Modules with every subgenerated module lifting

Kiyoichi Oshiro
Yamaguchi University
Yoshida, Yamaguchi, Japan 753
and
Robert Wisbauer
Mathematical Institute of the University
40225 Düsseldorf, Germany

Abstract

It was shown in Dung-Smith [2] that, for a module M, every module in $\sigma[M]$ is extending (CS module) if and only if every module in $\sigma[M]$ is a direct sum of indecomposable modules of length ≤ 2 or - equivalently - every module in $\sigma[M]$ is a direct sum of an M-injective module and a semisimple module. Here we characterize these modules by the fact that every module in $\sigma[M]$ is lifting or - equivalently - decompose as a direct sum of a semisimple module and a projective module in $\sigma[M]$. They are also determined by the functor ring of $\sigma[M]$ being a QF-2 ring with Jacobson radical square zero.

As a Corollary we obtain a result of Vanaja-Purav [8]: All (left) R-modules are lifting if and only if R is a generalized uniserial ring with Jacobson radical square zero.

1 Preliminaries

Let R denote an associative ring with unit, R-*Mod* the category of unital left R-modules, and M a left R-module. We call M *locally artinian, noetherian, of finite length* if every finitely generated submodule of M has the corresponding property. The notation $K \ll M$ means that K is a small (superfluous) submodule of M.

By $\sigma[M]$ we denote the full subcategory of *R-Mod* whose objects are submodules of M-generated modules.

For any R-module N, $E(N)$ will denote the injective hull of N in *R-Mod*. For $N \in \sigma[M]$, $\widehat{N}$ is the injective hull of N in $\sigma[M]$. $\widehat{N}$ is also called the *M-injective hull* of N and is isomorphic to the trace of M in $E(N)$.

$N \in \sigma[M]$ is injective in $\sigma[M]$ if and only if N is M-injective.

1.1 Functor ring. Denote by $\{U_\lambda\}_\Lambda$ a representing set of all finitely generated modules in $\sigma[M]$ and $U = \bigoplus_\Lambda U_\lambda$.

$$T := \widehat{End}_R(U) = \{f \in \mathrm{End}_R(U) \,|\, (U_\lambda)f = 0 \text{ almost everywhere } \}$$

is called the *functor ring* of $\sigma[M]$. T has no unit but has enough idempotents.

(1) *T is left perfect if and only if every module in $\sigma[M]$ is a direct sum of finitely generated modules. In this case M is called* pure semisimple *([10], 53.4).*

(2) *Assume M is locally of finite length. Then T is semiperfect ([10], 51.7).*

(3) *Assume for every primitive idempotent $e \in T$, Te is finitely cogenerated. Then M is locally artinian ([10], 52.1).*

A ring T with enough idempotents is called *semiperfect* if simple T-modules have projective covers (see [10], 49.10). T is said to be a *left (right) QF-2 ring* if it is semiperfect and, for every primitive idempotent $e \in T$, Te (resp. eT) has a simple essential socle (e.g., [3], section 4).

1.2 Theorem. *For an R-module M with functor ring T the following are equivalent:*

(a) *For some $k \in I\!N$, every module in $\sigma[M]$ is a direct sum of uniserial modules of length $\leq k$;*

(b) *T is a left and right QF-2 ring and $Jac(T)$ is nilpotent.*

Proof: Consider a representing set $\{U_\lambda\}_\Lambda$ of all finitely generated modules in $\sigma[M]$, $U = \bigoplus_\Lambda U_\lambda$ and $T = \widehat{End}_R(U)$.

$(a) \Rightarrow (b)$ By condition (a), U is a direct sum of indecomposable modules of bounded length. Hence, by the Harada-Sai Lemma (e.g., [10], 54.1), T is semiperfect and $Jac(T)$ is nilpotent.

Since M is locally of finite length we know from [10], 53.5 that U_T is T-injective. Now we can use the conclusions $(a) \Rightarrow (b) \Rightarrow (c)$ of [10], 55.15 to derive that T is left and right QF-2.

$(b) \Rightarrow (a)$ Assume T is a left and right QF-2 ring and $Jac(T)^n = 0$, for some $n \in I\!N$. Then M is pure semisimple and locally artinian (see 1.1) and hence locally of finite length. With the proof of $(c) \Rightarrow (a)$ in [10], 55.15 we see that indecomposable modules in $\sigma[M]$ are uniserial.

It remains to show that for every uniserial $N \in \sigma[M]$, length $N \leq n$. Assume N has a composition series

$$0 \neq N_1 \subset \ldots \subset N_n \subset N_{n+1} = N.$$

From this we obtain a sequence of n morphisms in $Jac(T)$,

$$N_n \to N \to N/N_1 \to \cdots \to N/N_{n-1},$$

whose product is not zero, contradicting $Jac(T)^n = 0$. $\square$

2 Lifting modules

An R-module M is called *extending* or *CS module* if every submodule is essential in a direct summand of M.

M is said to be *lifting* if every submodule $K \subset M$ lies above a direct summand, i.e., there is a direct summand $X \subset M$ with $X \subset K$ and $K/X \ll M/X$. For characterizations of this condition refer to [10], 41.11 and 41.12.

A family $\{N_\lambda\}_\Lambda$ of independent submodules of M is said to be a *local direct summand* of M if any finite (direct) sum of N_λ's is a direct summand in M, and we say it is a *direct summand* if $\bigoplus_\Lambda N_\lambda$ is a direct summand in M (see [4], Definition 2.15).

A module is called *continuous* if it is extending and direct injective. In particular, self-injective modules are continuous.

Recall two results about these modules:

2.1 Lemma. *Let M be a continuous R-module.*

(1) *Assume every local direct summand of M is a direct summand. Then M is a direct sum of indecomposable submodules.*

(2) *Assume M is lifting. Then local direct summands of M are direct summands.*

Proof: (1) See [5], Lemma 2.4 or [4], Theorem 2.17.

(2) This is shown in [5], Lemma 2.5. $\square$

A ring R is called a *left H-ring* if every injective module in R-*Mod* is lifting. Some of the characterizations of H-rings (see [5], Theorem 1) can be extended to modules. For this we need the

Definition. A module $K \in \sigma[M]$ is said to be *small in* $\sigma[M]$ if it is a small submodule in its M-injective hull, i.e., $K \ll \widehat{K}$.

2.2 Theorem *For any R-module M, the following are equivalent:*

(a) *Every injective module in $\sigma[M]$ is lifting;*

(b) *M is locally noetherian and every non-small module in $\sigma[M]$ contains an M-injective submodule;*

(c) *every module in $\sigma[M]$ is a direct sum of an M-injective module and a small module.*

Proof: $(a) \Rightarrow (b)$ By 2.1, every injective module in $\sigma[M]$ is a direct sum of indecomposable submodules. This implies that M is locally noetherian (see [10], 27.5).

Assume N is not small in its M-injective hull $\widehat{N}$. Since $\widehat{N}$ is lifting there is a direct summand $X \subset \widehat{N}$ with $X \subset N$ and $N/X \ll \widehat{N}/X$. By assumption, X is not zero.

$(b) \Rightarrow (a)$ Referring to [10], 27.3, apply the proof of Proposition 2.7 in [5].

$(a) \Rightarrow (c)$ Consider $N \in \sigma[M]$ with M-injective hull $\widehat{N}$. Since $\widehat{N}$ is lifting, by [10], 41.11, a direct summand $X \subset \widehat{N}$ is contained in N and $N = X + Y$ with $Y \ll \widehat{N}$. This implies that Y is small in $\sigma[M]$.

$(c) \Rightarrow (a)$ With respect to [10], 41.11, this is obvious. $\square$

It was pointed out in Osofsky [6], Lemma B (also in the proof $(1) \Rightarrow (3)$ of Vanaja-Purav, Proposition 2.13) that, for a uniserial module M with composition series $0 \neq V \subset U \subset M$, $M \oplus U/V$ is not an extending module. For the same situation we observe:

2.3 Lemma. *Assume M is a uniserial module with composition series $0 \neq V \subset U \subset M$. Then the module $M \oplus U/V$ is not lifting.*

Proof: Assume $M \oplus U/V$ is lifting. Then, by Theorem 1 in [1], U/V is M-projective. However, the diagram

$$\begin{array}{ccccc} & & U/V & & \\ & & \downarrow & & \\ M & \longrightarrow & M/V & \longrightarrow & 0 \end{array}$$

cannot be extended commutatively by any $h : U/V \to M$, since the image of such a morphism always is contained in V. $\square$

The main purpose of this note is to prove:

2.4 Theorem. *For any R-module M the following are equivalent:*

(a) *Every module in $\sigma[M]$ is lifting;*

(b) *every module in $\sigma[M]$ is a direct sum of a semisimple module and a projective module in $\sigma[M]$;*

(c) *every module in $\sigma[M]$ is a direct sum of modules of length ≤ 2;*

(d) *T is a left and right QF-2 ring and $Jac(T)^2 = 0$.*

If this conditions hold, there is a projective generator in $\sigma[M]$ and all indecomposable modules of length ≤ 2 are M-projective.

Proof: $(a)\Rightarrow(d)$ Assume every module in $\sigma[M]$ is lifting. Then by Theorem 2.2, M is locally noetherian. It is easy to see that finitely generated uniform lifting modules are local modules, i.e., their factor modules are indecomposable.

Consider an indecomposable injective module $Q \in \sigma[M]$. Then for any finitely generated submodule $K \subset Q$, $K/Rad(K)$ is simple and hence Q is uniserial (see [10], 55.1). In particular, every uniform module in $\sigma[M]$ is uniserial of length ≤ 2 (by Lemma 2.3). So the M-injective hull $\widehat{M}$ of M is a direct sum of modules of length ≤ 2 and hence $\widehat{M}$ (and M) is locally of finite length. This implies that every finitely generated module in $\sigma[M]$ is a direct sum of indecomposable modules (of length ≤ 2).

Denote by $\{U_\lambda\}_\Lambda$ a representing set of all finitely generated modules in $\sigma[M]$ and $U = \bigoplus_\Lambda U_\lambda$. By the Harada-Sai Lemma, the functor ring $T := \widehat{End}_R(U)$ has the properties that $T/Jac(T)$ is semisimple and $Jac(T)$ is nilpotent.

In particular, M is pure-semisimple, i.e., every module in $\sigma[M]$ is a direct sum of finitely generated modules and these are direct sums of uniserial submodules of length ≤ 2. Now the assertion follows from Theorem 1.2.

Since T is right perfect, there exists a projective generator in $\sigma[M]$ by [10], 51.13.

Consider an indecomposable module N of length 2. This is a factor module of a supplemented projective module in $\sigma[M]$ and hence has a projective cover P (see [10], 42.1), which again is indecomposable and hence of length ≤ 2. This implies $P = N$, i.e., N is M-projective.

$(c)\Leftrightarrow(d)$ This is clear by Theorem 1.2.

$(c)\Rightarrow(a)$ Consider any module $N = \bigoplus_A N_\alpha$ in $\sigma[M]$, with N_α uniserial of length ≤ 2. By Theorem 1 in [1], N is lifting if and only if $\{N_\alpha\}_A$ is locally semi T-nilpotent and N_α is almost N_β-projective for any $\alpha \neq \beta$ in A.

The first condition is satisfied by the Harada-Sai Lemma (see [10], 54.1). Any N_α of length 2 is projective in $\sigma[M]$ (as noted above) and hence is almost K-projective for any $K \in \sigma[M]$.

Assume N_α has length 1 and consider any diagram with exact line

$$\begin{array}{ccccc} & & N_\alpha & & \\ & & \downarrow_f & & \\ N_\beta & \xrightarrow{p} & L & \to & 0, \end{array}$$

with length $N_\beta \leq 2$. If p is not an isomorphism and $f \neq 0$, there exists an epimorphism $g : N_\beta \to N_\alpha$ with $p = gf$. From this we see that N_α is almost N_β-projective and N is lifting.

$(c)\Rightarrow(b)$ It is clear from the above that modules of length 2 are M-projective. Recall that finitely generated M-projective modules are projective in $\sigma[M]$. From this the assertion is obvious.

$(b)\Rightarrow(c)$ Consider a finitely generated $N \in \sigma[M]$. Then every factor module of N is a direct sum of a projective module and a noetherian module and hence N is noetherian by [7], section 3. This implies that M is locally noetherian.

Now let $K \in \sigma[M]$ be any indecomposable M-injective module. Assume K is not semisimple. Then it is projective in $\sigma[M]$. Since $\text{End}_R(K)$ is local, K is a local module, i.e., every factor module is indecomposable (see [10], 19.7) and hence simple. From this we deduce that K has length ≤ 2.

Since every M-injective module in $\sigma[M]$ is a direct sum of indecomposables the assertion follows. $\square$

From Theorem 2.4 together with Theorem 11 in Dung-Smith [2] we obtain a characterization of rings with all modules lifting which extends Proposition 2.13 in Vanaja-Purav [8]:

2.5 Corollary. *For any ring R the following are equivalent:*

(a) *Every left R-module is lifting;*

(b) *every left R-module is extending;*

(c) *every left R-module is a direct sum of a semisimple module and a projective module;*

(d) *every left R-module is a direct sum of modules of length ≤ 2;*

(e) *R is a generalized uniserial ring with $Jac(R)^2 = 0$.*

It follows from (e) that the conditions $(a) - (d)$ are left right symmetric.

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