M-density, M-adic completion and M-subgeneration

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Abstract

For left modules X, M over the unital ring R, the *M-adic topology* on X is defined by taking as a basis of open neighbourhoods of zero in X the kernels of all morphisms $X \to M^k$, $k \in \mathbb{N}$.

The aim of this paper is to study the relationship between the notions addressed in the title. We describe the M-adic completions of the modules X and $_RR$ and display some of their module theoretic properties.

Adopting ideas from Leptin [6], for a given filter basis $\mathcal{L}$ of submodules of M, we investigate the ring of endomorphisms f of M with $(L)f \subseteq L$ for all $L \in \mathcal{L}$. It is shown that this ring is complete in the point-wise convergence topology provided M is Hausdorff and complete in the topology determined by the filter basis $\mathcal{L}$. Taking for $\mathcal{L}$ the filter of all submodules of M we obtain information about alglat(M) (as considered in [2, 3, 4]).

The paper generalizes results from Fuller [2], Fuller-Nicholson-Watters [3], Hauger-Zimmermann [5], Leptin [6], Menini-Orsatti [8], Vámos [12], Wisbauer [13].

Introduction

Let M be a left R-module over the unital ring R, and denote by $\sigma[M]$ the full subcategory of R-Mod consisting of all M-subgenerated R-modules, a closed subcategory of R-Mod (see e.g. [13, Section 15]). Any closed subcategory of R-Mod is uniquely determined by a left linear topology on R and vice versa (cf. [11, p. 145] or [14]). In particular, $\sigma[M]$ defines the filter of left ideals of R,

$$F_M = \{\, I \leq_R R \,|\, R/I \in \sigma[M] \,\}\,,$$

which is the set of all open left ideals of R in the so called *M-adic topology* on R.

Similarly, an M-adic topology can be defined on any left R-module X, by taking as a basis of open neighbourhoods of zero in X the set of all submodules X' of X such that X/X' is finitely M-cogenerated. It is worth mentioning that for a two-sided ideal I of R and ${}_RX$ finitely generated (in particular, if $X = R$), the classical I-adic topology on X is precisely the M-adic topology on X, for $M = \bigoplus_{n\geq 1}(R/I^n)$.

In section 1 we collect some preliminaries. In section 2 we describe the M-adic completion of any R-module X (resp. ${}_RR$) and ask when this completion coincides with X^{**} (resp. R^{**}), the double dual with respect to the module M.

Applying these results we observe in section 3 that the M-adic completion $\widehat{R}$ is the 'largest' ring extension of R for which the categories $\sigma[{}_RM]$ and $\sigma[{}_{\widehat{R}}M]$ coincide.

In section 4 we study the R-module M with a given filter basis $\mathcal{L}$ of submodules of M. Adopting ideas from Leptin [6], we study the ring of $\mathbb{Z}$-endomorphisms f of M satisfying $(L)f \subseteq L$ for each $L \in \mathcal{L}$. We show that this ring is complete in the point-wise convergence topology (induced from M^M) if M is Hausdorff and complete in the topology determined by the filter basis $\mathcal{L}$.

Taking for $\mathcal{L}$ the filter of all submodules of M, the ring considered in section 4 yields alglat(M) as studied by Fuller, Nicholson and Watters (see [3, 4]). We extend their description of 'alglat' of finite direct sums to infinite direct sums. Applying results from the previous sections we provide more information about these notions. In particular, we observe that alglat$(M^{(\mathbb{N})})$ is isomorphic to the M-adic completion of R.

Our results generalize and subsume observations of Fuller [2], Fuller-Nicholson-Watters [3], Hauger-Zimmermann [5], Leptin [6], Menini-Orsatti [8], Vámos [12], Wisbauer [13].

1 Topological preliminaries

Throughout this paper R will denote an associative ring with nonzero identity, and R-Mod the category of all unital left R-modules. The notation ${}_RM$ (resp. M_R) will be used to emphasize that M is a left (resp. a right) R-module. Any unexplained terminology or notation can be found in [1] and [13].

Let ${}_RM$ be a fixed left R-module, and denote by $E = \mathrm{End}^{\ell}({}_{\mathbb{Z}}M)$ the ring of all endomorphisms of the underlying additive group of ${}_RM$ acting on M from the left, $S = \mathrm{End}({}_RM)$, and $B = \mathrm{Biend}({}_RM)$ the ring of biendomorphisms of ${}_RM$, i.e., the ring $\mathrm{End}(M_S)$. Module morphisms will be written as acting on the side opposite to scalar multiplication. All other maps will be written as acting on the left.

For any subsets $L, F \subseteq M$ we denote $(L : F) = \{ r \in R \mid rF \subseteq L \}$. In particular, for $I \subseteq R$ and $a \in R$, $(I : a) = \{ r \in R \mid ra \in I \}$.

1.1 Finite topology. If X and Y are two nonempty sets, then the *finite topology* of the set Y^X of all maps from X to Y, identified with the cartesian product Y^X, is the product topology on Y^X, where Y is endowed with the discrete topology. For an arbitrary $f \in Y^X$ a basis of open neighbourhoods of f consists of the sets

$$V_{\{x_1,\dots,x_n\}}(f) = \{ g \in Y^X \mid g(x_i) = f(x_i) \, , \ \forall i, \, 1 \le i \le n \} \, ,$$

where $\{x_1, \dots, x_n\}$ ranges over the finite subsets of X.

For any $Z \subseteq Y^X$, by the *finite topology* of Z we will understand the topology on Z induced by the finite topology on Y^X.

In particular, for $X = Y = M$ an R-module, we have the *finite topology* on the set M^M of all maps from M to M, and the finite topology of $E = \mathrm{End}^{\ell}({}_{\mathbb{Z}}M)$ is the induced topology on $E \subseteq M^M$.

1.2 Point-wise convergence topology. More generally, if Y is a nonempty topological space and X is any nonempty set, then the *point-wise convergence topology* of Y^X is the product topology on Y^X (Y with the given topology). When the topology on Y is discrete we obtain the finite topology on Y^X. By the *point-wise convergence topology* of any $Z \subseteq Y^X$ we will understand the topology on Z induced by the point-wise convergence topology on Y^X.

1.3 M-dense subrings. If A and C are two unital subrings of the ring $E = \mathrm{End}^{\ell}({}_{\mathbb{Z}}M)$, with $A \subseteq C$, we say that A is *M-dense* in C, and we write $C \subseteq \overline{A}$, if A is a dense subset of C endowed with the finite topology, where $\overline{A}$ means the closure of A in E. This means precisely that for every finite subset $\{x_1, \dots, x_n\}$ of M and for every $c \in C$ there exists an $a \in A$ such that

$$ax_i = cx_i \quad \text{for each } i \, , \ 1 \le i \le n \, .$$

1.4 M-(co-)generated modules. A left R-module X is said to be *M-generated* (resp. *M-cogenerated*) if there exists a set I and an epimorphism $M^{(I)} \longrightarrow X$ (resp. a monomorphism $X \longrightarrow M^I$); in case the set I is finite, then X is called *finitely M-generated* (resp. *finitely M-cogenerated*). The full subcategory of R-Mod consisting of all M-generated (resp. M-cogenerated) R-modules is denoted by $\mathrm{Gen}(M)$ (resp. $\mathrm{Cog}(M)$).

1.5 $\sigma[M]$ and M-adic topology on R. A left R-module X is called *M-subgenerated* if X is isomorphic to a submodule of an M-generated module, and the full subcategory of R-Mod consisting of all M-subgenerated R-modules is denoted by $\sigma[M]$. This is a Grothendieck category (see [13]) and it determines a filter of left ideals,

$$F_M = \{\, I \leq_R R \mid R/I \in \sigma[M]\,\}\,,$$

which is precisely the set of all open left ideals of R in the so called *M-adic topology* on R. A basis of open neighbourhoods of zero in this topology is

$$\mathcal{B}_M(R) = \{\, \mathrm{Ann}_R(F) \mid F \text{ a finite subset of } M\,\}\,.$$

It is easily verified that the inverse image under the canonical ring morphism

$$\lambda : R \longrightarrow E\,,\ \ \lambda(r)x = rx\,,\ \ r \in R\,,\ x \in M\,,$$

of the finite topology on E is the M-adic topology on R.

1.6 M-adic topology on X. More generally, for any ${}_RX$, the set

$$\mathcal{B}_M(X) = \{\, X' \mid X' \leq_R X \text{ and } X/X' \text{ is finitely } M\text{-cogenerated}\,\}$$

$$= \{\, \mathrm{Ker}(f) \mid f : X \to M^k\,,\ k \in \mathbb{N}\,\}$$

is a basis of open neighbourhoods of zero in the *M-adic topology* on X. This topology on X is Hausdorff separated if and only if $X \in \mathrm{Cog}(M)$.

In case ${}_RX$ is finitely generated and M-projective, the set of all open submodules of X in the M-adic topology is

$$\{\, \mathrm{Ker}(f) \mid f : X \to N\,,\ N \in \sigma[M]\,\}\,.$$

For ${}_RX = {}_RR$ we regain the descriptions of F_M and $\mathcal{B}_M(R)$.

1.7 Hausdorff completion. For any ${}_RX$ we shall denote by $\widehat{X}_M$ (or $\widehat{X}$ if no confusion occurs) the *Hausdorff completion*, or shortly, the *completion* of X in the M-adic topology,

$$\widehat{X} = \varprojlim_{X' \in \mathcal{B}_M(X)} X/X'\,.$$

The completion $\widehat{X}$ of X is a Hausdorff separated complete left linearly topologized R-module over the left linearly topologized ring R endowed with the M-adic topology, the canonical map $\eta_X : X \longrightarrow \widehat{X}$ is continuous, and $\mathrm{Im}(\eta_X)$ is dense in $\widehat{X}$.

Recall that if R is a topological ring and X is a topological left R-module, then X is said to be a *linearly topologized* R-module if there exists a basis of open neighbourhoods of zero in X consisting of submodules of X. In particular, a topological ring R is said to be *left linearly topologized* if ${}_RR$ is a linearly topologized R-module.

1.8 $\mathcal{L}$-topology. Let $\mathcal{L}$ be a *filter basis* (*inverse system*) of submodules of M, i.e., a nonempty set of submodules of ${}_RM$ such that for each $M_1, M_2 \in \mathcal{L}$ there exists an $M_0 \in \mathcal{L}$ with $M_0 \subseteq M_1 \cap M_2$.

Such an $\mathcal{L}$ defines a basis for the neighbourhoods of zero for a linear topology on ${}_RM$, called the $\mathcal{L}$-*topology* of M. The resulting topological space, in the sequel denoted by $(M, \mathcal{L})$, is Hausdorff if and only if $\bigcap_{L\in\mathcal{L}} L = 0$, which implies that M is cogenerated by the modules of the set $\{M/L \mid L \in \mathcal{L}\}$. The other implication is not true. Take for instance $R = \mathbb{Z}$, $M = \mathbb{Z}_{2^\infty}$, and $\mathcal{L} = \{T\}$, where T is the socle of M. Then $(M, \mathcal{L})$ is not Hausdorff, but M is M/T-cogenerated.

1.9 $(M, \mathcal{L})$-adic topology. Any filter basis $\mathcal{L}$ of M defines a topology on the ring R, called the $(M, \mathcal{L})$-*adic topology*, by taking as a basis of open neighbourhoods of zero the set of left ideals of R

$$\mathcal{B}_{M,\mathcal{L}}(R) = \{\, (L : F) \mid L \in \mathcal{L}\,,\ F \subseteq M \text{ finite} \,\}\,.$$

Since $((L : F) : r) = (L : rF)$ for any $L, F \subseteq M$ and $r \in R$, it follows that $\mathcal{B}_{M,\mathcal{L}}(R)$ is a basis of open neighbourhoods of zero for a left linear topology on the ring R.

Moreover, $(M, \mathcal{L})$ is a linearly topologized module over the ring R endowed with the $(M, \mathcal{L})$-adic topology. Note that this topology on R is the coarsest left linear topology which makes $(M, \mathcal{L})$ a linearly topologized left R-module.

More generally, for any left R-module X, the $(M, \mathcal{L})$-*adic topology* on X is defined by the set of submodules

$$\mathcal{C}_{M,\mathcal{L}}(X) := \{\mathrm{Ker}(f) \mid f : X \to \bigoplus_{i=1}^{n} M/L_i\,,\ L_1, \ldots, L_n \in \mathcal{L}\,,\ n \in \mathbb{N}\,\}$$

as a filter basis. This is a Hausdorff topology if and only if X is cogenerated by $\{M/L \mid L \in \mathcal{L}\}$. Note that for $\mathcal{L} = \{0\}$ we regain the M-adic topology on X.

In particular, for $X = {}_RR$ (or any finitely generated M-projective module), the $(M, \mathcal{L})$-adic topology coincides with the $\widetilde{M}$-adic topology, for $\widetilde{M} := \bigoplus_{L\in\mathcal{L}} (M/L)$. In this case the $(M, \mathcal{L})$-adic topology on X also coincides with the *weak topology of characters of* X (see [8]), i.e., the coarsest topology on X such that all elements of $\mathrm{Hom}_R(X, M)$ are continuous.

This observation does not apply to modules X which are not M-projective. Two different filter bases $\mathcal{L}_1$ and $\mathcal{L}_2$ of submodules of M generating the same topology on M may give rise to different $(M, \mathcal{L}_1)$-adic and $(M, \mathcal{L}_2)$-adic topologies on X. For example, take $R = M = \mathbb{Z}$, $X = \mathbb{Z}_2$, $\mathcal{L}_1 = \{0\}$ and for $\mathcal{L}_2$ the set of all submodules of $\mathbb{Z}$. Of course, $\mathcal{L}_1$ and $\mathcal{L}_2$ determine the same topologies on $\mathbb{Z}$. However, $\mathbb{Z}_2$ is Hausdorff separated in the $(\mathbb{Z}, \mathcal{L}_2)$-adic topology but is not Hausdorff separated in the $(\mathbb{Z}, \mathcal{L}_1)$-adic topology.

2 M-density and M-completion of modules

In this section we investigate the completion of an arbitrary left R-module X with respect to the M-adic topology.

Let ${}_RM$ be a fixed R-module and denote $S = \mathrm{End}({}_RM)$, $B = \mathrm{Biend}({}_RM)$. Then M becomes in a canonical way a bimodule ${}_RM_S$. For any module ${}_RX$ we use the notation

$$X_S^* = \mathrm{Hom}_R({}_RX\,,\,{}_RM_S) \text{ and } {}_RX^{**} = \mathrm{Hom}_S(X_S^*\,,\,{}_RM_S)\,.$$

By Φ_X we denote the canonical R-morphism

$$\Phi_X : X \longrightarrow X^{**}\ ,\ \ ((x)\Phi_X)(f) := (x)f\ ,\ \ x \in X,\ f \in X^*\,.$$

Note that for ${}_RX = {}_RR$ we have

$$R^{**} = \mathrm{Biend}({}_RM) = B \text{ and } \Phi_R = \mu\,,$$

where μ denotes the canonical ring morphism

$$\mu : R \longrightarrow B\ ,\ \ \mu(r)x = rx\,,\ \ r \in R\,,\ x \in M\,.$$

In the sequel we shall endow R and X with the M-adic topology, and X^{**} with the finite topology, by considering X^{**} as a subset of the topological space M^{X^*} endowed with the direct product topology, where M is endowed with the discrete topology. As mentioned before, a basis of open neighbourhoods of zero in the finite topology of X^{**} consists of the sets

$$V_F(0) = \{\ h \in \mathrm{Hom}_S(X^*,\,M) \mid \mathrm{Ker}(h) \supseteq F\ \}\,,$$

where F ranges over the finite subsets of X^*.

It is known (see e.g. [12, Proposition 1.5]) that R is a topological ring, X and X^{**} are topological R-modules, Φ_X is a continuous map, $\mathrm{Ker}(\Phi_X)$ is the closure of $0 \in X$ in the M-adic topology, and the topology on $(X)\Phi_X$ induced by the finite topology of X^{**} coincides with the direct image topology under Φ_X of the M-adic topology on X.

It is easily verified that the M-adic topology on X^{**} is finer than the finite topology on X^{**}. We do not know under which condition they coincide.

Recall that $\widehat{X}$ denotes the completion of an R-module ${}_RX$ in the M-adic topology. For any $Z \subseteq X^{**}$ we shall denote by $\overline{Z}$ the closure of Z in the topological space X^{**} endowed with the finite topology.

Proposition 2.1 *Let ${}_RX$ be a module. Then the M-adic completion of X is precisely the closure of $(X)\Phi_X$ in X^{**}, and is described explicitly as*

$$\widehat{X} = \{\ h \in X^{**} \,|\, \forall\, n \in I\!N,\ \forall\, f_1, \ldots, f_n \in X^*,\ \exists\, x \in X,\ h(f_i) = (x)f_i\,,\ \forall\ i\,,\ 1 \leq i \leq n\ \}\,.$$

Proof: First, note that the abelian group M^{X^*} endowed with the finite topology is a complete group, being a direct product of discrete topological groups. But $X^{**} \subseteq M^{X^*}$, and it is easily checked that X^{**} is a closed subset of M^{X^*}, which implies that X^{**} is complete. For another proof of the completeness of X^{**} see [12, Proposition 1.5 (iii)].

Denote by $\widetilde{X}$ the right part of the equality from the statement of the proposition. Looking at the form of elements of $\widetilde{X}$ it is clear that $\widetilde{X} \subseteq \overline{(X)\Phi_X}$. To prove the opposite inclusion, let $z \in \overline{(X)\Phi_X}$, and take finitely many $f_1, \dots, f_n \in X^*$. Then

$$U = \{\, h \in X^{**} \,|\, h(f_i) = z(f_i)\,,\ \forall\, i\,,\ 1 \le i \le n \,\}$$

is a neighbourhood of z, hence $U \cap (X)\Phi_X \neq \emptyset$, and so there exists an $x \in X$ such that $(x)\Phi_X \in U$. It follows that

$$((x)\Phi_X)(f_i) = (x)f_i = z(f_i)\,,\ \forall\, i\,,\ 1 \le i \le n\,,$$

which shows that $z \in \widetilde{X}$. Thus, we have proved that $\widetilde{X} = \overline{(X)\Phi_X}$. But, as we already have shown, X^{**} is complete, and consequently $\widetilde{X} = \widehat{X}$. □

Corollary 2.2 [5, 2.3] *The M-adic completion $\widehat{R}$ of the ring R is the subring*

$$\{\, b \in B \,|\, \forall\, n \in I\!N\,,\ \forall\, x_1, \dots, x_n \in M\,,\ \exists\, r \in R\,,\ bx_i = rx_i\,,\ \forall\, i\,,\ 1 \le i \le n \,\}$$

of $B = \mathrm{Biend}({}_RM)$. It coincides with the closure $\overline{\mu(R)}$ of $\mu(R)$ in B.
Hence $\widehat{R} = \mathrm{Biend}({}_RM)$ if and only if $\mu(R)$ is M-dense in B.

Proof: Apply 2.1 for ${}_RX = {}_RR$. □

In order to give some sufficient conditions on the given R-module M which ensure the M-density of $\mu(R)$ in B, we need some definitions:

Definitions 2.3 *The module ${}_RM$ is said to be a* self-generator (self-cogenerator) *if it generates all its submodules (cogenerates all its factor modules).*

${}_RM$ is said to be c-self-cogenerator *('c' from cyclic) if M^n/X is M-cogenerated for each $n \in I\!N$ and each cyclic submodule X of ${}_RM^n$.*

According to [13, 15.5] we have

$$\sigma[M] = \mathrm{Gen}(M) \Longleftrightarrow M^{(I\!N)} \text{ is a self-generator } \Longrightarrow M \text{ is a self-generator}\,.$$

An example, due to F. Dischinger, of a self-generator which is not a generator in $\sigma[M]$ can be found in [15, Example 1.2].

Notice that another notion of 'self-cogenerator', apparently different from those in 2.3, was introduced by Sandomierski in [9, Definition 3.1]: He calls ${}_RM$ a 'self-cogenerator' if for any $n \in I\!N$ and $X \le {}_RM^n$, $M^n/X \in \mathrm{Cog}(M)$. It is obvious that this condition implies that M is a self-cogenerator as well as a c-self-cogenerator.

According to [5, Satz 2.8] or [13, 15.7], $\mu(R)$ is M-dense in $B = \mathrm{Biend}({}_RM) = R^{**}$ if M is a generator in $\sigma[M]$ or M is a c-self-cogenerator. Hence we have from 2.2:

Corollary 2.4 [5, 2.9] *If M is either a generator in $\sigma[M]$ or a c-self-cogenerator, then* $\mathrm{Biend}({}_RM)$ *is the M-adic completion of R.*

We shall say that an R-module ${}_RX$ is *M-dense* if $(X)\Phi_X$ is a dense topological subspace of X^{**}. It is easily checked that this happens if and only if, in the terminology of [13], Φ_X is *dense*, that is, for any $h \in X^{**}$ and finitely many $f_1, \dots, f_n \in X^*$ there exists $x \in X$ such that

$$h(f_i) = ((x)\Phi_X)(f_i) = (x)f_i\,,\ \forall\, i\,,\ 1 \le i \le n\,.$$

As already noticed, for ${}_RX = {}_RR$, $R^{**} = \mathrm{Biend}({}_RM)$ and $\Phi_R = \mu$; thus ${}_RR$ is M-dense if and only if $\mu(R)$ is M-dense in B.

Lemma 2.5 [13] *Let X and M be left R-modules satisfying*

$$(\sharp) \quad \textit{for all } k \in I\!N \textit{ and } f \in \mathrm{Hom}_R(X, M^k)\,,\ \mathrm{Coker}(f) \in \mathrm{Cog}(M).$$

Then X is M-dense.

Proof: See [13, 47.7 (1)]. □

Remarks 2.6 (1) For ${}_RX = {}_RR$, condition $(\sharp)$ means that M is a c-self-cogenerator.

(2) The condition $(\sharp)$ is sufficient for ${}_RX$ to be M-dense (not necessary, see [13, 47.7]).

Corollary 2.7 *For any ${}_RX$, $\widehat{X} = X^{**}$ if and only if X is M-dense.*
In particular, $\widehat{X} = X^{**}$ whenever the condition $(\sharp)$ is satisfied.

Proof: Apply 2.1 and 2.5. □

The next observation relates the condition $(\sharp)$ to Vámos' condition in [12, Proposition 1.5 (iv)].

Corollary 2.8 *Suppose that each finitely M-generated module is Hausdorff separated in the M-adic topology. Then any left R-module X satisfies the condition $(\sharp)$, hence $\widehat{X} = X^{**}$.*

Proof: Let ${}_RX$ be an R-module, $k \in I\!N$, and $f \in \mathrm{Hom}_R(X, M^k)$. Then $\mathrm{Coker}(f)$ is a factor module of M^k, and so, by assumption, it is a Hausdorff separated space in the M-adic topology. But, as mentioned in section 1, a module ${}_RY$ is Hausdorff separated in the M-adic topology if and only if $Y \in \mathrm{Cog}(M)$. □

3 $\sigma[M]$ and M-density of R

Recall that for the left R-module ${}_RM$ we use the notation $E = \mathrm{End}^{\ell}({}_{\mathbb{Z}}M)$ and $B = \mathrm{Biend}({}_RM)$. The canonical left E-module structure of M and the scalar multiplication by R defines the ring morphism

$$\lambda : R \longrightarrow \mathrm{End}^{\ell}({}_{\mathbb{Z}}M)\,,\ \lambda(r)x = rx\,,\ \ r \in R\,,\ x \in M\,.$$

If A is an arbitrary unital subring of E containing $\lambda(R)$, then M has also a canonical structure of left A-module, and we shall denote by

$$\lambda^A : R \longrightarrow A$$

the corestriction of λ to A. Note that according to this notation, the canonical ring morphism $\mu : R \longrightarrow B$ considered in the preceding section is precisely λ^B.

For any $X \in A$-Mod we write $\lambda_*^A(X)$ for the left R-module obtained from ${}_AX$ by restriction of scalars via λ^A. If $\mathcal{X}$ is a nonempty class of left A-modules, then we shall also use the notation

$$\lambda_*^A(\mathcal{X}) = \{\, \lambda_*^A(X) \mid X \in \mathcal{X}\,\}\,.$$

In this section we show that for a unital subring A of the ring E containing $\lambda(R)$, any module in $\sigma[{}_RM]$ has a left A-module structure induced by the A-module structure of M if and only if $\lambda(R)$ is M-dense in A, or equivalently, if A is a subring of the M-adic completion $\widehat{R}$ of R.

Proposition 3.1 *Let ${}_RM$, $E = \mathrm{End}^{\ell}({}_{\mathbb{Z}}M)$, $\widehat{R}$ the completion of R in the M-adic topology, and A a unital subring of E containing $\lambda(R)$, with $\lambda : R \longrightarrow E$ defined above. Then*

$$\sigma[{}_RM] = \lambda_*^A(\sigma[{}_AM]) \iff A \subseteq \overline{\lambda(R)} = \widehat{R}\,.$$

In this case, for any $X, Y \in \sigma[{}_RM]$, $\mathrm{Hom}_R(X,Y) = \mathrm{Hom}_A(X,Y)$.

Proof: Suppose that $\sigma[{}_RM] = \lambda_*^A(\sigma[{}_AM])$, and let $\{x_1,\ldots,x_n\}$ be an arbitrary finite subset of M. Then $R(x_1,\ldots,x_n) \le {}_RM^n$, hence $R(x_1,\ldots,x_n) \in \sigma[{}_RM]$. By assumption, any module in $\sigma[{}_RM]$ has an A-module structure (induced by the left A-module structure of M), hence $A(x_1,\ldots,x_n) \subseteq R(x_1,\ldots,x_n)$, i.e., for any $a \in A$ there exists an $r \in R$ such that

$$ax_i = rx_i \ \text{ for each } i\,,\ 1 \le i \le n\,.$$

This means precisely that $\lambda(R)$ is M-dense in A.

Conversely, suppose that $\lambda(R)$ is M-dense in A, and let $U \le {}_RM^{(\Lambda)}$ for an arbitrary nonempty set Λ. But $M^{(\Lambda)}$ is also a left A-module, because M is so. Let $u \in U$. Then

$u = (x_i)_{i\in\Lambda}$, and $x_i = 0$ for all $i \in \Lambda \setminus F$, where F is a finite subset of Λ. Since $\lambda(R)$ is M–dense in A, we deduce that for any $a \in A$ there exists $r \in R$ such that

$$ax_i = rx_i \quad \text{for each } i \in F\,,$$

and so $au = ru \in U$, which shows that U is an A–submodule of ${}_AM^{(\Lambda)}$. Arbitrary R-modules in $\sigma[{}_RM]$ have the form U/V with $V \leq U$ R-submodules of ${}_RM^{(\Lambda)}$, for some set Λ. Because U and V are A-modules, so is also U/V. It follows that $\sigma[{}_RM] = \lambda_*^A(\sigma[{}_AM])$.

Let now $X, Y \in \sigma[{}_RM]$ and $f \in \mathrm{Hom}_R(X, Y)$. Then $X \oplus Y \in \sigma[{}_RM] = \lambda_*^A(\sigma[{}_AM])$, hence for any $x \in X$ one has $(x, (x)f) \in X \oplus Y$, and so, there exist $(x_1, \ldots, x_k) \in M^k$ and an A-morphism

$$\varphi : A(x_1, \ldots, x_k) \longrightarrow A(x, (x)f)\,, \;\; a(x_1, \ldots, x_k) \mapsto a(x, (x)f)\,, \;\; a \in A\,.$$

Since $\lambda(R)$ is dense in A, for any $a \in A$ there exists $r \in R$ such that

$$a(x_1, \ldots, x_k) = r(x_1, \ldots, x_k)\,.$$

Applying the A-morphism φ, we obtain

$$(a(x_1, \ldots, x_k))\varphi = a(x, (x)f) = r((x_1, \ldots, x_k)\varphi) = r(x, (x)f)\,,$$

and consequently

$$a((x)f) = r((x)f) = (rx)f = (ax)f\,,$$

which proves that $f \in \mathrm{Hom}_A(X, Y)$.

Finally, since B is a closed subset of E, the closure of $\mu(R)$ in B is the same as the closure of $\lambda(R)$ in E, and consequently, by 2.2, we obtain $\overline{\lambda(R)} = \overline{\mu(R)} = \widehat{R}$. □

Corollary 3.2 [13, 15.8] *For the R-module M, let $B =$ $\mathrm{Biend}({}_RM)$, and $\widehat{R}$ the completion of R in the M-adic topology. Then*

$$\sigma[{}_RM] = \lambda_*^B(\sigma[{}_BM]) \iff B = \overline{\lambda(R)} = \widehat{R}\,.$$

In this case, for any $X, Y \in \sigma[{}_RM]$, $\mathrm{Hom}_R(X, Y) = \mathrm{Hom}_B(X, Y)$.

Proof: Apply 3.1 for $A = \mathrm{Biend}({}_RM)$. □

Corollary 3.3 [8, 6.5] *For the R-module M, denote by $\widehat{R}$ the completion of R in the M-adic topology. Then*

$$\sigma[{}_RM] = \lambda_*^{\widehat{R}}(\sigma[{}_{\widehat{R}}M])\,,$$

and for any $X, Y \in \sigma[{}_RM]$, $\mathrm{Hom}_R(X, Y) = \mathrm{Hom}_{\widehat{R}}(X, Y)$.

Proof: By 2.2, $\widehat{R}$ is a subring of B containing $\lambda(R)$. Now apply 3.1 for $A = \widehat{R}$. □

Corollary 3.4 *For any R-module M, the ring $\widehat{R}$ is the largest unital subring A of* $\mathrm{End}^\ell({}_{\mathbb{Z}}M)$ *containing $\lambda(R)$ for which $\sigma[{}_RM] = \lambda_*^A(\sigma[{}_AM])$.*

Proof: Apply 3.1 and 3.3. □

4 $\mathcal{L}$-invariant endomorphisms

Motivated by ideas from Leptin [6], the aim of this section is to introduce and study the set of all $\mathcal{L}$-invariant endomorphisms of a bimodule ${}_RM_D$ with respect to a given filter basis $\mathcal{L}$ of R-submodules of M. Putting for $\mathcal{L}$ the set $\mathcal{L}({}_RM)$ of all submodules of ${}_RM$ we obtain $\mathrm{alglat}({}_RM_D)$ (of [2, 3]). In case $\mathcal{L}$ is the set of all open submodules of a linearly topologized R-module we obtain some of the results from [6].

Throughout this section we assume that M is an (R,D)-bimodule ${}_RM_D$ for some ring D (e.g. $D=\mathbb{Z}$), such that M_D is a topological module over the discrete ring D, having as a basis of neighbourhoods of zero the given filter basis $\mathcal{L}$ of R–submodules of M (this means precisely that for any $d \in D$ and any $L \in \mathcal{L}$ there exists $K \in \mathcal{L}$ such that $Kd \subseteq L$, in other words, for any $d \in D$ the map $\rho_d : M \longrightarrow M\,,\ x \mapsto xd$ is a continuous endomorphism of ${}_RM$ endowed with the $\mathcal{L}$-topology).

Endowing $\mathrm{End}(M_D)$ with the point-wise convergence topology, where M is considered as the topological space $(M,\mathcal{L})$ by means of the given filter basis $\mathcal{L}$, we have as a basis of open neighbourhoods of zero the subsets

$$W(F,L) = \{\,\alpha \in \mathrm{End}(M_D)\,|\,\alpha(F) \subseteq L\,\}\,,$$

where F is a finite subset of M and $L \in \mathcal{L}$. Denote by $\mathcal{W}$ the set of all these $W(F,L)$.

Observe that in case $(M,\mathcal{L})$ is Hausdorff, then $\mathrm{End}(M_D)$ is a closed subgroup of M^M endowed with the point-wise convergence topology.

We are interested in the ring

$$\mathcal{A}({}_RM_D,\mathcal{L}) := \{\,\alpha \in \mathrm{End}(M_D)\,|\,\alpha(L) \subseteq L \text{ for any } L \in \mathcal{L}\}\,.$$

These are the *hypercontinuous (hyperstetigen)* functions considered in Leptin [6, p.250]. Taking for $\mathcal{L}$ the set $\mathcal{L}({}_RM)$ of all submodules of ${}_RM$ we obtain $\mathrm{alglat}({}_RM_D)$ (of [2, 3]).

Notice that $\mathcal{A}({}_RM_D,\mathcal{L})$ depends on the filter $\mathcal{L}$ and not on the topology on M defined by this filter. For instance, the filters $\mathcal{L}_1 = \{0\}$ and $\mathcal{L}_2 = \mathcal{L}({}_RM)$ both define the discrete topology on M but in general

$$\mathcal{A}({}_RM_D,\mathcal{L}_1) = \mathrm{End}(M_D) \neq \mathrm{alglat}({}_RM_D) = \mathcal{A}({}_RM_D,\mathcal{L}_2)\,.$$

Clearly $\mathcal{A}({}_RM_D,\mathcal{L})$ is a unital subring of the ring $\mathrm{Cend}(M_D)$ of all endomorphisms of M_D which are continuous in the $\mathcal{L}$-topology of M.

For any finite subset F of M and any $L \in \mathcal{L}$ let us denote

$$\widetilde{W}(F,L) = \mathcal{A}({}_RM_D,\mathcal{L}) \cap W(F,L)\,.$$

Then, the set $\widetilde{\mathcal{W}} = \{\,\widetilde{W}(F,L)\,|\,W(F,L) \in \mathcal{W}\,\}$ is a basis of open neighbourhoods of zero in the topology on $\mathcal{A}({}_RM_D,\mathcal{L})$ induced by the point-wise convergence topology on $\mathrm{End}(M_D)$.

Moreover, $\widetilde{\mathcal{W}}$ consists of left ideals of $\mathcal{A}({}_RM_D,\mathcal{L})$. Indeed, if $\alpha \in \mathcal{A}({}_RM_D,\mathcal{L})$ and $\beta \in \widetilde{W}(F,L)$ then $(\alpha\circ\beta)(F) = \alpha(\beta(F)) \subseteq \alpha(L) \subseteq L$. Since for any $\widetilde{W}(F,L) \in \mathcal{W}$ and $\alpha \in \mathcal{A}({}_RM_D,\mathcal{L})$ one has $\widetilde{W}(\alpha F, L) \subseteq (\widetilde{W}(F,L) : \alpha)$ we deduce that $\mathcal{A}({}_RM_D,\mathcal{L})$ is a left linearly topologized ring.

Note that the canonical morphism

$$\nu : R \longrightarrow \mathcal{A}({}_RM_D,\mathcal{L})\ ,\ \ r \mapsto \nu_r\ ,$$

is a continuous ring morphism, where $\nu_r(x) = rx\ ,\ r \in R\ ,\ x \in M\ .$

Clearly M has a canonical left $\mathcal{A}({}_RM_D,\mathcal{L})$-module structure (since it is a left $\mathrm{End}(M_D)$-module), and any $L \in \mathcal{L}$ is an $\mathcal{A}({}_RM_D,\mathcal{L})$-submodule of M. It is easily verified that the map

$$\mathcal{A}({}_RM_D,\mathcal{L}) \times M \longrightarrow M\ ,\ \ (\alpha, x) \mapsto \alpha(x)\ ,\ \alpha \in \mathcal{A}({}_RM_D,\mathcal{L})\ ,\ x \in M\ ,$$

is continuous, so $(M,\mathcal{L})$ becomes a linearly topologized left $\mathcal{A}({}_RM_D,\mathcal{L})$-module.

We will consider the topologies on the subspaces induced by the product topology (i.e., point-wise convergence topology) on M^M,

$$\nu(R) \subseteq \mathcal{A}({}_RM_D,\mathcal{L}) \subseteq \mathcal{A}({}_RM_{\mathbb{Z}},\mathcal{L}) \subseteq M^M.$$

The next two results are similar to observations in Leptin [6].

Proposition 4.1 *If $(M,\mathcal{L})$ is Hausdorff separated and complete, then $\mathcal{A}({}_RM_D,\mathcal{L})$ is a complete topological ring in the point-wise convergence topology.*

Proof: It is sufficient to show that $\mathcal{A}({}_RM_D,\mathcal{L})$ is a closed subspace of the Hausdorff separated complete topological space M^M (or $\mathrm{End}(M_D)$, because this last one is a closed subspace of M^M, as we already have noted above). Put $C := \mathcal{A}({}_RM_D,\mathcal{L})$.

Let $\beta \in \overline{C}$, where $\overline{C}$ denotes the closure of C in $\mathrm{End}(M_D)$. Then, for any neighbourhood $W(\{x_1,\dots,x_n\},L)$ of zero in $\mathrm{End}(M_D)$, with $\{x_1,\dots,x_n\}$ an arbitrary finite subset of M and $L \in \mathcal{L}$, one has

$$(W(\{x_1,\dots,x_n\},L) + \beta) \cap C \neq \emptyset\,.$$

We show that $\beta \in C$. Let $L \in \mathcal{L}$ and $x \in L$. There exists $\gamma \in (W(\{x\},L) + \beta) \cap C$, that is, $(\gamma - \beta)(x) \in L$. But $\gamma(x) \in \gamma(L) \subseteq L$ since $\gamma \in C$ and $L \in \mathcal{L}$, hence $\beta(x) \in L$ for any $x \in L$, in other words, $\beta(L) \subseteq L$. This proves that $\beta \in C$. $\Box$

Recall that a linearly topologized left R-module N is said to be *linearly compact* if N has the following property: for any set $\mathcal{F}$ of closed cosets (i.e., cosets of closed submodules) in N having the finite intersection property (any finite number of elements of $\mathcal{F}$ has a nonempty intersection), the cosets in $\mathcal{F}$ have nonempty intersection (see e.g. [7]).

Proposition 4.2 *If $(M, \mathcal{L})$ is a Hausdorff separated linearly compact R-module, then $\mathcal{A}({}_RM_D, \mathcal{L})$ is a left linearly compact ring.*

Proof: We adopt ideas from the proof of [6, Satz 7]. First notice that any Hausdorff separated linearly compact module is complete (see e.g. [7, 3.11]), so in particular ${}_RM$ is a complete module. By 4.1, the ring $C := \mathcal{A}({}_RM_D, \mathcal{L})$ is complete, hence it is isomorphic to the direct limit of the family of left C-modules $C/\widetilde{W}(F,L)$, where F is a finite subset of M, $L \in \mathcal{L}$, and $\widetilde{W}(F,L) = C \cap W(F,L)$. It is well-known (see e.g. [7, 3.7]) that an inverse limit of linearly compact modules is also linearly compact, so it is sufficient to prove that any such discrete left C-module $C/\widetilde{W}(F,L)$ is linearly compact.

We know that M is a left C-module, and any $L \in \mathcal{L}$ is a C-submodule of ${}_CM$. For $F = \{x_1, \ldots, x_n\}$, consider the canonical C-morphism

$$\psi : C \longrightarrow (M/L)^n \ , \ \ \psi(\alpha) = (\alpha(x_1) + L, \ldots, \alpha(x_n) + L) \, ,$$

which has as kernel the left ideal $\widetilde{W}(F,L)$ of C. Thus, the left C-module $C/\widetilde{W}(F,L)$ is embedded into the discrete linearly compact C-module $(M/L)^n$. It follows that the discrete left C-module $C/\widetilde{W}(F,L)$ is linearly compact, which finishes the proof. □

Remark 4.3 As in [6], one can show that if $(M, \mathcal{L})$ is strictly linear compact, then so is also $\mathcal{A}({}_RM_D, \mathcal{L})$

5 alglat(M) and M-adic completion

In this section we present some connections between our results from the previous sections and the concept of alglat(M). First recall some definitions from [2, 3].

Definitions 5.1 *For any (R, D)-bimodule ${}_RM_D$ we define*

$$\mathrm{alglat}(M) = \mathrm{alglat}({}_RM_D) = \{\ \alpha \in \mathrm{End}(M_D) \mid \alpha(L) \subseteq L \ \textit{for all} \ L \leq {}_RM \ \} \, .$$

For $n \in I\!N$, denote by $\mathrm{alglat}_n(M)$ the set of all $\alpha \in \mathrm{alglat}(M)$ such that for all $(x_1, \ldots, x_n) \in M^n$ there exists an $r \in R$ with $\alpha(x_i) = rx_i$ for all i, $1 \leq i \leq n$; by [2], this is canonically isomorphic to $\mathrm{alglat}(M^n)$.

With our previous notation we have $\mathrm{alglat}({}_RM_D) = \mathcal{A}({}_RM_D, \mathcal{L}({}_RM))$.

The behaviour of 'alglat' with respect to finite direct sums was investigated in [2, Section 2]. The general case of arbitrary direct sums is considered below.

Lemma 5.2 *If $(M_i)_{i \in I}$ is an arbitrary family of (R, D)-bimodules, then for each $\alpha \in \mathrm{alglat}(\bigoplus_{i \in I} M_i)$ there exist $\alpha_i \in \mathrm{alglat}(M_i)$, $i \in I$ such that*

$$\alpha((x_i)_{i \in I}) = (\alpha_i(x_i))_{i \in I} \, ,$$

for each $(x_i)_{i\in I} \in \bigoplus_{i\in I} M_i$ *.*

In case $M_i = M$ *for each* $i \in I$*, then* $\alpha_i = \alpha_j$ *for all* $i, j \in I$*.*

Proof: For each $i \in I$ denote by $\varepsilon_i : M_i \longrightarrow \bigoplus_{j\in I} M_j$ the canonical injection and $M_i' = (M_i)\varepsilon_i$. Then $M_i' \simeq M_i$ as R-D bimodules, hence $\mathrm{alglat}(M_i) \simeq \mathrm{alglat}(M_i')$ and $(M_i')\alpha \subseteq M_i'$ for any $i \in I$ because $\alpha \in \mathrm{alglat}(\bigoplus_{i\in I} M_i)$. For any $i \in I$, denote by α_i the element in $\mathrm{alglat}(M_i)$ corresponding to $\alpha_i' = \alpha | M_i' \in \mathrm{alglat}(M_i')$ by the isomorphism $\mathrm{alglat}(M_i) \simeq \mathrm{alglat}(M_i')$. Then for any $(x_i)_{i\in I} \in \bigoplus_{i\in I} M_i$,

$$\alpha((x_i)_{i\in I}) = \alpha(\sum_{i\in I}(x_i)\varepsilon_i)) = \sum_{i\in I} \alpha_i'((x_i)\varepsilon_i)) = (\alpha_i(x_i))_{i\in I} .$$

Now consider the particular case when $M_\lambda = M$ for each $\lambda \in I$, and take arbitrary two elements $i \neq j$ in I. For an arbitrary $x \in M$ consider the element $(x_\lambda)_{\lambda\in I} \in M^{(I)}$ defined as follows: $x_i = x_j = x$ and $x_\lambda = 0$ for all $\lambda \in I \setminus \{i, j\}$. Since $\alpha \in \mathrm{alglat}(M^{(I)})$, we deduce that there exists an $r \in R$ such that

$$\alpha((x_\lambda)_{\lambda\in I}) = (\alpha_\lambda(x_\lambda))_{\lambda\in I} = r(x_\lambda)_{\lambda\in I} ,$$

and consequently we deduce that $\alpha_i(x) = \alpha_j(x) = rx$, i.e., $\alpha_i = \alpha_j$ for all $i, j \in I$. □

Corollary 5.3 *With the notation from 5.2, the map* $\alpha \longmapsto (\alpha_i)_{i\in I}$ *defines a ring monomorphism*

$$\mathrm{alglat}(\bigoplus_{i\in I} M_i) \longrightarrow \prod_{i\in I} \mathrm{alglat}(M_i) .$$

The next result is the discrete variant of Proposition 4.1. Note that for any bimodule ${}_RM_D$, $\mathrm{alglat}({}_RM_D)$ is a subring of $\mathrm{End}(M_D)$ which is a subset of M^M, so it makes sense to consider the finite topology on $\mathrm{alglat}({}_RM_D)$.

Proposition 5.4 *For any bimodule* ${}_RM_D$ *the ring* $\mathrm{alglat}({}_RM_D)$ *is complete in the finite topology.*

Proof: This is a special case of 4.1 for $\mathcal{L} = \mathcal{L}({}_RM)$. □

For an arbitrary R-module ${}_RN$, $\mathrm{alglat}(N)$ will denote throughout the remainder of this section $\mathrm{alglat}({}_RN_T)$, where $T = \mathrm{End}({}_RN)$.

Proposition 5.5 *For any module* ${}_RM$*,*

$$\widehat{R} = \bigcap_{n\geq 1} \mathrm{alglat}_n(M) \simeq \mathrm{alglat}(M^{(\mathbb{N})}) ,$$

where $\widehat{R}$ *is the* M*-adic completion of* R*.*

Proof: If we denote $D = \mathrm{End}({}_RM)$, then clearly $\mathrm{End}(M_D) = \mathrm{Biend}({}_RM) = B$, and $\bigcap_{n\geq 1} \mathrm{alglat}_n(M)$ is precisely the set

$$\{ b \in B \mid \forall n \in I\!N\,,\ \forall x_1, \ldots, x_n \in M\,,\ \exists r \in R\,,\ bx_i = rx_i\,,\ \forall i\,,\ 1 \leq i \leq n \}\,.$$

Apply 2.2 to conclude that $\bigcap_{n\geq 1} \mathrm{alglat}_n(M) = \widehat{R}$.

Let $b \in \bigcap_{n\geq 1} \mathrm{alglat}_n(M)$, and denote by $\overline{b}$ the endomorphism of the right D-module $M^{(I\!N)}$ defined by

$$\overline{b}((x_n)_{n\in I\!N}) = (bx_n)_{n\in I\!N}\,.$$

Since $x_n = 0$ for n sufficiently large, it follows that $\overline{b}((x_n)_{n\in I\!N}) = r(x_n)_{n\in I\!N}$ for some $r \in R$, hence $\overline{b} \in \mathrm{alglat}({}_RM_D^{(I\!N)})$. By 5.2 and 5.3, the map $b \longmapsto \overline{b}$ defines a ring isomorphism

$$\Theta : \bigcap_{n\geq 1} \mathrm{alglat}_n(M) \xrightarrow{\sim} \mathrm{alglat}({}_RM_D^{(I\!N)})\,.$$

It remains to prove that

$$\mathrm{alglat}({}_RM_D^{(I\!N)}) \simeq \mathrm{alglat}(M^{(I\!N)})\,.$$

Recall that $\mathrm{alglat}(M^{(I\!N)})$ denotes $\mathrm{alglat}({}_RM^{(I\!N)}{}_G)$, where $G = \mathrm{End}({}_RM^{(I\!N)})$. But $\mathrm{End}(M^{(I\!N)}{}_G) = \mathrm{Biend}({}_RM^{(I\!N)})$, and there is a ring isomorphism ([1, 4.2])

$$\rho : \mathrm{Biend}({}_RM) \longrightarrow \mathrm{Biend}({}_RM^{(I\!N)})\,,\quad \rho(b)((x_n)_{n\in I\!N}) = (bx_n)_{n\in I\!N}\,.$$

On the other hand,

$$\mathrm{alglat}({}_RM^{(I\!N)}{}_G) = \{ b \in \mathrm{Biend}({}_RM^{(I\!N)}) \mid bz \in Rz \text{ for all } z \in M^{(I\!N)} \}\,.$$

We conclude that the endomorphism $\overline{b}$ of the right D-module $M^{(I\!N)}$ corresponding to $b \in \bigcap_{n\geq 1} \mathrm{alglat}_n(M)$ by Θ, belongs to $\mathrm{alglat}(M^{(I\!N)})$, which finishes the proof.

Notice that the isomorphism stated in the proposition also follows from [10, Theorem 1]. □

Corollary 5.6 *The M-adic completion $\widehat{R}$ of R is a subring of* $\mathrm{alglat}(M)$.

Corollary 5.7 *If ${}_RR$ is M-dense (in particular, if M is generator in $\sigma[M]$ or a c-self-cogenerator) then*

$$\mathrm{alglat}(M^{(I\!N)}) \simeq \mathrm{Biend}({}_RM) = \widehat{R}\,.$$

Proof: Apply 2.4 and 5.5. □

Proposition 5.8 *Suppose the bimodule ${}_RM_D$ is such that ${}_RM$ is discrete linearly compact. Then* $\mathrm{alglat}({}_RM_D)$ *is a left linearly compact ring in the finite topology.*

Proof: Apply 4.1 for $\mathcal{L} = \mathcal{L}({}_RM)$. $\square$

Notice that by [13, 15.6], for any self-generator ${}_RM$, $\mathrm{alglat}(M) \simeq \mathrm{Biend}({}_RM)$. In view of 5.6 and 5.7 we end with the problem:

When is alglat(M) *the* M*-adic completion of* R*, or more generally, if* $\mathcal{L}$ *is a filter basis of submodules of* ${}_RM$*, when is* $\mathcal{A}({}_RM_D, \mathcal{L})$ *, with* $D = \mathrm{End}({}_RM)$*, the completion of* R *in the* $(M, \mathcal{L})$*-adic topology of the ring* R *?*

Acknowledgements

This paper was started while the first author was a Visiting Professor at the Department of Pure and Applied Mathematics of the University of Padova (April - May 1994) and continued when he was a Humboldt Fellow at the Mathematical Institute of the Heinrich-Heine-University in Düsseldorf (June - July 1994). He would like to thank both these Institutes for warm hospitality and also to thank the University of Padova and the Alexander von Humboldt Foundation for financial support. Last but not least, he is very indebted to Adalberto Orsatti for inviting him to Padova and for many stimulating conversations on topological aspects of the category $\sigma[M]$. In the final stage of the preparation of this paper the first author has been partially supported by the Grant 221/1995 of CNCSU (Romania).

The second author would like to thank John Clark and Kent Fuller for helpful discussions on the subject.

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