Duprime and Dusemiprime Modules

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Abstract

A *lattice ordered monoid* is a structure $\langle L; \oplus, 0_L; \leq\rangle$ where $\langle L; \oplus, 0_L\rangle$ is a monoid, $\langle L; \leq\rangle$ is a lattice and the binary operation $\oplus$ distributes over finite meets. If $M \in R$-Mod then the set $\mathbb{L}_M$ of all hereditary pretorsion classes of $\sigma[M]$ is a lattice ordered monoid with binary operation given by

$$\alpha :_M \beta := \{N \in \sigma[M] \mid \text{there exists } A \leq N \text{ such that } A \in \alpha \text{ and } N/A \in \beta\},$$

whenever $\alpha, \beta \in \mathbb{L}_M$ (the subscript in $:_M$ is omitted if $\sigma[M] = R$-Mod). $\sigma[M]$ is said to be *duprime* (resp. *dusemiprime*) if $M \in \alpha :_M \beta$ implies $M \in \alpha$ or $M \in \beta$ (resp. $M \in \alpha :_M \alpha$ implies $M \in \alpha$), for any $\alpha, \beta \in \mathbb{L}_M$. The main results characterize these notions in terms of properties of the subgenerator M. It is shown, for example, that M is duprime (resp. dusemiprime) if M is strongly prime (resp. strongly semiprime). The converse is not true in general, but holds if M is polyform or projective in $\sigma[M]$. The notions duprime and dusemiprime are also investigated in conjunction with finiteness conditions on $\mathbb{L}_M$, such as coatomicity and compactness.

Introduction

A classical example of lattice ordered monoid is given by the set of all ideals $\operatorname{Id} R$ of an arbitrary ring R with identity. Here, the lattice structure is induced by the relation of *reverse* set inclusion with ideal multiplication the binary operation. Several ring theoretic notions are characterizable as sentences in the language of the lattice ordered monoid $\operatorname{Id} R$. Primeness and semiprimeness are two examples. An ideal P of a ring R is prime if and only if for any $I, J \in \operatorname{Id} R$, $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$, and semiprime if and only if for any $I \in \operatorname{Id} R$, $I^2 \subseteq P$ implies $I \subseteq P$.

$\operatorname{Id} R$ is embeddable in a larger lattice ordered monoid comprising the set of all hereditary pretorsion classes of R-Mod (denoted $\mathbb{L}_R$) via the mapping

$$\eta : \operatorname{Id} R \to \mathbb{L}_R,\ I \mapsto \eta(I) := \{M \in R\text{-Mod} \mid IM = 0\}.$$

The embedding η allows us to express the notions prime and semiprime, for example, in terms of hereditary pretorsion classes thus: $P \in \mathrm{Id}\, R$ is prime if and only if for all $I, J \in \mathrm{Id}\, R$, $\eta(I) : \eta(J) \supseteq \eta(P)$ implies $\eta(I) \supseteq \eta(P)$ or $\eta(J) \supseteq \eta(P)$ and P is semiprime if and only if for all $I \in \mathrm{Id}\, R$, $\eta(I) : \eta(I) \supseteq \eta(P)$ implies $\eta(I) \supseteq \eta(P)$. This observation motivates the introduction of a notion of 'primeness' and 'semiprimeness' in $\mathbb{L}_R$. We call $\gamma \in \mathbb{L}_R$ *dual prime*, henceforth to be abbreviated *duprime*, if for all $\alpha, \beta \in \mathbb{L}_R$, $\alpha : \beta \supseteq \gamma$ implies $\alpha \supseteq \gamma$ or $\beta \supseteq \gamma$, and γ is said to be *dual semiprime*, henceforth *dusemiprime*, if for all $\alpha \in \mathbb{L}_R$, $\alpha : \alpha \supseteq \gamma$ implies $\alpha \supseteq \gamma$. (The prefix 'dual' is explained by the fact that the above sentence corresponds with the usual notion of primeness (resp. semiprimeness) if interpreted in the order dual of $\mathrm{Id}\, R$.)

Insofar as $\mathbb{L}_R$ may be viewed as a structure which properly contains $\mathrm{Id}\, R$ (via the embedding η), it is not difficult to see that P will be a prime ideal of R if $\eta(P)$ is duprime in $\mathbb{L}_R$. The latter condition is thus at least as strong as the former. In particular, taking P to be the zero ideal, R will be a prime ring if the hereditary pretorsion class consisting of all left R-modules, namely R-Mod, is duprime in $\mathbb{L}_R$. It is shown in [11, Theorem 26 and Remark 27] that the rings R for which R-Mod is duprime are precisely the left strongly prime rings of Handelman and Lawrence [7]. It is shown similarly [11, Theorem 32 and Remark 33] that R-Mod is dusemiprime if and only if R is left strongly semiprime in the sense of Handelman [6].

Viewing R-Mod as the hereditary pretorsion class subgenerated by the module ${}_RR$, these results can be seen as an attempt to characterize duprimeness and dusemiprimeness of $\sigma[{}_RR]$ in terms of properties of the subgenerator ${}_RR$. This paper addresses the following natural generalization: if M is an arbitary module, characterize duprimeness and dusemiprimeness of the hereditary pretorsion class $\sigma[M]$ in terms of properties of the subgenerator M.

Results do not generalize easily from ${}_RR$ to a general M, for the module ${}_RR$ is finitely generated and projective. These rather special properties impart a type of finiteness to R-Mod which is absent in the case of a general $\sigma[M]$. Every strongly prime module, in the sense of [1], subgenerates a duprime hereditary pretorsion class. But the converse turns out to be false, in general.

Results in this paper have a mixed flavour; they make use of standard module theoretic techniques, but are also reliant on the body of theory on lattice ordered monoids developed in [11].

1 Preliminaries

The symbol $\subseteq$ denotes containment and $\subset$ proper containment for sets. Throughout the paper R will denote an associative ring with identity, R-Mod the category of unital left R-modules, and M any object in R-Mod. If N is a submodule (resp. essential submodule) of M we write $N \leq M$ (resp. $N \trianglelefteq M$). We denote the left annihilator of a subset X of M by $(0 : X)$. We call M *cofaithful* if $(0 : X) = 0$ for some finite subset X of M.

1.1 Hereditary pretorsion classes. Let $\mathcal{A}$ be a nonempty class of modules in R-Mod. We introduce the following abbreviations:

$$\begin{aligned}
\mathbf{P}(\mathcal{A}) &= \{M \in R\text{-Mod} \mid M \text{ is a product of modules in } \mathcal{A}\},\\
\mathbf{C}(\mathcal{A}) &= \{M \in R\text{-Mod} \mid M \text{ is a direct sum of modules in } \mathcal{A}\},\\
\mathbf{S}(\mathcal{A}) &= \{M \in R\text{-Mod} \mid M \text{ is a submodule of some module in } \mathcal{A}\},\\
\mathbf{E}(\mathcal{A}) &= \{M \in R\text{-Mod} \mid M \text{ is an injective hull of some module in } \mathcal{A}\},\\
\mathbf{H}(\mathcal{A}) &= \{M \in R\text{-Mod} \mid M \text{ is a homomorphic image of some module in } \mathcal{A}\}.
\end{aligned}$$

We say $B \in R$-Mod is *subgenerated* by $\mathcal{A}$ if $B \in \mathbf{SHC}(\mathcal{A}) = \mathbf{HSC}(\mathcal{A})$ and *cogenerated* by $\mathcal{A}$ if $B \in \mathbf{SP}(\mathcal{A})$. A nonempty class in R-Mod which is closed under direct sums, homomorphic images and submodules, is called a *hereditary pretorsion class*. $\mathbf{SHC}(\mathcal{A})$ is the smallest hereditary pretorsion class of R-Mod containing $\mathcal{A}$. Dually, a nonempty class in R-Mod which is closed under submodules, products and the taking of injective hulls, is called a *torsion-free class*. $\mathbf{SPE}(\mathcal{A})$ is the smallest torsion-free class in R-Mod containing $\mathcal{A}$ [3, Corollary 1.8(ii)].

If $\mathcal{A} = \{M\}$ is a singleton, we write $\sigma[M]$ in place of $\mathbf{SHC}(\mathcal{A})$. Every hereditary pretorsion class $\mathcal{C}$ has this form for it is easily shown that if M is the direct sum of a representative set of cyclic modules in $\mathcal{C}$, then $\mathcal{C} = \sigma[M]$.

We shall not distinguish notationally between $\sigma[M]$ and the full subcategory of R-Mod whose class of objects is $\sigma[M]$.

Associated with any hereditary pretorsion class $\sigma[M]$, there is a *left exact preradical* (also called *torsion preradical* or *kernel functor*)

$$\mathcal{T}^M : R\text{-Mod} \to \sigma[M],\ N \mapsto \mathcal{T}^M(N) := \mathrm{Tr}(\sigma[M], N),$$

where $\mathrm{Tr}(\sigma[M], N)$ denotes the trace of the class $\sigma[M]$ in N. $\mathrm{Tr}(\sigma[M], N)$ corresponds with the unique largest submodule of N contained in $\sigma[M]$. It follows from properties of injectives that $\mathrm{Tr}(\sigma[M], N) = \mathrm{Tr}(M, N)$ whenever N is injective in $\sigma[M]$.

The collection of all hereditary pretorsion classes of R-Mod is a set [9, Proposition VI.4.2, p. 145] whose elements we shall denote by $\alpha, \beta, \ldots$, or by $\sigma[M]$ if we wish to refer to a specific subgenerator. We shall, for notational convenience, identify a hereditary pretorsion class α with its associated left exact preradical and write $\alpha(N)$ in place of $\mathrm{Tr}(\alpha, N)$ whenever $N \in R$-Mod. We call $K \leq M$ a *pretorsion submodule* of M if $K = \alpha(M)$, for some hereditary pretorsion class α. Every pretorsion submodule of M is fully invariant in M. If M is injective in $\sigma[M]$, then the converse is also true, for if U is a fully invariant submodule of M and $\alpha = \sigma[U]$, then $\alpha(M) = U$.

1.2 The Grothendieck category $\sigma[M]$. Coproducts, quotient objects and subobjects in $\sigma[M]$ are the same as in R-Mod because of the defining closure properties of a hereditary pretorsion class [18, 15.1((1),(2)), p. 118]. It follows that the hereditary pretorsion classes of $\sigma[M]$ are precisely the hereditary pretorsion classes of R-Mod which are contained in $\sigma[M]$. For the most part, these shall be our objects of study.

Put $\alpha = \sigma[M]$. If $\{N_i \mid i \in \Gamma\}$ is a family of modules in α then $\prod^{\alpha}_{i\in\Gamma} N_i := \alpha(\prod_{i\in\Gamma} N_i)$ is the product of $\{N_i \mid i \in \Gamma\}$ in α [18, 15.1(6), p. 118], and $E^{\alpha}(N) := \alpha(E(N))$ is the injective hull of N in α [18, 17.9(2), p. 141]. If $\mathcal{A}$ is a nonempty class of modules in R-Mod we introduce two abbreviations:

$$\begin{aligned}\mathbf{P}_\alpha(\mathcal{A}) &= \{N \in \alpha \mid N = \alpha(\textstyle\prod_{i\in\Gamma} A_i), \text{ for some family } \{A_i \mid i \in \Gamma\} \text{ in } \mathcal{A}\},\\ \mathbf{E}_\alpha(\mathcal{A}) &= \{N \in \alpha \mid N = \alpha(E(A)) \text{ for some } A \in \mathcal{A}\}.\end{aligned}$$

We claim that

$$\alpha \cap \mathbf{SPE}(\mathcal{A}) = \mathbf{SP}_\alpha\mathbf{E}_\alpha(\mathcal{A}).$$

Since $\mathbf{SPE}(\mathcal{A})$ is a torsion-free class in R-Mod containing $\mathcal{A}$, it follows that $\mathbf{SP}_\alpha\mathbf{E}_\alpha(\mathcal{A}) \subseteq \mathbf{SPE}(\mathcal{A})$. The containment in one direction follows. The reverse containment follows since $\alpha \cap \mathbf{SPE}(\mathcal{A}) \subseteq \mathbf{SP}_\alpha\mathbf{E}(\mathcal{A}) = \mathbf{SP}_\alpha\mathbf{E}_\alpha(\mathcal{A})$. Observe that if $\mathcal{A} \subseteq \alpha$ then $\alpha \cap \mathbf{SPE}(\mathcal{A}) = \mathbf{SP}_\alpha\mathbf{E}_\alpha(\mathcal{A})$ is the smallest torsion-free class of α containing $\mathcal{A}$.

1.3 The lattice $\mathbb{L}_M$. We shall denote by $\mathbb{L}_M$ the set of all hereditary pretorsion classes of $\sigma[M]$. $\mathbb{L}_R$ is thus the set of all hereditary pretorsion classes of R-Mod. $\mathbb{L}_R$ is partially ordered by inclusion and is a complete lattice under the operations:

$$\begin{aligned}\textstyle\bigvee_{i\in\Lambda} \sigma[K_i] &= \sigma[\textstyle\bigoplus_{i\in\Lambda} K_i],\\ \textstyle\bigwedge_{i\in\Lambda} \sigma[K_i] &= \textstyle\bigcap_{i\in\Lambda} \sigma[K_i].\end{aligned}$$

Observe that $\mathbb{L}_M$ is just the interval $\{\alpha \in \mathbb{L}_R \mid \alpha \subseteq \sigma[M]\}$ of $\mathbb{L}_R$. It follows from the description of the join above, that if N is injective in $\bigvee_{i\in\Lambda} \sigma[K_i]$, then $\mathrm{Tr}(\bigvee_{i\in\Lambda} \sigma[K_i], N) = \sum_{i\in\Lambda} \mathrm{Tr}(K_i, N)$. It follows from the description of the meet, that for every M, the set of pretorsion submodules of M is a meet subsemilattice of the submodule lattice of M.

Recall that an element x of a complete lattice $\mathbb{L}$ is said to be *compact* if, whenever $X \subseteq \mathbb{L}$ is such that $x \leq \bigvee X$, we also have $x \leq \bigvee X'$ for some finite $X' \subseteq X$. The lattice $\mathbb{L}$ is said to be *compact* if it has compact top element, and *algebraic* (or *compactly generated*) if each of its elements is the join of a set of compact elements.

A complete lattice $\mathbb{L}$ is said to be *uniquely pseudocomplemented* if, for each $x \in \mathbb{L}$, the set $\{y \in \mathbb{L} \mid x \wedge y = 0_{\mathbb{L}}\}$ has a unique largest element.

α is a compact element of $\mathbb{L}_R$ if and only if $\alpha = \sigma[M]$ for some finitely generated M. (In fact, M can be chosen to be cyclic [5, Proposition 2.16, p. 21].) The lattice $\mathbb{L}_R$ is known to be atomic, coatomic (because $\mathbb{L}_R$ is compact), algebraic, modular and uniquely pseudocomplemented. Proofs establishing algebraicity and atomicity may be found in [5, Corollaries 2.17, p. 22 and 2.24, p. 24] and modularity in [10, Proposition II.1.6, p. 68]. It is proved in [8, Corollary 17] that $\mathbb{L}_M$ is uniquely pseudocomplemented for all M.

$\mathbb{L}_M$, being an interval in $\mathbb{L}_R$, inherits much from $\mathbb{L}_R$. It is atomic, algebraic, modular and uniquely pseudocomplemented. In general, $\mathbb{L}_M$ is not coatomic. $\mathbb{L}_M$ will be compact precisely if $\sigma[M]$ is a compact element in $\mathbb{L}_R$. This is a consequence of the fact that $\mathbb{L}_R$ is upper continuous [9, Proposition III.5.3, p. 73].

1.4 Extension of hereditary pretorsion classes. If $\alpha, \beta \in \mathbb{L}_R$, the *extension of* β *by* α is defined[1] as

$$\alpha : \beta := \{N \in R\text{-Mod} \mid \text{there exists } A \leq N \text{ such that } A \in \alpha \text{ and } N/A \in \beta\}.$$

It is easily verified that $\alpha : \beta \in \mathbb{L}_R$ and $(\alpha : \beta)(M)/\alpha(M) = \beta(M/\alpha(M))$ for all M. Note that $\alpha : \beta \geq \alpha \vee \beta$.

Observe that α is idempotent in the sense that $\alpha : \alpha = \alpha$ precisely if α is closed under extensions and thus a hereditary torsion class. The structure $\langle \mathbb{L}_R; :, \{0\}; \subseteq\rangle$ (here, $\{0\}$ denotes the bottom element of $\mathbb{L}_R$) is a lattice ordered monoid because:

(1) $\langle \mathbb{L}_R; :, \{0\}\rangle$ is a monoid;
(2) $\langle \mathbb{L}_R; \subseteq\rangle$ is a lattice; and
(3) $\alpha : (\beta \wedge \gamma) = (\alpha : \beta) \wedge (\alpha : \gamma)$ and $(\alpha \wedge \beta) : \gamma = (\alpha : \gamma) \wedge (\beta : \gamma)$, for all $\alpha, \beta, \gamma \in \mathbb{L}_R$ [5, Proposition 4.1, p. 43].

$\langle \mathbb{L}_R; :, \{0\}; \subseteq\rangle$ is said to be *integral* because the bottom element $\{0\}$ of $\mathbb{L}_R$ coincides with the monoid identity.

The interval $\mathbb{L}_M$ of $\mathbb{L}_R$ is, in general, not closed under the operation ':'. Nevertheless, we can define an associative operation '$:_M$' on $\mathbb{L}_M$ by truncating at the top element of $\mathbb{L}_M$. If $\alpha, \beta \in \mathbb{L}_M$,

$$\begin{aligned} \alpha :_M \beta &:= (\alpha : \beta) \cap \sigma[M] \\ &= \{N \in \sigma[M] \mid \text{there exists } A \leq N \text{ such that } A \in \alpha \text{ and } N/A \in \beta\}. \end{aligned}$$

$\langle \mathbb{L}_M; :_M, \{0\}; \subseteq\rangle$ is thus an integral lattice ordered monoid for all M.

We warn the reader that, inasmuch as the operations $:_M$ and ':' differ, an idempotent element of $\mathbb{L}_M$, i.e., hereditary torsion class of $\sigma[M]$, need not be idempotent in $\mathbb{L}_R$.

1.5 The monus operation. For any $\alpha, \beta \in \mathbb{L}_R$, the set

$$\{\gamma \in \mathbb{L}_R \mid \beta : \gamma \geq \alpha\}$$

[1] Notice that the operation ':' defined here is opposite to the multiplication operation introduced in [12], [13] and [11]. Consequently, properties which are prefixed with 'left' in the aforementioned papers, become 'right' in this paper.

has a unique smallest element [5, p. 44] called α *monus* β and written $\alpha \dot{-} \beta$. The existence of such a unique smallest element can be shown to be equivalent to the identity

$$\alpha : (\bigwedge_{i\in\Lambda} \beta_i) = \bigwedge_{i\in\Lambda} (\alpha : \beta_i)$$

holding in $\mathbb{L}_R$ [11, Proposition 3]. It should be noted that in [8, Definition 9] the element $\alpha \dot{-} \beta$ is referred to as the *right supplement* of β in α. A lattice ordered monoid is said to be *right residuated* if it satisfies the above identity, and thus admits a monus operation defined in the above manner. Thus $\langle \mathbb{L}_R; :, \{0\}; \subseteq\rangle$ is right residuated.

It is easily shown that $\mathbb{L}_M$ is closed under the monus operation. Hence if $\alpha, \beta \in \mathbb{L}_M$, then $\alpha \dot{-} \beta$ is the unique smallest element of $\{\gamma \in \mathbb{L}_M \mid \beta :_M \gamma \geq \alpha\}$. Thus the monus operation, unlike the operation $:_M$, is independent of M. We conclude that $\langle \mathbb{L}_M; :_M, \{0\}; \subseteq\rangle$ is a *lattice ordered right residuated integral monoid* (abbreviated *lorrim*) for all M.

1.6 Lemma. *[8, Theorem 10((1)⇔(4))] For any left R-module M and $\beta \in \mathbb{L}_R$, $\sigma[M] \dot{-} \beta = \sigma[M/\beta(M)]$.*

1.7 Special subgenerators. Since $\sigma[M]$ is a Grothendieck category we can always find an injective subgenerator for $\sigma[M]$, for example the M-injective hull $\widehat{M}$ of M. We can even find an injective cogenerator for $\sigma[M]$ which is also a subgenerator for $\sigma[M]$, for example $\widehat{M} \oplus Q$, where Q is any injective cogenerator for $\sigma[M]$. Notice that not every cogenerator is a subgenerator. For example, $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator but not a subgenerator for $\mathbb{Z}$-Mod since $\sigma[\mathbb{Q}/\mathbb{Z}]$ is just the class of torsion $\mathbb{Z}$-modules.

Let $Q \in R$-Mod be injective in $\sigma[M]$. We call Q a *big cogenerator* for $\sigma[M]$ if $\mathbf{SC}(\{Q\})$ contains all finitely generated modules in $\sigma[M]$. Big cogenerators are important because they are both subgenerators and cogenerators. The former property is a consequence of the fact that the hereditary pretorsion class $\mathbf{HSC}(\{Q\}) \supseteq \mathbf{SC}(\{Q\})$ contains all finitely generated modules in $\sigma[M]$, whence $\mathbf{HSC}(\{Q\}) = \sigma[M]$. To see the latter property, observe that if Q is a big cogenerator for $\alpha = \sigma[M]$, then the torsion-free class $\mathbf{SPE}(\{Q\})$ of R-Mod contains every finitely generated module in α, whence $\mathbf{SPE}(\{Q\}) \supseteq \alpha$ and so $\alpha = \alpha \cap \mathbf{SPE}(\{Q\}) = \mathbf{SP}_\alpha\mathbf{E}_\alpha(\{Q\})$. Since Q is injective in α, $\alpha = \mathbf{SP}_\alpha(\{Q\}) = \mathbf{SP}_\alpha\mathbf{E}_\alpha(\{Q\})$. We conclude that Q is a cogenerator for $\sigma[M]$.

For example, if M is locally noetherian, then the direct sum of a representative set of indecomposable (uniform) injective modules in $\sigma[M]$ is an (injective) big cogenerator for $\sigma[M]$. If M is locally of finite length (i.e., locally artinian and noetherian) then every injective cogenerator for $\sigma[M]$ is a big cogenerator for $\sigma[M]$.

1.8 The Lambek torsion class. A nonempty class $\mathcal{C}$ of modules in $\sigma[M]$ is a hereditary torsion class of $\sigma[M]$ if and only if $\mathcal{C}$ has the form

$$\mathcal{C} = \{N \in \sigma[M] \mid \mathrm{Hom}_R(N, E) = 0\},$$

for some module E which is injective in $\sigma[M]$ [19, 9.5, p. 59]. It is easily shown that such a class $\mathcal{C}$ is the unique largest element of $\mathbb{L}_M$ whose corresponding torsion-free class contains E. In particular, taking E to be $\widehat{M}$ we obtain the *M-Lambek torsion class* λ_M. Thus

$$\lambda_M := \{N \in \sigma[M] \mid \mathrm{Hom}_R(N, \widehat{M}) = 0\}.$$

Note that $\lambda_M(M) = 0$ and $\lambda_M \geq \alpha$ whenever $\alpha \in \mathbb{L}_M$ and $\alpha(M) = 0$.

1.9 Correspondence Theorem. Suppose M is injective in $\sigma[M]$. Put $H = \mathrm{End}_R M$ and let $\mathcal{L}\{{}_RM_H\}$ denote the lattice of all fully invariant submodules (i.e., (R, H)-submodules) of M. Consider the interval $[\lambda_M, \sigma[M]]$ of $\mathbb{L}_M$. We define a map:

$$\Theta : [\lambda_M, \sigma[M]] \to \mathcal{L}\{{}_RM_H\},\ \alpha \mapsto \Theta(\alpha) := \alpha(M).$$

If $U \in \mathcal{L}\{{}_RM_H\}$ and $\alpha = \sigma[U]$ it can be shown, using the injectivity of M, that $\alpha(M) = U$. Thus Θ is onto. It is easily shown that Θ preserves arbitrary meets and joins and is thus a complete lattice epimorphism.

Now suppose M is a big (injective) cogenerator for $\sigma[M]$. Let $\alpha \in [\lambda_M, \sigma[M]]$. Note that if $N \in \mathbf{SC}(\{M\})$ then $\alpha(N) \in \mathbf{SC}(\{\alpha(M)\})$. It follows that $\sigma[\alpha(M)] \supseteq \alpha \cap \mathbf{SC}(\{M\})$. Since M is a big cogenerator, $\alpha \cap \mathbf{SC}(\{M\})$ contains every finitely generated member of α, so $\sigma[\alpha(M)] = \alpha$. This shows that α may be recovered from its image under Θ, whence Θ is one-to-one. Moreover, since M is a cogenerator for $\sigma[M]$ we must have $\lambda_M = \{0\}$. We conclude that $\Theta : \mathbb{L}_M \to \mathcal{L}\{{}_RM_H\}$ is a lattice isomorphism.

2 Basic observations

If $\mathbb{L}$ is a lattice with top element $1_{\mathbb{L}}$, then $\alpha \in \mathbb{L}$ is called *small* provided $\alpha \vee \beta = 1_{\mathbb{L}}$ implies $\beta = 1_{\mathbb{L}}$ whenever $\beta \in \mathbb{L}$.

2.1 Lemma. *For any left R-module M, the M-Lambek torsion class λ_M is small in $\mathbb{L}_M$.*

Proof. Let $\lambda_M = \sigma[K]$ for some $K \in \sigma[M]$ and assume $\sigma[K] \vee \sigma[L] = \sigma[K \oplus L] = \sigma[M]$ for some $L \in \sigma[M]$. Consider the M-injective hull $\widehat{M}$ of M. Observe that $\widehat{M}$ is $(K \oplus L)$-generated and $\mathrm{Tr}(K, \widehat{M}) = \mathrm{Tr}(\lambda_M, \widehat{M}) = 0$. Consequently,

$$\widehat{M} = \mathrm{Tr}(K \oplus L, \widehat{M}) = \mathrm{Tr}(K, \widehat{M}) + \mathrm{Tr}(L, \widehat{M}) = \mathrm{Tr}(L, \widehat{M}),$$

implying that $\sigma[K]$ is small in $\mathbb{L}_M$. □

Recall that $N \in \sigma[M]$ is called *singular in* $\sigma[M]$ (or M*-singular*) provided $N \simeq L/K$ for some $L \in \sigma[M]$ and essential submodule K of L. This notion is strongly dependent on the category $\sigma[M]$. We define

$$\delta_M = \{N \in \sigma[M] \mid N \text{ is } M\text{-singular}\}.$$

It is known that $\delta_M \in \mathbb{L}_M$ [18, 17.3, p. 138 and 17.4, p. 139] and $\delta_M \supseteq \lambda_M$ [19, 10.2, p. 72]. We call M *polyform* (or *non-*M*-singular*) if $\delta_M(M) = 0$. Observe that M will be polyform precisely if $\delta_M = \lambda_M$.

It was remarked in [11, Proposition 29] that the class of singular modules in R-Mod is small in $\mathbb{L}_R$. In general, however, δ_M need not be small in $\mathbb{L}_M$. For example, in $\mathbb{Z}$-Mod, if $M = \mathbb{Q}/\mathbb{Z}$ then $\sigma[M]$ consists of all torsion $\mathbb{Z}$-modules and these are precisely the M-singular modules. See also Examples 3.4 and 3.5.

2.2 Lemma. *Assume M is projective in $\sigma[M]$ or M is polyform. Then δ_M is small in $\mathbb{L}_M$.*

Proof. Write $\delta_M = \sigma[K]$ for some $K \in \sigma[M]$ and assume $\sigma[K] \vee \sigma[L] = \sigma[K \oplus L] = \sigma[M]$ for some $L \in \sigma[M]$. Then there exists a monomorphism $f : M \to K' \oplus L'$ where $K' \in \sigma[K]$ and $L' \in \sigma[L]$.

Composition with the canonical projections yields two maps:

$$f_K : M \xrightarrow{f} K' \oplus L' \xrightarrow{\pi'_K} K' \text{ and } f_L : M \xrightarrow{f} K' \oplus L' \xrightarrow{\pi'_L} L',$$

where $\mathrm{Ke}\, f = \mathrm{Ke}\, f_K \cap \mathrm{Ke}\, f_L = 0$.

Since M is projective in $\sigma[M]$ (or polyform) and K' is M-singular, $\mathrm{Ke}\, f_K \trianglelefteq M$. This implies $\mathrm{Ke}\, f_L = 0$ and $M \in \sigma[L]$.

We point out that the polyform case ($\delta_M = \lambda_M$) also follows from Lemma 2.1. □

It is an elementary fact that for any ideal I of R, R/I-Mod $= R$-Mod if and only if $I = 0$. If U is a fully invariant submodule of M then the statement $\sigma[M/U] = \sigma[M]$ implies $U = 0$, does not hold in general. It does, however, hold if M is projective in $\sigma[M]$, as shown in [17, Lemma 2.8, p. 3623]. Lemma 2.4 below identifies another condition sufficient for the implication to hold. We first require a preliminary result.

Recall that the smallest hereditary torsion class of $\sigma[M]$ containing δ_M is called the M*-Goldie torsion class*. It is shown in [19, 10.5, p. 74] that the M-Goldie torsion class coincides with $\delta_M :_M \delta_M$.

2.3 Lemma. *Let $U = \alpha(M)$ where α is a hereditary torsion class of $\sigma[M]$. If $\sigma[M/U] = \sigma[M]$, then U belongs to the M-Goldie torsion class.*

Proof. Let γ denote the M-Goldie torsion class and put $V = U/\gamma(U)$. Take $f \in \mathrm{Hom}_R(M/U, E(V))$. If $\mathrm{Ke}\, f$ is not essential in M/U, then M/U contains a nonzero submodule of $E(V)$. Since $\alpha(M/U) = 0$ and $V \in \alpha$, this is not possible. Thus $\mathrm{Ke}\, f \trianglelefteq M/U$, whence $\mathrm{Im}\, f \in \delta_M \subseteq \gamma$. But $\gamma(V) = 0$, so $\mathrm{Im}\, f = 0$. It follows that $\mathrm{Hom}_R(M/U, E(V)) = 0$. Since $V \in \sigma[M] = \sigma[M/U]$ we must have $\mathrm{Hom}_R(V, E(V)) = 0$, whence $V = U/\gamma(U) = 0$. We conclude that $U \in \gamma$, as required. □

2.4 Lemma. *Suppose M is polyform and $U = \alpha(M)$ for some hereditary torsion class α of $\sigma[M]$. Then $\sigma[M] = \sigma[M/U]$ if and only if $U = 0$.*

Proof. The implication in one direction is obvious. Suppose $\sigma[M] = \sigma[M/U]$ and let γ denote the M-Goldie torsion class. By Lemma 2.3, $U \in \gamma$. But M is polyform so $\delta_M(M) = 0$. Since δ_M and γ have the same associated torsion-free class, we must have $\gamma(U) \subseteq \gamma(M) = 0$. We conclude that $U = 0$, as required. □

3 Duprime modules

Interpreting [11, Theorem 14] in the case where the lattice ordered monoid is chosen to be $\mathbb{L}_M$, we obtain:

3.1 Theorem. *The following assertions are equivalent for a left R-module M:*

(a) *if $\alpha : \beta \supseteq \sigma[M]$ for $\alpha, \beta \in \mathbb{L}_R$, then $\alpha \supseteq \sigma[M]$ or $\beta \supseteq \sigma[M]$;*

(b) *if $\alpha :_M \beta = \sigma[M]$ for $\alpha, \beta \in \mathbb{L}_M$, then $\alpha = \sigma[M]$ or $\beta = \sigma[M]$;*

(c) *if $M \in \sigma[K] :_M \sigma[L]$ for $K, L \in \sigma[M]$, then $M \in \sigma[K]$ or $M \in \sigma[L]$;*

(d) *for any $\alpha \in \mathbb{L}_M$, $\alpha = \sigma[M]$ or $\sigma[M] \dot{-} \alpha = \sigma[M]$;*

(e) *for any $\alpha \in \mathbb{L}_M$, $\sigma[M/\alpha(M)]$ is equal to $\{0\}$ or $\sigma[M]$;*

(f) *for any submodule K of M, $M \in \sigma[K]$ or $M \in \sigma[M/K]$;*

(g) *for any fully invariant submodule K of M, $M \in \sigma[K]$ or $M \in \sigma[M/K]$;*

(h) *for any pretorsion submodule K of M, $M \in \sigma[K]$ or $M \in \sigma[M/K]$.*

We call M *duprime* if it satisfies the above equivalent conditions.

Proof. (a)⇒(b) is clear since $\alpha :_M \beta = (\alpha : \beta) \cap \sigma[M]$ for all $\alpha, \beta \in \mathbb{L}_M$.

(b)⇒(a) Suppose $\alpha : \beta \supseteq \sigma[M]$ with $\alpha, \beta \in \mathbb{L}_R$. Then

$$\begin{aligned} (\alpha \cap \sigma[M]) : (\beta \cap \sigma[M]) &= (\alpha : \beta) \cap (\alpha : \sigma[M]) \cap (\sigma[M] : \beta) \cap (\sigma[M] : \sigma[M]) \\ &\supseteq (\alpha : \beta) \cap \sigma[M] = \sigma[M]. \end{aligned}$$

Since $\alpha \cap \sigma[M], \beta \cap \sigma[M] \in \mathbb{L}_M$, it follows from (b) that $\alpha \cap \sigma[M] = \sigma[M]$ in which case $\alpha \supseteq \sigma[M]$, or $\beta \cap \sigma[M] = \sigma[M]$ in which case $\beta \supseteq \sigma[M]$.

(b)$\Leftrightarrow$(c) is clear since every $\alpha \in \mathbb{L}_M$ is of the form $\sigma[K]$ for some $K \in \sigma[M]$.

(b)$\Leftrightarrow$(d) is a direct consequence of [11, Theorem 14((i)$\Leftrightarrow$(iii))].

(d)$\Leftrightarrow$(e) follows from Lemma 1.6 and the fact that $\sigma[M] \dot{-} \alpha = \{0\}$ if and only if $\sigma[M] = \alpha$.

(e)$\Rightarrow$(f) Let $K \leq M$ and put $\alpha = \sigma[K]$. By hypothesis, $\sigma[M/\alpha(M)] = \{0\}$ or $\sigma[M]$. The former implies $M \in \alpha = \sigma[K]$. The latter implies $M \in \sigma[M/\alpha(M)] \subseteq \sigma[M/K]$ (because $\alpha(M) \supseteq K$).

(f)$\Rightarrow$(g)$\Rightarrow$(h) is obvious.

(h)$\Rightarrow$(e) Let $\alpha \in \mathbb{L}_M$. By hypothesis, $M \in \sigma[M/\alpha(M)]$ or $M \in \sigma[\alpha(M)] \subseteq \alpha$. The latter implies $\sigma[M/\alpha(M)] = \{0\}$. □

The results which follow reveal a rich variety of characterizations of duprime modules in the case where a finiteness condition is imposed on the lattice $\mathbb{L}_M$.

Recall that M is said to be *strongly prime* if $\alpha(M) = 0$ or $\alpha(M) = M$ for all $\alpha \in \mathbb{L}_R$. The study of strongly prime modules was initiated in Beachy-Blair [1]. It is clear from the definition that M will be strongly prime if and only if every proper element of $\mathbb{L}_M$ is contained in λ_M. Further characterizations of strongly prime modules may be found in [19, 13.3, p. 96].

It is an immediate consequence of Theorem 3.1 that every strongly prime module is duprime. In Example 3.4 we exhibit a module which is duprime but not strongly prime. Thus duprimeness is a strictly weaker notion. The reader will observe that the duprimeness of M depends only on properties of the lorrim $\mathbb{L}_M$, and in fact, if M is duprime then every subgenerator of $\sigma[M]$ inherits the same property. In contrast, strong primeness is an intrinsic property of the module M. If M is strongly prime it is not necessarily the case that every subgenerator for $\sigma[M]$ is strongly prime. However, as Theorem 3.3 shows, if M is duprime then every projective or polyform subgenerator for $\sigma[M]$ is strongly prime.

3.2 Theorem. *The following assertions are equivalent for a nonzero left R-module M:*

(a) *M is duprime and $\mathbb{L}_M$ is compact, i.e., $\sigma[M]$ has a finitely generated subgenerator;*

(b) *M is duprime and $\mathbb{L}_M$ contains coatoms;*

(c) *$\mathbb{L}_M$ is coatomic with a unique coatom and the coatom is idempotent;*

(d) *there is an idempotent $\gamma \in \mathbb{L}_M$ such that $M/\gamma(M)$ is strongly prime and subgenerates $\sigma[M]$;*

(e) *$\sigma[M]$ has a strongly prime subgenerator.*

Proof. (a)⇒(b) A routine application of Zorn's Lemma shows that every nontrivial compact lattice has coatoms.

(b)⇒(c) Let γ be a coatom of $\mathbb{L}_M$. Suppose $\alpha \in \mathbb{L}_M$ and $\alpha \not\subseteq \gamma$. It follows from the maximality of γ that $\alpha :_M \gamma \supseteq \alpha \vee \gamma = \sigma[M]$. Since M is duprime it follows from Theorem 3.1(b) that $\alpha = \sigma[M]$. This shows that $\mathbb{L}_M$ is coatomic with a unique coatom. Since M is duprime $\gamma :_M \gamma \neq \sigma[M]$. Hence $\gamma :_M \gamma = \gamma$, i.e., γ is idempotent.

(c)⇒(d) Let γ be the unique coatom of $\mathbb{L}_M$. If α is a proper element of $\mathbb{L}_M$ then $\alpha \subseteq \gamma$ and so $\alpha(M/\gamma(M)) \subseteq \gamma(M/\gamma(M)) = 0$. This shows that $M/\gamma(M)$ is strongly prime. We must also have $\sigma[M/\gamma(M)] \not\subseteq \gamma$ (because $\gamma(M/\gamma(M)) = 0$). Hence $\sigma[M/\gamma(M)] = \sigma[M]$.

(d)⇒(e) is obvious.

(e)⇒(a) Let N be a strongly prime subgenerator for $\sigma[M]$. Certainly, N is duprime and since $\sigma[N] = \sigma[M]$, M must be duprime. Let K be any nonzero finitely generated submodule of N. Since N is strongly prime, K is a subgenerator for $\sigma[N] = \sigma[M]$. We conclude that $\sigma[M]$ is compact. □

In general, the conditions: (1) M is projective in $\sigma[M]$, and (2) M is polyform, are independent. If M is duprime then condition (1) is stronger than (2). To see this, suppose M is duprime and projective in $\sigma[M]$. Note that $\sigma[M/\delta_M(M)] = \{0\}$ or $\sigma[M]$ by Theorem 3.1(e). The former implies $M/\delta_M(M) = 0$, whence $M \in \delta_M$. But this contradicts the fact that δ_M is small in $\mathbb{L}_M$ (Lemma 2.2). Consequently, we must have $\sigma[M/\delta_M(M)] = \sigma[M]$. This implies $\delta_M(M) = 0$, i.e., M is polyform, as noted in the discussion preceding Lemma 2.3. In Section 4 we shall improve on this result by showing that (1) implies (2) under conditions weaker than M duprime.

3.3 Theorem. *Assume M is projective in $\sigma[M]$ or M is polyform. Then the following assertions are equivalent:*

(a) *M is duprime;*

(b) *M is strongly prime.*

Proof. (b)⇒(a) holds with no assumption on M.

(a)⇒(b) Since M is by hypothesis duprime, M will be polyform if M is projective in $\sigma[M]$. It therefore suffices to establish (b) in the case where M is polyform.

Suppose $U = \alpha(M)$ is a proper pretorsion submodule of M. To establish the strong primeness of M we must show that $U = 0$. Since U is a proper submodule of M we cannot have $M \in \sigma[U]$. It follows from Theorem 3.1(h), that $M \in \sigma[M/U]$.

Assume U is essential in M. Then M/U is M-singular, but M is non-M-singular, so we cannot have $M \in \sigma[M/U]$, a contradiction. We conclude that U is not essential in M. Let $\overline{\alpha}$ denote the unique smallest hereditary torsion class containing α. Inasmuch as α and $\overline{\alpha}$ have the same associated torsion-free class, $\overline{\alpha}(M) \neq M$. By Theorem 3.1(h), $M \in \sigma[M/\overline{\alpha}(M)]$. Since M is polyform it follows from Lemma 2.4 that $U \subseteq \overline{\alpha}(M) = 0$, as required. □

Taking $M = {}_RR$ in Theorem 3.3 we see that ${}_RR$ is duprime precisely if R is a left strongly prime ring. This fact was observed in [11, Theorem 26 and Remark 27].

If M is duprime and polyform, in particular, if M is projective in $\sigma[M]$, then by Theorem 3.3, $\sigma[M]$ satisfies the equivalent finiteness conditions listed in Theorem 3.2.

3.4 Example. *Consider the Prüfer group $M = \mathbb{Z}_{p^\infty} \in \mathbb{Z}$-Mod, p any prime. Put*
$\alpha = \{N \in \mathbb{Z}\text{-}Mod \mid pN = 0\}$. *Then*
$2\alpha = \alpha : \alpha = \{N \in \mathbb{Z}\text{-}Mod \mid p^2N = 0\}$,
$3\alpha = \alpha : \alpha : \alpha = \{N \in \mathbb{Z}\text{-}Mod \mid p^3N = 0\}$, *etc, and*
$\sigma[M] = \bigvee_{n=1}^{\infty} n\alpha = \{N \in \mathbb{Z}\text{-}Mod \mid \forall x \in N \; \exists n \in \mathbb{N} \text{ such that } p^n x = 0\}$.
Moreover, every nonzero proper element of $\mathbb{L}_M$ is of the form $n\alpha$ for some $n \in \mathbb{N}$. The lattice $\mathbb{L}_M$ is thus a chain, isomorphic to the ordinal $\omega + 1$. It is clear that the set of proper elements of $\mathbb{L}_M$ is closed under the operation ':' so M is duprime by Theorem 3.1. Observe that M does not satisfy the finiteness conditions of Theorem 3.2 because $\mathbb{L}_M$ has no coatom. From this we can infer that $\sigma[M]$ has no projective subgenerator.

Alternatively, it is possible to deduce that M is duprime by considering only the submodule structure of M: if $K < M$ then $M/K \simeq M$, whence $\sigma[M/K] = \sigma[M]$. It follows from Theorem 3.1(f) that M is duprime.

Observe that $\delta_M = \sigma[M]$ and $\lambda_M = \{0\}$ since M cogenerates $\sigma[M]$.

3.5 Example. *It is known [16, Lemma 6, p. 24] that if R is an arbitrary left chain ring then every $\alpha \in \mathbb{L}_R$ has one of two forms:*

$$\begin{aligned} \alpha &= \{N \in R\text{-Mod} \mid IN = 0\}; \text{ or} \\ \alpha &= \{N \in R\text{-Mod} \mid (0 : x) \supset I \text{ for all } x \in N\} \end{aligned}$$

for some ideal I of R. The elements of $\mathbb{L}_R$ thus constitute a chain. Furthermore, if R is a domain and every ideal of R is idempotent, then every member of $\mathbb{L}_R$ is in fact a hereditary torsion class [13, Theorem 28, p. 5539].

Now suppose that R is a left chain domain whose only proper nonzero ideal is the Jacobson radical $J(R)$. (The existence of such rings is established in [15, Proposition

16, p. 1112] and [14, Theorem 9, p. 104].) It follows that $\mathbb{L}_R$ *contains exactly two nonzero proper members:*

$$\begin{aligned} \alpha &= \{N \in R\text{-Mod} \mid J(R)N = 0\}, \textit{ and} \\ \beta &= \{N \in R\text{-Mod} \mid (0:x) \neq 0 \textit{ for all } x \in N\}. \end{aligned}$$

Observe that α *consists of all the semisimple modules in* R-Mod *while* β *consists of all modules in* R-Mod *which are not cofaithful. Since* $\mathbb{L}_R$ *is a finite (4-element to be precise) chain all of whose members are idempotent, every nonzero left* R*-module is duprime and satisfies the finiteness conditions of Theorem 3.2.*

If M *is nonzero and semisimple then* $\sigma[M] = \alpha$ *is the unique atom of* $\mathbb{L}_R$*. In this case,* $\delta_M = \lambda_M = \{0\}$*.*

If M *is neither semisimple nor cofaithful (for example, if* $M = {}_R(R/K)$ *where* K *is a left ideal of* R *such that* $0 \neq K \subset J(R)$*), then* $\sigma[M] = \beta$*. In this case* $\delta_M = \beta$ *and if* $\alpha(M) = 0$ *then* $\lambda_M = \alpha$*, otherwise* $\lambda_M = \{0\}$*.*

4 Dusemiprime modules

Interpreting [11, Theorem 13] in the case where the lattice ordered monoid is chosen to be $\mathbb{L}_M$, we obtain the following analogue of Theorem 3.1.

4.1 Theorem. *The following assertions are equivalent for a left* R*-module* M*:*

(a) *if* $\alpha : \alpha \supseteq \sigma[M]$ *for* $\alpha \in \mathbb{L}_R$*, then* $\alpha \supseteq \sigma[M]$*;*

(b) *if* $\alpha :_M \alpha = \sigma[M]$ *for* $\alpha \in \mathbb{L}_M$*, then* $\alpha = \sigma[M]$*;*

(c) *if* $M \in \sigma[K] :_M \sigma[K]$ *for* $K \in \sigma[M]$*, then* $M \in \sigma[K]$*;*

(d) *for any* $K, L \in \sigma[M]$*,* $M \in \sigma[K] :_M \sigma[L]$ *if and only if* $M \in \sigma[K \oplus L]$*;*

(e) *for any submodule* K *of* M*,* $M \in \sigma[K \oplus M/K]$*;*

(f) *for any fully invariant submodule* K *of* M*,* $M \in \sigma[K \oplus M/K]$*;*

(g) *for any pretorsion submodule* K *of* M*,* $M \in \sigma[K \oplus M/K]$.

We call M *dusemiprime* if it satisfies the above equivalent conditions.

Proof. $(a) \Rightarrow (b)$ is clear since $\alpha :_M \alpha = (\alpha : \alpha) \cap \sigma[M]$, for all $\alpha \in \mathbb{L}_M$.

$(b) \Rightarrow (c)$ is obvious.

$(c) \Leftrightarrow (d)$ is a direct consequence of [11, Theorem 13((i)⇔(ii))]. Notice that $\sigma[K \oplus L] = \sigma[K] \vee \sigma[L]$.

$(d) \Rightarrow (e)$ Let $K \leq M$. Certainly, $K, M/K \in \sigma[M]$. Inasmuch as M is an extension of K by M/K, we must have $M \in \sigma[K] :_M \sigma[M/K]$. By hypothesis, $M \in \sigma[K \oplus M/K]$.

(e)⇒(f)⇒(g) is obvious.

(g)⇒(a) Let $\alpha \in \mathbb{L}_R$ and suppose $\alpha : \alpha \supseteq \sigma[M]$. There must exist a short exact sequence $0 \to A \to M \to B \to 0$ where $A, B \in \alpha$. Since $A \subseteq \alpha(M)$ it follows that $M/\alpha(M) \in \sigma[M/A] = \sigma[B] \subseteq \alpha$. By hypothesis, $M \in \sigma[\alpha(M) \oplus M/\alpha(M)]$. Since $\alpha(M), M/\alpha(M) \in \alpha$ we must have $M \in \alpha$, i.e., $\alpha \supseteq \sigma[M]$. □

The following result allows us to generate new examples of dusemiprime modules from old.

4.2 Proposition. *Any direct sum of dusemiprime modules is dusemiprime.*

Proof. Suppose $\{M_i \mid i \in \Gamma\}$ is a family of dusemiprime left R-modules and put $M = \bigoplus_{i\in\Gamma} M_i$. Let $\alpha \in \mathbb{L}_R$. Then $\alpha(M) = \bigoplus_{i\in\Gamma} \alpha(M_i)$ and $M/\alpha(M) \simeq \bigoplus_{i\in\Gamma}(M_i/\alpha(M_i))$. Since each M_i is dusemiprime, it follows from Theorem 4.1(g) that $M_i \in \sigma[\alpha(M_i) \oplus M_i/\alpha(M_i)]$ for all $i \in \Gamma$. Hence $M \in \sigma[\bigoplus_{i\in\Gamma} M_i] = \sigma[\bigoplus_{i\in\Gamma}(\alpha(M_i) \oplus M_i/\alpha(M_i))] = \sigma[\alpha(M) \oplus M/\alpha(M)]$. We conclude from Theorem 4.1(g) that M is dusemiprime. □

4.3 Example. *In Example 3.4 it was shown that for each prime p the Prüfer group $M = \mathbb{Z}_{p^\infty} \in \mathbb{Z}$-Mod is duprime and hence dusemiprime. Any direct sum of Prüfer groups is dusemiprime by Proposition 4.2. In particular, the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z} = \bigoplus_{\text{primes } p} \mathbb{Z}_{p^\infty}$ is dusemiprime.*

Consider the $\mathbb{Z}$-module $M = \mathbb{Q}/\mathbb{Z}$. The fully invariant submodules of M are precisely those submodules of the form $\bigoplus_{\text{primes } p} N_p$ where, for each prime p, $N_p \leq \mathbb{Z}_{p^\infty}$. (This is deduced easily from the fact that the fully invariant submodules of M coincide with the pretorsion submodules of M because M is injective.) Specifically, if U is a fully invariant submodule which is small in the lattice of submodules of M, then $U = \bigoplus_{\text{primes } p} N_p$ where, for each prime p, $N_p < \mathbb{Z}_{p^\infty}$. Observe that $M/U \simeq M$ because $\mathbb{Z}_{p^\infty}/N_p \simeq \mathbb{Z}_{p^\infty}$ whenever $N_p < \mathbb{Z}_{p^\infty}$. Consequently, $\sigma[M] = \sigma[M/U]$. The situation described here is a special case of the following more general result:

4.4 Proposition. *Let M be a self-injective dusemiprime left R-module. If U is any small, fully invariant submodule of M, then $M \in \sigma[M/U]$.*

Proof. By Theorem 4.1(f), $M \in \sigma[U \oplus M/U]$. Since M is injective in $\sigma[M]$,

$$M = \mathrm{Tr}(U \oplus M/U, M) = \mathrm{Tr}(U, M) + \mathrm{Tr}(M/U, M) = U + \mathrm{Tr}(M/U, M).$$

But U is small in M, so $U + \mathrm{Tr}(M/U, M) = \mathrm{Tr}(M/U, M)$. Hence $M \in \sigma[M/U]$. □

As was the case with duprime modules, a variety of characterizations of dusemiprime modules is obtained if a finiteness condition is imposed on $\mathbb{L}_M$.

We recall the notion of a strongly semiprime module introduced by Beidar-Wisbauer [2]. Put $H = \mathrm{End}_R \widehat{M}$. We call M *strongly semiprime* if ${}_R\widehat{M}_H$ is semisimple as an (R,H)-bimodule. We noted in the previous section that M is strongly prime if and only if M is duprime and every proper element of $\mathbb{L}_M$ is contained in λ_M, which is to say, $\mathbb{L}_M$ is coatomic and λ_M is the unique coatom of $\mathbb{L}_M$. Assertion (d) of Theorem 4.6 below provides us with an analogous characterization for strongly semiprime modules. We first recall some elementary facts on lattices.

If $\mathbb{L}$ is a complete lattice then $\mathrm{Rad}\,\mathbb{L}$ denotes the meet of all coatoms of $\mathbb{L}$. If s denotes the join of all small elements of $\mathbb{L}$ then $s \le \mathrm{Rad}\,\mathbb{L}$. Equality holds if $\mathbb{L}$ is coatomic or if $\mathbb{L}$ is modular, algebraic and for each $a \in \mathbb{L}$ the set $\{b \in \mathbb{L} \mid a \vee b = 1_{\mathbb{L}}\}$ has a unique smallest element. To see the latter, consider the interval $[s, 1_{\mathbb{L}}]$ of $\mathbb{L}$. Take $a \in [s, 1_{\mathbb{L}}]$ and let b be the unique smallest element of $\{b \in \mathbb{L} \mid a \vee b = 1_{\mathbb{L}}\}$. Using the modularity of $\mathbb{L}$ it can be shown that $a \wedge b$ is small in $\mathbb{L}$ and that $b \vee s$ is the unique complement of a in the lattice $[s, 1_{\mathbb{L}}]$. Thus $[s, 1_{\mathbb{L}}]$ is boolean. Since $\mathbb{L}$ is algebraic, every element in $[s, 1_{\mathbb{L}}]$ is a join of atoms of the lattice $[s, 1_{\mathbb{L}}]$ [9, Proposition III.5.5, p. 74]. By duality, every element in $[s, 1_{\mathbb{L}}]$ is a meet of coatoms of $[s, 1_{\mathbb{L}}]$. In particular, s is a meet of coatoms of $[s, 1_{\mathbb{L}}]$. But every coatom of $[s, 1_{\mathbb{L}}]$ is a coatom of $\mathbb{L}$, so $s \ge \mathrm{Rad}\,\mathbb{L}$, whence equality.

4.5 Proposition. *If M is dusemiprime and s denotes the join of all small elements of $\mathbb{L}$, then:*

(1) $s = \mathrm{Rad}\,\mathbb{L}_M$*; and*

(2) *if α and β are small elements of $\mathbb{L}_M$ then so is $\alpha :_M \beta$.*

Proof. (1) In view of the preceding paragraph, it suffices to show that for any $\alpha \in \mathbb{L}_M$ the set $\{\beta \in \mathbb{L}_M \mid \alpha \vee \beta = \sigma[M]\}$ has a unique smallest element. Such an element is given by $\sigma[M] \dot{-} \alpha$ because, by Theorem 4.1(d), $\alpha \vee \beta = \sigma[M]$ if and only if $\alpha :_M \beta = \sigma[M]$.

(2) Suppose $(\alpha :_M \beta) \vee \gamma = \sigma[M]$ for some $\gamma \in \mathbb{L}_M$. Certainly, $(\alpha :_M \beta) :_M \gamma = \sigma[M]$. By associativity of the operation '$:_M$', we have $\alpha :_M (\beta :_M \gamma) = \sigma[M]$. Since M is dusemiprime this implies $\alpha \vee (\beta :_M \gamma) = \sigma[M]$, whence $\beta :_M \gamma = \sigma[M]$, because α is small. Again, it follows from dusemiprimeness and the smallness of β that $\gamma = \sigma[M]$. We conclude that $\alpha :_M \beta$ is small in $\mathbb{L}_M$. □

4.6 Theorem. *The following assertions are equivalent for a left R-module M:*

(a) *M is strongly semiprime;*

(b) *the lattice of all fully invariant submodules of $\widehat{M}$ contains no proper essential element;*

(c) *the meet subsemilattice of all pretorsion submodules of* M *contains no proper essential element;*

(d) M *is dusemiprime and* $Rad\, \mathbb{L}_M = \lambda_M$.

Proof. (a)⇔(b) Let $H = \mathrm{End}_R\widehat{M}$ and $\mathcal{L}\{_R\widehat{M}_H\}$ denote the lattice of all (R,H)-submodules of $_R\widehat{M}_H$. Observe that $\mathcal{L}\{_R\widehat{M}_H\}$ coincides with the lattice of all fully invariant submodules of $\widehat{M}$. Since $\mathcal{L}\{_R\widehat{M}_H\}$ is a modular algebraic lattice, the join of all atoms of $\mathcal{L}\{_R\widehat{M}_H\}$ is equal to the meet of all essential elements of $\mathcal{L}\{_R\widehat{M}_H\}$ [9, Proposition III.6.7, p75]. Hence (b) is equivalent to the requirement that $_R\widehat{M}_H$ be a sum of simple (R,H)-submodules. The equivalence of (a) and (b) follows.

(c)⇒(b) Let U be an essential element in $\mathcal{L}\{_R\widehat{M}_H\}$. Since $\widehat{M}$ is injective in $\sigma[M]$, the pretorsion submodules and fully invariant submodules of $\widehat{M}$ coincide. Consequently, $U = \alpha(\widehat{M})$ for some $\alpha \in \mathbb{L}_M$. Note that $\alpha(M) = M \cap \alpha(\widehat{M}) = M \cap U$. If $\beta \in \mathbb{L}_M$ and $\alpha(M) \cap \beta(M) = 0$ then

$$M \cap [\beta(\widehat{M}) \cap U] = [M \cap \beta(\widehat{M})] \cap [M \cap U] = \beta(M) \cap \alpha(M) = 0.$$

Since M is an essential submodule of $\widehat{M}$, this entails $\beta(\widehat{M}) \cap U = 0$. By hypothesis, we must have $\beta(\widehat{M}) = 0$, whence $\beta(M) = 0$. It follows that $\alpha(M)$ is essential in the meet subsemilattice of all pretorsion submodules of M. By (c) we must have $\alpha(M) = M \cap U = M$, whence $U \supseteq M$ and so $\alpha \supseteq \sigma[U] = \sigma[M] = \sigma[\widehat{M}]$. It follows that $\widehat{M} \in \alpha$, whence $U = \alpha(\widehat{M}) = \widehat{M}$.

(d)⇒(c) Let $\{\rho_i \mid i \in \Lambda\}$ be the set of all coatoms of $\mathbb{L}_M$. Since λ_M is small in $\mathbb{L}_M$ (Lemma 2.1), it follows from (d) that $\mathrm{Rad}\, \mathbb{L}_M$ is small in $\mathbb{L}_M$, whence $\mathbb{L}_M$ is coatomic. Suppose $\alpha(M) \subset M$ where $\alpha \in \mathbb{L}_M$. We shall demonstrate that $\alpha(M) \cap N = 0$ for some nonzero pretorsion submodule N of M. Since $\mathbb{L}_M$ is coatomic, $\alpha \subseteq \rho_j$ for some $j \in \Lambda$. Taking $N = (\bigcap_{i \in \Lambda \setminus \{j\}} \rho_i)(M)$, we have

$$\begin{aligned} \alpha(M) \cap N &\subseteq \rho_j(M) \cap N = (\textstyle\bigcap_{i\in\Lambda} \rho_i)(M) \\ &= (\mathrm{Rad}\, \mathbb{L}_M)(M) = \lambda_M(M) = 0, \end{aligned}$$

as required.

(a)⇒(d) We first show that M is dusemiprime. Let K be a fully invariant submodule of $\widehat{M}$. Since $_R\widehat{M}_H$ is semisimple as an (R,H)-module, the lattice $\mathcal{L}\{_R\widehat{M}_H\}$ is complemented. We can therefore choose a fully invariant $L \leq \widehat{M}$ such that $\widehat{M} = K \oplus L$. It follows that $\sigma[\widehat{M}] = \sigma[K \oplus L] = \sigma[K \oplus \widehat{M}/K]$. The dusemiprimeness of M now follows from Theorem 4.1(f).

We now show that $\mathbb{L}_M$ is coatomic. Since λ_M is small in $\mathbb{L}_M$ (Lemma 2.1) it suffices to show that every proper element of the interval $[\lambda_M, \sigma[M]]$ is contained in a coatom of $\mathbb{L}_M$. By the Correspondence Theorem there is a lattice epimorphism Θ from $[\lambda_M, \sigma[M]]$ onto $\mathcal{L}\{_R\widehat{M}_H\}$. Let α be a proper element of $[\lambda_M, \sigma[M]]$. Since $\sigma[M]$

is the only element of $[\lambda_M, \sigma[M]]$ which has image $\widehat{M}$ under Θ, it follows that $\Theta(\alpha)$ is a proper element of $\mathcal{L}\{_R\widehat{M}_H\}$. Since M is strongly semiprime, $_R\widehat{M}_H$ is semisimple so $\mathcal{L}\{_R\widehat{M}_H\}$ is coatomic. Choose a coatom U of $\mathcal{L}\{_R\widehat{M}_H\}$ such that $\Theta(\alpha) \subseteq U$ and consider $\beta = \bigvee \Theta^{-1}(U) \in [\lambda_M, \sigma[M]]$. Note that $\Theta(\alpha \vee \beta) = \Theta(\alpha) + \Theta(\beta) = \Theta(\alpha) + U = U$, whence $\alpha \vee \beta \in \Theta^{-1}(U)$ and so $\alpha \vee \beta \subseteq \beta$. Hence $\alpha \subseteq \beta$. We show now that β is a coatom of $[\lambda_M, \sigma[M]]$. Suppose $\gamma \in [\lambda_M, \sigma[M]]$ and $\gamma \supsetneq \beta$. Then $\Theta(\gamma) \supseteq \Theta(\beta) = \sum \Theta[\Theta^{-1}(U)] = U$. Inasmuch as U is a coatom of $\mathcal{L}\{_R\widehat{M}_H\}$ this implies $\Theta(\gamma) = \widehat{M}$ in which case $\gamma = \sigma[M]$, or $\Theta(\gamma) = U$ in which case $\gamma \in \Theta^{-1}(U)$ and $\gamma \subseteq \beta$. We conclude that $[\lambda_M, \sigma[M]]$ and hence $\mathbb{L}_M$ is coatomic.

It remains to show that $\operatorname{Rad} \mathbb{L}_M = \lambda_M$. Certainly, since λ_M is small in $\mathbb{L}_M$ (Lemma 2.1), $\operatorname{Rad} \mathbb{L}_M \supseteq \lambda_M$. Since $\mathbb{L}_M$ is coatomic, $\operatorname{Rad} \mathbb{L}_M$ is small in $\mathbb{L}_M$ and hence in $[\lambda_M, \sigma[M]]$. Using the fact that $\sigma[M]$ is the only element of $[\lambda_M, \sigma[M]]$ which has image $\widehat{M}$ under Θ, it can be shown that $\Theta(\operatorname{Rad} \mathbb{L}_M)$ is small in $\mathcal{L}\{_R\widehat{M}_H\}$. But $\mathcal{L}\{_R\widehat{M}_H\}$ being complemented, has no nonzero small elements. Therefore $\Theta(\operatorname{Rad} \mathbb{L}_M) = 0$, whence $\operatorname{Rad} \mathbb{L}_M = \lambda_M$. □

The equivalence of assertions (a) and (d) in Theorem 4.6 can be used to show that every strongly semiprime duprime module is strongly prime. It follows that the Prüfer group $\mathbb{Z}_{p^\infty}$ of Example 3.4, being duprime but not strongly prime, cannot be strongly semiprime. This shows that the notion strongly semiprime is strictly stronger than dusemiprime.

If M is semisimple then N is a pretorsion submodule of M if and only if N is a direct sum of homogeneous components of M. From this it can be seen that the meet subsemilattice of all pretorsion submodules of M is complemented and thus satisfies Theorem 4.6(c). It follows that every semisimple module is strongly semiprime. In an attempt to generalize this result it is natural to ask whether every direct sum of strongly prime modules is strongly semiprime. In general, the answer to this question is no. As counter-example take $M = S \oplus {_R}R$ where R is a left strongly prime ring which is not semisimple and S a nonzero simple left R-module. Clearly S is the only nonzero proper pretorsion submodule of M. Hence M is not strongly semiprime by Theorem 4.6(c).

Interpreting Theorem 4.6(c) in the case where M is chosen to be a direct sum of strongly prime modules we obtain the following:

4.7 Corollary. *The following assertions are equivalent for a family $\{N_i \mid i \in \Gamma\}$ of strongly prime left R-modules:*

(a) $\bigoplus_{i\in\Gamma} N_i$ *is strongly semiprime;*

(b) *if $\alpha \in \mathbb{L}_R$ and $\alpha(\bigoplus_{i\in\Gamma} N_i) = \bigoplus_{i\in\Gamma'} N_i$ with $\Gamma' \subset \Gamma$, then there exists $\beta \in \mathbb{L}_R$ such that $\beta(\bigoplus_{i\in\Gamma} N_i) = \bigoplus_{i\in\Gamma''} N_i$ where $\emptyset \neq \Gamma'' \subseteq \Gamma \backslash \Gamma'$.*

The next theorem is the dusemiprime analogue of Theorem 3.2.

4.8 Theorem. *The following are equivalent for a nonzero left R-module M:*

(a) *M is dusemiprime and $\mathbb{L}_M$ is coatomic;*

(b) *$\mathbb{L}_M$ is coatomic and for any $\alpha \in \mathbb{L}_M$, if $\sigma[M/\alpha(M)]$ is small in $\mathbb{L}_M$ then $\alpha = \sigma[M]$;*

(c) *$\mathbb{L}_M$ is coatomic and every coatom of $\mathbb{L}_M$ is idempotent;*

(d) *if $\{\gamma_i \mid i \in \Lambda\}$ is the family of all coatoms of $\mathbb{L}_M$ then each γ_i is idempotent, $M/\gamma_i(M)$ is strongly prime and $\{M/\gamma_i(M) \mid i \in \Lambda\}$ subgenerates $\sigma[M]$;*

(e) *$\operatorname{Rad} \mathbb{L}_M$ is idempotent and $M/(\operatorname{Rad} \mathbb{L}_M)(M)$ is strongly semiprime and subgenerates $\sigma[M]$;*

(f) *$\sigma[M]$ has a strongly semiprime subgenerator.*

Proof. (a)⇔(b)⇔(c) follows directly from [11, Theorem 13((i)⇔(iv)⇔(v))]. Note that in (b), $\sigma[M/\alpha(M)]$ can be replaced by $\sigma[M] \dot{-} \alpha$ in view of Lemma 1.6.

(c)⇒(d) By hypothesis, each γ_i is idempotent. Take $i \in \Lambda$ and let $\alpha \in \mathbb{L}_M$. If $\alpha \subseteq \gamma_i$ then $\alpha(M/\gamma_i(M)) \subseteq \gamma_i(M/\gamma_i(M)) = 0$. If $\alpha \subsetneq \gamma_i$ then $\gamma_i :_M \alpha \supseteq \gamma_i \vee \alpha = \sigma[M]$ (because γ_i is a coatom). Hence $\alpha(M/\gamma_i(M)) = (\gamma_i :_M \alpha)(M)/\gamma_i(M) = M/\gamma_i(M)$. This shows that each $M/\gamma_i(M)$ is strongly prime.

We now show that $\{M/\gamma_i(M) \mid i \in \Lambda\}$ subgenerates $\sigma[M]$. If the contrary were true then $\bigvee_{i\in\Lambda} \sigma[M/\gamma_i(M)] = \bigvee_{i\in\Lambda}(\sigma[M] \dot{-} \gamma_i) \subseteq \gamma_j$ for some $j \in \Lambda$, since $\mathbb{L}_M$ is coatomic. In particular, $\gamma_j \supseteq \sigma[M] \dot{-} \gamma_j$, whence $\gamma_j :_M \gamma_j = \gamma_j = \sigma[M]$, a contradiction.

(d)⇒(e) Since the set of idempotent elements of $\mathbb{L}_M$ is closed under arbitrary meets, it follows that $\operatorname{Rad} \mathbb{L}_M = \bigcap_{i\in\Lambda} \gamma_i$ is idempotent.

Put $\overline{M} = M/(\operatorname{Rad} \mathbb{L}_M)(M)$. Inasmuch as $\gamma_i(M) \supseteq (\operatorname{Rad} \mathbb{L}_M)(M)$ for all $i \in \Lambda$ we must have $\sigma[M/\gamma_i(M)] \subseteq \sigma[\overline{M}]$ for all $i \in \Lambda$. Since $\{M/\gamma_i(M) \mid i \in \Lambda\}$ subgenerates $\sigma[M]$, it follows that $\overline{M}$ subgenerates $\sigma[M]$.

It remains to show that $\overline{M}$ is strongly semiprime. Observe that $\bigoplus_{i\in\Lambda} M/\gamma_i(M)$ is dusemiprime (by Proposition 4.2) and a subgenerator for $\sigma[M]$. Therefore $\overline{M}$ is dusemiprime. Since $\lambda_{\overline{M}}$ is small (Lemma 2.1) it follows from Proposition 4.5 that $\lambda_{\overline{M}} \subseteq \operatorname{Rad} \mathbb{L}_{\overline{M}}$. But $\operatorname{Rad} \mathbb{L}_{\overline{M}} = \operatorname{Rad} \mathbb{L}_M$ is idempotent, whence $(\operatorname{Rad} \mathbb{L}_{\overline{M}})(\overline{M}) = 0$ and so $\operatorname{Rad} \mathbb{L}_{\overline{M}} \subseteq \lambda_{\overline{M}}$. We conclude that $\operatorname{Rad} \mathbb{L}_{\overline{M}} = \lambda_{\overline{M}}$. By Theorem 4.6((a)⇐(d)), $\overline{M}$ is strongly semiprime.

(e)⇒(f) is obvious.

(f)⇒(a) Let N be a strongly semiprime subgenerator for $\sigma[M]$. By Theorem 4.6((a)⇒(d)), N is dusemiprime and $\operatorname{Rad} \mathbb{L}_N = \lambda_N$. By Lemma 2.1, $\operatorname{Rad} \mathbb{L}_N$ is small in $\mathbb{L}_N$, whence $\mathbb{L}_N$ is coatomic. We conclude that M is dusemiprime and $\mathbb{L}_M$ is coatomic. □

Observe that the equivalent assertions listed in Theorem 4.9 below are stronger than those of Theorem 4.8.

4.9 Theorem. *The following assertions are equivalent for a nonzero left R-module M:*

(a) *M is dusemiprime and $\mathbb{L}_M$ is compact;*

(b) *$\mathbb{L}_M$ is coatomic and $\mathbb{L}_M$ has finitely many coatoms and each coatom of $\mathbb{L}_M$ is idempotent;*

(c) *$\sigma[M]$ has a finitely generated strongly semiprime subgenerator.*

Proof. (a)⇒(b) Since $\mathbb{L}_M$ is compact, $\mathbb{L}_M$ is coatomic. The remainder of assertion (b) follows from [11, Theorem 24((i)⇒(iii))] taking the lattice ordered monoid to be $\mathbb{L}_M$.

(b)⇒(c) A routine exercise shows that every coatomic lattice with only finitely many coatoms is compact. It follows that $\sigma[M]$ has a finitely generated subgenerator N, say. Since $\mathbb{L}_N$ is compact, $\mathbb{L}_N$ is coatomic. Clearly, $N/(\operatorname{Rad}\mathbb{L}_N)(N)$ is finitely generated and also a strongly semiprime subgenerator for $\sigma[N]$ by Theorem 4.8((a)⇒(e)).

(c)⇒(a) Inasmuch as $\sigma[M]$ has a finitely generated subgenerator, $\mathbb{L}_M$ is compact. The dusemiprimeness of M follows from Theorem 4.8((a)⇐(f)). □

We noted earlier that if $\{N_i \mid i \in \Gamma\}$ is a family of strongly prime modules then $N = \bigoplus_{i\in\Gamma} N_i$ need not be strongly semiprime. In fact, as the following example shows, even the weaker coatomicity of $\mathbb{L}_N$ is not guaranteed.

4.10 Example. *Let R be a left chain domain all of whose ideals are idempotent and with the property that R contains no smallest nonzero ideal. Such a ring R exists by [14, Theorem 9, p104]. As noted in Example 3.5, $\mathbb{L}_R$ is a chain with unique coatom $\beta = \{N \in R\text{-Mod} \mid (0 : x) \neq 0 \text{ for all } x \in N\}$. Since R contains no smallest nonzero ideal it is easily seen that β has no predecessor in $\mathbb{L}_R$.*

We show now that β is subgenerated by a direct sum of strongly prime modules. Let $\mathcal{I}$ be the set of all proper nonzero ideals of R. By [13, Proposition 27, p5539] every element of $\mathcal{I}$ is completely prime. It follows that R/I is a domain for all $I \in \mathcal{I}$. This means that the ring R/I is left (and right) strongly prime, whence ${}_R(R/I)$ is a strongly prime module for all $I \in \mathcal{I}$. Put $N = \bigoplus_{I\in\mathcal{I}} {}_R(R/I)$. Since N is not cofaithful, $N \in \beta$. Since $\mathcal{I}$ contains no smallest element, N subgenerates β.

Observe that $\mathbb{L}_N$ is not coatomic because $\sigma[N]$ has no predecessor in $\mathbb{L}_R$.

4.11 Lemma. *Suppose δ_M, the M-singular hereditary pretorsion class, is small in $\mathbb{L}_M$. Then the following assertions are equivalent:*

(a) *M is dusemiprime;*

(b) *for all essential submodules K of M, $\sigma[K] = \sigma[M]$.*

Proof. (a)⇒(b) Let $K \trianglelefteq M$. Then $M/K \in \delta_M$. Clearly

$$M \in \sigma[K] :_M \sigma[M/K] \subseteq \sigma[K] :_M \delta_M.$$

By Theorem 4.1(d), $\sigma[M] = \sigma[K] \vee \delta_M$. By hypothesis, δ_M is small, so $\sigma[K] = \sigma[M]$.

(b)⇒(a) Let $K \leq M$. Choose $L \leq M$ maximal such that $K \cap L = 0$. Then

$$\sigma[K \oplus M/K] = \sigma[K] \vee \sigma[M/K] \geq \sigma[K] \vee \sigma[L] = \sigma[K \oplus L] = \sigma[M]$$

(by (b)). It follows from Theorem 4.1(e) that M is dusemiprime. □

4.12 Lemma. *If M is dusemiprime and projective in $\sigma[M]$ then M is polyform.*

Proof. By Theorem 4.1, $\sigma[M] = \sigma[\delta_M(M) \oplus M/\delta_M(M)] \subseteq \delta_M \vee \sigma[M/\delta_M(M)]$. But δ_M is small in $\mathbb{L}_M$ by Lemma 2.2, so $\sigma[M] = \sigma[M/\delta_M(M)]$. Inasmuch as M is projective in $\sigma[M]$, this implies $\delta_M(M) = 0$, i.e., M is polyform [17, Lemma 2.8, p. 3623]. □

4.13 Theorem. *Assume M is projective in $\sigma[M]$ or M is polyform. Then the following assertions are equivalent:*

(a) *M is strongly semiprime;*

(b) *M is dusemiprime;*

(c) *for all essential submodules K of M, $\sigma[K] = \sigma[M]$.*

Proof. (a)⇒(b) is obvious.

(b)⇔(c) follows from Lemma 4.11 since δ_M is small in $\mathbb{L}_M$ by Lemma 2.2.

(c)⇒(a) It is clear from the proof of Lemma 4.11((b)⇒(a)), that (c) is at least as strong as (b) in the absence of any assumption about M. Consequently, by Lemma 4.12, M projective in $\sigma[M]$ implies M is polyform. It suffices therefore to establish (a) in the case where M is assumed to be polyform.

Let α be a small element of $\mathbb{L}_M$. Choose $K \leq M$ maximal such that $\alpha(M) \cap K = 0$. By (c), $\sigma[M] = \sigma[\alpha(M) \oplus K] \subseteq \alpha \vee \sigma[K]$. But α is small in $\mathbb{L}_M$ so we must have $\sigma[M] = \sigma[K]$. Let $\overline{\alpha}$ denote the smallest hereditary torsion class containing α. Inasmuch as α and $\overline{\alpha}$ have the same associated torsion-free class, it follows that $\overline{\alpha}(M) \cap K = 0$. Hence $\sigma[M] = \sigma[K] \subseteq \sigma[M/\overline{\alpha}(M)]$. By Lemma 2.4, $\alpha(M) \subseteq \overline{\alpha}(M) = 0$. We conclude that $\alpha \subseteq \lambda_M$.

Since λ_M contains every small element of $\mathbb{L}_M$ it follows from Proposition 4.5 that $\operatorname{Rad} \mathbb{L}_M = \lambda_M$. By Theorem 4.6((a)⇐(d)), M is strongly semiprime. □

Taking $M = {}_RR$ in Theorem 4.13, we see that ${}_RR$ is dusemiprime precisely if R is a left strongly semiprime ring. This fact was observed in [11, Theorem 32 and Remark 33].

4.14 Example. *Let A be any (nonassociative) algebra and consider it as a left module over its multiplication algebra $M(A)$ [19, p. 6]. Consider the subcategory $\sigma[{}_{M(A)}A]$ of $M(A)$-*Mod.

(a) *If A is semiprime then ${}_{M(A)}A$ is polyform [19, 32.1, p. 262] and so, by Theorem 4.13, ${}_{M(A)}A$ is dusemiprime if and only if ${}_{M(A)}A$ is strongly semiprime.*

(b) *If A is a direct sum of (possibly nilpotent) simple algebras then ${}_{M(A)}A$, being semisimple, is necessarily dusemiprime. Observe that A, regarded as an algebra, is not necessarily semiprime.*

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