IKEDA-NAKAYAMA MODULES

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Abstract. Let ${}_SM_R$ be an (S,R)-bimodule and denote $\mathbf{l}_S(A) = \{s \in S : sA = 0\}$ for any submodule A of M_R. Extending the notion of an *Ikeda-Nakayama ring*, we investigate the condition $\mathbf{l}_S(A \cap B) = \mathbf{l}_S(A) + \mathbf{l}_S(B)$ for any submodules A, B of M_R. Various characterizations and properties are derived for modules with this property. In particular, for $S = End(M_R)$, the π-injective modules are those modules M_R for which $S = \mathbf{l}_S(A) + \mathbf{l}_S(B)$ whenever $A \cap B = 0$, and our techniques also lead to some new results on these modules.

1 Annihilator conditions

Let R and S be rings and ${}_SM_R$ be a bimodule. For any $X \subseteq M$ and any $T \subseteq S$, denote

$$\mathbf{l}_S(X) = \{s \in S : sX = 0\} \quad \text{and} \quad \mathbf{r}_M(T) = \{m \in M : Tm = 0\}.$$

There is a canonical ring homomorphism $\lambda : S \longrightarrow End(M_R)$ given by $\lambda(s)(x) = sx$ for $x \in M$ and $s \in S$. For any submodules A and B of M_R and any $t \in \mathbf{l}_S(A \cap B)$, define

$$\alpha_t : A + B \to M, \quad a + b \mapsto ta.$$

Clearly, α_t is a well-defined R-homomorphism.

Lemma 1. *Let ${}_SM_R$ be a bimodule and A, B be submodules of M_R. The following are equivalent:*

1. $\mathbf{l}_S(A \cap B) = \mathbf{l}_S(A) + \mathbf{l}_S(B)$.
2. *For any $t \in \mathbf{l}_S(A \cap B)$, the diagram*

$$\begin{array}{ccccc} 0 & \to & A+B & \to & M \\ & & \downarrow \alpha_t & & \\ & & M & & \end{array}$$

can be extended commutatively by $\lambda(s)$, for some $s \in S$.

Proof. (1) $\Rightarrow$ (2). Suppose (1) holds. For A, B, t given as in (2), write $t = u + v$ where $u \in \mathbf{l}_S(A)$ and $v \in \mathbf{l}_S(B)$. Then, for all $a \in A$ and $b \in B$,

$$\alpha_t(a + b) = ta = (u + v)a = va = v(a + b) = \lambda(v)(a + b).$$

(2) $\Rightarrow$ (1). It is clear that $\mathbf{l}_S(A \cap B) \supseteq \mathbf{l}_S(A) + \mathbf{l}_S(B)$. Let $t \in \mathbf{l}_S(A \cap B)$. Define $\alpha_t : A + B \longrightarrow M$ as above. By (2), there exists $s \in S$ such that $\lambda(s)$ extends α_t .

1991 *Mathematics Subject Classification.* Primary 16D50; secondary 16L60.
The research was supported in part by NSERC of Canada and a grant from Ohio State University.

Thus, for all $a \in A$ and $b \in B$, $ta = \alpha_t(a+b) = \lambda(s)(a+b) = s(a+b)$. It follows that $(t-s)a + (-s)b = 0$ for all $a \in A$ and $b \in B$. So, $t - s \in \mathbf{l}_S(A)$ and $-s \in \mathbf{l}_S(B)$, and hence $t = (t-s) - (-s) \in \mathbf{l}_S(A) + \mathbf{l}_S(B)$. □

Lemma 2. *Let ${}_SM_R$ be a bimodule and A, B be submodules of M_R such that $A \cap B = 0$. The following are equivalent:*

1. $S = \mathbf{l}_S(A) + \mathbf{l}_S(B)$.
2. *The diagram*

$$\begin{array}{ccccc} 0 & \rightarrow & A+B & \rightarrow & M \\ & & \downarrow \alpha_1 & & \\ & & M & & \end{array}$$

can be extended commutatively by $\lambda(s)$, for some $s \in S$.

Proof. (1) ⇒ (2). Apply Lemma 1 with $t = 1$.

(2) ⇒ (1). It suffices to show that $1 \in \mathbf{l}_S(A) + \mathbf{l}_S(B)$. Note that $\alpha_1 : A + B \longrightarrow M$ is given by $\alpha_1(a+b) = a$ ($a \in A$ and $b \in B$). By (2), there exists $s \in S$ such that $\lambda(s)$ extends α_1 . Arguing as in the proof of '(2) ⇒ (1)' of Lemma 1, we have $1 = (1-s) - (-s) \in \mathbf{l}_S(A) + \mathbf{l}_S(B)$. □

Lemma 3. *Let ${}_SM_R$ be a bimodule such that ${}_SM$ is faithful and A, B be complements of each other in M_R. The following are equivalent:*

1. $S = \mathbf{l}_S(A) + \mathbf{l}_S(B)$.
2. $S = \mathbf{l}_S(A) \oplus \mathbf{l}_S(B)$.
3. $M = A \oplus B$ *and, for the projection f of M onto A along B, $f = \lambda(s)$ for some $s \in S$.*

Proof. (1) ⇒ (3). By (1), we have $S = \mathbf{l}_S(A) + \mathbf{l}_S(B)$. Write $1_S = u + v$ where $u \in \mathbf{l}_S(A)$ and $v \in \mathbf{l}_S(B)$. It follows that $a = va$ for all $a \in A$, $b = ub$ for all $b \in B$ and $vB = uA = 0$. Thus, $B \subseteq \mathbf{r}_M(v) \subseteq \mathbf{r}_M(v^2)$ and $\mathbf{r}_M(v^2) \cap A = 0$. Since B is complement of A in M_R, we have $B = \mathbf{r}_M(v) = \mathbf{r}_M(v^2)$. Similarly, $A = \mathbf{r}_M(u) = \mathbf{r}_M(u^2)$. Next we show that $(vu)M \cap (A+B) = 0$. For any $z \in (vu)M \cap (A+B)$, write $z = vux = a + b$, where $x \in M$, $a \in A$ and $b \in B$. Noting that $vu = uv$, we have that $(v^2u^2)x = (vu)(a+b) = 0$. So, $u^2x \in \mathbf{r}_M(v^2) = \mathbf{r}_M(v)$, and this gives that $u^2vx = vu^2x = 0$. So, $vx \in \mathbf{r}_M(u^2) = \mathbf{r}_M(u)$. Thus, $z = vux = uvx = 0$. So, $(vu)M \cap (A+B) = 0$. Since $A + B$ is essential in M_R, $(vu)M = 0$, and hence $vu = 0$ since ${}_SM$ is faithful. So, $uM \subseteq \mathbf{r}_M(v) = B$ and $vM \subseteq \mathbf{r}_M(u) = A$, and hence $M = vM + uM = A + B = A \oplus B$.

Let f be the projection of M onto A along B. Then $f(M) = A$ and $(1-f)(M) = B$. Noting that ${}_SM$ is faithful, we have $\mathbf{l}_S(A) = \mathbf{l}_S(f(M)) = \{s \in S : \lambda(s)f(M) = 0\} = \{s \in S : \lambda(s)f = 0\}$ and $\mathbf{l}_S(B) = \mathbf{l}_S((1-f)(M)) = \{s \in S : \lambda(s)(1-f) = 0\}$. Thus, $\lambda(u)f = 0$ and $\lambda(v)(1-f) = 0$. It follows that

$$0 = \lambda(v)(1-f) = \lambda(1-u)(1-f) = (1-\lambda(u))(1-f) = 1 - f - \lambda(u),$$

and thus $f = 1 - \lambda(u) = \lambda(1-u) = \lambda(v)$.

(3) ⇒ (2). By (3), $M = A \oplus B$. Let f be the projection of M onto A along B. Then $f^2 = f \in End(M_R)$, $A = f(M)$ and $B = (1-f)(M)$. By (3), $f = \lambda(s)$ for

some $s \in S$. It follows that $(s^2 - s)M = \lambda(s^2 - s)(M) = (f^2 - f)(M) = 0$. So, $s^2 = s$, since ${}_SM$ is faithful. And so,

$$\mathbf{l}_S(A) = \mathbf{l}_S(f(M)) = \mathbf{l}_S(sM) = \mathbf{l}_S(s) = S(1 - s),$$

and, similarly, $\mathbf{l}_S(B) = Ss$. Thus, $S = \mathbf{l}_S(A) \oplus \mathbf{l}_S(B)$.

(2) $\Rightarrow$ (1). Obvious. □

A module M_R is called *π-injective* (or *quasi-continuous*) if every submodule is essential in a direct summand (C1) and, for any two direct summands M_1, M_2 with $M_1 \cap M_2 = 0$, $M_1 \oplus M_2$ is also a direct summand (C3) (see [8]). It is known that M_R is π-injective if and only if $M = A \oplus B$ whenever A and B are complements of each other in M_R (see [8, Theorem 2.8]).

Corollary 4. *Let ${}_SM_R$ be a bimodule such that ${}_SM$ is faithful. The following are equivalent:*

1. *For any submodules A and B of M_R with $A \cap B = 0$, $S = \mathbf{l}_S(A) + \mathbf{l}_S(B)$.*
2. *If A and B are complements of each other in M_R, then $S = \mathbf{l}_S(A) + \mathbf{l}_S(B)$.*
3. *If A and B are complements of each other in M_R, then $S = \mathbf{l}_S(A) \oplus \mathbf{l}_S(B)$.*
4. *M is π-injective and, for any $f^2 = f \in End(M_R)$, $f = \lambda(s)$ for some $s \in S$.*

Proof. (1) $\Leftrightarrow$ (2) is obvious, and (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) is by Lemma 3. □

For submodules A, B of M_R, let

$$\pi : M/(A \cap B) \to M/A \oplus M/B, \quad m + (A \cap B) \mapsto (m + A, m + B)$$

be the canonical R-homomorphism. The next Lemma can easily be verified.

Lemma 5. *Let M_R be an R-module with $S = End(M_R)$ and A, B be submodules of M_R. The following are equivalent:*

1. $\mathbf{l}_S(A \cap B) = \mathbf{l}_S(A) + \mathbf{l}_S(B)$.
2. *For any R-homomorphism $f : M/(A \cap B) \longrightarrow M$, the diagram*

$$\begin{array}{ccccc} 0 & \to & M/(A \cap B) & \xrightarrow{\pi} & M/A \oplus M/B \\ & & \downarrow f & & \\ & & M & & \end{array}$$

can be extended commutatively by some $g : M/A \oplus M/B \longrightarrow M$.

2 Ikeda-Nakayama modules

A well known result of Ikeda and Nakayama [6] says that every right self-injective ring R satisfies the so called *Ikeda-Nakayama annihilator condition*, i.e., $\mathbf{l}_R(A \cap B) = \mathbf{l}_R(A) + \mathbf{l}_R(B)$ for all right ideals A, B of R. Rings with the Ikeda-Nakayama annihilator condition, called *right Ikeda-Nakayama rings*, were studied in [2]. Extending this notion we call M_R an *Ikeda-Nakayama module* (*IN-module*) if

$$\mathbf{l}_S(A \cap B) = \mathbf{l}_S(A) + \mathbf{l}_S(B)$$

for any submodules A and B of M_R where $S = End(M_R)$. Clearly, every quasi-injective module is an IN-module (Lemma 1) and every IN-module is π-injective (Corollary 4).

Proposition 6. *The following are equivalent for a module M_R with $S = End(M_R)$:*

1. *M_R is an IN-module.*
2. *For any finite set $\{A_i : i = 1, \cdots, n\}$ of submodules of M_R,*
$$\mathbf{l}_S(A_1 \cap \cdots \cap A_n) = \mathbf{l}_S(A_1) + \cdots + \mathbf{l}_S(A_n).$$
3. *For any submodules A, B of M_R and any $f \in S$ with $f(A \cap B) = 0$, the diagram*
$$\begin{array}{ccccc} 0 & \to & A+B & \to & M \\ & & \downarrow \alpha_f & & \\ & & M & & \end{array}$$
can be extended commutatively by some $g : M \longrightarrow M$.
4. *For any submodules A, B of M_R and any R-homomorphism $f : M/(A \cap B) \longrightarrow M$, the diagram*
$$\begin{array}{ccccc} 0 & \to & M/(A \cap B) & \xrightarrow{\pi} & M/A \oplus M/B \\ & & \downarrow f & & \\ & & M & & \end{array}$$
can be extended commutatively by some $g : M/A \oplus M/B \longrightarrow M$.

Proof. (1) $\Rightarrow$ (2) can be easily proved by using induction on n; (2) $\Rightarrow$ (1) is obvious; (1) $\Leftrightarrow$ (3) is by Lemma 1; and (1) $\Leftrightarrow$ (4) is by Lemma 5. $\square$

Remark 7. The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) in Proposition 6 can be proved to hold for an arbitrary bimodule ${}_SM_R$.

Many characterizations of π-injective modules are given in [13, 41.21& 41.23]. In particular, the equivalence "(1) $\Leftrightarrow$ (2)" of the next theorem is contained in [13, 41.21].

Theorem 8. *The following are equivalent for a module M_R with $S = End(M_R)$:*

1. *M is π-injective.*
2. *For any submodules A and B of M_R with $A \cap B = 0$, $S = \mathbf{l}_S(A) + \mathbf{l}_S(B)$.*
3. *For any submodules A and B of M_R with $A \cap B = 0$ and any $f \in S$, the diagram*
$$\begin{array}{ccccc} 0 & \to & A+B & \to & M \\ & & \downarrow \alpha_f & & \\ & & M & & \end{array}$$
can be extended commutatively by some $g : M \longrightarrow M$.
4. *For any submodules A, B of M_R with $A \cap B = 0$, the diagram*
$$\begin{array}{ccccc} 0 & \to & A+B & \to & M \\ & & \downarrow \alpha_1 & & \\ & & M & & \end{array}$$
can be extended commutatively by some $g : M \longrightarrow M$.

5. *For any submodules* A, B *of* M_R *with* $A \cap B = 0$ *and any* $f \in S$*, the diagram*

$$\begin{array}{cccc} 0 & \rightarrow & M & \xrightarrow{\pi} & M/A \oplus M/B \\ & & \downarrow f & & \\ & & M & & \end{array}$$

can be extended commutatively by some $g : M/A \oplus M/B \longrightarrow M$.

6. *For any submodules* A *and* B *of* M_R *with* $A \cap B = 0$, $S_0 = \mathbf{l}_{S_0}(A) + \mathbf{l}_{S_0}(B)$ *where* S_0 *is the subring of* S *generated by all idempotents of* S.
7. *If* A *and* B *are complements of each other in* M_R*, then* $S = \mathbf{l}_S(A) \oplus \mathbf{l}_S(B)$.

In each of the conditions (2)-(6), the pair A, B *of submodules with* $A \cap B = 0$ *can be replaced by a pair* A, B *of submodules such that they are complements of each other in* M_R.

Proof. (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5). By Lemmas 1, 2 and 5.

(1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (7). By Corollary 4.

(1) $\Leftrightarrow$ (6). Apply Corollary 4 to the bimodule ${}_{S_0}M_R$. □

One condition in the equivalence list of Theorem 8 says that, if A, B are complements of each other in M_R, then the map $\alpha_1 : A \oplus B \longrightarrow M$ given by $\alpha_1(a + b) = a$ extends to M. This is an improvement of a result of Smith and Tercan [11, Thm.4] where it was proved that M_R is π-injective if and only if M satisfies (P_2), i.e., if A and B are complement submodules of M with $A \cap B = 0$, then every map from $A \oplus B$ to M extends to M.

Remark 9. Two modules X and Y are said to be *orthogonal* and written $X \perp Y$ if they have no nonzero isomorphic submodules. A submodule N of the module M is called a *type submodule* if, whenever $N \subset P \subseteq M$, there exists $0 \neq X \subseteq P$ such that $N \perp X$. Two submodules X and Y of M are said to be *type complements of each other in* M if they are complements of each other in M such that $X \perp Y$. The module M is called TS if each of its type submodules is a direct summand of M. The module M is said to satisfy (T_3) if, whenever X and Y are type submodules as well as direct summands such that $X \oplus Y$ is essential in M, $X \oplus Y = M$. As shown in [14], a module M satisfies both TS and (T_3) if and only if, whenever A, B are type complements of each other in M, $M = A \oplus B$. The module satisfying TS and (T_3) can be regarded as the 'type' analogue of the notion of π-injective modules. Several characterizations of this 'type' analogue of π-injective modules have been obtained in [14]. Some new characterizations of this notion can be obtained by restating Theorem 8 with '$A \cap B = 0$' being replaced by '$A \perp B$', 'A, B are complements of each other in M' replaced by 'A, B are type complements of each other in M', and "all idempotents of S" by "all idempotents f with $f(M) \perp Ker(f)$".

Proposition 10. *Let* C *be the center of* $End(M_R)$*. The following are equivalent:*

1. *For any submodules* A, B *of* M_R *with* $A \cap B = 0$, $C = \mathbf{l}_C(A) + \mathbf{l}_C(B)$.
2. M_R *is* π*-injective and every idempotent of* $End(M_R)$ *is central.*
3. M_R *is* π*-injective and every direct summand of* M_R *is fully invariant.*

Proof. (1) $\Leftrightarrow$ (2). Apply Corollary 4 to the bimodule ${}_CM_R$.

(2) $\Rightarrow$ (3). Let X be a direct summand of M_R. Then $X = f(M)$ for some $f^2 = f \in End(M_R)$. For any $g \in End(M_R)$, since f is central by (2), $g(X) = g(f(M)) = f(g(M)) \subseteq f(M) = X$. This shows that X is a fully invariant submodule of M_R.

(3) $\Rightarrow$ (2). Let $f, g \in End(M_R)$ with $f^2 = f$. By (3), $g(f(M)) \subseteq f(M)$ and $g((1-f)(M)) \subseteq (1-f)(M)$. It follows that $fgf = gf$ and $(1-f)g(1-f) = g(1-f)$. Thus, $g - gf = g(1-f) = (1-f)g(1-f) = g - gf - fg + fgf = g - gf - fg + gf = g - fg$. This shows that $fg = gf$. $\square$

3 Applications

In the rest of the paper, we discuss some applications of Theorem 8. Recall that a module M is called *continuous* if (C1) holds and every submodule isomorphic to a direct summand is itself a direct summand of M (C2). As a generalization of (C2)-condition, a module M_R is called *GC2* if, for any submodule N of M_R with $N \cong M$, N is a summand of M. Note that if R is the 2×2 upper triangular matrix ring over a field, then R_R satisfies both (C1) and (GC2) but it does not satisfy (C3).

Proposition 11. *Let M_R be a module with $S = End(M_R)$. The following are equivalent:*

1. *For any family $\{A_i : i \in I\}$ of submodules of M_R with $\cap_{i\in I} A_i = 0$, $S = \Sigma_{i\in I}\mathbf{l}_S(A_i)$.*
2. *M_R is finitely cogenerated and, for any finite family $\{A_i : i = 1, \cdots, n\}$ of submodules of M_R with $\cap_{i=1}^n A_i = 0$, the map*
$$M \xrightarrow{h} \oplus_{i=1}^n M/A_i, \quad m \mapsto (m + A_1, \cdots, m + A_n),$$
splits.
3. *M_R is finitely cogenerated and, for any finite family $\{A_i : i = 1, \cdots, n\}$ of submodules of M_R with $\cap_{i=1}^n A_i = 0$, $S = \Sigma_{i=1}^n \mathbf{l}_S(A_i)$.*

If M_R satisfies both (1) *and (GC2), then M_R is continuous and S is semiperfect.*

Proof. It is straightforward to verify the equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

Suppose that M_R satisfies both (1) and (GC2). By Theorem 8, M_R is π-injective. Thus, by [8, Lemma 3.14], M is continuous. To show that S is semilocal, let $\sigma : M \longrightarrow M$ be a monomorphism. Then $M = \sigma(M) \oplus N$ for some $N \subseteq M$ (by the GC2-condition). It must be that $N = 0$ since M is finite dimensional (indeed, finitely cogenerated). So, σ is an isomorphism. Therefore, M satisfies the assumptions in Camps-Dicks [3, Thm.5], and so $End(M)$ is semilocal. But, by [8, Prop.3.5 & Lemma 3.7], idempotents of $S/J(S)$ lift to idempotents of S, and thus S is semiperfect. $\square$

A ring R is called *right Kasch* if every simple right R-module embeds in R_R, or equivalently if $\mathbf{l}(I) \neq 0$ for any maximal right ideal I of R.

Corollary 12. *If R satisfies the condition that, for any set $\{A_i : i \in I\}$ of right ideals such that $\cap_{i\in I} A_i = 0$, $R = \Sigma_{i\in I}\mathbf{l}_R(A_i)$ and R_R satisfies (GC2), then R is a semiperfect right continuous ring with a finitely generated essential right socle. In particular, R is left and right Kasch.*

Proof. The first part follows from Theorem 11. The second part is by [9, Lemma 4.16]. □

A ring R is called *strongly right IN* if, for any set $\{A_i : i \in I\}$ of right ideals, $\mathbf{l}_R(\cap_{i\in I} A_i) = \Sigma_{i\in I}\mathbf{l}_R(A_i)$. The ring R is called *right dual* if every right ideal of R is a right annihilator. It is well-known that every two-sided dual ring is strongly left and right IN.

Corollary 13. *The following are equivalent for a ring R:*

1. *R is a two-sided dual ring.*
2. *R is strongly left and right IN, and left (or right) GC2.*
3. *R is left and right finitely cogenerated, left and right IN, and left (or right) GC2.*

Proof. (1) $\Rightarrow$ (2). Obvious.

(2) $\Rightarrow$ (3). It is clear by Corollary 12.

(3) $\Rightarrow$ (1). Suppose $\cap_{i\in I} A_i = 0$ where all A_i are right ideals R. Since R is right finitely cogenerated, $\cap_{i\in F} A_i = 0$ where F is a finite subset of I. Thus, $R = \mathbf{l}_R(\cap_{i\in F} A_i) = \Sigma_{i\in F}\mathbf{l}_R(A_i)$ because of the IN-condition, and hence $R = \Sigma_{i\in I}\mathbf{l}_R(A_i)$. By Corollary 12, R is left and right Kasch. Since R is left and right IN, it follows from [2, Lemma 9] that R is a two-sided dual ring. □

The GC2-condition in Corollary 12 and in Corollary 13(3) can not be removed. To see this, let R be the trivial extension of $\mathbb{Z}$ and the $\mathbb{Z}$-module $\mathbb{Z}_{2^\infty}$. Then R has an essential minimal ideal, so R is finitely cogenerated and, for any set $\{A_i : i \in I\}$ of right ideals of R, $R = \Sigma_{i\in I}\mathbf{l}_R(A_i)$. Moreover, R is IN. But R contains non-zero divisors which are not invertible, so R is not GC2. Clearly, R is not Kasch, so it is not semiperfect by Corollary 12. We do not know if the GC2-condition can be removed in Corollary 13(2).

Proposition 14. *Suppose every finitely generated left ideal of R is a left annihilator. Then the following are equivalent:*

1. *Every closed right ideal of R is a right annihilator of a finite subset of R.*
2. *R_R satisfies (C1).*
3. *R is right continuous.*

Proof. (3) $\Rightarrow$ (2). Obvious.

(2) $\Rightarrow$ (1). If I_R is closed in R_R, then $I = eR$ for some $e^2 = e \in R$. Hence $I = \mathbf{r}(1-e)$.

(1) $\Rightarrow$ (2). Let I_R and K_R be complements of each other in R_R. Then, by (1), $I = \mathbf{r}_R(a_1, \cdots, a_n)$ and $K = \mathbf{r}_R(b_1, \cdots, b_m)$ where $a_i, b_j \in R$. Thus,

$$\begin{aligned} R &= \mathbf{l}_R(I \cap K) = \mathbf{l}_R[\mathbf{r}_R(a_1, \cdots, a_n) \cap \mathbf{r}_R(b_1, \cdots, b_m)] \\ &= \mathbf{l}_R(\mathbf{r}_R(\Sigma_{i=1}^n Ra_i + \Sigma_{j=1}^m Rb_j)) = \Sigma_{i=1}^n Ra_i + \Sigma_{j=1}^m Rb_j \\ &= \mathbf{l}_R(I) + \mathbf{l}_R(K). \end{aligned}$$

Thus, by Theorem 8, R_R is π-injective, and in particular R_R satisfies (C1).

(2) $\Rightarrow$ (3). Since $\mathbf{r}_R(\mathbf{l}_R(F)) = F$ for all finitely generated left ideals F of R, R is right P-injective, and hence satisfies the right C2-condition. Thus, R is right continuous. $\square$

A ring R is called a *right CF-ring* (resp. *right FGF-ring*) if every cyclic (resp. finitely generated) right R-module embeds in a free module. The ring R is called *right FP-injective* if every R-homomorphism from a finitely generated submodule of a free right R-module F into R extends to F. Note that every right self-injective ring is right FP-injective, but not conversely. Also every finitely generated left ideal of a right FP-injective ring is a left annihilator (see [7]). The well known FGF problem asks whether every right FGF-ring is QF. It is known that every right self-injective, right FGF-ring is QF. In fact, Björk [1] and Tolskaya [12] independently proved that every right self-injective, right CF-ring is QF. On the other hand, Nicholson-Yousif [10, Theorem 4.3] shows that every right FP-injective ring for which every 2-generated right module embeds in a free module is QF. Our next Corollary extends the two results.

Corollary 15. *Suppose R is a right CF-ring such that every finitely generated left ideal is a left annihilator. Then R is a QF-ring.*

Proof. Since R is right CF, every right ideal is a right annihilator of a finite subset of R. By Proposition 14, R_R is π-injective. Then, by [5, Corollary 2.9], R is right artinian. Clearly, R is two-sided mininjective. So, R is QF by [9, Cor.4.8]. $\square$

Corollary 16. *Every right CF, right FP-injective ring is QF. In particular, every right FGF, right FP-injective ring is QF.*

A ring R is called *right FPF-ring* if every finitely generated faithful right R-module is a generator of Mod-R, the category of all right R-modules. A ring is *left (resp. right) duo* if every left (resp. right) ideal is two sided. We conclude by noticing that every right FPF-ring which is left or right duo is π-injective. The next Corollary follows from Theorem 8 and the proof of [4, 3.1A2, p.3.2].

Corollary 17. *Let R be a right FPF-ring. If R is a left or right duo ring, then R_R is π-injective. In particular, every commutative FPF-ring is π-injective.*

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