma[M] = \bigoplus\nolimits_\Lambda \sigma[M_\lambda],$$

where each $\sigma[M_\lambda]$ is σ-indecomposable.

(1) $\sigma[M]$ is σ-indecomposable if and only if for any indecomposable injectives $K, L \in \sigma[M]$, $K \approx L$ (as defined in 2.9).

(2) If M has locally finite length, then $\sigma[M]$ is σ-indecomposable if and only if for any simple modules $S_1, S_2 \in \sigma[M]$, $\widehat{S}_1 \approx \widehat{S}_2$ (M-injective hulls).

Proof. Let Q be an injective cogenerator which is also a subgenerator in $\sigma[M]$. Then Q is a direct sum of indecomposable M-injective modules and this is a decomposition which complements direct summands (by 2.11). By Lemma 2.10, Q has a fully invariant decomposition $Q = \bigoplus_\Lambda Q_\lambda$ such that Q_λ has no non-trivial fully invariant decomposition. Now the assertions follow from Corollaries 2.6, 2.7, and 2.5.

(1) This is clear by the above proof.

(2) By our assumption every indecomposable M-injective module is an M-injective hull of some simple module in $\sigma[M]$. $\square$

2.13 σ-decomposition for semiperfect generators. *If M is a projective generator which is semiperfect in $\sigma[M]$, then M has a σ-decomposition $M = \bigoplus_\Lambda M_\lambda$, where each M_λ is σ-indecomposable.*

In particular, every semiperfect ring A has a σ-decomposition $A = Ae_1 \oplus \cdots \oplus Ae_k$, where the e_i are central idempotents of A which are not a sum of non-zero orthogonal central idempotents.

Proof. By [11, 42.5] and 2.11, M has a decomposition which complements direct summands. By Lemma 2.10, M has a fully invariant decomposition and the assertions follow from the Corollaries 2.6, 2.7, and 2.5. □

3 Coalgebras and comodules

We recall some basic definitions for coalgebras and comodules.

An R-module C is an *R-coalgebra* if there is an R-linear map (*comultiplication*)

$$\Delta : C \to C \otimes_R C, \text{ with } (id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta.$$

An R-linear map $\varepsilon : C \to R$ is a *counit* if $(id \otimes \varepsilon) \circ \Delta$ and $(\varepsilon \otimes id) \circ \Delta$ yield the canonical isomorphism $C \simeq C \otimes_R R$.

Henceforth C will denote a coalgebra with counit (C, Δ, ε) and we assume that C is flat as an R-module.

The R-dual of C, $C^* = \mathrm{Hom}_R(C, R)$, is an associative R-algebra with unit ε where the multiplication of $f, g \in C^*$ is defined by

$$(f * g)(c) = (f \otimes g)(\Delta(c)), \text{ for } c \in C.$$

An R-submodule $D \subset C$ is a *left coideal* if $\Delta(D) \subset C \otimes_R D$, a *right coideal* if $\Delta(D) \subset D \otimes_R C$, and a *sub-coalgebra* if $\Delta(D) \subset D \otimes_R D$ and D is pure in C.

An R-module M is called a *right C-comodule* if there exists an R-linear map $\varrho : M \to M \otimes_R C$ such that $(id \otimes \Delta) \circ \varrho = (\varrho \otimes id) \circ \varrho$, and $(id \otimes \varepsilon) \circ \varrho$ yields the canonical isomorphism $M \simeq M \otimes_R R$. An R-submodule $N \subset M$ is called *C-sub-comodule* if $\varrho(N) \subset N \otimes_R C$.

Left C-comodules are defined similarly. Clearly C is a right and left C-comodule, and right (left) sub-comodules of C are right (left) coideals.

An R-linear map $f : M \to M'$ between right comodules is a *comodule morphism* provided $\varrho' \circ f = (f \otimes id) \circ \varrho$.

The right (left) C-comodules and the comodule morphisms form a category which we denote by Comod-C (C-Comod). These are Grothendieck categories (remember that we assume ${}_RC$ to be flat). The close connection between comodules and modules is based on the following facts which are proved in [12, Section 3,4].

3.1 C**-comodules and** C^***-modules.** *Assume* ${}_RC$ *to be projective and let* $\varrho : M \to M \otimes_R C$ *be any right* C*-comodule. Then* M *is a left* C^**-module by*

$$\psi : C^* \otimes_R M \to M, \quad f \otimes m \mapsto ((id\underline{\otimes} f) \circ \varrho)(m).$$

(1) *An* R*-submodule* $U \subset M$ *is a sub-comodule if and only if it is a* C^**-submodule.*

(2) *Any* R*-linear map between right comodules is a comodule morphism if and only if it is* C^**-linear.*

(3) *The category of right* C*-comodules can be identified with* $\sigma[{}_{C^*}C]$*, the full subcategory of* C^*-Mod*, subgenerated by* ${}_{C^*}C$*.*

(4) C *is a balanced* (C^*, C^*)*-bimodule and the subcoalgebras of* C *correspond to the* (C^*, C^*)*-sub-bimodules.*

The properties of the comodule C are strongly influenced by the properties of the ring R (see [12, 4.9]).

3.2 Coalgebras over special rings. *Let* ${}_RC$ *be projective.*

(1) *If* R *is noetherian, then* C *is a locally noetherian* C^**-module and direct sums of injectives are injective in* $\sigma[{}_{C^*}C]$*.*

(2) *If* R *is artinian, then every finitely generated module in* $\sigma[{}_{C^*}C]$ *has finite length.*

(3) *If* R *is injective, then* C *is injective in* $\sigma[{}_{C^*}C]$*.*

Applying our results on decompositions of closed subcategories we obtain

3.3 σ**-decomposition of coalgebras.** *Let* C *be a coalgebra over a noetherian ring* R *with* C_R *projective.*

(1) *There exist a* σ*-decomposition* $C = \bigoplus_\Lambda C_\lambda$*, and a family of orthogonal central idempotents* $\{e_\lambda\}_\Lambda$ *in* C^**, with* $C_\lambda = C \cdot e_\lambda$*, for each* $\lambda \in \Lambda$*.*

(2) $\sigma[{}_{C^*}C] = \bigoplus_\Lambda \sigma[{}_{C^*}C_\lambda]$*.*

(3) *Each* C_λ *is a sub-coalgebra of* C*,* $C^*_\lambda \simeq C^* * e_\lambda$*, and* $\sigma[{}_{C^*}C_\lambda] = \sigma[{}_{C^*_\lambda}C_\lambda]$*.*

(4) $\sigma[{}_{C^*}C]$ is indecomposable if and only if, for any two injective uniform $L, N \in \sigma[{}_{C^*}C]$, we have $L \approx N$.

(5) Assume R to be artinian. Then $\sigma[{}_{C^*}C]$ is indecomposable if and only if for any two simple $E_1, E_2 \in \sigma[{}_{C^*}C]$, we have $\widehat{E}_1 \approx \widehat{E}_2$.

Proof. (1),(2) By 3.2, C is a locally noetherian C^*-module. Now the decomposition of $\sigma_{C^*}[C]$ follows from 2.12. Clearly the resulting σ-decomposition of C is a fully invariant decomposition and hence it can be described by central idempotents in the endomorphism ring ($= C^*$, see 3.1).

(3) The fully invariant submodules $C_\lambda \subset C$ are in particular R-direct summands in C and hence are sub-coalgebras (by [12, 4.4]). It is straightforward to verify that $\mathrm{Hom}_R(C_\lambda, R) = C^*_\lambda \simeq C^* * e_\lambda$ is an algebra isomorphism.

(4) is a special case of 2.12(2).

(5) follows from 2.12(3). Notice that $\widehat{E}_1 \approx \widehat{E}_2$ can be described by extensions of simple modules (see 2.9). The assertion means that the Ext-quiver of the simple modules in $\sigma_{C^*}[C]$ is connected. □

3.4 Corollary. *Let R be a QF ring and C an R-coalgebra with C_R projective. Then:*

(1) C has fully invariant decompositions with σ-indecomposable summands.

(2) Each fully invariant decomposition is a σ-decomposition.

(3) C is σ-indecomposable if and only if C has no non-trivial fully invariant decomposition.

(4) If C is cocommutative then $C = \bigoplus_\Lambda \widehat{E}_\lambda$ is a fully invariant decomposition, where $\{E_\lambda\}_\Lambda$ is a minimal representing set of simple modules.

Proof. By [12, 6.1], C is an injective cogenerator in $\sigma_{C^*}[C]$ and so the assertions (1)-(3) follow from 2.6 and 3.3.

(4) Our assumption implies that C^* is a commutative algebra and so the assertion follows by 2.8. □

For coalgebras C over QF rings we have a bijective correspondence between closed subcategories of $\sigma[_{C^*}C]$ and (C^*, C^*)-submodules in C. However the latter need not be pure R-submodules of C and hence they may not be sub-coalgebras.

3.5 Correspondence relations. *Let R be a QF ring and C an R-coalgebra with C_R projective. Then*

(1) for every $N \in \sigma[_{C^*}C]$, $\sigma[N] = \sigma[\mathrm{Tr}(N, C)]$;

(2) the map $\sigma[N] \to \mathrm{Tr}(N, C)$ yields a bijective correspondence between the closed subcategories of $\sigma[_{C^*}C]$ and the (C^*, C^*)-submodules of C;

(3) $\sigma[N]$ is closed under essential extensions (injective hulls) in $\sigma[_{C^*}C]$ if and only if $\mathrm{Tr}(N, C)$ is a C^*-direct summand of $_{C^*}C$. In this case $\mathrm{Tr}(N, C)$ is a sub-coalgebra of C.

(4) $N \in \sigma[_{C^*}C]$ is semisimple if and only if $\mathrm{Tr}(N, C) \subset \mathrm{Soc}_{C^*}(Q)$;

(5) If R is semisimple, then all $\mathrm{Tr}(N, C)$ are sub-coalgebras of C.

Proof. Since R is a QF ring, $_{C^*}C$ has locally finite length and is an injective cogenerator of $\sigma[_{C^*}C]$. Hence (1)-(4) follow from 2.2.

(5) For R semisimple all (C^*, C^*)-submodules $\mathrm{Tr}(\sigma[N], C)$ are direct summands as R-modules in C and so they are sub-coalgebras by [12, 4.4]. □

Remarks. 3.3 and 3.4 extend decomposition results for coalgebras over fields to coalgebras over noetherian (QF) rings. It was shown in Kaplansky [6] that any coalgebra C over a field K is a direct sum of indecomposable coalgebras, and that for C cocommutative, these components are even irreducible. In Montgomery [7, Theorem 2.1], a direct proof was given to show that C is a direct sum of link-indecomposable components. It is easy to see that the link-indecomposable components are just the σ-indecomposable components of C (see remark in proof of 3.3(5)). As outlined in [7, Theorem 1.7] this relationship can also be described by using the "wedge". In this context another proof of the decomposition theorem is given in Shudo-Miyamoto [9, Theorem]. These techniques are also used in Zhang [13]. We refer to García-Jara-Merino [3, 4] for a detailed description of the corresponding constructions.

In Green [5], for every C-comodule M, the coefficient space $C(M)$ was defined as the smallest sub-coalgebra $C(M) \subset C$ such that M is a $C(M)$-comodule. The definition heavily relies on the existence of a K-basis for comodules. In the more general correspondence theorem 3.5, the $C(M)$ are replaced by $\mathrm{Tr}(M, C)$. For coalgebras over fields, $C(M)$ and $\mathrm{Tr}(M, C)$ coincide and 3.5 yields [5, 1.3d], [4, Proposition 7], and [7, Lemma 1.8]. Notice that in [5] closed subcategories in $\sigma[_{C^*}C]$ are called *pseudovarieties.*

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