Weak Corings

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Abstract

Entwined structures (A, C, ψ) were introduced by Brzeziński and Majid to study the interdependence of an R-algebra A and an R-coalgebra C, R a commutative ring. It turned out that this relationship can also be expressed by the fact that $A \otimes_R C$ has a canonical A-coring structure. More generally *weak entwined structures* and their modules were studied by Caenepeel and Groot and it was suggested by Caenepeel to relate these to *pre-corings*. Slightly modifying this notion we introduce *weak corings* and develop a general theory of comodules over such corings. In particular we obtain that (A, C, ψ) is a weak entwined structure if and only if $A \otimes_R C$ is a weak A-coring (with canonical structure maps). *Weak bialgebras* in the sense of Böhm-Nill-Szlachányi are characterized as R-modules with an algebra and coalgebra structure (B, μ, Δ) such that $B \otimes_R B$ is a weak coring for the various coring structures induced by μ, $\mu \circ \tau$, Δ and $\tau \circ \Delta$. Moreover we will characterize *weak Hopf algebras* as those weak bialgebras B, which are generators for the comodules over $(B \otimes_R B) \cdot 1$.

Introduction

Throughout the paper R will be an associative commutative ring with unit.

An R-algebra (A, μ, ι) and an R-coalgebra (C, Δ, ε) are said to be *entwined*, and (A, C, ψ) is said to be an *entwining structure* if there exists an R-linear map

$$\psi : C \otimes_R A \to A \otimes_R C,$$

such that

$$\begin{array}{ll} \psi \circ (I \otimes \mu) = (\mu \otimes I) \circ (I \otimes \psi) \circ (\psi \otimes I), & \psi \circ (I \otimes \iota) = \iota \otimes I, \\ (I \otimes \Delta) \circ \psi = (\psi \otimes I) \circ (I \otimes \psi) \circ (\Delta \otimes I), & (I \otimes \varepsilon) \circ \psi = \varepsilon \otimes I, \end{array}$$

where I denotes the appropriate identity maps. In [4] these conditions are displayed in a nice bow-tie diagram. A similar "entwining" of two algebras is considered in Tambara [12].

Entwining structures are introduced in Brzeziński-Majid [2] to develop a theory of "coalgebra principal bundles" and the associated modules are defined in Brzeziński [3] as right A-modules with a coaction $\varrho : M \to M \otimes_R C$ such that

$$\varrho(m \cdot a) = \sum m_{\underline{0}} \psi(m_{\underline{1}} \otimes a), \quad \text{for } m \in M,\, a \in A.$$

Although these structures are very useful and managable there is no immediate evidence from the algebraic point of view why they are of such interest. This evidence is provided in [5] by the observation that (A, C, ψ) is an entwining structure if and only if $A \otimes_R C$ has an A-coring structure given by the comultiplication

$$\underline{\Delta} := I \otimes \Delta : A \otimes_R C \to A \otimes_R C \otimes_R C \simeq (A \otimes_R C) \otimes_A (A \otimes_R C),$$

and the counit $\underline{\varepsilon} := I \otimes \varepsilon : A \otimes_R C \to A$, where $A \otimes_R C$ has the canonical A-module structure on the left, and the right A-action

$$(1 \otimes c) \cdot a = \psi(c \otimes a), \qquad \text{for } a \in A,\, c \in C.$$

In particular, an R-module B with an algebra and a coalgebra structure is a *bialgebra* if and only if the construction just described makes $B \otimes_R B$ a B-coring (resp. (B, B, ψ) an entwining structure), where the right B-action is

$$(1 \otimes c) \cdot b = (1 \otimes c)\Delta(b) \; (= \psi(b \otimes c)), \qquad \text{for } b \in B,\, c \in C.$$

Motivated by problems in quantum field theory and operator algebras the notion of bialgebras was extended to *weak bialgebras* by Böhm, Nill and Szlachányi [10, 1]. To relate these with the notions mentioned before, *weak entwining structures* (A, C, ψ) and their (co-)modules were introduced and investigated in Caenepeel-Groot [6]. It is pointed out in Brzeziński [5] that the category of (co-)modules over weak entwining structures can be identified with the category of comodules over a suitable coring.

By ideas of Caenepeel (see [5, Section 6]) the interpretation of entwining structures as corings can be extended to weak entwining structures and *pre-corings*: These are (A, A)-bimodules $\mathcal{C}$, unital as left A-module, with an (A, A)-bimodule map $\Delta : \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}$ satisfying the coassociativity condition, and a left A-module map $\varepsilon : \mathcal{C} \to A$ with the property $\varepsilon(c \cdot a) = \varepsilon(c \cdot 1)a$, for $a \in A$, $c \in \mathcal{C}$.

Because of the obvious importance of pre-corings it is suggested in [5] to study the general properties of these structures. This is the motivation for the present paper.

Slightly modifying the definition of pre-corings we introduce, in Section 1, *weak A-corings* $\mathcal{C}$ where "weak" indicates the fact that $\mathcal{C}$ need not be unital as A-module - neither on the left nor on the right side. The corresponding notion of *weak comodules* is defined and their category is considered.

A weak A-coring $\mathcal{C}$ which happens to be unital as left A-module is (essentially) a pre-coring (as defined above), and $\mathcal{C}$ is a coring provided it is unital both as left and right A-module. In the definition of (right) weak $\mathcal{C}$-comodules M, we allow M to be non-unital as A-module and hence we will have $A\mathcal{C}$ as a right weak $\mathcal{C}$-comodule. This differs from the approach in [6] and [5].

In Section 2 we ask when A itself is a comodule over the A-coring $\mathcal{C}$. This is the case if and only if there exists a group-like element in $A\mathcal{C}A$, and the *coinvariants* of any weak $\mathcal{C}$-comodule M are introduced as the images of 1 under the comodule morphisms $A \to M$. The notion of a *Galois weak A-coring* is defined and it is shown how these are related to equivalences between the comodules over $A\mathcal{C}A$ and the modules over the coinvariants (see 2.5).

As for coalgebras and for corings, the dual algebra ${}^*\mathcal{C} = \mathrm{Hom}_{A-}(\mathcal{C}, A)$ plays a prominent role for weak corings. This is investigated in Section 3. Every right $\mathcal{C}$-comodule may be considered as right ${}^*\mathcal{C}$-module and in case $A\mathcal{C}$ is projective as a left A-module, for any right $\mathcal{C}$-comodule the $\mathcal{C}$-comodule structure and the ${}^*\mathcal{C}$-module structure coincide. Some results shown for coalgebras in [14] are extended and a *finiteness theorem* for weak comodules is proved (see 3.8). Notice that here ${}^*\mathcal{C}$ need not have a unit.

Given an R-algebra A and an R-coalgebra C, a comultiplication is defined on $A \otimes_R C$ in a canonical way (see Section 4) and it is shown that this yields a weak A-coring if and only if there exists a weak entwining map $\psi : C \otimes_R A \to A \otimes_R C$ (as considered in Caenepeel-Groot [6]). In this case the dual algebra ${}^*(A \otimes_R C) \simeq \mathrm{Hom}_R(C, A)$ yields the (Doi-Koppinen) smash product (see 4.2).

In Section 5 we finally consider an R-module B which is an algebra and a coalgebra $\Delta : B \to B \otimes_R B$, with $\Delta(ab) = \Delta(a)\Delta(b)$, for $a, b \in B$. We show that B is a *weak bialgebra* (in the sense of Böhm, Nill, Szlachányi [1]) if and only if $B \otimes_R B$ is a weak B-coring both with respect to Δ and $\tau \circ \Delta$ (where τ is the twist map). Moreover *weak Hopf algebras* are characterized as those bialgebras B, which are generators in the category of right comodules over $(B \otimes_R B) \cdot 1$ (see 5.12).

The papers on weak Hopf algebras mostly consider finite dimensional algebras over fields. Here we are working with algebras and coalgebras over any commutative ring R without finiteness conditions. For explicit examples and applications we refer to [5], [1], [6], and the references given there.

1 Weak corings

Throughout A will be an associative ring with unit 1 (or 1_A). In module theory usually the category of unital A-modules is considered. It has turned out that for some applications non-unital modules are of interest and hence we recall some elementary properties of non-unital modules over unital rings.

1.1. Non-unital modules. By $\tilde{\mathcal{M}}_A$ (resp. ${}_A\tilde{\mathcal{M}}$) we denote the category of all (not necessarily unital) right (left) A-modules while $\mathcal{M}_A$ and ${}_A\mathcal{M}$ denote the corresponding subcategories of unital A-modules. For any module M the identity map is denoted by I_M or just by I if no confusion arises.

We write ${}_A\tilde{\mathcal{M}}_B$ for the category of (A,B)-bimodules, B an associative ring, which need not be unital neither on the left nor on the right, i.e., for any $M \in {}_A\tilde{\mathcal{M}}_B$ and $m \in M$, $a \in A$, $b \in B$, we have $(am)b = a(mb)$ but possibly $m1_B \neq m$ or $1_A m \neq m$. The subcategory of those bimodules which are left and right unital is denoted by ${}_A\mathcal{M}_B$.

For $M, N \in {}_A\tilde{\mathcal{M}}_B$, the set of bimodule morphisms $M \to N$ will be denoted by $\mathrm{Hom}_{AB}(M,N)$ and we will write $\mathrm{Hom}_{A-}(M,N)$ or $\mathrm{Hom}_{-B}(M,N)$ for the left A-module or right B-module morphisms, respectively.

For any $M \in \tilde{\mathcal{M}}_A$ there is a splitting A-epimorphism

$$-\otimes 1 : M \to M \otimes_A A, \quad m \mapsto m \otimes 1,$$

which is injective (bijective) if and only if M is a unital A-module. We have canonical isomorphisms

$$\begin{array}{rcl} M \otimes_A A & \to & MA, \quad m \otimes a \mapsto ma, \text{ and} \\ \mathrm{Hom}_A(A, M) & \to & MA, \quad f \mapsto f(1), \end{array}$$

and we will identify these modules if appropriate. In particular, $MA = M1$.

For any A-module morphism $f : M \to N$, the map $f \otimes I : M \otimes_A A \to N \otimes_A A$ can be identified with the restriction $f\,|_{MA}: MA \to NA$ which we will usually also denote by the symbol f. We have a functor

$$- \otimes_A A : \tilde{\mathcal{M}}_A \to \mathcal{M}_A \subset \tilde{\mathcal{M}}_A, \quad M \mapsto M \otimes_A A, \quad f \mapsto f \otimes I,$$

which is left (right) adjoint to itself, i.e., for any $M, N \in \tilde{\mathcal{M}}_A$,

$$\mathrm{Hom}_A(M \otimes A, N) \simeq \mathrm{Hom}_A(M \otimes A, N \otimes A) \simeq \mathrm{Hom}_A(M, N \otimes A).$$

Since A is a unital A-module this implies $\mathrm{Hom}_A(M, A) \simeq \mathrm{Hom}_A(MA, A)$.

Of course we have - and will use - the corresponding properties for $A\otimes_A -$ and left A-modules. For any $M \in {}_A\tilde{\mathcal{M}}_A$, this induces a splitting (A,A)-morphism

$$1_\otimes - {}_\otimes 1 : M \to A\otimes_A M \otimes_A A \simeq AMA, \quad m \mapsto 1_\otimes m_\otimes 1 \quad (= 1m1),$$

and the isomorphisms

$$\mathrm{Hom}_{AA}(M,A) \simeq \mathrm{Hom}_{AA}(MA,A) \simeq \mathrm{Hom}_{AA}(AMA,A).$$

1.2. Weak A-corings. Let $\mathcal{C}$ be an (A,A)-bimodule. An (A,A)-bilinear map

$$\underline{\Delta} : \mathcal{C} \to \mathcal{C}\otimes_A A \otimes_A \mathcal{C}$$

is called a *weak comultiplication.* For $c \in \mathcal{C}$ we write $\underline{\Delta}(c) = \sum c_{\underline{1}} \otimes 1 \otimes c_{\underline{2}}$.

An (A,A)-bilinear map $\underline{\varepsilon} : \mathcal{C} \to A$ is called *weak counit* (for $\underline{\Delta}$) provided we have a commutative diagram

$$\begin{tikzcd} & \mathcal{C} \arrow[dl, "\underline{\Delta}"'] \arrow[dd, "1\otimes - \otimes 1"] \arrow[dr, "\underline{\Delta}"] & \\ \mathcal{C}\otimes_A A\otimes_A \mathcal{C} \arrow[dr, "\underline{\varepsilon}\otimes I"'] & & \mathcal{C}\otimes_A A\otimes_A \mathcal{C} \arrow[dl, "I\otimes\underline{\varepsilon}"] \\ & \mathcal{C} & \end{tikzcd}.$$

In our notation this means

$$1c1 = \sum \underline{\varepsilon}(c_{\underline{1}})c_{\underline{2}} = \sum c_{\underline{1}}\underline{\varepsilon}(c_{\underline{2}}).$$

We call $\mathcal{C}$ a *weak coring* provided it has a weak comultiplication $\underline{\Delta}$ and a weak counit $\underline{\varepsilon}$.

An (A,A)-submodule $D \subset \mathcal{C}$ which is pure as a left and right A-submodule is called a *weak subcoring* provided $\underline{\Delta}(D) \subset D\otimes_A A\otimes_A D$.

The weak comultiplication $\underline{\Delta}$ is *coassociative* if we have a commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\underline{\Delta}"] \arrow[d, "\underline{\Delta}"'] & \mathcal{C}\otimes_A A\otimes_A \mathcal{C} \arrow[d, "I\otimes I\otimes\underline{\Delta}"] \\ \mathcal{C}\otimes_A A\otimes_A \mathcal{C} \arrow[r, "\underline{\Delta}\otimes I\otimes I"] & \mathcal{C}\otimes_A A\otimes_A \mathcal{C}\otimes_A A\otimes_A \mathcal{C}, \end{tikzcd}$$

which is expressed by the equality

$$\sum c_{\underline{1}\,\underline{1}} \otimes 1 \otimes c_{\underline{1}\,\underline{2}} \otimes 1 \otimes c_{\underline{2}} = \sum c_{\underline{1}} \otimes 1 \otimes c_{\underline{2}\,\underline{1}} \otimes 1 \otimes c_{\underline{2}\,\underline{2}}.$$

A weak A-coring $\mathcal{C}$ is said to be *right (left) unital* provided $\mathcal{C}$ is unital as a right (left) A-module, and $\mathcal{C}$ is called A-*coring* provided $\mathcal{C}$ is unital both as a left and right A-module. In this case $\mathcal{C} \otimes_A A \otimes_A \mathcal{C} \simeq \mathcal{C} \otimes_A \mathcal{C}$ as bimodules, we have the (more familiar) notation $\underline{\Delta} : \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}$ for the comultiplication, and the diagram for the counit simplifies to

$$\begin{tikzcd} & \mathcal{C} \arrow[dl, "\underline{\Delta}"'] \arrow[dd, "I_{\mathcal{C}}"] \arrow[dr, "\underline{\Delta}"] & \\ \mathcal{C} \otimes_A \mathcal{C} \arrow[dr, "\underline{\varepsilon} \otimes I"'] & & \mathcal{C} \otimes_A \mathcal{C} \arrow[dl, "I \otimes \underline{\varepsilon}"] \\ & \mathcal{C} & \end{tikzcd}.$$

This shows that for any A-coring $\mathcal{C}$, $\underline{\Delta}$ splits as an (A, A)-bimodule morphism.

An A-coring is said to be an A-*coalgebra* if A is commutative and the left and right action of A on $\mathcal{C}$ coincide (i.e., $ca = ac$ for all $c \in \mathcal{C}$, $a \in A$).

Notice that left unital A-corings are essentially the A-*pre-corings* introduced by S. Caenepeel (see [5, Section 6]).

The following observations are immediate consequences of the definitions.

1.3. Proposition. *Let $(\mathcal{C}, \underline{\Delta}, \underline{\varepsilon})$ be a weak A-coring. Then*

(1) $(\mathcal{C}A, \underline{\Delta}, \underline{\varepsilon})$ is a (right unital) weak A-coring;

(2) $(A\mathcal{C}, \underline{\Delta}, \underline{\varepsilon})$ is a (left unital) weak A-coring;

(3) $(A\mathcal{C}A, \underline{\Delta}, \underline{\varepsilon})$ is an A-coring.

For any weak A-coring $\mathcal{C}$, the A-linear maps $\mathcal{C} \to A$ have ring structures which we are going to describe now. Notice the canonical isomorphisms

$$\begin{array}{rcll} \mathcal{C}^* & := & \mathrm{Hom}_{-A}(\mathcal{C}, A) & \simeq \mathrm{Hom}_{-A}(\mathcal{C}A, A), \\ (A\mathcal{C})^* & := & \mathrm{Hom}_{-A}(A\mathcal{C}, A) & \simeq \mathrm{Hom}_{-A}(A\mathcal{C}A, A), \\ {}^*\mathcal{C} & := & \mathrm{Hom}_{A-}(\mathcal{C}, A) & \simeq \mathrm{Hom}_{A-}(A\mathcal{C}, A), \\ {}^*(\mathcal{C}A) & := & \mathrm{Hom}_{A-}(\mathcal{C}A, A) & \simeq \mathrm{Hom}_{A-}(A\mathcal{C}A, A), \\ {}^*\mathcal{C}^* & := & \mathrm{Hom}_{AA}(\mathcal{C}, A) & \simeq \mathrm{Hom}_{AA}(A\mathcal{C}A, A) = {}^*\mathcal{C} \cap \mathcal{C}^*. \end{array}$$

1.4. Multiplication on $\mathrm{Hom}_A(\mathcal{C}, A)$**.** *Let $\mathcal{C}$ be a weak A-coring.*

(1) $\mathcal{C}^*$ has a ring structure given by the (convolution) product, for $f, g \in \mathcal{C}^*$,

$$f *_r g \ : \ \mathcal{C} \xrightarrow{\underline{\Delta}} \mathcal{C}A \otimes_A \mathcal{C} \xrightarrow{g \otimes I} A \otimes_A \mathcal{C} \simeq A\mathcal{C} \xrightarrow{f} A,$$

i.e., $f *_r g(c) = \sum f(g(c_{\underline{1}})c_{\underline{2}})$.
$\underline{\varepsilon}$ is a central idempotent in $\mathcal{C}^*$ and $(A\mathcal{C})^* = \underline{\varepsilon} *_r \mathcal{C}^*$.

(2) ${}^*\mathcal{C}$ *has a ring structure given by the product, for* $f, g \in \mathcal{C}^*$*,*

$$f *_l g \ : \ \mathcal{C} \xrightarrow{\underline{\Delta}} \mathcal{C} \otimes_A A\mathcal{C} \xrightarrow{I \otimes f} \mathcal{C} \otimes_A A \simeq \mathcal{C}A \xrightarrow{g} A,$$

i.e., $f *_l g(c) = \sum g(c_{\underline{1}} f(c_{\underline{2}}))$*.*
$\underline{\varepsilon}$ *is a central idempotent in* ${}^*\mathcal{C}$ *and* ${}^*(\mathcal{C}A) \simeq \underline{\varepsilon} *_l {}^*\mathcal{C}$*.*

(3) ${}^*\mathcal{C}^*$ *is a ring with multiplication, for* $f, g \in {}^*\mathcal{C}^*$*,*

$$f * g \ : \ \mathcal{C} \xrightarrow{\underline{\Delta}} \mathcal{C} \otimes_A A \otimes_A \mathcal{C} \xrightarrow{g \otimes I \otimes f} A,$$

i.e., $f * g(c) = \sum g(c_{\underline{1}}) f(c_{\underline{2}})$*, with unit* $\underline{\varepsilon}$*.*

(4) If $\mathcal{C}$ *is a coassociative weak A-coring, then all these rings are associative.*

Proof. (1) For any $f \in \mathcal{C}^*$ and $c \in \mathcal{C}$,

$$\begin{aligned} f *_r \underline{\varepsilon}(c) &= \sum f(\underline{\varepsilon}(c_{\underline{1}}) c_{\underline{2}}) = f(1c1), \quad \text{and} \\ \underline{\varepsilon} *_r f(c) &= \sum \underline{\varepsilon}(f(c_{\underline{1}}) c_{\underline{2}}) = \sum f(c_{\underline{1}}) \underline{\varepsilon}(c_{\underline{2}}) = \sum f(c_{\underline{1}} \underline{\varepsilon}(c_{\underline{2}})) = f(1c1). \end{aligned}$$

(2) is symmetric to (1), and (3) follows from (1) and (2).
(4) This can be verified by direct computation. □

So for any A-coring $\mathcal{C}$, the rings $\mathcal{C}^*$, ${}^*\mathcal{C}$ and ${}^*\mathcal{C}^*$ have unit $\underline{\varepsilon}$. This was already observed in [11, Proposition 3.2]. In case $\mathcal{C}$ is an A-coalgebra (A commutative) we have ${}^*\mathcal{C} = \mathcal{C}^*$ and the above results are well known facts about the dual algebra of a coalgebra.

1.5. Weak comodules. Let $\mathcal{C}$ be a weak A-coring and $M \in \tilde{\mathcal{M}}_A$. An A-linear map $\varrho_M : M \to M \otimes_A A \otimes_A \mathcal{C}$ is called a *weak coaction* on M, and it is said to be *weakly counital*, provided the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\varrho_M"] \arrow[dr, "-\otimes 1"'] & M \otimes_A A\mathcal{C} \arrow[d, "I \otimes \underline{\varepsilon}"] \\ & M \otimes_A A. \end{tikzcd}$$

ϱ_M is said to be *coassociative* if the diagram

$$\begin{tikzcd} M \arrow[r, "\varrho_M"] \arrow[d, "\varrho_M"] & M \otimes_A A\mathcal{C} \arrow[d, "I \otimes \underline{\Delta}"] \\ M \otimes_A A\mathcal{C} \arrow[r, "\varrho_M \otimes I"] & M \otimes_A A\mathcal{C} \otimes_A A\mathcal{C} \end{tikzcd}$$

is commutative. For $m \in M$ we write $\varrho_M(m) = \sum m_{\underline{0}} \otimes 1 \otimes m_{\underline{1}}$.

With this notation coassociativity of ϱ_M corresponds to the equality

$$\sum m_{\underline{0}} \otimes 1 \otimes \underline{\Delta}(m_{\underline{1}}) = \sum \varrho_M(m_{\underline{0}}) \otimes 1 \otimes m_{\underline{1}},$$

and weak counitality of ϱ_M is expressed by

$$m1 = \sum m_{\underline{0}}\, \underline{\varepsilon}(m_{\underline{1}})\,.$$

Clearly, in case M is a unital A-module we have $(I_M \otimes \underline{\varepsilon}) \circ \varrho_M = I_M$.

For a coassociative weak A-coring $\mathcal{C}$, an (non-unital) A-module M with a counital coassociative coaction is called a *right (weak) $\mathcal{C}$-comodule.*

An A-submodule $K \subset M$ is a *weak subcomodule* if

$$\varrho_M(K) \subset K \otimes_A A \otimes_A \mathcal{C} \subset M \otimes_A A \otimes_A \mathcal{C}.$$

Left weak coactions and *left weak $\mathcal{C}$-comodules* etc. are defined in a symmetric way.

Notice that any weak A-coring $\mathcal{C}$ has a left and a right coaction (by $\underline{\Delta}$) which, however, need not be weakly counital. On the other side, it is easy to see that the obvious right (left) $\mathcal{C}$-coaction on $A\mathcal{C}$ (on $\mathcal{C}A$) is weakly counital. In particular, for any coassociative weak A-coring, $A\mathcal{C}$ and $\mathcal{C}A$ are right and left weak $\mathcal{C}$-comodules, respectively.

Let $\mathcal{C}$ be an A-coring. Then a right weak $\mathcal{C}$-comodule M is called a *right $\mathcal{C}$-comodule* provided $MA = M$, i.e., M is a unital right A-module. As mentioned above, this implies $(I_M \otimes \underline{\varepsilon}) \circ \varrho_M = I_M$.

1.6. Proposition. *Let M be a right weak comodule over the coassociative weak A-coring $\mathcal{C}$. Then:*

(1) MA is a weak comodule over $\mathcal{C}$;

(2) MA is a weak comodule over the (left unital) weak A-coring $A\mathcal{C}$;

(3) MA is a weak comodule over the (right unital) weak A-coring $\mathcal{C}A$;

(4) MA is a comodule over the A-coring $A\mathcal{C}A$.

Notice that - in contrast to comodules - the structure map $\varrho_M : M \to M \otimes_A A \otimes_A \mathcal{C}$ of weak comodules need not be injective even if $\mathcal{C}$ is a coring. For example, considering A as an A-coring (by $\underline{\Delta} : A \simeq A \otimes_A A$, $\underline{\varepsilon} = I_A$), every right A-module M is a weak A-comodule by the map $-\otimes 1 : M \to M \otimes_A A$, which is not injective unless M is unital.

1.7. Morphisms. A *morphism* of modules with weak coaction $f : M \to N$ is an A-linear map such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow \rho_M & & \downarrow \rho_N \\ M \otimes_A A\mathcal{C} & \xrightarrow{f\otimes I} & N \otimes_A A\mathcal{C} \end{array}$$

commutes, which means $\varrho_N \circ f = (f{\otimes}I) \circ \varrho_M$.

The set $\mathrm{Hom}^{\mathcal{C}}(M,N)$ of morphisms of modules with weak coaction is an abelian group, and by definition it is determined by the exact sequence

$$0 \to \mathrm{Hom}^{\mathcal{C}}(M,N) \to \mathrm{Hom}_A(M,N) \xrightarrow{\gamma} \mathrm{Hom}_A(M, N \otimes_A A\mathcal{C}),$$

where $\gamma(f) := \rho_N \circ f - (f{\otimes}I) \circ \rho_M$.

For weak comodules, morphisms respecting the coactions are called *comodule morphisms.* The following observations are easy to verify.

1.8. Weak coaction and tensor products. *Let X be any unital right A-module. Let $M \in {}_A\tilde{\mathcal{M}}_A$ with a right weak $\mathcal{C}$-coaction $\varrho_M : M \to M \otimes_A A\mathcal{C}$.*

(1) $X \otimes_A M$ has a right weak $\mathcal{C}$-coaction

$$I{\otimes}\varrho_M : X \otimes_A M \longrightarrow X \otimes_A M \otimes_A A\mathcal{C}\,,$$

and for any A-module morphism $f : X \to Y$,

$$f{\otimes}I : X \otimes_A M \to Y \otimes_A M$$

is a morphism of modules with weak $\mathcal{C}$-coaction.

(2) In particular, $X \otimes_A \mathcal{C}$ is a right $\mathcal{C}$-comodule by

$$I{\otimes}\underline{\Delta} : X \otimes_A \mathcal{C} \simeq X \otimes_A A\mathcal{C} \longrightarrow X \otimes_A A\mathcal{C} \otimes_A A\mathcal{C}\,,$$

and $f{\otimes}I : X \otimes_A \mathcal{C} \to Y \otimes_A \mathcal{C}$ is a morphism of modules with weak $\mathcal{C}$-coaction.

(3) For any index set Λ, the module with right weak $\mathcal{C}$-coaction $A^{(\Lambda)} \otimes_A A\mathcal{C}$ is isomorphic to $A\mathcal{C}^{(\Lambda)}$.

(4) Assume $\mathcal{C}$ and ϱ_M to be coassociative. Then $X \otimes_A \mathcal{C}$ and $X \otimes_A M$ are right weak $\mathcal{C}$-comodules and ϱ_M is a comodule morphism.

1.9. Kernels and cokernels. Let $f : K \to M$ a be a morphism of right A-modules with weak coaction. So we have an exact commutative diagram in $\tilde{\mathcal{M}}_A$,

$$\begin{array}{ccccccc}
K & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0 \\
\downarrow \rho_K & & \downarrow \rho_M & & & & \\
K \otimes_A A\mathcal{C} & \xrightarrow{f \otimes I} & M \otimes_A A\mathcal{C} & \xrightarrow{g \otimes I} & N \otimes_A A\mathcal{C} & \longrightarrow & 0.
\end{array}$$

By the cokernel property of N in $\tilde{\mathcal{M}}_A$, this can be completed commutatively by some A-linear map $\varrho_N : N \to N \otimes_A A\mathcal{C}$, i.e., we have a weak $\mathcal{C}$-coaction on N, and - by construction - g is a morphisms for modules with weak $\mathcal{C}$-coaction. This shows that f has a kokernel which is a morphism of modules with weak coaction.

The existence of a kernel of f can be shown in a similar way provided the functor $- \otimes_A A\mathcal{C}$ respects monomorphisms, i.e., $A\mathcal{C}$ is flat as a left A-module.

For a coassociative weak A-coring $\mathcal{C}$, the class of weak $\mathcal{C}$-comodules together with the $\mathcal{C}$-comodule morphisms form an additive category which we denote by $\tilde{\mathcal{M}}^{\mathcal{C}}$.

For a coassociative A-coring $\mathcal{C}$ we only consider (weak) comodules which are unital as A-modules and the category of these is denoted by $\mathcal{M}^{\mathcal{C}}$.

We summarize the above observations.

1.10. The category $\tilde{\mathcal{M}}^{\mathcal{C}}$. *Let $\mathcal{C}$ be a coassociative weak A-coring.*

(1) The category $\tilde{\mathcal{M}}^{\mathcal{C}}$ has direct sums and cokernels.
It has kernels provided $A\mathcal{C}$ is flat as a left A-module.

(2) For the functor $- \otimes_A \mathcal{C} : \mathcal{M}_A \to \tilde{\mathcal{M}}^{\mathcal{C}}$ we have the natural isomorphism

$$\mathrm{Hom}^{\mathcal{C}}(MA, X \otimes_A \mathcal{C}) \to \mathrm{Hom}_A(MA, X), \quad f \mapsto (I \otimes \underline{\varepsilon}) \circ f,$$

for $M \in \tilde{\mathcal{M}}^{\mathcal{C}}$, $X \in \mathcal{M}_A$, with inverse map $h \mapsto (h \otimes I) \circ \varrho_M$.

(3) The functor $- \otimes_A \mathcal{C}A : \mathcal{M}_A \to \tilde{\mathcal{M}}^{\mathcal{C}}$ is right adjoint to $- \otimes_A A : \tilde{\mathcal{M}}^{\mathcal{C}} \to \mathcal{M}_A$.

(4) If $\mathcal{C}$ is a coring, then $- \otimes_A \mathcal{C} : \mathcal{M}_A \to \mathcal{M}^{\mathcal{C}}$ is right adjoint to the forgetful functor $\mathcal{M}^{\mathcal{C}} \to \mathcal{M}_A$.

Proof. (1) It is easy to check that coproducts in $\tilde{\mathcal{M}}_A$ yield coproducts in $\tilde{\mathcal{M}}^{\mathcal{C}}$ in an obvious way. The rest is clear by the preceding remarks.

(2) For $h \in \mathrm{Hom}_A(MA, X)$, the composition

$$MA \xrightarrow{\varrho_M} MA \otimes_A \mathcal{C} \xrightarrow{h \otimes I} X \otimes_A \mathcal{C} \xrightarrow{I \otimes \underline{\varepsilon}} X$$

yields the map h.

Let $f \in \mathrm{Hom}^{\mathcal{C}}(MA, X \otimes_A \mathcal{C})$ and put $h = (I \underline{\otimes} \underline{\varepsilon}) \circ f$. Then the composition

$$MA \xrightarrow{\varrho_M} MA \otimes_A \mathcal{C} \xrightarrow{h \otimes I} X \otimes_A \mathcal{C}$$

yields the map f. Thus the given assignments are inverse to each other.

Any A-morphism $M \to N$ of right A-modules induces a morphism $MA \to NA$ and so it is easy to see that the isomorphism is natural in both arguments.

(3) This follows from (2) by the isomorphism

$$\mathrm{Hom}^{\mathcal{C}}(MA, X \otimes_A \mathcal{C}) \simeq \mathrm{Hom}^{\mathcal{C}}(M, X \otimes_A \mathcal{C}A).$$

(4) is a consequence of (3). It is also shown in [5, Lemma 3.1]. □

Putting $X = A$ and $M = A\mathcal{C}$ we obtain the

1.11. Corollary. *For any weak A-coring $\mathcal{C}$, there are ring isomorphisms*

$$\mathrm{End}^{-\mathcal{C}}(A\mathcal{C}A) \simeq (A\mathcal{C})^*, \quad \mathrm{End}^{\mathcal{C}-}(A\mathcal{C}A) \simeq {}^*(\mathcal{C}A),$$

which are both given by $f \mapsto \underline{\varepsilon} \circ f$.

Proof. By 1.10, the map

$$\mathrm{End}^{-\mathcal{C}}(A\mathcal{C}A) \simeq \mathrm{Hom}_{-A}(A\mathcal{C}A, A) = (A\mathcal{C})^*, \quad f \mapsto \underline{\varepsilon} \circ f,$$

is an isomorphism of abelian groups. Moreover, for $f, g \in \mathrm{End}^{-\mathcal{C}}(A\mathcal{C}A)$ and $c \in A\mathcal{C}A$,

$$\begin{aligned}
(\underline{\varepsilon} \circ f) *_r (\underline{\varepsilon} \circ g)(c) &= \underline{\varepsilon} \circ f(\underline{\varepsilon} \circ g(c_{\underline{1}})c_{\underline{2}}) \\
&= \underline{\varepsilon} \circ f((\underline{\varepsilon} \underline{\otimes} I) \circ (g \underline{\otimes} I) \circ \underline{\Delta}(c)) \\
&= \underline{\varepsilon} \circ f((\underline{\varepsilon} \underline{\otimes} I) \circ \underline{\Delta} \circ g(c)) \\
&= \underline{\varepsilon} \circ f(g(c)) = \underline{\varepsilon} \circ (f \circ g)(c).
\end{aligned}$$

□

To end this section we notice some elementary properties of the $\mathrm{Hom}^{\mathcal{C}}$-functors.

1.12. Exactness of the $\mathrm{Hom}^{\mathcal{C}}$-functor. *Let ${}_A\mathcal{C}$ be flat and $M, N \in \tilde{\mathcal{M}}^{\mathcal{C}}$. Then:*

(1) $\mathrm{Hom}^{\mathcal{C}}(-, N) : \tilde{\mathcal{M}}^{\mathcal{C}} \to \mathbb{Z}\text{-Mod}$ *is left exact.*

(2) $\mathrm{Hom}^{\mathcal{C}}(M, -) : \tilde{\mathcal{M}}^{\mathcal{C}} \to \mathbb{Z}\text{-Mod}$ *is left exact.*

(3) If A is right A-injective then $\mathrm{Hom}^{\mathcal{C}}(-, A\mathcal{C}A) : \tilde{\mathcal{M}}^{\mathcal{C}} \to \mathbb{Z}\text{-Mod}$ *is exact.*

Proof. (1) Any exact sequence $X \to Y \to Z \to 0$ in $\tilde{\mathcal{M}}^{\mathcal{C}}$ yields a commutative diagram with exact columns,

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \to & \mathrm{Hom}^{\mathcal{C}}(Z,N) & \to & \mathrm{Hom}^{\mathcal{C}}(Y,N) & \to & \mathrm{Hom}^{\mathcal{C}}(X,N) \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \to & \mathrm{Hom}_A(Z,N) & \to & \mathrm{Hom}_A(Y,N) & \to & \mathrm{Hom}_A(X,N) \\
 & \downarrow & & \downarrow & & \downarrow \\
0 \to & \mathrm{Hom}_A(Z,N\otimes_A A\mathcal{C}) & \to & \mathrm{Hom}_A(Y,N\otimes_A A\mathcal{C}) & \to & \mathrm{Hom}_A(X,N\otimes_A A\mathcal{C}).
\end{array}$$

The second and third row are exact because of the exactness properties of Hom_A. Now diagram lemmata imply exactness of the first row.

(2) is shown with a similar diagram.

(3) This is a consequence of the functorial isomorphism in 1.10. □

2 A as weak $\mathcal{C}$-comodule, coinvariants

For a given A-coring $\mathcal{C}$, in general A need not be a weak comodule over $\mathcal{C}$. If this is the case it will be of special interest when A is a generator in $\tilde{\mathcal{M}}^{\mathcal{C}}$. First we describe the general situation.

2.1. A as weak comodule. *For any weak A-coring $\mathcal{C}$, the following are equivalent:*

(a) A is a right $\mathcal{C}$-comodule;

(b) A is a right $A\mathcal{C}A$-comodule;

(c) there exists a group-like element $g \in A\mathcal{C}A$ (i.e., $\underline{\Delta}(g) = g \otimes_A g$ and $\underline{\varepsilon}(g) = 1$).

Proof. (*a*) ⇔ (*b*) Let $\varrho_A : A \to A \otimes_A \mathcal{C}$ be a coaction which makes A a right $\mathcal{C}$-comodule. Then $\mathrm{Im}\,\varrho_A \subset A\mathcal{C}A$ and so A is a right $A\mathcal{C}A$-comodule.

The converse implication is trivial.

(*b*) ⇔ (*c*) Since $A\mathcal{C}A$ is an A-coring the assertion follows by [5, Lemma 5.1]. Notice that for a group-like $g \in \mathcal{C}$, the coaction on A is given by

$$\varrho_A : A \to A \otimes_A \mathcal{C}, \quad a \mapsto 1 \otimes g \cdot a \; (= g \cdot a).$$

□

If $A, M \in \tilde{\mathcal{M}}^{\mathcal{C}}$, any comodule morphism $f : A \to M$ is uniquely determined by the image of $1_A \in A$ and this explains the importance of the

2.2. Coinvariants. *Let $\mathcal{C}$ be a weak A-coring with group-like element $g \in A\mathcal{C}A$.*

(1) The coinvariants *of any $M \in \tilde{\mathcal{M}}^{\mathcal{C}}$ are defined by*

$$M^{co\mathcal{C}} = \{f(1) \mid f \in \mathrm{Hom}^{\mathcal{C}}(A, M)\} = \{m \in MA \mid \varrho_M(m) = m \otimes 1 \otimes g\}.$$

(2) In particular, for $M = A$ we have a subring

$$A^{co\mathcal{C}} = \{f(1) \mid f \in \mathrm{End}^{\mathcal{C}}(A)\} = \{a \in A \mid g \cdot a = a \cdot g\} \subset A.$$

(3) The map $\mathrm{End}^{\mathcal{C}}(A) \to A^{co\mathcal{C}}$, $f \mapsto f(1)$, *is a ring isomorphism, and*

$$\mathrm{Hom}^{\mathcal{C}}(A, M) \to M^{co\mathcal{C}}, \quad f \mapsto f(1),$$

is a right $A^{co\mathcal{C}}$-module isomorphism, for $M \in \tilde{\mathcal{M}}^{\mathcal{C}}$.

(4) $(N \otimes_A A\mathcal{C})^{co\mathcal{C}} \simeq \mathrm{Hom}^{\mathcal{C}}(A, N \otimes_A A\mathcal{C}A) \simeq \mathrm{Hom}_A(A, NA) \simeq NA,$
for any $N \in \tilde{\mathcal{M}}_A$, with the maps

$$\begin{array}{rcccc} \varphi_N : \mathrm{Hom}^{\mathcal{C}}(A, N \otimes_A A\mathcal{C}A) \to & \mathrm{Hom}_A(A, NA) \to & NA, \\ f \quad \mapsto & (I \otimes \underline{\varepsilon}) \circ f \quad \mapsto & (I \otimes \underline{\varepsilon}) \circ f(1). \end{array}$$

(5) $(A\mathcal{C})^{co\mathcal{C}} \simeq \mathrm{Hom}^{\mathcal{C}}(A, A\mathcal{C}A) \simeq \mathrm{Hom}_A(A, A) \simeq A,$ *with the maps*

$$\varphi_A : \mathrm{Hom}^{\mathcal{C}}(A, A \otimes_A A\mathcal{C}A) \to \mathrm{Hom}_A(A, A) \to A, \quad f \mapsto \underline{\varepsilon} \circ f \mapsto \underline{\varepsilon} \circ f(1).$$

Proof. Most of these assertions are obvious. To prove (4) we refer to 1.10. □

The standard Hom-tensor relation yields (compare [5, Proposition 5.2]):

2.3. The coinvariant functor. *Let $\mathcal{C}$ be a weak A-coring and A a right $\mathcal{C}$-comodule. Putting $B = A^{co\mathcal{C}}$, for any $N \in \mathcal{M}_B$ and $M \in \tilde{\mathcal{M}}^{\mathcal{C}}$, there is a natural isomorphism*

$$\mathrm{Hom}^{\mathcal{C}}(N \otimes_B A, M) \simeq \mathrm{Hom}_B(N, \mathrm{Hom}^{\mathcal{C}}(A, M)),$$

showing that the functor

$$(-)^{co\mathcal{C}} = \mathrm{Hom}^{\mathcal{C}}(A, -) : \tilde{\mathcal{M}}^{\mathcal{C}} \to \mathcal{M}_B, \quad M \mapsto M^{co\mathcal{C}},$$

is right adjoint to the induction functor $- \otimes_B A : \mathcal{M}_B \to \tilde{\mathcal{M}}^{\mathcal{C}}$, where the $\mathcal{C}$-comodule structure of $N \otimes_B A$ is given by $I \otimes \varrho_A$.

Clearly, if ${}_A\mathcal{C}$ is flat, then $(-)^{co\mathcal{C}}$ is an exact functor if and only if A is a projective object in $\tilde{\mathcal{M}}^{\mathcal{C}}$.

2.4. Galois A-corings. Let $\mathcal{C}$ be a weak A-coring with group-like element $g \in A\mathcal{C}A$, and put $B = A^{co\mathcal{C}}$. Then $\mathcal{C}$ is said to be *right Galois* if the canonical map

$$\mathrm{Hom}^{\mathcal{C}}(A, A\mathcal{C}) \otimes_B A \to A\mathcal{C}A, \quad f{\otimes}a \mapsto f(a),$$

is an isomorphism.

By the isomorphisms considered in 2.2(5), the diagram

$$\begin{array}{ccccc} \mathrm{Hom}^{\mathcal{C}}(A, A \otimes_A A\mathcal{C}A) \otimes_B A & \to & A\mathcal{C}A & f{\otimes}b & \mapsto & f(b) \\ \downarrow {\scriptstyle \varphi_A \otimes I} & & \| & \downarrow & & \| \\ A \otimes_B A & \to & A\mathcal{C}A, & \underline{\varepsilon} \circ f(1){\otimes}b & \mapsto & \underline{\varepsilon} \circ f(1) \cdot g \cdot b, \end{array}$$

is commutative since (recall that $g = \varrho_A(1)$)

$$\begin{array}{rcl} \underline{\varepsilon} \circ f(1) \cdot g \cdot b & = & \underline{\varepsilon} \circ f(1) \cdot \varrho_A(b) = \underline{\varepsilon} \circ f(b_{\underline{0}}) b_{\underline{1}} \\ & = & (\underline{\varepsilon}{\otimes}I) \circ (f{\otimes}I)\varrho_A(b) \\ {\scriptstyle property\ of\ f} & = & (\underline{\varepsilon}{\otimes}I) \circ \underline{\Delta}(f(b)) = f(b). \end{array}$$

Hence $\mathcal{C}$ is right Galois if and only if the canonical map

$$\gamma : A \otimes_B A \to A\mathcal{C}A, \quad a{\otimes}b \mapsto a \cdot \varrho_A(1) \cdot b,$$

is an isomorphism. It is obvious from this definition that the weak A-coring $\mathcal{C}$ is right Galois if and only if the A-coring $A\mathcal{C}A$ is right Galois and this condition coincides with Definition 5.3 in [5].

Notice that $A \otimes_B A$ may be considered as an A-coring in a canonical way and it is straightforward to verify that the canonical map γ is in fact an A-coring morphism (see [11, Example 1.2, Definition 1.3]).

The interest in Galois objects lies in the following observation.

2.5. A as a (projective) generator in $\mathcal{M}^{A\mathcal{C}A}$. *Let $\mathcal{C}$ be a weak A-coring with group-like element $g \in A\mathcal{C}A$ and put $B = A^{co\mathcal{C}}$.*

(1) The following are equivalent:

(a) $\mathcal{C}$ is right Galois, and A is flat as left B-module;

(b) $A\mathcal{C}$ is flat as left A-module, and A is a generator in $\mathcal{M}^{A\mathcal{C}A}$;

(c) $\mathcal{M}^{A\mathcal{C}A}$ is a Grothendieck category, and $\mathrm{Hom}^{\mathcal{C}}(A, -) : \mathcal{M}^{A\mathcal{C}A} \to \text{Mod-}B$ *is a faithful functor;*

(d) $A\mathcal{C}$ is flat as left A-module, and for any $M \in \mathcal{M}^{A\mathcal{C}A}$, the map

$$M^{co\mathcal{C}} \otimes_B A \to M, \quad m{\otimes}a \mapsto ma,$$

is an isomorphism.

(2) The following are equivalent:

(a) $\mathcal{C}$ is right Galois, and A is faithfully flat as left B-module;

(b) $A\mathcal{C}$ is flat as left A-module, and A is a projective generator in $\mathcal{M}^{A\mathcal{C}A}$;

(c) $\mathcal{M}^{A\mathcal{C}A}$ is a Grothendieck category, and $\mathrm{Hom}^{\mathcal{C}}(A,-) : \mathcal{M}^{A\mathcal{C}A} \to \text{Mod-}B$ is an equivalence.

Proof. (1) $(a) \Rightarrow (b)$ If ${}_BA$ is flat then the functor $-\otimes_A (A \otimes_B A) \simeq -\otimes_A A\mathcal{C}A$ is exact, i.e., $A\mathcal{C}$ is flat as left A-module. The first part of the proof of [5, Theorem 5.6] (also [8, 2.5]) shows that A is a generator in $\mathcal{M}^{A\mathcal{C}A}$.

$(b) \Leftrightarrow (c)$ This is a well-known characterization of generators in any catgeory. $A\mathcal{C}$ flat as A-module implies that $\mathcal{M}^{A\mathcal{C}A}$ is a Grothendieck category (see 1.10).

$(d) \Rightarrow (a)$ In a Grothendieck category any generator is flat as module over its endomorphism ring (e.g., [13, 15.9]). In particular A is a flat B-module.

$(b) \Leftrightarrow (d)$ This is easily shown by standard arguments.

(2) By (1), $\mathcal{M}^{A\mathcal{C}A}$ is a Grothendieck category. Therefore a finitely generated generator P in $\mathcal{M}^{A\mathcal{C}A}$ is projective in $\mathcal{M}^{A\mathcal{C}A}$ if and only if P is faithfully flat as module over its endomorphism ring (e.g., [13, 18.5]). Moreover, for such modules P, $\mathrm{Hom}^{A\mathcal{C}A}(P,-)$ induces an equivalence (e.g., [13, 46.2]). □

3 $\mathcal{C}$-comodules and ${}^*\mathcal{C}$-modules

For any coalgebra $\mathcal{C}$, $\mathcal{C}$-comodules are closely related to modules over the dual algebra of $\mathcal{C}$. To a certain extent this transfers to weak corings and comodules. Before studying this we recall some basic facts.

3.1. Canonical maps. For any left A-module K and right A-module N, consider the canonical map

$$\alpha'_{N,K} : N \otimes_A K \to \mathrm{Hom}_{\mathbb{Z}}(K^*, N), \quad n\otimes k \mapsto [f \mapsto nf(k)].$$

It is easy to see that this map factors through $N \otimes_A AK$ yielding a map

$$\alpha_{N,K} : N \otimes_A AK \to \mathrm{Hom}_{\mathbb{Z}}(K^*, NA).$$

(1) The following are equivalent:

(a) $\alpha_{N,K}$ is injective;

(b) for $u \in N \otimes_A AK$, $(I\otimes f)(u) = 0$ for all $f \in K^*$, implies $u = 0$.

(2) *If $\alpha_{N,K}$ is injective for each right A-module N, then AK is flat and cogenerated by A.*

(3) *If AK is a projective A-module, then $\alpha_{N,K}$ is injective, for each $N \in \tilde{\mathcal{M}}_A$.*

Proof. (1) Let $u = \sum n_i \otimes k_i \in N \otimes_A AK$. Then $(I \otimes f)(u) = \sum n_i f(k_i) = 0$, for all $f \in K^*$, if and only if $u \in \operatorname{Ke} \alpha_{N,K}$.

(2) For any exact sequence $0 \to N \to M$ of unital right A-modules, we have the commutative diagram

$$\begin{array}{ccccc} 0 & \to & N \otimes_A AK & \to & M \otimes_A AK \\ & & \downarrow \alpha_{N,K} & & \downarrow \alpha_{M,K} \\ 0 & \to & \operatorname{Hom}_{\mathbb{Z}}(K^*, N) & \to & \operatorname{Hom}_{\mathbb{Z}}(K^*, M). \end{array}$$

The exactness of the second line implies exactness of the first line thus showing that AK is flat.

Notice that $A \otimes_A AK \xrightarrow{\alpha_{A,K}} Hom_{\mathbb{Z}}(K^*, A) \subset A^{K^*}$.

(3) For a dual basis $\{(p_i, k_i) \,|\, p_i \in (AK)^*,\ k_i \in AK\}_I$, let $\sum_i n_i \otimes k_i \in \operatorname{Ke} \alpha_{N,K}$. Then

$$\sum_i n_i \otimes k_i = \sum_i n_i \otimes \sum_l p_l(k_i) k_l = \sum_l (\sum_i n_i\, p_l(k_i)) \otimes k_l = 0,$$

since $\sum_i n_i\, p_l(k_i) = 0$, for each l, showing that $\alpha_{N,K}$ is injective. □

To transfer properties of ${}^*\mathcal{C}$-modules to weak $\mathcal{C}$-comodules the following conditions on the A-module structure of $\mathcal{C}$ is necessary.

3.2. α-condition for weak corings. We say that a weak A-coring $\mathcal{C}$ *satisfies the left (right) α-condition* if the map

$$\alpha_{N,\mathcal{C}} : N \otimes_A A\mathcal{C} \to \operatorname{Hom}_{\mathbb{Z}}({}^*\mathcal{C}, NA), \qquad n \otimes c \mapsto [f \mapsto n f(c)],$$
$$(\alpha_{\mathcal{C},L} : \mathcal{C}A \otimes_A L \to \operatorname{Hom}_{\mathbb{Z}}(\mathcal{C}^*, AL), \qquad c \otimes l \mapsto [g \mapsto g(c) l],)$$

is injective for every right A-module N (left A-module L).

By 3.1(3), $\mathcal{C}$ satifies the left (right) α-condition provided $A\mathcal{C}$ (resp. $\mathcal{C}A$) is projective as a left (right) A-module.

3.3. $\mathcal{C}$-coaction and ${}^*\mathcal{C}$-action.

(1) *Let $\varrho_M : M \to M \otimes_A A \otimes_A \mathcal{C}$ be a weak coaction. Then*

$$\leftharpoondown : M \otimes_A {}^*\mathcal{C} \to M, \quad m \otimes f \mapsto (I \otimes f) \circ \varrho(m),$$

defines a right ${}^*\mathcal{C}$-action on M.

(2) Every A-submodule $K \subset M$ with coaction is a submodule with ${}^*\mathcal{C}$-action.

(3) If $\mathcal{C}$ satisfies the left α-condition, then every submodule closed under ${}^*\mathcal{C}$-action has $\mathcal{C}$-coaction.

(4) Let $h : M \to N$ be an A-linear map of modules with right $\mathcal{C}$-coaction.

(i) If h is a morphism for right $\mathcal{C}$-coaction, then h is a morphism for right ${}^*\mathcal{C}$-action.

(ii) If $\mathcal{C}$ satifies the left α-condition and h is a morphism for left ${}^*\mathcal{C}$-action, then h is a morphism for right $\mathcal{C}$-coaction.

Proof. The assertions in (1) and (2) are straightforward to verify.

(3) Let $K \subset M$ be a submodule with ${}^*\mathcal{C}$-action and consider the map

$$\beta_K : K \to \mathrm{Hom}_{\mathbb{Z}}({}^*\mathcal{C}, K),\ k \mapsto [f \mapsto kf].$$

Notice that $\beta_M = \alpha_{M,\mathcal{C}} \circ \varrho_M$. We have the commutative diagram with exact lines

$$\begin{array}{ccccccc}
0 \to & K & \xrightarrow{i} & M & \xrightarrow{p} & M/K & \to 0 \\
 & \vdots & & \downarrow \varrho_M & & & \\
0 \to & K \otimes_A A\mathcal{C} & \xrightarrow{i\otimes I} & M \otimes_A A\mathcal{C} & \xrightarrow{p\otimes I} & M/K \otimes_A A\mathcal{C} & \to 0 \\
 & \downarrow \alpha_{K,\mathcal{C}} & & \downarrow \alpha_{M,\mathcal{C}} & & \downarrow \alpha_{M/K,\mathcal{C}} & \\
0 \to & \mathrm{Hom}_{\mathbb{Z}}({}^*\mathcal{C}, K) & \xrightarrow{\mathrm{Hom}({}^*\mathcal{C},i)} & \mathrm{Hom}_{\mathbb{Z}}({}^*\mathcal{C}, M) & \to & \mathrm{Hom}_{\mathbb{Z}}({}^*\mathcal{C}, M/K) & \to 0,
\end{array}$$

where all the α's are injective and $\mathrm{Hom}({}^*\mathcal{C}, i) \circ \beta_K = \alpha_{M,\mathcal{C}} \circ \varrho \circ i$. This implies that $(p\otimes I)\circ\varrho_M\circ i = 0$, and by the kernel property (in $\tilde{\mathcal{M}}_A$), $\varrho_M\circ i$ factors through $K\otimes_A A\mathcal{C}$, i.e., we have a coaction $K \to K \otimes_A A\mathcal{C}$.

Obviously the diagram yields a coaction on M/K, too.

(4) Consider the diagram

$$\begin{array}{ccc}
M & \xrightarrow{h} & N \\
\downarrow \varrho_M & & \downarrow \varrho_N \\
M \otimes_A A\mathcal{C} & \xrightarrow{h\otimes I} & N \otimes_A A\mathcal{C} \\
\downarrow \alpha_{M,\mathcal{C}} & & \downarrow \alpha_{N,\mathcal{C}} \\
\mathrm{Hom}_{\mathbb{Z}}({}^*\mathcal{C}, M) & \xrightarrow{\mathrm{Hom}({}^*\mathcal{C},h)} & \mathrm{Hom}_{\mathbb{Z}}({}^*\mathcal{C}, N),
\end{array}$$

in which the lower square is always commutative.

If h is a comodule map, then the upper square is also commutative and so is the outer rectangle. It is straightforward to see that this is equivalent to h respecting ${}^*\mathcal{C}$-action thus showing (i).

Now assume the outer rectangle to be commutative. By assumtion $\alpha_{N,\mathcal{C}}$ is injective and this implies that the upper square is also commutative proving (ii). $\square$

3.4. $\mathcal{C}$-comodules and $^*\mathcal{C}$-modules. *Let $\mathcal{C}$ be a coassociative weak A-coring, $\varrho_M : M \to M \otimes_A A \otimes_A \mathcal{C}$ a right weak coaction and $\leftharpoonup : M \otimes_A {}^*\mathcal{C} \longrightarrow MA \subset M$ the corresponding action.*

(1) If ϱ_M is coassocciative then $\leftharpoonup$ makes M a right $^*\mathcal{C}$-module and $\underline{\varepsilon}$ acts as identity on MA.

(2) If $\mathcal{C}$ satisfies the left α-condition and M is a right $^*\mathcal{C}$-module by $\leftharpoonup$, then ϱ_M is coassociative and every $^*\mathcal{C}$-submodule of M is a weak $\mathcal{C}$-sub-comodule.

Proof. (1) If ϱ_M is coassociative we have the commutative diagram, for $f, g \in {}^*\mathcal{C}$,

$$\begin{tikzcd}
 & M \otimes_A A\mathcal{C} \arrow[dr, "I\otimes\underline{\Delta}"] & & & \\
M \arrow[ur, "\varrho_M"] \arrow[dr, "\varrho_M"'] & & M \otimes_A A\mathcal{C} \otimes_A A\mathcal{C} \arrow[r, "I\otimes I\otimes f"] & M \otimes_A A\mathcal{C}A \arrow[r, "I\otimes g"] & MA\,. \\
 & M \otimes_A A\mathcal{C} \arrow[ur, "\varrho_M\otimes I"'] & & &
\end{tikzcd}$$

For any $m \in M$ the upper path yields $m\leftharpoonup(f *_l g)$ while the lower path yields $(m\leftharpoonup f)\leftharpoonup g$. This implies our first assertion.

Since M is weakly counital, for any $m \in M$, $m1\leftharpoonup\underline{\varepsilon} = \sum m_{\underline{0}}\underline{\varepsilon}(m_{\underline{1}}) = m1$.

(2) If M is a $^*\mathcal{C}$-module by $\leftharpoonup$, then $m\leftharpoonup(f *_l g) = (m\leftharpoonup f)\leftharpoonup g$ for all $f, g \in {}^*\mathcal{C}$ and the left α-condition implies commutativity of the rectangle in the above diagram.

The second assertion follows from 3.3. $\square$

By 3.3 we have the following relationship between

3.5. $\mathcal{C}$-comodule and $^*\mathcal{C}$-module morphisms. *Let M and N be right weak $\mathcal{C}$-comodules and $h : M \to N$ an A-linear map.*

(1) If h is a $\mathcal{C}$-comodule morphism then h is a $^*\mathcal{C}$-module morphism.

(2) If $\mathcal{C}$ satisfies the left α-condition and h is a $^*\mathcal{C}$-module morphism, then h is a $\mathcal{C}$-comodule morphism, i.e.,

$$\mathrm{Hom}^{\mathcal{C}}(M,N) = \mathrm{Hom}_{^*\mathcal{C}}(M,N), \textit{ for any } M, N \in \tilde{\mathcal{M}}^{\mathcal{C}}.$$

In a similar way left weak coactions on a left A-module M yield left actions of $\mathcal{C}^*$ on M. In particular we have for $\mathcal{C}$ itself:

3.6. ${}^*\mathcal{C}$- and $\mathcal{C}^*$-actions on $\mathcal{C}$. *For any coassociative weak A-coring $\mathcal{C}$ there are actions*

$$\begin{array}{l} \leftharpoondown : \mathcal{C} \otimes_A {}^*\mathcal{C} \to \mathcal{C}A, \quad c{\otimes}f \mapsto (I{\otimes}I{\otimes}f)\circ\underline{\Delta}(c), \\ \rightharpoondown : \mathcal{C}^* \otimes_A \mathcal{C} \to A\mathcal{C}, \quad g{\otimes}c \mapsto (g{\otimes}I{\otimes}I)\circ\underline{\Delta}(c)\,. \end{array}$$

(1) For any $f \in {}^*\mathcal{C}$, $g \in \mathcal{C}^*$, and $c \in \mathcal{C}$, $\quad (g\rightharpoondown c)\leftharpoondown f = g\rightharpoondown(c\leftharpoondown f)$.

(2) For any $f \in {}^*\mathcal{C}$, $h \in {}^*\mathcal{C}^*$, and $c \in \mathcal{C}$, $\quad f{*_l}h(c) = f(h\rightharpoondown c) = h(c\leftharpoondown f)$.

(3) For any $c \in \mathcal{C}$, $\quad c\leftharpoondown\underline{\varepsilon} = 1c1 = \underline{\varepsilon}\rightharpoondown c$.
${}^*(A\mathcal{C}A)$ and $(A\mathcal{C}A)^*$ act faithfully on $A\mathcal{C}A$.

(4) If $\mathcal{C}$ satisfies the left α-condition, then any right A-submodule $D \subset \mathcal{C}A$ which is closed under right ${}^*\mathcal{C}$-action has right weak coaction.

(5) Let $\mathcal{C}$ satisfy the left and right α-condition, and consider any (A, A)-submodule $D \subset A\mathcal{C}A$ which is pure as left and right A-submodule. Then D is a weak sub-coring if and only if D is closed under left $\mathcal{C}^*$-action and right ${}^*\mathcal{C}$-action.

Proof. (1) By definition,

$$(g\rightharpoondown c)\leftharpoondown f = \sum g(c_{\underline{1}})c_{\underline{2}\underline{1}}f(c_{\underline{2}\underline{2}}) = \sum g(c_{\underline{1}\underline{1}})c_{\underline{1}\underline{2}}f(c_{\underline{2}}) = g\rightharpoondown(c\leftharpoondown f).$$

(2) By definition,

$$\begin{array}{rcl} f{*_l}h(c) & = & \sum h(c_{\underline{1}}f(c_2)) = h(c\leftharpoondown f) \\ & = & \sum h(c_{\underline{1}})f(c_2) = f(h\rightharpoondown c)\,. \end{array}$$

(3) is clear by weak counitality of $\underline{\varepsilon}$ and 1.11; (4) follows from 3.3.

(5) Clearly every weak sub-coring D is closed under left $\mathcal{C}^*$-action and right ${}^*\mathcal{C}$-action.

Let $D \subset \mathcal{C}$ be an (A, A)-submodule with the purity condition which is closed under left $\mathcal{C}^*$-action and right ${}^*\mathcal{C}$-action. Then the restriction of $\underline{\Delta}$ yields a left and right $\mathcal{C}$-coaction on D and

$$\underline{\Delta}(D) \subset D \otimes_A A \otimes_A \mathcal{C} \;\cap\; \mathcal{C} \otimes_A A \otimes_A D = D \otimes_A A \otimes_A D\,.$$

The first inclusion follows from 3.3. For the equality consider the commutative and exact diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & D \otimes_A A \otimes_A D & \to & D \otimes_A A \otimes_A \mathcal{C} & \to & D \otimes_A A \otimes_A \mathcal{C}/D & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{C} \otimes_A A \otimes_A D & \to & \mathcal{C} \otimes_A A \otimes_A \mathcal{C} & \to & \mathcal{C} \otimes_A A \otimes_A \mathcal{C}/D & \to & 0\,. \end{array}$$

Since the left square is a pullback (e.g., [13, 10.3]), we can make the identification stated. This shows that D is a weak subcoring. $\square$

Writing morphisms of left (co-) modules on the right side of the argument and vice versa, the following is now obvious:

3.7. Coassociative A-corings. *Let $\mathcal{C}$ be a coassociative A-coring.*

(1) ${}^*\mathcal{C}$ *and* $\mathcal{C}^*$ *are associative rings with unit.*

(2) The actions $\leftharpoonup$ *and* $\rightharpoonup$ *make* $\mathcal{C}$ *a* $(\mathcal{C}^*, {}^*\mathcal{C})$*-bimodule which is faithful on the left and on the right.*

(3) $\mathrm{End}^{-\mathcal{C}}(\mathcal{C}) \simeq \mathcal{C}^*$ *and* $\mathrm{End}^{\mathcal{C}-}(\mathcal{C}) \simeq {}^*\mathcal{C}$.

(4) If $\mathcal{C}$ *satisfies the left (right)* α*-condition then*

$$\mathrm{End}_{-{}^*\mathcal{C}}(\mathcal{C}) = \mathrm{End}^{-\mathcal{C}}(\mathcal{C}) \simeq \mathcal{C}^*, \quad (\textit{ resp.,}\ \mathrm{End}_{\mathcal{C}^*-}(\mathcal{C}) = \mathrm{End}^{\mathcal{C}-}(\mathcal{C}) \simeq {}^*\mathcal{C}).$$

The preceding observations yield a close relationship between weak $\mathcal{C}$-comodules and ${}^*\mathcal{C}$-modules and we obtain a general form of the finiteness theorem for coalgebras.

3.8. The category of weak comodules. *Let $\mathcal{C}$ be a coassociative weak A-coring satisfying the left α-condition.*

(1) $\tilde{\mathcal{M}}^{\mathcal{C}}$ *is a full subcategory of* $\tilde{\mathcal{M}}_{{}^*\mathcal{C}}$.

(2) For every $M \in \tilde{\mathcal{M}}^{\mathcal{C}}$, $M \otimes_A A\mathcal{C}$ *is generated (and* MA *is subgenerated) by the right* $\mathcal{C}$*-comodule* $A\mathcal{C}$.

(3) For every $M \in \tilde{\mathcal{M}}^{\mathcal{C}}$*, finitely generated* ${}^*\mathcal{C}$*-submodules of* MA *are finitely generated as (right) A-modules.*

(4) If $A\mathcal{C}A$ *is finitely generated as left* $\mathcal{C}^*$*-module (left A-module), then* ${}^*(A\mathcal{C}A) \in \tilde{\mathcal{M}}^{\mathcal{C}}$.

Proof. (1) This is clear by 3.4 and 3.5.

(2) We have an epimorphism $A^{(\Lambda)} \to M \otimes_A A$ of right A-modules. By 1.8 this yields an epimorphism $(A \otimes_A \mathcal{C})^{(\Lambda)} \simeq A^{(\Lambda)} \otimes_A \mathcal{C} \to M \otimes_A A\mathcal{C}$ in $\tilde{\mathcal{M}}^{\mathcal{C}}$.

Notice that ϱ_M is a comodule morphism but need not be injective. However the restriction to $MA \subset M$ is injective and hence MA is a subcomodule of $M \otimes_A A\mathcal{C}$.

(3) For $k \in MA$ consider the cyclic submodule $K := k\,{}^*\mathcal{C} \subset MA$. By 3.4, there exists a weak coaction $\varrho_K : K \to K \otimes_A A\mathcal{C}$ and we have $\varrho_K(k) = \sum_{i=1}^r k_i \otimes c_i$, where $k_i \in K$, $c_i \in \mathcal{C}$. So for any $f \in {}^*\mathcal{C}$, $k \leftharpoonup f = \sum_{i=1}^r k_i f(c_i)$ which shows that K is finitely generated by $k_1, \ldots, k_r$ as right A-module.

(4) Let $A\mathcal{C}A$ be finitely generated as left $\mathcal{C}^*$-module (or A-module) by $a_1, \ldots, a_r \in A\mathcal{C}A$ and consider the map

$$^*(A\mathcal{C}A) \to (a_1, \ldots, a_r)\, ^*(A\mathcal{C}A) \subset (A\mathcal{C}A)^r \subset (A\mathcal{C})^r, \quad f \mapsto (a_1, \ldots, a_r) \leftharpoonup f.$$

Since $^*(A\mathcal{C}A)$ acts faithfully on $A\mathcal{C}A$ this is a monomorphism of right $^*(A\mathcal{C}A)$-modules. So $^*(A\mathcal{C}A)$ is a submodule of the weak comodule $(A\mathcal{C})^r$ and hence is a right weak $\mathcal{C}$-subcomodule (by 3.4). □

The proof shows that under the given conditions $^*(A\mathcal{C}A)$ is in fact a comodule over the coring $A\mathcal{C}A$. For corings the situation simplifies to the following. Notice that assertion (3) was already observed in [5, Lemma 4.3].

3.9. The category of comodules. *Let $\mathcal{C}$ be a coassociative A-coring satisfying the left α-condition.*

(1) $\mathcal{C}$ is a subgenerator in $\mathcal{M}^{\mathcal{C}}$ and $\mathcal{M}^{\mathcal{C}} = \sigma[\mathcal{C}_{^*\mathcal{C}}]$ is a full subcategory of $\mathcal{M}_{^*\mathcal{C}}$.

(2) For every $M \in \mathcal{M}^{\mathcal{C}}$, any finitely many elements of M are contained in a subcomodule ($^*\mathcal{C}$-submodule) which is finitely generated as A-module.

(3) If $\mathcal{C}$ is finitely generated as left $\mathcal{C}^*$-module or left A-module, then $\mathcal{M}^{\mathcal{C}} = \mathcal{M}_{^*\mathcal{C}}$.

(4) For a left noetherian ring A, the following are equivalent:

  (a) $\mathcal{C}$ is finitely generated as left A-module;

  (b) $\mathcal{C}$ is finitely generated as left $\mathcal{C}^*$-module;

  (c) $\mathcal{M}^{\mathcal{C}} = \mathcal{M}_{^*\mathcal{C}}$.

Proof. (1), (2) and (3) follow immediately from 3.8.

(4) $(a) \Rightarrow (b) \Rightarrow (c)$ are clear by 3.8.

$(c) \Rightarrow (a)$ By (2) and (3), $^*\mathcal{C}$ is finitely generated as right A-module and hence $\mathcal{C}^{**}$ is a finitely generated (noetherian) left A-module. By the left α-condition, $_A\mathcal{C}$ is cogenerated by A and so $_A\mathcal{C}$ is a submodule of $\mathcal{C}^{**}$ and hence finitely generated. □

4 Entwining structures

For the history and importance of (weak) entwining structures and their (co)modules we refer to Caenepeel-Groot [6] and Brzeziński [5]. Here we show how this theory can be derived and interpreted by using weak corings studied in the preceding sections thus providing alternative proofs of related results in [6].

Let R be a commutative associative ring with unit, $\mu : A \otimes_R A \to A$ an R-algebra with unit $\iota : R \to A$, and $\Delta : C \to C \otimes_R C$ an R-coalgebra with counit $\varepsilon : C \to R$.

We are interested in the interaction between the algebra A and the coalgebra C. For this we ask for possible structures of $A \otimes_R C$. The following result was essentially announced in [6] and [5].

4.1. $A \otimes_R C$ as an A-coring. *Consider $A \otimes_R C$ as a left A-module canonically.*

(1) Assume there exists a right A-action $\cdot$ on $A \otimes_R C$ and define the R-linear map

$$\psi : C \otimes_R A \to A \otimes_R C, \quad c \otimes a \mapsto (1 \otimes c) \cdot a,$$

writing $\psi(c \otimes a) = \sum a_\psi \otimes c^\psi$, for suitable $a_\psi \in A$, $c^\psi \in C$.
Moreover, consider the maps

$$\begin{array}{ccccccc} \underline{\Delta} & : & A \otimes_R C & \to & (A \otimes_R C) \otimes_A (A \otimes_R C) & \simeq & (A \otimes_R C) \cdot 1 \otimes_R C, \\ & & a \otimes c & \mapsto & \sum (a \otimes c_{\underline{1}}) \otimes_A (1 \otimes c_{\underline{2}}) & \mapsto & \sum (a \otimes c_{\underline{1}}) \cdot 1 \otimes c_{\underline{2}}, \\ \underline{\varepsilon} & : & A \otimes_R C & \to & (A \otimes_R C) \cdot 1 & \to & A\ , \\ & & a \otimes c & \mapsto & (a \otimes c) \cdot 1 & \mapsto & (I \otimes \varepsilon)((a \otimes c) \cdot 1), \end{array}$$

where $\Delta(c) = \sum c_{\underline{1}} \otimes c_{\underline{2}}$, for $c \in C$. Then:

(i) If $(A \otimes_R C, \underline{\Delta}, \underline{\varepsilon})$ is an A-coring, then

(1.1) $\sum (ab)_\psi \otimes c^\psi = \sum a_\psi b_\varphi \otimes c^{\psi\varphi}$.

(1.2) $\sum a_\psi \otimes c^\psi{}_{\underline{1}} \otimes c^\psi{}_{\underline{2}} = \sum a_{\psi\varphi} \otimes c_{\underline{1}}^{\varphi} \otimes c_{\underline{2}}^{\psi}$.

(1.3) $\sum a_\psi \varepsilon(c^\psi) = \varepsilon(c) a$.

(1.4) $\sum 1_\psi \otimes c^\psi = 1 \otimes c$.

(ii) If $(A \otimes_R C, \underline{\Delta}, \underline{\varepsilon}))$ is a weak A-coring, then (1.1) holds and

(1.2)$'$ $\sum a_\psi \psi(c^\psi{}_{\underline{1}} \otimes 1) \otimes c^\psi{}_{\underline{2}} = \sum a_{\psi\varphi} \otimes c_{\underline{1}}^{\varphi} \otimes c_{\underline{2}}^{\psi}$.

(1.3)$'$ $\sum a_\psi \varepsilon(c^\psi) = \sum \varepsilon(c^\psi) 1_\psi a$.

(1.4)$'$ $\sum 1_\psi \otimes c^\psi = \sum \varepsilon(c_{\underline{1}}^{\psi}) 1_\psi \otimes c_{\underline{2}}$.

(2) Assume there exists an R-linear map $\psi : C \otimes_R A \to A \otimes_R C$ and define a right A-action on $A \otimes_R C$ by

$$(A \otimes_R C) \otimes_R A \to A \otimes_R C, \quad (a \otimes c) \otimes b \mapsto a\,\psi(c \otimes b).$$

If ψ satisfies $(1.1) - (1.4)$, *then $A \otimes_R C$ is an A-coring.*
If ψ satisfies $(1.1), (1.2)', (1.3)', (1.4)'$, *then $A \otimes_R C$ is a (left unital) weak A-coring.*

In the first case (A, C, ψ) is called an *entwining structure*, in the second case (A, C, ψ) is called a *weak entwinig structure*. Notice that $(1.2)'$ differs slightly from the corresponding condition in [6].

Proof. (1) (*i*) (1.1) Associativity of right multiplication yields

$$\sum (ab)_\psi \otimes c^\psi = (1 \otimes c) \cdot ab = (1 \otimes c) \cdot a \cdot b = \sum a_\psi b_\varphi \otimes c^{\psi\varphi}.$$

(1.2) By definition we have

$$\begin{aligned} \underline{\Delta}((1 \otimes c) \cdot a) &= \underline{\Delta}(\sum a_\psi \otimes c^\psi) \\ &= \sum a_\psi \otimes c^\psi{}_{\underline{1}} \otimes c^\psi{}_{\underline{2}}, \quad \text{and} \\ \underline{\Delta}(1 \otimes c) \cdot a &= \sum (1 \otimes c_{\underline{1}}) \otimes_A (1 \otimes c_{\underline{2}}) \cdot a \\ &= \sum (1 \otimes c_{\underline{1}}) \otimes_A (\sum a_\psi \otimes c_{\underline{2}}^\psi) \\ &= \sum a_{\psi\varphi} \otimes c_{\underline{1}}^\varphi \otimes c_{\underline{2}}^\psi \,. \end{aligned}$$

If $\underline{\Delta}$ is a right A-module morphism the two expressions are the same.

(1.3) $\underline{\varepsilon}$ is a right A-module morphism, so $I \otimes \varepsilon((1 \otimes c) \cdot a) = \varepsilon(c)a$.

(1.4) $A \otimes_R C$ is a unital right module, so $1 \otimes c = (1 \otimes c) \cdot 1 = \sum 1_\psi \otimes c^\psi$.

(*ii*) $(1.2)'$ One expression from (1.2) remains unchanged, for the other we get

$$\begin{aligned} \underline{\Delta}((1 \otimes c) \cdot a) &= \underline{\Delta}(\sum a_\psi \otimes c^\psi) \\ &= \sum (a_\psi \otimes c^\psi{}_{\underline{1}}) \cdot 1 \otimes c^\psi{}_{\underline{2}} \\ &= \sum a_\psi \psi(c^\psi{}_{\underline{1}} \otimes 1) \otimes c^\psi{}_{\underline{2}} \,. \end{aligned}$$

$(1.3)'$ $\underline{\varepsilon}$ is a right A-module morphism, so

$$\begin{aligned} \sum a_\psi \varepsilon(c^\psi) &= I \otimes \varepsilon((1 \otimes c) \cdot a) \\ &= (I \otimes \varepsilon((1 \otimes c) \cdot 1)) \cdot a \\ &= (I \otimes \varepsilon(\sum 1_\psi \otimes c^\psi)) \cdot a \\ &= \sum \varepsilon(c^\psi) 1_\psi a. \end{aligned}$$

$(1.4)'$ $\underline{\varepsilon}$ is weakly counitary, so

$$\begin{aligned} \sum 1_\psi \otimes c^\psi &= (1 \otimes c) \cdot 1 \\ {\scriptstyle counital} &= (\underline{\varepsilon} \otimes I) \circ \underline{\Delta}(1 \otimes c) \\ &= I \otimes \varepsilon \otimes I(\sum 1_\psi \otimes c_{\underline{1}}^\psi \otimes c_{\underline{2}}) \\ &= \sum \varepsilon(c_{\underline{1}}^\psi) 1_\psi \otimes c_{\underline{2}}. \end{aligned}$$

(2) Given the map ψ with the corresponding properties the assertions can be verified along the same lines. $\square$

4.2. Dual algebra and smash product. Let $A \otimes_R C$ be a weak A-coring (as in 4.1). Then the canonical R-module isomorphism

$$\mathrm{Hom}_{A-}(A \otimes_R C, A) \to \mathrm{Hom}_R(C, A), \quad h \mapsto h \circ (1\otimes -),$$

induces an associative algebra structure on $\mathrm{Hom}_R(C, A)$ with multiplication

$$f *_l g(c) = \sum f(c_{\underline{2}})_\psi g(c_{\underline{1}}^{\psi}), \qquad \text{for } f, g \in \mathrm{Hom}_R(C, A),\ c \in C.$$

We call this algebra the *smash product* of A and C and denote it by $\#(C, A)$.

$\#(C, A)$ contains a central idempotent e defined by

$$e(c) := \underline{\varepsilon}(1\otimes c) = I\otimes\varepsilon((1\otimes c)\cdot 1),\ \textit{for } c \in C.$$

Assume C to be projective as an R-module. Then:

(1) The category $\tilde{\mathcal{M}}^{A\otimes_R C}$ of right weak $A \otimes_R C$-comodules is a full subcategory of $\mathrm{Mod}\text{-}\#(C, A)$.

(2) $A \otimes_R C$ subgenerates all weak right $A \otimes_R C$-comodules which are unital right A-modules.

(3) If C is finitely generated as R-module, then $\#(C, A) *_l e \in \tilde{\mathcal{M}}^{A\otimes_R C}$.

Proof. For $\tilde{f}, \tilde{g} \in \mathrm{Hom}_{A-}(A \otimes_R C, A)$ we have (see 1.4)

$$\tilde{f} *_l \tilde{g} = \sum \tilde{g}((1\otimes c_{\underline{1}}) \cdot \tilde{f}(1\otimes c_{\underline{2}})) = \sum \tilde{g}(\tilde{f}(1\otimes c_{\underline{2}})_\psi \otimes c_1^{\psi}) = \sum \tilde{f}(1\otimes c_{\underline{2}})_\psi\, \tilde{g}(1\otimes c_1^{\psi}),$$

and this induces the multiplication suggested for $\mathrm{Hom}_R(C, A)$.

$\underline{\varepsilon}$ is a central idempotent in $\mathrm{Hom}_{A-}(A\otimes_R C, A) = {}^*(A \otimes_R C)$ (see 1.4) and - under the isomorphism under consideration - e is the image of $\underline{\varepsilon}$.

If C is projective as an R-module then $A\otimes_R C$ is a projective A-module and hence satisfies the α-condition. So (1) and (2) are special cases of 3.8.

Moreover, if C is finitely generated as an R-module then $A \otimes_R C$ is finitely generated as an A-module, and so is its homomorphic image $(A \otimes_R C) \cdot A$. Now 3.8(4) implies that ${}^*((A\otimes_R C)\cdot A) \simeq {}^*(A \otimes_R C) *_l \underline{\varepsilon}$ is in $\tilde{\mathcal{M}}^{A\otimes_R C}$ and this ring is isomorphic to $\#(C, A) *_l e$. □

The above observations are variations and refinements of what is called the *weak Koppinen smash product* in [6, Section 3.2]. Of course the situation simplifies for corings (compare [5, Lemma 4.3]):

4.3. Smash product of corings. *Let $A\otimes_R C$ be an A-coring (as in 4.1) and assume C to be projective as an R-module. Then:*

(1) $\#(C, A)$ has a unit and $A\otimes_R C$ is a subgenerator in $\mathcal{M}^{A\otimes_R C} = \sigma[(A\otimes_R C)_{\#(C,A)}]$.

(2) If C is finitely generated as R-module, then $\mathcal{M}^{A\otimes_R C} = \mathrm{Mod}\text{-}\#(C, A)$.

5 Weak bialgebras

Weak bialgebras are studied in Böhm-Nill-Szlachányi [1] and their relations to weak entwining structures are displayed in Caenepeel-Groot [6]. Here we give a characterization of weak bialgebras in terms of related weak corings thus showing that (part of) the theory is covered by our techniques.

Throughout this section (B,μ,Δ) will denote an R-module B which is an associative R-algebra with multiplication μ and unit 1 as well as a coassociative coalgebra with comultiplication Δ and counit ε, such that

$$\Delta(ab)=\Delta(a)\Delta(b), \quad \text{for all } a,b\in B.$$

With the twist map τ we can form another mutliplication $\mu^\tau := \mu\circ\tau$ and another comultiplication $\Delta^\tau := \tau\circ\Delta$ for B, and the resulting structures

$$(B,\mu^\tau,\Delta^\tau), \quad (B,\mu^\tau,\Delta), \quad (B,\mu,\Delta^\tau)$$

are again algebras and coalgebras with multiplicative comultiplication.

Based on any of these data we have canonical multiplications with unit $1\otimes 1$ on $B\otimes_R B$ and we will define comultiplications with counits on $B\otimes_R B$. For a *(weak) bialgebra* we expect that $B\otimes_R B$ becomes a (weak) B-coring in each of the four cases. As we shall see, for *bialgebras* it will be enough to check one of the cases whereas for *weak bialgebras* we have to check two (suitable) cases.

5.1. Comultiplications on $B\otimes_R B$. *Given (B,μ,Δ), we consider $B\otimes_R B$ as a (B,B)-bimodule with right and (unital) left B-actions*

$$\begin{array}{rcl} (a\otimes b)\cdot c &=& (a\otimes b)\Delta(c)=\sum ac_{\underline{1}}\otimes bc_{\underline{2}},\\ a(b\otimes c) &=& ab\otimes c\,, \qquad\quad \textit{for all } a,b,c\in B\,. \end{array}$$

(1) For (B,μ,Δ) define the maps

$$\begin{array}{rcccccc} \underline{\Delta} &:& B\otimes_R B &\to& (B\otimes_R B)\otimes_B(B\otimes_R B) &\simeq& (B\otimes_R B)\cdot 1\otimes_R B,\\ && a\otimes b &\mapsto& \sum(a\otimes b_{\underline{1}})\otimes_B(1\otimes b_{\underline{2}}) &\mapsto& \sum a1_{\underline{1}}\otimes b_{\underline{1}}1_{\underline{2}}\otimes b_{\underline{2}},\\ \underline{\varepsilon} &:& B\otimes_R B &\to& (B\otimes_R B)\cdot 1 &\xrightarrow{I\otimes\varepsilon}& B\,,\\ && a\otimes b &\mapsto& (a\otimes b)\cdot 1 &\mapsto& \sum a1_{\underline{1}}\varepsilon(b1_{\underline{2}}). \end{array}$$

(2) For (B,μ^τ,Δ^τ) we consider the maps

$$\underline{\Delta}^\tau : a\otimes b\mapsto \sum(a\otimes b_{\underline{2}})\otimes_B(1\otimes b_{\underline{1}}), \quad {}^\tau\underline{\varepsilon}^\tau : a\otimes b\mapsto\sum 1_{\underline{2}}a\varepsilon(1_{\underline{1}}b).$$

The module $B\otimes_R B$ with these maps we denote by $B\otimes_R^o B$.

(3) For (B,μ^τ,Δ) we consider the maps

$$\underline{\Delta}: a\otimes b \mapsto \sum (a\otimes b_{\underline{1}})\otimes_B(1\otimes b_{\underline{2}}), \quad \underline{\varepsilon}^\tau : a\otimes b \mapsto \sum 1_{\underline{1}} a\varepsilon(1_{\underline{2}} b).$$

(4) For (B,μ,Δ^τ) we consider the maps

$$\underline{\Delta}^\tau: a\otimes b \mapsto \sum (a\otimes b_{\underline{2}})\otimes_B(1\otimes b_{\underline{1}}), \quad {}^\tau\underline{\varepsilon} : a\otimes b \mapsto \sum a 1_{\underline{2}} \varepsilon(b 1_{\underline{1}}).$$

Then all the $\underline{\Delta}$'s are coassociative weak comultiplications on $B\otimes_R B$ and the $\underline{\varepsilon}$'s are left B-module morphism with

$$(a\otimes b)\cdot 1 = (I\otimes\underline{\varepsilon})\circ\underline{\Delta}(a\otimes b), \qquad \textit{for all } a,b\in B.$$

Proof. (1) Clearly $\underline{\Delta}$ is a left B-module morphism. For $a,b,c\in B$ we have

$$\begin{aligned}
\underline{\Delta}((1\otimes b)\cdot c) &= \sum (c_{\underline{1}}\otimes(bc_{\underline{2}})_{\underline{1}})\otimes_B(1\otimes(bc_{\underline{2}})_{\underline{2}} \\
&= \sum c_{\underline{1}} 1_{\underline{1}}\otimes b_{\underline{1}} c_{\underline{2}\,\underline{1}} 1_{\underline{2}}\otimes b_{\underline{2}} c_{\underline{2}\,\underline{2}} \\
&= \sum c_{\underline{1}\,\underline{1}}\otimes b_{\underline{1}} c_{\underline{1}\,\underline{2}}\otimes b_{\underline{2}} c_{\underline{2}}\ ; \\
\underline{\Delta}(1\otimes b)\cdot c &= \sum (1\otimes b_{\underline{1}})\otimes_B(1\otimes b_{\underline{2}})\cdot c \\
&= \sum (1\otimes b_{\underline{1}})\otimes_B(c_{\underline{1}}\otimes b_{\underline{2}} c_{\underline{2}}) \\
&= \sum c_{\underline{1}\,\underline{1}}\otimes b_{\underline{1}} c_{\underline{1}\,\underline{2}}\otimes b_{\underline{2}} c_{\underline{2}}.
\end{aligned}$$

This shows that $\underline{\Delta}$ is right B-linear. Coassociativity of $\underline{\Delta}$ follows easily from the coassociativity of Δ.

Clearly $\underline{\varepsilon}$ is left B-linear. Moreover, for $a,b\in B$,

$$\begin{aligned}
(I\otimes\underline{\varepsilon})\underline{\Delta}(a\otimes b) &= \sum (a\otimes b_{\underline{1}})\otimes_B 1_{\underline{1}}\varepsilon(b_{\underline{2}} 1_{\underline{2}}) \\
&= \sum a 1_{\underline{1}\,\underline{1}}\otimes b_{\underline{1}} 1_{\underline{1}\,\underline{2}}\varepsilon(b_{\underline{2}} 1_{\underline{2}}) \\
&= \sum a 1_{\underline{1}}\otimes b_{\underline{1}} 1_{\underline{2}\,\underline{1}}\varepsilon(b_{\underline{2}} 1_{\underline{2}\,\underline{2}}) \\
&= \sum a 1_{\underline{1}}\otimes b 1_{\underline{2}} = (a\otimes b)\cdot 1.
\end{aligned}$$

The proofs for (2), (3) and (4) follow by the same pattern. □

In general the properties of $\underline{\Delta}$ and $\underline{\varepsilon}$ are not sufficient to make $B\otimes_R B$ a coring. $\underline{\varepsilon}$ need neither be right B-linear nor $(\underline{\varepsilon}\otimes I)\circ\underline{\Delta}(a\otimes b) = (a\otimes b)\cdot 1$. To ensure these properties we have to pose additional conditions on ε and Δ.

We say that (B,μ,Δ) *induces a (weak) coring structure* on $B\otimes_R B$ if the latter is a (weak) B-coring with the maps defined in 5.1.

Recall that (B,μ,Δ) is said to be a *bialgebra* provided Δ and ε are unital algebra morphisms.

5.2. $B \otimes_R B$ **as coring.** *The followig are equivalent:*

(a) (B, μ, Δ) *induces a coring structure on* $B \otimes_R B$*;*

(b) $(B, \mu^\tau, \Delta^\tau)$ *induces a coring structure on* $B \otimes_R B$*;*

(c) (B, μ^τ, Δ) *induces a coring structure on* $B \otimes_R B$*;*

(d) (B, μ, Δ^τ) *induces a coring structure on* $B \otimes_R B$*;*

(e) B *is a bialgebra, i.e.,*

$(B.1)$ $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$, *for* $a, b \in B$.

$(B.2)$ $\Delta(1) = 1 \otimes 1$.

Proof. $(a) \Rightarrow (e)$ Assume $B \otimes_R B$ to be a B-coring. Then $B \otimes_R B$ is a unital right B-module, e.g.,

$$1 \otimes 1 = (1 \otimes 1) \cdot 1 = (1 \otimes 1)\Delta(1) = \Delta(1),$$

and $\underline{\varepsilon}$ is right B-linear, i.e.,

$$\underline{\varepsilon}((1 \otimes a) \cdot b) = \sum b_{\underline{1}} \varepsilon(a b_{\underline{2}}) = \varepsilon(a) b.$$

Applying ε we get

$$\begin{array}{rcccl} \sum \varepsilon(b_{\underline{1}} \varepsilon(a b_{\underline{2}})) & = & \sum \varepsilon(a \varepsilon(b_{\underline{1}}) b_{\underline{2}}) & = & \varepsilon(ab) \\ & = & \varepsilon(\varepsilon(a) b) & = & \varepsilon(a)\varepsilon(b)\,. \end{array}$$

$(e) \Rightarrow (a)$ If $(B.1)$ and $(B.2)$ are satisfied, then $B \otimes_R B$ is a unital right B-module and

$$\underline{\varepsilon}((a \otimes b) \cdot c) = \sum a b_{\underline{1}} \varepsilon(b c_{\underline{2}}) = \sum a c_{\underline{1}} \varepsilon(b) \varepsilon(c_{\underline{2}}) = a \varepsilon(b) c = \underline{\varepsilon}(a \otimes b) c,$$

showing that $\underline{\varepsilon}$ is right B-linear and so $B \otimes_R B$ is a B-coring.

The other implications are shown similarly. □

Part of the symmetry is lost in the case of weak corings.

5.3. $B \otimes_R B$ **as weak coring.**

(1) The following are equivalent:

(a) (B, μ, Δ) *induces a weak coring structure on* $B \otimes_R B$*;*

(b) $(B, \mu^\tau, \Delta^\tau)$ *induces a weak coring structure on* $B \otimes_R B$*;*

(c) $(W.1)$ $\varepsilon(abc) = \sum \varepsilon(a b_{\underline{2}}) \varepsilon(b_{\underline{1}} c)$, *for* $a, b, c \in B$*;*
$(W.2)$ $(I \otimes \Delta) \circ \Delta(1) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1)$ $(= \sum 1_{\underline{1}} \otimes 1_{\underline{1}'} 1_{\underline{2}} \otimes 1_{\underline{2}'})$.

(2) The following are equivalent:

(a) (B, μ, Δ^τ) *induces a weak coring structure on* $B \otimes_R B$*;*

(b) (B, μ^τ, Δ) *induces a weak coring structure on* $B \otimes_R B$*;*

(c) $(W^\tau.1)$ $\quad \varepsilon(abc) = \sum \varepsilon(ab_{\underline{1}})\varepsilon(b_{\underline{2}}c)$, *for* $a, b, c \in B$*;*
$(W^\tau.2)$ $\quad (I{\otimes}\Delta) \circ \Delta(1) = (\Delta(1){\otimes}1)(1{\otimes}\Delta(1))$ $(= \sum 1_{\underline{1}}{\otimes}1_{\underline{2}}1_{\underline{1}'}{\otimes}1_{\underline{2}'})$.

Proof. (1) $(a) \Rightarrow (c)$ Assume $B \otimes_R B$ to be a weak B-coring. Then $\underline{\varepsilon}$ is right B-linear,

$$\begin{aligned} \underline{\varepsilon}((1{\otimes}a) \cdot b \cdot c) &= \underline{\varepsilon}((1{\otimes}a) \cdot b)c &&= \sum b_{\underline{1}}\varepsilon(ab_{\underline{2}})c \\ &= \underline{\varepsilon}((1{\otimes}a) \cdot (bc)) &&= \sum (bc)_{\underline{1}}\varepsilon(a(bc)_{\underline{2}}), \end{aligned}$$

and applying ε yields

$$\sum \varepsilon(ab_{\underline{2}})\varepsilon(b_{\underline{1}}c) = \sum \varepsilon((bc)_{\underline{1}})\varepsilon(a(bc)_{\underline{2}}) = \sum \varepsilon(a\varepsilon((bc)_{\underline{1}})(bc)_{\underline{2}})) = \varepsilon(abc).$$

$\underline{\varepsilon}$ being weakly counital implies

$$(1{\otimes}a) \cdot 1 = \sum \underline{\varepsilon}(1{\otimes}a_{\underline{1}}){\otimes}a_{\underline{2}} = \sum 1_{\underline{1}}\varepsilon(a_{\underline{1}}1_{\underline{2}}){\otimes}a_{\underline{2}},$$

and replacing a by $1_{\underline{1}'}$ or 1, respectively, we have

$$\begin{aligned} (1{\otimes}1_{\underline{1}'})\Delta(1) &= \sum 1_{\underline{1}}\varepsilon(1_{\underline{1}'\underline{1}}1_{\underline{2}}){\otimes}1_{\underline{1}'\underline{2}}, \qquad \text{and} \\ \Delta(1) &= \sum 1_{\underline{1}}\varepsilon(1_{\underline{1}'}1_{\underline{1}'\underline{2}}){\otimes}1_{\underline{2}'}. \end{aligned}$$

Applying $I{\otimes}\Delta$ to the second equality yields

$$\begin{aligned} (I{\otimes}\Delta) \circ \Delta(1) &= \sum 1_{\underline{1}}\varepsilon(1_{\underline{1}'}1_{\underline{2}}){\otimes}1_{\underline{2}'\underline{1}}{\otimes}1_{\underline{2}'\underline{2}} \\ &= \sum 1_{\underline{1}}\varepsilon(1_{\underline{1}'\underline{1}}1_{\underline{2}}){\otimes}1_{\underline{1}'\underline{2}}{\otimes}1_{\underline{2}'} \\ &= \sum 1_{\underline{1}}{\otimes}1_{\underline{1}'}1_{\underline{2}}{\otimes}1_{\underline{2}'}\,. \end{aligned}$$

$(c) \Rightarrow (a)$ Suppose $(W.1)$ and $(W.2)$ are satisfied.
$(W.1)$ implies that $\underline{\varepsilon}$ is right B-linear by the following computation, for $a, b \in B$,

$$\begin{aligned} \underline{\varepsilon}((1{\otimes}a) \cdot 1 \cdot b) &= \sum (I{\otimes}\varepsilon)(1_{\underline{1}}b_{\underline{1}}{\otimes}a1_{\underline{2}}b_{\underline{2}}) \\ &= \sum 1_{\underline{1}}b_{\underline{1}}\,\varepsilon(a1_{\underline{2}}b_{\underline{2}}) \\ \scriptstyle (W.1) \quad &= \sum 1_{\underline{1}}b_{\underline{1}}\,\varepsilon(a1_{\underline{2}\,\underline{2}})\,\varepsilon(1_{\underline{2}\,\underline{1}}b_{\underline{2}}) \\ \scriptstyle (W.1) \quad &= \sum 1_{\underline{1}}b_{\underline{1}}\,\varepsilon(a1_{\underline{222}})\,\varepsilon(1_{\underline{221}})\,\varepsilon(1_{\underline{2}\,\underline{1}}b_{\underline{2}}) \\ \scriptstyle coass. \quad &= \sum 1_{\underline{1}\,\underline{1}}b_{\underline{1}}\,\varepsilon(a1_{\underline{2}\,\underline{2}})\,\varepsilon(1_{\underline{2}\,\underline{1}})\,\varepsilon(1_{\underline{1}\,\underline{2}}b_{\underline{2}}) \\ &= \sum 1_{\underline{1}\,\underline{1}}b_{\underline{1}}\,\varepsilon(1_{\underline{1}\,\underline{2}}b_{\underline{2}})\,\varepsilon(a\,\varepsilon(1_{\underline{2}\,\underline{1}})1_{\underline{2}\,\underline{2}}) \\ &= \sum 1_{\underline{1}}b\,\varepsilon(a1_{\underline{2}}) \\ &= \underline{\varepsilon}(1{\otimes}a)b\,. \end{aligned}$$

By $(W.2)$ we have, for $a \in B$,

$$\begin{array}{rcl}
\sum \underline{\varepsilon}(1 \otimes a_{\underline{1}}) \otimes a_{\underline{2}} & = & \sum \underline{\varepsilon}(1 \otimes (a1)_{\underline{1}}) \otimes (a1)_{\underline{2}} \\
& = & \sum (I \otimes \varepsilon)(1_{\underline{1}} \otimes a_{\underline{1}} 1_{\underline{1}'} 1_{\underline{2}}) \otimes a_{\underline{2}} 1_{\underline{2}'} \\
{\scriptstyle (W.2)} & = & \sum (I \otimes \varepsilon)(1_{\underline{1}} \otimes a_{\underline{1}} 1_{\underline{2}\,\underline{1}}) \otimes a_{\underline{2}} 1_{\underline{2}\,\underline{2}} \\
& = & \sum 1_{\underline{1}} \varepsilon(a_{\underline{1}} 1_{\underline{2}\,\underline{1}}) \otimes a_{\underline{2}} 1_{\underline{2}\,\underline{2}} \\
& = & \sum 1_{\underline{1}} \otimes (\varepsilon \otimes I) \Delta(a 1_{\underline{2}}) \\
& = & \sum 1_{\underline{1}} \otimes a 1_{\underline{2}} = (1 \otimes a) \Delta(1) = (1 \otimes a) \cdot 1\,,
\end{array}$$

which shows that $\underline{\varepsilon}$ is weakly counital.

$(b) \Leftrightarrow (c)$ is shown with a similar computation.

(2) The proof is similar to the proof of (1). □

5.4. Group-like elements. *Assume that (B, μ, Δ) induces a weak coring structure on $B \otimes_R B$. Then $\Delta(1)$ and $\Delta^\tau(1)$ are group-like elements for $B \otimes_R B$ and $B \otimes_R^o B$, respectively.*

(1) B is a right $B \otimes_R B$-comodule and for any $M \in \tilde{\mathcal{M}}^{(B \otimes_R B)}$, the coinvariants are

$$\begin{array}{rcl}
M^{co(B \otimes_R B)} & = & \{m \in MB \mid \varrho_M(m) = \sum m 1_{\underline{1}} \otimes 1_{\underline{2}})\}, \quad \textit{and} \\
B^{co(B \otimes_R B)} & = & \{a \in B \mid \Delta(a) = \sum a 1_{\underline{1}} \otimes 1_{\underline{2}}\}.
\end{array}$$

(2) B is a right $B \otimes_R^o B$-comodule and for any $M \in \tilde{\mathcal{M}}^{(B \otimes_R^o B)}$, the coinvariants are

$$\begin{array}{rcl}
M^{co(B \otimes_R^o B)} & = & \{m \in MB \mid \varrho'_M(m) = \sum m 1_{\underline{2}} \otimes 1_{\underline{1}}\}, \quad \textit{and} \\
B^{co(B \otimes_R^o B)} & = & \{a \in B \mid \Delta(a) = \sum 1_{\underline{2}} a \otimes 1_{\underline{1}}\}.
\end{array}$$

Proof. $\Delta(1)$ is a group-like element for $B \otimes_R B$ since

$$\begin{array}{rcl}
\underline{\Delta}(\Delta(1)) & = & \sum (1_{\underline{1}} \otimes 1_{\underline{2}\,\underline{1}}) \otimes_B (1 \otimes 1_{\underline{2}\,\underline{2}}) = \sum (1_{\underline{1}\,\underline{1}} \otimes 1_{\underline{1}\,\underline{2}}) \otimes_B (1 \otimes 1_{\underline{2}}) \\
& = & \sum (\Delta(1)) \otimes_B (1_{\underline{1}} \otimes 1_{\underline{2}}) = \Delta(1) \otimes_B \Delta(1)\,, \quad \text{and} \\
\underline{\varepsilon}(\Delta(1)) & = & (I \otimes \varepsilon)(\Delta(1) \cdot 1) = \sum 1_{\underline{1}}\, \varepsilon(1_{\underline{2}}) = 1.
\end{array}$$

Similarly we get that $\Delta^\tau(1)$ is a group-like element for $B \otimes_R^o B$.

(1) By 2.1, B is a right $B \otimes_R B$-comodule and 2.2(1) yields the given characterization of the coinvariants.

(2) This follows with the same proof as (1). □

Following Böhm-Nill-Szlachányi [1, Definition 2.1], we call B a *weak R-bialgebra* provided (B, μ, Δ), $(B, \mu^\tau, \Delta^\tau)$, (B, μ^τ, Δ) and (B, μ, Δ^τ) all induce coring structures on $B \otimes_R B$. From 5.3 we immediately obtain:

5.5. Weak bialgebras. *The following are equivalent:*

(a) *B is a weak R-bialgebra;*

(b) *(B, μ, Δ) and (B, μ, Δ^τ) induce coring structures on $B \otimes_R B$;*

(c) *$(B, \mu^\tau, \Delta^\tau)$ and (B, μ^τ, Δ) induce coring structures on $B \otimes_R B$;*

(d) *the conditions $(W.1)$, $(W.2)$, $(W^\tau.1)$ and $(W^\tau.2)$ are satisfied (see 5.3).*

Notice that 5.5 corresponds to the characterization of weak bialgebras by properties of entwining structures in [6, Section 4.1].

In case $(B \otimes_R B, \underline{\Delta}, \underline{\varepsilon})$ is a B-coring the condition $b \otimes 1 = \Delta(b)$ implies $b = \varepsilon(b)1$, which means $B^{co(B\otimes_R B)} = R1_B$ and R is an R-direct summand in B. This is no longer true in the weak case but some results in this direction still hold.

5.6. Coinvariants in weak bialgebras. *Let B be a weak bialgebra.*

(1) *For $a \in B$ the following are equivalent:*

   (a) *$\Delta(a) = \sum a1_{\underline{1}} \otimes 1_{\underline{2}}$ (i.e., $a \in B^{co(B\otimes_R B)}$);*

   (b) *$\Delta(a) = \sum 1_{\underline{1}} a \otimes 1_{\underline{2}}$;*

   (c) *$a = \sum \varepsilon(a1_{\underline{1}})1_{\underline{2}}$;*

   (d) *$a = \sum \varepsilon(1_{\underline{1}}a)1_{\underline{2}}$.*

(2) *For $a \in B$ the following are equivalent:*

   (a) *$\Delta(a) = \sum 1_{\underline{1}} \otimes 1_{\underline{2}}a$ (i.e., $a \in B^{co(B\otimes_R^o B)}$);*

   (b) *$\Delta(a) = \sum 1_{\underline{1}} \otimes a1_{\underline{2}}$;*

   (c) *$a = \sum 1_{\underline{1}}\varepsilon(1_{\underline{2}}a)$;*

   (d) *$a = \sum 1_{\underline{1}}\varepsilon(a1_{\underline{2}})$.*

Proof. (1) $(a) \Rightarrow (c)$, $(b) \Rightarrow (d)$ Apply $\varepsilon \otimes I$ to the equality in (a) and (b), respectively.

$(c) \Rightarrow (a), (b)$ Assume $a = \sum \varepsilon(a1_{\underline{1}})1_{\underline{2}}$. Then

$$\Delta(a) = \sum \varepsilon(a1_{\underline{1}})1_{\underline{2}\,\underline{1}} \otimes 1_{\underline{2}\,\underline{2}} \overset{(W^\tau.2)}{=} \sum \varepsilon(a1_{\underline{1}})1_{\underline{2}}1_{\underline{1}'} \otimes 1_{\underline{2}'} = \sum a1_{\underline{1}} \otimes 1_{\underline{2}}\,,$$

and similarly

$$\Delta(a) = \sum \varepsilon(a1_{\underline{1}})1_{\underline{2}\,\underline{1}} \otimes 1_{\underline{2}\,\underline{2}} \overset{(W.2)}{=} \sum \varepsilon(a1_{\underline{1}})1_{\underline{1}'}1_{\underline{2}} \otimes 1_{\underline{2}'} = \sum 1_{\underline{1}}a \otimes 1_{\underline{2}}\,.$$

$(d) \Rightarrow (a)$ is shown similarly.

(2) The proof goes along the lines of the proof of (1). □

5.7. The ring $(\mathrm{End}_R(B), *)$. Given (B, μ, Δ) the (usual) convolution product is defined on $\mathrm{End}_R(B)$ by

$$f * g = \mu \circ (f \otimes g) \circ \Delta, \text{ for } f, g \in \mathrm{End}_R(B),$$

and $(\mathrm{End}_R(B), *)$ is an associative R-algebra with unit $\varepsilon_B := \iota \circ \varepsilon$, i.e., $\varepsilon_B(b) = \varepsilon(b)1_B$, for any $b \in B$.

Besides ε_B there are other maps which are of particular interest for weak bialgebras and which coincide with ε_B for bialgebras.

5.8. The maps π^L **and** π^R. *Assume that* (B, μ, Δ) *induces a weak coring structure on* $B \otimes_R B$. *Define the maps*

$$\pi^R : B \xrightarrow{1\otimes -} B \otimes_R B \xrightarrow{\varepsilon} B, \quad b \mapsto \sum 1_{\underline{1}}\varepsilon(b1_{\underline{2}}),$$
$$\pi^L : B \xrightarrow{1\otimes -} B \otimes_R B \xrightarrow{\varepsilon^o} B, \quad b \mapsto \sum \varepsilon(1_{\underline{1}}b)1_{\underline{2}},$$

which obviously satisfy $\quad \pi^L * I = I = I * \pi^R$.

(1) *For* π^L *we have (where* $a, b \in B$*):*
   (i) $\sum b_{\underline{1}} \otimes \pi^L(b_{\underline{2}}) = \sum 1_{\underline{1}} b \otimes 1_{\underline{2}}$;
   (ii) $a\pi^L(b) = \sum \pi^L(a_{\underline{1}}b)a_{\underline{2}}$ $(= \sum \varepsilon(a_{\underline{1}}b)a_{\underline{2}})$;
   (iii) $f * \pi^L(b) = \sum f(1_{\underline{1}}b)1_{\underline{2}}$, *for any* $f \in \mathrm{End}_R(B)$;
   (iv) $\pi^L \circ \pi^L = \pi^L$;
   (v) $\varepsilon(ab) = \varepsilon(a\pi^L(b))$ *and* $\pi^L(ab) = \pi^L(a\pi^L(b))$;
   (vi) $\pi^L(a)\pi^L(b) = \pi^L(\pi^L(a)b)$.

   So $B^L := \pi^L(B)$ *is a subring of* B *and* π^L *is a left* B^L*-module morphism.*

(2) *For* π^R *we have (where* $a, b \in B$*):*
   (i) $\sum \pi^R(b_{\underline{1}}) \otimes b_{\underline{2}} = \sum 1_{\underline{1}} \otimes b1_{\underline{2}}$;
   (ii) $\pi^R(b)a = \sum a_{\underline{1}}\pi^R(ba_{\underline{2}})$ $(= \sum a_{\underline{1}}\varepsilon(ba_{\underline{2}}))$;
   (iii) $\pi^R * g(b) = \sum 1_{\underline{1}}g(b1_{\underline{2}})$, *for any* $g \in \mathrm{End}_R(B)$;
   (iv) $\pi^R \circ \pi^R = \pi^R$;
   (v) $\varepsilon(ab) = \varepsilon(\pi^R(a)b)$ *and* $\pi^R(ab) = \pi^R(\pi^R(a)b)$;
   (vi) $\pi^R(a)\pi^R(b) = \pi^R(a\pi^R(b))$.

   So $B^R := \pi^R(B)$ *is a subring of* B *and* π^R *is a right* B^R*-module morphism.*

(3) *Assume that* B *is a weak bialgebra. Then*

(i) $B^{co(B\otimes_R B)} = B^L$ *and* B^L *is a direct summand of* B *as left* B^L*-module.*

(ii) $B^{co(B\otimes^o_R B)} = B^R$ *and* B^R *is a direct summand of* B *as right* B^R*-module.*

Proof. (1) (i), (ii) follow directly from $(W.1)$ and $(W.2)$; (iii) is a consequence of (i).

(iv) and (v) follow from the equalities

$$\begin{array}{rcl} \pi^L(\pi^L(a)) & = & \sum \varepsilon(\varepsilon(1_{\underline{1}}a)1_{\underline{1}'}1_{\underline{2}})1_{\underline{2}'} \quad = \sum \varepsilon(1_{\underline{1}}a)\,\varepsilon(1_{\underline{1}'}1_{\underline{2}})1_{\underline{2}'} \\ {\scriptstyle (W.1)} & = & \sum \varepsilon(1_{\underline{1}'}a)1_{\underline{2}'} = \pi^L(a)\,, \quad \text{and} \\ \varepsilon(a\pi^L(b)) & = & \sum \varepsilon(a\varepsilon(1_{\underline{1}}b)1_{\underline{2}}) \\ {\scriptstyle (W.1)} & = & \sum \varepsilon(a1_{\underline{2}})\varepsilon(1_{\underline{1}}b) = \varepsilon(ab). \end{array}$$

(vi) We have $\Delta(\pi^L(a)) = \sum 1_{\underline{1}}\,\pi^L(a)\otimes 1_{\underline{2}}$, and hence by (ii),

$$\pi^L(\pi^L(a)\pi^L(b)) = \sum \varepsilon(1_{\underline{1}}\pi^L(a)b)1_{\underline{2}} = \pi^L(\pi^L(a)b).$$

(2) If (B, μ, Δ) induces a weak coring structure on $B \otimes_R B$ then this is also true for $(B, \mu^\tau, \Delta^\tau)$ (see 5.3) and the proof is similar to the proof of (1).

(3) This follows by 5.4, 5.6 and (1), resp. (2). $\square$

Notice that most of the identities considered in 5.8 and later on are already familiar from [10] and [1, Section 2.2]. Since we do not consider (finite dimensional) algebras over fields the (duality) arguments used there are not always available here and hence we prefer to indicate proofs if appropriate.

5.9. Antipodes. An element $S \in \mathrm{End}_R(B)$ is called a *left antipode* if $S * I = \pi^R$ and $S * \pi^L = S$, i.e., for $b \in B$,

$$\sum (Sb_{\underline{1}})\, b_{\underline{2}} = \sum 1_{\underline{1}}\varepsilon(b1_{\underline{2}}) \quad \text{and} \quad \sum S(1_{\underline{1}}b)1_{\underline{2}} = S(b),$$

a *right antipode* provided $I * S = \pi^L$ and $\pi^R * S = S$, i.e.,

$$\sum b_{\underline{1}}(Sb_{\underline{2}}) = \sum \varepsilon(1_{\underline{1}}b)1_{\underline{2}} \quad \text{and} \quad \sum 1_{\underline{1}}S(b1_{\underline{2}}) = S(b),$$

an *antipode* if S is both a left and a right antipode.

In view of the properties of π^L and π^R we have the following result which shows that our notion of an antipode coincides with the antipodes in [1, 2.1].

The following are equivalent for $S \in \mathrm{End}_R(B)$*:*

(a) S *is an antipode;*

(b) S *satifies* $S * I = \pi^R$, $I * S = \pi^L$ *and* $S * I * S = S$.

A weak bialgebra B with an antipode is called *weak Hopf algebra* (see [1]).

It is straightforward to see that the antipode of a weak bialgebra has the usual properties of the antipode in case B is a bialgebra (then π^L and π^R coincide with ε_B).

Notice that our antipodes satify $S * I * S = S$ and $I * S * I = I$, the conditions used in Fang Li [7] to define his "weak Hopf algebras".

5.10. Galois corings. Let B be a weak bialgebra. Then the B-coring $B \otimes_R B$ is *right Galois* (Definition 2.4) if the canonical map

$$\gamma_B : B \otimes_{B^L} B \to (B \otimes_R B) \cdot 1, \quad a{\otimes}b \mapsto (a{\otimes}1)\Delta(b),$$

is an isomorphism. Obviously γ_B is a left B-module morphism.

The following observation generalizes [9, Theorem 1.1].

5.11. Existence of antipodes. *Let B be a weak bialgebra. Then:*

*(1) B has a right antipode if and only if γ_B has a left inverse in B-*Mod.

(2) γ_B is an isomorphism if and only if B has an antipode.

Proof. (1) ($\Leftarrow$) Let β be a left inverse of γ_B. Then $1 \otimes_{B^L} b = \beta \circ \gamma(1{\otimes}_{B^L}b) = \beta(\Delta b)$, and applying $I{\otimes}\pi^L$ we get

$$\pi^L(b) = (I{\otimes}\pi^L) \circ \beta(\Delta b).$$

Then the composition

$$S : B \xrightarrow{1\otimes -} B \otimes_R B \xrightarrow{-\cdot 1} (B \otimes_R B) \cdot 1 \xrightarrow{\beta} B \otimes_{B^L} B \xrightarrow{I\otimes\pi^L} B,$$

is a right antipode since

$$\mu \circ (id{\otimes}S) \circ \Delta(b) = \sum b_{\underline{1}}((I{\otimes}\pi^L)\beta(1_{\underline{1}} \otimes b_{\underline{2}}1_{\underline{2}})) = (I{\otimes}\pi^L) \circ \beta(\Delta b) = \pi^L(b), \text{ and}$$

$$\pi^R * S(b) = \sum 1_{\underline{1}}S(b1_{\underline{2}}) = \sum (I{\otimes}\pi^L) \circ \beta(1_{\underline{1}}{\otimes}b1_{\underline{2}}) = S(b).$$

($\Rightarrow$) Now assume $S : B \to B$ to be a right antipode and consider the map

$$\beta : B \otimes_R B \to B \otimes_{B^L} B, \quad a \otimes b \mapsto \sum aS(b_{\underline{1}}){\otimes}_{B^L}b_{\underline{2}}.$$

By the property

$$\begin{aligned} \beta((a{\otimes}b)\Delta(1)) &= \sum a1_{\underline{1}}S(b_{\underline{1}}1_{\underline{2}\,\underline{1}}){\otimes}_{B^L}b_{\underline{2}}1_{\underline{2}\,\underline{2}} \\ \overset{(W.2)}{=}& \sum a1_{\underline{1}}S(b_{\underline{1}}1_{\underline{1}'}1_{\underline{2}}){\otimes}_{B^L}b_{\underline{2}}1_{\underline{2}'} \\ &= \sum aS(b_{\underline{1}}1_{\underline{1}'}){\otimes}_{B^L}b_{\underline{2}}1_{\underline{2}'} = \beta(a{\otimes}b), \end{aligned}$$

it induces a map $\beta : (B \otimes_R B) \cdot 1 \to B \otimes_{B^L} B$, which is a left inverse of γ_B since, for any $b \in B$,

$$\begin{array}{rclcl} \beta \circ \gamma(1 \otimes_{B^L} b) = \beta(\Delta b) & = & \sum b_{\underline{1}} S(b_{\underline{2}\underline{1}}) \otimes_{B^L} b_{\underline{2}\underline{2}} & = & \sum b_{\underline{1}\underline{1}} S(b_{\underline{1}\underline{2}}) \otimes_{B^L} b_{\underline{2}} \\ & = & \sum \pi^L(b_{\underline{1}}) \otimes_{B^L} b_{\underline{2}} & = & 1 \otimes_{B^L} b \,. \end{array}$$

(2) ($\Rightarrow$) Assume γ_B to be bijective. By (1), there exists a right antipode S and so we have $I * S * I = \pi^L * I = I$.

Any element in $(B \otimes B) \cdot 1$ can be written as $\sum_i a_i \Delta c_i$, for some $a_i, c_i \in B$, and

$$\sum_i \mu \circ (id \otimes (S * I - \varepsilon_B))(a_i \Delta c_i) = \sum_i a_i (I * S * I - I * \varepsilon_B)(c_i) = 0.$$

This implies for $(1 \otimes b)\Delta(1) \in (B \otimes B) \cdot 1$, where $b \in B$,

$$\begin{array}{rclcl} \pi^R(b) = \sum 1_{\underline{1}} \varepsilon(b 1_{\underline{2}}) & = & \sum 1_{\underline{1}} \, S * I(b 1_{\underline{2}}) & = & \sum 1_{\underline{1}} \, S(b_{\underline{1}} 1_{\underline{2}\underline{1}}) \, b_{\underline{2}} 1_{\underline{2}\underline{2}} \\ {\scriptstyle (W.2)} & = & \sum 1_{\underline{1}} \, S(b_{\underline{1}} 1_{\underline{1}'} 1_{\underline{2}}) \, b_{\underline{2}} 1_{\underline{2}'} & = & \sum S(b_{\underline{1}} 1_{\underline{1}'}) \, b_{\underline{2}} 1_{\underline{2}'} = S * I(b). \end{array}$$

Moreover, $\pi^R * S = S * I * S = S * \pi^L = S$ showing that S is a right antipode.

($\Leftarrow$) For the β defined in (1) we already know that $\beta \circ \gamma_B = I$.

For any $a, b \in B$ we have

$$\begin{array}{rclcl} \gamma_B \circ \beta((a \otimes b) \cdot 1) & = & \sum (a S(b_{\underline{1}}) \otimes 1) \Delta(b_{\underline{2}}) & = & \sum a \, S(b_{\underline{1}}) b_{\underline{2}\underline{1}} \otimes b_{\underline{2}\underline{2}} \\ & = & \sum a \, S(b_{\underline{1}\underline{1}}) b_{\underline{1}\underline{2}} \otimes b_{\underline{2}} & = & a \sum \pi^R(b_{\underline{1}}) \otimes b_{\underline{2}} \\ {\scriptstyle 5.8(1)(i)} & = & a(1 \otimes b) \cdot 1 & = & (a \otimes b) \cdot 1 \,, \end{array}$$

which shows $\gamma_B \circ \beta = I$ and hence γ is an isomorphism. □

Recall that the category of comodules over a coring $B \otimes_R B$ is Grothendieck provided $B \otimes_R B$ is flat as left B-module (see 1.10).

It follows from 5.8(3) that any weak bialgebra B has B^L as a direct summand which means that B is flat as a left B^L-module if and only if it is faithfully flat. Hence the characterization of a ring as a generator for related comodules in 2.5 immediately implies:

5.12. Fundamental theorem for weak Hopf algebras. *For a weak R-bialgebra B the following are equivalent:*

(a) B is a weak Hopf algebra, and B is flat as left B^L-module;

(b) $B \otimes_R B$ is flat as left B-module, and B is a (projective) generator in $\mathcal{M}^{(B \otimes_R B) \cdot 1}$;

(c) $\mathcal{M}^{(B\otimes_R B)\cdot 1}$ *is a Grothendieck category and*

$$\mathrm{Hom}^{B\otimes_R B}(B,-) : \mathcal{M}^{(B\otimes_R B)\cdot 1} \to \text{Mod-}B^L$$

is an equivalence (with inverse $-\otimes_{B^L} B$*);*

(d) $B\otimes_R B$ *is flat as left* B*-module, and for every* $M \in \mathcal{M}^{(B\otimes_R B)\cdot 1}$*,*

$$M^{coB} \otimes_{B^L} B \to M,\ m\otimes b \mapsto mb,$$

is an isomorphism.

Notice that $B\otimes_R B$ is flat (projective) as left B-module provided B is flat (projective) as R-module. Of course this is always the case if R is a field. For this situation a direct proof of the implication $(a) \Rightarrow (d)$ is given in [1, Theorem 3.9].

5.13. Remark. We can follow the proof of [1, Lemma 3.7] to show: *If* B *is a weak Hopf algebra with antipode* S*, then for any right* $B\otimes_R B$*-comodule* M*, the map*

$$(I{\otimes}S)\circ \varrho_M : M \to M^{co(B\otimes B)}$$

is a splitting B^L*-morphism.*

This entails that the first part of the proof of [5, Theorem 5.6] can be applied here without the initial condition that B is flat as left B^L-module. Therefore we can add as additional equivalent conditon in 5.12:

(e) B *is a weak Hopf algebra, and* $B\otimes_R B$ *is flat as left* B*-module.*

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