Static modules and equivalences

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Abstract

By a well known theorem of K. Morita, any equivalence between full module categories over rings R and S, are given by a bimodule ${}_RP_S$, such that ${}_RP$ is a finitely generated projective generator in R-*Mod* and $S = \mathrm{End}_R(P)$. There are various papers which describe equivalences between certain subcategories of R-*Mod* and S-*Mod* in a similar way with suitable properties of ${}_RP_S$. Here we start from the other side: Given any bimodule ${}_RP_S$ we ask for the subcategories which are equivalent to each other by the functor $\mathrm{Hom}_R(P, -)$. In R-*Mod* these are the *P-static (= P-solvable) modules.* In this context properties of s-Σ-quasi-projective, w-Σ-quasi-projective and (self-) tilting modules ${}_RP$ are reconsidered as well as Mittag-Leffler properties of P_S. Moreover for any ring extension $R \to A$ related properties of the A-module $A \otimes_R P$ are investigated.

1 Introduction

It was noticed by K. Morita that an R-Module P is a finitely generated, projective generator P in R-*Mod*, if and only if the functor

$$\mathrm{Hom}_R(P, -) : R\text{-}Mod \to S\text{-}Mod,$$

defines a category equivalence, where $S = \mathrm{End}_R(P)$.

Many authors have worked on generalizations of this setting by looking at representable equivalences between proper subcategories. Imposing various conditions on these subcategories, such as closure under submodules, factor modules or extensions, the problem was to find a module P with suitable properties to characterize the equivalence under consideration.

For this purpose notions like quasi-projective, s-Σ-quasi-projective, w-$\sum$-quasi-projective modules were introduced, and the generator property was replaced by weaker conditions. We refer to the papers of U. Albrecht, R. Colpi, T.G. Faticoni, K.R. Fuller, A.I. Kashu, T. Kato, C. Menini, T. Onodera, A. Orsatti, M. Sato and others for this approach. It should be mentioned that the importance of *tilting modules* in representation theory gave a new impact to this kind of investigation.

Here we suggest to put the question the other way round. We do not assume the subcategories to be given but we start with any R-module P and $S = \mathrm{End}_R(P)$. Then we ask if there are any non-trivial subcategories $\mathcal{C} \subset R\text{-}Mod$ and $\mathcal{D} \subset S\text{-}Mod$ for which $\mathrm{Hom}_R(P, -)$ provides an equivalence. Since the functor $P \otimes_S -$ is left adjoint to $\mathrm{Hom}_R(P, -)$ we know that the modules in $\mathcal{C}$ must be "invariant" under $P \otimes_S \mathrm{Hom}_R(P, -)$. Following Nauman [20] and Alperin [3] we call these modules *P-static*. Other names in the literature are *reflexive* (see [6]) or *P-solvable* or *P-coreflexive modules* (see [12, p. 75]), and in Ulmer [26] the class of static modules is called the *fixpoint category*.

Of course P itself and every finite direct sum of copies of P are P-static. We will say P is *$\sum$-self-static* if any direct sum of copies of P is P-static. Under this condition we obtain a straightforward characterization of P-static modules (in 3.7) which shows the importance of the projectivity notions mentioned above.

Section 4 will mainly be concerned with the interplay of conditions imposed on the category of static modules (or on the image of $\mathrm{Hom}_R(P, -)$) and properties of the module P. This includes characterizations of self-tilting modules and related equivalences.

It is well known that generators in a full module category can be characterized by properties over their endomorphism and biendomorphism rings. In Section 5 we provide similar characterizations for w-Σ-quasi-projective and self-tilting modules. In particular it turns out that, for a faithful $\sum$-self-static self-tilting R-module P, the ring R is dense in the biendomorphism ring of P, and P_S has P-dcc in the sense of Zimmermann [31], a property which makes P_S a certain Mittag-Leffler module. Similar Mittag-Leffler properties for P_S are observed for the case that the category of P-static modules is closed under products in the category of P-generated modules.

There is another topic considered in the paper of Nauman. Ring extensions $R \to A$ are studied and the transfer of properties from an R-module P to the A-module $A \otimes_R P$. Exploiting his basic ideas we give an account of this relationship in Section 6 thus extending results of Fuller [13] in this direction.

The main concern of this note is to generalize known results and provide simple proofs by relating papers which were written independently. Various results scattered around in the literature are gathered under a common point of view. In particular it should be mentioned that our techniques also apply to modules P which are not self-small thus including interesting examples from abelian group theory (e.g., [30, 4.12, 5.8]).

2 Preliminaries

Let R be an associative ring with unit and R-*Mod* the category of unital left R-modules. Homomorphisms of modules will be written on the opposite side of the scalars. For unexplained notation we refer to [27].

Throughout the paper P will be a left R-module and $S := \mathrm{End}_R(P)$.

An R-module N is *P-generated* if there exists an exact sequence

$$0 \to K \to P^{(\Lambda)} \to N \to 0, \quad \Lambda \text{ some set},$$

and N is *P-presented* if there exists such a sequence where K is P-generated.

$\mathrm{Gen}(P)$, $\mathrm{Pres}(P)$ and $\sigma[P]$ will denote the full subcategories of R-*Mod* whose objects are P-generated, P-presented or submodules of P-generated modules, respectively. $\sigma[P]$ is closed under direct sums, factor modules and submodules in R-*Mod* and hence is a Grothendieck category. Recall that for $Q \in \sigma[P]$, $Q|_P^\Lambda$ denotes the product of Λ copies of Q in $\sigma[P]$ (e.g., [30]). If Q is P-injective then $Q|_P^\Lambda = \mathrm{Tr}(P, Q^\Lambda)$ (the trace of P in Q^Λ).

An R-module N is *P-cogenerated* if there exists an exact sequence

$$0 \to N \to P^{\Lambda} \to L \to 0, \quad \Lambda \text{ some set},$$

and N is *P-copresented* if there exists such a sequence where L is P-cogenerated.

By $\mathrm{Cog}(P)$ and $\mathrm{Cop}(P)$ we denote the full subcategories of R-*Mod* consisting of P-cogenerated, resp., P-copresented modules.

$\mathrm{Add}\,(P)$ (resp. $\mathrm{add}\,(P)$) stands for the class of modules which are direct summands of (finite) direct sums of copies of P.

2.1 Canonical functors. Related to ${}_RP_S$ we have the adjoint pair of functors

$$\mathrm{Hom}_R(P, -) : R\text{-}Mod \to S\text{-}Mod, \quad P \otimes_S - : S\text{-}Mod \to R\text{-}Mod,$$

and for any $N \in R$-*Mod* and $X \in S$-*Mod*, the canonical morphisms

$$\begin{aligned} \mu_N &: P \otimes_S \mathrm{Hom}_R(P, N) \to N, \quad p \otimes f \mapsto (p)f, \\ \nu_X &: X \to \mathrm{Hom}_R(P, P \otimes_S X), \quad x \mapsto [p \mapsto p \otimes x]. \end{aligned}$$

We recall the following useful properties (e.g., [27, 45.8]).

2.2 Proposition. *Consider any $N \in R$-Mod and $X \in S$-Mod.*

(1) *Each of the following compositions of maps yield the identity:*

$$\mathrm{Hom}_R(P, N) \xrightarrow{\nu_{\mathrm{Hom}(P,N)}} \mathrm{Hom}_R(P, P \otimes_S \mathrm{Hom}_R(P, N)) \xrightarrow{\mathrm{Hom}(P, \mu_N)} \mathrm{Hom}_R(P, N),$$

$$P \otimes_S X \xrightarrow{id \otimes \nu_X} P \otimes_S \mathrm{Hom}_R(P, P \otimes_S X) \xrightarrow{\mu_{P \otimes X}} P \otimes_S X.$$

(2) $\mathrm{Coke}(\nu_{\mathrm{Hom}(P,N)}) \simeq \mathrm{Hom}_R(P, \mathrm{Ke}(\mu_N))$ *and* $\mathrm{Ke}(\mu_{P\otimes X}) \simeq P \otimes_S \mathrm{Coke}(\nu_X)$.

2.3 Static and adstatic modules. An R-module N is called *P-static* if μ_N is an isomorphism and the class of all P-static R-modules is denoted by $\mathrm{Stat}(P)$.

An S-module X is called *P-adstatic* if ν_X is an isomorphism and we denote the class of all P-adstatic S-modules by $\mathrm{Adst}(P)$.

The name *P-static* was used in Alperin [3] and Nauman [20] and the name *P-adstatic* should remind that we have an adjoint situation. It is easy to see that, for every P-static module N, $\mathrm{Hom}_R(P, N)$ is P-adstatic, and for any P-adstatic module X, $P \otimes_R X$ is P-static. In fact we have the following (e.g., Onodera [21, Theorem 1], Alperin [3, Lemma], Nauman [20, Theorem 2.5], Faticoni [12, Proposition 6.3.2]):

2.4 Basic equivalence. *For any R-module P, the functor*

$$\mathrm{Hom}_R(P, -) : \mathrm{Stat}(P) \to \mathrm{Adst}(P)$$

defines an equivalence with inverse $P \otimes_S -$.

3 $\sum$-self-static and pseudo-finite modules

Clearly the module P and finite direct sums P^k are P-static. Moreover, for P finitely generated any direct sum $P^{(\Lambda)}$ is P-static. This also holds more generally when P is *self-small*, i.e., if for any set Λ, the canonical map

$$\mathrm{Hom}_R(P, P)^{(\Lambda)} \to \mathrm{Hom}_R(P, P^{(\Lambda)})$$

is an isomorphism. However this condition is not necessary for $P^{(\Lambda)}$ to be P-static. Because of the importance of this property we give it its own name.

3.1 Definition. We say that P is *$\sum$-self-static* if, for any set Λ, $P^{(\Lambda)}$ is P-static, i.e., we have an isomorphism

$$\mu_{P^{(\Lambda)}} : P \otimes_S \mathrm{Hom}_R(P, P^{(\Lambda)}) \to P^{(\Lambda)}.$$

As we will see soon $\sum$-self-static modules can be far from being finitely generated. Nevertheless many examples of $\sum$-self-static modules have a property which is familiar from finitely generated modules. Again we suggest a name for this.

3.2 Definition. We call a module P *pseudo-finite* if, for any set Λ, and any morphisms

$$P \xrightarrow{g} P^{(\Lambda)} \xrightarrow{h} N,$$

where $gh \neq 0$, there exists a morphism $\bar{g} : P \to \mathrm{Im}(g) \cap P^{\Lambda_o}$, for some finite subset $\Lambda_o \subset \Lambda$, such that $\bar{g}h \neq 0$. These maps are displayed in the diagram

$$\begin{array}{ccccc} P & \xrightarrow{\bar{g}} & P^{\Lambda_o} & & \\ & & \downarrow \varepsilon & & \\ P & \xrightarrow{g} & P^{(\Lambda)} & \xrightarrow{h} & N, \end{array}$$

where ε denotes the canonical inclusion.

Clearly every self-small module is pseudo-finite, and it is easy to see that any direct summand of a direct sum of finitely generated modules is pseudo-finite. In particular P is pseudo-finite provided it is projective in $\sigma[P]$. Moreover, if P is a generator in $\sigma[P]$ it is also pseudo-finite.

We do not expect that every pseudo-finite module is $\sum$-self-static. However the last two examples mentioned share a generalized projectivity condition which makes them $\sum$-self-static as we will prove in our next propositon.

Recall that P is *self-pseudo-projective* in $\sigma[P]$ if any diagram with exact sequence

$$\begin{array}{ccccccccc} & & & & P & \overset{\alpha}{\cdots} & P & & \\ & & & & \beta \vdots & & \downarrow & & \\ 0 & \to & K & \to & L & \to & N & \to & 0, \end{array}$$

where $K \in \mathrm{Gen}(P)$ and $L \in \sigma[P]$, can be non-trivially commutatively extended by some $\alpha : P \to P$, $\beta : P \to L$. This condition is equivalent to $\mathrm{Gen}(P)$ being closed under extensions in $\sigma[P]$, and also to the fact that $\mathrm{Hom}_R(P, -)$ respects exactness of sequences of the form (see [17, Proposition 2.2])

$$0 \to \mathrm{Tr}(P, L) \to L \to L/\mathrm{Tr}(P, L) \to 0, \text{ for any } L \in \sigma[P].$$

3.3 Pseudo-finite self-pseudo-projective modules. *Let P be pseudo-finite and self-pseudo-projective. Then:*

(1) *For any $N \in R$-Mod,* $\quad \mathrm{Hom}_R(P, \mathrm{Ke}(\mu_N)) = 0$.

(2) *P is $\sum$-self-static.*

Proof. (1) Let $\{f_\lambda\}_\Lambda$ be a generating set of the S-module $\mathrm{Hom}_R(P, N)$ and consider the canonical map

$$S^{(\Lambda)} \to \mathrm{Hom}_R(P, N), \quad s_\lambda \mapsto s_\lambda f_\lambda.$$

Tensoring with P_S we obtain the morphism

$$h : P^{(\Lambda)} \simeq P \otimes S^{(\Lambda)} \to P \otimes_S \mathrm{Hom}_R(P, N), \quad p_\lambda \mapsto p_\lambda \otimes f_\lambda,$$

where the kernel of h is P-generated. By our projectivity condition, for every non-zero map $g : P \to P \otimes_S \mathrm{Hom}_R(P, N)$, we may construct a commutative diagram

$$\begin{array}{ccccc} P & \xrightarrow{\alpha} & P & & \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle g} & & \\ P^{(\Lambda)} & \xrightarrow{h} & P \otimes_S \mathrm{Hom}_R(P, N) & \to & 0 \\ & {\scriptstyle\bar{h}}\searrow & \downarrow{\scriptstyle \mu_N} & & \\ & & N\,, & & \end{array}$$

where $\alpha g = \beta h \neq 0$. By our finiteness condition we may assume that $\mathrm{Im}\,\beta$ is contained in a finite partial sum of $P^{\Lambda_o} \subset P^{(\Lambda)}$ and $\beta h \neq 0$. With the canonical projections π_λ related to $P^{(\Lambda)}$, and $\Lambda_o = \{\lambda_1, \ldots, \lambda_k\}$, we have for any $p \in P$,

$$(p)\beta h = \sum_{i=1}^{k} (p)\beta\pi_{\lambda_i} \otimes f_{\lambda_i} = p \otimes \sum_{i=1}^{k} \beta\pi_{\lambda_i} f_{\lambda_i} = p \otimes \beta\bar{h}.$$

Now assume $0 \neq \mathrm{Im}\, g \subset \mathrm{Ke}\,\mu_N$. Then $\beta\bar{h} = 0$ and hence $\beta h = 0$, contradicting our assumption. So we have $\mathrm{Hom}_R(P, \mathrm{Ke}(\mu_N)) = 0$.

The map $\mu_{P^{(\Lambda)}} : P \otimes_S \mathrm{Hom}_R(P, P^{(\Lambda)}) \to P^{(\Lambda)}$ is surjective and is split by the map

$$P^{(\Lambda)} \to P \otimes_S \mathrm{Hom}_R(P, P^{(\Lambda)}), \quad p_\lambda \mapsto p_\lambda \otimes \varepsilon_\lambda.$$

Hence $\mathrm{Ke}\,(\mu_{P^{(\Lambda)}})$ is a direct summand and so it is P-generated. Now (1) implies that $\mu_{P^{(\Lambda)}}$ is injective. □

The above proposition subsumes several well known results:

3.4 Corollary. *Let P be an R-module and $T = \mathrm{Tr}(P, R)$.*

(1) *Assume P is projective in $\sigma[P]$ or $P = TP$. Then $\mathrm{Hom}_R(P, \mathrm{Ke}(\mu_N)) = 0$ and P is $\sum$-self-static.*

(2) *If P is a generator in $\sigma[P]$ then $\sigma[P] = \mathrm{Stat}(P)$ (and P is $\sum$-self-static).*

Proof. (1) Assume P is projective in $\sigma[P]$. Then obviously P is self-pseudo-projective and is a direct summand of a direct sum of finitely generated modules, hence pseudo-finite.

If $P = TP$ then every P-generated module is T-generated and vice versa. From this it is easy to see that $\mathrm{Gen}(P)$ is closed under extensions in $\sigma[P]$ (even in *R-Mod*) and so P is self-pseudo-projective. Consider any morphisms

$$P \xrightarrow{g} P^{(\Lambda)} \xrightarrow{h} N,$$

where $gh \neq 0$. Choose some $t \in T$, $a \in P$ with $(ta)gh \neq 0$. Then by restriction we have a non-zero map

$$Ta \to Ra \xrightarrow{g} R(a)g \subset (P)g \cap P^{\Lambda_o} \xrightarrow{h} N,$$

for some finite $\Lambda_o \subset \Lambda$. Since Ta is P-generated, there exists $\bar{g} : P \to (P)g \cap P^{\Lambda_o}$ with $\bar{g}h \neq 0$, showing that P is pseudo-finite.

Now the assertion follows from 3.3.

(2) Let P be a generator in $\sigma[P]$. Then trivially P is self-pseudo-projective. Arguments similar to those used in the proof of (1) show that P is pseudo-finite. Hence by 3.3, for any R-module N, $\mathrm{Hom}_R(P, \mathrm{Ke}(\mu_N)) = 0$ and hence $\mathrm{Ke}(\mu_N) = 0$ □

Remarks. The second case in 3.4(1) was shown in Onodera [21, Theorem 2]. Notice that for P projective in *R-Mod*, $TP = P$. Assertion (2) was proved in Zimmermann-Huisgen [32, Lemma 1.3].

More examples of $\sum$-self-static and pseudo-finite modules are provided by our next result.

3.5 Non-singular noetherian rings. *Let R be a left noetherian ring with injective hull $E(R)$.*

(1) *If R is left non-singular, then $E(R)$ is $\sum$-self-static.*

(2) *If R is left hereditary, then $E(R)$ is pseudo-finite and self-pseudo-projective.*

Proof. (1) By our assumptions we have the S-module isomorphisms

$$E(R) \simeq \mathrm{Hom}_R(R, E(R)) \simeq \mathrm{End}_R(E(R)) =: S,$$

and for every non-singular injective R-module V,

$$E(R) \otimes_S \mathrm{Hom}_R(E(R), V) \simeq \mathrm{Hom}_R(E(R), V) \simeq V,$$

showing that V is $E(R)$-static. In particular $E(R)$ is $\sum$-self-static.

(2) Let $E(R) \xrightarrow{g} E(R)^{(\Lambda)} \xrightarrow{h} N$ be any morphisms with $gh \neq 0$. Choose any $a \in E(R)$ with $(a)gh \neq 0$. Then $\mathrm{Im}\, g$ contains an injective hull L of Ra. By the uniqueness of maximal essential extensions in $E(R)^{(\Lambda)}$, L is contained in a finite partial sum of $E(R)^{(\Lambda)}$. Since L is generated by $E(R)$ there exists some $\bar{g} : E(R) \to L$ with $\bar{g}h \neq 0$. □

As a special case we conclude from (1) that the rationals $\mathbb{Q}$ are $\sum$-self-static. This is not a surprise since it is shown in Arnold-Murley [4, Corollary 1.4] that any module with countable endomorphism ring is in fact self-small.

The following observation is due to D.K. Harrison (see [14, Proposition 2.1]):

3.6 Properties of $\mathbb{Q}/\mathbb{Z}$. *Put $S = \mathrm{End}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z})$ and let V be any divisible torsion $\mathbb{Z}$-module. Then we have an isomorphism*

$$\mu_V : \mathbb{Q}/\mathbb{Z} \otimes_S \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, V) \to V.$$

So in particular $\mathbb{Q}/\mathbb{Z}$ is $\sum$-self-static. However $\mathbb{Q}/\mathbb{Z}$ is not a self-small $\mathbb{Z}$-module since self-small torsion $\mathbb{Z}$-modules are finite (by [4, Proposition 3.1]).

For $\sum$-self-static R-modules we have the following.

3.7 Characterization of static modules. *For R-modules N, P consider the following statements:*

(i) *N is P-static;*

(ii) *there exists an exact sequence $P^{(\Lambda')} \to P^{(\Lambda)} \to N \to 0$ in R-Mod, which stays exact under* $\mathrm{Hom}_R(P, -)$*;*

(iii) *there exists an exact sequence $0 \to K \to P^{(\Lambda)} \to N \to 0$ in R-Mod with* $K \in \mathrm{Gen}(P)$, *which stays exact under* $\mathrm{Hom}_R(P, -)$.

For any P, $(i) \Rightarrow (ii) \Leftrightarrow (iii)$.
If P is a $\sum$-self-static R-module, then $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$.

Proof. Recall that for any P-generated N and $\Lambda = \mathrm{Hom}_R(P, N)$, the canonical exact sequence $P^{(\Lambda)} \to N \to 0$ remains exact under $\mathrm{Hom}_R(P, -)$.

$(ii) \Rightarrow (iii)$ This is obvious since $\mathrm{Hom}_R(P, -)$ is left exact.

$(iii) \Rightarrow (ii)$ Given the sequence in (iii), put $\Lambda' = \mathrm{Hom}_R(P, K)$. Now the assertion follows by the preceding remark.

$(i) \Rightarrow (iii)$ From any exact sequence $0 \to K \to P^{(\Lambda)} \to N \to 0$ we construct the commutative diagram

$$\begin{array}{ccccccc}
 & P \otimes_S \mathrm{Hom}_R(P, K) & \to & P \otimes_S \mathrm{Hom}_R(P, P^{(\Lambda)}) & \to & P \otimes_S \mathrm{Hom}_R(P, N) & \to 0 \\
 & \downarrow \mu_K & & \downarrow \mu_{P^{(\Lambda)}} & & \downarrow \mu_N & \\
0 \to & K & \to & P^{(\Lambda)} & \to & N & \to 0.
\end{array}$$

Now assume (i), put $\Lambda = \mathrm{Hom}_R(P, N)$ and consider the canonical epimorphism. Then the upper sequence in the diagram is exact and $\mu_{P^{(\Lambda)}}$ is an epimorphism. Hence $\mathrm{Coke}(\mu_K) \simeq \mathrm{Ke}(\mu_N) = 0$ and so K is P-generated.

$(iii) \Rightarrow (i)$ Let P be $\sum$-self-static. For the diagram above, assume that K is P-generated and $\mathrm{Hom}_R(P, -)$ is exact on the given sequence. Then again the upper sequence in the diagram is exact and since $\mu_{P^{(\Lambda)}}$ is an isomorphism, $0 = \mathrm{Coke}(\mu_K) \simeq \mathrm{Ke}(\mu_N)$. Hence μ_N is an isomorphism. □

Remarks. In Alperin [3], modules with property 3.7(ii) are called *Auslander with respect to P* and the equivalence $(i) \Leftrightarrow (ii)$ is asserted in Lemma 2 without any further condition. However the proof given there only holds for self-small P.

The implication $(i) \Rightarrow (iii)$ was also observed in Faticoni [12, Corollary 6.1.9].

4 Static modules and equivalences

4.1 Classes of modules related to P. From the preceding definitions we have the following chain of subclasses

$$\text{add}\,(P) \subset \text{Stat}(P) \subset \text{Pres}(P) \subset \text{Gen}(P) \subset \sigma[P] \subset R\text{-}Mod.$$

Since any direct summand of a P-static module is again P-static we have that

$$\text{Add}\,(P) \subset \text{Stat}(P) \text{ if and only if } P \text{ is } \textstyle\sum\text{-self-static.}$$

Our investigations will be concerned with the problem when some of these classes coincide. For example, P is a generator in $\sigma[P]$ if and only if $\text{Stat}(P) = \sigma[P]$. If P is a semisimple module, then clearly $\text{Add}\,(P) = \sigma[P]$, and if P is locally noetherian and cohereditary in $\sigma[P]$, then $\text{Add}\,(P) = \text{Pres}(P)$ (see [30]).

For classes $\mathcal{C} \subset R\text{-}Mod$ and $\mathcal{D} \subset S\text{-}Mod$ we use the notation

$$\begin{array}{ll} H_P(\mathcal{C}) & = \{X \in S\text{-}Mod \mid X \simeq \text{Hom}_R(P, C) \;\text{ for some } C \in \mathcal{C}\}; \\ P_S(\mathcal{D}) & = \{N \in R\text{-}Mod \mid N \simeq P \otimes_S D \;\text{ for some } D \in \mathcal{D}\}. \end{array}$$

Throughout this section let Q be any injective cogenerator in $\sigma[P]$ and $P^* = \text{Hom}_R(P, Q)$. Then we have the chain of classes of S-modules

$$\text{add}({}_SS) \subset \text{Adst}(P) \subset H_P(\text{Gen}(P)) \subset \text{Cop}(P^*) \subset \text{Cog}(P^*) \subset S\text{-}Mod.$$

With this notation we collect some elementary properties.

4.2 Fundamental relationships.

(1) $H_P(R\text{-}Mod) = H_P(\text{Gen}(P)) \subset \text{Cop}(P^*)$.

(2) *If* $(P^*)^\Lambda \in \text{Adst}(P)$*, for any set* Λ*, then* $H_P(\text{Gen}(P)) = \text{Cop}(P^*)$.

(3) *If P is $\sum$-self-static, then* $P_S(S\text{-}Mod) = \text{Pres}(P)$.

(4) *P is self-small if and only if* $\text{Add}({}_SS) \subset \text{Adst}(P)$.
Then $\text{Hom}_R(P, -) : \text{Add}(P) \to \text{Add}({}_SS)$ *is an equivalence.*

(5) *If P is $\sum$-self-static, then P is self-small provided* $\text{Adst}(P)$ *is closed under submodules.*

Proof. (1) The first equality follows from $\text{Hom}_R(P, L) = \text{Hom}_R(P, \text{Tr}(P, L))$. Since Q is a cogenerator in $\sigma[M]$, for any $N \in \sigma[P]$ we have an exact sequence

$$0 \to N \to Q|_P^\Lambda \to Q|_P^{\Lambda'}.$$

Applying $\text{Hom}_R(P, -)$ we obtain $\text{Hom}_R(P, N) \in \text{Cop}(P^*)$.

(2) For $X \in \mathrm{Cop}(P^*)$ we have an exact sequence in S-*Mod*,

$$0 \to X \to (P^*)^\Lambda \xrightarrow{g} (P^*)^{\Lambda'}.$$

Applying $P \otimes_S -$ and $\mathrm{Hom}_R(P, -)$ we obtain the commutative exact diagram

$$\begin{array}{ccccccc}
0 \to & X & \to & (P^*)^\Lambda & \xrightarrow{g} & (P^*)^{\Lambda'} \\
& \downarrow \alpha & & \downarrow \simeq & & \downarrow \simeq \\
0 \to & \mathrm{Hom}_R(P, P \otimes_S K) & \to & \mathrm{Hom}_R(P, P \otimes_S (P^*)^\Lambda) & \to & \mathrm{Hom}_R(P, P \otimes_S (P^*)^{\Lambda'}),
\end{array}$$

where $K = \mathrm{Ke}\,(id_P \otimes g)$ and the isomorphisms are given by our assumption. Hence α is an isomorphisms showing $X \in H_P(\mathrm{Gen}(P))$.

(3) $P_S(S\text{-}Mod) \subset \mathrm{Pres}(P)$ always holds.

Let $P^{(\Lambda')} \xrightarrow{f} P^{(\Lambda)} \to N \to 0$ be a P-presentation for $N \in \mathrm{Pres}(P)$. Applying $\mathrm{Hom}_R(P, -)$ we obtain an exact sequence

$$\mathrm{Hom}_R(P, P^{(\Lambda')}) \to \mathrm{Hom}_R(P, P^{(\Lambda)}) \to X \to 0,$$

where $X = \mathrm{Coke}(\mathrm{Hom}_R(P, f))$. Tensoring with P_S we obtain the commutative exact diagram

$$\begin{array}{ccccccc}
P \otimes_S \mathrm{Hom}_R(P, P^{(\Lambda')}) & \to & P \otimes_S \mathrm{Hom}_R(P, P^{(\Lambda)}) & \to & P \otimes_S X & \to 0 \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow & \\
P^{(\Lambda')} & \to & P^{(\Lambda)} & \to & N & \to 0,
\end{array}$$

from which we see that $N \simeq P \otimes_S X$ and hence $\mathrm{Pres}(P) \subset P_S(S\text{-}Mod)$.

(4) One implication follows directly from the definition. If $S^{(\Lambda)} \in \mathrm{Adst}(P)$ then

$$\mathrm{Hom}_R(P, P)^{(\Lambda)} \simeq S^{(\Lambda)} \simeq Hom_R(P, P \otimes_S S^{(\Lambda)}) \simeq \mathrm{Hom}_R(P, P^{(\Lambda)}),$$

showing that P is self-small.

(5) Clearly $S^{(\Lambda)} \subset \mathrm{Hom}_R(P, P^{(\Lambda)})$ and $\mathrm{Hom}_R(P, P^{(\Lambda)}) \in \mathrm{Adst}(P)$. If $\mathrm{Adst}(P)$ is closed under submodules then $S^{(\Lambda)} \in \mathrm{Adst}(P)$ and hence $\mathrm{Add}({}_S S) \subset \mathrm{Adst}(P)$. $\square$

The next result considers the case when the image of $\mathrm{Hom}_R(P, -)$ is contained in $\mathrm{Adst}(P)$ and was proved in Sato [25, Theorem] and Kashu [15, Proposition 9.5].

4.3 Conditions on the image of $\mathrm{Hom}_R(P, -)$. *The following are equivalent:*

(a) $H_P(\mathrm{Gen}(P)) = \mathrm{Adst}(P)$;

(b) $\mathrm{Cop}(P^*) = \mathrm{Adst}(P)$;

(c) $P_S(S\text{-}Mod) = \mathrm{Stat}(P)$;

(d) $\mathrm{Pres}(P) = \mathrm{Stat}(P)$;

(e) $\mathrm{Hom}_R(P, \mathrm{Ke}\,\mu_N) = 0$, *for every* $N \in \mathrm{Gen}(P)$;

(f) $P \otimes_S \mathrm{Coke}\,\nu_X = 0$, *for every* $X \in S$-*Mod;*

(g) $\mathrm{Hom}_R(P, -) : \mathrm{Pres}(P) \to \mathrm{Cop}(P^*)$ *is an equivalence (with inverse* $P \otimes_S -$*).*

Proof. $(a) \Leftrightarrow (b)$ Under the given conditions we have for any set Λ,

$$(P^*)^\Lambda \simeq \mathrm{Hom}_R(P, Q^\Lambda) \simeq \mathrm{Hom}_R(P, \mathrm{Tr}(P, Q^\Lambda)) \in \mathrm{Adst}(P).$$

Hence by 4.2, $H_P(\mathrm{Gen}(P)) = \mathrm{Cop}(P^*)$.

$(c) \Leftrightarrow (d)$ Under the given conditions, P is $\sum$-self-static and hence $P_S(S\text{-}Mod) = \mathrm{Pres}(P)$ (by 4.2).

$(a) \Leftrightarrow (e)$ Assume for any $N \in \mathrm{Gen}(P)$ that $\mathrm{Hom}_R(P, N)$ is P-adstatic. Then by 2.2, $0 = \mathrm{Coke}(\nu_{Hom(P,N)}) \simeq \mathrm{Hom}_R(P, \mathrm{Ke}(\mu_N))$.

The same formula yields the converse conclusion.

$(c) \Leftrightarrow (f)$ Assume (c). Then for an $X \in S\text{-}Mod$, $\mu_{P \otimes X}$ is an isomorphism, and by 2.2, $0 = \mathrm{Ke}(\mu_{P \otimes X}) \simeq P \otimes_S \mathrm{Coke}(\nu_X)$.

Again the same formula yields the converse conclusion.

$(e) \Rightarrow (f)$ We have $0 = \mathrm{Hom}_R(P, \mathrm{Ke}(\mu_{P \otimes X}) \simeq \mathrm{Hom}_R(P, P \otimes_S \mathrm{Coke}(\nu_X))$, and hence $P \otimes_S \mathrm{Coke}(\nu_X) = 0$.

$(f) \Rightarrow (e)$ By assumption, $0 = P \otimes_S \mathrm{Coke}(\nu_{Hom(P,N)}) \simeq P \otimes_S \mathrm{Hom}_R(P, \mathrm{Ke}(\mu_N))$, and so $\mathrm{Hom}_R(P, \mathrm{Ke}(\mu_N)) = 0$.

$(a) \Leftrightarrow (g)$ This follows from the basic equivalence 2.4. □

It was shown in 3.3 that pseudo-finite self-pseudo-projective modules satisfy the conditions in 4.3.

Interesting cases arise imposing conditions on the categories $\mathrm{Stat}(P)$ and $\mathrm{Adst}(P)$.

4.4 Conditions on $\mathrm{Stat}(P)$**.**

(1) *The following are equivalent for the R-module P:*

  (a) P *is* $\sum$*-self-static and* $\mathrm{Stat}(P)$ *is closed under factor modules;*

  (b) $\mathrm{Stat}(P) = \mathrm{Gen}(P)$;

  (c) $\mathrm{Hom}_R(P, -) : \mathrm{Gen}(P) \to \mathrm{Cop}(P^*)$ *is an equivalence.*

(2) *The following are equivalent for the R-module P:*

  (a) $\mathrm{Stat}(P)$ *is closed under submodules;*

  (b) $\mathrm{Gen}(P) = \sigma[P]$;

  (c) $\mathrm{Stat}(P) = \sigma[P]$;

(d) $\mathrm{Hom}_R(P,-):\sigma[P]\to \mathrm{Cop}(P^*)$ *is an equivalence.*

Proof. (1) $(a)\Leftrightarrow(b)$ is clear by the fact that the $P^{(\Lambda)}$'s belong to $\mathrm{Stat}(P)$.

$(b)\Leftrightarrow(c)$ From 4.3 we know that $H_P(\mathrm{Gen}(P)) = \mathrm{Cop}(P^*)$. Now the assertion follows from the basic equivalence 2.4.

(2) $(a)\Rightarrow(b)\Rightarrow(c)$ Since any $P^k \in \mathrm{Stat}(P)$, for any $k \in I\!N$, (a) implies that all submodules of P^k are P-generated and hence $\sigma[P] = \mathrm{Gen}(P)$. By 3.4(2) this implies $\mathrm{Stat}(P) = \sigma[P]$.

The other implications are obvious (by (1)). $\square$

Remarks. The modules described in 4.4(1) were named *W_o-modules* and those in 4.4(2) are called *W-modules* in Orsatti [22]. In fact 4.4 is a refinement of Teorema 3.2 and Proposizione 5.1 given there. Moreover it is pointed out in [22, 3.5] that $P := \mathbb{Z}_{p^\infty}$ (Prüfer p-group) is a W_o-module over $\mathbb{Z}$ with $\mathrm{Cop}(P^*) \neq \mathrm{Cog}(P^*)$.

Recall that a module P is *s-Σ-quasi-projective* if $\mathrm{Hom}_R(P,-)$ respects exactness of sequences

$$P^{(\Lambda')} \to P^{(\Lambda)} \to N \to 0, \text{ where } \Lambda', \Lambda \text{ are any sets.}$$

P is *w-Σ-quasi-projective* if $\mathrm{Hom}_R(P,-)$ respects exactness of sequences

$$0 \to K \to P^{(\Lambda)} \to N \to 0, \text{ where } K \in \mathrm{Gen}(P) \text{ and } \Lambda \text{ is any set.}$$

P is called *self-tilting* (in [30]) if P is w-Σ-quasi-projective and $\mathrm{Gen}(P) = \mathrm{Pres}(P)$.

4.5 Implications from projectivity. *Let P be $\sum$-self-static.*

(1) *If P is w-Σ-quasi-projective or s-Σ-quasi-projective, then* $\mathrm{Pres}(P) = \mathrm{Stat}(P)$.

(2) *If P is self-tilting, then* $\mathrm{Gen}(P) = \mathrm{Stat}(P)$ *and* $\mathrm{Adst}(P) = \mathrm{Cop}(P^*)$.

Proof. (1) This is obvious from 3.7.

(2) By [30, 3.2 and 3.3], self-tilting modules are self-pseudo-projective and hence $\mathrm{Adst}(P) = H_P(\mathrm{Gen}(P)) = \mathrm{Cop}(P^*)$ follows from 4.3. $\square$

The following results are shown in Colpi [6], Sato [24] and Faticoni [12].

4.6 Conditions on $\mathrm{Adst}(P)$.

(1) *The following are equivalent for an R-module P:*

(a) $\mathrm{Adst}(P) = \mathrm{Cog}(P^*)$*;*

(b) *P is self-small and w-Σ-quasi-projective;*

(c) $\mathrm{Hom}_R(P,-) : \mathrm{Pres}(P) \to \mathrm{Cog}(P^*)$ *is an equivalence.*

(2) *The following are equivalent for P:*

(a) $\mathrm{Adst}(P) = S$-*Mod;*

(b) *P is self-small and s-Σ-quasi-projective;*

(c) $\mathrm{Hom}_R(P, -) : \mathrm{Pres}(P) \to S$-*Mod is an equivalence.*

Proof. (1) See Colpi [6, Proposition 3.7] and Faticoni [12, Theorem 6.1.9],

(2) See Sato [24, Theorem 2.1] and Faticoni [12, Theorem 6.1.14]. □

Self-small self-tilting modules are also known as $*$-modules (see [30]). Combining the preceding propositions we obtain a characterization of $*$-modules given in Colpi [6, Theorem 4.1]:

4.7 Corollary. *The following are equivalent for an R-module P:*

(a) $\mathrm{Gen}(P) = \mathrm{Stat}(P)$ *and* $\mathrm{Adst}(P) = \mathrm{Cog}(P^*)$*;*

(b) *P is self-small and self-tilting;*

(c) $\mathrm{Hom}_R(P, -) : \mathrm{Gen}(P) \to \mathrm{Cog}(P^*)$ *is an equivalence.*

Remarks. s(emi)-Σ-quasi-projective modules were defined in Sato [24] and the notion of w(eakly)-Σ-quasi-projective modules was introduced in the study of $*$-modules (see Colpi [6]). Notice that the condition $\mathrm{Gen}(P) = \mathrm{Pres}(P)$ in the definition of self-tilting modules was already considered by Onodera in [21]. However, he combined it with projectivity of P such yielding a projective self-generator (see [21, Theorem 5]).

Obviously if P is a generator in $\sigma[P]$ then P is self-pseudo-projective in $\sigma[P]$ but need neither be s-Σ-quasi-projective nor w-Σ-quasi-projective.

5 Properties over the (bi)endomorphism ring

Let P be an R-module, $S = \mathrm{End}_R(P)$ and $B = \mathrm{End}_S(P) = \mathrm{Biend}_R(P)$, the biendomorphism ring. There is a remarkable interplay between the properties of P as a module over R, B and S, and we begin with considering properties of P as an S-module.

Recall that P is said to be *direct projective* if for every direct summand $X \subset P$, any epimorphism $P \to X$ splits (see [27, 41.18]), and P is $\sum$-*direct projective* if any direct sum of copies of P is direct projective. The latter means that P is a projective object in $\mathrm{Add}(P)$ and can be characterized by the fact that for $L \in \mathrm{Gen}(P)$ and $N \in \mathrm{Add}(P)$, any epimorphism $L \to N$ splits (see [11, Lemma 11.2]). For abelian groups this is known as the *Baer splitting property* (see [12, 7.1]).

The following is a variation and extension of a result in Ulmer [26] and the Theorems 2.2 and 2.5 in Albrecht [1].

5.1 P_S (faithfully) flat. *We keep the notation above.*

(1) *The following are equivalent:*

   (a) *P_S is a flat module;*

   (b) *for any $k, l \in \mathbb{N}$, the kernel of any $f : P^k \to P^l$ is P-generated;*

   (c) $\mathrm{Stat}(P)$ *is closed under kernels.*

   In this case $\mathrm{Stat}(P)$ *is also closed under P-generated submodules.*

(2) *If P is self-small the following are equivalent:*

   (a) *P_S is a faithfully flat module;*

   (b) $\mathrm{Stat}(P)$ *is closed under kernels, and* $\mathrm{Hom}_R(P,-)$ *is exact on short exact sequences in* $\mathrm{Stat}(P)$*;*

   (c) $\mathrm{Stat}(P)$ *is closed under kernels, and P is $\sum$-direct projective;*

   (d) $\mathrm{Stat}(P)$ *is closed under kernels, and for any left ideal $I \subset S$,* $I = \mathrm{Hom}_R(P, PI)$*;*

   (e) $\mathrm{Stat}(P)$ *is closed under kernels and for any left ideal $I \subset S$, $PI \neq P$.*

Proof. (1) $(a) \Leftrightarrow (b)$ is well known (e.g., [27, 15.9]).

$(a) \Rightarrow (c)$ Consider an exact sequence $0 \to K \to L \to N$, where $L, N \in \mathrm{Stat}(P)$. By our assumptions we may construct an exact commutative diagram

$$\begin{array}{cccccc} 0 \to & P \otimes_S \mathrm{Hom}_R(P,K) & \to & P \otimes_S \mathrm{Hom}_R(P,L) & \to & P \otimes_S \mathrm{Hom}_R(P,N) \\ & \downarrow \mu_K & & \downarrow \simeq & & \downarrow \simeq \\ 0 \to & K & \to & L & \to & N \end{array} ,$$

showing that μ_K is an isomorphism.

$(c) \Rightarrow (b)$ is obvious.

Assume (a) holds. Let K be a P-generated submodule of some $L \in \mathrm{Stat}(P)$. An argument similar to that in $(a) \Rightarrow (c)$ shows $K \in \mathrm{Stat}(P)$.

(2) $(a) \Rightarrow (b)$ Consider an exact sequence $L \xrightarrow{f} N \to 0$, where $L, N \in \mathrm{Stat}(P)$. We construct an exact commutative diagram

$$\begin{array}{ccccc} P \otimes \mathrm{Hom}_R(P,L) & \to & P \otimes \mathrm{Hom}_R(P,N) & \to & P \otimes_S X \ \to 0 \\ \downarrow \simeq & & \downarrow \simeq & & \\ L & \to & N & \to & 0, \end{array}$$

where $X = \mathrm{Coke}\,(\mathrm{Hom}_R(P,f))$. This implies $P \otimes_S X = 0$ and hence $X = 0$.

$(b) \Rightarrow (c)$ For every $N \in \mathrm{Add}(P)$, $\mathrm{Hom}_R(N,-)$ is exact on short exact sequences in $\mathrm{Stat}(P)$ and so every epimorphism $P^{(\Lambda)} \to N$ splits, i.e., P is $\sum$-direct projective.

$(c) \Rightarrow (b)$ Let $L \to N \to 0$ be an exact sequence, where $L, N \in \mathrm{Stat}(P)$. For any $P \to N$ we obtain, by a pullback construction, a commutative diagram

$$\begin{array}{ccccc} U & \to & P & \to & 0 \\ \downarrow & & \downarrow & & \\ L & \to & N & \to & 0. \end{array}$$

As a kernel of a morphism $L \oplus P \to N$, U is P-generated and hence $U \to P$ splits (since P is $\sum$-direct projective).

$(b) \Rightarrow (d)$ Let $I \subset S$ be a left ideal. By (1), $PI \subset P$ is P-static and for a generating set $\{\gamma_\lambda\}_\Lambda$ of I, we have an exact sequence of modules in $\mathrm{Stat}(P)$,

$$0 \to K \to P^{(\Lambda)} \xrightarrow{\sum \gamma_\lambda} PI \to 0,$$

and so $\mathrm{Hom}_R(P, -)$ is exact on this sequence. By standard arguments this implies $I = \mathrm{Hom}_R(P, PI)$.

$(d) \Rightarrow (e) \Rightarrow (a)$ is clear. □

Notice that for P finitely generated, $I = \mathrm{Hom}_R(P, PI)$ for every left ideal $I \subset S$, if and only if P is *intrinsically projective* (see [28, 5.7]).

Next we recall some well known cases of special interest.

5.2 Proposition. *We keep the notation above.*

(1) *P is a generator in R-Mod if and only if $R \simeq B$ and P_S is finitely generated and projective.*

(2) *P is a progenerator in R-Mod if and only if $R \simeq B$ and P_S is a progenerator in Mod-S.*

(3) *P is self-small and tilting in R-Mod if and only if $R \simeq B$ and P_S is self-small and tilting in Mod-S.*

(4) *If P is faithful and a generator in $\sigma[P]$, then R is dense in B and P_S is flat.*

(5) *If P is self-tilting and self-small, then $P \otimes_S -$ is exact on short exact sequences with modules from $H_P(\mathrm{Gen}(P))$.*

Proof. For (1),(2) and (4) we refer to [27, 18.8 and 15.7]. (3) is proved in Colby-Fuller [5, Proposition 1.1].

(5) By 4.7 we have that $\mathrm{Hom}_R(P, -) : \mathrm{Gen}(P) \to \mathrm{Cog}(P^*) = H_P(\mathrm{Gen}(P))$ is an equivalence. Now the assertion follows from Colpi-Menini [9, Proposition 1.1]. □

We have a striking left right symmetry in (2) and (3) for the properties in *R-Mod* but there are only implications in one direction for the properties in $\sigma[P]$ considered in (4) and (5). The question arises which property of P_S would guarantee the converse implication in (4). To answer the corresponding question for (5) also some form of density property is needed. Our next result relates to the latter problem.

5.3 Proposition. *Let P be an R-module and $B = \mathrm{Biend}_R(P)$.*

(1) *Every P-presented R-module is a (P-presented) B-module.*

(2) *If P is w-Σ-quasi-projective then $\mathrm{Hom}_R(P,N) = \mathrm{Hom}_B(P,N)$, for every $N \in \mathrm{Pres}(P)$.*

(3) *If P is $\sum$-self-static then $\mathrm{Hom}_R(P,N) = \mathrm{Hom}_B(P,N)$, for every $N \in \mathrm{Stat}(P)$.*

(4) *If P is $\sum$-self-static the following are equivalent:*

  (a) *${}_RP$ is w-Σ-quasi-projective;*

  (b) *${}_BP$ is w-Σ-quasi-projective and $\mathrm{Hom}_R(P,N) = \mathrm{Hom}_B(P,N)$, for every $N \in \mathrm{Pres}(P)$.*

(5) *If P is faithful and $\sum$-self-static the following are equivalent:*

  (a) *${}_RP$ is self-tilting;*

  (b) *${}_BP$ is self-tilting and R is dense in B.*

Proof. (1) For $N \in \mathrm{Pres}(P)$ we have a short exact sequence

$$P^{(\Lambda')} \xrightarrow{f} P^{(\Lambda)} \xrightarrow{g} N \to 0, \text{ where } \Lambda',\ \Lambda \text{ are any sets.}$$

Since every P-generated submodule of $P^{(\Lambda)}$ is a B-submodule (see [27, 15.6]), $\mathrm{Ke}\, g = \mathrm{Im}\, f$ is a B-module. Hence N is a B-module and g is a B-morphism.

(2) We have an exact sequence

$$\mathrm{Hom}_R(P, P^{(\Lambda')}) \to \mathrm{Hom}_R(P, P^{(\Lambda)}) \to \mathrm{Hom}_R(P,N) \to 0.$$

It is easy to see that $\mathrm{Hom}_R(P, P^{(\Lambda)}) = \mathrm{Hom}_B(P, P^{(\Lambda)})$ and this implies $\mathrm{Hom}_R(P,N) = \mathrm{Hom}_B(P,N)$.

(3) In view of 3.7 the same proof as in (2) applies.

(4) This follows easily by (1) and (2).

(5) *(a)*$\Rightarrow$*(b)* Let P be self-tilting. Then for any submodule $K \subset P^n$, $n \in I\!N$, the factor module $N = P^n/K \in \mathrm{Gen}(P) = \mathrm{Pres}(P)$ is a B-module and by (4), $\mathrm{Hom}_R(P,N) = \mathrm{Hom}_B(P,N)$. In particular the canonical projection $P^n \to N$ is a B-morphism and hence its kernel K is a B-submodule of P^n. This implies that R is dense in B (e.g., [27, 15.7]).

(b)$\Rightarrow$*(a)* is obvious since R dense in B implies $\sigma[{}_RP] = \sigma[{}_BP]$ (see [27, 15.8]). $\square$

Let $\mathrm{Inj}\,(P)$ denote the class of all injectives in $\sigma[P]$. Since $\mathrm{Inj}\,(P) \subset \mathrm{Gen}(P)$, for any self-small self-tilting module P, $\mathrm{Inj}\,(P) \subset \mathrm{Stat}(P)$. More generally we may ask for which P the latter inclusion holds. Before answering this let us recall the canonical map

$$\alpha_{L,P} : L \otimes_S Hom_R(P,V) \to \mathrm{Hom}_R(\mathrm{Hom}_S(L,P), V), \quad l \otimes f \mapsto [g \mapsto (g(l))f],$$

which is an isomorphism provided L_S is finitely presented and V is P-injective (e.g., [27, 25.5]).

Following Zimmermann [31, 3.2], we say that P_S has *L-dcc* if $\alpha_{L,P}$ is a monomorphism for all $V \in \operatorname{Inj}(P)$. The notation was chosen to indicate that the condition is related to descending chain conditions on certain matrix subgroups of P.

Now let Q be an injective cogenerator in $\sigma[P]$. Then the above condition is equivalent to

$$\alpha_{L,P} : L \otimes_S Hom_R(P, Q^\Lambda) \to \mathrm{Hom}_R(\mathrm{Hom}_S(L, P), Q^\Lambda)$$

being a monomorphism. This indicates the relationship to certain Mittag-Leffler modules (e.g., Albrecht [2], Rothmaler [23]).

Let $\mathcal{X}$ be a class of left S-modules. P_S is called an *$\mathcal{X}$-Mittag-Leffler* or *$\mathcal{X}$-ML module* if, for any family $\{X_\lambda\}_\Lambda$ of modules in $\mathcal{X}$, the canonical map

$$P \otimes_S \prod_\Lambda X_\lambda \to \prod_\Lambda (P \otimes_S X_\lambda),$$

is injective. In particular, for $\mathcal{X} = \{X\}$, P_S is *X-Mittag-Leffler* or *X-ML*, if the canonical map

$$P \otimes_S X^\Lambda \to (P \otimes_S X)^\Lambda,$$

is injective for any index set Λ.

The connection of these notions with static modules becomes obvious if we put $L = P$ and assume that P is balanced (i.e. $R \simeq B$). Then P_S has P-dcc implies an isomorphism

$$P \otimes_S Hom_R(P, V) \to \mathrm{Hom}_R(\mathrm{Hom}_S(P, P), V) \simeq V,$$

for all injective $V \in \sigma[P]$, since they are P-generated. For the injective cogenerator $Q \in \sigma[P]$ and $P^* := Hom_R(P, Q)$ this corresponds to the condition that

$$P \otimes_S (P^*)^\Lambda \simeq (P \otimes_S P^*)^\Lambda,$$

is a monomorphism for any set Λ (i.e., P_S is P^*-ML), and this is equivalent to

$$P \otimes_S (P^*)^\Lambda \simeq P \otimes_S Hom_R(P, Q|_P^\Lambda) \to Q|_P^\Lambda,$$

being an isomorphism for any set Λ.

Summarizing these remarks and referring to the basic equivalence 2.4 we have (see [31, Corollary 3.10]):

5.4 P balanced with P_S-dcc. *For a balanced bimodule ${}_RP_S$, the following are equivalent:*

(a) $\mathrm{Inj}\,(P) \subset \mathrm{Stat}(P)$;

(b) *P_S has P_S-dcc;*

(c) *$Q \in \mathrm{Stat}(P)$ and P_S is a P^*-ML module;*

(d) *for every set Λ, $Q|_P^\Lambda \in \mathrm{Stat}(P)$;*

(e) *$\mathrm{Hom}_R(P,-) : \mathrm{Inj}\,(P) \to H_P(\mathrm{Inj}\,(P))$ is an equivalence, (and*

$$H_P(\mathrm{Inj}\,(P)) = \{X \in S\text{-}Mod \,|\, X \text{ is a direct summand of } (P^*)^\Lambda,\ \Lambda \text{ some set}\}).$$

Combining this with our previous observations on density properties (in 5.3) we are now able to describe when P-injectives are P-static.

5.5 Injective and static modules. *For a $\sum$-self-static R-module P, the following are equivalent:*

(a) $\mathrm{Inj}\,(P) \subset \mathrm{Stat}(P)$;

(b) *for every set Λ, $Q|_P^\Lambda \in \mathrm{Stat}(P)$;*

(c) *$\mathrm{Hom}_R(P,-) : \mathrm{Inj}\,(P) \to H_P(\mathrm{Inj}\,(P))$ is an equivalence (with inverse $P \otimes_S -$);*

(d) *P_S has P_S-dcc, $\mathrm{Inj}\,(P) \subset \mathrm{Pres}(P)$ and $\mathrm{Hom}_R(P,V) = \mathrm{Hom}_B(P,V)$, for all $V \in \mathrm{Inj}\,(P)$.*

Proof. $(a) \Leftrightarrow (b) \Leftrightarrow (c)$ is clear by the observations preceding 5.4.

$(a) \Rightarrow (d)$ Clearly $\mathrm{Inj}\,(P) \subset \mathrm{Pres}(P)$, and by 5.3(3), for every injective $V \in \sigma[P]$, $\mathrm{Hom}_R(P,V) = \mathrm{Hom}_B(P,V)$. The last equality implies

$$\mathrm{Hom}_R(L,V) = \mathrm{Hom}_B(L,V), \text{ for any } P\text{-generated } B\text{-module } L.$$

Indeed, for $f \in \mathrm{Hom}_R(L,V)$ let $l \in L$, $b \in B$. There exists $g \in \mathrm{Hom}_B(P,L)$ and $p \in P$ with $(p)g = l$, implying $(bl)f = (bp)gf = b((p)gf) = b(l)f$. This shows $f \in \mathrm{Hom}_B(L,V)$.

Now let W be an injective module in $\sigma[{}_BP] \subset \sigma[P]$ and $\alpha : W \to V$ an R-monomorphism for some injective $V \in \sigma[P]$. Then $\alpha \in \mathrm{Hom}_B(W,V)$ and hence it splits proving that W is injective in $\sigma[P]$.

This implies $\mathrm{Inj}\,({}_BP) \subset \mathrm{Stat}({}_BP)$ and P_S has P_S-dcc by 5.4.

$(d) \Rightarrow (a)$ Any injective $V \in \sigma[P]$ is a B-module and hence there exists a B-monomorphism $\beta : V \to V'$, for some injective B-module V' in $\sigma[{}_BP]$. This is R-split by some $\beta' : \mathrm{Hom}_R(V',V) = \mathrm{Hom}_B(V',V)$ (see proof above) and hence V is injective in $\sigma[{}_BP]$. Now 5.4 implies $\mathrm{Inj}\,(P) = \mathrm{Inj}\,({}_BP) \subset \mathrm{Stat}({}_BP) = \mathrm{Stat}(P)$. $\square$

If P is a cogenerator in $\sigma[P]$ then it satisfies the density property (see [27, 15.7]) and so we have:

5.6 Corollary. *If P is an injective cogenerator in $\sigma[P]$ the following are equivalent:*

(a) $\mathrm{Inj}\,(P) \subset \mathrm{Stat}(P)$;

(b) *P_S has P_S-dcc;*

(c) *P_S is an S-ML module;*

(d) *for every set Λ, $\mathrm{Tr}(P, P^\Lambda) \in \mathrm{Stat}(P)$.*

If P satisfies (a) and is $\sum$-pure-injective in $\sigma[P]$, then P is $\sum$-self-static.

Notice that the case $\mathrm{Inj}\,(P) = \mathrm{Stat}(P)$ is described in [30, 6.5]. This condition characterizes locally noetherian cohereditary modules P for which every (injective) module in $\sigma[P]$ is embedded in some $P^{(\Lambda)}$.

Since the inclusion functor $\mathrm{Gen}(P) \to R\text{-}Mod$ is left adjoint to the trace functor $\mathrm{Tr}(P,-) : R\text{-}Mod \to \mathrm{Gen}(P)$, the product of any family $\{N_\lambda\}_\Lambda$ in $\mathrm{Gen}(P)$ is just $\mathrm{Tr}(P, \prod_\Lambda N_\lambda)$ (see [27, 45.11]). By this we may describe when $\mathrm{Stat}(P)$ is closed under certain products. A special case of the next proposition is considered in Albrecht [2, Theorem 3.2].

5.7 $\mathrm{Stat}(P)$ **closed under products in** $\mathrm{Gen}(P)$**.**

(1) *The following are equivalent:*

  (a) $\mathrm{Stat}(P)$ *is closed under products in* $\mathrm{Gen}(P)$;

  (b) *for any family* $\{N_\lambda\}_\Lambda$ *in* $\mathrm{Stat}(P)$, $P \otimes_S \mathrm{Hom}_R(P, \prod_\Lambda N_\lambda) \simeq \mathrm{Tr}(P, \prod_\Lambda N_\lambda)$;

  (c) *P_S is an* $\mathrm{Adst}(P)$*-ML module.*

(2) *Assume that P_S is flat and an* $\mathrm{Adst}(P)$*-ML module. Then* $\mathrm{Stat}(P)$ *has inverse limits.*

Proof. (1) $(a) \Leftrightarrow (b)$ is clear by the definitions.

$(b) \Rightarrow (c)$ Consider a family $\{X_\lambda\}_\Lambda$ of modules in $\mathrm{Adst}(P)$. Then $P \otimes_S X_\lambda \in \mathrm{Stat}(P)$ and

$$\begin{aligned} P \otimes_S \textstyle\prod_\Lambda X_\lambda &\simeq P \otimes_S \textstyle\prod_\Lambda \mathrm{Hom}_R(P, P \otimes_S X_\lambda) \\ &\simeq P \otimes_S \mathrm{Hom}_R(P, \textstyle\prod_\Lambda (P \otimes_S X_\lambda)) \simeq \mathrm{Tr}(P, \textstyle\prod_\Lambda (P \otimes_S X_\lambda)), \end{aligned}$$

showing that P_S is an $\mathrm{Adst}(P)$-ML module.

$(c) \Rightarrow (b)$ Take any family $\{N_\lambda\}_\Lambda$ of modules in $\mathrm{Stat}(P)$. Then $\mathrm{Hom}_R(P, N_\lambda) \in \mathrm{Adst}(P)$ and therefore we have a monomorphism

$$P \otimes_S \mathrm{Hom}_R(P, \prod_\Lambda N_\lambda) \simeq P \otimes_S \prod_\Lambda \mathrm{Hom}_R(P, N_\lambda) \to \prod_\Lambda N_\lambda,$$

and this implies $P \otimes_S \mathrm{Hom}_R(P, \prod_\Lambda N_\lambda) \simeq \mathrm{Tr}(P, \prod_\Lambda N_\lambda)$.

(2) Any category has inverse limits provided it has products and kernels. By (1), the ML-property implies that $\mathrm{Stat}(P)$ has products. As shown in 5.1, P_S flat implies that $\mathrm{Stat}(P)$ has kernels. $\square$

Notice that in particular $\mathrm{Stat}(P) = \mathrm{Gen}(P)$ implies that P_S is an $\mathrm{Adst}(P)$-ML module (by 5.7(1)).

We are now in a position to get new characterizations for generators and tilting modules in $\sigma[M]$.

5.8 Corollary. *Let P be a faithful R-module.*

(1) *The following are equivalent:*
   (a) *P is a generator in $\sigma[P]$;*
   (b) *R is dense in B, P_S is flat and has P_S-dcc.*

(2) *If P is finitely generated the following are equivalent:*
   (a) *P is a projective generator in $\sigma[P]$;*
   (b) *R is dense in B, P_S is faithfully flat and has P_S-dcc.*

(3) *If P is self-small then the following are equivalent:*
   (a) *P is self-tilting;*
   (b) *R is dense in B, ${}_BP$ is w-Σ-quasi-projective, P_S has P_S-dcc, and $P \otimes_S -$ is exact on short exact sequences with modules from $H_P(\mathrm{Gen}(P))$.*

(4) *For P self-small the following are equivalent:*
   (a) *P is self-tilting and P_S is flat;*
   (b) *P is a projective generator in $\sigma[P]$.*

Proof. (1) $(a) \Rightarrow (b)$ follows from 5.2(4) and 5.5.

$(b) \Rightarrow (a)$ By 5.4, $\mathrm{Inj}\,(P) \subset \mathrm{Stat}(P)$. For any $K \in \sigma[P]$, there exists an exact sequence $0 \to K \to Q_1 \to Q_2$, where Q_1, Q_2 are injectives in $\sigma[P]$. By 5.1 this implies that K is P-generated.

(2) By [27, 18.5], any finitely generated generator in $\sigma[P]$ is projective in $\sigma[P]$ if and only if it is faithfully flat over its endomorphism ring.

(3) $(a) \Rightarrow (b)$ follows from 5.2(5), 5.3(5) and 5.5.

$(b) \Rightarrow (a)$ By 5.5, $\mathrm{Inj}\,(P) \subset \mathrm{Stat}(P)$. Let K be P-generated and consider an exact sequence $0 \to K \to \widehat{K} \to N \to 0$, where $\widehat{K}$ is the P-injective hull of K. ${}_BP$ being w-Σ-quasi-projective the functor $\mathrm{Hom}_B(P, -)$ is exact on this sequence and we obtain an exact commutative diagram

$$\begin{array}{ccccccccc} 0 \to & P \otimes_S \mathrm{Hom}_B(P, K) & \to & P \otimes_S \mathrm{Hom}_B(P, \widehat{K}) & \to & P \otimes_S \mathrm{Hom}_B(P, N) & \to 0 \\ & \downarrow {\scriptstyle \mu_K} & & \downarrow \simeq & & \downarrow {\scriptstyle \mu_N} & \\ 0 \to & K & \to & \widehat{K} & \to & N & \to 0, \end{array}$$

showing that μ_K is an isomorphism and hence $K \in \mathrm{Stat}({}_BP)$. So ${}_BP$ is self-tilting and by density ${}_RP$ is self-tilting.

(4) $(a) \Rightarrow (b)$ Let P be self-tilting with P_S flat. Then by (1) and (3), P is a generator in $\sigma[P]$. By [30, Proposition 4.1], any self-tilting module which is a generator in $\sigma[P]$ is projective in $\sigma[P]$.

$(b) \Rightarrow (a)$ is trivial. $\square$

Remark. Abelian group theorists have been mainly interested in modules which are (faithfully) flat over their endomorphism rings while in representation theory (self-) tilting modules have received much attention. From 5.8(4) we see that these notions generalize projective generators in different directions.

6 Ring extensions and equivalences

In this section we investigate the behaviour of equivalences as considered in the previous sections under ring extensions.

6.1 Ring extensions. Let $\alpha : R \to A$ be a morphism of associative rings with units. Related to it we have the *induction functor*

$$R\text{-}Mod \to A\text{-}Mod, \quad M \mapsto A \otimes_R M,$$

and the *restriction functor*

$$A\text{-}Mod \to R\text{-}Mod, \quad {}_AN \mapsto {}_RN.$$

For a given R-module P, putting $S = \mathrm{End}_R(P)$ and $T = \mathrm{End}_A(A \otimes_R P)$ we have the ring morphism

$$\beta : S \to T, \quad f \mapsto id \otimes f.$$

We will be interested in A-modules (resp., T-modules) which have certain properties as R-modules (resp., S-modules). Refining the notations introduced before we set

$$\begin{array}{rcl}
\mathrm{add}_R^A(P) & = & \{V \in A\text{-}Mod \mid {}_RV \in \mathrm{add}(P)\}, \\
\mathrm{Add}_R^A(P) & = & \{V \in A\text{-}Mod \mid {}_RV \in \mathrm{Add}(P)\}, \\
\mathrm{Gen}_R^A(P) & = & \{V \in A\text{-}Mod \mid {}_RV \in \mathrm{Gen}(P)\}, \\
\mathrm{Pres}_R^A(P) & = & \{V \in A\text{-}Mod \mid {}_RV \in \mathrm{Pres}(P)\}, \\
\mathrm{Stat}_R^A(P) & = & \{V \in A\text{-}Mod \mid {}_RV \in \mathrm{Stat}_R(P)\}, \\
\mathrm{Adst}_S^T(P) & = & \{X \in T\text{-}Mod \mid {}_SX \in \mathrm{Adst}(P)\}\,.
\end{array}$$

It is easy to see (e.g., [13, Lemma 1.2]) that

$$A \otimes_R P \in \mathrm{Gen}(P) \text{ if and only if } \mathrm{Gen}_A(A \otimes_R P) = \mathrm{Gen}_R^A(P).$$

From Hom-tensor relations (e.g., [28, 15.6]) we have the

6.2 Basic isomorphisms.

(1) *For any $V \in A\text{-}Mod$ there is a functorial S-module isomorphism*

$$\varphi : \mathrm{Hom}_R(P, V) \to \mathrm{Hom}_A(A \otimes_R P, V), \quad f \mapsto [a \otimes p \mapsto a \cdot (p)f].$$

(2) *For $V = A \otimes_R P$ we get $\mathrm{Hom}_R(P, A \otimes_R P) \simeq T$ and the R-module isomorphism*

$$id \otimes \varphi : P \otimes_S \mathrm{Hom}_R(P, A \otimes_R P) \to P \otimes_S T.$$

Our main interest is to transfer properties of the R-module P to the A-module $A \otimes_R P$. It turns out that the conditions we are looking at can be transferred provided $A \otimes_R P$ is P-static as an R-module. With the above preparations we can prove the following crucial result (see [20, Theorem 4.9, 5.5]):

6.3 Related equivalences. *Assume $A \otimes_R P \in \mathrm{Stat}_R^A(P)$. Then:*

(1) $\mathrm{Stat}_R^A(P) = \mathrm{Stat}_A(A \otimes_R P)$.

(2) $\mathrm{Adst}_S^T(P) = \mathrm{Adst}(A \otimes_R P)$ *and there is an equivalence*

$$\mathrm{Hom}_A(A \otimes_R P, -) : \mathrm{Stat}_R^A(P) \to \mathrm{Adst}_S^T(P).$$

(3) *If P is self-small then we have an equivalence*

$$\mathrm{Hom}_A(A \otimes_R P, -) : \mathrm{Add}_R^A(P) \to \mathrm{Add}_S^T(S).$$

In each case the inverse functor is $(A \otimes_R P) \otimes_T -$.

Proof. $A \otimes_R P \in \mathrm{Stat}_R^A(P)$ means $P \otimes_S \mathrm{Hom}_R(P, A \otimes_R P) \simeq A \otimes_R P$.

(1) Combined with an isomorphism from 6.2 we get the R-module isomorphism

$$P \otimes_S T \simeq A \otimes_R P.$$

This implies the isomorphisms for any $V \in A\text{-}Mod$,

$$\begin{aligned}(A \otimes_R P) \otimes_T \mathrm{Hom}_A(A \otimes_R P, V) &\simeq (P \otimes_S T) \otimes_T \mathrm{Hom}_A(A \otimes_R P, V) \\ &\simeq P \otimes_S \mathrm{Hom}_A(A \otimes_R P, V) \\ &\simeq P \otimes_S \mathrm{Hom}_R(P, V).\end{aligned}$$

Now $V \in \mathrm{Stat}_R^A(P)$ means by definition that the last expression in this chain is isomorphic to V, whereas $V \in \mathrm{Stat}_A(A \otimes_R P)$ means that the first expression is isomorphic to V.

(2) First notice the R-module isomorphisms for any T-module V,

$$P \otimes_S V \simeq (P \otimes_S T) \otimes_T V \simeq (A \otimes_R P) \otimes_T V.$$

Assume $V \in \mathrm{Adst}_S^T(P)$. Then we have the isomorphisms

$$\begin{aligned}V \simeq \mathrm{Hom}_R(P, P \otimes_S V) &\simeq \mathrm{Hom}_R(P, (A \otimes_R P) \otimes_T V) \\ &\simeq \mathrm{Hom}_A(A \otimes_R P, (A \otimes_R P) \otimes_T V),\end{aligned}$$

proving $V \in \mathrm{Adst}_T(A \otimes_R P)$.

The same chain of isomorphisms shows $\mathrm{Adst}_T(A \otimes_R P) \subset \mathrm{Adst}_S^T(P)$.

In view of (1) and the first part of (2) the final assertion about the equivalence follows from the basic equivalence 2.4 applied to $A \otimes_R P$.

(3) We know that $\mathrm{Add}_R^A(P) \subset \mathrm{Stat}_R^A(P) = \mathrm{Stat}_A(A \otimes_R P)$, and it is easy to verify that $H_P(\mathrm{Add}_R^A(P)) \subset \mathrm{Add}_S^T(S) \subset \mathrm{Adst}_S^T(P)$. $\square$

Combining the preceding observations we obtain:

6.4 w-Σ-quasi-projective modules. *Let P be a w-Σ-quasi-projective R-module and assume $A \otimes_R P \in \mathrm{Pres}(P)$. Then:*

(1) *$A \otimes_R P$ is a w-Σ-quasi-projective A-module.*

(2) *If P is $\sum$-self-static we have an equivalence*

$$\mathrm{Hom}_A(A \otimes_R P, -) : \mathrm{Pres}_R^A(P) \to \mathrm{Adst}_S^T(P),$$

with inverse functor $(A \otimes_R P) \otimes_T -$.

(3) *If P is a self-small R-module then $A \otimes_R P$ is a self-small A-module.*

Proof. (1) By [30, 3.2], factor modules of P-presented modules by P-generated modules are P-presented. This implies

$$\mathrm{Pres}_A(A \otimes_R P) \subset \mathrm{Pres}_R^A(P),$$

and by the functorial isomorphism in 6.2 we conclude that $A \otimes_R P$ is w-Σ-quasi-projective as an A-module.

(2) By 4.5, P-presented modules are P-static and so $A \otimes_R P \in \mathrm{Pres}(P)$ implies $A \otimes_R P \in \mathrm{Stat}_R^A(P)$. Hence by 6.3, we have the inclusions

$$\mathrm{Pres}_A(A \otimes_R P) \subset \mathrm{Pres}_R^A(P) = \mathrm{Stat}_R^A(P) = \mathrm{Stat}_A(A \otimes_R P) \subset \mathrm{Pres}_A(A \otimes_R P).$$

From this and 2.4 we obtain the equivalence as given.

(3) By [30, 5.1], for P self-small and w-Σ-quasi-projective, $\mathrm{Hom}_R(P, -)$ commutes with direct limits of P-presented modules. Since $A \otimes_R P \in \mathrm{Pres}(P)$, any infinite direct sum $(A \otimes_R P)^{(\Lambda)}$ is the direct limit of its finite partial sums and hence

$$\begin{aligned}\mathrm{Hom}_A(A \otimes_R P, (A \otimes_R P)^{(\Lambda)}) &\simeq \mathrm{Hom}_R(P, (A \otimes_R P)^{(\Lambda)}) \\ &\simeq \mathrm{Hom}_R(P, A \otimes_R P)^{(\Lambda)} \\ &\simeq \mathrm{Hom}_A(A \otimes_R P, A \otimes_R P)^{(\Lambda)}.\end{aligned}$$

□

6.5 Self-tilting modules. *Let P be a self-tilting and $\sum$-self-static R-module, and assume $A \otimes_R P \in \mathrm{Gen}(P)$. Then $A \otimes_R P$ is a self-tilting A-module and we have an equivalence*

$$\mathrm{Hom}_A(A \otimes_R P, -) : \mathrm{Gen}_R^A(P) \to \mathrm{Adst}_S^T(P),$$

with inverse functor $(A \otimes_R P) \otimes_T -$.

Proof. By 6.4, we have the equalities

$$\mathrm{Gen}_A(A \otimes_R P) = \mathrm{Gen}_R^A(P) = \mathrm{Pres}_R^A(P) = \mathrm{Pres}_A(A \otimes_R P) = \mathrm{Stat}_A(A \otimes_R P).$$

□

Recall that P is a tilting module in *R-Mod* if it is self-tilting and a subgenerator in *R-Mod*. We consider some special cases of this.

6.6 Corollary. *Assume $A \otimes_R P \in \mathrm{Gen}(P)$. Then:*

1. *If P is a projective generator in $\sigma[P]$, then $A \otimes_R P$ is a projective generator in $\sigma_A[A \otimes_R P]$.*
2. *If P is self-tilting and $\mathrm{Gen}(P)$ is closed under extensions in R-Mod, then $A \otimes_R P$ is self-tilting and $\mathrm{Gen}_A(A \otimes_R P)$ is closed under extensions in A-Mod.*

(3) *Assume that P is $\sum$-self-static and (i) A_R is flat or (ii) ${}_AA$ is finitely cogenerated and $A \otimes_R P$ is a faithful A-module.*
If P is tilting in R-Mod, then $A \otimes_R P$ is tilting in A-Mod.

Proof. (1) Recall that P is a projective generator in $\sigma[P]$ if and only if it is self-tilting and a generator in $\sigma[P]$.

Let P be a projective generator in $\sigma[P]$. Then P is $\sum$-self-static by 3.4, and by 6.5, $A \otimes_R P$ is self-tilting. Since $\sigma_A[A \otimes_R P] \subset \mathrm{Gen}_R^A(P) = \mathrm{Gen}_A(A \otimes_R P)$ we see that $A \otimes_R P$ is a generator in $\sigma_A[A \otimes_R P]$.

(2) Suppose $\mathrm{Gen}(P)$ is closed under extensions in R-*Mod*. Since $\mathrm{Gen}_A(A \otimes_R P) = \mathrm{Gen}_R^A(P)$ this implies that $\mathrm{Gen}_A(A \otimes_R P)$ is closed under extensions in A-*Mod*.

(3) Suppose $R \in \sigma[P]$, i.e., there is a monomorphism $R \to P^k$, for some $k \in \mathbb{N}$. If A_R is flat, then $A \simeq A \otimes_R R \to A \otimes_R P^k$ is a monomorphism and hence $A \otimes_R P$ is a subgenerator in A-*Mod*. If ${}_AA$ is finitely cogenerated and $A \otimes_R P$ is faithful, then $A \subset (A \otimes_R P)^k$, for some $k \in \mathbb{N}$. Now the assertions follow from 6.5. $\square$

Remarks. Let P be finitely generated. If P is self-tilting then it is a $*$-module and 6.5 implies Fuller [13, Theorem 2.2]. Condition (1) in 6.6 corresponds to [13, Corollary 2.4]. If P satisfies the condition in 6.6(2) then P is called *quasi-tilting in R-Mod* (see [8]) and we obtain [13, Corollary 2.5].

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