ON RATIONAL PAIRINGS OF FUNCTORS

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Abstract. In the theory of coalgebras C over a ring R, the rational functor relates the category ${}_{C^*}\mathbb{M}$ of modules over the algebra C^* (with convolution product) with the category ${}^{C}\mathbb{M}$ of comodules over C. This is based on the pairing of the algebra C^* with the coalgebra C provided by the evaluation map $\mathrm{ev} : C^* \otimes_R C \to R$. The (rationality) condition under consideration ensures that ${}^{C}\mathbb{M}$ becomes a coreflective full subcategory of ${}_{C^*}\mathbb{M}$.

We generalise this situation by defining a *pairing* between endofunctors T and G on any category $\mathbb{A}$ as a map, natural in $a, b \in \mathbb{A}$,

$$\beta_{a,b} : \mathbb{A}(a, G(b)) \to \mathbb{A}(T(a), b),$$

and we call it *rational* if these all are injective. In case $\mathbf{T} = (T, m_T, e_T)$ is a monad and $\mathbf{G} = (G, \delta_G, \varepsilon_G)$ is a comonad on $\mathbb{A}$, additional compatibility conditions are imposed on a pairing between $\mathbf{T}$ and $\mathbf{G}$. If such a pairing is given and is rational, and $\mathbf{T}$ has a right adjoint monad $\mathbf{T}^\diamond$, we construct a *rational functor* as the functor-part of an idempotent comonad on the $\mathbf{T}$-modules $\mathbb{A}_T$ which generalises the crucial properties of the rational functor for coalgebras. As a special case we consider pairings on monoidal categories.

Contents

1. Introduction

The pairing of a k-vector space V with its dual space $V^* = \mathrm{Hom}(V, k)$ provided by the evaluation map $V^* \otimes_k V \to k$ can be extended from base fields k to arbitrary base rings A. Then it can be applied to the study of A-corings C to obtain a faithful functor from the category of C-comodules to the category of C^*-modules. The purpose of this paper it to extend these results to (endo)functors on arbitrary categories. We begin by recalling some facts from module theory.

1.1. **Pairing of modules.** Let C be a bimodule over a ring A and $C^* = \mathrm{Hom}_A(C, A)$ the right dual. Then $C \otimes_A -$ and $C^* \otimes_A -$ are endofunctors on the category ${}_A\mathbb{M}$ of left A-modules and the evaluation

$$\mathrm{ev} : C^* \otimes_A C \to A, \quad f \otimes c \mapsto f(c),$$

induces a pairing between these functors. For left A-modules X and Y, the map

$$\alpha_Y : C \otimes_A Y \to {}_A\mathrm{Hom}(C^*, Y), \quad c \otimes y \mapsto [f \mapsto f(c)y],$$

induces the map

$$\begin{array}{rcl}\beta_{X,Y} : {}_A\mathrm{Hom}(X, C \otimes_A Y) & \longrightarrow & {}_A\mathrm{Hom}(X, {}_A\mathrm{Hom}(C^*, Y)),\\ X \xrightarrow{f} C \otimes_A Y & \longmapsto & X \xrightarrow{f} C \otimes_A Y \xrightarrow{\alpha_Y} {}_A\mathrm{Hom}(C^*, Y).\end{array}$$

Clearly $\beta_{X,Y}$ is injective for all left A-modules X, Y if and only if α_Y is a monomorphism (injective) for any left A-module Y, that is, C_A is locally projective (see [1], [8, 42.10]).

Now consider the situation above with some additional structure.

1.2. **Pairings for corings.** Let $\mathbf{C} = (C, \Delta, \varepsilon)$ be a coring over the ring A, that is, C is an A-bimodule with bimodule morphisms coproduct $\Delta : C \to C \otimes_A C$ and counit $\varepsilon : C \to A$. Then the right dual $C^* = \mathrm{Hom}_A(C, A)$ has a ring structure by the convolution product for $f, g \in C^*$, $f * g = f \circ (g \otimes_A I_C) \circ \Delta$ (convention opposite to [8, 17.8]) with unit ε, and we have a pairing between the comonad $C \otimes_A -$ and the monad $C^* \otimes_A -$ on ${}_A\mathbb{M}$. In this case, $\mathrm{Hom}_A(C^*, -)$ is a comonad on ${}_A\mathbb{M}$ and α_Y considered in 1.1 induces a comonad morphism $\alpha : C \otimes - \to \mathrm{Hom}_A(C^*, -)$. We have the commutative diagrams

$$\begin{tikzcd} C^* \otimes_A C^* \otimes_A C \arrow[r, "I\otimes I\otimes\Delta"] \arrow[d, "*\otimes I"'] & C^* \otimes_A C^* \otimes_A C \otimes_A C \arrow[r, "I\otimes \mathrm{ev}\otimes I"] & C^* \otimes_A C \arrow[d, "\mathrm{ev}"] & C \arrow[l, "\varepsilon\otimes I"'] \arrow[dl, "\varepsilon"] \\ C^* \otimes_A C \arrow[rr, "\mathrm{ev}"] & & A & \end{tikzcd} \tag{1.1}$$

The Eilenberg-Moore category $\mathbb{M}^{\mathrm{Hom}(C^*,-)}$ of $\mathrm{Hom}(C^*, -)$-comodules is equivalent to the category ${}_{C^*}\mathbb{M}$ of left C^*-modules (see 3.3) and thus α induces a functor

$${}^C\mathbb{M} \to \mathbb{M}^{\mathrm{Hom}(C^*,-)} \simeq {}_{C^*}\mathbb{M}$$

which is fully faithful if and only if the pairing (C^*, C, ev) is rational, that is, α_Y is monomorph for all $Y \in {}_A\mathbb{M}$ (see [8, 19.2 and 19.3]). Moreover, α is an isomorphism if and only if the categories ${}^C\mathbb{M}$ and ${}_{C^*}\mathbb{M}$ are equivalent and this is tantamount to C_A being finitely generated and projective.

Note that for any A-coring C, the right adjoint functor $\mathrm{Hom}_A(C, -)$ is a monad on ${}_A\mathbb{M}$ and the Kleisli category of C-comodules is equivalent to the Kleisli category of the $\mathrm{Hom}_A(C, -)$-modules (contramodules, see [11], [5]). This certainly relates comodules and modules - but the modules in this context are over the monad $\mathrm{Hom}_A(C, -)$ whereas algebraists prefer to have modules over a tensor monad (algebra). The rationality condition imposed on the functor $C \otimes_A -$, that is, conditions on the A-module structure of C, is modelled to match C-comodules with C^*-modules. It does not enforce C_A to be finitely generated (or projective) but allows to describe C-comodules as special C^*-modules.

To the content of the paper. In Section 2 we recall the notions and some basic facts on natural transformations between endofunctors needed for our investigations.

Weakening the conditions for an adjoint pair of functors, a *pairing* of two functors $T : \mathbb{A} \to \mathbb{B}$ and $G : \mathbb{B} \to \mathbb{A}$ is defined in Section 3 (see 3.1) as a map $\beta_{a,b} : \mathbb{A}(a, G(b)) \to \mathbb{B}(T(a), b)$, natural in $a \in \mathbb{A}$, $b \in \mathbb{B}$, and it is called *rational* if all the $\beta_{a,b}$ are injective maps. For pairings of monads $\mathbf{T}$ with comonads $\mathbf{G}$ on a category $\mathbb{A}$, additional conditions are imposed on the defining natural transformations (see 3.2). These imply the existence of a functor $\Phi^{\mathcal{P}} : \mathbb{A}^G \to \mathbb{A}_T$ from the $\mathbf{G}$-comodules to the $\mathbf{T}$-modules (see 3.6), which is full and faithful provided the pairing is rational (see 3.8). Of special interest is the situation that the monad $\mathbf{T}$ has a right adjoint $\mathbf{T}^\diamond$ and the last part of Section 3 is dealing with this case.

Referring to these results, a *rational functor* $\mathrm{Rat}^{\mathcal{P}} : \mathbb{A}_T \to \mathbb{A}_T$ is associated with any rational pairing in Section 4. This leads to the definition of *rational* T-modules and under some additional conditions they form a coreflective subcategory of $\mathbb{A}_T$ (see 4.8).

The application of the general notions of pairings to monoidal categories is outlined in Section 5. The resulting formalism is very close to the module case considered in 1.2.

In Section 6, we apply our results to entwining structures $(\mathbf{A}, \mathbf{C}, \lambda)$ on monoidal categories $\mathcal{V} = (\mathbb{V}, \otimes, \mathbb{I})$. The objects A and C induce a functor $\mathbb{V}(- \otimes C, A) : \mathbb{V}^{op} \to \mathrm{Set}$ and if this is representable we call the entwining *representable*, that is if $\mathbb{V}(- \otimes C, A) \simeq \mathbb{V}(-, E)$ for some object $E \in \mathbb{V}$. This E allows for an algebra structure, an algebra morphism $A \to E$ (see Proposition 6.3), and a functor from the category ${}^{C}_{A}\mathbb{V}(\lambda)$ of entwined modules to the category ${}_{E}\mathbb{V}$ of left E-modules. In case the tensor functors have right adjoints, a pairing on $\mathbb{V}$ is related to the entwining (see 6.8) and its properties are studied. Several results known for the rational functors for ordinary entwined modules (see, for example, [1], [12] and [13]) can be obtained as corollaries from the main result of this section (see Theorem 6.9).

2. Preliminaries

In this section we recall some notation and basic facts from category theory. Throughout $\mathbb{A}$ and $\mathbb{B}$ will denote any categories. By I_a, $I_{\mathbb{A}}$ or just by I we denote the identity morphism of an object $a \in \mathbb{A}$, respectively the identity functor of a category $\mathbb{A}$.

2.1. Monads and comonads. For a monad $\mathbf{T} = (T, m_T, e_T)$ on $\mathbb{A}$, we write

$\mathbb{A}_T$ for the Eilenberg-Moore category of $\mathbf{T}$-modules;

$U_T : \mathbb{A}_T \to \mathbb{A},\ (a, h_a) \to a$, for the underlying (forgetful) functor;

$\phi_T : \mathbb{A} \to \mathbb{A}_T,\ a \to (T(a), (m_T)_a)$, for the free $\mathbf{T}$-module functor, and

$\eta_T, \varepsilon_T : \phi_T \dashv U_T : \mathbb{A}_T \to \mathbb{A}$ for the forgetful-free adjunction.

Dually, if $\mathbf{G} = (G, \delta_G, \varepsilon_G)$ is a comonad on $\mathbb{A}$, we write

$\mathbb{A}^G$ for the category of the Eilenberg-Moore category of $\mathbf{G}$-comodules;

$U^G : \mathbb{A}^G \to \mathbb{A},\ (a, \theta_a) \to a$, for the forgetful functor;

$\phi^G : \mathbb{A} \to \mathbb{A}^G,\ a \to (G(a), (\delta_G)_a)$, for the cofree $\mathbf{G}$-comodule functor, and

$\eta^G, \varepsilon^G : U^G \dashv \phi^G : \mathbb{A} \to \mathbb{A}^G$ for the forgetful-cofree adjunction.

2.2. Idempotent comonads. A comonad $\mathbf{H} = (H, \varepsilon, \delta)$ is said to be *idempotent* if one of the following equivalent conditions is satisfied (see, for example, [9]):

(a) the forgetful functor $U^H : \mathbb{A}^H \to \mathbb{A}$ is full and faithful;

(b) the unit $\eta^H : I \to \phi^H U^H$ of the adjunction $U^H \dashv \phi^H$ is an isomorphism;

(c) the natural transformation $\delta : H \to HH$ is an isomorphism;

(d) for any $(a, \vartheta_a) \in \mathbb{A}^H$, the morphism $\vartheta_a : a \to H(a)$ is an isomorphism and $(\vartheta_a)^{-1} = \varepsilon_a$;

(e) $H\varepsilon$ (or εH) is an isomorphism.

When $\mathbf{H}$ is an idempotent comonad, then an object $a \in \mathbb{A}$ is the carrier of an H-comodule if and only if there exists an isomorphism $H(a) \simeq a$, or, equivalently, if and only if the morphism $\varepsilon_a : H(a) \to a$ is an isomorphism. In this case, the pair $(a, (\varepsilon_a)^{-1})$ is an H-comodule. In fact, every H-comodule is of this form. Thus the category $\mathbb{A}^H$ is isomorphic to the full subcategory of $\mathbb{A}$ generated by those objects for which there exists an isomorphism $H(a) \simeq a$.

In particular, any comonad $\mathbf{H} = (H, \varepsilon, \delta)$ with ε a componentwise monomorphism is idempotent (e.g. [9]). In this case, there is at most one morphism from any comonad $\mathbf{H}'$ to $\mathbf{H}$. When $\mathbf{H}'$ is also an idempotent comonad, then a natural transformation $\tau : \mathbf{H}' \to \mathbf{H}$ is a morphism of comonads if and only if $\varepsilon \cdot \tau = \varepsilon'$, where ε' is the counit of the comonad $\mathbf{H}'$.

We will need the following result whose proof is an easy diagram chase:

2.3. Lemma. *Suppose that in the commutative diagram*

$$\begin{tikzcd}
a \arrow[r, "k"] \arrow[d, "h_1"'] & b \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "h_2"'] & c \arrow[d, "h_3"] \\
a' \arrow[r, "k'"'] & b' \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] & c',
\end{tikzcd}$$

the bottom row is an equaliser and the morphism h_3 is a monomorphism. Then the left square in the diagram is a pullback if and only if the top row is an equaliser diagram.

2.4. **Proposition.** *Let $t : G \to R$ be a natural transformation between endofunctors of $\mathbb{A}$ with componentwise monomorphisms and assume that R preserves equalisers. Then the following are equivalent:*

(a) *the functor G preserves equalisers;*

(b) *for any regular monomorphism $i : a_0 \to a$ in $\mathbb{A}$, the following square is a pullback:*

$$\begin{tikzcd} G(a_0) \arrow[r, "G(i)"] \arrow[d, "t_{a_0}"'] & G(a) \arrow[d, "t_a"] \\ R(a_0) \arrow[r, "R(i)"'] & R(a). \end{tikzcd}$$

When $\mathbb{A}$ admits and G preserves pushouts, (a) *and* (b) *are equivalent to:*

(c) *G preserves regular monomorphisms, i.e. G takes a regular monomorphism into a regular monomorphism.*

Proof. (b)⇒(a) Let

$$\begin{tikzcd} a \arrow[r, "k"] & b \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & c \end{tikzcd}$$

be an equaliser diagram in $\mathbb{A}$ and consider the commutative diagram

$$\begin{tikzcd} G(a) \arrow[r, "G(k)"] \arrow[d, "t_a"'] & G(b) \arrow[r, shift left, "G(f)"] \arrow[r, shift right, "G(g)"'] \arrow[d, "t_b"'] & G(c) \arrow[d, "t_c"] \\ R(a) \arrow[r, "R(k)"'] & R(b) \arrow[r, shift left, "R(f)"] \arrow[r, shift right, "R(g)"'] & R(c)\,. \end{tikzcd}$$

Since R preserves equalisers, the bottom row of this diagram is an equaliser. By (b), the left square in the diagram is a pullback. t_c being a monomorphism, it follows from Lemma 2.3 that the top row of the diagram is an equaliser. Thus G preserves equalisers.

(a)⇒(b) Reconsider the diagram in the above proof and apply Lemma 2.3.

Suppose now that $\mathbb{A}$ admits and G preserves pushouts. The implication (a)⇒(c) always holds and so it remains to show

(c)⇒(b) Consider an arbitrary regular monomorphism $i : a_0 \to a$ in $\mathbb{A}$. Since $\mathbb{A}$ admits pushouts and i is a regular monomorphism in $\mathbb{A}$, the diagram

$$\begin{tikzcd} a_0 \arrow[r, "i"] & a \arrow[r, shift left, "i_1"] \arrow[r, shift right, "i_2"'] & a \sqcup_{a_0} a\,, \end{tikzcd}$$

where i_1 and i_2 are the canonical injections into the pushout, is an equaliser diagram (e.g. [2, Proposition 11.22]). Consider now the commutative diagram

$$\begin{tikzcd} G(a_0) \arrow[r, "G(i)"] \arrow[d, "t_{a_0}"'] & G(a) \arrow[r, shift left, "G(i_1)"] \arrow[r, shift right, "G(i_2)"'] \arrow[d, "t_a"'] & G(a \sqcup_{a_0} a) \arrow[d, "t_{a \sqcup_{a_0} a}"] \\ R(a_0) \arrow[r, "R(i)"'] & R(a) \arrow[r, shift left, "R(i_1)"] \arrow[r, shift right, "R(i_2)"'] & R(a \sqcup_{a_0} a)\,, \end{tikzcd}$$

in which the bottom row is an equaliser diagram since R preserves equalisers. Since G takes regular monomorphisms into regular monomorphisms and G preserves pushouts, the top row of the diagram is also an equaliser diagram. Now, using that $t_{a \sqcup_{a_0} a}$ is a monomorphism, one can apply Lemma 2.3 to conclude that the square in the diagram is a pullback showing (c)⇒(b). □

3. Pairings of functors

Generalising the results sketched in the introduction we define the notion of pairings of functors on arbitrary categories.

3.1. **Pairing of functors.** For any functors $T : \mathbb{A} \to \mathbb{B}$ and $G : \mathbb{B} \to \mathbb{A}$, there is a bijection (e.g. [21, 2.1]) between (the class of) natural transformations between functors $\mathbb{A}^{op} \times \mathbb{B} \to \mathrm{Set}$,

$$\beta_{a,b} : \mathbb{A}(a, G(b)) \to \mathbb{B}(T(a), b), \quad a \in \mathbb{A},\, b \in \mathbb{B},$$

and natural transformations $\sigma : TG \to I_{\mathbb{B}}$, with $\sigma_a := \beta_{G(a),a}(I_{G(a)}) : TG(a) \to a$ and

$$\beta_{a,b} : a \xrightarrow{f} G(b) \;\longmapsto\; T(a) \xrightarrow{T(f)} TG(b) \xrightarrow{\sigma_b} b.$$

We call (T, G, σ) a *pairing* between the functors T and G and name it a *rational pairing* provided the $\beta_{a,b}$ are monomorphisms for all $a \in \mathbb{A}$ and $b \in \mathbb{B}$.

Clearly, if all the $\beta_{a,b}$ are isomorphisms, then we have an *adjoint pairing*, that is, the functor G is right adjoint to T and σ is just the counit of the adjunction. Thus rational pairings generalise adjointness. We mention that, given a pairing (T, G, σ), Medvedev [18] calls T a *left semiadjoint* to G provided there is a natural transformation $\varphi : I_A \to GT$ such that $\sigma T \circ T\varphi = I_T$. This means that $\beta_{-,-}$ is a bifunctorial coretraction (dual of [18, Proposition 1]) in which case σ is a rational pairing. In case all $\beta_{a,b}$ are epimorphisms, the functor G is said to be *a weak right adjoint* to T in Kainen [15]. For more about weakened forms of adjointness we refer to Börger and Tholen [6].

Similar to the condition on an adjoint pair of a monad and a comonad (see [11]) we define

3.2. **Pairing of monads and comonads.** A *pairing* $\mathcal{P} = (\mathbf{T}, \mathbf{G}, \sigma)$ between a *monad* $\mathbf{T} = (T, m, e)$ and a *comonad* $\mathbf{G} = (G, \delta, \varepsilon)$ on a category $\mathbb{A}$ is a pairing $\sigma : TG \to I$ between the functors T and G inducing - for $a, b \in \mathbb{A}$ - commutativity of the diagrams

$$\begin{tikzcd}
 & \mathbb{A}(a,b) & \\
\mathbb{A}(a,G(b)) \arrow[ur, "\mathbb{A}(a,\varepsilon_b)"] \arrow[rr, "\beta^{\mathcal{P}}_{a,b}"] \arrow[d, "\mathbb{A}(a,\delta_b)"'] & & \mathbb{A}(T(a),b) \arrow[ul, "\mathbb{A}(e_a,b)"'] \arrow[d, "\mathbb{A}(m_a,b)"] \\
\mathbb{A}(a,G^2(b)) \arrow[r, "\beta^{\mathcal{P}}_{a,G(b)}"'] & \mathbb{A}(T(a),G(b)) \arrow[r, "\beta^{\mathcal{P}}_{T(a),b}"'] & \mathbb{A}(T^2(a),b),
\end{tikzcd} \tag{3.1}$$

where

$$\beta^{\mathcal{P}}_{a,b} : \mathbb{A}(a, G(b)) \to \mathbb{A}(T(a), b), \quad f : a \to G(b) \;\mapsto\; \sigma_b \cdot T(f) : T(a) \to b. \tag{3.2}$$

The pairing $\mathcal{P} = (\mathbf{T}, \mathbf{G}, \sigma)$ is said to be *rational* if $\beta^{\mathcal{P}}_{a,b}$ is injective for any $a, b \in \mathbb{A}$.

By the Yoneda Lemma, commutativity of the diagrams in (3.1) correspond to commutativity of the diagrams

$$\begin{tikzcd}
G \arrow[r, "eG"] \arrow[dr, "\varepsilon"'] & TG \arrow[d, "\sigma"] \\
 & I ,
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2G \arrow[r, "T^2\delta"] \arrow[d, "mG"'] & T^2G^2 \arrow[r, "T\sigma G"] & TG \arrow[d, "\sigma"] \\
TG \arrow[rr, "\sigma"'] & & I .
\end{tikzcd} \tag{3.3}$$

3.3. **Adjoints of monads.** Consider a monad $\mathbf{T} = (T, m, e)$ and an endofunctor G on $\mathbb{A}$ that is right adjoint to T with unit $\eta : I \to GT$ and counit $\sigma : TG \to I$. Then there is a unique way to make G into a comonad $\mathbf{G} = (G, \delta, \varepsilon)$ such that $\mathbf{G}$ is right adjoint to the monad $\mathbf{T}$. In this case, the functor $K_{T,G} : \mathbb{A}_T \to \mathbb{A}^G$ that takes $(a, h) \in \mathbb{A}_T$ to $(a, G(h) \cdot \eta_a) \in \mathbb{A}^G$ is an isomorphism of categories inducing a commutative diagram

$$\begin{tikzcd}
\mathbb{A}_T \arrow[rr, "K_{T,G}"] \arrow[dr, "U_T"'] & & \mathbb{A}^G \arrow[dl, "U^G"] \\
& \mathbb{A} &
\end{tikzcd}$$

(e.g. [11], [5, Section 3]). Writing $\beta_{a,b} : \mathbb{A}(T(a), b) \to \mathbb{A}(a, G(b))$ for the adjunction bijection, it follows that the triple $\mathcal{P} = (\mathbf{T}, \mathbf{G}, \sigma)$ is a rational pairing with $\beta^{\mathcal{P}} = \beta^{-1}$ (see commutative diagrams (1) and (2) in [16]).

3.4. **Pairings and morphisms.** *Let $\mathcal{P} = (\mathbf{T}, \mathbf{G}, \sigma)$ be a pairing, $\mathbf{T}' = (T', m', e')$ any monad, and $t : \mathbf{T}' \to \mathbf{T}$ a monad morphism.*

(i) *The triple $\mathcal{P}' = (\mathbf{T}', \mathbf{G}, \sigma' := \sigma \cdot tG)$ is also a pairing.*

(ii) *If $\mathcal{P}$ is rational, then $\mathcal{P}'$ is also rational provided the natural transformation t is a componentwise epimorphism.*

Proof. (i). The diagram

$$\begin{tikzcd}
G \arrow[r, "e'G"] \arrow[d, "\varepsilon"'] \arrow[dr, "eG"] & T'G \arrow[d, "tG"] \\
I & TG \arrow[l, "\sigma"]
\end{tikzcd}$$

commutes since t is a monad morphism (thus $t \cdot e' = e$) and because of the commutativity of the triangle in the diagram (3.3.) Thus $\sigma' \cdot e'G = \sigma \cdot tG \cdot e'G = \sigma \cdot eG = \varepsilon$.

Consider now the diagram

$$\begin{tikzcd}
& T'T'G \arrow[r, "T'T'\delta"] \arrow[dl, "m'G"'] \arrow[d, "ttG"] & T'T'GG \arrow[rr, "T'tGG"] \arrow[dr, "ttGG"'] \arrow[d, phantom, "(2)"] & & T'TGG \arrow[rr, "T'\sigma G"] \arrow[dl, "tTGG"] \arrow[d, phantom, "(3)"] & \arrow[d, phantom, "(4)"] & T'G \arrow[dl, "tG"] \\
T'G \arrow[r, phantom, "(1)"] \arrow[dr, "tG"'] & TTG \arrow[rr, "TT\delta"'] \arrow[d, "mG"] & \arrow[d, phantom, "(5)"] & TTGG \arrow[rr, "T\sigma G"'] & & TG \arrow[dl, "\sigma"] & \\
& TG \arrow[rrr, "\sigma"'] & & & I\,, & &
\end{tikzcd}$$

in which diagram (1) commutes since t is a morphism of monads, diagrams (2), (3) and (4) commute by naturality of composition, and diagram (5) commutes by commutativity of the rectangle in (3.3). It then follows that

$$\sigma' \cdot T\sigma'G \cdot T'T'\delta = \sigma \cdot tG \cdot T'\sigma G \cdot T'tGG \cdot T'T'\delta = \sigma \cdot tG \cdot m'G = \sigma' \cdot m'G,$$

proving that the triple $\mathcal{P}' = (\mathbf{T}', \mathbf{G}, \sigma' = \sigma \cdot tG)$ is a pairing.

(ii). It is easy to check that the composite

$$\mathbb{A}(a, G(b)) \xrightarrow{\beta^{\mathcal{P}}_{a,b}} \mathbb{A}(T(a), b) \xrightarrow{\mathbb{A}(t_a, b)} \mathbb{A}(T'(a), b)$$

takes $f : a \to G(b)$ to $\sigma'_b \cdot T'(f) : T'(a) \to b$ and thus – since $T(f) \cdot t_a = t_{G(a)} \cdot T'(f)$ by naturality of t – $\beta^{\mathcal{P}'}_{a,b} = \mathbb{A}(t_a, b) \cdot \beta^{\mathcal{P}}_{a,b}$. If t is a componentwise epimorphism, then t_a is an epimorphism, and then the map $\mathbb{A}(t_a, b)$ is injective. It follows that $\beta^{\mathcal{P}'}_{a,b}$ is also injective provided that $\beta^{\mathcal{P}}_{a,b}$ is injective (i.e. the pairing $\mathcal{P} = (\mathbf{T}, \mathbf{G}, \sigma)$ is rational). $\square$

Dually, one has

3.5. **Proposition.** *Let $\mathcal{P} = (\mathbf{T}, \mathbf{G}, \sigma)$ be a pairing, $\mathbf{G}' = (G', \delta', \varepsilon')$ any comonad, and $t : \mathbf{G}' \to \mathbf{G}$ a comonad morphism.*

(i) *The triple $\mathcal{P}' = (\mathbf{T}, \mathbf{G}', \sigma' := \sigma \cdot Tt)$ is also a pairing.*

(ii) *If $\mathcal{P}$ is rational, then $\mathcal{P}'$ is also rational provided the natural transformation t is a componentwise monomorphism.*

3.6. **Functors induced by pairings.** *Let $\mathcal{P} = (\mathbf{T}, \mathbf{G}, \sigma)$ be a pairing on a category $\mathbb{A}$ with $\beta^{\mathcal{P}}_{a,\,b} : \mathbb{A}(a, G(b)) \to \mathbb{A}(T(a), b)$ (see (3.2).*

(1) *If $(a, \theta_a) \in \mathbb{A}^G$, then $(a, \beta^{\mathcal{P}}_{a,a}(\theta_a)) = (a, \sigma_a \cdot T(\theta_a)) \in \mathbb{A}_T$.*

(2) *The assignments* $\begin{array}{rcl} (a, \theta_a) & \longmapsto & (a, \sigma_a \cdot T(\theta_a)), \\ f : a \to b & \longmapsto & f : a \to b, \end{array}$

yield a conservative functor $\Phi^{\mathcal{P}} : \mathbb{A}^G \to \mathbb{A}_T$ inducing a commutative diagram

(3.4)
$$\begin{tikzcd} \mathbb{A}^G \arrow[rr, "\Phi^{\mathcal{P}}"] \arrow[dr, "U^G"'] & & \mathbb{A}_T \arrow[dl, "U_T"] \\ & \mathbb{A}\,. & \end{tikzcd}$$

Proof. (1) We have to show that the diagrams

$$\begin{tikzcd} a \arrow[r, "e_a"] \arrow[dr, equal] & T(a) \arrow[d, "\beta^{\mathcal{P}}_{a,a}(\theta_a)"] \\ & a, \end{tikzcd} \qquad \begin{tikzcd} T^2(a) \arrow[r, "m_a"] \arrow[d, "T(\beta^{\mathcal{P}}_{a,a}(\theta_a))"'] & T(a) \arrow[d, "\beta^{\mathcal{P}}_{a,a}(\theta_a)"] \\ T(a) \arrow[r, "\beta^{\mathcal{P}}_{a,a}(\theta_a)"'] & a \end{tikzcd}$$

are commutative. In the diagram

$$\begin{tikzcd} a \arrow[r, "\theta_a"] \arrow[d, "e_a"'] & G(a) \arrow[d, "e_{G(a)}"'] \arrow[dr, "\varepsilon_a"] & \\ T(a) \arrow[r, "T(\theta_a)"'] & TG(a) \arrow[r, "\sigma_a"'] & a \end{tikzcd}$$

the square commutes by naturality of $e : I \to T$, while the triangle commutes by (3.3). Thus $\beta^{\mathcal{P}}_{a,a}(\theta_a) \cdot e_a = \sigma_a \cdot T(\theta_a) \cdot e_a = \varepsilon_a \cdot \theta_a$. But $\varepsilon_a \cdot \theta_a = I_a$ since $(a, \theta_a) \in \mathbb{A}^G$, implying that $\beta^{\mathcal{P}}_{a,a}(\theta_a) \cdot e_a = I_a$. This shows that the left hand diagram commutes.

Since $\beta^{\mathcal{P}}_{a,a}(\theta_a) = \sigma_a \cdot T(\theta_a)$, and $T(\theta_a) \cdot T(\sigma_a) = T(\sigma_{G(a)}) \cdot T^2G(\theta_a)$ by naturality of σ, the right hand diagram can be rewritten as

$$\begin{tikzcd} T^2(a) \arrow[r, "m_a"] \arrow[d, "T^2(\theta_a)"'] & T(a) \arrow[r, "T(\theta_a)"] & TG(a) \arrow[dr, "\sigma_a"] & \\ T^2G(a) \arrow[r, "T^2G(\theta_a)"'] & T^2G^2(a) \arrow[r, "T(\sigma_{G(a)})"'] & TG(a) \arrow[r, "\sigma_a"'] & a\,. \end{tikzcd}$$

It is easy to see that $\sigma_a \cdot T(\theta_a) \cdot m_a = (\mathbb{A}(m_a, a) \cdot \beta^{\mathcal{P}}_{a,a})(\theta_a)$ and it follows from the commutativity of the bottom diagram in (3.1) that

$$\begin{aligned} \sigma_a \cdot T(\theta_a) \cdot m_a &= (\beta^{\mathcal{P}}_{T(a),a} \cdot \beta^{\mathcal{P}}_{a,G(a)} \cdot \mathbb{A}(a, \delta_a))(\theta_a) \\ &= \sigma_a \cdot T(\sigma_{G(a)}) \cdot T^2(\delta_a) \cdot T^2(\theta_a). \end{aligned}$$

Recalling that $\delta_a \cdot \theta_a = G(\theta_a) \cdot \theta_a$ since $(a, \theta_a) \in \mathbb{A}^G$, we get

$$\sigma_a \cdot T(\theta_a) \cdot m_a = \sigma_a \cdot T(\sigma_{G(a)}) \cdot T^2 G(\theta_a) \cdot T^2(\theta_a),$$

proving that the right hand diagram is commutative. Thus $(a, \beta^{\mathcal{P}}_{a,a}(\theta_a)) \in \mathbb{A}_T$.

(2) By (1), it suffices to show that if $f : (a, \theta_a) \to (b, \theta_b)$ is a morphism in $\mathbb{A}^G$, then f is a morphism in $\mathbb{A}_T$ from the $\mathbf{T}$-module $(a, \sigma_a \cdot T(\theta_a))$ to the $\mathbf{T}$-module $(b, \sigma_b \cdot T(\theta_b))$. To say that $f : a \to b$ is a morphism in $\mathbb{A}_T$ is to say that the outer diagram of

$$\begin{tikzcd}
T(a) \arrow[r, "T(\theta_a)"] \arrow[d, "T(f)"'] & TG(a) \arrow[r, "\sigma_a"] \arrow[d, "TG(f)"'] & a \arrow[d, "f"] \\
T(b) \arrow[r, "T(\theta_b)"'] & TG(b) \arrow[r, "\sigma_b"'] & b
\end{tikzcd}$$

is commutative, which is indeed the case since the left square commutes because f is a morphism in $\mathbb{A}^G$, while the right square commutes by naturality of σ. Clearly $\Phi^{\mathcal{P}}$ is conservative.

The commutativity of the diagram of functors is obvious. □

3.7. Properties of the functor $\Phi^{\mathcal{P}}$. *Let $\mathcal{P} = (\mathbf{T}, \mathbf{G}, \sigma)$ be a pairing on a category $\mathbb{A}$ with induced functor $\Phi^{\mathcal{P}} : \mathbb{A}^G \to \mathbb{A}_T$ (see 3.6).*

(1) *The functor $\Phi^{\mathcal{P}}$ is comonadic if and only if it has a right adjoint.*

(2) *Let $\mathbb{C}$ be a small category such that any functor $\mathbb{C} \to \mathbb{A}$ has a colimit that is preserved by T. Then the functor $\Phi^{\mathcal{P}}$ preserves the colimit of any functor $\mathbb{C} \to \mathbb{A}^G$.*

(3) *When $\mathbb{A}$ admits and T preserves all small colimits, the category $\mathbb{A}^G$ has and the functor $\Phi^{\mathcal{P}}$ preserves all small colimits.*

(4) *Let $\mathbb{A}$ be a locally presentable category, suppose that G preserves filtered colimits and T preserves all small colimits. Then $\Phi^{\mathcal{P}}$ is comonadic.*

Proof. (1) Since the functors U^G and U_T both reflect isomorphisms, it follows from commutativity of Diagram (3.4) that $\Phi^{\mathcal{P}}$ also reflects isomorphisms. Moreover, since U^G creates equalisers of G-split pairs and since U_T creates all equalisers that exist in $\mathbb{A}$, $\mathbb{A}^G$ admits equalisers of $\Phi^{\mathcal{P}}$-split pairs and $\Phi^{\mathcal{P}}$ preserves them. The result now follows from the dual of Beck's Precise Tripleability Theorem [3, Theorem 3.3.10].

(2) Since T preserves the colimit of any functor $\mathbb{C} \to \mathbb{A}$, the category $\mathbb{A}_T$ admits and the functor U_T preserves the colimit of any functor $\mathbb{C} \to \mathbb{A}_T$ (see [7]). Now, if $F : \mathbb{C} \to \mathbb{A}^G$ is an arbitrary functor, then the composition $\Phi^{\mathcal{P}} \cdot F : \mathbb{C} \to \mathbb{A}_T$ has a colimit in $\mathbb{A}_T$. Since $U_T \cdot \Phi^{\mathcal{P}} = U^G$ and U^G preserves all colimits that exist in $\mathbb{A}^G$, it follows that the functor U_T preserves the colimit of the composite $\Phi^{\mathcal{P}} \cdot F$. Now the assertion follows from the fact that an arbitrary conservative functor reflects such colimits as it preserves.

(3) is a corollary of (2).

(4) Note first that Adámek and Rosický proved in [2] that the Eilenberg-Moore category with respect to a filtered-colimit preserving monad on a locally presentable category is locally presentable. This proof can be adopted to show that if G preserves filtered colimits and $\mathbb{A}$ is locally presentable, then $\mathbb{A}^G$ is also locally presentable. Therefore $\mathbb{A}^G$ is finitely complete, cocomplete, co-wellpowered and has a small set of generators (see [2]). Since the functor $\Phi^{\mathcal{P}}$ preserves all small colimits by (3), it follows from the (dual of the) Special Adjoint Functor Theorem (see [17]) that $\Phi^{\mathcal{P}}$ admits a right adjoint functor. Combining this with (1.ii) gives that the functor $\Phi^{\mathcal{P}}$ is comonadic. □

3.8. Properties of rational pairings. *Let $\mathcal{P} = (\mathbf{T}, \mathbf{G}, \sigma)$ be rational pairing on $\mathbb{A}$.*

(1) *The functor $\Phi^{\mathcal{P}} : \mathbb{A}^G \to \mathbb{A}_T$ is full and faithful.*

(2) *The functor G preserves monomorphisms.*

(3) *If $\mathbb{A}$ is abelian and G is right exact, then G is also left exact, hence exact.*

Proof. (1) Obviously, $\Phi^{\mathcal{P}}$ is faithful. Let $(a, \theta_a), (b, \theta_b) \in \mathbb{A}^G$ and let

$$f : \Phi^{\mathcal{P}}(a, \theta_a) = (a, \sigma_a \cdot T(\theta_a)) \to (b, \sigma_b \cdot T(\theta_b)) = \Phi^{\mathcal{P}}(b, \theta_b)$$

be a morphism in $\mathbb{A}_T$. We have to show that the diagram

(3.5)
$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "\theta_a"'] & b \arrow[d, "\theta_b"] \\ G(a) \arrow[r, "G(f)"'] & G(b) \end{tikzcd}$$

commutes. Since f is a morphism in $\mathbb{A}_T$, the diagram

(3.6)
$$\begin{tikzcd} T(a) \arrow[r, "T(\theta_a)"] \arrow[d, "T(f)"'] & TG(a) \arrow[r, "\sigma_a"] & a \arrow[d, "f"] \\ T(b) \arrow[r, "T(\theta_b)"'] & TG(b) \arrow[r, "\sigma_b"'] & b \end{tikzcd}$$

commutes and we have

$$\begin{aligned} \beta^{\mathcal{P}}_{a,b}(\theta_b \cdot f) &= \sigma_b \cdot T(\theta_b \cdot f) = \sigma_b \cdot T(\theta_b) \cdot T(f) \\ {\scriptstyle \text{by (3.6)}} &= f \cdot \sigma_a \cdot T(\theta_a) \\ {\scriptstyle \sigma \text{ is a natural}} &= \sigma_b \cdot TG(f) \cdot T(\theta_a) = \sigma_b \cdot T(G(f) \cdot \theta_a) = \beta^{\mathcal{P}}_{a,b}(G(f) \cdot \theta_a). \end{aligned}$$

Since $\beta^{\mathcal{P}}_{a,b}$ is injective by assumption, it follows that $G(f) \cdot \theta_a = \theta_b \cdot f$, that is, (3.5) commutes.

(2) Let $f : a \to b$ be a monomorphism in $\mathbb{A}$. Then for any $x \in \mathbb{A}$, the map

$$\mathbb{A}(T(x), f) : \mathbb{A}(T(x), a) \to \mathbb{A}(T(x), b)$$

is injective. Considering the commutative diagram

$$\begin{tikzcd} \mathbb{A}(x, G(a)) \arrow[r, "\beta^{\mathcal{P}}_{x,a}"] \arrow[d, "\mathbb{A}(x,G(f))"'] & \mathbb{A}(T(x), a) \arrow[d, "\mathbb{A}(T(x),f)"] \\ \mathbb{A}(x, G(b)) \arrow[r, "\beta^{\mathcal{P}}_{x,b}"'] & \mathbb{A}(T(x), b)\,, \end{tikzcd}$$

one sees that the map $\mathbb{A}(x, G(f))$ is injective for all $x \in \mathbb{A}$, proving that $G(f)$ is a monomorphism in $\mathbb{A}$.

(3) Here a right exact functor preserving monomorphisms is exact. □

3.9. **Comonad morphisms.** Recall (e.g. [3]) that a morphism of comonads $t : \mathbf{G} \to \mathbf{G}'$ induces a functor

$$t_* : \mathbb{A}^G \to \mathbb{A}^{G'}, \quad (a, \theta_a) \mapsto (a, t_a \cdot \theta_a).$$

The passage $t \to t_*$ yields a bijection between comonad morphisms $\mathbf{G} \to \mathbf{G}'$ and functors $V : \mathbb{A}^G \to \mathbb{A}^{G'}$ with $U^{G'}V = U^G$.

If $V : \mathbb{A}^G \to \mathbb{A}^{G'}$ is such a functor, then the image of any cofree $\mathbf{G}$ -comodule $(G(a), \delta_a)$ under V has the form $(G(a), s_a)$ for some $s_a : G(a) \to G'G(a)$. Then the collection $\{s_a \mid a \in \mathbb{A}\}$ constitute a natural transformation $s : G \to G'G$ such that $G'\varepsilon \cdot s : G \to G'$ is a comonad morphism.

3.10**. Right adjoint for** T**.** Let $\mathcal{P} = (\mathbf{T}, \mathbf{G}, \sigma)$ be a pairing and suppose that there exists an endofunctor $T^\diamond$ right adjoint to T with unit $\overline{\eta} : 1 \to T^\diamond T$. Then we obtain a comonad $\mathbf{T}^\diamond = (T^\diamond, \delta^\diamond, \varepsilon^\diamond)$ that is right adjoint to the monad $\mathbf{T}$ (see 3.3). In this situation, the functor $K_{T,T^\diamond} : \mathbb{A}_T \to \mathbb{A}^{T^\diamond}$ that takes any $(a, h_a) \in \mathbb{A}_T$ to $(a, T^\diamond(h_a) \cdot \overline{\eta}_a)$, is an isomorphism of categories and $U^{T^\diamond} K_{T,T^\diamond} = U_T$. Since $U_T \Phi^{\mathcal{P}} = U^G$, one gets the commutative diagram

$$\begin{tikzcd}
\mathbb{A}^G \arrow[r, "\Phi^{\mathcal{P}}"] \arrow[dr, "U^G"'] & \mathbb{A}_T \arrow[r, "K_{T,T^\diamond}"] & \mathbb{A}^{T^\diamond} \arrow[dl, "U^{T^\diamond}"] \\
& \mathbb{A} &
\end{tikzcd}.$$

It follows that there is a morphism of comonads $t : \mathbf{G} \to \mathbf{T}^\diamond$ with $K_{T,T^\diamond} \Phi^{\mathcal{P}} = t_*$.

3.11. **Lemma.** *In the situation given in 3.10, t is the composite*

$$G \xrightarrow{\overline{\eta}G} T^\diamond TG \xrightarrow{T^\diamond \sigma} T^\diamond .$$

Proof. Since

- for any $(a, \theta_a) \in \mathbb{A}^G$, $\Phi^{\mathcal{P}}(a, \theta_a) = (a, \sigma_a \cdot T(\theta_a))$, and
- for any $(a, h_a) \in \mathbb{A}_T$, $K_{T,T^\diamond}(a, h_a) = (a, T^\diamond(h_a) \cdot \overline{\eta}_a)$,

it follows that for any cofree **G**-comodule $(G(a), \delta_a)$,

$$K_{T,T^\diamond} \Phi^{\mathcal{P}}(G(a), \delta_a) = (G(a), T^\diamond(\sigma_{G(a)}) \cdot T^\diamond T(\delta_a) \cdot \overline{\eta}_{G(a)}),$$

thus

$$t_a = T^\diamond(\varepsilon_a) \cdot T^\diamond(\sigma_{G(a)}) \cdot T^\diamond T(\delta_a) \cdot \overline{\eta}_{G(a)}.$$

But since

$$T^\diamond(\varepsilon_a) \cdot T^\diamond(\sigma_{G(a)}) = T^\diamond(\sigma_a) \cdot T^\diamond TG(\varepsilon_a)$$

by naturality of σ and $G(\varepsilon_a) \cdot \delta_a = I_{G(a)}$, one has

$$t_a = T^\diamond(\sigma_a) \cdot T^\diamond TG(\varepsilon_a) \cdot T^\diamond T(\delta_a) \cdot \overline{\eta}_{G(a)} = T^\diamond(\sigma_a) \cdot \overline{\eta}_{G(a)}.$$

□

3.12**. Right adjoint functor of** t_***.** In the setting of 3.10, suppose that the category $\mathbb{A}^G$ admits equalisers. Then it is well-known (e.g. [10]) that for any comonad morphism $t : G \to T^\diamond$, the functor $t_* : \mathbb{A}^G \to \mathbb{A}^{T^\diamond}$ admits a right adjoint $t^* : \mathbb{A}^{T^\diamond} \to \mathbb{A}^G$, which can be calculated as follows. Recall from 2.1 that, for any comonad G, we denote by η^G and ε^G the unit and counit of the adjoint pair (U^G, ϕ^G). Writing α_t for the composite

$$t_* \phi^G \xrightarrow{\eta^{T^\diamond} t_* \phi^G} \phi^{T^\diamond} U^{T^\diamond} t_* \phi^G = \phi^{T^\diamond} U^G \phi^G \xrightarrow{\phi^{T^\diamond} \varepsilon^G} \phi^{T^\diamond},$$

and β_t for the composite

$$\phi^G U^{T^\diamond} \xrightarrow{\eta^G \phi^G U^{T^\diamond}} \phi^G U^G \phi^G U^{T^\diamond} = \phi^G U^{T^\diamond} t_* \phi^G U^{T^\diamond} \xrightarrow{\phi^G U^{T^\diamond} \alpha_t U^{T^\diamond}} \phi^G U^{T^\diamond} \phi^{T^\diamond} U^{T^\diamond},$$

then t^* is the equaliser (we assumed that $\mathbb{A}^G$ has equalisers)

$$t^* \xrightarrow{i_t} \phi^G U^{T^\diamond} \underset{\beta_t}{\overset{\phi^G U^{T^\diamond} \eta^{T^\diamond}}{\rightrightarrows}} \phi^G U^{T^\diamond} \phi^{T^\diamond} U^{T^\diamond}.$$

Note that the counit $\varepsilon_t : t_*t^* \to I$ of the adjunction $t_* \dashv t^*$ is the unique natural transformation that makes the square in the following diagram commute,

(3.7)
$$\begin{tikzcd}
t_*t^* \arrow[r, "t_*i_t"] \arrow[d, dotted, "\varepsilon_t"'] & t_*\phi^G U^{T^\diamond} \arrow[r, shift left, "t_*\phi^G U^{T^\diamond}\eta^{T^\diamond}"] \arrow[r, shift right, "t_*\beta_t"'] \arrow[d, "\alpha_t U^{T^\diamond}"'] & t_*\phi^G U^{T^\diamond}\phi^{T^\diamond}U^{T^\diamond} \arrow[d, "\alpha_t U^{T^\diamond}\phi^{T^\diamond}U^{T^\diamond}"] \\
I \arrow[r, "\eta^{T^\diamond}"'] & \phi^{T^\diamond}U^{T^\diamond} \arrow[r, shift left, "\phi^{T^\diamond}U^{T^\diamond}\eta^{T^\diamond}"] \arrow[r, shift right, "\eta^{T^\diamond}\phi^{T^\diamond}U^{T^\diamond}"'] & \phi^{T^\diamond}U^{T^\diamond}\phi^{T^\diamond}U^{T^\diamond}.
\end{tikzcd}$$

It is not hard to see that for any $a \in \mathbb{A}$, the a-component of the natural transformation α_t is just the morphism $t_a : G(a) \to T^\diamond(a)$ seen as a morphism

$$t_*\phi^G(a) = (G(a), t_{G(a)} \cdot \delta_a) \to \phi^{T^\diamond}(a) = (T^\diamond(a), \delta^\diamond_a)$$

in $\mathbb{A}^{T^\diamond}$. Indeed, since for any $a \in A$, $(\varepsilon^G)_a = \varepsilon_a$, while for any $(a, \nu_a) \in \mathbb{A}^{T^\diamond}$, $(\eta^{T^\diamond})_{(a,\nu_a)} = \nu_a$, $(\alpha_t)_a$ is the composite $T^\diamond(\varepsilon_a) \cdot t_{G(a)} \cdot \delta_a$. Considering now the diagram

$$\begin{tikzcd}
G(a) \arrow[r, "\delta_a"] \arrow[dr, equal] & GG(a) \arrow[r, "t_{G(a)}"] \arrow[d, "G(\varepsilon_a)"] & T^\diamond G(a) \arrow[d, "T^\diamond(\varepsilon_a)"] \\
& G(a) \arrow[r, "t_a"'] & T^\diamond(a)\,,
\end{tikzcd}$$

in which the square commutes by naturality of composition, while the triangle commutes by the definition of a comonad, one sees that $(\alpha_t)_a = t_a$.

3.13. **Theorem.** *Let $\mathcal{P} = (\mathbf{T}, \mathbf{G}, \sigma)$ be a monad-comonad pairing on $\mathbb{A}$. Suppose that $\mathbb{A}^G$ admits equalisers and that the monad $\mathbf{T}$ has a right adjoint comonad $\mathbf{T}^\diamond$. Then*

(1) *the functor $\Phi^{\mathcal{P}} : \mathbb{A}^G \to \mathbb{A}_T$ (and hence also $t_* : \mathbb{A}^G \to \mathbb{A}^{T^\diamond}$) is comonadic;*

(2) *if the pairing $\mathcal{P}$ is rational, then $\mathbb{A}^G$ is equivalent to a reflective subcategory of $\mathbb{A}_T$.*

Proof. By 3.7(1), $\Phi^{\mathcal{P}}$ is comonadic if and only if it has a right adjoint.

(1) Since the category $\mathbb{A}^G$ admits equalisers, the functor $t_* : \mathbb{A}^G \to \mathbb{A}^{T^\diamond}$ has a right adjoint $t^* : \mathbb{A}^{T^*} \to \mathbb{A}^G$ (by 3.12). Then evidently the functor $t^*K_{T,G}$ is right adjoint to the functor $\Phi^{\mathcal{P}}$. Since $K_{T,T^\diamond}\Phi^{\mathcal{P}} = t_*$, it is clear that t_* is also comonadic. This completes the proof of the first part.

(2) If $\mathcal{P}$ is rational, $\Phi^{\mathcal{P}}$ is full and faithful by 3.8(1), and when $\Phi^{\mathcal{P}}$ has a right adjoint, the unit of the adjunction is a componentwise isomorphism (see [17]). □

Given two functors $F, F' : \mathbb{A} \to \mathbb{B}$, we write $\mathrm{Nat}(F, F')$ for the collection of all natural transformations from F to F'. As a consequence of the Yoneda Lemma recall:

3.14. **Lemma.** *Let $\mathbf{T} = (T, m, e)$ be a monad on the category $\mathbb{A}$ with right adjoint comonad $\mathbf{T}^\diamond = (T^\diamond, \delta^\diamond, \varepsilon^\diamond)$, unit $\overline{\eta} : I \to T^\diamond T$ and counit $\overline{\varepsilon} : TT^\diamond \to I$. Then for any endofunctor $G : \mathbb{A} \to \mathbb{A}$, there is a bijection*

$$\chi : \mathrm{Nat}(TG, I) \to \mathrm{Nat}(G, T^\diamond), \quad TG \xrightarrow{\sigma} I \;\longmapsto\; G \xrightarrow{\overline{\eta}G} T^\diamond TG \xrightarrow{T^\diamond\sigma} T^\diamond,$$

with the inverse given by the assignment

$$G \xrightarrow{s} T^\diamond \;\longmapsto\; TG \xrightarrow{Ts} TT^\diamond \xrightarrow{\overline{\varepsilon}} I.$$

3.15. **Proposition.** *Let $\mathbf{T}$ be a monad on $\mathbb{A}$ with right adjoint comonad $\mathbf{T}^\diamond$ (as in 3.14) and let $\mathbf{G} = (G, \delta, \varepsilon)$ be a comonad on $\mathbb{A}$.*

(1) *There exists a bijection between*

(i) *natural transformations $\sigma : TG \to I$ for which the triple $\mathcal{P} = (\mathbf{T}, \mathbf{G}, \sigma)$ is a pairing;*

(ii) *natural transformations* $\sigma : TG \to I$ *for which* $\chi(\sigma) : G \to T^\diamond$ *is a morphism of comonads;*

(iii) *functors* $V : \mathbb{A}^G \to \mathbb{A}_T$ *such that* $U_T V = U^G$.

(2) *A functor* $V : \mathbb{A}^G \to \mathbb{A}_T$ *with* $U_T V = U^G$ *is an isomorphism if and only if there exists an isomorphism of comonads* $\mathbf{G} \simeq \mathbf{T}^\diamond$.

Proof. (1) In the light of (3.3) and Lemma 3.14, the bijection between (i) and (ii) follows from Lemma 3.11 and the dual of 3.4.

It remains to show that there is a bijective correspondence between comonad morphisms $\mathbf{G} \to \mathbf{T}^\diamond$ and functors $V : \mathbb{A}^G \to \mathbb{A}_T$ such that $U_T V = U^G$. But to give a comonad morphism $\mathbf{G} \to \mathbf{T}^\diamond$ is to give a functor $W : \mathbb{A}^G \to \mathbb{A}^{T^\diamond}$ with $U^{T^\diamond} W = U^G$, which is in turn equivalent - since $K^{-1}_{T,T^\diamond}$ is an isomorphism of categories with $U_T K^{-1}_{T,T^\diamond} = U^{T^\diamond}$ (see 3.10) - to giving a functor $V : \mathbb{A}^G \to \mathbb{A}_T$ with $U_T V = U^G$.

(2) According to (1), to give a functor $V : \mathbb{A}^G \to \mathbb{A}_T$ with $U_T V = U^G$ is to give a comonad morphism $t : \mathbf{G} \to \mathbf{T}^\diamond$ such that $t_* = K_{T,T^\diamond} V$. It follows - since $K_{T,T^\diamond}$ is an isomorphism of categories - that V is an isomorphism of categories if and only if t_* is, or, equivalently, if t is an isomorphism of comonads. □

3.16. **Proposition.** *Let* $\mathcal{P} = (\mathbf{T}, \mathbf{G}, \sigma)$ *be a monad-comonad pairing on a category* $\mathbb{A}$ *and* $\mathbf{T}^\diamond = (T^\diamond, \delta^\diamond, \varepsilon^\diamond)$ *a comonad right adjoint to* $\mathbf{T}$. *Consider the statements:*

(i) *the pairing* $\mathcal{P}$ *is rational;*

(ii) $\chi(\sigma) : G \to T^\diamond$ *is componentwise a monomorphism;*

(iii) *the functor* $\Phi^{\mathcal{P}} : \mathbb{A}^G \to \mathbb{A}_T$ *is full and faithful.*

Then one has the implications (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii).

Proof. (i) $\Rightarrow$ (iii) is just 3.8(1).

(i) $\Leftrightarrow$ (ii) For any $a \in \mathbb{A}$, consider the natural transformation

$$\mathbb{A}(a, G(-)) \xrightarrow{\beta^{\mathcal{P}}_{a,-}} \mathbb{A}(T(a), -) \xrightarrow{\alpha_{a,-}} \mathbb{A}(a, T^\diamond(-)) ,$$

where α denotes the isomorphism of the adjunction $T \dashv T^\diamond$. It is easy to see that the induced natural transformation $G \to T^\diamond$ is just $\chi(\sigma)$. Since each component of $\alpha_{a,-}$ is a bijection, it follows that $\chi(\sigma)$ is a componentwise monomorphism if and only if each component of $\beta^{\mathcal{P}}_{a,-}$ is monomorphism for all $a \in \mathbb{A}$. □

To illustrate the situation considered above we describe the details for corings.

3.17. **Rational pairing for corings.** Let C be an A-coring (as in 1.1). As described in 1.2, this induces a pairing $(C^* \otimes_A -, C \otimes_A -, \mathrm{ev})$. From Lemma 3.14 we obtain that for $X \in {}_A\mathbb{M}$,

$$(\chi(\mathrm{ev}))_X = \alpha_X : C \otimes_A X \to \mathrm{Hom}_A(C^*, X),\ c \otimes x \mapsto [f \mapsto f(c)x],$$

(with α from 1.1) and Proposition 3.16 says that for the assertions

(i) $(C^* \otimes_A -, C \otimes_A -, \mathrm{ev})$ *is a rational pairing,*

(ii) $\alpha_X : C \otimes_A X \to \mathrm{Hom}_A(C^*, X)$ *is injective for any left A-module* X,

(iii) *the functor* ${}^C\mathbb{M} \to {}_{C^*}\mathbb{M}$ *is full and faithful,*

we have the implications (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii).

Condition (ii) is sometimes called the *(right) α-condition* for corings (e.g. [8, 19.2]). It is equivalent to C_A being locally projective and implies in particular that C_A is flat. For corings all the assertions (i), (ii) and (iii) are equivalent (e.g. [8, 19.3]).

4. Rational functors

Let $\mathbb{A}$ be an arbitrary category admitting pullbacks.

4.1. Throughout this section we fix a rational pairing $\mathcal{P} = (\mathbf{T}, \mathbf{G}, \sigma)$ on the category $\mathbb{A}$ with a comonad $\mathbf{T}^\diamond = (T^\diamond, \delta^\diamond, \varepsilon^\diamond)$ right adjoint to $\mathbf{T}$, unit $\overline{\eta} : 1 \to T^\diamond T$ and counit $\overline{\varepsilon} : TT^\diamond \to I$. Let $t = \chi(\sigma)$ (see Lemma 3.14).

For any $(a, \vartheta_a) \in \mathbb{A}^{\mathbf{T}^\diamond}$, write $\Upsilon(a, \vartheta_a)$ for a chosen pullback

$$\begin{tikzcd} \Upsilon(a,\vartheta_a) \arrow[r, "p_2"] \arrow[d, "p_1"'] & G(a) \arrow[d, "t_a"] \\ a \arrow[r, "\vartheta_a"'] & T^\diamond(a)\,. \end{tikzcd} \tag{4.1}$$

Since monomorphisms are stable under pullbacks in any category and since t_a and ϑ_a are both monomorphisms, it follows that p_1 and p_2 are also monomorphisms.

As a right adjoint functor, $T^\diamond$ preserves all limits existing in $\mathbb{A}$ and thus the category $\mathbb{A}^{\mathbf{T}^\diamond}$ admits those limits existing in $\mathbb{A}$. Moreover, the forgetful functor $U^{T^\diamond} : \mathbb{A}^{T^\diamond} \to \mathbb{A}$ creates them; hence these limits (in particular pullbacks) can be computed in $\mathbb{A}$.

Now, $\vartheta_a : a \to T^\diamond(a)$ is the (a, ϑ_a)-component of the unit $\eta^{T^\diamond} : I \to \phi^{T^\diamond} U^{T^\diamond}$ and thus it can be seen as a morphism in $\mathbb{A}^{\mathbf{T}^\diamond}$ from (a, ϑ_a) to $(T^\diamond(a), (\delta^\diamond)_a)$, while $t_a : G(a) \to T^\diamond(a)$ is the $U^{T^\diamond}(a, \vartheta_a)$-component of the natural transformation $\alpha_t : t_* \phi^{T^\diamond} \to \phi^{T^\diamond}$ (see 3.12) and thus it can be seen as a morphism in $\mathbb{A}^{\mathbf{T}^\diamond}$ from $t_*(G(a), \delta_a) = (G(a), t_{G(a)} \cdot \delta_a)$ to $(T^\diamond(a), \delta^\diamond_a)$. It follows that the diagram (4.1) underlies a pullback in $\mathbb{A}^{\mathbf{T}^\diamond}$. In other words, there exists exactly one $T^\diamond$-coalgebra structure $\vartheta_{\Upsilon(a,\vartheta_a)} : \Upsilon(a, \vartheta_a) \to T^\diamond(\Upsilon(a, \vartheta_a))$ on $\Upsilon(a, \vartheta_a)$ making the diagram

$$\begin{tikzcd} (\Upsilon(a,\vartheta_a), \vartheta_{\Upsilon(a,\vartheta_a)}) \arrow[r, "p_2"] \arrow[d, "p_1"'] & (G(a), t_{G(a)} \cdot (\delta_G)_a) \arrow[d, "t_a"] \\ (a,\vartheta_a) \arrow[r, "\vartheta_a"'] & (T^\diamond(a), (\delta^\diamond)_a) \end{tikzcd} \tag{4.2}$$

a pullback in $\mathbb{A}^{\mathbf{T}^\diamond}$. Moreover, since in any functor category, pullbacks are computed componentwise, it follows that the diagram (4.2) is the (a, ϑ_a)-component of a pullback diagram in $\mathbb{A}^{T^\diamond}$,

$$\begin{tikzcd} \Upsilon \arrow[r, "P_2"] \arrow[d, "P_1"'] & t_*\phi^G U^{T^\diamond} \arrow[d, "\alpha_t U^{T^\diamond}"] \\ I \arrow[r, "\eta^{T^\diamond}"'] & \phi^{T^\diamond} U^{T^\diamond}. \end{tikzcd} \tag{4.3}$$

Since the forgetful functor $U_T : \mathbb{A}_T \to \mathbb{A}$ respects monomorphisms and $U^{T^\diamond} K_{T,T^\diamond} = U_T$, it follows that the forgetful functor $U^{T^\diamond} : \mathbb{A}^{T^\diamond} \to \mathbb{A}$ also respects monomorphisms. Thus, the natural transformations $\alpha_t U^{T^\diamond}$ and $\eta^{T^\diamond}$ are both componentwise monomorphisms and hence so too is the natural transformation $P_1 : \Upsilon \to I$.

Summing up, we have seen that for any $(a, \vartheta_a) \in \mathbb{A}^{\mathbf{T}^\diamond}$, $\Upsilon(a, \vartheta_a)$ is an object of $\mathbb{A}^{T^\diamond}$ yielding an endofunctor

$$\Upsilon : \mathbb{A}^{T^\diamond} \to \mathbb{A}^{T^\diamond}, \quad (a, \vartheta_a) \mapsto \Upsilon(a, \vartheta_a).$$

As we shall see later on, the endofunctor Υ is - under some assumptions - the functor-part of an idempotent comonad on $\mathbb{A}^{T^\diamond}$.

4.2. **Proposition.** *Under the assumptions from 4.1, suppose that* $\mathbb{A}$ *admits and* G *preserves equalisers. Then the category* $\mathbb{A}^G$ *also admits equalisers and the functor* $t_* : \mathbb{A}^G \to \mathbb{A}^{T^\diamond}$ *preserves them.*

Proof. Since the functor G preserves equalisers, the forgetful functor $U^G : \mathbb{A}^G \to \mathbb{A}$ creates and preserves equalisers. Thus $\mathbb{A}^G$ admits equalisers.

Since the forgetful functor $U^{T^\diamond} : \mathbb{A}^{T^\diamond} \to \mathbb{A}$ also creates and preserves equalisers, it follows from the commutativity of the diagram

$$\begin{tikzcd} \mathbb{A}^G \arrow[r, "t_*"] \arrow[dr, "U^G"'] & \mathbb{A}^{T^\diamond} \arrow[d, "U^{T^\diamond}"] \\ & \mathbb{A} \end{tikzcd}$$

that the functor $t_* : \mathbb{A}^G \to \mathbb{A}^{T^\diamond}$ preserves equalisers. □

Note that when $\mathbb{A}$ is an abelian category, and G is right exact, then G is left exact by Proposition 2.4 and by 3.8(3).

Note also that it follows from the previous proposition that if $\mathbb{A}$ admits and G preserves equalisers, then the category $\mathbb{A}^G$ admits equalisers. In view of Proposition 4.2, it then follows from 3.12 that the functor $t_* : \mathbb{A}^G \to \mathbb{A}^{T^\diamond}$ has a right adjoint $t^* : \mathbb{A}^{T^\diamond} \to \mathbb{A}^G$.

4.3. **Proposition.** *Under the conditions of Proposition 4.2, the functor* $\Upsilon : \mathbb{A}^{T^\diamond} \to \mathbb{A}^{T^\diamond}$ *is isomorphic to the functor part* t_*t^* *of the* $\mathbb{A}^{T^\diamond}$*-comonad generated by the adjunction* $t_* \dashv t^* : \mathbb{A}^{T^\diamond} \to \mathbb{A}^G$.

Proof. Since the functor G preserves equalisers, the functor $t_* : \mathbb{A}^G \to \mathbb{A}^{T^\diamond}$ also preserves equalisers by Proposition 4.2. Then the top row in the diagram (3.7) is an equaliser, and since the bottom row is also an equaliser and the natural transformation $\alpha_t : t_*\phi^G \to \phi^{T^\diamond}$ is a componentwise monomorphism (since $(\alpha)_a = t_a$ for all $a \in \mathbb{A}$, see 3.12), it follows from Lemma 2.3 that the square in the diagram (3.7) is a pullback. Comparing this pullback with (4.3), one sees that Υ is isomorphic to t_*t^* (and P_1 to ε_t). □

Note that in the situation of the previous proposition, $\varepsilon_t : \Upsilon \to I$ is componentwise a monomorphism.

For the next results we will assume that the $\mathbb{A}$ has and G preserves equalisers. This implies in particular that the category $\mathbb{A}^G$ admits equalisers.

4.4. **Proposition.** *With the data given in 4.1 assume that* $\mathbb{A}$ *has and* G *preserves equalisers. Let* (a, ϑ_a) *be an any object of* $\mathbb{A}^{\mathbf{T}^\diamond}$*. Then*

(i) Υ *is idempotent, that is,* $\Upsilon(\Upsilon(a, \vartheta_a)) \simeq \Upsilon(a, \vartheta_a)$.

(ii) *For every regular* $\mathbf{T}^\diamond$*-subcomodule* (a_0, ϑ_{a_0}) *of* (a, ϑ_a)*, the following diagram is a pullback,*

$$\begin{tikzcd} \Upsilon(a_0, \vartheta a_0) \arrow[r, "\Upsilon(i)"] \arrow[d, "(\varepsilon_t)_{(a_0,\vartheta a_0)}"'] & \Upsilon(a, \vartheta_{a,\vartheta_a}) \arrow[d, "(\varepsilon_t)_{(a,\vartheta_a)}"] \\ a_0 \arrow[r, "i"'] & a. \end{tikzcd}$$

Proof. (i) As observed after Proposition 4.3, the natural transformation $\varepsilon_t : \Upsilon \to I$ is componentwise monomorph. Thus Υ is an idempotent endofunctor.

(ii) For any regular monomorphism $i : (a_0, \vartheta_{a_0}) \to (a, \vartheta_a)$ in $\mathbb{A}^{\mathbf{T}^\diamond}$, consider the commutative diagram

(4.4)
$$\begin{tikzcd}
\Upsilon(a_0,\vartheta_{a_0}) \arrow[rr, "\Upsilon(i)"] \arrow[dr, "(i_t)_{(a_0,\vartheta_{a_0})}"] \arrow[dd, "(\varepsilon_t)_{(a_0,\vartheta_{a_0})}"'] & & \Upsilon(a,\vartheta_a) \arrow[dr, "(i_t)_{(a,\vartheta_a)}"] \arrow[dd, dotted, "(\varepsilon_t)_{(a,\vartheta_a)}" near end] & \\
& G(a_0) \arrow[rr, "G(i)"] \arrow[dd, "t_{a_0}" near end] & & G(a) \arrow[dd, "t_a"] \\
a_0 \arrow[rr, dotted, "i" near start] \arrow[dr, "\vartheta_{a_0}"'] & & a \arrow[dr, dotted, "\vartheta_a"] & \\
& T^\diamond(a_0) \arrow[rr, "T^\diamond(i)"] & & T^\diamond(a)\,.
\end{tikzcd}$$

We claim that the square $(\varepsilon_t)_{(a,\vartheta_a)} \cdot \Upsilon(i) = i \cdot (\varepsilon_t)_{(a_0,\vartheta_{a_0})}$ is a pullback. Indeed, if $f : x \to a_0$ and $g : x \to \Upsilon(a, \vartheta_a)$ are morphisms such that $i \cdot f = (\varepsilon_t)_{(a_0,\vartheta_{a_0})} \cdot g$, then we have

$$T^\diamond(i) \cdot \vartheta_{a_0} \cdot f = \vartheta_a \cdot i \cdot f = \vartheta_a \cdot (\varepsilon_t)_{(a_0,\vartheta_{a_0})} \cdot g = t_a \cdot (i_t)_{(a,\vartheta_a)} \cdot g,$$

and since the square $t_a \cdot G(i) = T^\diamond(i) \cdot t_{a_0}$ is a pullback, there exists a unique morphism $k : x \to G(a_0)$ with $t_{a_0} \cdot k = \vartheta_{a_0} \cdot f$ and $(i_t)_{(a,\vartheta_a)} \cdot g = G(i) \cdot k$. Since the functor $T^\diamond$, as a right adjoint functor, preserves regular monomorphisms, the forgetful functor $U^{T^\diamond} : \mathbb{A}^{T^\diamond} \to \mathbb{A}$ also preserves regular monomorphisms. Thus $i : a_0 \to a$ is a regular monomorphism in $\mathbb{A}$. Then the square $t_{a_0} \cdot (i_t)_{(a_0,\vartheta_{a_0})} = \vartheta_{a_0} \cdot (\varepsilon_t)_{(a_0,\vartheta_{a_0})}$ is a pullback by Proposition 2.4. Therefore, there exists a unique morphism $k' : x \to \Upsilon(a_0, \vartheta_{a_0})$ with $k = (i_t)_{(a_0,\vartheta_{a_0})} \cdot k'$ and $(\varepsilon_t)_{(a_0,\vartheta_{a_0})} \cdot k' = f$. To show that $\Upsilon(i) \cdot k' = g$, consider the composite

$$(\varepsilon_t)_{(a,\vartheta_a)} \cdot \Upsilon(i) \cdot k' = i \cdot (\varepsilon_t)_{(a_0,\vartheta_{a_0})} \cdot k' = i \cdot f = (\varepsilon_t)_{(a,\vartheta_a)} \cdot g.$$

Since $(\varepsilon_t)_{(a,\vartheta_a)}$ is a monomorphism, we get $k \cdot k' = g$. This completes the proof of the fact that the square $(\varepsilon_t)_{(a,\vartheta_a)} \cdot \Upsilon(i) = i \cdot (\varepsilon_t)_{(a_0,\vartheta_{a_0})}$ is a pullback. □

4.5. **Proposition.** *Assume the same conditions as in Proposition 4.4. The functor $t_* : \mathbb{A}^G \to \mathbb{A}^{T^\diamond}$ corestricts to an equivalence between $\mathbb{A}^G$ and the full subcategory of $\mathbb{A}^{T^\diamond}$ generated by those $T^\diamond$-coalgebras (a, ϑ_a) for which there exists an isomorphism $\Upsilon(a, \vartheta_a) \simeq (a, \vartheta_a)$. This holds if and only if there is a (necessarily unique) morphism $x : a \to G(a)$ with $t_a \cdot x = \vartheta_a$.*

Proof. Since the category $\mathbb{A}^G$ admits equalisers, it follows from Theorem 3.13 that the functor t_* is comonadic. Thus $\mathbb{A}^G$ is equivalent to $(\mathbb{A}^{T^\diamond})^\Upsilon$. But since $\varepsilon_t : \Upsilon \to I$ is a componentwise monomorphism, the result follows from 2.2.

Next, $\Upsilon(a, \vartheta_a) \simeq (a, \vartheta_a)$ if and only if the morphism p_1 in the pullback diagram (4.2) is an isomorphism. In this case the composite $x = p_2 \cdot (p_1)^{-1} : a \to G(a)$ satisfies the condition of the proposition. Since t_a is a monomorphism, it is clear that such an x is unique.

Conversely, suppose that there is a morphism $x : a \to G(a)$ with $t_a \cdot x = \vartheta_a$. Then it is easy to see – using that t_a is a monomorphism– that the square

$$\begin{tikzcd}
a \arrow[r, "x"] \arrow[d, "I"'] & G(a) \arrow[d, "t_a"] \\
a \arrow[r, "\vartheta_a"'] & T^\diamond(a)
\end{tikzcd}$$

is a pullback. It follows that $\Upsilon(a, \vartheta_a) \simeq (a, \vartheta_a)$. □

4.6. **Rational functor.** *Assume the data from 4.1 to be given and that* $\mathbb{A}$ *has and* G *preserves equalisers. Define the functor*

$$\mathrm{Rat}^{\mathcal{P}} : \ \mathbb{A}_T \xrightarrow{K_{T,T^\diamond}} \mathbb{A}^{\mathbf{T}^\diamond} \xrightarrow{\Upsilon} \mathbb{A}^{\mathbf{T}^\diamond} \xrightarrow{K^{-1}_{T,T^\diamond}} \mathbb{A}_T \,.$$

Then the triple $(\mathrm{Rat}^{\mathcal{P}}, \varepsilon^{\mathcal{P}}, \delta^{\mathcal{P}})$*, where* $\varepsilon^{\mathcal{P}} = K^{-1}_{T,T^\diamond} \cdot \varepsilon_t \cdot K_{T,T^\diamond}$ *and* $\delta^{\mathcal{P}} = K^{-1}_{T,T^\diamond} \cdot \delta_t \cdot K_{T,T^\diamond}$*, is a comonad on* $\mathbb{A}_T$*. Moreover, for any object* (a, h_a) *of* $\mathbb{A}_T$*,*

(i) $\mathrm{Rat}^{\mathcal{P}}(\mathrm{Rat}^{\mathcal{P}}(a, h_a)) \simeq \mathrm{Rat}^{\mathcal{P}}(a, h_a)$;

(ii) *for any regular* T*-submodule* (a_0, h_{a_0}) *of* (a, h_a)*, the following diagram is a pullback,*

$$\begin{tikzcd}
\mathrm{Rat}^{\mathcal{P}}(a_0, \vartheta a_0) \arrow[r, "\mathrm{Rat}^{\mathcal{P}}(i)"] \arrow[d, "(\varepsilon^{\mathcal{P}})_{(a_0, \vartheta a_0)}"'] & \mathrm{Rat}^{\mathcal{P}}(a, \vartheta_{a,\vartheta_a}) \arrow[d, "(\varepsilon^{\mathcal{P}})_{(a,\vartheta_a)}"] \\
a_0 \arrow[r, "i"'] & a.
\end{tikzcd}$$

Proof. Observing that $(\mathrm{Rat}^{\mathcal{P}}, \varepsilon^{\mathcal{P}}, \delta^{\mathcal{P}})$ is the comonad obtained from the comonad $(\Upsilon, \varepsilon_t, \delta_t)$ along the isomorphism $K^{-1}_{T,T^\diamond} : \mathbb{A}^{T^\diamond} \to \mathbb{A}_T$ (see [23]), the results follow from Proposition 4.4. □

We call a T-module (a, h_a) *rational* if $\mathrm{Rat}^{\mathcal{P}}(a, h_a) \simeq (a, h_a)$ (which is the case if and only if $(\varepsilon^{\mathcal{P}})_{(a,h_a)}$ is an isomorphism, see 2.2), and write $\mathrm{Rat}^{\mathcal{P}}(\mathbf{T})$ for the corresponding full subcategory of $\mathbb{A}_T$. Applying Proposition 4.5 gives:

4.7. **Proposition.** *Under the assumptions of Proposition 4.6, let* $(a, h_a) \in \mathbb{A}_T$*. Then* $(a, h_a) \in \mathrm{Rat}^{\mathcal{P}}(\mathbf{T})$ *if and only if there exists a (necessarily unique) morphism* $x : a \to G(a)$ *inducing commutativity of the diagram*

$$\begin{tikzcd}
 & G(a) \arrow[d, "t_a"] \\
a \arrow[ur, "x"] \arrow[r, "\vartheta_a"'] & T^\diamond(a).
\end{tikzcd}$$

Putting together the information obtained so far, we obtain as main result of this section:

4.8. **Theorem.** *Let* $\mathcal{P} = (\mathbf{T}, \mathbf{G}, \sigma)$ *be a rational pairing on a category* $\mathbb{A}$ *with a comonad* $\mathbf{T}^\diamond = (T^\diamond, \delta^\diamond, \varepsilon^\diamond)$ *right adjoint to* $\mathbf{T}$*. Suppose that* $\mathbb{A}$ *admits and* G *preserves equalisers.*

(1) $\mathrm{Rat}^{\mathcal{P}}(\mathbf{T})$ *is a coreflective subcategory of* $\mathbb{A}_T$*, i.e. the inclusion* $i_{\mathcal{P}} : \mathrm{Rat}^{\mathcal{P}}(\mathbf{T}) \to \mathbb{A}_T$ *has a right adjoint* $\mathrm{rat}^{\mathcal{P}} : \mathbb{A}_T \to \mathrm{Rat}^{\mathcal{P}}(\mathbf{T})$.

(2) *The idempotent comonad on* $\mathbb{A}_T$ *generated by the adjunction* $i_{\mathcal{P}} \dashv \mathrm{rat}^{\mathcal{P}}$ *is just the idempotent monad* $(\mathrm{Rat}^{\mathcal{P}}, \varepsilon^{\mathcal{P}})$.

(3) *The functor*

$$\Phi^{\mathcal{P}} : \mathbb{A}^G \to \mathbb{A}_T, \quad (a, \vartheta_a) \mapsto (a, \sigma_a \cdot T(\vartheta_a)),$$

corestricts to an equivalence of categories $R^{\mathcal{P}} : \mathbb{A}^G \to \mathrm{Rat}^{\mathcal{P}}(\mathbf{T})$.

For illustration we write this theorem out for the case of corings.

4.9. **Rational functor for corings.** For an A-coring C, there is a pairing $\mathcal{P} = (C^*, C, \mathrm{ev})$ on the category ${}_A\mathbb{M}$. Assume this to be rational. Then $C \otimes_A -$ is an exact functor and hence preserves equalisers. Hence, by Theorem 4.8,

(1) $\mathrm{Rat}^{\mathcal{P}}(C^* \otimes_A -)$ *is a coreflective subcategory of* ${}_{C^*}\mathbb{M}$*, i.e. the inclusion* $i_{\mathcal{P}} : \mathrm{Rat}^{\mathcal{P}}(C^* \otimes_A -) \to {}_{C^*}\mathbb{M}$ *has a right adjoint* $\mathrm{rat}^{\mathcal{P}} : {}_{C^*}\mathbb{M} \to \mathrm{Rat}^{\mathcal{P}}(C^* \otimes_A -)$.

(2) *The idempotent comonad on* ${}_{C^*}\mathbb{M}$ *generated by the adjunction* $i_{\mathcal{P}} \dashv \mathrm{rat}^{\mathcal{P}}$ *is just the idempotent monad* $(\mathrm{Rat}^{\mathcal{P}}, \varepsilon^{\mathcal{P}})$.

(3) *The functor*

$$\Phi^{\mathcal{P}} : {}^{C}\mathbb{M} \to {}_{C^*}\mathbb{M}, \quad (M \xrightarrow{\vartheta} C \otimes_A M) \longmapsto (C^* \otimes_A M \xrightarrow{I \otimes \vartheta} C^* \otimes_A C \otimes_A M \xrightarrow{\mathrm{ev} \otimes I} M),$$

leaving the morphisms unchanged, corestricts to an equivalence of categories

$$R^{\mathcal{P}} : {}^{C}\mathbb{M} \to \mathrm{Rat}^{\mathcal{P}}(C^* \otimes_A -).$$

It is straightforward to see, that every C-comodule is a subcomodule of a C-generated comodule, that is, C is a subgenerator in ${}^{C}\mathbb{M}$ (e.g. [8, 18.9]). As a consequence, if $\mathcal{P}$ is rational, then the left C-comodules are precisely the C^*-modules subgenerated by C, and this subcategory is denoted by $\sigma[{}_{C^*}C]$ (e.g. [8, 19.3]). (This type of subcategories of module categories over any ring are thoroughly studied in [24].) Thus the rational functor can be written as a trace functor, that is, for $M \in {}_A\mathbb{M}$,

$$\mathrm{rat}^{\mathcal{P}}(M) = \sum \{\mathrm{Im}\, f \mid f \in \mathrm{Hom}(N, M),\ N \in \sigma[{}_{C^*}C]\}.$$

5. Pairings in monoidal categories

We begin by reviewing some standard definitions associated with monoidal categories.

Let $\mathcal{V} = (\mathbb{V}, \otimes, \mathbb{I})$ be a monoidal category with tensor product $\otimes$ and unit object $\mathbb{I}$. We will freely appeal to MacLane's coherence theorem (see [17]); in particular, we write as if the associativity and unitality isomorphisms were identities. Thus $X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$ and $\mathbb{I} \otimes X = X \otimes \mathbb{I}$ for all $X, Y, X \in \mathbb{V}$. We sometimes collapse $\otimes$ to concatenation, to save space.

Recall that an algebra $\mathbf{A}$ in $\mathcal{V}$ (or $\mathcal{V}$-algebra) is an object A of $\mathbb{V}$ equipped with a multiplication $m_A : A \otimes A \to A$ and a unit $e_A : \mathbb{I} \to A$ satisfying associativity and unitality conditions.

Dually, a coalgebra $\mathbf{C}$ in $\mathcal{V}$ (or $\mathcal{V}$-coalgebra) is an object C of $\mathbb{V}$ equipped with a comultiplication $\delta_C : C \to C \otimes C$, a counit $\varepsilon_C : C \to \mathbb{I}$ subject to coassociativity and counitality conditions.

5.1. **Definition.** A triple $\mathcal{P} = (\mathbf{A}, \mathbf{C}, t)$ consisting of a $\mathcal{V}$-algebra $\mathbf{A}$, a $\mathcal{V}$-coalgebra $\mathbf{C}$ and a morphism $t : A \otimes C \to \mathbb{I}$, for which the diagrams

$$\begin{tikzcd}
A \otimes A \otimes C \arrow[r, "A \otimes A \otimes \delta_C"] \arrow[d, "m_A \otimes C"'] & A \otimes A \otimes C \otimes C \arrow[r, "A \otimes t \otimes C"] & A \otimes C \arrow[d, "t"'] & C \arrow[l, "e_A \otimes C"'] \arrow[dl, "\varepsilon_C"] \\
A \otimes C \arrow[rr, "t"'] & & \mathbb{I} &
\end{tikzcd} \tag{5.1}$$

commute, is called a *left pairing*.

A *left action* of a monoidal category $\mathcal{V} = (\mathbb{V}, \otimes, \mathbb{I})$ on a category $\mathbb{X}$ is a functor

$$-\Diamond- : \mathbb{V} \times \mathbb{X} \to \mathbb{X},$$

called the *action of* $\mathbb{V}$ *on* $\mathbb{X}$, along with invertible natural transformations

$$\alpha_{A,B,X} : (A \otimes B)\Diamond X \to A\Diamond(B\Diamond X) \text{ and } \lambda_X : \mathbb{I}\Diamond X \to X,$$

called the *associativity* and *unit* isomorphisms, respectively, satisfying two coherence axioms (see Bénabou [4]). Again we write as if α and λ were identities.

5.2. **Example.** Recall that if $\mathcal{B}$ is a bicategory (in the sense of Bénabou [4]), then for any $A \in \mathrm{Ob}(\mathcal{B})$, the triple $(\mathcal{B}(A, A), \circ, I_A)$, where $\circ$ denotes the horizontal composition operation, is a monoidal category, and that, for an arbitrary $B \in \mathcal{B}$, there is a canonical left action of $\mathcal{B}(A, A)$ on $\mathcal{B}(B, A)$, given by $f\Diamond g = f \circ g$ for all $f \in \mathcal{B}(A, A)$ and all $g \in \mathcal{B}(B, A)$. In particular, since monoidal categories are nothing but bicategories with exactly one object, any monoidal category $\mathcal{V} = (\mathbb{V}, \otimes, \mathbb{I})$ has a canonical (left) action on the category $\mathbb{V}$, given by $A\Diamond B = A \otimes B$.

5.3. **Actions and pairings.** Given a left action $-\Diamond- : \mathbb{V}\times\mathbb{X}\to\mathbb{X}$ of a monoidal category $\mathcal{V}$ on a category $\mathbb{X}$ and an algebra $\mathbf{A}=(A,e_A,m_A)$ in $\mathcal{V}$, one has a monad $\mathbf{T}_{\mathbf{A}}^{\mathbb{X}}$ on $\mathbb{V}$ defined on any $X\in\mathcal{V}$ by

- $\mathbf{T}_{\mathbf{A}}^{\mathbb{X}}(X)=A\Diamond X$,
- $(e_{\mathbf{T}_{\mathbf{A}}^{\mathbb{X}}})_X=e_A\Diamond X : X=\mathbb{I}\Diamond X\to A\Diamond X=\mathbf{T}_{\mathbf{A}}^{\mathbb{X}}(X)$,
- $(m_{T_{\mathbf{A}}^{\mathbb{X}}})_X=m_A\Diamond X : \mathbf{T}_{\mathbf{A}}^{\mathbb{X}}(\mathbf{T}_{\mathbf{A}}^{\mathbb{X}}(X))=A\Diamond(A\Diamond X)=(A\otimes A)\Diamond X\to A\Diamond X=\mathbf{T}_{\mathbf{A}}^{\mathbb{X}}(X)$,

and we write ${}_{\mathbf{A}}\mathbb{X}$ for the Eilenberg-Moore category of $\mathbf{T}_{\mathbf{A}}$-algebras. Note that in the case of the canonical left action of $\mathcal{V}$ on itself, ${}_{\mathbf{A}}\mathcal{V}$ is just the category of (left) $\mathbf{A}$-modules.

Dually, for a $\mathcal{V}$-coalgebra $\mathbf{C}=(C,\varepsilon_C,\delta_C)$, a comonad $\mathbf{G}_{\mathbb{X}}^{\mathbf{C}}=(C\Diamond-,\,\varepsilon_C\Diamond-,\,\delta_C\Diamond-)$ is defined on $\mathbb{X}$ and one has the corresponding Eilenberg-Moore category ${}^{\mathbf{C}}\mathbb{X}$; for $\mathbb{X}=\mathcal{V}$ this is just the category of (left) $\mathbf{C}$-comodules.

It is easy to see that if $\mathcal{P}=(\mathbf{A},\mathbf{C},t)$ is a left pairing in $\mathcal{V}$, then the triple $\mathcal{P}_{\mathbb{X}}=(\mathbf{T}_{\mathbf{A}}^{\mathbb{X}},\mathbf{G}_{\mathbb{X}}^{\mathbf{C}},\sigma_t)$, where σ_t is the natural transformation

$$t\Diamond- : \mathbf{T}_{\mathbf{A}}\cdot\mathbf{G}^{\mathbf{C}}=A\Diamond(C\Diamond-)=(A\otimes C)\Diamond-\to\mathbb{I}\otimes-=I,$$

is a pairing between the monad $\mathbf{T}_{\mathbf{A}}^{\mathbb{X}}$ and comonad $\mathbf{G}_{\mathbb{X}}^{\mathbf{C}}$.

We say that a left pairing $(\mathbf{A},\mathbf{B},\sigma)$ is $\mathbb{X}$-*rational*, if the corresponding pairing $\mathcal{P}_{\mathbb{X}}$ is rational, i.e. if the map

$$\beta_{X,Y}^{\mathcal{P}_{\mathbb{X}}} : \mathbb{X}(X,C\Diamond Y)\to\mathbb{X}(A\Diamond X,Y)$$

taking $f:X\to C\Diamond Y$ to the composite

$$A\Diamond X\xrightarrow{A\Diamond f}A\Diamond(C\Diamond Y)=(A\otimes C)\Diamond Y\xrightarrow{\sigma\Diamond Y}Y,$$

is injective.

We will generally drop the $\mathbb{X}$ from the notations $\mathbf{T}_{\mathbf{A}}^{\mathbb{X}}$, $\mathbf{G}_{\mathbb{X}}^{\mathbf{C}}$ and $\mathcal{P}_{\mathbb{X}}$ when there is no danger of confusion.

5.4. **Closed categories.** A monoidal category $\mathcal{V}=(\mathbb{V},\otimes,\tau,\mathbb{I})$ is said to be *right closed* if each functor $-\otimes X:\mathbb{V}\to\mathbb{V}$ has a right adjoint $[X,-]:\mathbb{V}\to\mathbb{V}$. So there is a bijection

$$\pi_{Y,X,Z} : \mathbb{V}(Y\otimes X,Z)\simeq\mathbb{V}(Y,[X,Z]),\tag{5.2}$$

with unit $\eta_X^Y : X\to[Y,X\otimes Y]$ and counit $e_Z^Y:[Y,Z]\otimes Y\to Z$.

We write $(-)^*$ for the functor $[-,\mathbb{I}]:\mathbb{V}^{op}\to\mathbb{V}$ that takes $X\in\mathbb{V}$ to $[X,\mathbb{I}]$ and $f:Y\to X$ to the morphism $[f,\mathbb{I}]:[X,\mathbb{I}]\to[Y,\mathbb{I}]$ that corresponds under the bijection (5.2) to the composite

$$[X,\mathbb{I}]\otimes Y\xrightarrow{[X,\mathbb{I}]\otimes f}[X,\mathbb{I}]\otimes X\xrightarrow{e_{\mathbb{I}}^X}\mathbb{I}.$$

Symmetrically, a monoidal category $\mathcal{V}=(\mathbb{V},\otimes,\tau,\mathbb{I})$ is said to be *left closed* if each functor $X\otimes-:\mathbb{V}\to\mathbb{V}$ has a right adjoint $\{X,-\}:\mathbb{V}\to\mathbb{V}$. We write $\overline{\eta}_X^Y : X\to[Y,Y\otimes X]$ and $\overline{e}_Z^Y : Y\otimes\{Y,Z\}\to Z$ for the unit and counit of the adjunction $X\otimes-\dashv\{X,-\}$. One calls a monoidal category *closed* when it is both left and right closed. A typical example is the category of bimodules over a non-commutative ring R, with $\otimes_R$ as $\otimes$.

5.5. **Actions with right adjoints.** Suppose now that $-\Diamond- : \mathbb{V}\times\mathbb{X}\to\mathbb{X}$ is a left action of a monoidal category $\mathcal{V}$ on a category $\mathbb{X}$ and that $\mathbf{A}=(A,e_A,m_A)$ is an algebra in $\mathcal{V}$ such that the functor $A\Diamond-:\mathbb{X}\to\mathbb{X}$ has a right adjoint $\{A,-\}_{\mathbb{X}}:\mathbb{X}\to\mathbb{X}$ (as it surely is when $\mathcal{V}=\mathcal{B}(A,A)$ and $\mathbb{X}=\mathcal{B}(B,A)$ for some objects A,B of a *closed* bicategory $\mathcal{B}$, see Example 5.2.)

Since the functor $\{A,-\}_{\mathbb{X}}:\mathbb{X}\to\mathbb{X}$ is right adjoint to the functor $A\Diamond-:\mathbb{X}\to\mathbb{X}$ and since $A\Diamond-$ is the functor-part of the monad $\mathbf{T}_{\mathbf{A}}$, there is a unique way to make $\{A,-\}_{\mathbb{X}}$ into a comonad $\mathbf{G}(\mathbf{A})=(\{A,-\}_{\mathbb{X}},\,\delta_{\mathbf{G}(\mathbf{A})},\,\varepsilon_{\mathbf{G}(\mathbf{A})})$ such that the comonad $\mathbf{G}(\mathbf{A})$ is right adjoint to the monad $\mathbf{T}_{\mathbf{A}}$ (see 3.3).

Dually, for any coalgebra $\mathbf{C} = (C, \delta_C, \varepsilon_C)$ in $\mathcal{V}$ such that the functor $C \diamondsuit - : \mathbb{X} \to \mathbb{X}$ has a right adjoint $\{C, -\}_{\mathbb{X}} : \mathbb{X} \to \mathbb{X}$, there exists a monad $\mathbf{T}(\mathbf{C})$ whose functor-part is $\{C, -\}_{\mathbb{X}}$ and which is right adjoint to the comonad $\mathbf{G}^{\mathbf{C}}$.

5.6. **Pairings with right adjoints.** Consider a left action of a monoidal category $\mathcal{V}$ on a category $\mathbb{X}$ and a left pairing $\mathcal{P} = (\mathbf{A}, \mathbf{C}, \sigma)$ in $\mathcal{V}$, such that there exist adjunctions

$$A\diamondsuit - \dashv \{A, -\}_{\mathbb{X}}, \quad - \diamondsuit A \dashv [A, -]_{\mathbb{X}}, \quad C\diamondsuit - \dashv \{C, -\}_{\mathbb{X}}, \quad \text{and } - \diamondsuit C \dashv [C, -]_{\mathbb{X}}.$$

Since the comonad $\mathbf{G}(\mathbf{A})$ is right adjoint to the monad $\mathbf{T}_{\mathbf{A}}$, the following are equivalent by Proposition 3.16:

(a) the pairing $\mathcal{P}$ is $\mathbb{X}$-rational;

(b) for every $X \in \mathbb{X}$, the composite

$$\alpha_X^{\mathcal{P}} : C\diamondsuit X \xrightarrow{\overline{\eta}^A_{C\diamondsuit X}} \{A,\, A\diamondsuit(C\diamondsuit X)\}_{\mathbb{X}} = \{A,\, (A \otimes C)\diamondsuit X\}_{\mathbb{X}} \xrightarrow{\{A,\, t\diamondsuit X\}_{\mathbb{X}}} \{A, X\}_{\mathbb{X}},$$

where $\overline{\eta}^A_{C\diamondsuit X}$ is the $C\diamondsuit X$-component of the unit of the adjunction $A\diamondsuit - \dashv \{A, -\}_{\mathbb{X}}$, is a monomorphism.

If these conditions hold, then the functor

$$\Phi^{\mathcal{P}} : {}^{\mathbf{C}}\mathbb{X} \to {}_{\mathbf{A}}\mathbb{X},\ (X, X \xrightarrow{\theta_X} C\diamondsuit X) \longmapsto (X, A\diamondsuit X \xrightarrow{A\diamondsuit\theta_X} A\diamondsuit(C\diamondsuit X) = (A\otimes C)\diamondsuit X \xrightarrow{t\diamondsuit X} X)$$

is full and faithful.

Writing $\mathrm{Rat}^{\mathcal{P}}$ (resp. $\mathrm{Rat}^{\mathcal{P}}(\mathbf{A})$) for the functor $\mathrm{Rat}^{\mathcal{P}_{\mathbb{X}}}$ (resp. the category $\mathrm{Rat}^{\mathcal{P}_{\mathbb{X}}}(\mathbf{T}_{\mathbf{A}})$), one gets:

5.7. **Proposition.** *Under the conditions given in 5.6, assume the category $\mathbb{X}$ to admit equalisers. If $\mathcal{P}$ is an $\mathbb{X}$-rational pairing in $\mathcal{V}$ such that either*

(i) *the functor $C\diamondsuit - : \mathbb{X} \to \mathbb{X}$ preserves equalisers, or*

(ii) *$\mathbb{X}$ admits pushouts and the functor $C\diamondsuit - : \mathbb{X} \to \mathbb{X}$ preserves regular monomorphisms, or*

(iii) *$\mathbb{X}$ admits pushouts and every monomorphism in $\mathbb{X}$ is regular,*

then

(1) *the inclusion $i_{\mathcal{P}} : \mathrm{Rat}^{\mathcal{P}}(\mathbf{A}) \to {}_{\mathbf{A}}\mathbb{X}$ has a right adjoint $\mathrm{rat}^{\mathcal{P}} : {}_{\mathbf{A}}\mathbb{X} \to \mathrm{Rat}^{\mathcal{P}}(\mathbf{A})$;*

(2) *the idempotent comonad on ${}_{\mathbf{A}}\mathbb{X}$ generated by the adjunction is just $\mathrm{Rat}^{\mathcal{P}} : {}_{\mathbf{A}}\mathbb{X} \to {}_{\mathbf{A}}\mathbb{X}$;*

(3) *the functor $\Phi^{\mathcal{P}} : {}^{\mathbf{C}}\mathbb{X} \to {}_{\mathbf{A}}\mathbb{X}$ corestricts to an equivalence $R^{\mathcal{P}} : \mathrm{Rat}^{\mathcal{P}}(\mathbf{A}) \to {}^{\mathbf{C}}\mathbb{X}$.*

Proof. For condition (i), the assertion follows from Theorem 4.8. We show that (iii) is a particular case of (ii), while (ii) is itself a particular case of (i). Indeed, since $\mathcal{P}$ is an $\mathbb{X}$-rational pairing in $\mathcal{V}$, it follows from 3.8(2) that the functor $C\diamondsuit - : \mathbb{X} \to \mathbb{X}$ preserves monomorphisms, and if every monomorphism in $\mathbb{X}$ is regular, then $C\diamondsuit - : \mathbb{X} \to \mathbb{X}$ clearly preserves regular monomorphisms. Next, since the functor $C\diamondsuit - : \mathbb{X} \to \mathbb{X}$ admits a right adjoint, it preserves pushouts, and then it follows from Proposition 2.4 that $C\diamondsuit - : \mathbb{X} \to \mathbb{X}$ preserves equalisers. This completes the proof. □

5.8. **Nuclear objects.** We call an object $V \in \mathbb{V}$ is (left) $\mathbb{X}$-*prenuclear* (resp. $\mathbb{X}$-*nuclear*) if

- the functor $- \otimes V : \mathbb{V} \to \mathbb{V}$ has a right adjoint $[V, -] : \mathbb{V} \to \mathbb{V}$,
- the functor $V^*\diamondsuit - : \mathbb{X} \to \mathbb{X}$, with $V^* = [V, \mathbb{I}]$, has a right adjoint $\{V^*, -\}_{\mathbb{X}} : \mathbb{X} \to \mathbb{X}$, and
- the composite

$$\alpha_X : V\diamondsuit X \xrightarrow{(\eta_{\mathbb{X}})_{V\diamondsuit X}} \{V^*,\, V^*\diamondsuit(V\diamondsuit X)\} = \{V^*,\, (V^* \otimes V)\diamondsuit X\} \xrightarrow{\{V^*,\, e^V_{\mathbb{I}}\diamondsuit X\}} \{V^*, X\},$$

is a monomorphism (resp. an isomorphism), where $((\eta)_{\mathbb{X}})_{V\diamondsuit X}$ is the $(V\diamondsuit X)$-component of the unit $\eta_{\mathbb{X}} : - \to \{V^*, V^*\diamondsuit -\}$ of the adjunction $V^*\diamondsuit - \dashv \{V^*, -\}$.

Note that the morphism $\alpha_X : V \diamondsuit X \to \{V^*, X\}$ is the transpose of the morphism $e_{\mathbb{I}}^V \diamondsuit X : (V^* \otimes V) \diamondsuit X \to X$ under the adjunction $V \diamondsuit - \dashv \{V, -\}_{\mathbb{X}}$.

Applying Proposition 5.7 and 6.10, we get:

5.9. **Proposition.** *Let $\mathcal{V}$ be a monoidal closed category and $\mathbf{C} = (C, \varepsilon_C, \delta_C)$ a $\mathcal{V}$-coalgebra with C $\mathbb{X}$-prenuclear, and assume $\mathbb{X}$ to admit equalisers. If either*

(i) *the functor $C \diamondsuit - : \mathbb{X} \to \mathbb{X}$ preserves equalisers, or*

(ii) *$\mathbb{X}$ admits pushouts and the functor $C \diamondsuit - : \mathbb{X} \to \mathbb{X}$ preserves regular monomorphisms, or*

(iii) *$\mathbb{X}$ admits pushouts and every monomorphism in $\mathbb{X}$ is regular,*

then $\mathrm{Rat}^{\mathcal{P}(\mathbf{C})}(\mathbf{C}^*)$ is a full coreflective subcategory of $_{\mathbf{C}^*}\mathbb{X}$ and the functor $\Phi^{\mathcal{P}(\mathbf{C})} : {}^{\mathbf{C}}\mathbb{X} \to {}_{\mathbf{C}^*}\mathbb{X}$ corestricts to an equivalence $R^{\mathcal{P}(\mathbf{C})} : {}^{\mathbf{C}}\mathbb{X} \to \mathrm{Rat}^{\mathcal{P}(\mathbf{C})}(\mathbf{C}^*)$.

Specialising the previous result to the case of the left action of the monoidal category $_{C^*}\mathbb{M}_{C^*}$ of C^*-bimodules on the category of left C^*-modules, one sees that the equivalence of the category of comodules $^C\mathbb{M}$ and a full subcategory of $_{C^*}\mathbb{M}$ for $_AC$ locally projective addressed in 1.2 is a special case of the preceding theorem.

Recall (e.g. from [17]) that a monoidal category $\mathcal{V} = (\mathbb{V}, \otimes, \mathbb{I})$ is said to be *symmetric* if for all $X, Y \in \mathbb{V}$, there exists functorial isomorphisms $\tau_{X,Y} : X \otimes Y \to Y \otimes X$ obeying certain identities. It is clear that if $\mathbb{V}$ is closed, then $\{X, -\} \simeq [X, -]$ for all $X \in \mathbb{V}$.

5.10**. Pairings in symmetric monoidal closed categories.** Let $\mathcal{V} = (\mathbb{V}, \otimes, \mathbb{I}, \tau, [-,-])$ be a symmetric monoidal closed category and $\mathcal{P} = (\mathbf{A}, \mathbf{C}, t)$ a left pairing in $\mathcal{V}$. For any $X \in \mathcal{V}$, we write

$$\gamma_X : A^* \otimes X \to [A, X]$$

for the morphism that corresponds under π (see (5.2)) to the composition

$$A^* \otimes X \otimes A \xrightarrow{\tau_{A^*,\, X} \otimes A} X \otimes A^* \otimes A \xrightarrow{e_{\mathbb{I}}^A} X\,.$$

Since the functor $[A, -]$, as a left adjoint, preserves colimits, it follows from [14, Theorem 2.3] that there exists a unique morphism $\gamma(t) : C \to A^*$ such that the diagram

(5.3)
$$\begin{tikzcd} C \otimes X \arrow[r, "\gamma(t) \otimes X"] \arrow[dr, "\alpha_X"'] & A^* \otimes X \arrow[d, "\gamma_X"] \\ & {[A, X]} \end{tikzcd}$$

commutes. Hereby the morphism $\gamma(t)$ corresponds under π to the composite $\sigma \cdot \tau_{C,\,A}$.

5.11. **Proposition.** *Consider the situation given in 5.10.*

(1) *If $\mathcal{P}$ is a rational pairing, then the morphism $\gamma(t) : C \to A^*$ is pure, that is, for any $X \in \mathcal{V}$, the morphism $\gamma(t) \otimes X$ is a monomorphism.*

(2) *If A is $\mathcal{V}$-nuclear, then $\mathcal{P}$ is rational if and only if $\gamma(t) : C \to A^*$ is pure.*

(3) *$\Phi^{\mathcal{P}} : {}^{\mathbf{C}}\mathcal{V} \to {}_{\mathbf{A}}\mathcal{V}$ is an isomorphism if and only if A is $\mathcal{V}$-nuclear and $\gamma(t) : C \to A^*$ is an isomorphism. In this case C is also $\mathcal{V}$-nuclear.*

Proof. (1) To say that $\mathcal{P}$ is a rational pairing is to say that α_X is a monomorphism for all $X \in \mathcal{V}$ (see Proposition 5.6). Then it follows from the commutativity of the diagram (5.3) that $\gamma(t) \otimes X$ is also a monomorphism, thus $\gamma(t) : C \to A^*$ is pure.

(2) follows from the commutativity of diagram (5.3).

(3) One direction is clear, so suppose that the functor Φ is an isomorphism of categories. Then it follows from Proposition 3.15(2) that α_X is an isomorphism for all $X \in \mathcal{V}$. It implies that the functor $[A, -]$ preserves colimits and thus the morphism γ_X is an isomorphism (see [14, Theorem 2.3]). Therefore A is $\mathcal{V}$-nuclear. Since α_X and γ_X are both isomorphisms,

it follows from the commutativity of diagram (5.3) that the morphism $\gamma(t) \otimes X$ is also an isomorphism. This clearly implies that $\gamma(t) : C \to A^*$ is an isomorphism.

It is proved in [22, Corollary 2.2] that if A is $\mathcal{V}$-nuclear, then so is A^*. □

6. Entwinings in monoidal categories

Recall (for example, from [19]) that an entwining in a monoidal category $\mathcal{V} = (\mathbb{V}, \otimes, \mathbb{I})$ is a triple $(\mathbf{A}, \mathbf{C}, \lambda)$, where $\mathbf{A} = (A, e_A, m_A)$ is a $\mathcal{V}$-algebra, $\mathbf{C} = (C, \varepsilon_C, \delta_C)$ is a $\mathcal{V}$-coalgebra, and $\lambda : A \otimes C \to C \otimes A$ is a morphism inducing commutativity of the diagrams

$$\begin{tikzcd} C \arrow[d, "e_A \otimes C"'] \arrow[dr, "C \otimes e_A"] & \\ A \otimes C \arrow[r, "\lambda"'] & C \otimes A\,, \end{tikzcd} \qquad \begin{tikzcd} A \otimes C \arrow[d, "\lambda"'] \arrow[dr, "A \otimes \varepsilon_{\mathbf{C}}"] & \\ C \otimes A \arrow[r, "\varepsilon_{\mathbf{C}} \otimes A"'] & A, \end{tikzcd}$$

$$\begin{tikzcd} A \otimes C \arrow[r, "A \otimes \delta_C"] \arrow[d, "\lambda"'] & A \otimes C \otimes C \arrow[r, "\lambda \otimes C"] & C \otimes A \otimes C \arrow[d, "C \otimes \lambda"] \\ C \otimes A \arrow[rr, "\delta_C \otimes A"'] & & C \otimes C \otimes A, \end{tikzcd} \qquad \begin{tikzcd} A \otimes A \otimes C \arrow[r, "A \otimes \lambda"] \arrow[d, "m_A \otimes C"'] & A \otimes C \otimes A \arrow[r, "\lambda \otimes A"] & C \otimes A \otimes A \arrow[d, "C \otimes m_A"] \\ A \otimes C \arrow[rr, "\lambda"'] & & C \otimes A\,. \end{tikzcd}$$

For any entwining $(\mathbf{A}, \mathbf{C}, \lambda)$, the natural transformation

$$\lambda' = \lambda \otimes - : \mathrm{T}_{\mathbf{A}} \circ \mathrm{G}^{\mathbf{C}} = A \otimes C \otimes - \to C \otimes A \otimes - = \mathrm{G}^{\mathbf{C}} \circ \mathrm{T}_{\mathbf{A}}$$

is a mixed distributive law from the monad $\mathrm{T}_{\mathbf{A}}$ to the comonad $\mathrm{G}^{\mathbf{C}}$. We write $\widetilde{\mathbf{C}}$ for the ${}_A\mathbb{V}$-comonad $\widetilde{\mathrm{G}^{\mathbf{C}}}$, that is, for any $(V, h_V) \in {}_A\mathbb{V}$,

$$\widetilde{\mathbf{C}}(V, h_V) = (C \otimes V, A \otimes C \otimes V \xrightarrow{\lambda \otimes V} C \otimes A \otimes V \xrightarrow{C \otimes h_V} C \otimes V),$$

and write ${}^C_A\mathbb{V}(\lambda)$ for the category $\mathbb{V}^{\mathrm{G}^{\mathbf{C}}}_{\mathrm{T}_{\mathbf{A}}}(\lambda')$. An object of this category is a three-tuple (V, θ_V, h_V), where $(V, \theta_V) \in {}^C\mathbb{V}$ and $(V, h_V) \in {}_A\mathbb{V}$, with commuting diagram

$$\begin{tikzcd} A \otimes V \arrow[r, "h_V"] \arrow[d, "A \otimes \theta_V"'] & V \arrow[r, "\theta_V"] & C \otimes V \\ A \otimes C \otimes V \arrow[rr, "\lambda \otimes V"'] & & C \otimes A \otimes V. \arrow[u, "C \otimes h_V"'] \end{tikzcd} \tag{6.1}$$

The assignment $((V, h_V), \theta_{(V,h_V)}) \longmapsto (V, \theta_{(V,h_V)}, h_V)$ yields an isomorphism of categories

$$\Lambda : ({}_A\mathbb{V})^{\widetilde{\mathbf{C}}} \to {}^C_A\mathbb{V}(\lambda).$$

6.1. **Representable entwinings.** For objects A, C in a monoidal category $\mathcal{V} = (\mathbb{V}, \otimes, \mathbb{I})$, consider the functor

$$\mathbb{V}(- \otimes C, A) : \mathbb{V}^{op} \to \mathrm{Set}$$

taking an arbitrary object $V \in \mathbb{V}$ to the set $\mathbb{V}(V \otimes C, A)$. Suppose there is an object $E \in \mathbb{V}$ that represents the functor, i.e. there is a natural bijection

$$\omega : \mathbb{V}(- \otimes C, A) \simeq \mathbb{V}(-, E).$$

Writing $\beta : E \otimes C \to A$ for the morphism $\omega^{-1}(I_E)$, it follows that for any object $V \in \mathbb{V}$ and any morphism $f : V \otimes C \to A$, there exists a unique $\beta_f : V \to E$ making the diagram

$$\begin{tikzcd} V \otimes C \arrow[rr, "f"] \arrow[dr, "\beta_f \otimes C"'] & & A \\ & E \otimes C \arrow[ur, "\beta"'] & \end{tikzcd}$$

commute. It is clear that $\omega^{-1}(\beta_f) = f$.

We call an entwining $(\mathbf{A}, \mathbf{C}, \lambda)$ *representable* if the functor $\mathbb{V}(- \otimes C, A) : \mathbb{V}^{op} \to \mathrm{Set}$ is representable.

6.2. **Examples.**

(i) If the functor $- \otimes C : \mathbb{V} \to \mathbb{V}$ has a right adjoint $[C, -] : \mathbb{V} \to \mathbb{V}$, then it follows from the bijection $\mathbb{V}(V \otimes C, A) \simeq \mathbb{V}(V, [C, A])$ that the object $[C, A]$ represents the functor $\mathbb{V}(- \otimes C, A)$. In particular, when $\mathcal{V}$ is right closed, each entwining in $\mathcal{V}$ is representable.

(ii) If C is a right $\mathcal{V}$-nuclear object, the functor $\mathbb{V}(- \otimes C, A) : \mathbb{V}^{op} \to \mathrm{Set}$ is representable. Indeed, to say that C is right $\mathcal{V}$-nuclear is to say that the functor $C \otimes - : \mathbb{V} \to \mathbb{V}$ has a right adjoint $\{C, -\} : \mathbb{V} \to \mathbb{V}$, the functor $- \otimes {}^*C = - \otimes \{C, \mathbb{I}\} : \mathbb{V} \to \mathbb{V}$ also has a right adjoint $[{}^*C, -] : \mathbb{V} \to \mathbb{V}$, and the morphism

$$\alpha'_V : V \otimes C \to [{}^*C, V]$$

is a natural isomorphism. Considering then the composition of bijections

$$\mathbb{V}(- \otimes {}^*C, ?) \simeq \mathbb{V}(-, [{}^*C, ?]) \simeq \mathbb{V}(-, ? \otimes C),$$

one sees that the functor $- \otimes C$ admits the functor $- \otimes {}^*C$ as a left adjoint. The same arguments as in the proof of Theorem X.7.2 in [17] then show that there is a natural bijection

$$\mathbb{V}(- \otimes C, A) \simeq \mathbb{V}(-, A \otimes {}^*C).$$

Thus the object $A \otimes {}^*C$ represents the functor $\mathbb{V}(- \otimes C, A) : \mathbb{V}^{op} \to \mathrm{Set}$.

6.3. **Proposition.** *Let $(\mathbf{A}, \mathbf{C}, \lambda)$ be a representable entwining in $\mathcal{V} = (\mathbb{V}, \otimes, \mathbb{I})$ with representing object E (see 6.1). Let $e_E = \beta_\tau : \mathbb{I} \to E$ and $m_E = \beta_\varrho : E \otimes E \to E$, where τ is the composite*

$$\mathbb{I} \otimes C \simeq C \xrightarrow{\varepsilon_C} \mathbb{I} \xrightarrow{e_A} A,$$

while ϱ is the composite

$$E \otimes E \otimes C \xrightarrow{E \otimes E \otimes \delta_C} E \otimes E \otimes C \otimes C \xrightarrow{E \otimes \beta \otimes C} E \otimes A \otimes C \xrightarrow{E \otimes \lambda} E \otimes C \otimes A \xrightarrow{\beta \otimes A} A \otimes A \xrightarrow{m_A} A.$$

(i) *The triple (E, e_E, m_E) is a $\mathcal{V}$-algebra.*

(ii) *The morphism $i := \beta_{A \otimes \varepsilon_C} : A \to E$ is a morphism of $\mathcal{V}$-algebras.*

Proof. (i) First observe commutativity of the diagrams

$$\begin{tikzcd} E \otimes E \otimes C \arrow[d, "m_E \otimes C"'] \arrow[dr, "\varrho"] & \\ E \otimes C \arrow[r, "\beta"'] & A, \end{tikzcd} \qquad \begin{tikzcd} \mathbb{I} \otimes C = C \arrow[d, "e_E \otimes C"'] \arrow[dr, "e_A \cdot \varepsilon_C"] & \\ E \otimes C \arrow[r, "\beta"'] & A. \end{tikzcd} \tag{6.2}$$

To prove that m_E is associative, i.e. $m_E \cdot (m_E \otimes E) = m_E \cdot (E \otimes m_E)$, it is to show

$$\beta \cdot (m_E \otimes C) \cdot (m_E \otimes E \otimes C) = \beta \cdot (m_E \otimes C) \cdot (E \otimes m_E \otimes C).$$

This can be achieved by standard arguments which are left to the reader.

To prove that e_E is the unit for the multiplication m_E, it is to show

$$\beta \cdot (m_E \otimes C) \cdot (e_E \otimes E \otimes C) = \beta \quad \text{and} \quad \beta \cdot (m_E \otimes C) \cdot (E \otimes e_E \otimes C) = \beta.$$

Again this can be done by straightforward computation completing the proof of (i).

(ii) We have to show $i \cdot e_A = i_E$ and $m_E \cdot (i \otimes i) = i \cdot m_A$. For this consider the diagram

$$\begin{tikzcd} C \arrow[r, "e_A \otimes C"] \arrow[d, "\varepsilon_C"'] & A \otimes C \arrow[r, "i \otimes C"] \arrow[d, "A \otimes \varepsilon_C"'] & E \otimes C \arrow[dl, "\beta"] \\ \mathbb{I} \arrow[r, "e_A"'] & A & \end{tikzcd}$$

Since the square commutes by naturality of composition, while the triangle commutes by definition of i, the outer diagram is also commutative, meaning $\beta_{e_A \cdot \varepsilon_C} = i \cdot e_A$. But $\beta_{e_A \cdot \varepsilon_C} = e_E$. Thus $i \cdot e_A = e_E$.

Next, consider the diagram

$$\begin{tikzcd}
 & A \otimes A \otimes C \otimes C \arrow[r, "A\otimes A\otimes\varepsilon_C\otimes C"] \arrow[d, "i\otimes A\otimes C\otimes C"] & A \otimes A \otimes C \arrow[r, "A\otimes\lambda"] \arrow[dd, "i\otimes A\otimes C"] & A \otimes C \otimes A \arrow[dd, "i\otimes C\otimes A"'] \arrow[ddr, "A\otimes\varepsilon_C\otimes A"] & \\
A \otimes A \otimes C \arrow[ur, "A\otimes A\otimes\delta_C"] \arrow[d, "i\otimes i\otimes C"'] & E \otimes A \otimes C \otimes C \arrow[d, "E\otimes i\otimes C\otimes C"'] \arrow[dr, "E\otimes A\otimes\varepsilon_C\otimes C"] & & & \\
E \otimes E \otimes C \arrow[r, "E\otimes E\otimes\delta_C"'] & E \otimes E \otimes C \otimes C \arrow[r, "E\otimes\beta\otimes C"'] & E \otimes A \otimes C \arrow[r, "E\otimes\lambda"'] & E \otimes C \otimes A \arrow[r, "\beta\otimes A"'] & A \otimes A
\end{tikzcd}$$

in which the two triangles commute by definition of i, and the quadrangles commute by naturality of composition. We have

$$\begin{array}{rcl}
\omega^{-1}(m_E \cdot (i \otimes i)) & = & m_A \cdot (\beta \otimes A) \cdot (E \otimes \lambda) \cdot (E \otimes \beta \otimes C) \cdot (E \otimes E \otimes \delta_C) \cdot (i \otimes i \otimes C) \\
 & = & m_A \cdot (A \otimes \varepsilon_C \otimes A) \cdot (A \otimes \lambda) \cdot (A \otimes A \otimes \varepsilon_C \otimes C) \cdot (A \otimes A \otimes \delta_C) \\
\text{since } (\varepsilon_C \otimes C) \cdot \delta_C = I & = & m_A \cdot (A \otimes \varepsilon_C \otimes A) \cdot (A \otimes \lambda) \\
\text{by definition of } \lambda & = & m_A \cdot (A \otimes A \otimes \varepsilon_C) \\
\text{nat. of composition} & = & (A \otimes \varepsilon_C) \cdot (m_A \otimes C) \\
\text{since } A \otimes \varepsilon_C = \beta \cdot (i \otimes C) & = & \beta \cdot (i \otimes C) \cdot (m_A \otimes C) \\
 & = & \beta \cdot ((i \cdot m_A) \otimes C) \\
 & = & \omega^{-1}(i \cdot m_A).
\end{array}$$

Thus $m_E \cdot (i \otimes i) = i \cdot m_A$. This completes the proof. □

6.4. **Proposition.** *Assume the data from Proposition 6.3 to be given,*

(1) *For any $(V, \theta_V, h_V) \in {}^C_A\mathbb{V}(\lambda)$, the composite*

$$\iota_V : E \otimes V \xrightarrow{E\otimes\theta_V} E \otimes C \otimes V \xrightarrow{\beta\otimes V} A \otimes V \xrightarrow{h_V} V$$

provides an E-module structure on V.

(2) *There is a functor*

$$\Xi : {}^C_A\mathbb{V}(\lambda) \to {}_E\mathbb{V}, \quad (V, \theta_V, h_V) \mapsto (V, \iota_V),$$

leaving the morphisms unchanged.

(3) *The diagram*

$$\begin{tikzcd}
{}^C_A\mathbb{V}(\lambda) \arrow[rr, "\Xi"] \arrow[dr, "{}^C_AU"'] & & {}_E\mathbb{V} \arrow[dl, "{}_EU"] \\
 & \mathbb{V}, &
\end{tikzcd} \tag{6.3}$$

is commutative where ${}^C_AU : {}^C_A\mathbb{V}(\lambda) \to \mathbb{V}$ is the evident forgetful functor.

Proof. (1) can be verified by direct computation.

(2) Let $f : (V, \theta_V, h_V) \to (V', \theta_{V'}, h_{V'})$ be in ${}^C_A\mathbb{V}(\lambda)$. Then the diagram

$$\begin{tikzcd}
A \otimes V \arrow[r, "h_V"] \arrow[d, "A\otimes f"'] & V \arrow[r, "\theta_V"] \arrow[d, "f"'] & C \otimes V \arrow[d, "C\otimes f"] \\
A \otimes V' \arrow[r, "h_{V'}"'] & V' \arrow[r, "\theta_{V'}"'] & C \otimes V'
\end{tikzcd} \tag{6.4}$$

is commutative. It is to show that f is also a morphism in ${}_E\mathbb{V}$. In the diagram

$$\begin{tikzcd}
E\otimes V \arrow[r, "E\otimes\theta_V"] \arrow[d, "E\otimes f"'] & E\otimes C\otimes V \arrow[r, "\beta\otimes V"] \arrow[d, "E\otimes C\otimes f"'] & A\otimes V \arrow[r, "h_V"] \arrow[d, "A\otimes f"'] & V \arrow[d, "f"] \\
E\otimes V' \arrow[r, "E\otimes\theta_{V'}"'] & E\otimes C\otimes V' \arrow[r, "\beta\otimes V'"'] & A\otimes V' \arrow[r, "h_{V'}"'] & V',
\end{tikzcd}$$

the middle square commutes by naturality of composition, while the other squares commute by (6.4). This proves our claim and shows that the assignment given in (1) yields a functor $\Xi : {}^C_A\mathbb{V}(\lambda) \to {}_E\mathbb{V}$.

(3) It is clear that Ξ makes the diagram (6.3) commute. □

In order to proceed, we need the following result.

6.5. **Lemma.** *Let $\mathbf{T} = (T, e_T, m_T)$ and $\boldsymbol{H} = (H, e_H, m_H)$ be monads on a category $\mathbb{A}$, $i : T \to H$ a monad morphism and $i_* : \mathbb{A}_H \to \mathbb{A}_T$ the functor that takes an H-algebra (a, h_A) to the T-algebra $(a, h_a \cdot i_a)$. Suppose that $\mathbf{T}^\diamond = (T^\diamond, e^\diamond, m^\diamond)$ (resp. $\boldsymbol{H}^\diamond = (H^\diamond, e_{H^\diamond}, m_{H^\diamond})$) is a comonad that is right adjoint to $\mathbf{T}$ (resp. $\boldsymbol{H}$). Write $\bar{i} : H^\diamond \to T^\diamond$ for the mate of i. Then*

(1) *$\bar{i}$ is a morphism of comonads.*

(2) *We have commutativity of the diagram*

(6.5)
$$\begin{tikzcd}
\mathbb{A}_H \arrow[r, "i_*"] \arrow[d, "K_{H,H^\diamond}"'] & \mathbb{A}_T \arrow[d, "K_{T,T^\diamond}"] \\
\mathbb{A}^{H^\diamond} \arrow[r, "(\bar{i})_*"'] & \mathbb{A}^{T^\diamond}.
\end{tikzcd}$$

(3) *If $\mathbb{A}$ admits both equalisers and coequalisers, then*

(i) *the functors i_* and $(\bar{i})_*$ admit both right and left adjoints;*

(ii) *i_* is monadic and $(\bar{i})_*$ is comonadic.*

Proof. (1) This follows from the properties of mates (see, for example, [20]).

(2) An easy calculation shows that for any $(a, h_a) \in \mathbb{A}_H$, one has

$$(\bar{i})_* \circ K_{H,H^\diamond}(a, h_a) = (a, (\bar{i})_a \cdot H^\diamond(h_a) \cdot \sigma_a) \quad \text{and}$$
$$K_{T,T^\diamond} \circ i_*(a, h_a) = (a, T^\diamond(h_a) \cdot T^\diamond(i_a) \cdot \tau_a),$$

where $\sigma : I \to H^\diamond H$ is the unit of the adjunction $H \dashv H^\diamond$, while $\tau : I \to T^\diamond T$ is the unit of the adjunction $T \dashv T^\diamond$. Considering the diagram

$$\begin{tikzcd}
a \arrow[r, "\sigma_a"] \arrow[d, "\tau_a"'] & H^\diamond H(a) \arrow[r, "H^\diamond(h_a)"] \arrow[d, "\bar{i}_{H(a)}"] & H^\diamond(a) \arrow[d, "(\bar{i})_a"] \\
T^\diamond T(a) \arrow[r, "T^\diamond(i_a)"'] & T^\diamond H(a) \arrow[r, "T^\diamond(h_a)"'] & T^\diamond(a),
\end{tikzcd}$$

in which the right square commutes by naturality of $\bar{i}$, while the left one commutes by Theorem IV.7.2 of [17], since $\bar{i}$ is the mate of i. Thus $(\bar{i})_a \cdot H^\diamond(h_a) \cdot \sigma_a = T^\diamond(h_a) \cdot T^\diamond(i_a) \cdot \tau_a$, and hence $(\bar{i})_* \circ K_{H,H^\diamond} = K_{T,T^\diamond} \circ i_*$.

(3)(i) If the category $\mathbb{A}$ admits both equalisers and coequalisers, then $\mathbb{A}_H$ admits equalisers, while $A^{H^\diamond}$ admits coequalisers. Since $K_{H,H^\diamond}$ is an isomorphism of categories, it follows that the category $\mathbb{A}_H$ as well as the category $A^{H^\diamond}$ admit both equalisers and coequalisers. Then, according to 3.12 and its dual, the functor $(\bar{i})_*$ admits a right adjoint $(\bar{i})^*$, while the functor i_* admits a left adjoint i^*. Then clearly the composite $i_! = (K_{H,H^\diamond})^{-1} \cdot (\bar{i})^* \cdot K_{T,T^\diamond}$ is right adjoint to i_*, while the composite $(\bar{i})_! = K_{H,H^\diamond} \cdot i^* \cdot (K_{T,T^\diamond})^{-1}$ is left adjoint to $(\bar{i})_*$.

(3)(ii) Since the functors i_* and $(\bar{i})_*$ are clearly conservative, the assertion follows by a simple application of Beck's monadicity theorem (see, [17]) and its dual. $\square$

6.6. Left and right adjoints to the functor i_*. In the setting considered in Proposition 6.3, suppose now that $\mathbb{V}$ admits both equalisers and coequalisers and that there are adjunctions $A \otimes - \dashv \{A, -\} : \mathbb{V} \to \mathbb{V}$ and $E \otimes - \dashv \{E, -\} : \mathbb{V} \to \mathbb{V}$. Then, according to Lemma 6.5, the functor $i_* : {}_E\mathbb{V} \to {}_A\mathbb{V}$ has both left and right adjoints i^* and $i_!$. It follows from 3.12 and its dual that for any $(V, h_V) \in {}_A\mathbb{V}$, $i^*(V, h_V)$ is the coequaliser

$$E \otimes A \otimes V \underset{E \otimes h_V}{\overset{h_E^r \otimes V}{\rightrightarrows}} E \otimes V \xrightarrow{q_{(V,h_V)}} i^*(V, h_V)\,,$$

while $i_!(V, h_V)$ is the equaliser

$$i_!(V, h_V) \xrightarrow{e_{(V,h_V)}} \{E, V\} \underset{k}{\overset{\{h_E^l, V\}}{\rightrightarrows}} \{A \otimes E, V\}\,, \tag{6.6}$$

where $h_E^r = m_E \cdot (E \otimes i)$, $h_E^l = m_E \cdot (i \otimes E)$ and k is the transpose of the composition

$$A \otimes E \otimes \{E, V\} \xrightarrow{A \otimes \overline{e}_V^E} A \otimes V \xrightarrow{h_V} V.$$

We write $E \otimes_A -$ for the functor i^* (as well as for the ${}_A\mathbb{V}$-monad generated by the adjunction $i^* \dashv i_*$) and write $\{E, -\}_A$ for the functor $i_!$ (as well as for the ${}_A\mathbb{V}$-comonad generated by the adjunction $i_* \dashv i_!$).

According to Lemma 6.5, the comparison functor $K_{i^*} : {}_E\mathbb{V} \to ({}_A\mathbb{V})_{E \otimes_A -}$ is an equivalence of categories. It is easy to check that for any $(V, h_V) \in {}_E\mathbb{V}$, $K_{i^*}(V, h_V) = ((V, \nu_V), \kappa_{(V,h_V)})$, where $\nu_V = h_V \cdot (i \otimes V)$, while $\kappa_{(V,h_V)} : E \otimes_A V \to V$ is the unique morphism with commutative diagram

$$\begin{tikzcd} E \otimes V \arrow[rr, "q_{(V,\nu_V)}"] \arrow[dr, "h_V"'] & & E \otimes_A V \arrow[dl, "\kappa_{(V,h_V)}"] \\ & V & \end{tikzcd} \tag{6.7}$$

Such a unique morphism exists because the morphism $h_V : E \otimes V \to V$ coequalises the pair of morphisms $(h_E \otimes V, E \otimes \nu_V)$.

6.7. Pairing induced by the adjunction $E \otimes_A - \dashv \{E, -\}_A$. We refer to the setting considered in 6.6. Writing $\mathfrak{F}$ for the composition

$$({}_A\mathbb{V})^{\widetilde{\mathbf{C}}} \xrightarrow{\Lambda} {}_A^C\mathbb{V}(\lambda) \xrightarrow{\Xi} {}_E\mathbb{V} \xrightarrow{K_{i^*}} ({}_A\mathbb{V})_{E \otimes_A -} \xrightarrow{K_{E \otimes_A -, \{E,-\}_A}} ({}_A\mathbb{V})^{\{E,-\}_A},$$

one easily sees that $\mathfrak{F}$ makes the diagram

$$\begin{tikzcd} ({}_A\mathbb{V})^{\widetilde{\mathbf{C}}} \arrow[rr, "\mathfrak{F}"] \arrow[dr, "U^{\widetilde{\mathbf{C}}}"'] & & ({}_A\mathbb{V})^{\{E,-\}_A} \arrow[dl, "{}_{\{E,-\}_A}U"] \\ & {}_A\mathbb{V} & \end{tikzcd}$$

commute, were $U^{\widetilde{\mathbf{C}}}$ is the evident forgetful functor. Then, according to 3.9, there is a unique morphism of comonads $\alpha : \widetilde{\mathbf{C}} \to \{E, -\}_A$ such that $\alpha_* = \mathfrak{F}$.

Since the triple $(E \otimes_A -, \{E, -\}_A, \widehat{e}_-^E)$, where $\widehat{e}_-^E$ is the counit of the adjunction $E \otimes_A - \dashv \{E, -\}_A$, is a pairing (see 3.1), it follows from Proposition 3.5 that the triple

$$\mathcal{P}(\lambda) = (E \otimes_A -, \{E, -\}_A, \sigma := \widehat{e}_-^E \cdot (E \otimes_A \alpha)) \tag{6.8}$$

is a pairing on the category ${}_A\mathbb{V}$.

A direct inspection shows that for any $((V,\nu_V),\theta_{(V,\nu_V)}) \in ({}_A\mathbb{V})^{\widetilde{\mathbf{C}}}$, $\Xi\Lambda((V,\nu_V),\theta_{(V,\nu_V)}) = (V,\xi)$, where ξ is the composite

$$E\otimes V \xrightarrow{E\otimes\theta_{(V,\nu_V)}} E\otimes C\otimes V \xrightarrow{\beta\otimes V} A\otimes V \xrightarrow{\nu_V} V.$$

Then

$$K_{i^*}\Xi\Lambda((V,\nu_V),\theta_{(V,\nu_V)}) = K_{i^*}(V,\xi) = ((V,\nu_V),\kappa_{(V,\xi)} : E\otimes_A V \to V),$$

and thus

$$\mathfrak{F}((V,\nu_V),\theta_{(V,\nu_V)}) = K_{E\otimes_A -,\,\{E,-\}_A}((V,\nu_V),\kappa_{(V,\xi)}) = ((V,\nu_V),\overline{\theta}_{(V,\nu_V)}),$$

where $\overline{\theta}_{(V,\nu_V)}$ is the composite

$$V \xrightarrow{\widehat{\eta}^E_V} \{E, E\otimes_A V\}_A \xrightarrow{\{E,\kappa_{(V,\xi)}\}_A} \{E,V\}_A.$$

Here $\widehat{\eta}^E_- : I \to \{E, E\otimes_A -\}_A$ is the unit of the adjunction $E\otimes_A - \dashv \{E,-\}_A$.

Since for any object $(V,\nu_V)\in \mathbb{V}_A$, the pair

$$(\widetilde{C}(V,\nu_V),(\delta_{\widetilde{C}})_{(V,\nu_V)}) = ((C\otimes V,h),\delta_C\otimes V),$$

where h is the composite $(C\otimes\nu_V)\cdot(\lambda\otimes V) : A\otimes C\otimes V \to C\otimes V$, is an object of the category $({}_A\mathbb{V})^{\widetilde{C}}$, one has

$$\mathfrak{F}((C\otimes V,h),\delta_C\otimes V) = ((C\otimes V,(C\otimes\nu_V)\cdot(\lambda\otimes V)),\overline{\theta}_{(C\otimes V,h)}),$$

where $\overline{\theta}_{(C\otimes V,h)} : C\otimes V \to \{E, E\otimes_A V\}_A$ is the composite

$$C\otimes V \xrightarrow{\widehat{\eta}^E_{C\otimes V}} \{E, E\otimes_A (C\otimes V)\}_A \xrightarrow{\{E,\kappa_{(C\otimes V,\xi)}\}} \{E, C\otimes V\}_A.$$

Here ξ is the composite

$$E\otimes C\otimes V \xrightarrow{E\otimes\delta_C\otimes V} E\otimes C\otimes C\otimes V \xrightarrow{\beta\otimes C\otimes V} A\otimes C\otimes V \xrightarrow{\lambda\otimes V} C\otimes A\otimes V \xrightarrow{C\otimes\nu_V} C\otimes V.$$

Now, according to 3.9, the (V,ν_V)-component $\alpha_{(V,\nu_V)}$ of the comonad morphism $\alpha : \widetilde{C} \to \{E,-\}_A$ is the composite

$$C\otimes V \xrightarrow{\widehat{\eta}^E_{C\otimes V}} \{E, E\otimes_A (C\otimes V)\}_A \xrightarrow{\{E,\kappa_{(C\otimes V,\xi)}\}} \{E, C\otimes V\}_A \xrightarrow{\{E,\varepsilon_C\otimes V\}_A} \{E,V\}_A,$$

i.e. $\alpha_{(V,\nu_V)}$ is the transpose of the composite $(\varepsilon_C\otimes V)\cdot\kappa_{(C\otimes V,\xi)}$. By (6.7), the diagram

$$\begin{tikzcd}
E\otimes C\otimes V \arrow[d, "q_{(C\otimes CV,\xi)}"'] \arrow[dr, "\xi"] & & \\
E\otimes_A (C\otimes V) \arrow[r, "\kappa_{(C\otimes V,\xi)}"'] & C\otimes V \arrow[r, "\varepsilon_C\otimes V"'] & V
\end{tikzcd}$$

commutes. In the diagram

$$\begin{tikzcd}
E\otimes C\otimes V \arrow[r, "E\otimes\delta_c\otimes V"] \arrow[dr, equal] & E\otimes C\otimes C\otimes V \arrow[r, "\beta\otimes C\otimes V"] \arrow[d, "E\otimes\varepsilon_C\otimes C\otimes V"] & A\otimes C\otimes V \arrow[r, "\lambda\otimes V"] \arrow[dr, "A\otimes\varepsilon_C\otimes C"'] & C\otimes A\otimes V \arrow[r, "C\otimes\nu_V"] \arrow[d, "\varepsilon_C\otimes A\otimes V"] & C\otimes V \arrow[d, "\varepsilon_C\otimes V"] \\
 & E\otimes C\otimes V \arrow[rr, "\beta\otimes V"'] & & A\otimes V \arrow[r, "\nu_V"'] & V\,,
\end{tikzcd}$$

the left triangle commutes since $\mathbf{C}$ is a coalgebra in $\mathcal{V}$, the middle triangle commutes since the triple $(\mathbf{A},\mathbf{C},\lambda)$ is an entwining, and the trapeze and the rectangle commute by naturality of composition, hence $(\varepsilon_C \otimes V)\cdot\xi = \nu_V \cdot (\beta\otimes V)$. Thus, $\kappa_{(C\otimes V,\xi)}$ is the unique morphism that makes the diagram

$$\begin{tikzcd} E\otimes C\otimes V \arrow[r, "\beta\otimes V"] \arrow[d, "q_{(C\otimes V,h)}"'] & A\otimes V \arrow[d, "\nu_V"] \\ E\otimes_A (C\otimes V) \arrow[r, "(\varepsilon_C\otimes V)\cdot\kappa_{(C\otimes V,h)}"'] & V \end{tikzcd} \tag{6.9}$$

commute. (Recall that here $h = (C\otimes\nu_V)\cdot(\lambda\otimes V) : A\otimes C\otimes V \to C\otimes V$.)

6.8. **Proposition.** *For any $(V,\nu_V)\in {}_A\mathbb{V}$, consider the morphism $\alpha'_{(V,\nu_V)} : C\otimes V \to \{E,V\}$ that is the transpose of the composition $E\otimes C\otimes V \xrightarrow{\beta\otimes V} A\otimes V \xrightarrow{\nu_V} V$. Then the diagram*

$$\begin{tikzcd} \{E,V\}_A \arrow[rr, "e_{(V,\nu_V)}"] & & \{E,V\} \\ & C\otimes V \arrow[ul, "\alpha_{(V,\nu_V)}"] \arrow[ur, "\alpha'_{(V,\nu_V)}"'] & \end{tikzcd}$$

is commutative.

Proof. We show first that

$$\{h_E, V\}\cdot\alpha'_{(V,\nu_V)} = k\cdot\alpha'_{(V,\nu_V)} \tag{6.10}$$

(see the equaliser diagram (6.6)). Since the transpose of the morphism k is the composite

$$A\otimes E\otimes\{E,V\} \xrightarrow{A\otimes\overline{e}^E_V} A\otimes V \xrightarrow{h_V} V,$$

while the transpose of the morphism $\{h^l_E, V\}$ is the composite

$$A\otimes E\otimes\{E,V\} \xrightarrow{A\otimes E\otimes\{h^l_E,V\}} A\otimes E\otimes\{A\otimes E,V\} \xrightarrow{\overline{e}^{A\otimes E}_V} V$$

and this is easily seen to be the composite

$$A\otimes E\otimes\{E,V\} \xrightarrow{h^l_E\otimes\{E,V\}} E\otimes\{E,V\} \xrightarrow{\overline{e}^E_V} V.$$

Thus, since

$$(h^l_E\otimes\{E,V\})\cdot(A\otimes E\otimes\alpha'_{(V,\nu_V)}) = (E\otimes\alpha'_{(V,\nu_V)})\cdot(h^l_E\otimes C\otimes V)$$

by naturality of composition, it is enough to show that the diagram

$$\begin{tikzcd} A\otimes E\otimes C\otimes V \arrow[r, "A\otimes E\otimes\alpha'_{(V,\nu_V)}"] \arrow[d, "h^l_E\otimes C\otimes V"'] & A\otimes E\otimes\{E,V\} \arrow[r, "A\otimes\overline{e}^E_V"] \arrow[d, dashed, "h^l_E\otimes\{E,V\}"'] & A\otimes V \arrow[d, "\nu_V"] \\ E\otimes C\otimes V \arrow[r, "E\otimes\alpha'_{(V,\nu_V)}"'] & E\otimes\{E,V\} \arrow[r, "\overline{e}^E_V"'] & V \end{tikzcd}$$

is commutative. Since $\overline{e}^E_V\cdot(E\otimes\alpha'_{(V,\nu_V)}) = \nu_V\cdot(\beta\otimes V)$, the diagram can be rewritten as

$$\begin{tikzcd} A\otimes E\otimes C\otimes V \arrow[r, "A\otimes(\nu_V\cdot(\beta\otimes V))"] \arrow[d, "h^l_E\otimes C\otimes V"'] & A\otimes V \arrow[d, "\nu_V"] \\ E\otimes C\otimes V \arrow[r, "\nu_V\cdot(\beta\otimes V)"'] & V\,. \end{tikzcd} \tag{6.11}$$

Consider now the diagram

$$\begin{tikzcd}
A\otimes E\otimes C \arrow[r, "A\otimes E\otimes\delta_C"] \arrow[d, "i\otimes E\otimes C"'] & A\otimes E\otimes C\otimes C \arrow[r, "A\otimes\beta\otimes C"] \arrow[d, "i\otimes E\otimes C\otimes C"'] & A\otimes A\otimes C \arrow[r, "A\otimes\lambda"] \arrow[d, "i\otimes A\otimes C"'] & A\otimes C\otimes A \arrow[d, "i\otimes C\otimes A"'] \arrow[dr, "A\otimes\varepsilon_C\otimes A"] & \\
E\otimes E\otimes C \arrow[r, "E\otimes E\otimes\delta_C"'] & E\otimes E\otimes C\otimes C \arrow[r, "E\otimes\beta\otimes C"'] & E\otimes A\otimes C \arrow[r, "E\otimes\lambda"'] & E\otimes C\otimes A \arrow[r, "\beta\otimes A"'] & A\otimes A\,,
\end{tikzcd}$$

in which the three rectangles commute by naturality of composition, while the triangle commutes since $\omega^{-1}(A\otimes\varepsilon_C)=\beta\cdot(i\otimes C)$. Recalling now that $h^l_E=m_E\cdot(i\otimes E)$, we have

$$\begin{aligned}
\beta\cdot(h^l_E\otimes C) &= \beta\cdot(m_E\otimes C)\cdot(i\otimes E\otimes C)\\
&= m_A\cdot(\beta\otimes A)\cdot(E\otimes\lambda)\cdot(E\otimes\beta\otimes C)\cdot(E\otimes E\otimes\delta_C)\cdot(i\otimes E\otimes C)\\
&= m_A\cdot(A\otimes\varepsilon_C\otimes A)\cdot(A\otimes\lambda)\cdot(A\otimes\beta\otimes C)\cdot(A\otimes E\otimes\delta_C)\\
\text{since } \lambda \text{ is an entwining} &= m_A\cdot(A\otimes A\otimes\varepsilon_C)\cdot(A\otimes\beta\otimes C)\cdot(A\otimes E\otimes\delta_C)\\
\text{nat. of composition} &= m_A\cdot(A\otimes\beta)\cdot(A\otimes E\otimes C\otimes\varepsilon_C)\cdot(A\otimes E\otimes\delta_C)\\
\text{since } (C\otimes\varepsilon_C)\cdot\delta_C=I_C &= m_A\cdot(A\otimes\beta).
\end{aligned}$$

Therefore, $m_A\cdot(A\otimes\beta)=\beta\cdot(h^l_E\otimes C)$, and hence

$$(m_A\otimes V)\cdot(A\otimes\beta\otimes V)=(\beta\otimes V)\cdot(h^l_E\otimes C\otimes V).$$

Using now that $\nu_V\cdot(A\otimes\nu_V)=\nu_V\cdot(m_A\otimes V)$, since $(V,\nu_V)\in{}_A\mathbb{V}$, one has

$$\nu_V\cdot(A\otimes\nu_V)\cdot(A\otimes\beta\otimes V)=\nu_V\cdot(\beta\otimes V)\cdot(h^l_E\otimes C\otimes V).$$

Thus the diagram (6.11) commutes. It follows that $\{h^l_E,V\}\cdot\alpha'_{(V,\nu_V)}=k\cdot\alpha'_{(V,\nu_V)}$, and since the diagram (6.6) is an equaliser, there exists a unique morphism $\gamma_{(V,\nu_V)}:C\otimes V\to\{E,V\}_A$ that makes the diagram

$$\begin{tikzcd}
\{E,V\}_A \arrow[rr, "e_{(V,\nu_V)}"] & & \{E,V\} \\
& C\otimes V \arrow[ul, "\gamma_{(V,\nu_V)}"] \arrow[ur, "\alpha'_{(V,\nu_V)}"'] &
\end{tikzcd}$$

commute. We claim that $\gamma_{(V,\nu_V)}=\alpha_{(V,\nu_V)}$. To see this, consider the diagram

$$\begin{tikzcd}
E\otimes C\otimes V \arrow[r, "E\otimes\gamma_{(V,\nu_V)}"] \arrow[d, "q_{(C\otimes V,h)}"'] & E\otimes\{E,V\}_A \arrow[r, "E\otimes e_{(V,\nu_V)}"] \arrow[d, "q_{\{E,V\}_A}"'] & E\otimes\{E,V\} \arrow[d, "\overline{e}^E_V"] \\
E\otimes_A(C\otimes V) \arrow[r, "E\otimes_A\gamma_{(V,\nu_V)}"'] & E\otimes_A\{E,V\}_A \arrow[r, "\widehat{e}^E_V"'] & V\,,
\end{tikzcd}$$

where $\widehat{e}^E_-$ is the counit of the adjunction $-\otimes_A E\dashv[E,-]_A$. In this diagram, the left rectangle commutes by naturality of q (recall that $\gamma_{(V,\nu_V)}:C\otimes V\to\{E,V\}_A$ is a morphism in ${}_A\mathbb{V}$), while the right one commutes by definition of $\widehat{e}^E_-$. Since

$$\overline{e}^E_V\cdot(E\otimes e_{(V,\nu_V)})\cdot(E\otimes\gamma_{(V,\nu_V)})=\overline{e}^E_V\cdot(E\otimes(e_{(V,\nu_V)}\cdot\gamma_{(V,\nu_V)}))=\overline{e}^E_V\cdot(E\otimes\alpha'_{(V,\nu_V)}),$$

and since $\overline{e}^E_V\cdot(E\otimes\alpha'_{(V,\nu_V)})=\nu_V\cdot(\beta\otimes V)$, it follows that the diagram

$$\begin{tikzcd}
E\otimes C\otimes V \arrow[r, "\beta\otimes V"] \arrow[d, "q_{(C\otimes V,h)}"'] & A\otimes V \arrow[d, "\nu_V"] \\
E\otimes_A(V\otimes C) \arrow[r, "\widehat{e}^E_V\cdot(E\otimes_A\gamma_{(V,\nu_V)})"'] & V
\end{tikzcd}$$

commutes. Comparing this diagram with (6.9), one sees that

$$\widehat{e}^E_V\cdot(E\otimes_A\gamma_{(V,\nu_V)})=(\varepsilon_C\otimes V)\cdot\kappa_{(C\otimes V,h)}.$$

Thus $\gamma_{(V,\nu_V)} : C \otimes V \to \{E, V\}_A$ is the transpose of the morphism $(\varepsilon_C \otimes V) \cdot \kappa_{(C\otimes V,h)}$. Thus $\gamma_{(V,\nu_V)}$ is just $\alpha_{(V,\nu_V)}$. This completes the proof. $\square$

When the pairing $\mathcal{P}(\lambda)$ (6.8) is rational, we write $\mathrm{Rat}^{\mathcal{P}(\lambda)}(E)$ for the full subcategory of the category ${}_E\mathbb{V}$ generated by those objects whose images under the functor K_{i^*} lie in the category $\mathrm{Rat}^{\mathcal{P}(\lambda)}(E \otimes_A -)$.

The following result extends [1, Theorem 3.10], [12, Proposition 2.1], and [13, Theorem 2.6] from module categories to monoidal categories.

6.9. **Theorem.** *Let $\mathcal{V} = (\mathbb{V}, \otimes, \mathbb{I})$ be a monoidal category with $\mathbb{V}$ admitting both equalisers and coequalisers, and $(\mathbf{A}, \mathbf{C}, \lambda)$ a representable entwining with representable object E. Suppose that*

(1) *the functors $A \otimes -$, $E \otimes - : \mathbb{V} \to \mathbb{V}$ have right adjoints $\{A, -\}$ and $\{E, -\}$,*
(2) *for any $(V, \nu_V) \in {}_A\mathbb{V}$, the transpose $\alpha'_{(V,\nu_V)} : C \otimes V \to \{E, V\}$ of the composite $E \otimes C \otimes V \xrightarrow{\beta\otimes V} A \otimes V \xrightarrow{\nu_V} V$ is a monomorphism, and*
(3) (i) *the functor $C \otimes - : \mathbb{V} \to \mathbb{V}$ preserves equalisers, or*
   (ii) *the category $\mathbb{V}$ admits pushouts and the functor $C \otimes - : \mathbb{V} \to \mathbb{V}$ preserves regular monomorphisms and has a right adjoint, or*
   (iii) *the category $\mathbb{V}$ admits pushouts, every monomorphism in $\mathbb{V}$ is regular and the functor $C \otimes - : \mathbb{V} \to \mathbb{V}$ has a right adjoint.*

Then the pairing $\mathcal{P}(\lambda)$ is rational and there is an equivalence of categories

$$ {}^C_A\mathbb{V}(\lambda) \simeq \mathrm{Rat}^{\mathcal{P}(\lambda)}(E). $$

Proof. Since $e_{(V,\nu_V)} : \{E, V\}_A \to \{E, V\}$ is an equaliser for all $(V, \nu_V) \in {}_A\mathbb{V}$, $\alpha_{(V,\nu_V)}$ is a monomorphism if and only if $\alpha'_{(V,\nu_V)}$ is so. Thus, the pairing $\mathcal{P}(\lambda)$ is rational if and only if for any $(V, \nu_V) \in {}_A\mathbb{V}$, the morphism $\alpha'_{(V,\nu_V)} : C \otimes V \to \{E, V\}$ is a monomorphism. Thus, condition (2) implies that the pairing $\mathcal{P}(\lambda)$ is rational.

Next, since the forgetful functor ${}_AU : {}_A\mathbb{V} \to \mathbb{V}$ preserves and creates equalisers, the functor $\widetilde{C} : {}_A\mathbb{V} \to {}_A\mathbb{V}$ preserves equalisers if and only if the composite ${}_AU\widetilde{C} : {}_A\mathbb{V} \to \mathbb{V}$ does so. But for any $(V, \nu_V) \in {}_A\mathbb{V}$, ${}_AU\widetilde{C}(V, \nu_V) = C \otimes V$. It follows that if the functor $C \otimes - : \mathbb{V} \to \mathbb{V}$ preserves equalisers, then the functor $\widetilde{C} : {}_A\mathbb{V} \to {}_A\mathbb{V}$ does so. Since each of the conditions in (3) implies that the functor $C \otimes - : \mathbb{V} \to \mathbb{V}$ preserves equalisers (see the proof of Proposition 5.7) and since the functor $E \otimes_A - : {}_A\mathbb{V} \to {}_E\mathbb{V}$ has a right adjoint $\{E, -\}_A : {}_A\mathbb{V} \to {}_E\mathbb{V}$, one can apply Theorem 4.8 to get the desired result. $\square$

Since for any $\mathcal{V}$-coalgebra $\mathbf{C} = (C, \varepsilon_C, \delta_C)$, the identity morphism $I_C : C \otimes \mathbb{I} = C \to C = \mathbb{I} \otimes C$ is an entwining from the trivial $\mathcal{V}$-algebra $\mathcal{I} = (\mathbb{I}, I_{\mathbb{I}}, I_{\mathbb{I}})$ to the $\mathcal{V}$-coalgebra $\mathbf{C}$, it follows from Example 6.2(1) that this entwining is representable with representable object $C^* = [C, \mathbb{I}]$. Applying Proposition 6.3 gives:

6.10**. Coalgebras in monoidal closed categories.** *Assume the monoidal category $\mathcal{V}$ to be closed and consider any $\mathcal{V}$-coalgebra $\mathbf{C} = (C, \varepsilon_C, \delta_C)$. Then the triple*

$$ \mathbf{C}^* = (C^* = [C, \mathbb{I}], e_{C^*}, m_{C^*}) $$

is a $\mathcal{V}$-algebra, where $e_{C^*} = \pi(\varepsilon_C)$, while m_{C^*} is the morphism $C^* \otimes C^* \to C^*$ that corresponds to the composite

$$ C^* \otimes C^* \otimes C \xrightarrow{C^*\otimes C^*\otimes\delta_C} C^* \otimes C^* \otimes C \otimes C \xrightarrow{C^*\otimes e^C_{\mathbb{I}}\otimes C} C^* \otimes C \xrightarrow{e^C_{\mathbb{I}}} \mathbb{I} $$

under the bijection (see (5.2))

$$ \pi = \pi_{C^*\otimes C^*,C,\mathbb{I}} : \mathbb{V}(C^* \otimes C^* \otimes C, \mathbb{I}) \simeq \mathbb{V}(C^* \otimes C^*, C^*). $$

6.11. **Proposition.** *In the situation of 6.10, the triple*

$$\mathcal{P}(\mathbf{C}) = (\mathbf{C}^*, \mathbf{C}, t = e_{\mathbb{I}}^C : C^* \otimes C \to \mathbb{I})$$

is a left pairing in $\mathcal{V}$.

Proof. We just note that the equalities $\pi^{-1}(e_{C^*}) = \varepsilon_C$ and

$$\pi^{-1}(m_{C^*}) = e_{\mathbb{I}}^C \cdot (C^* \otimes e_{\mathbb{I}}^C \otimes C) \cdot (C^* \otimes C^* \otimes \delta_C)$$

imply commutativity of the diagrams

$$\begin{tikzcd}
C^* \otimes C^* \otimes C \arrow[r, "C^* \otimes C^* \otimes \delta_C"] \arrow[d, "m_{C^*} \otimes C"'] & C^* \otimes C^* \otimes C \otimes C \arrow[r, "C^* \otimes e_{\mathbb{I}}^C \otimes C"] & C^* \otimes C \arrow[d, "e_{\mathbb{I}}^C"] & C \arrow[l, "e_{C^*} \otimes C"'] \arrow[dl, "\varepsilon_C"] \\
C^* \otimes C \arrow[rr, "e_{\mathbb{I}}^C"'] & & \mathbb{I}. &
\end{tikzcd}$$

□

Applying now either Proposition 5.9 or Proposition 6.11 yields

6.12. **Theorem.** *Let Let $\mathcal{V} = (\mathbb{V}, \otimes, \mathbb{I}, [-,-])$ be a monoidal closed category, $\mathbf{C} = (C, \varepsilon_C, \delta_C)$ a $\mathcal{V}$-coalgebra with C $\mathcal{V}$-prenuclear, and assume $\mathbb{V}$ to admit equalisers. If either*

(i) *the functor $C \otimes - : \mathbb{V} \to \mathbb{V}$ preserves equalisers, or*

(ii) *$\mathbb{V}$ admits pushouts and the functor $C \otimes - : \mathbb{V} \to \mathbb{V}$ preserves regular monomorphisms, or*

(iii) *$\mathbb{V}$ admits pushouts and every monomorphism in $\mathbb{V}$ is regular,*

then $\mathrm{Rat}^{\mathcal{P}(\mathbf{C})}(\mathbf{C}^*)$ is a full coreflective subcategory of ${}_{\mathbf{C}^*}\mathbb{V}$ and the functor $\Phi^{\mathcal{P}(\mathbf{C})} : {}^{\mathbf{C}}\mathbb{V} \to {}_{\mathbf{C}^*}\mathbb{V}$ corestricts to an equivalence $R^{\mathcal{P}(\mathbf{C})} : {}^{\mathbf{C}}\mathbb{V} \to \mathrm{Rat}(\mathbf{C}^*)$.

A special case of the situation described in Theorem 6.12 is given by a locally projective A-coalgebra C over a commutative ring A and $\mathbb{V}$ the category of A-modules (see also (4.9)).

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