Comparing Grothendieck-Witt Groups of a Complex Variety to its Real Topological K-Groups

Smith-Knight/Rayleigh-Knight Essay[1]

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[1] The essay reflects my current research. Section 1 contains a substantial amount of review material, with my own work concentrated in 1.3, 1.4 and 1.7. Section 2 is original except where stated otherwise. The main new result is given in Section 3. In the example that follows I compare already existing calculations.

Abstract

On a complex variety X, two different approaches to K-theory are available: the algebraic K-theory of the variety, and the topological K-theory of the underlying topological space $X_{\mathbb{C}}$. In this context, the algebraic variant known as Hermitian K-theory corresponds to topological KO-theory. Our aim is to compare the two approaches.

We start by constructing a comparison map from certain Hermitian K-groups of X to the KO-groups of $X_{\mathbb{C}}$. It is clear what this map must be on groups in degree zero, but the definitions of relative and higher groups differ widely in the algebraic and the topological setting. This difficulty can be overcome by viewing relative and higher groups as subgroups of degree zero groups of certain auxiliary spaces.

Once the definition of our comparison map is in place, we prove a number of fundamental properties, in particular compatibility with pushforwards along closed embeddings. We also show how we can use it to compare an exact sequence relating usual algebraic K-theory to Hermitian K-theory with a portion of the Bott sequence in topology. This finally allows us to deduce that the map is an isomorphism on smooth cellular varieties. We conclude with some details concerning projective spaces, for which independent computations of the algebraic and the topological groups exist.

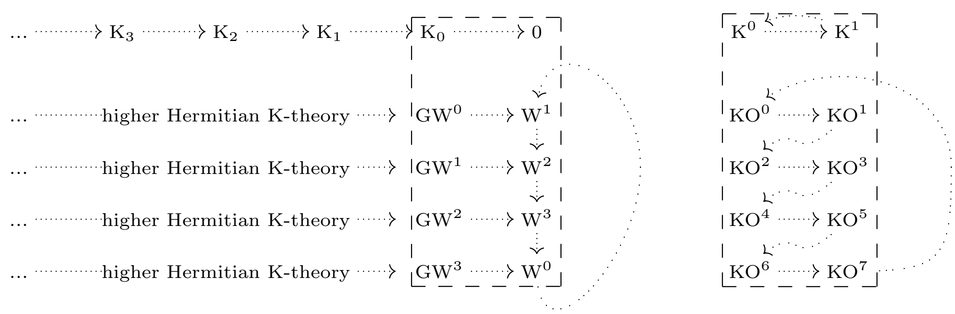
Contents

0 Introduction

K-theory is an approach to the classification problem of vector bundles on a given space that has been studied intensely in both algebraic geometry and topology. In the latter context, the K-groups of complex and real vector bundles give rise to generalised cohomology theories, known as topological K- and KO-theory, respectively. These theories are well-understood. For example, the K-groups of a topological space are 2-periodic, and both groups are known in many examples. In algebraic geometry the picture is more complicated. In particular, the cohomology theory associated to the K-theory of algebraic vector bundles on a variety, usually referred to as "higher algebraic K-theory", is not periodic, and hardly any higher groups K_i are known.

Closely related to K-theory are the Grothendieck-Witt and Witt groups of a space, which, rather than classifying plain vector bundles, classify vector bundles equipped with non-degenerate symmetric bilinear forms. As we will see, the Grothendieck-Witt group of complex vector bundles on a topological space is nothing but its KO-group. In algebraic geometry, Grothendieck-Witt groups lead to "Hermitian K-theory". The Witt group of a space can be defined as a quotient of its Grothendieck-Witt group by its K-group. Witt groups appear as certain Hermitian K-groups in negative degrees.

Recently, Paul Balmer introduced "shifted" versions of Witt groups, W^i, for $i \in \mathbb{Z}_4$ [4, 5]. Interestingly, not only does each of these give rise to a shifted Hermitian K-theory, but also the shifted Witt groups form a 4-periodic cohomology theory on smooth varieties in themselves. Denoting the corresponding Grothendieck-Witt groups by GW^i, the picture could be tentatively summarised by the left half of the following diagram, in which the dotted arrows indicate the direction of the boundary maps in the corresponding long exact cohomological sequences. More precise information can be found in [16], [17] and [15], for example.



For a complex variety X we can simultaneously consider algebraic K-theory and the topological K-theory of its underlying topological space $X_{\mathbb{C}}$, displayed in the right half of the diagram. In this essay, we construct a comparison map w from the algebraic groups inside the dotted box on the left to the groups on the right. Our main result, Theorem 3.1, says that for smooth cellular varieties this map is an isomorphism. Though this was already known for the plain K-groups K_0 and K^0, the result seems to be new for Grothendieck-Witt groups and KO-theory.

1 Construction of a comparison map

1.1 Basic definitions

K-groups. We start by recalling some basic definitions. The isomorphism classes of vector bundles on a fixed space form an abelian monoid under the operation of direct sum. The standard way of turning such a monoid into a group is the following construction: the Grothendieck group of an abelian monoid $(M, \oplus)$ is the free abelian group on the elements of M modulo the relation $[x] + [y] = [x \oplus y]$. So the obvious candidate for the K-group of a space is the Grothendieck group of the abelian monoid of vector bundles over it. Indeed, in the case of a paracompact Hausdorff space X, for example, this is exactly what K^0X is. In general, the K-group is only a quotient of this Grothendieck group:

Definition 1.1 ([1, § 2])**.** Let $\mathcal{A}$ be an exact category. Its K-group $\mathrm{K}\mathcal{A}$ is the Grothendieck group of isomorphism classes of objects of $\mathcal{A}$ modulo the equivalence relation generated by the following identification: for any exact sequence $E \rightarrowtail F \twoheadrightarrow G$ in $\mathcal{A}$, we have $[F] = [E] + [G]$ in $\mathrm{K}\mathcal{A}$.

For a general topological space X, we obtain K^0X by taking $\mathcal{A}$ to be the category of continuous complex vector bundles $\mathrm{Vect}_{\mathbb{C}}X$ over X. Note that when X is paracompact and Hausdorff, $\mathrm{Vect}_{\mathbb{C}}X$ is split exact, so the relation in the definition becomes vacuous. Thus, two vector bundles E and F on X define the same class in K^0X if and only if they are stably isomorphic, i.e. if $E \oplus \mathbb{C}^n \cong F \oplus \mathbb{C}^n$ for some trivial bundle $\mathbb{C}^n$.

For an algebraic variety, we take $\mathcal{A}$ to be the category of algebraic vector bundles (or equivalently of locally free sheaves of finite rank) $\mathrm{Vect}X$ on X. The corresponding K-groups will be denoted by K_0X. In general, $\mathrm{Vect}X$ is not split exact, so much more information is lost by passing to K_0X.

Grothendieck-Witt groups. The definition of Grothendieck-Witt groups requires some more preliminaries. In general, they can be defined for any exact category with duality, i.e. any exact category $\mathcal{A}$ equipped with an exact functor ${}^\vee\colon \mathcal{A} \to \mathcal{A}^{\mathrm{op}}$ (mapping objects and morphisms to their "duals") and a natural isomorphism $\eta\colon \mathrm{id} \to {}^{\vee\vee}$ satisfying $(\eta_E)^\vee \circ \eta_{E^\vee} = \mathrm{id}_{E^\vee}$ (the "double-dual identification"). A symmetric space over such a category is an object E of $\mathcal{A}$ equipped with a symmetric isomorphism $\epsilon\colon E \to E^\vee$, where symmetric means that $\epsilon^\vee \circ \eta_E = \epsilon$. Two such spaces (E, ϵ) and (F, ϕ) are considered isometric if there exists an isomorphism i from E to F compatible with the symmetric forms in the sense that $i^\vee \circ \phi \circ i = \epsilon$. The isometry classes form an abelian monoid under the operation of orthogonal sum: $(E, \epsilon) \perp (F, \phi) := (E \oplus F, \epsilon \oplus \phi)$.

For example, the usual duality on vector bundles makes both of the categories above into exact categories with duality. Here, a symmetric space corresponds to a vector bundle E with a non-degenerate symmetric bilinear form $E \otimes E \to \mathbb{C}$, and the notions of isometry and orthogonal sum are the usual ones.

Returning to the general context, from any object E of $\mathcal{A}$ we can construct a symmetric space $H(E) := (E \oplus E^\vee, \left(\begin{smallmatrix} 0 & \mathrm{id} \\ \eta & 0 \end{smallmatrix}\right))$, the hyperbolic space over E. These spaces are the simplest examples of the more general class of metabolic spaces:

a symmetric space (M, μ) is metabolic if it has a subobject $j\colon L \rightarrowtail M$, called Lagrangian of M, such that

$$L \overset{j}{\rightarrowtail} M \overset{j^\vee \circ \mu}{\twoheadrightarrow} L^\vee$$

is exact.

Definition 1.2 ([12, 2.9])**.** For an exact category $\mathcal{A}$ with duality, the Grothendieck-Witt group $\mathrm{GW}\mathcal{A}$ is the Grothendieck group of isometry classes of symmetric spaces modulo the equivalence relation generated by the following identification: for any metabolic space (M, μ) with Lagrangian L, we have $[M, \mu] = [H(L)]$ in $\mathrm{GW}\mathcal{A}$.

If the category $\mathcal{A}$ is split exact, then for any metabolic space (M, μ) as above we can find another metabolic space (M', μ') such that the orthogonal sum $(M, \mu) \perp (M', \mu')$ is isometric to $(H(L), \mu) \perp (M', \mu')$, so the relation on the Grothendieck group is trivial [12, Corollary 2.12]. In fact, if $\mathcal{A}$ moreover contains $\frac{1}{2}$, then (M, μ) is itself isometric to $H(L)$.

Of course, if X is an algebraic variety, $\mathrm{GW}^0 X$ is the Grothendieck-Witt group of $\mathrm{Vect}X$ equipped with the usual duality and double-dual identification, and if X is a topological space, we consider $\mathrm{GW}(\mathrm{Vect}_{\mathbb{C}} X)$. If X is paracompact and Hausdorff, this is in fact equal to the K-group of real vector bundles on X, $\mathrm{KO}^0 X$. As both $\mathrm{Vect}_{\mathbb{C}} X$ and $\mathrm{Vect}_{\mathbb{R}} X$ are split exact in this case, this is saying that the Grothendieck group of isometry classes of complex vector bundles with a non-degenerate symmetric bilinear form is isomorphic to the Grothendieck group of real vector bundles. In fact, this is already true on the level of monoids:

Lemma 1.3. *For a paracompact Hausdorff space X, the abelian monoid of isomorphism classes of continuous real vector bundles is isomorphic to the monoid of isometry classes of continuous complex vector bundles equipped with non-degenerate symmetric bilinear forms.*

Proof. On any real vector bundle E over a paracompact Hausdorff space, we may choose some inner product σ. Then the $\mathbb{C}$-bilinear extension $\sigma_{\mathbb{C}}$ determines a non-degenerate symmetric bilinear form on the complex vector bundle $E \oplus iE$. The inner product σ on E is unique up to isometry: Given any inner product δ on $E^\vee$, there exists a unique positive definite symmetric $g\colon E \to E^\vee$ defining an isometry between (E, σ) and $(E^\vee, \delta)$. This can be checked locally. So if σ' is another inner product on E, the symmetric spaces (E, σ) and (E, σ') are isometric because they are both isometric to $(E^\vee, \delta)$. An isometry between (E, σ) and (E, σ') extends $\mathbb{C}$-linearly to an isometry between $(E \oplus iE, \sigma_{\mathbb{C}})$ and $(E \oplus iE, \sigma'_{\mathbb{C}})$. So we obtain a well-defined map by sending E to $(E \oplus iE, \sigma_{\mathbb{C}})$.

Conversely, any complex vector space V with a non-degenerate symmetric bilinear form ϕ can be decomposed as $V = W \oplus iW$ such that the real part of ϕ, $\Re(\phi)\colon W \otimes W \to \mathbb{R}$, is positive definite on W and negative definite on iW. Given a complex vector bundle F with such a form, it is in fact possible to choose fibre-wise decompositions in such a way that the positive definite subspaces form a real subbundle $E \subset F$, see for example [2, Chapter V, § 2]. As is also shown there, this subbundle is uniquely determined up to isomorphism: if $F = E' \oplus iE'$ is another such decomposition, then E cannot intersect iE', whence the composition $E \to F \to \frac{F}{iE'} \cong E'$ is an isomorphism. So we obtain another well-defined map by sending (F, ϕ) to E.

The two mappings described are clearly morphisms of monoids, and they are mutually inverse. □

We will use the identification of $\mathrm{KO}^0 X$ with $\mathrm{GW}(\mathrm{Vect}_{\mathbb{C}} X)$ implicitly in all that follows.

Suppose now that X is an algebraic variety over $\mathbb{C}$. Its underlying topological space $X_{\mathbb{C}}$ (i.e. its set of closed points equipped with the complex topology) is a paracompact Hausdorff space. Moreover, the forgetful functor $\mathrm{Vect}X \to \mathrm{Vect}_{\mathbb{C}}(X_{\mathbb{C}})$ sending a vector bundle over X to the underlying continuous complex vector bundle on $X_{\mathbb{C}}$ is exact and preserves duality, so it induces comparison maps

$$w\colon\ \mathrm{K}_0 X \to \mathrm{K}^0(X_{\mathbb{C}})$$
$$w\colon\ \mathrm{GW}^0 X \to \mathrm{KO}^0(X_{\mathbb{C}})$$

Witt-groups. For any exact category $\mathcal{A}$ with duality, there is a forgetful map

$$F\colon \mathrm{GW}\mathcal{A} \to \mathrm{K}\mathcal{A},$$

sending $[E,\epsilon]$ to $[E]$. In the other direction, we have already seen how to associate a hyperbolic space $H(E)$ to any object E of $\mathcal{A}$. This induces a map

$$H\colon \mathrm{K}\mathcal{A} \to \mathrm{GW}\mathcal{A}.$$

Definition 1.4 ([12, 2.11])**.** The Witt group $\mathrm{W}\mathcal{A}$ of an exact category $\mathcal{A}$ is the quotient of $\mathrm{GW}\mathcal{A}$ by the image of H.

Under the identification of $\mathrm{KO}^0 X$ with $\mathrm{GW}(\mathrm{Vect}_{\mathbb{C}} X)$, F corresponds to the complexification map

$$c\colon \mathrm{KO}^0 X \to \mathrm{K}^0 X,$$

sending $[E]$ to $[E \otimes \mathbb{C}]$, and H corresponds to the realification map

$$\rho\colon \mathrm{K}^0 X \to \mathrm{KO}^0 X,$$

sending the class of a complex vector bundle to the class of its underlying real vector bundle. Thus, the two versions of w above induce a third version,

$$w\colon \mathrm{W}^0 X \to \tfrac{\mathrm{KO}}{\mathrm{K}}^0(X_{\mathbb{C}}),$$

where $\frac{\mathrm{KO}}{\mathrm{K}}^0(X_{\mathbb{C}})$ denotes the quotient of $\mathrm{KO}^0(X_{\mathbb{C}})$ by the image of $\mathrm{K}^0(X_{\mathbb{C}})$ under realification.

Twisted groups. The usual duality on $\mathrm{Vect}X$ is not the only possible one. For example, if $\mathcal{L}$ is a line bundle on X, then $E \mapsto E^\vee \otimes \mathcal{L}$ defines an alternative duality. Note that $(E^\vee \otimes \mathcal{L})^\vee \otimes \mathcal{L}$ is canonically isomorphic to E, so we have a natural double-dual identification $\eta_{\mathcal{L}}$.

Definition 1.5. For an algebraic variety X with a line bundle $\mathcal{L}$, the Grothendieck-Witt group of X with coefficients in $\mathcal{L}$ is

$$\mathrm{GW}^0(X;\mathcal{L}) := \mathrm{GW}(\mathrm{Vect}X, {}^\vee \otimes \mathcal{L}, \eta_{\mathcal{L}}).$$

Likewise, for a topological space X with a complex line bundle $\mathcal{L}$,

$$\mathrm{KO}^0(X;\mathcal{L}) := \mathrm{GW}(\mathrm{Vect}_{\mathbb{C}}X, {}^{\vee}\otimes\mathcal{L}, \eta_{\mathcal{L}}).$$

All maps discussed above extend to this twisted context.

1.2 Algebraic K-groups with support

Given an algebraic variety X with a closed subvariety Z, the K-groups of X and of the complement of Z fit into an exact sequence

$$\mathrm{K}_0^Z X \to \mathrm{K}_0 X \to \mathrm{K}_0(X - Z).$$

The group on the left is the "K-group of X with support on Z". Similarly, we have notions of (Grothendieck-)Witt groups with support, $\mathrm{GW}^0_Z X$ and $\mathrm{W}^0_Z X$. In order to define these, one usually passes from a category of vector bundles to some category of complexes of these: Given any exact category $\mathcal{A}$, the bounded chain complexes over $\mathcal{A}$ form an exact category with weak equivalences $(\mathrm{Ch}^b\mathcal{A}, \mathrm{quis})$, where the exact sequences are those which are degreewise exact in $\mathcal{A}$, and the class of weak equivalences is the class quis of quasi-isomorphisms [14, Definition 2.8 and Example 2.9]. Moreover, a duality on $\mathcal{A}$ induces a duality on $\mathrm{Ch}^b\mathcal{A}$. The definitions of K and GW can be extended to this context such that the inclusion $\mathcal{A} \hookrightarrow \mathrm{Ch}^b\mathcal{A}$ in degree zero induces isomorphisms

$$\begin{aligned} i_0\colon\ \mathrm{K}\mathcal{A} &\xrightarrow{\cong} \mathrm{K}(\mathrm{Ch}^b\mathcal{A}, \mathrm{quis}) \\ i_0\colon\ \mathrm{GW}(\mathcal{A}, {}^{\vee}, \eta) &\xrightarrow{\cong} \mathrm{GW}(\mathrm{Ch}^b\mathcal{A}, {}^{\vee}, \eta, \mathrm{quis}) \end{aligned} \tag{1}$$

[14, Theorem 2.16; 13, Proposition 6.4]. Thus, for a variety X, the groups $\mathrm{K}_0 X$ and $\mathrm{GW}(X;\mathcal{L})$ can be expressed in terms of the category $\mathrm{Ch}^b(\mathrm{Vect}X)$, and groups with support can be defined by using the full subcategory $\mathrm{Ch}^b_Z(\mathrm{Vect}X)$ of bounded chain complexes with homology supported on the closed subvariety Z.

We include here the complete definitions of the K- and Grothendieck-Witt groups of an exact category with weak equivalences as a convenient reference for the following section, cf. [13, 2.4].

Definition 1.6. Let $(\mathcal{C}, \omega)$ be an exact category with weak equivalences. $\mathrm{K}(\mathcal{C}, \omega)$ is the Grothendieck group of isomorphism classes of $\mathcal{C}$ modulo the equivalence relation generated by the following identifications:

- For any exact sequence $E \rightarrowtail F \twoheadrightarrow G$ in $\mathcal{C}$, we have $[F] = [E] + [G]$.
- For any weak equivalence $E \xrightarrow{\simeq} F$ in ω, we have $[E] = [F]$.

For an exact category with weak equivalences and a duality $(\mathcal{C}, {}^{\vee}, \eta, \omega)$, the Grothendieck-Witt group $\mathrm{GW}(\mathcal{C}, {}^{\vee}, \eta, \omega)$ is the Grothendieck group of isometry classes of symmetric spaces over $(\mathcal{C}, {}^{\vee}, \eta, \omega)$ modulo the following identifications:

- For any symmetric space (F, ϕ) over $\mathcal{C}$ and any weak equivalence $g\colon E \xrightarrow{\simeq} F$ in ω, we have $[F, \phi] = [E, g^{\vee} \circ \phi \circ g]$.

- Suppose a symmetric space (F, ϕ) fits into a commutative diagram

$$\begin{array}{ccccc} E & \rightarrowtail & F & \twoheadrightarrow & G \\ {\scriptstyle \gamma^\vee \circ \eta}\big\downarrow \simeq & & {\scriptstyle \phi}\big\downarrow \simeq & & {\scriptstyle \gamma}\big\downarrow \simeq \\ G^\vee & \rightarrowtail & F^\vee & \twoheadrightarrow & E^\vee \end{array}$$

  such that the rows are exact, the second row is the dual of the first, and γ is a weak equivalence. Then $[F, \phi] = [E \oplus G, \left(\begin{smallmatrix} 0 & \gamma \\ \gamma^\vee \eta & 0 \end{smallmatrix}\right)]$ in $\mathrm{GW}(\mathcal{C}, ^\vee, \eta, \omega)$.

1.3 Relative topological K-groups

In topological K-theory, the existence of relative groups $\mathrm{K}(X, A)$ of a pair of spaces (X, A) is a natural outcome of the general machinery that extends K-groups to a cohomology theory. The key fact we need for the moment is that when X retracts onto A we have short exact sequences:

$$\begin{aligned} \mathrm{K}^0(X, A) &\rightarrowtail \mathrm{K}^0 X \twoheadrightarrow \mathrm{K}^0 A \\ \mathrm{KO}^0(X, A; \mathcal{L}) &\rightarrowtail \mathrm{KO}^0(X; \mathcal{L}) \twoheadrightarrow \mathrm{KO}^0(A; \mathcal{L}|_A) \end{aligned} \tag{2}$$

In general, we only have exactness in the middle, and higher and lower K-groups to the left and to the right. We will come back to this in Section 1.5.

In order to extend the map w above to a relative context, it would be convenient to have a description of (relative) topological K- and KO-groups in terms of complexes. At least for K^0, several similar such descriptions exist in the literature, e.g. in [22] and [29]. The approach taken here differs slightly from these in that no "geometric" notion of homotopy is used in defining relations on the complexes. Instead, the constructions employed in the algebraic context are carried over verbatim. Indeed, for any space X which is paracompact and Hausdorff, $\mathrm{K}^0 X$ is simply $\mathrm{K}(\mathrm{Vect}_{\mathbb{C}} X) \cong \mathrm{K}(\mathrm{Ch}^b(\mathrm{Vect}_{\mathbb{C}} X))$, and $\mathrm{KO}^0 X$ is $\mathrm{GW}(\mathrm{Vect}_{\mathbb{C}} X) \cong \mathrm{GW}(\mathrm{Ch}^b(\mathrm{Vect}_{\mathbb{C}} X))$. Unfortunately, for relative groups corresponding identities are less obvious. The result given here may in itself be rather unsatisfactory, but it serves our purpose.

For any topological pair (X, A), let $\mathrm{Ch}^b(X, A)$ denote the category $\mathrm{Ch}^b_{X-A}(\mathrm{Vect}_{\mathbb{C}} X)$ of bounded chain complexes of continuous complex vector bundles over X that are acyclic over A.

Proposition 1.7. *For any paracompact Hausdorff space X with a closed subspace A, there exist natural maps*

$$\begin{aligned} u\colon\ \mathrm{K}(\mathrm{Ch}^b(X, A)) &\to \mathrm{K}^0(X, A) \quad \textit{and} \\ u\colon\ \mathrm{GW}(\mathrm{Ch}^b(X, A)) &\to \mathrm{KO}^0(X, A), \end{aligned}$$

inverse to the isomorphisms i_0 in (1) *for empty A.*

The proof will take up the remainder of this section. We follow the strategy employed in [29]. In fact, in the case of K-theory, hardly any alterations to the proofs given there are necessary. To fix notation, let X and Y be paracompact Hausdorff spaces which intersect in a common closed subspace A. Our intermediate aim is to show that the excision isomorphism

$\mathrm{K}^0(X, A) \cong \mathrm{K}^0(X \cup_A Y, Y)$ has an analogue for $\mathrm{K}(\mathrm{Ch}^b(X, A))$, and likewise for KO-theory. We need a series of lemmas.

Lemma 1.8 (Extension of complexes). *For any complex $E_\bullet$ in $\mathrm{Ch}^b(X, A)$, there exists a complex $\widetilde{E}_\bullet$ in $\mathrm{Ch}^b(X \cup_A Y, Y)$ such that $\widetilde{E}_\bullet|_X$ is quasi-isomorphic to $E_\bullet$ and contains $E_\bullet$ as a direct summand.*

Proof. cf. [29, Appendix, proof of Lemma A.2] □

Lemma 1.9 (Extension of quasi-isomorphisms). *For complexes $E_\bullet$ and $F_\bullet$ in $\mathrm{Ch}^b(X \cup_A Y, Y)$, any quasi-isomorphism $f\colon E_\bullet|_X \to F_\bullet|_X$ extends to a quasi-isomorphism $E_\bullet \to F_\bullet$.*

Proof. cf. [29, loc. cit.]: As $E_\bullet$ and $F_\bullet$ are acyclic over A, $f|_A$ is homotopic to 0. So $f_i|_A = d_{i+1}h_i + h_{i-1}d_i$ for some continuous functions $h_i\colon E_i|_A \to F_{i+1}|_A$ defined on A. As Y is paracompact and Hausdorff, these can be extended to functions $\widetilde{h}_i\colon E_i|_Y \to F_{i+1}|_Y$. An extension $\widetilde{f}$ of f is then obtained by defining $\widetilde{f}$ to be $d_{i+1}\widetilde{h}_i + \widetilde{h}_{i-1}d_i$ on Y and f on X. □

Given a short exact sequence of complexes of vector bundles on X, we know that the maps split in each degree. However, in general it is not clear whether the sequence also splits as a sequence of complexes. The next lemma gives a sufficient condition for the existence of such a splitting in the category of chain complexes $\mathrm{Ch}^b\mathcal{A}$ over any split exact category $\mathcal{A}$. Call a monomorphism (epimorphism) admissible if it appears as the first (respectively second) map in a short exact sequence. Then the lemma can be stated as follows:

Lemma 1.10 (Splittings). *Let $\mathcal{A}$ be a split exact category and $\mathrm{Ch}^b\mathcal{A}$ the exact category of bounded chain complexes over $\mathcal{A}$, such that the admissible morphisms in $\mathrm{Ch}^b\mathcal{A}$ are those which are degreewise split. If $E_\bullet$ and $F_\bullet$ in $\mathrm{Ch}^b\mathcal{A}$ are acyclic, then any admissible monomorphism (epimorphism) $f\colon E_\bullet \to F_\bullet$ splits as a map of complexes: there is a morphism of complexes $s\colon E_\bullet \leftarrow F_\bullet$ such that $s \circ f = \mathrm{id}$ ($f \circ s = \mathrm{id}$).*

Proof. As $E_\bullet$ and $F_\bullet$ are acyclic and $\mathcal{A}$ is split exact, we can write

$$\begin{aligned} E_n &= E'_{n+1} \oplus E'_n \\ F_n &= F'_{n+1} \oplus F'_n \end{aligned}$$

such that all differentials have the form $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. It then follows from the commutativity of

$$\begin{array}{ccc} E_n & \xrightarrow{f_n} & F_n \\ \downarrow & & \downarrow \\ E_{n-1} & \xrightarrow{f_{n-1}} & F_{n-1} \end{array}$$

that each f_n has the form

$$E'_{n+1} \oplus E'_n \xrightarrow{\left(\begin{smallmatrix} f'_{n+1} & g'_n \\ 0 & f'_n \end{smallmatrix}\right)} F'_{n+1} \oplus F'_n$$

for some g'_n. Now, if each f_n is a split monomorphism with splitting $\begin{pmatrix} s'_{n+1} & t'_n \\ u'_{n+1} & v'_n \end{pmatrix}$, then s'_{n+1} is a splitting of f'_{n+1} and $s_n := \begin{pmatrix} s'_{n+1} & t'_n \\ 0 & s'_n \end{pmatrix}$ defines the desired splitting of f that commutes with the differentials. The case of a split epimorphism works analogously. $\square$

Proposition 1.11. *The restriction map $E_\bullet \mapsto E_\bullet|_X$ induces an isomorphism*

$$\mathrm{K}(\mathrm{Ch}^b(X \cup_A Y, Y), \mathrm{quis}) \xrightarrow{\cong} \mathrm{K}(\mathrm{Ch}^b(X, A), \mathrm{quis}).$$

Proof. Define a map i in the other direction by mapping the class of a complex $E_\bullet$ in $\mathrm{Ch}^b(X, A)$ to the class of an extension $\widetilde{E}_\bullet$ as described in Lemma 1.8. This map is well-defined:

- If $\widetilde{E}_\bullet$ and $\bar{E}_\bullet$ are two different extensions of $E_\bullet$, they are quasi-isomorphic over X and hence also over $X \cup_A Y$, by Lemma 1.9.
- Likewise, if $E_\bullet \simeq F_\bullet$ then $\widetilde{E}_\bullet \simeq \widetilde{F}_\bullet$. (Note that the notions of quasi-isomorphism and chain homotopy equivalence coincide due to split exactness of $\mathrm{Vect}_{\mathbb{C}}(\cdot)$, so $\simeq$ is an equivalence relation.)
- Given a short exact sequence $E_\bullet \overset{j}{\rightarrowtail} F_\bullet \overset{p}{\twoheadrightarrow} G_\bullet$ on X, choose $\widetilde{E}_\bullet$, $\widetilde{F}_\bullet$ and $\widetilde{G}_\bullet$ in the following way:

  Start with any $\widetilde{E}_\bullet$. According to Lemma 1.8, we can write $\widetilde{E}_\bullet|_X$ as $E_\bullet \oplus E^e_\bullet$, for some acyclic complex $E^e_\bullet$ on X. Now, add a copy of $E^e_\bullet$ to $F_\bullet$. Extend the monomorphism j as the identity, and extend the epimorphism p as zero. Then repeat this process for $G_\bullet$: write $\widetilde{G}_\bullet|_X = G_\bullet \oplus G^e_\bullet$, add a copy of $G^e_\bullet$ to $F_\bullet$, and extend the morphisms to obtain

  $$\widetilde{E}_\bullet|_X = E_\bullet \oplus E^e_\bullet \overset{\left(\begin{smallmatrix} j & 0 \\ 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)}{\rightarrowtail} F_\bullet \oplus E^e_\bullet \oplus G^e_\bullet \overset{\left(\begin{smallmatrix} p & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)}{\twoheadrightarrow} G_\bullet \oplus G^e_\bullet = \widetilde{G}_\bullet|_X.$$

  The sequence is still exact. Moreover, all complexes involved are acyclic over A. So by Lemma 1.10 the sequence splits over A, i.e.

  $$(F_\bullet \oplus E^e_\bullet \oplus G^e_\bullet)\,|_A \cong \widetilde{E}_\bullet|_A \oplus \widetilde{G}_\bullet|_A.$$

  The right-hand side extends to Y by construction, so we may take $\widetilde{F}_\bullet$ to be $F_\bullet \oplus E^e_\bullet \oplus G^e_\bullet$ on X and $\widetilde{E}_\bullet \oplus \widetilde{G}_\bullet$ on Y. With this choice,

  $$\widetilde{E}_\bullet \rightarrowtail \widetilde{F}_\bullet \twoheadrightarrow \widetilde{G}_\bullet$$

  is an exact sequence on $X \cup_A Y$.

$\square$

Proposition 1.12. *Let $\mathcal{L}$ be a complex line bundle on $X \cup_A Y$. The restriction map $(E_\bullet, \phi) \mapsto (E_\bullet|_X, \phi|_X)$ induces an isomorphism*

$$\mathrm{GW}(\mathrm{Ch}^b(X \cup_A Y, Y), \mathrm{quis}, {}^\vee\!\otimes\mathcal{L}, \eta_{\mathcal{L}}) \xrightarrow{\cong} \mathrm{GW}(\mathrm{Ch}^b(X, A), \mathrm{quis}, {}^\vee\!\otimes\mathcal{L}|_X, \eta_{\mathcal{L}}).$$

Proof. To define an inverse i of the restriction, send $[E_\bullet,\phi]$ to $[\widetilde{E_\bullet},\widetilde{\phi}]$, where $\widetilde{E_\bullet}$ is defined as above and $\widetilde{\phi}$ is constructed as follows:

Using the same notation as in the previous proof, extend ϕ to $\widetilde{E_\bullet}|_X$ by defining it to be zero on $E^e_\bullet$.

$$\begin{array}{ccc} \widetilde{E_\bullet}|_X = E_\bullet \oplus E^e_\bullet & \xrightarrow{\left(\begin{smallmatrix}\phi&0\\0&0\end{smallmatrix}\right)} & (E_\bullet^\vee\otimes\mathcal{L}|_X)\oplus(E^{e\vee}_\bullet\otimes\mathcal{L}|_X) \\ \simeq\Big\uparrow\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) & & \simeq\Big\downarrow(1\ 0) \\ E_\bullet & \xrightarrow[\simeq]{\phi} & E_\bullet^\vee\otimes\mathcal{L}|_X \end{array}$$

As $Y\subset X\cup_A Y$ is closed, $\widetilde{E_\bullet}$ is acyclic not only on Y but also on some open neighbourhood O of Y in $X\cup_A Y$. Choose a continuous function $\rho\colon X\cup_A Y\to[0,1]$ such that $\rho=1$ on $X\cup_A Y-O$ and $\rho=0$ on Y. Define

$$\widetilde{\phi}:=\rho^2\cdot\left(\begin{smallmatrix}\phi&0\\0&0\end{smallmatrix}\right).$$

(The product is well-defined on all of $X\cup_A Y$.) Then $\widetilde{\phi}=(\widetilde{\phi}^\vee\otimes\mathrm{id}_{\mathcal{L}})\circ\eta_{\mathcal{L}}$ and $\widetilde{\phi}$ is a quasi-isomorphism as $\widetilde{\phi}$ agrees with ϕ everywhere where $\widetilde{E_\bullet}$ has non-zero homology.

Suppose for the moment that i is thus well-defined. Then i is indeed inverse to the restriction r: The composition $r\circ i$ is the identity on $\mathrm{GW}(\mathrm{Ch}^b(X,A))$ because, for any symmetric space $(E_\bullet,\phi)$ on X with $E_\bullet$ acyclic over A, both $(E_\bullet,\phi)$ and $(\widetilde{E_\bullet}|_X,\widetilde{\phi}|_X)$ are isometric to $(E_\bullet,\rho|_X^2\cdot\phi)$:

$$\begin{array}{ccc} E_\bullet & \xrightarrow{\phi} & E_\bullet^\vee\otimes\mathcal{L}|_X \\ \simeq\Big\uparrow\cdot\rho|_X & & \simeq\Big\downarrow(\cdot\rho)|_X^\vee\otimes\mathrm{id}_{\mathcal{L}|_X}=(\cdot\rho)|_X \\ E_\bullet & \xrightarrow{\rho|_X^2\cdot\phi} & E_\bullet^\vee\otimes\mathcal{L}|_X \end{array}
\qquad
\begin{array}{ccc} \widetilde{E_\bullet}|_X & \xrightarrow{\widetilde{\phi}|_X} & \widetilde{E_\bullet}|_X^\vee\otimes\mathcal{L}|_X \\ \simeq\Big\uparrow\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) & & \simeq\Big\downarrow(1\ 0) \\ E_\bullet & \xrightarrow{\rho|_X^2\cdot\phi} & E_\bullet^\vee\otimes\mathcal{L}|_X \end{array}$$

Likewise, the composition $i\circ r$ is the identity on $\mathrm{GW}(\mathrm{Ch}^b(X\cup_A Y,Y))$ as, for any symmetric space $(E_\bullet,\phi)$ on $X\cup_A Y$ with $E_\bullet$ acyclic over Y, both $(E_\bullet,\phi)$ and $(\widetilde{E_\bullet|_X},\widetilde{\phi|_X})$ are isometric to $(E_\bullet,\rho^2\cdot\phi)$:

$$\begin{array}{ccc} E_\bullet & \xrightarrow{\phi} & E_\bullet^\vee\otimes\mathcal{L} \\ \simeq\Big\uparrow\cdot\rho & & \simeq\Big\downarrow\cdot\rho \\ E_\bullet & \xrightarrow{\rho^2\cdot\phi} & E_\bullet^\vee\otimes\mathcal{L} \end{array}
\qquad
\begin{array}{ccc} \widetilde{E_\bullet|_X} & \xrightarrow{\widetilde{\phi|_X}} & \widetilde{E_\bullet|_X}^\vee\otimes\mathcal{L} \\ \simeq\Big\uparrow\widetilde{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} & & \simeq\Big\downarrow\widetilde{(1\ 0)}\otimes\mathrm{id}_{\mathcal{L}} \\ E_\bullet & \xrightarrow{\rho^2\cdot\phi} & E_\bullet^\vee\otimes\mathcal{L} \end{array}$$

Here, $\widetilde{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}$ denotes an arbitrary extension of the morphism $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, and $\widetilde{(1\ 0)}$ denotes its dual.

It remains to check that i is well-defined, i.e. that the class of $(\widetilde{E_\bullet},\widetilde{\phi})$ is independent of the choices made and that i preserves the relations stated in Definition 1.6.

- Suppose that $\sigma\colon X \cup_A Y \to [0,1]$ is another function with the same properties as ρ, and that $[\widetilde{E}_\bullet, \rho^2 \left(\begin{smallmatrix} \phi & 0 \\ 0 & 0 \end{smallmatrix}\right)]$ and $[\bar{E}_\bullet, \sigma^2 \left(\begin{smallmatrix} \phi & 0 \\ 0 & 0 \end{smallmatrix}\right)]$ are two different choices for $i[E_\bullet, \phi]$. Then $(\bar{E}_\bullet, \sigma^2 \left(\begin{smallmatrix} \phi & 0 \\ 0 & 0 \end{smallmatrix}\right))$ is isometric to $(\widetilde{E}_\bullet, \sigma^2 \left(\begin{smallmatrix} \phi & 0 \\ 0 & 0 \end{smallmatrix}\right))$:

$$\begin{array}{ccc} \bar{E}_\bullet & \xrightarrow{\sigma^2 \left(\begin{smallmatrix} \phi & 0 \\ 0 & 0 \end{smallmatrix}\right)} & \bar{E}_\bullet^\vee \otimes \mathcal{L} \\ \widetilde{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)} \Big\uparrow \simeq & & \simeq \Big\downarrow \widetilde{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)}^\vee \otimes \mathrm{id}_{\mathcal{L}} \\ \widetilde{E}_\bullet & \xrightarrow{\sigma^2 \left(\begin{smallmatrix} \phi & 0 \\ 0 & 0 \end{smallmatrix}\right)} & \widetilde{E}_\bullet^\vee \otimes \mathcal{L} \end{array}$$

  So it suffices to check that $[\widetilde{E}, \rho^2 \left(\begin{smallmatrix} \phi & 0 \\ 0 & 0 \end{smallmatrix}\right)] = [\widetilde{E}, \sigma^2 \left(\begin{smallmatrix} \phi & 0 \\ 0 & 0 \end{smallmatrix}\right)]$. This is true because both representatives are isometric to $(\widetilde{E}, \rho^2\sigma^2 \left(\begin{smallmatrix} \phi & 0 \\ 0 & 0 \end{smallmatrix}\right))$.

- Suppose we have a quasi-isomorphism $g\colon E_\bullet \xrightarrow{\simeq} F_\bullet$ on X, so that for any non-degenerate symmetric form ψ on $F_\bullet$ we identify $[E_\bullet, (g^\vee \otimes \mathrm{id}_{\mathcal{L}|_X})\psi g]$ with $[F_\bullet, \psi]$ in $\mathrm{GW}(\mathrm{Ch}^b(X, A))$. Then, for any choice of $\widetilde{E}_\bullet$ and $\widetilde{F}_\bullet$, both complexes are acyclic over the same open neighbourhood O of Y. Let $\widetilde{g}$ be the morphism $\rho \left(\begin{smallmatrix} g & 0 \\ 0 & 0 \end{smallmatrix}\right) : \widetilde{E}_\bullet \to \widetilde{F}_\bullet$. This $\widetilde{g}$ defines an isometry between $\left(\widetilde{F}_\bullet, \rho^2 \left(\begin{smallmatrix} \psi & 0 \\ 0 & 0 \end{smallmatrix}\right)\right)$ and $\left(\widetilde{E}_\bullet, \rho^4 \left(\begin{smallmatrix} (g^\vee \otimes \mathrm{id}_{\mathcal{L}})\psi g & 0 \\ 0 & 0 \end{smallmatrix}\right)\right)$. As the latter is isometric to $\left(\widetilde{E}_\bullet, \rho^2 \left(\begin{smallmatrix} (g^\vee \otimes \mathrm{id}_{\mathcal{L}})\psi g & 0 \\ 0 & 0 \end{smallmatrix}\right)\right)$, we can identify $\left[\widetilde{E}_\bullet, (g^\vee \widetilde{\otimes \mathrm{id}_{\mathcal{L}}})\psi g\right]$ with $[\widetilde{F}_\bullet, \widetilde{\psi}]$ in $\mathrm{GW}(\mathrm{Ch}^b(X \cup_A Y, Y))$.

- Finally, suppose we have the following row-exact commutative diagram of complexes on X:

$$\begin{array}{ccccc} E_\bullet & \overset{j}{\rightarrowtail} & F_\bullet & \overset{p}{\twoheadrightarrow} & G_\bullet \\ \simeq \Big\downarrow (\gamma^\vee \otimes \mathrm{id}_{\mathcal{L}}) \circ \eta_{\mathcal{L}} & & \simeq \Big\downarrow \psi = (\psi^\vee \otimes \mathrm{id}_{\mathcal{L}}) \circ \eta_{\mathcal{L}} & & \simeq \Big\downarrow \gamma \\ G_\bullet^\vee \otimes \mathcal{L}|_X & \underset{p^\vee \otimes \mathrm{id}_{\mathcal{L}}}{\rightarrowtail} & F_\bullet^\vee \otimes \mathcal{L}|_X & \underset{j^\vee \otimes \mathrm{id}_{\mathcal{L}}}{\twoheadrightarrow} & E_\bullet^\vee \otimes \mathcal{L}|_X \end{array}$$

  Construct $\widetilde{F}_\bullet$ as in the proof of Proposition 1.11, such that

$$\widetilde{E}_\bullet|_X \overset{\left(\begin{smallmatrix} j & 0 \\ 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)}{\rightarrowtail} \widetilde{F}_\bullet|_X \overset{\left(\begin{smallmatrix} p & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)}{\twoheadrightarrow} \widetilde{G}_\bullet|_X$$

  is split exact over A and extends to $X \cup_A Y$. There is an open neighbourhood O of Y in $X \cup_A Y$ over which all three complexes $\widetilde{E}_\bullet$, $\widetilde{F}_\bullet$ and $\widetilde{G}_\bullet$ are acyclic. Using this neighbourhood to define ρ as before, γ may be extended as $\rho^2 \left(\begin{smallmatrix} \gamma & 0 \\ 0 & 0 \end{smallmatrix}\right) \colon \widetilde{G}_\bullet \to \widetilde{E}_\bullet^\vee \otimes \mathcal{L}$, and likewise for ψ. This yields an extension of the whole diagram to $X \cup_A Y$.

□

We can now prove Proposition 1.7. The isomorphism i constructed in Proposition 1.11 can be used to connect the relative group $\mathrm{K}(\mathrm{Ch}^b(X,A))$ defined in terms of complexes with the classical relative group $\mathrm{K}^0(X,A)$:

$$\begin{array}{ccccc}
\mathrm{K}(\mathrm{Ch}^b(X,A)) & & & & \\
\downarrow{\scriptstyle i}\,{\scriptstyle \cong} & & & & \\
\mathrm{K}(\mathrm{Ch}^b(X\cup_A X, X)) & \longrightarrow & \mathrm{K}(\mathrm{Ch}^b(X\cup_A X)) & \twoheadrightarrow & \mathrm{K}(\mathrm{Ch}^b X) \\
{\scriptstyle u'}\,\vdots & & \downarrow{\scriptstyle i_0^{-1}}\,{\scriptstyle \cong} & & \downarrow{\scriptstyle i_0^{-1}}\,{\scriptstyle \cong} \\
\mathrm{K}^0(X\cup_A X, X) & \rightarrowtail & \mathrm{K}^0(X\cup_A X) & \twoheadrightarrow & \mathrm{K}^0 X \\
\downarrow{\scriptstyle \text{excision}}\,{\scriptstyle \cong} & & & & \\
\mathrm{K}^0(X,A) & & & &
\end{array}$$

As X is a retract of $X\cup_A X$, the lower row of K^0-groups is a short exact sequence (cf. (2)). The row above it may not be exact, but at least the composition of the two maps must be zero. So we obtain the factorization u' of i_0^{-1} indicated by the broken arrow. The map u promised in Proposition 1.7 is the vertical composition.

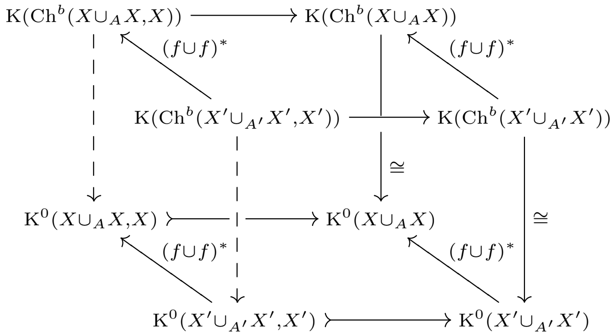
The same construction works for KO-groups, using Proposition 1.12: for $\mathrm{KO}^0(X,A;\mathcal{L})$, consider $X\cup_A X$ with the line bundle $r^*\mathcal{L}$, where r is the canonical retraction $X\cup_A X\to X$.

Lemma 1.13 (Naturality of u)**.** *The map u is natural with respect to continuous maps of pairs $f\colon (X,A)\to(X',A')$.*

Proof. For notational convenience, we concentrate on the version of u defined on K-groups. The proof for GW-/KO-groups is identical up to obvious notational alterations. First, note that $f(A)\subset A'$ implies that $f^{-1}(X'-A')\subset X-A$, so the pullback $f^*\colon \mathrm{K}(\mathrm{Ch}^b(X,A))\leftarrow \mathrm{K}(\mathrm{Ch}^b(X',A'))$ is well-defined. As f induces a map $f\cup f\colon X\cup_A X\to X'\cup_{A'} X'$, we can check naturality step by step:

$$\begin{array}{ccc}
\mathrm{K}(\mathrm{Ch}^b(X,A)) & \xleftarrow{\;f^*\;} & \mathrm{K}(\mathrm{Ch}^b(X',A')) \\
\downarrow & & \downarrow \\
\mathrm{K}(\mathrm{Ch}^b(X\cup_A X,X)) & \xleftarrow{(f\cup f)^*} & \mathrm{K}(\mathrm{Ch}^b(X'\cup_{A'} X',X')) \\
\vdots & & \vdots \\
\mathrm{K}^0(X\cup_A X,X) & \xleftarrow{(f\cup f)^*} & \mathrm{K}^0(X'\cup_{A'} X',X') \\
\downarrow & & \downarrow \\
\mathrm{K}^0(X,A) & \xleftarrow{\;f^*\;} & \mathrm{K}^0(X',A')
\end{array}$$

Commutativity of the top and bottom squares follows from the fact that f and $f \cup f$ commute with the restrictions. The central square is the left face of the following cube:



All other faces of this cube commute, and the lower horizontal maps are injections, so the left face commutes as well. □

The tensor product of (complexes of) vector bundles induces multiplications on K-, KO- and GW-groups (cf. Sections 1.5 and 1.6). These multiplicative structures are clearly respected by the isomorphisms i_0 of (1), so they are also respected by $u = i_0^{-1}$ on non-relative groups. It follows that u respects multiplication in general:

Lemma 1.14. *The map u respects multiplication, i.e. we have commutative diagrams*

$$\begin{array}{ccc} \mathrm{K}(\mathrm{Ch}^b X) \otimes \mathrm{K}(\mathrm{Ch}^b(X,A)) & \xrightarrow{\text{multiplication}} & \mathrm{K}(\mathrm{Ch}^b(X,A)) \\ \downarrow{\scriptstyle u\otimes u} & & \downarrow{\scriptstyle u} \\ \mathrm{K}^0 X \otimes \mathrm{K}^0(X,A) & \longrightarrow & \mathrm{K}^0(X,A) \end{array}$$

and similarly for GW-/KO-*groups.*

Proof. Again, we concentrate on the case of K-groups. To check that u' respects multiplication, take $x \in \mathrm{K}(\mathrm{Ch}^b(X \cup_A X))$ and $y \in \mathrm{K}(\mathrm{Ch}^b(X \cup_A X, X))$, and consider

$$\begin{array}{ccccc} \mathrm{K}(\mathrm{Ch}^b(X\cup_A X,X)) & \xrightarrow{j^*} & \mathrm{K}(\mathrm{Ch}^b(X\cup_A X)) & \twoheadrightarrow & \ldots \\ \downarrow{\scriptstyle u'} & & {\scriptstyle i_0^{-1}}\downarrow{\scriptstyle \cong} & & \\ \mathrm{K}^0(X\cup_A X,X) & \overset{j^*}{\rightarrowtail} & \mathrm{K}^0(X\cup_A X) & \twoheadrightarrow & \ldots \end{array}$$

We have

$$\begin{aligned} j^* u'(x \cdot y) &= i_0^{-1} j^*(x \cdot y) = i_0^{-1}(x \cdot j^*(y)) \\ &= i_0^{-1}(x) \cdot i_0^{-1}(j^* y) = i_0^{-1}(x) \cdot j^*(u'(y)) \\ &= j^*(i_0^{-1}(x) \cdot u'(y)), \end{aligned}$$

so by injectivity of the lower j^*, we have $u'(x \cdot y) = i_0^{-1}(x) \cdot u'(y)$. The result for u follows as the map i and the excision isomorphism appearing in the definition of u are both multiplicative. □

1.4 Extending the map to relative groups

We are now in a position to extend the definition of $w\colon \mathrm{K}_0 X \to \mathrm{K}^0(X_{\mathbb{C}})$ to relative groups. More precisely, for a topological space X with subspace Z, let $\mathrm{K}^0_Z X$ denote $\mathrm{K}^0(X, X - Z)$. Then, for any complex variety X with a closed subvariety Z, we will construct $w\colon \mathrm{K}_0^Z X \to \mathrm{K}^0_{Z_{\mathbb{C}}}(X_{\mathbb{C}})$.

The only remaining technical difficulty is that, in order to use the results of the preceding section, we have to approximate the open set $X_{\mathbb{C}} - Z_{\mathbb{C}}$ by some closed subset A of $X_{\mathbb{C}}$. For this, we use the notion of regular neighbourhoods. Recall that any compact complex variety may be triangulated [19]. So given any such variety X with a closed subvariety Z, we can view $X_{\mathbb{C}}$ as a (finite) simplicial complex with $Z_{\mathbb{C}}$ as a subcomplex. It follows from standard results in piecewise-linear topology that $Z_{\mathbb{C}}$ has a regular neighbourhood N which is essentially unique: given any two triangulations of $(X_{\mathbb{C}}, Z_{\mathbb{C}})$ and any two regular neighbourhoods N_1 and N_2 arising from these, there exists a homeomorphism $h\colon X_{\mathbb{C}} \to X_{\mathbb{C}}$ carrying N_1 to N_2 and restricting to the identity on $Z_{\mathbb{C}}$ [31, Theorem 3.8]. This homeomorphism is homotopic to the identity as a map of pairs $(X_{\mathbb{C}}, X_{\mathbb{C}} - Z_{\mathbb{C}}) \to (X_{\mathbb{C}}, X_{\mathbb{C}} - Z_{\mathbb{C}})$. In particular, if j_1 and j_2 denote the inclusions of $(X_{\mathbb{C}}, X_{\mathbb{C}} - \mathring{N}_1)$ and $(X_{\mathbb{C}}, X_{\mathbb{C}} - \mathring{N}_2)$ into $(X_{\mathbb{C}}, X_{\mathbb{C}} - Z_{\mathbb{C}})$, respectively, $j_2 \circ h$ is homotopic to j_1. So we have a commutative triangle:

$$\begin{array}{ccc} \mathrm{K}^0(X_{\mathbb{C}}, X_{\mathbb{C}} - \mathring{N}_1) & \xleftarrow{\;j_1^*\;} & \\ {\scriptstyle h^*}\uparrow & & \mathrm{K}^0_{Z_{\mathbb{C}}}(X_{\mathbb{C}}) \\ \mathrm{K}^0(X_{\mathbb{C}}, X_{\mathbb{C}} - \mathring{N}_2) & \xleftarrow{\;j_2^*\;} & \end{array} \tag{3}$$

Moreover, for any regular neighbourhood N of $Z_{\mathbb{C}}$ in $X_{\mathbb{C}}$, the inclusion of $X_{\mathbb{C}} - \mathring{N}$ into $X_{\mathbb{C}} - Z_{\mathbb{C}}$ is a homotopy equivalence, implying that the pullback $\mathrm{K}^0(X_{\mathbb{C}}, X_{\mathbb{C}} - \mathring{N}) \underset{j^*}{\longleftarrow} \mathrm{K}^0_{Z_{\mathbb{C}}}(X_{\mathbb{C}})$ is an isomorphism.

This allows us to define w in the following manner: First, as in the non-relative case, we forget all algebraic structure.

$$\mathrm{K}_0^Z X = \mathrm{K}(\mathrm{Ch}^b_Z(\mathrm{Vect} X)) \xrightarrow{\text{forget}} \mathrm{K}(\mathrm{Ch}^b_{Z_{\mathbb{C}}}(\mathrm{Vect}_{\mathbb{C}} X_{\mathbb{C}}))$$

Next, we choose a regular neighbourhood N of $Z_{\mathbb{C}}$ in $X_{\mathbb{C}}$ and approximate the open set $X_{\mathbb{C}} - Z_{\mathbb{C}}$ by the closed subset $X_{\mathbb{C}} - \mathring{N}$ of $X_{\mathbb{C}}$.

$$\mathrm{K}(\mathrm{Ch}^b_{Z_{\mathbb{C}}}(\mathrm{Vect}_{\mathbb{C}} X_{\mathbb{C}})) = \mathrm{K}(\mathrm{Ch}^b(X_{\mathbb{C}}, X_{\mathbb{C}} - Z_{\mathbb{C}})) \xrightarrow{\text{forget}} \mathrm{K}(\mathrm{Ch}^b(X_{\mathbb{C}}, X_{\mathbb{C}} - \mathring{N}))$$

The map here simply forgets acyclicity on $\mathring{N} - Z_{\mathbb{C}}$. Now we can apply the map u of the preceding section and compose with the inverse of j^*.

$$\mathrm{K}(\mathrm{Ch}^b(X_{\mathbb{C}}, X_{\mathbb{C}} - \mathring{N})) \xrightarrow{\;u\;} \mathrm{K}^0(X_{\mathbb{C}}, X_{\mathbb{C}} - \mathring{N}) \xrightarrow{(j^*)^{-1}} \mathrm{K}^0_{Z_{\mathbb{C}}}(X_{\mathbb{C}})$$

By naturality of u (Lemma 1.13) and diagram (3), this definition is independent of the choice of N.

In general, for any complex variety X, we can find an open immersion into a compact variety $\bar{X}$ [20, 21]. Then the complement of X in $\bar{X}$ is also compact, so we can view $X_{\mathbb{C}}$ as a finite simplicial complex with a subcomplex S removed.

Any closed subvariety Z of X can be written as $\bar{Z} \cap X$, where $\bar{Z}$ is the closure of Z in $\bar{X}$. So we can define regular neighbourhoods of $Z_{\mathbb{C}}$ in $X_{\mathbb{C}}$ to be intersections $N := \bar{N} \cap X_{\mathbb{C}}$, where $\bar{N}$ is a regular neighbourhood of $\bar{Z}_{\mathbb{C}}$ in $\bar{X}_{\mathbb{C}}$. All maps considered above restrict to this context. In particular, w can be defined in the same way as before.

Definition 1.15. Given a complex variety X with a closed subvariety Z, $w\colon \mathrm{K}_0^Z X \to \mathrm{K}^0_{Z_{\mathbb{C}}}(X_{\mathbb{C}})$ is the composition

$$\mathrm{K}_0^Z X \xrightarrow{\text{forget}} \mathrm{K}(\mathrm{Ch}^b(X_{\mathbb{C}}, X_{\mathbb{C}} - \mathring{N})) \xrightarrow{u} \mathrm{K}^0(X_{\mathbb{C}}, X_{\mathbb{C}} - \mathring{N}) \xrightarrow{(j^*)^{-1}} \mathrm{K}^0_{Z_{\mathbb{C}}}(X_{\mathbb{C}}),$$

where N is any regular neighbourhood of $Z_{\mathbb{C}}$ in $X_{\mathbb{C}}$.

Naturality. A morphism of varieties $f\colon X \to X'$ satisfying $f^{-1}(Z') \subset Z$ for closed subvarieties $Z \subset X$ and $Z' \subset X'$ induces pullbacks

$$\mathrm{K}_0^Z X \xleftarrow{f^*} \mathrm{K}_0^{Z'} X'$$
$$\mathrm{K}^0_{Z_{\mathbb{C}}}(X_{\mathbb{C}}) \xleftarrow{f^*} \mathrm{K}^0_{Z'_{\mathbb{C}}}(X'_{\mathbb{C}})$$

It is immediate from Lemma 1.13 that w commutes with these whenever regular neighbourhoods N of $Z_{\mathbb{C}}$ and N' of $Z'_{\mathbb{C}}$ can be chosen in such a way that f maps $X_{\mathbb{C}} - \mathring{N}$ to $X'_{\mathbb{C}} - \mathring{N}'$. In particular, taking $Z = X$, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{K}_0 X & \xleftarrow{f^*} & \mathrm{K}_0^{Z'} X' \\ \downarrow w & & \downarrow w \\ \mathrm{K}^0(X_{\mathbb{C}}) & \xleftarrow{f^*} & \mathrm{K}^0_{Z'_{\mathbb{C}}}(X'_{\mathbb{C}}) \end{array}$$

for any morphism f and any closed subvariety Z' of X'. We record two simple cases in which we may also restrict the support of the K-groups of X.

Lemma 1.16. *The map w commutes with the pullbacks along a closed embedding $i\colon X \hookrightarrow X'$: for any closed subvariety Z of X and any closed subvariety Z' of X' satisfying $i^{-1}(Z') \subset Z$, we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{K}_0^Z X & \xleftarrow{i^*} & \mathrm{K}_0^{Z'} X' \\ \downarrow w & & \downarrow w \\ \mathrm{K}^0_{Z_{\mathbb{C}}}(X_{\mathbb{C}}) & \xleftarrow{i^*} & \mathrm{K}^0_{Z'_{\mathbb{C}}}(X'_{\mathbb{C}}) \end{array}$$

Proof. First, choose a regular neighbourhood N of $Z_{\mathbb{C}}$ in $X_{\mathbb{C}}$. As i is closed, $i(X_{\mathbb{C}} - \mathring{N})$ is closed in $X'_{\mathbb{C}}$. Moreover, $i(X_{\mathbb{C}} - \mathring{N})$ does not intersect $Z'_{\mathbb{C}}$ as $i^{-1}(Z') \subset Z$. This allows us to choose a regular neighbourhood $N' \subset X'_{\mathbb{C}} - i(X_{\mathbb{C}} - \mathring{N})$. Then $i(X_{\mathbb{C}} - \mathring{N}) \subset X'_{\mathbb{C}} - \mathring{N}'$. □

Lemma 1.17. *The map w commutes with the pullbacks of a projection $\pi_X : X \times Y \twoheadrightarrow X$ in the sense that, for any closed subvariety Z of X, we have a commutative diagram*

$$\begin{array}{ccc} \mathrm{K}_0^{Z\times Y}(X\times Y) & \xleftarrow{\pi_X^*} & \mathrm{K}_0^{Z} X \\ \downarrow w & & \downarrow w \\ \mathrm{K}^0_{Z_{\mathbb{C}}\times Y_{\mathbb{C}}}(X_{\mathbb{C}}\times Y_{\mathbb{C}}) & \xleftarrow{\pi_X^*} & \mathrm{K}^0_{Z_{\mathbb{C}}}(X_{\mathbb{C}}) \end{array}$$

Proof. If N is a regular neighbourhood of $Z_{\mathbb{C}}$ in $X_{\mathbb{C}}$, then $N \times Y$ is a regular neighbourhood of $Z_{\mathbb{C}} \times Y_{\mathbb{C}}$ in $X_{\mathbb{C}} \times Y_{\mathbb{C}}$, and $\pi_X(X_{\mathbb{C}} \times Y_{\mathbb{C}} - \mathring{N} \times Y_{\mathbb{C}}) = X_{\mathbb{C}} - \mathring{N}$. □

All considerations of this section generalise to Grothendieck-Witt groups without any problems. Altogether, for a complex variety X with a closed subvariety Z and a line bundle $\mathcal{L}$, we have maps

$$\begin{aligned} \mathrm{K}_0^Z X &\to \mathrm{K}^0_{Z_{\mathbb{C}}}(X_{\mathbb{C}}) \\ \mathrm{GW}^0_Z(X;\mathcal{L}) &\to \mathrm{KO}^0_{Z_{\mathbb{C}}}(X_{\mathbb{C}};\mathcal{L}) \\ \mathrm{W}^0_Z(X;\mathcal{L}) &\to \tfrac{\mathrm{KO}}{\mathrm{K}}{}^0_{Z_{\mathbb{C}}}(X_{\mathbb{C}};\mathcal{L}) \end{aligned}$$

all of which we will call w.

1.5 Representable topological K-theory

Complex K-theory. Topological K-theory can be constructed as a "generalised cohomology theory" by a standard method. The starting point is that complex vector bundles over a space are classified by homotopy classes of maps into the classifying space BU of the infinite unitary group U. In fact, for a CW complex X, the usual definition of $\mathrm{K}^0 X$ is

$$\mathrm{K}^0 X := [X, \mathrm{BU}\times\mathbb{Z}]\,.$$

This agrees with our earlier definition when X is a finite-dimensional CW complex (e.g. [30, p. 204]). To turn this into a cohomology theory on CW complexes, one considers the iterated loop spaces $\Omega^i(\mathrm{BU})$. Bott periodicity implies that the twofold loop space $\Omega^2(\mathrm{BU}\times\mathbb{Z})$ is again equivalent to $\mathrm{BU}\times\mathbb{Z}$, so we obtain a 2-periodic spectrum, with all even terms equivalent to $\mathrm{BU}\times\mathbb{Z}$, and odd terms equivalent to $\Omega(\mathrm{BU}\times\mathbb{Z}) = \mathrm{U}$. The associated 2-periodic generalised cohomology theory is known as representable topological K-theory. Once the theory for CW complexes is established, it can easily be generalised to arbitrary spaces by considering CW approximations. See [30] for a detailed exposition of spectra and generalised cohomology theories.

The upshot is that we have essentially two different K-groups for a space, the usual group $\mathrm{K}^0 X$ and a higher group $\mathrm{K}^1 X$; that we moreover have relative groups $\mathrm{K}^i(X, A)$ for pairs of spaces (X, A); and that all of these fit into a 6-term exact sequence

$$\begin{array}{ccccc} \mathrm{K}^0(X,A) & \longrightarrow & \mathrm{K}^0 X & \longrightarrow & \mathrm{K}^0 A \\ \uparrow & & & & \downarrow \\ \mathrm{K}^1 A & \longleftarrow & \mathrm{K}^1 X & \longleftarrow & K^1(X,A) \end{array} \qquad (4)$$

As before, we will let $\mathrm{K}^i_Z X$ denote the group $\mathrm{K}^i(X, X - Z)$.

It is sometimes convenient to consider the "reduced" group $\widetilde{\mathrm{K}}^i X$, defined as the kernel of the restriction map $\mathrm{K}^i X \to \mathrm{K}^i(\text{point})$. For example, using this notation we have

$$\mathrm{K}^i(X, A) = \widetilde{\mathrm{K}}^i(X/A)$$

for any CW pair (X, A). Also, in negative degrees $-i$ it is possible to give a slightly more explicit description of the groups $\mathrm{K}^{-i} X$ using suspensions of X, since the suspension functor is the left adjoint of the loop functor:

$$\mathrm{K}^{-i} X = \widetilde{\mathrm{K}}^0(S^i(X \sqcup \text{point})),$$

the reduced group of the i^{th} suspension of the union of X with a disjoint base point.

We will make extensive use of the fact that K-theory is multiplicative, i.e. that we have internal and external products

$$\begin{aligned} \cdot\colon \ \mathrm{K}^i_Z X \otimes \mathrm{K}^j X &\to \mathrm{K}^{i+j}_Z X \\ \times\colon \ \mathrm{K}^i_Z X \otimes \mathrm{K}^j Y &\to \mathrm{K}^{i+j}_{Z \times Y}(X \times Y) \end{aligned}$$

(In degree zero, these products are induced by the tensor product of vector bundles.) In particular, if we let $\mathrm{K}^* X$ denote the total K-group $\bigoplus_{i \in \mathbb{Z}} \mathrm{K}^i X$, $\mathrm{K}^* X$ becomes a module over $\mathrm{K}^*(\text{point})$.

This last group, $\mathrm{K}^*(\text{point})$, is relatively simple. $\mathrm{K}^0(\text{point})$ is isomorphic to $\mathbb{Z}$, with the class of the trivial line bundle as a generator. $\mathrm{K}^{-2}(\text{point})$ is of course also isomorphic to $\mathbb{Z}$; under the isomorphism of $\mathrm{K}^{-2}(\text{point})$ with $\widetilde{\mathrm{K}}^0(S^2)$, its generator g corresponds to $\mathcal{O}(-1) - \mathbb{C}$, the Hopf bundle minus the trivial line bundle. All the other even groups are generated by powers of g, whereas the odd K-groups of a point are all zero. So we have

$$\mathrm{K}^*(\text{point}) = \mathbb{Z}[g, g^{-1}] \tag{5}$$

with g of degree -2.

More generally, $\mathrm{K}^1 X$ vanishes for any CW complex which has cells only in even dimensions, and, in that case, $\mathrm{K}^0 X$ is the free abelian group on these cells. For example, we have $\mathrm{K}^1(\mathbb{C}P^n) = 0$ and $\mathrm{K}^0(\mathbb{C}P^n) = \mathbb{Z}^{n+1}$.

KO-theory. For KO-theory, we may proceed similarly. Here, we have $\mathrm{KO}^0 X = [X, \mathrm{BO} \times \mathbb{Z}]$, where BO is the classifying space of the infinite orthogonal group. However, in this case the multiplicative cohomology theory we obtain is 8-periodic. The KO-groups of a point are already more complicated than in the complex case.

i	0	-1	-2	-3	-4	-5	-6	-7
$\mathrm{KO}^{\mathbf{i}}(\text{point})$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0

The ring structure can be given as

$$\mathrm{KO}^*(\text{point}) = \mathbb{Z}[\eta, \alpha, \lambda, \lambda^{-1}] \tag{6}$$

with η, α and λ of degrees -1, -4 and -8, respectively, subject to the relations $2\eta = 0$, $\eta^3 = 0$ and $\alpha^2 = 4\lambda$ [23].

KO-groups twisted by line bundles. Twisted groups $\mathrm{KO}^i(X;\mathcal{L})$ are less common in the literature, so we include here a brief discussion of their behaviour. In Definition 1.5, we introduced $\mathrm{KO}^0(X;\mathcal{L})$ as a Grothendieck-Witt group of vector bundles with a "twisted symmetric form", $\mathrm{GW}(\mathrm{Vect}_{\mathbb{C}}X, {}^\vee\otimes\mathcal{L}, \eta_{\mathcal{L}})$. The key fact is that this group can alternatively be expressed as the usual reduced group $\widetilde{\mathrm{KO}}^2(\mathrm{Th}_X\mathcal{L})$ of the Thom space of $\mathcal{L}$ over X [25, 3.8]. This allows a straightforward generalisation to relative groups and higher degrees:

Definition 1.18. For a CW complex X with a subcomplex A and a complex line bundle $\mathcal{L}$,

$$\mathrm{KO}^i(X;\mathcal{L}) := \widetilde{\mathrm{KO}}^{i+2}(\mathrm{Th}_X\mathcal{L})$$
$$\mathrm{KO}^i(X,A;\mathcal{L}) := \mathrm{KO}^{i+2}(\mathrm{Th}_X\mathcal{L}, \mathrm{Th}_A(\mathcal{L}|_A))$$

Equivalently, if we choose any metric on $\mathcal{L}$ and write $D_X\mathcal{L}$ and $S_X\mathcal{L}$ for the associated disk and sphere bundle, respectively, $\mathrm{KO}^i(X,A;\mathcal{L})$ is equal to $\mathrm{KO}^{i+2}(D_X\mathcal{L}, S_X\mathcal{L}\cup D_A(\mathcal{L}|_A))$. Of course, for an arbitrary pair (X,A) with a line bundle $\mathcal{L}$ on X, we consider a CW approximation $f\colon (X',A')\to(X,A)$ and the line bundle $f^*\mathcal{L}$.

Basic properties of the twisted groups such as the existence of a long exact cohomology sequence follow directly from the corresponding properties of the usual groups. However, multiplication needs a little care: the usual product on the KO-groups of Thom spaces does not immediately give a sensible multiplicative structure on twisted groups, for example, the products have wrong degrees. The quickest way of arriving at a multiplicative structure suitable for our purpose seems to be the following. We first consider non-relative groups in degree zero, interpreted as Grothendieck-Witt groups as above. As in the untwisted case, the tensor product induces a multiplication, e.g. a cross product

$$\times_{\mathrm{twist}}\colon \mathrm{KO}^0(X;\mathcal{L})\otimes\mathrm{KO}^0(Y;\mathcal{M})\to\mathrm{KO}^0(X\times Y;\pi_X^*\mathcal{L}\otimes\pi_Y^*\mathcal{M}).$$

We can extend this to a product

$$\times_{\mathrm{twist}}\colon \mathrm{KO}^0(X,A;\mathcal{L})\otimes\mathrm{KO}^0(Y;\mathcal{M})\to\mathrm{KO}^0(X\times Y, A\times Y;\pi_X^*\mathcal{L}\otimes\pi_Y^*\mathcal{M})$$

by using the identification of $\mathrm{KO}^0(X,A;\mathcal{L})$ with a direct summand of the non-relative group $\mathrm{KO}^0(X\cup_A X; r^*\mathcal{L})$ as in Section 1.3. To obtain an extension to arbitrary degrees, we introduce here the technique that will later also allow us to extend the map w:

Let $e_n\in\mathrm{KO}^n(S^n)$ be the element corresponding to $1\in\mathrm{KO}^0(\text{point})\cong\mathbb{Z}$ under the suspension isomorphism $\mathrm{KO}^0(\text{point})\cong\widetilde{\mathrm{KO}}^n(S^n)$. It is a general property of multiplicative generalised cohomology theories that we have isomorphisms

$$\mathrm{KO}^i(X,A)\oplus\mathrm{KO}^{i-n}(X,A)\to\mathrm{KO}^i(X\times S^n, A\times S^n)$$
$$(x,y)\mapsto\pi_X^*x+y\times e_n$$

This generalises directly to twisted groups:
Taking $\mathrm{KO}^i(X,A;\mathcal{L})\oplus\mathrm{KO}^{i-n}(X,A;\mathcal{L})$ on the left, we have

$$\begin{aligned}&\mathrm{KO}^{i+2}(D_X\mathcal{L}\times S^n, (S_X\mathcal{L}\cup D_A(\mathcal{L}|_A))\times S^n)\\ =&\mathrm{KO}^{i+2}(D_{X\times S^n}(\pi_X^*\mathcal{L}), S_{X\times S^n}(\pi_X^*\mathcal{L})\cup D_{A\times S^n}((\pi_X^*\mathcal{L})|_{A\times S^n}))\\ \overset{\text{def}}{=}&\mathrm{KO}^i(X\times S^n, A\times S^n;\pi_X^*\mathcal{L})\end{aligned}$$

on the right. In particular, cross product with e_n induces injections

$$\mathrm{KO}^{-n}(X, A; \mathcal{L}) \to \mathrm{KO}^0(X \times S^n, A \times S^n; \pi_X^* \mathcal{L}).$$

Analogous injections exist for the KO-groups of $S^n \times X$. We extend the definition of multiplication by forcing it to be compatible with these. Explicitly, given x in $\mathrm{KO}^{-i}(X, A; \mathcal{L})$ and y in $\mathrm{KO}^{-j}(Y; \mathcal{M})$, we define $x \times_{\text{twist}} y$ to be the unique element in $\mathrm{KO}^{-i-j}(X \times Y, A \times Y; \pi_X^* \mathcal{L} \otimes \pi_Y^* \mathcal{M})$ satisfying $e_i \times (x \times_{\text{twist}} y) \times e_j = (e_i \times x) \times_{\text{twist}} (y \times e_j)$. Periodicity makes this into a multiplicative structure on twisted KO-groups of arbitrary degrees. We will write $\times$ for $\times_{\text{twist}}$ from now on.

1.6 Shifted (Grothendieck-)Witt groups

The groups that we would like to compare the higher topological KO-groups of a variety with are its "shifted" (Grothendieck-)Witt groups. Recall from Section 1.2 that the usual Grothendieck-Witt group of an exact category $\mathcal{A}$ with a duality can alternatively be seen as the Grothendieck-Witt group of the category of bounded chain complexes $\mathrm{Ch}^b\mathcal{A}$. Then a symmetric space is a chain complex $E_\bullet$ together with a symmetric map $\phi\colon E_\bullet \to E_\bullet^\vee$. The idea of shifted groups is to consider maps

$$E_\bullet \to E_\bullet^\vee[i]$$

with an adapted notion of symmetry instead, where $[i]$ denotes an i-fold shift of the chain complex to the left.

This idea was made precise by Paul Balmer in [4] and [5], where he develops a general theory of Witt groups of triangulated categories $(\mathcal{D}, ^\vee, \eta)$ equipped with an adequate duality functor $^\vee$ and a double-dual identification η. Once we have some notion of the "usual" Witt group of such a category, we may equally well consider shifted groups

$$\mathrm{W}^i\mathcal{D} := \mathrm{W}(\mathcal{D}, ^\vee[i], (-1)^{\frac{i(i+1)}{2}}\eta).$$

So elements of $\mathrm{W}^i\mathcal{D}$ are represented by objects E of $\mathcal{D}$ with isomorphisms $\phi\colon E \to E^\vee[i]$ such that $\phi^\vee = (-1)^{\frac{i(i+1)}{2}} \cdot \phi^\vee[i] \circ \eta$. The mysterious sign comes in because there are two different kinds of dualities on triangulated categories, $+$-dualities and $-$-dualities, and the sign of the duality is not preserved under odd shifts. For even shifts everything is fine. In fact, shifting twice induces an isomorphism $\mathrm{W}^i\mathcal{D} \cong \mathrm{W}^{i+4}\mathcal{D}$.

Starting from the category $\mathrm{Ch}^b\mathcal{A}$ of bounded chain complexes over an exact category $\mathcal{A}$, we can easily obtain a triangulated category by inverting all quasi-isomorphisms, i.e. we consider the bounded derived category $\mathcal{D}^b\mathcal{A}$ of $\mathcal{A}$. A duality on $\mathcal{A}$ induces a duality on $\mathcal{D}^b\mathcal{A}$, so we can consider the "derived Witt groups" $\mathrm{W}^i(\mathcal{D}^b\mathcal{A})$. For $i = 0$, the inclusion $\mathcal{A} \subset \mathcal{D}^b\mathcal{A}$ induces an isomorphism with $\mathrm{W}\mathcal{A}$, but for other values of $i \in \mathbb{Z}_4$ we obtain something new. For example, $\mathrm{W}^2(\mathcal{D}^b\mathcal{A})$ is the Witt group of skew-symmetric spaces over $\mathcal{A}$.

Naturally enough, for a complex variety X with a line bundle $\mathcal{L}$, the shifted Witt groups $\mathrm{W}^i(X; \mathcal{L})$ are defined as the derived groups of $(\mathrm{Vect}X, ^\vee \otimes \mathcal{L}, \eta_{\mathcal{L}})$. The remarkable property of these shifted groups is that, when X is smooth, they fit into a long exact localization sequence

$$\cdots \to \mathrm{W}^i_Z X \to \mathrm{W}^i X \to \mathrm{W}^i(X - Z) \to \mathrm{W}^{i+1}_Z X \to \dots \qquad (7)$$

for any closed subvariety Z of X. As the Witt groups are 4-periodic, this sequence may be thought of as a long exact polygon with 12 vertices.

All the definitions work equally well for Grothendieck-Witt groups [11]. However, we do not obtain as nice a localization sequence. Rather, for every $i \in \mathbb{Z}_4$, we have an exact sequence

$$\mathrm{GW}^i_Z X \to \mathrm{GW}^i X \to \mathrm{GW}^i(X-Z) \to \mathrm{W}^{i+1}_Z X \to \mathrm{W}^{i+1} X \to \dots \tag{8}$$

i.e. the localization sequences for Grothendieck-Witt groups merge into the localization sequence of Witt groups after just three terms. The reason behind this behaviour is that Grothendieck-Witt and Witt groups are special cases of Hermitian K-groups, and the sequences above are only portions of localization sequences of Hermitian K-theory; we could extend them to the left using higher Hermitian K-groups [15, 16, 17]. The situation is summarised by the picture in the introduction. However, higher Hermitian K-groups are much more mysterious. Unlike the Grothendieck-Witt and Witt groups, they do not have a description in purely algebraic terms.

Example 1.19. For a non-trivial example of a shifted symmetric space, consider the complex

$$\dots \longrightarrow 0 \longrightarrow \mathcal{O}(-1) \xrightarrow{\cdot z_0} \mathcal{O} \longrightarrow 0 \longrightarrow \dots$$

on the complex projective line $\mathbb{P}^1$, where the only non-zero map is given by multiplication with z_0 in coordinates $[z_0 : z_1]$. If we arrange $\mathcal{O}$ to be in degree zero, multiplication by z_1 induces a symmetric quasi-isomorphism with the dual complex shifted one to the left:

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & 0 & \longrightarrow & \mathcal{O}(-1) & \xrightarrow{\cdot z_0} & \mathcal{O} & \longrightarrow & 0 & \longrightarrow & \dots \\ & & & & \downarrow{\scriptstyle \cdot z_1} & & \downarrow{\scriptstyle \cdot(-z_1)} & & & & \\ \dots & \longrightarrow & 0 & \longrightarrow & \mathcal{O} & \xrightarrow{\cdot(-z_0)} & \mathcal{O}(1) & \longrightarrow & 0 & \longrightarrow & \dots \end{array}$$

It turns out that this symmetric space represents a generator of $\mathrm{GW}^1(\mathbb{P}^1) \cong \mathbb{Z}$ (e.g. [10, proof of Theorem 1.5]). We will denote its negative by ε.

As in KO-theory, the tensor product of vector bundles induces a multiplication on GW- and W-groups [8; 10, p. 8]. This is usually given in the form

$$\cdot\colon\ \mathrm{GW}^i_Z(X;\mathcal{L}) \otimes \mathrm{GW}^j(X;\mathcal{M}) \to \mathrm{GW}^{i+j}_Z(X;\mathcal{L}\otimes\mathcal{M}).$$

To simplify the notation and some of the calculations in what follows, we construct from this a cross product on Grothendieck-Witt groups analogous to the cross product we have on topological cohomology theories, i.e.

$$\begin{aligned} \times\colon\ \mathrm{GW}^i_Z(X;\mathcal{L}) \otimes \mathrm{GW}^j(Y;\mathcal{M}) &\to \mathrm{GW}^{i+j}_{Z\times Y}(X\times Y;\mathcal{L}\times\mathcal{M}) \\ (x,y) &\mapsto \pi_X^*(x)\cdot\pi_Y^*(y) \end{aligned}$$

Here, $\mathcal{L}\times\mathcal{M}$ denotes the line bundle $\pi_X^*\mathcal{L}\otimes\pi_Y^*\mathcal{M}$ on the product. The properties of this cross product can be derived from those of the ordinary product in the usual way. In particular, it is associative and commutative [10, p. 8]. (Note that over $\mathbb{C}$, the symmetric space $\langle -1\rangle = (\mathbb{C}, -\mathrm{id})$ over a point is isometric to $\langle 1\rangle = (\mathbb{C}, \mathrm{id})$ via multiplication by i, so the commutativity is not graded or twisted in any way.) The usual product can be completely recovered from the cross product using a diagonal map.

1.7 Extending the map to lower degrees

We would now like to extend our definition of $w_0 := w$ from $\mathrm{GW}^0_Z(X;\mathcal{L})$ to $\mathrm{KO}^0_{Z_\mathbb{C}}(X_\mathbb{C};\mathcal{L})$ to maps

$$w_i\colon\ \mathrm{GW}^i_Z(X;\mathcal{L}) \to \mathrm{KO}^{2i}_{Z_\mathbb{C}}(X_\mathbb{C};\mathcal{L})$$

for all $i \in \mathbb{Z}$. For the moment, we will restrict ourselves to negative i, but in Section 2.4 we will see that the maps obtained respect the periodicity isomorphisms of Grothendieck-Witt groups and of KO-theory, yielding an extension of w to all degrees.

Our starting point is the following observation: Let ε be the generator of $\mathrm{GW}^1(\mathbb{P}^1)$ described in Example 1.19. Multiplication with this element induces isomorphisms

$$\begin{aligned}\mathrm{GW}^i_Z(X;\mathcal{L}) \oplus \mathrm{GW}^{i-1}_Z(X;\mathcal{L}) &\xrightarrow{\cong} \mathrm{GW}^i_{Z\times\mathbb{P}^1}(X\times\mathbb{P}^1;\pi_X^*\mathcal{L})\\ (x,y) &\mapsto \pi_X^*x + y\times\varepsilon\end{aligned}$$

(These isomorphisms follow directly from the versions without supports described in [10, Theorem 1.5].) Likewise, as we have already seen in Section 1.5, multiplication with the element $e \in \mathrm{KO}^2(S^2)$ corresponding to $1 \in \mathrm{KO}^0(\text{point})$ under twofold suspension yields isomorphisms

$$\begin{aligned}\mathrm{KO}^j_Z(X;\mathcal{L}) \oplus \mathrm{KO}^{j-2}_Z(X;\mathcal{L}) &\xrightarrow{\cong} \mathrm{KO}^j_{Z\times S^2}(X\times S^2;\pi_X^*\mathcal{L})\\ (x,y) &\mapsto \pi_X^*x + y\times e\end{aligned}$$

This means that we can define w_{-i} for $i \in \mathbb{N}$ inductively:

Definition 1.20. For $x \in \mathrm{GW}^{-i-1}_Z(X;\mathcal{L})$, the image $w_{-i-1}(x)$ is the unique element of $\mathrm{KO}^{-2i-2}_{Z_\mathbb{C}}(X_\mathbb{C};\mathcal{L})$ satisfying

$$w_{-i-1}(x)\times e = w_{-i}(x\times\varepsilon)$$

in $\mathrm{KO}^{-2i}_{Z\times S^2}(X\times S^2;\pi_X^*\mathcal{L})$.

2 Properties of the comparison map

We begin with two basic properties that can be deduced from those of w_0 by simple calculations.

2.1 Naturality

Let a pair of varieties (X,Z) denote a complex variety X with a closed subvariety Z, and let a morphism of pairs be a morphism

$$X \xrightarrow{f} X'$$

satisfying $f^{-1}(Z') \subset Z$. (Note that this convention differs from the usual notion of a morphism of pairs in topology.) As we have seen, such a morphism induces pullbacks of the form

$$\mathrm{K}_0^Z X \xleftarrow{f^*} \mathrm{K}_0^{Z'} X'.$$

The considerations at the end of Section 1.4 imply the following naturality property of w.

Proposition 2.1. *The maps w_{-i} commute with the pullbacks of*

- *arbitrary morphisms $X \to (X', Z')$, where we write X for the pair (X, X),*
- *closed embeddings $(X, Z) \hookrightarrow (X', Z')$, and*
- *projections $(X \times Y, Z \times Y) \twoheadrightarrow (X, Z)$.*

Proof. If $f\colon (X, Z) \to (X', Z')$ falls into the list above, so does the morphism $f \times \mathrm{id}\colon (X \times \mathbb{P}^1, Z \times \mathbb{P}^1) \to (X' \times \mathbb{P}^1, Z' \times \mathbb{P}^1)$. Assume by induction that w_{-i} satisfies the proposition for some i. Then so does w_{-i-1}:

$$\begin{aligned} f^* w_{-i-1}(x) \times e &= (f \times \mathrm{id})^*(w_{-i-1}(x) \times e) \\ &= w_{-i}((f \times \mathrm{id})^*(x \times \varepsilon)) && \text{by naturality of } w_{-i} \\ &= w_{-i}(f^* x \times \varepsilon) \\ &= w_{-i}(f^* x) \times e \end{aligned}$$

So $f^* w_{-i-1}(x) = w_{-i-1}(f^* x)$. □

2.2 Multiplication

Proposition 2.2. *The maps w_{-i} respect multiplication, i.e. we have commutative diagrams*

$$\begin{array}{ccc} \mathrm{GW}^{-i}_Z(X;\mathcal{L}) \otimes \mathrm{GW}^{-j}(Y;\mathcal{M}) & \xrightarrow{\times} & \mathrm{GW}^{-i-j}_{Z\times Y}(X \times Y; \mathcal{L} \times \mathcal{M}) \\ \downarrow{\scriptstyle w_{-i}\otimes w_{-j}} & & \downarrow{\scriptstyle w_{-i-j}} \\ \mathrm{KO}^{-2i}_{Z_{\mathbb{C}}}(X_{\mathbb{C}};\mathcal{L}) \otimes \mathrm{KO}^{-2j}(Y_{\mathbb{C}};\mathcal{M}) & \xrightarrow{\times} & \mathrm{KO}^{-2i-2j}_{Z_{\mathbb{C}}\times Y_{\mathbb{C}}}(X_{\mathbb{C}} \times Y_{\mathbb{C}}; \mathcal{L} \times \mathcal{M}) \end{array}$$

Proof. It follows from Lemma 1.14 that w_0 respects the usual multiplication in degree zero. As w_0 is natural with respect to projections, it also commutes with the cross product. Now take $x \in \mathrm{GW}^{-i}_Z(X;\mathcal{L})$ and $y \in \mathrm{GW}^{-j}(Y;\mathcal{M})$. Then

$$\begin{aligned} w_{-i}(x) \times w_{-j}(y) \times e^{i+j} &= w_{-i}(x) \times e^i \times w_{-j}(y) \times e^j \\ &\quad \text{(No signs occur as all classes are of even degree.)} \\ &= w_0(x \times \varepsilon^i) \times w_0(y \times \varepsilon^j) \\ &= w_0(x \times \varepsilon^i \times y \times \varepsilon^j) \\ &\quad \text{(as } w_0 \text{ respects multiplication)} \\ &= w_0(x \times y \times \varepsilon^{i+j}) \\ &= w_{-i-j}(x \times y) \times e^{i+j}. \end{aligned}$$

So $w_{-i}(x) \times w_{-j}(y) = w_{-i-j}(x \times y)$. □

2.3 Realification and complexification

Just as the usual Witt group W^0, the shifted groups W^i can be written as quotients

$$\mathrm{K}_0X \xrightarrow{H_i} \mathrm{GW}^iX \longrightarrow \mathrm{W}^iX \longrightarrow 0,$$

where H_i is the "i^{th} hyperbolic map", sending the class of a complex $E_\bullet$ to the class of the symmetric i-space $\left(E_\bullet \oplus E_\bullet^\vee[i], \begin{pmatrix} 0 & \mathrm{id} \\ (-\mathrm{id})^{\frac{i(i+1)}{2}} & 0 \end{pmatrix}\right)$. In general, H_i is not injective. However, using the forgetful maps $F\colon \mathrm{GW}^{i-1}X \to \mathrm{K}_0X$, we can extend the sequences one term to the left, so that we have exact sequences

$$\mathrm{GW}^{i-1}X \xrightarrow{F} \mathrm{K}_0X \xrightarrow{H_i} \mathrm{GW}^iX \longrightarrow \mathrm{W}^iX \longrightarrow 0 \tag{9}$$

(cf. [11]). A sort of analogue in topological K-theory is the "Bott sequence" [23], by which we mean the following long exact sequence:

$$\begin{aligned} \ldots \to \mathrm{KO}^{2i-1}X \to \mathrm{KO}^{2i-2}X \to \mathrm{K}^0X \to \mathrm{KO}^{2i}X \to \mathrm{KO}^{2i-1}X \to \mathrm{K}^1X \\ \to \mathrm{KO}^{2i+1}X \to \mathrm{KO}^{2i}X \to \mathrm{K}^0X \to \mathrm{KO}^{2i+2}X \to \mathrm{KO}^{2i+1}X \to \mathrm{K}^1X \to \ldots \end{aligned} \tag{10}$$

Here, the maps into K^d (for $d = 0, 1$) are the composites of complexification and multiplication with an appropriate power of $g \in \mathrm{K}^{-2}(\text{point})$, the maps starting from K^d are the composites of multiplication with some power of g and realification, and the maps between the KO-groups are given by multiplication with $\eta \in \mathrm{KO}^{-1}(\text{point})$.

We have already seen that, in degree zero, H and F correspond to realification and complexification. The aim of this section is to show that, more generally, the corresponding parts of the above two exact sequences form a commutative diagram

$$\begin{array}{ccccccccc} \mathrm{GW}^{i-1}X & \xrightarrow{F} & \mathrm{K}_0X & \xrightarrow{H_i} & \mathrm{GW}^iX & \longrightarrow & \mathrm{W}^iX & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \mathrm{KO}^{2i-2}(X_\mathbb{C}) & \longrightarrow & \mathrm{K}^0(X_\mathbb{C}) & \longrightarrow & \mathrm{KO}^{2i}(X_\mathbb{C}) & \longrightarrow & \frac{\mathrm{KO}}{\mathrm{K}}^{2i}(X_\mathbb{C}) & \longrightarrow & 0 \end{array} \tag{11}$$

This is shown in Propositions 2.4 and 2.5 below. As a first step towards their proofs, we compare the multiplicative behaviour of the maps involved. F and c are both multiplicative, but for ρ we only have $\rho(c(x) \cdot y) = x \cdot \rho(y)$. A corresponding equation holds for H_i:

Lemma 2.3. *For $x \in \mathrm{GW}^kX$ and $y \in \mathrm{K}_0X$ we have $H_i(Fx \cdot y) = x \cdot H_{i-k}(y)$.*

Proof. It suffices to show this on the level of chain complexes. So let $F_\bullet$ be a chain complex representing y, and let $(E_\bullet, \phi)$ be a chain complex with a symmetric form $\phi\colon E_\bullet \to E_\bullet^\vee[k]$ representing x. The idea of the proof is that, for i and k equal to zero, the left-hand side of the equation is represented by $[(E_\bullet \otimes F_\bullet) \oplus (E_\bullet^\vee \otimes F_\bullet^\vee), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)]$, the right-hand side is represented by $\left[(E_\bullet \otimes F_\bullet) \oplus (E_\bullet \otimes F_\bullet^\vee), \left(\begin{smallmatrix} 0 & \phi \\ \phi & 0 \end{smallmatrix}\right)\right]$, and that these two representatives are isometric via $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \phi \end{smallmatrix}\right)$.

However, there is some work involved in taking care of arbitrary shifts. The problem is that these do not commute with the tensor product, i.e. we do not have $E_\bullet[i] \otimes F_\bullet[j] = (E_\bullet \otimes F_\bullet)[i+j]$ in general. Of course, the complexes are equal in each degree, but the differentials differ by signs. We do, however, have natural isomorphisms

$$l^j : E_\bullet[j] \otimes F_\bullet \to (E_\bullet \otimes F_\bullet)[j] \quad \text{and} \quad r^j : E_\bullet \otimes F_\bullet[j] \to (E_\bullet \otimes F_\bullet)[j].$$

Using these, the form on the product of a symmetric i-space $(E_\bullet, \alpha)$ with a symmetric j-space $(F_\bullet, \beta)$ may be written as $(-1)^{ij}$ times the composition

$$\begin{aligned} &E_\bullet \otimes F_\bullet \xrightarrow{\mathrm{id}\otimes\beta} (E_\bullet \otimes F_\bullet^\vee[j]) \xrightarrow{r^j} (E_\bullet \otimes F_\bullet^\vee)[j] \\ &\xrightarrow{(\alpha\otimes\mathrm{id})[j]} (E_\bullet^\vee[i] \otimes F_\bullet^\vee)[j] \xrightarrow{l^i[j]} (E_\bullet^\vee \otimes F_\bullet^\vee)[i][j] \cong (E_\bullet \otimes F_\bullet)^\vee[i+j] \end{aligned}$$

– cf. [8], especially Example 1.4 and Theorem 2.9. We will use the following identities (cf. Lemmas 1.3 and 1.12 of the same paper):

- $(l^j)^\vee[j] = l^j$
- $(r^j)^\vee[j] = r^j$
- $r^i[j] \circ l^j = (-1)^{ij} \cdot l^j[i] \circ r^i$

Given these, the calculation runs as follows: On the one hand we have

$$H_i\left(F(E_\bullet)\cdot F_\bullet\right) = \left[(E_\bullet \otimes F_\bullet) \oplus (E_\bullet^\vee \otimes F_\bullet^\vee)[i], \begin{pmatrix} 0 & \mathrm{id} \\ (-\mathrm{id})^{\frac{i(i+1)}{2}} & 0 \end{pmatrix}\right],$$

whereas on the other hand we have

$$\begin{aligned} &E_\bullet \cdot H_{i-k}(F_\bullet) = \left[(E_\bullet, \phi)\cdot\left(F_\bullet \oplus F_\bullet^\vee[i-k], \begin{pmatrix} 0 & \mathrm{id} \\ (-\mathrm{id})^{\frac{(i-k)(i-k+1)}{2}} & 0 \end{pmatrix}\right)\right] \\ &= \left[E_\bullet \otimes (F_\bullet \oplus F_\bullet^\vee[i-k]), (-1)^{k(i-k)} \cdot l^k[i-k] \circ (\phi\otimes\mathrm{id})[i-k] \circ r^{i-k} \circ \left(\mathrm{id} \otimes \begin{pmatrix} 0 & \mathrm{id} \\ (-\mathrm{id})^{\frac{(i-k)(i-k+1)}{2}} & 0 \end{pmatrix}\right)\right] \\ &= \left[(E_\bullet \otimes F_\bullet) \oplus (E_\bullet \otimes F_\bullet^\vee[i-k]), \begin{pmatrix} 0 & \psi \\ (-1)^{\frac{(i-k)(i-k+1)}{2}} \cdot \psi & 0 \end{pmatrix}\right], \end{aligned}$$

where

$$\psi := (-1)^{k(i-k)} \cdot l^k[i-k] \circ (\phi \otimes \mathrm{id})[i-k] \circ r^{i-k}.$$

We claim that these two representatives are isometric via $g := \begin{pmatrix} \mathrm{id} & 0 \\ 0 & \psi \end{pmatrix}$, i.e. that

$$\begin{aligned} g^\vee[i] \begin{pmatrix} 0 & \mathrm{id} \\ (-\mathrm{id})^{\frac{i(i+1)}{2}} & 0 \end{pmatrix} g &= \begin{pmatrix} \mathrm{id} & 0 \\ 0 & \psi^\vee[i] \end{pmatrix} \begin{pmatrix} 0 & \mathrm{id} \\ (-\mathrm{id})^{\frac{i(i+1)}{2}} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{id} & 0 \\ 0 & \psi \end{pmatrix} \\ &= \begin{pmatrix} 0 & \psi \\ (-1)^{\frac{i(i+1)}{2}} \psi^\vee[i] & 0 \end{pmatrix} \end{aligned}$$

is equal to $\begin{pmatrix} 0 & \psi \\ (-1)^{\frac{(i-k)(i-k+1)}{2}} \cdot \psi & 0 \end{pmatrix}$.

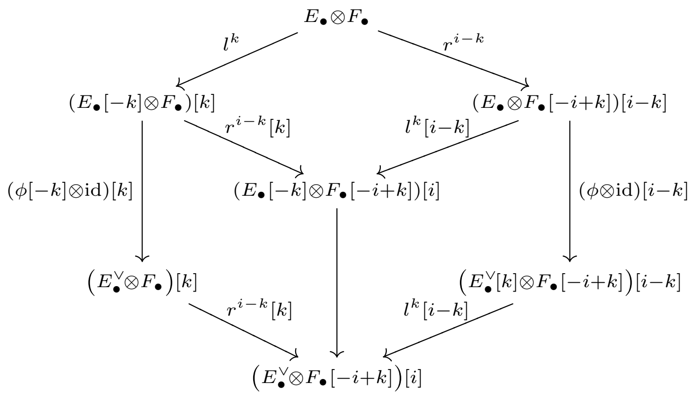
We check the bottom left entry. As ϕ is a symmetric k-form,

$$\phi^\vee = \phi^\vee[k][-k] = (-1)^{\frac{k(k+1)}{2}} \phi[-k].$$

Using this and the first two of the three identities above, we see that

$$\begin{aligned} \psi^\vee[i] &= (-1)^{k(i-k)} \cdot (r^{i-k})^\vee[i] \circ (\phi^\vee \otimes \mathrm{id})[k] \circ (l^k)^\vee[k] \\ &= (-1)^{k(i-k)+\frac{k(k+1)}{2}} \cdot r^{i-k}[k] \circ (\phi[-k] \otimes \mathrm{id})[k] \circ l^k. \end{aligned}$$

The composition of maps involved here is the left path of the following diagram.



The tilted square at the is top $(-1)^{k(i-k)}$-commutative by the third of the three identities above, and the other two squares commute by naturality of r and l, respectively. So $\psi^\vee[i]$ can alternatively be written in terms of the right path of the diagram:

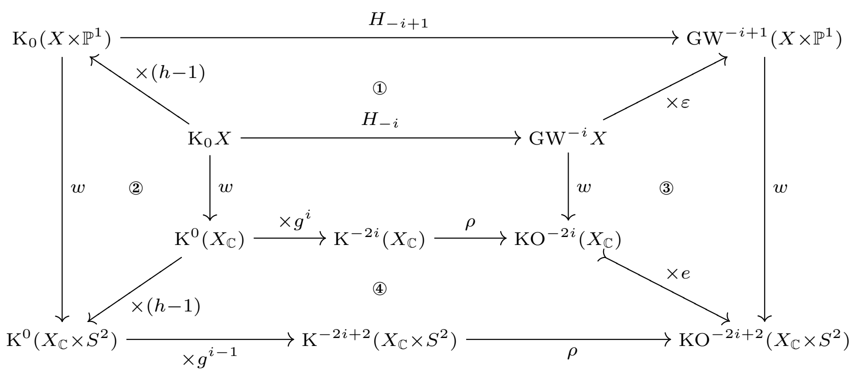
$$\begin{aligned}\psi^\vee[i] &= (-1)^{k(i-k)+\frac{k(k+1)}{2}} \cdot (-1)^{k(i-k)} \cdot l^k[i-k] \circ (\phi \otimes \mathrm{id})[i-k] \circ r^{i-k} \\ &= (-1)^{k(i-k)+\frac{k(k+1)}{2}} \cdot \psi\end{aligned}$$

The sign here multiplied by $(-1)^{\frac{i(i+1)}{2}}$ is in fact equal to $(-1)^{\frac{(i-k)(i-k+1)}{2}}$. This finishes the proof. □

Proposition 2.4. *Let $g \in \mathrm{K}^{-2}(\mathrm{point}) \cong \widetilde{\mathrm{K}}^0(S^2)$ be the generator corresponding to the reduced Hopf bundle $\mathcal{O}(-1) - \mathbb{C}$ (cf. (5)). We have commutative diagrams*

$$\begin{array}{ccccc} \mathrm{K}_0 X & & \xrightarrow{\quad H_{-i} \quad} & & \mathrm{GW}^{-i} X \\ \downarrow w & & & & \downarrow w \\ \mathrm{K}^0(X_{\mathbb{C}}) & \xrightarrow{\times g^i} & \mathrm{K}^{-2i}(X_{\mathbb{C}}) & \xrightarrow{\rho} & \mathrm{KO}^{-2i}(X_{\mathbb{C}}) \end{array}$$

Proof. We have already seen this for $i = 0$. Now we proceed by induction. Assume the proposition holds for some $i-1$. We can fit the i^{th} square for X into the $(i-1)^{th}$ square for $X \times \mathbb{P}^1$:



Here, h and 1 denote the elements of $\mathrm{K}_0(\mathbb{P}^1)$ and $\mathrm{K}^0(S^2)$ given by the Hopf bundle and the trivial line bundle, respectively. By the induction hypothesis, the outer square commutes. Also, ② commutes as w maps the Hopf bundle to the Hopf bundle, and ③ commutes by the very definition of w in lower degrees.

To see that ① commutes, recall from Example 1.19 that $F\varepsilon = (h-1)$. Therefore, for any $x \in \mathrm{K}_0 X$, we can use Lemma 2.3 to check that

$$H_{-i+1}(x \times (h-1)) = H_{-i+1}(x \times F\varepsilon) = H_{-i}(x) \times \varepsilon.$$

For ④, we use that, in general, $\rho(x) \times y = \rho(x \times c(y))$. Moreover, the class $c(e) \in \widetilde{\mathrm{K}}^0(S^2)$ corresponds to $1 \in \mathrm{K}^0(\text{point})$ under twofold suspension, whence the inverse of this twofold suspension isomorphism is given by cross product with $c(e)$,

$$\widetilde{\mathrm{K}}^j X \xrightarrow[\times c(e)]{\cong} \widetilde{\mathrm{K}}^{j+2}(S^2 X),$$

for any space X. In particular, by definition of g,

$$(h-1) = g \times c(e) \in \mathrm{K}^0(S^2).$$

So

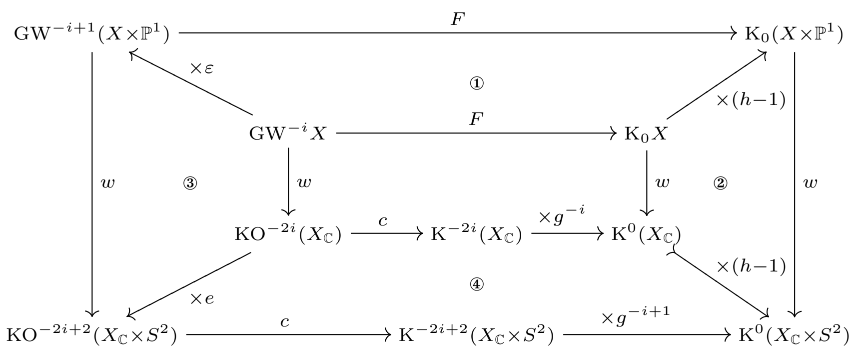
$$\begin{aligned} \rho(x \times g^i) \times e &= \rho(x \times g^i \times c(e)) \\ &= \rho(x \times g^{i-1} \times (h-1)) \\ &= \rho(x \times (h-1) \times g^{i-1}). \end{aligned}$$

As the multiplication $\times e$ is injective, it follows that the inner square must also commute. □

Proposition 2.5. *The forgetful map* $\mathrm{GW}^{-i}X \to \mathrm{K}_0 X$ *corresponds to complexification, i.e. with notation as before, we have commutative diagrams*

$$\begin{array}{ccccc} \mathrm{GW}^{-i}X & & \xrightarrow{\quad F \quad} & & \mathrm{K}_0 X \\ \downarrow w & & & & \downarrow w \\ \mathrm{KO}^{-2i}(X_{\mathbb{C}}) & \xrightarrow{c} & \mathrm{K}^{-2i}(X_{\mathbb{C}}) & \xrightarrow{\times g^i} & \mathrm{K}^0(X_{\mathbb{C}}) \end{array}$$

Proof. The proof is completely analogous to the preceding one. The statement is clear for $i = 0$, and for the induction step we study the same kind of diagram:



Again, the outer square commutes by the induction hypothesis, and ② and ③ commute for the same reasons as before. Square ① commutes as the forgetful map F respects multiplication, so $F(x \times \varepsilon) = F(x) \times F(\varepsilon) = F(x) \times (h-1)$. Commutativity of ④ is again due to the fact that $(h-1) = g \times c(e)$:

$$\begin{aligned} c(x) \times g^{-i} \times (h-1) &= c(x) \times g^{-i+1} \times c(e) \\ &= c(x) \times c(e) \times g^{-i+1} \\ &= c(x \times e) \times g^{-i+1} \end{aligned}$$

Lastly, note that cross product with $(h-1) \in \mathrm{K}^0(S^2)$ is injective: The element is invertible, so if X is any space and σ is the projection $X \times S^2 \twoheadrightarrow S^2$, cup product with $\sigma^*(h-1)$ is an isomorphism. Hence we have

$$\begin{array}{ccc} \mathrm{K}^0 X & \xrightarrow{\times(h-1)} & \mathrm{K}^0(X \times S^2) \\ \big\downarrow \pi^* & \overset{\cong}{\nearrow}{\scriptstyle \cup\sigma^*(h-1)} & \\ \mathrm{K}^0(X \times S^2) & & \end{array}$$

□

2.4 Periodicity

Proposition 2.6. *The map* $w\colon \mathrm{GW}^i X \to \mathrm{KO}^{2i}(X_{\mathbb{C}})$ *respects the periodicity isomorphisms* $\mathrm{GW}^i X \cong \mathrm{GW}^{i-4} X$ *and* $\mathrm{KO}^j(X_{\mathbb{C}}) \cong \mathrm{KO}^{j-8}(X_{\mathbb{C}})$.

Proof. The periodicity isomorphism for Grothendieck-Witt groups is induced by shifting complexes two to the right: if $(E_\bullet, \phi)$ is an i-space, $E_\bullet[-2]$ carries a natural symmetric $(i-4)$-form. Alternatively, the periodicity isomorphism can be interpreted as cross product with $\Lambda := [\mathbb{C}[-2], \mathrm{id}] \in \mathrm{GW}^{-4}(\text{point})$, where $\mathbb{C}[-2]$ denotes the complex consisting of the trivial line bundle concentrated in degree -2.

In topology, recall that $\mathrm{KO}^*(\text{point}) = \mathbb{Z}[\eta, \alpha, \lambda, \lambda^{-1}]$, with η, α and λ of degrees -1, -4 and -8, respectively, subject to the relations $2\eta = 0$, $\eta^3 = 0$ and $\alpha^2 = 4\lambda$. The periodicity isomorphism is given by multiplication with λ. Note that the last relation fixes the sign of this element.

Of course, we would now like to show that w maps Λ to λ. Using the results of the previous section, we have a row-exact commutative diagram

$$\begin{array}{ccccccc} \mathrm{GW}^{-5}(\text{point}) & \longrightarrow & \mathrm{K}_0(\text{point}) & \xrightarrow{H_{-4}} & \mathrm{GW}^{-4}(\text{point}) & \twoheadrightarrow & \mathrm{W}^{-4}(\text{point}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{KO}^{-10}(\text{point}) & \longrightarrow & \mathrm{K}^0(\text{point}) & \xrightarrow{\rho(\cdot \times g^4)} & \mathrm{KO}^{-8}(\text{point}) & \twoheadrightarrow & \frac{\mathrm{KO}}{\mathrm{K}}^{-8}(\text{point}) \end{array}$$

$\mathrm{GW}^{-5}(\text{point})$ and $\mathrm{KO}^{-10}(\text{point})$ are both isomorphic to $\mathbb{Z}_2$. The K-groups of a point are $\mathbb{Z}$ in degree zero, so each of the two horizontal maps on the far left must be zero. Calculating the other groups, we obtain

$$\begin{array}{ccccc} \mathbb{Z}\cdot 1 & \overset{H_{-4}}{\rightarrowtail} & \mathbb{Z}\cdot\Lambda & \twoheadrightarrow & \mathbb{Z}_2 \\ \cdot 1 \big\downarrow \cong & & \cdot a \big\downarrow & & \cdot\bar{a} \big\downarrow \\ \mathbb{Z}\cdot 1 & \overset{\rho(\cdot\times g^4)}{\rightarrowtail} & \mathbb{Z}\cdot\lambda & \twoheadrightarrow & \mathbb{Z}_2 \end{array}$$

So $H_{-4}(1) = \pm 2\Lambda$, $\rho(g^4) = \pm 2\lambda$, and w maps Λ to $\pm\lambda$. It remains to show that all signs are positive.

For Λ we have $H_{-4}(1) = H_{-4}(F(\Lambda)) = 2\Lambda$. To calculate $\rho(g^4)$, consider the following part of the Bott sequence (10) of a point:

$$\begin{array}{ccccccc} \mathrm{KO}^{-4}(\text{point}) & \xrightarrow{c(\cdot)\times g^{-2}} & \mathrm{K}^0(\text{point}) & \xrightarrow{\rho(\cdot\times g)} & \mathrm{KO}^{-2}(\text{point}) & \longrightarrow & \mathrm{KO}^{-3}(\text{point}) \\ \| & & \| & & \| & & \| \\ \mathbb{Z}\cdot\alpha & & \mathbb{Z}\cdot 1 & & \mathbb{Z}_2 & & 0 \end{array}$$

We see that $c(\alpha) = \pm 2g^2$. Therefore, $8\lambda = 2\alpha^2 = \rho(c(\alpha^2)) = \rho(4g^4)$. So $\rho(g^4) = 2\lambda$, and w maps Λ to λ. □

2.5 Thom isomorphisms

In algebraic K-theory, groups supported in a subvariety can often be expressed as the groups of the support itself. This is true in particular when Z is a smooth closed subvariety of a smooth complex variety X. We then have a "dévissage isomorphism" $\mathrm{K}_0 Z \to \mathrm{K}_0^Z X$ (e.g. [14, 1.20 and 1.23]). On the level of vector bundles and complexes, this is given by pushing forward a vector bundle on Z to X and then taking a free resolution. For Grothendieck-Witt groups, the situation is slightly more complicated, as shifts and twists have to be taken into account. We will state the precise form of the corresponding isomorphisms in the next section, where we will show that our map w is compatible with these. As a first step towards this goal, we consider here the special case that X is a vector bundle and Z is its base space.

Proposition 2.7. *Let $p\colon \mathcal{E} \to Z$ be a vector bundle over a smooth complex variety Z. Then w commutes with the Thom isomorphisms*

$$\begin{array}{cc} \mathrm{K}_0 Z \to \mathrm{K}_0^Z \mathcal{E} & \mathrm{K}^0(Z_{\mathbb{C}}) \to \mathrm{K}^0_{Z_{\mathbb{C}}}(\mathcal{E}_{\mathbb{C}}) \\ \mathrm{GW}^i(Z;\mathcal{L}) \to \mathrm{GW}^i_Z(\mathcal{E};\mathcal{L}') & \mathrm{KO}^{2i}(Z_{\mathbb{C}};\mathcal{L}) \to \mathrm{KO}^{2i}_{Z_{\mathbb{C}}}(\mathcal{E}_{\mathbb{C}};\mathcal{L}') \\ \mathrm{W}^i(Z;\mathcal{L}) \to \mathrm{W}^i_Z(\mathcal{E};\mathcal{L}') & \frac{\mathrm{KO}}{\mathrm{K}}^{2i}(Z_{\mathbb{C}};\mathcal{L}) \to \frac{\mathrm{KO}}{\mathrm{K}}^{2i}_{Z_{\mathbb{C}}}(\mathcal{E}_{\mathbb{C}};\mathcal{L}') \end{array}$$

Here, $\mathcal{L}$ is an arbitrary line bundle on Z, and $\mathcal{L}' := p^*\mathcal{L} \otimes p^*\det \mathcal{E}^\vee$.

In each case, the isomorphism is the composite of the pullback by p with multiplication by an element in the cohomology of $\mathcal{E}$ supported on Z, called "Thom class" of $\mathcal{E}$. The pullback by p is an isomorphism by homotopy invariance, and it commutes with w as it is inverse to the pullback along a closed embedding. So if suffices to concentrate on the second step.

We begin by looking at a Thom class for K_0. Given any vector bundle $p\colon \mathcal{E} \to Z$ with a section $s\colon \mathcal{O}_Z \to \mathcal{E}$, we can form the Koszul complex $K_\bullet(s)$,

$$0 \to \bigwedge^{\mathrm{rk}\,\mathcal{E}} \mathcal{E}^\vee \to \bigwedge^{\mathrm{rk}\,\mathcal{E}-1} \mathcal{E}^\vee \to \cdots \to \bigwedge^{2} \mathcal{E}^\vee \to \mathcal{E}^\vee \xrightarrow{s^\vee} \mathcal{O}_Z \to 0,$$

with $\mathcal{O}_Z$ in degree zero. The support S of $K_\bullet(s)$ is the zero-set of s. Moreover, if s is a regular section, $K_\bullet(s)$ is a resolution of the pushforward of $\mathcal{O}_S$ to Z (cf. Fulton [18, App. B]). In particular, if we consider the pullback $p^*\mathcal{E}$ of $\mathcal{E}$ to itself, the canonical section yields a complex $K_\bullet\mathcal{E}$ defining an element in $\mathrm{K}_0^Z\mathcal{E}$. As $K_\bullet\mathcal{E}$ is a resolution of the pushforward of $\mathcal{O}_Z$ to $\mathcal{E}$, this element is the image of

$1 \in \mathrm{K}_0 Z$ under the dévissage isomorphism $\mathrm{K}_0 Z \to \mathrm{K}_0^Z \mathcal{E}$. In fact, if we identify $\mathrm{K}_0 Z$ with $\mathrm{K}_0 \mathcal{E}$ via p^*, the dévissage isomorphism becomes multiplication by $K_\bullet \mathcal{E}$. So $K_\bullet \mathcal{E}$ represents a Thom class for $\mathcal{E}$.

In topology, given an even-dimensional complex vector bundle $\mathcal{E}$ over a space Z, any element $\theta \in \mathrm{K}^0_Z \mathcal{E}$ that restricts to a generator of $K^0_{\{z\}}(\mathcal{E}_z)$ for each fibre $\mathcal{E}_z$ is a Thom class inducing an isomorphism $\mathrm{K}^0 \mathcal{E} \to \mathrm{K}^0_Z \mathcal{E}$ (eg. [24, § 12]). The first part of Proposition 2.7 now follows from the following observation.

Proposition 2.8. *The map w sends the class of the Koszul complex of an algebraic vector bundle to a Thom class of the underlying complex continuous bundle.*

Proof. Let $\mathcal{E}$ be a vector bundle of rank r. We have to show that the restriction of $w(K_\bullet \mathcal{E}) \in \mathrm{K}^0_{Z_\mathbb{C}}(\mathcal{E}_\mathbb{C})$ to any fibre is a generator of $\mathrm{K}^0_{\{0\}}\, \mathbb{C}^r = \widetilde{\mathrm{K}}^0(S^{2r}) \cong \mathbb{Z}$. The inclusion $i_x \colon \mathcal{E}_x \hookrightarrow \mathcal{E}$ of the fibre over some point $x \in Z$ is a closed embedding, so w commutes with its pullback and the following square is commutative.

$$\begin{array}{ccc}
\mathrm{K}_0^{\{0\}} \mathbb{A}^r & \xleftarrow{\;i_x^*\;} & \mathrm{K}_0^Z \mathcal{E} \\
\downarrow w & & \downarrow w \\
\mathrm{K}^0_{\{0\}} \mathbb{C}^r & \xleftarrow{\;i_x^*\;} & \mathrm{K}^0_{Z_\mathbb{C}}(\mathcal{E}_\mathbb{C})
\end{array}$$

So we only have to check that the Koszul complex of the trivial complex vector bundle of rank r is a generator of $\mathrm{K}^0_{\{0\}}\, \mathbb{C}^r$. This is done in [22, § 2.6, p. 99]. In fact, both of the groups on the left of the above diagram are equal to $\mathbb{Z}$, generated by the Koszul complex, and $w \colon \mathrm{K}_0^{\{0\}} \mathbb{A}^r \to \mathrm{K}^0_{\{0\}} \mathbb{C}^r$ is an isomorphism. $\square$

We proceed in the same way for Grothendieck-Witt groups. For any vector bundle $p \colon \mathcal{E} \to Z$ of rank r, the Koszul complex $K_\bullet \mathcal{E}$ is canonically isomorphic to $(K_\bullet \mathcal{E}^\vee \otimes \det p^* \mathcal{E}^\vee)[r]$, so we obtain a "symmetric Koszul space" $(K_\bullet \mathcal{E}, \kappa_\mathcal{E})$. This defines an element in each of the groups $\mathrm{GW}^r_Z(\mathcal{E}; \det p^* \mathcal{E}^\vee)$ and $\mathrm{W}^r_Z(\mathcal{E}; \det p^* \mathcal{E}^\vee)$, inducing maps

$$\mathrm{GW}^i(\mathcal{E}; \mathcal{L}) \to \mathrm{GW}^{i+r}_Z(\mathcal{E}; \mathcal{L} \otimes \det p^* \mathcal{E}^\vee)$$
$$\mathrm{W}^i(\mathcal{E}; \mathcal{L}) \to \mathrm{W}^{i+r}_Z(\mathcal{E}; \mathcal{L} \otimes \det p^* \mathcal{E}^\vee)$$

Nenashev shows in [9, Theorem 2.5] that the map on Witt groups is an isomorphism. As we already know that multiplication with $K_\bullet \mathcal{E}$ is an isomorphism on K-groups, we can deduce that the map on Grothendieck-Witt groups is also an isomorphism from the exact sequence (9) in Section 2.3 and Lemma 2.3.

In order to carry Proposition 2.8 over to Grothendieck-Witt groups, we begin by computing the GW-groups of a fibre supported in a point explicitly.

Lemma 2.9. *Let r denote the trivial bundle of rank r over $\mathbb{A}^r$, and let $(K_\bullet(r), \kappa)$ be its symmetric Koszul space. Then*

$$\mathrm{GW}^i_{\{0\}} \mathbb{A}^r = \begin{cases} \mathbb{Z} \cdot (K_\bullet(r), \kappa) & \text{if } i \equiv r \ (\text{mod } 4) \\ 0 & \text{if } i \equiv r+1 \\ \mathbb{Z} \cdot H_i\,(K_\bullet(r)) & \text{if } i \equiv r+2 \\ \mathbb{Z}_2 \cdot H_i\,(K_\bullet(r)) & \text{if } i \equiv r+3 \end{cases}$$

Proof. We have already seen that

$$\mathrm{K}_0^{\{0\}}\mathbb{A}^r = \mathbb{Z}\cdot K_\bullet(r).$$

The Witt groups are computed in [6, Theorem 8.2]:

$$\mathrm{W}^i_{\{0\}}\mathbb{A}^r = \begin{cases} \mathbb{Z}_2\cdot (K_\bullet(r),\kappa) & \text{if } i\equiv r \pmod 4\\ 0 & \text{otherwise}\end{cases}$$

The idea is now to use the exact sequence (9):

$$\mathrm{GW}^{i-1}_{\{0\}}\mathbb{A}^r \xrightarrow{F} \mathrm{K}_0^{\{0\}}\mathbb{A}^r \xrightarrow{H_i} \mathrm{GW}^i_{\{0\}}\mathbb{A}^r \twoheadrightarrow \mathrm{W}^i_{\{0\}}\mathbb{A}^r$$

First, taking $i=r$, we see that $(K_\bullet(r),\kappa)$ must represent a non-zero class in $\mathrm{GW}^r_{\{0\}}\mathbb{A}^r$. Taking $i=r+1$, it follows that $F\colon \mathrm{GW}^r_{\{0\}}\mathbb{A}^r\to \mathrm{K}_0^{\{0\}}\mathbb{A}^r$ is a surjection and $\mathrm{GW}^{r+1}_{\{0\}}\mathbb{A}^r$ is zero. For $i=r+2$, the sequence now reduces to an isomorphism between K_0 and GW^{r+2}. So indeed $\mathrm{GW}^{r+2}_{\{0\}}\mathbb{A}^r$ is $\mathbb{Z}\cdot H_{r+2}\left(K_\bullet(r)\right)$.

To get any further, we need to calculate $F\left(H_{r+2}(K_\bullet(r))\right)$. Note that $K_\bullet(r)$ and $K_\bullet(r)^\vee$ are isomorphic except for a shift by r, so they only differ by the sign $(-1)^r$ in $\mathrm{K}_0^{\{0\}}\mathbb{A}^r$. This allows us to compute $FH_j(K_\bullet(r))$ for any j explicitly:

$$\begin{aligned} FH_j(K_\bullet(r)) &= K_\bullet(r) + K_\bullet(r)^\vee[j] = K_\bullet(r) + (-1)^{r+j}K_\bullet(r)\\ &= \begin{cases} 2\cdot K_\bullet(r) & \text{if } r+j \text{ is even}\\ 0 & \text{if } r+j \text{ is odd}\end{cases}\end{aligned}$$

In particular, it follows that $F\colon \mathrm{GW}^{r+2}_{\{0\}}\mathbb{A}^r\to \mathrm{K}_0^{\{0\}}\mathbb{A}^r$ in the sequence with $i=r+3$ is multiplication by 2. So $\mathrm{GW}^{r+3}_{\{0\}}\mathbb{A}^r$ is $\mathbb{Z}_2\cdot H_{r+3}(K_\bullet(r))$. Finally, for $i=r$ again, F is zero, yielding a short exact sequence

$$\mathbb{Z}\cdot K_\bullet(r) \xrightarrow{H_r} \mathrm{GW}^r_{\{0\}}\mathbb{A}^r \twoheadrightarrow \mathbb{Z}_2\cdot (K_\bullet(r),\kappa)\,.$$

This leaves two possibilities for GW^r: $\mathbb{Z}\cdot(K_\bullet(r),\kappa)$ and $\mathbb{Z}\cdot H_r(K_\bullet(r))\oplus\mathbb{Z}_2$. However, $H_r(K_\bullet(r)) = H_rF(K_\bullet(r),\kappa) = 2\cdot (K_\bullet(r),\kappa)$, which would be impossible in the latter case. So $\mathrm{GW}^r_{\{0\}}\mathbb{A}^r = \mathbb{Z}\cdot(K_\bullet(r),\kappa)$, as claimed. □

Proposition 2.10. *The map w sends the class of the symmetric Koszul space of an algebraic vector bundle to a Thom class of the underlying real continuous bundle.*

Proof. Again, let $\mathcal{E}$ be a vector bundle of rank r. This time we have to show that the restriction of $w(K_\bullet\mathcal{E},\kappa_{\mathcal{E}})\in \mathrm{KO}^{2r}_{Z_{\mathbb{C}}}(\mathcal{E}_{\mathbb{C}};\det(p^*\mathcal{E}^\vee))$ to any fibre is a generator of $\mathrm{KO}^{2r}_{\{0\}}\mathbb{C}^r = \widetilde{\mathrm{KO}}^{2r}(S^{2r})\cong\mathbb{Z}$. As before, w commutes with this restriction:

$$\begin{array}{ccc} \mathrm{GW}^r_{\{0\}}\mathbb{A}^r & \xleftarrow[i_x^*]{} & \mathrm{GW}^r_Z(\mathcal{E};\det(p^*\mathcal{E}^\vee))\\ \downarrow w & & \downarrow w\\ \mathrm{KO}^{2r}_{\{0\}}\mathbb{C}^r & \xleftarrow[i_x^*]{} & \mathrm{KO}^{2r}_{Z_{\mathbb{C}}}(\mathcal{E}_{\mathbb{C}};\det(p^*\mathcal{E}^\vee)) \end{array}$$

As the restriction of $(K_\bullet\mathcal{E}, \kappa_\mathcal{E})$ is the generator $(K_\bullet(r), \kappa)$ of $\mathrm{GW}^r_{\{0\}}\,\mathbb{A}^r$, it suffices to show that w defines an isomorphism $\mathrm{GW}^r_{\{0\}} \to \mathrm{KO}^{2r}_{\{0\}}\mathbb{C}^r$.

To see this, note that it follows from the previous lemma that the forgetful map $F\colon \mathrm{GW}^r_{\{0\}}\mathbb{A}^r \to \mathrm{K}_0^{\{0\}}\mathbb{A}^r$ is an isomorphism. If we look at the Bott sequence (10) for $(\mathbb{C}^r, \mathbb{C}^r - \{0\})$, we see that the corresponding map from $\mathrm{KO}^{2r}_{\{0\}}\mathbb{C}^r$ to $\mathrm{K}^0_{\{0\}}\mathbb{C}^r$ is also an isomorphism:

$$\mathrm{KO}^{2r+1}_{\{0\}}\mathbb{C}^r \longrightarrow \mathrm{KO}^{2r}_{\{0\}}\mathbb{C}^r \longrightarrow \mathrm{K}^0_{\{0\}}\mathbb{C}^r \longrightarrow \mathrm{KO}^{2r+2}_{\{0\}}\mathbb{C}^r$$

The group on the far left is isomorphic to $\widetilde{\mathrm{KO}}^{2r+1}(S^{2r}) = \mathrm{KO}^1(\text{point})$ and the group on the far right is isomorphic to $\widetilde{\mathrm{KO}}^{2r+2}(S^{2r}) = \mathrm{KO}^2(\text{point})$, both of which are trivial.

So it follows from Proposition 2.5 and the fact that w is an isomorphism on $\mathrm{K}_0^{\{0\}}\mathbb{A}^r$ that it is also an isomorphism on $\mathrm{GW}^r_{\{0\}}\mathbb{A}^r$. □

2.6 Pushforwards

Proposition 2.11. *Let X be a smooth variety, and let $Z \subset X$ be a smooth closed subvariety of codimension c, with normal bundle $\mathcal{N}$. Given a line bundle $\mathcal{L}$ on X, let $\mathcal{L}'$ denote the line bundle $\mathcal{L}|_Z \otimes \det \mathcal{N}$ on Z. Then w commutes with the isomorphisms given by pushing forward along the closed embedding $Z \hookrightarrow X$:*

$$\begin{array}{cc}
\mathrm{K}_0 Z \to \mathrm{K}_0^Z X & \mathrm{K}^0(Z_\mathbb{C}) \to \mathrm{K}^0_{Z_\mathbb{C}}(X_\mathbb{C}) \\
\mathrm{GW}^i(Z;\mathcal{L}') \to \mathrm{GW}^{i+c}_Z(X;\mathcal{L}) & \mathrm{KO}^{2i}(Z_\mathbb{C};\mathcal{L}') \to \mathrm{KO}^{2i+2c}_{Z_\mathbb{C}}(X_\mathbb{C};\mathcal{L}) \\
\mathrm{W}^i(Z;\mathcal{L}') \to \mathrm{W}^{i+c}_Z(X;\mathcal{L}) & \frac{\mathrm{KO}}{\mathrm{K}}^{2i}(Z_\mathbb{C};\mathcal{L}') \to \frac{\mathrm{KO}}{\mathrm{K}}^{2i+2c}_{Z_\mathbb{C}}(X_\mathbb{C};\mathcal{L})
\end{array}$$

Proof. All of the isomorphisms are well-known. In the case of Witt groups, Nenashev gives a possible construction of such an isomorphism in [9]. Inspection shows that this same construction also works in the other cases, and that w is compatible with each of the steps involved.

In slightly more detail, Nenashev considers the composition

$$\mathrm{W}^i(Z;\mathcal{L}') \xrightarrow[\text{Thom}]{\cong} \mathrm{W}^{i+c}_Z(\mathcal{N}; p^*(\mathcal{L}|_Z)) \xrightarrow[d]{\cong} \mathrm{W}^{i+c}_Z(X;\mathcal{L}).$$

The first map is the Thom isomorphism of the preceding section. The second isomorphism, d, is (the inverse of) the so-called deformation to the normal bundle, as described in [9, Section 3]. Briefly, it is obtained as follows: From the smooth pair (X, Z), one constructs a smooth variety D commonly known as deformation space, containing $Z \times \mathbb{A}^1$ as a closed subspace and fibred over $\mathbb{A}^1$ such that

- the fibre over 1 is isomorphic to X, and the inclusion i_1 maps $Z \subset X$ to $Z \times 1 \subset Z \times \mathbb{A}^1$, whereas
- the fibre over 0 is isomorphic to $\mathcal{N}$, and the inclusion i_0 maps $Z \subset \mathcal{N}$ to $Z \times 0 \subset Z \times \mathbb{A}^1$.

Moreover, for any line bundle $\mathcal{L}$ on X, there exists a line bundle $\widetilde{\mathcal{L}}$ on D restricting to $\mathcal{L}$ over X and to $p^*\mathcal{L}|_Z$ over $\mathcal{N}$. Thus, we can consider the pullbacks

$$\mathrm{W}^i_Z(\mathcal{N}; p^*(\mathcal{L}|_Z)) \xleftarrow{i_0^*} \mathrm{W}^i_{Z\times\mathbb{A}^1}(D;\widetilde{\mathcal{L}}) \xrightarrow{i_1^*} \mathrm{W}^i_Z(X;\mathcal{L}).$$

It turns out that both pullbacks are isomorphisms on the Witt groups, so d can be defined as the inverse of i_0^* followed by i_1^*.

We claim that this approach also works in the other cases: Regarding Witt groups as a cohomology theory on smooth pairs, the key properties needed to prove that the pullbacks are isomorphisms are homotopy invariance, Mayer-Vietoris and Nisnevich excision. All of these properties are satisfied by (higher) algebraic K-theory, see for example [7]. For Grothendieck-Witt groups, we may deduce the fact that i_0 and i_1 induce isomorphisms from sequence (9). Moreover, generalised cohomology theories in topology always satisfy homotopy invariance, Mayer-Vietoris and Nisnevich excision, so the same construction works for the K- and KO-groups of a smooth complex variety. Then, as w is natural with respect to closed embeddings, it commutes with d. □

3 Application to smooth cellular varieties

3.1 The main theorem

We have now assembled all the tools we need to show that, in the special case of smooth cellular complex varieties, all versions of w are isomorphisms. So fix one such variety X. By definition, X has a filtration by closed subvarieties $\emptyset = Z_0 \subset Z_1 \subset Z_2 \cdots \subset Z_N = X$ such that each Z_{k+1} can be written as the disjoint union of Z_k and an open cell C_k isomorphic to $\mathbb{A}^{n_k}$ for some n_k. In general, the subvarieties Z_k will not be smooth. Their complements $X_k := X - Z_k$, however, are always smooth as they are open in X. So we obtain a filtration $X = X_0 \supset X_1 \supset X_2 \cdots \supset X_N = \emptyset$ of X by smooth open subvarieties X_k, each of which is a disjoint union of an open subvariety X_{k+1} and a closed cell C_k.

Theorem 3.1. *For a smooth cellular complex variety X, all versions of w are isomorphisms:*

$$\begin{aligned} \mathrm{K}_0 X &\xrightarrow{\cong} \mathrm{K}^0(X_{\mathbb{C}}) \\ \mathrm{GW}^i X &\xrightarrow{\cong} \mathrm{KO}^{2i}(X_{\mathbb{C}}) \\ \mathrm{W}^i X &\xrightarrow{\cong} \tfrac{\mathrm{KO}}{\mathrm{K}}^{2i}(X_{\mathbb{C}}) \cong \mathrm{KO}^{2i-1}(X_{\mathbb{C}}) \end{aligned}$$

This remains true for twisted groups.

Proof. If X is a single cell, the pullback along the projection to a point induces isomorphisms on all the groups above. As a map from $\mathrm{K}_0(\text{point})$ to $\mathrm{K}^0(\text{point})$ and as a map from $\mathrm{W}^i(\text{point})$ to $\frac{\mathrm{KO}}{\mathrm{K}}^{2i}(\text{point})$, w is an isomorphism. Hence it is also an isomorphism on the Grothendieck-Witt groups of a point (cf. diagram (11)). So w is an isomorphism on single cells.

More generally, the relative groups of any space supported on a closed cell are isomorphic to the corresponding groups of the cell itself via the pushforward along the closed embedding of that cell. The map w commutes with this pushforward by Proposition 2.11, so it is an isomorphism on all groups supported on a single closed cell.

Now we can induct on the number of cells using the filtration X_k described above: Given that we have isomorphisms on X_{k+1}, we have to show that we also have isomorphisms on X_k. For K-theory, this follows by applying the snake lemma to the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & \mathrm{K}_0^{C_k}(X_k) & \longrightarrow & \mathrm{K}_0 X_k & \longrightarrow & \mathrm{K}_0 X_{k+1} & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow w & & \downarrow \cong \text{ by induction} & & \\
0 & \longrightarrow & \mathrm{K}^0_{C_k}(X_k) & \longrightarrow & \mathrm{K}^0 X_k & \longrightarrow & \mathrm{K}^0 X_{k+1} & \longrightarrow & 0
\end{array}$$

Here, the first line is exact because C_k and X_k are smooth. Exactness of the bottom line comes from the fact that the K^1-groups of a CW complex with only even-dimensional cells are always zero. For the same reason, the Bott sequence (10) of such a complex falls apart into seven-term exact sequences ending in

$$\ldots \longrightarrow \mathrm{K}^0 \longrightarrow \mathrm{KO}^{2i} \xrightarrow{\eta} \mathrm{KO}^{2i-1} \longrightarrow 0\,.$$

Consequently, multiplication by η induces isomorphisms $\frac{\mathrm{KO}}{\mathrm{K}}^{2i} \xrightarrow{\cong} \mathrm{KO}^{2i-1}$. We can therefore rewrite parts of the long exact sequence of KO-groups of the pair (X_k, X_{k+1}) as

$$\ldots \longrightarrow \mathrm{KO}^{2i}_{C_k}(X_k) \longrightarrow \mathrm{KO}^{2i} X_k \longrightarrow \mathrm{KO}^{2i} X_{k+1}$$
$$\xrightarrow{\partial'} \frac{\mathrm{KO}}{\mathrm{K}}^{2i+2}_{C_k}(X_k) \to \frac{\mathrm{KO}}{\mathrm{K}}^{2i+2} X_k \to \frac{\mathrm{KO}}{\mathrm{K}}^{2i+2} X_{k+1} \to \ldots$$

These sequences can be compared to the localization sequences of the Grothendieck-Witt groups of $C_{k+1} \subset X_k$ (cf. (8) in Section 1.6):

$$\begin{array}{ccccccccccccc}
\mathrm{GW}^i_{C_k}(X_k) & \to & \mathrm{GW}^i X_k & \to & \mathrm{GW}^i X_{k+1} & \xrightarrow{\partial} & \mathrm{W}^{i+1}_{C_k}(X_k) & \longrightarrow & \mathrm{W}^{i+1} X_k & \longrightarrow & \mathrm{W}^{i+1} X_{k+1} \\
\downarrow \cong & & \downarrow w & & \downarrow \cong & ? & \downarrow \cong & & \downarrow w & & \downarrow \cong \\
\mathrm{KO}^{2i}_{C_k}(X_k) & \to & \mathrm{KO}^{2i} X_k & \to & \mathrm{KO}^{2i} X_{k+1} & \xrightarrow{\partial'} & \frac{\mathrm{KO}}{\mathrm{K}}^{2i+2}_{C_k}(X_k) & \to & \frac{\mathrm{KO}}{\mathrm{K}}^{2i+2} X_k & \to & \frac{\mathrm{KO}}{\mathrm{K}}^{2i+2} X_{k+1}
\end{array}$$

Commutativity of these diagrams is clear everywhere except for the squares with the boundary maps. In general, the question whether the two boundary maps ∂ and ∂' are compatible is non-trivial; their definitions have very little in common. However, in this particular case commutativity follows from rather simple considerations: $\mathrm{W}^{i+1}_{C_k}(X_k)$ is isomorphic to some Witt group of a point, so it is either 0 or $\mathbb{Z}_2$. If it is 0, the diagram commutes trivially. If it is $\mathbb{Z}_2$, there are again two possibilities:

If ∂' is zero, we have an injection $\frac{\mathrm{KO}}{\mathrm{K}}^{2i+2}_{C_k}(X_k) \rightarrowtail \frac{\mathrm{KO}}{\mathrm{K}}^{2i+2} X_k$. This implies that $\mathrm{W}^{i+1}_{C_k}(X_k) \to \mathrm{W}^{i+1} X_k$ is also an injection, whence ∂ must also be zero. So again the square commutes trivially.

If ∂' is non-zero, then $\mathrm{KO}^{2i} X_k \to \mathrm{KO}^{2i} X_{k+1}$ cannot be surjective. Hence $\mathrm{GW}^i X_k \to \mathrm{GW}^i X_{k+1}$ cannot be surjective either, so ∂ must also be non-zero. It follows that $w \circ \partial$ and $\partial' \circ w$ are two surjections onto $\mathbb{Z}_2$ such that $\partial' \circ w$ vanishes on the kernel of $w \circ \partial$. So both maps factor as $\mathrm{GW}^i X_{k+1} \twoheadrightarrow \frac{\mathrm{GW}^i X_{k+1}}{\ker \partial}$ followed by an isomorphism of $\frac{\mathrm{GW}^i X_{k+1}}{\ker \partial}$ with $\mathbb{Z}_2$. As there can only be one such isomorphism, $w \circ \partial$ and $\partial' \circ w$ must agree.

Given commutativity, the five lemma implies that we have surjections $\mathrm{GW}^i X_k \twoheadrightarrow \mathrm{KO}^{2i} X_k$ and injections $\mathrm{W}^i X_k \rightarrowtail \frac{\mathrm{KO}}{\mathrm{K}}^{2i} X_k$, for all i. Combining these with the isomorphisms on K-groups which we established at the beginning, we obtain diagrams of the form

$$\begin{array}{ccccccccc}
\mathrm{GW}^i X_k & \longrightarrow & \mathrm{K}_0 X_k & \longrightarrow & \mathrm{GW}^{i+1} X_k & \twoheadrightarrow & \mathrm{W}^{i+1} X_k & \longrightarrow & 0 \\
\downarrow\!\!\downarrow & & \downarrow \cong & & \downarrow\!\!\downarrow & & \downarrow & & \\
\mathrm{KO}^{2i} X_k & \longrightarrow & \mathrm{K}^0 X_k & \longrightarrow & \mathrm{KO}^{2i+2} X_k & \twoheadrightarrow & \frac{\mathrm{KO}}{\mathrm{K}}^{2i+2} X_k & \longrightarrow & 0
\end{array}$$

Applying the five lemma twice for different i or otherwise, it follows that, in fact, all of the vertical maps are isomorphisms. □

3.2 An example: projective spaces

Finally, we would like to demonstrate that our theorem is non-vacuous by considering the complex projective spaces $\mathbb{C}P^n$. The Picard-group of $\mathbb{C}P^n$ is isomorphic to $\mathbb{Z}$, generated by the hyperplane bundle $\mathcal{O}(1)$. This gives infinitely many groups $\mathrm{GW}^i(\mathbb{C}P^n; \mathcal{O}(l))$ and $\mathrm{W}^i(\mathbb{C}P^n; \mathcal{O}(l))$, one for each $l \in \mathbb{Z}$. However, in general, given a variety X with line bundles $\mathcal{L}$ and $\mathcal{M}$, the tensor product with $\mathcal{L}$ induces isomorphisms

$$\mathrm{GW}^i(X; \mathcal{M}) \cong \mathrm{GW}^i(X; \mathcal{M} \otimes \mathcal{L} \otimes \mathcal{L})$$
$$\mathrm{W}^i(X; \mathcal{M}) \cong \mathrm{W}^i(X; \mathcal{M} \otimes \mathcal{L} \otimes \mathcal{L})$$

So we need only consider values of l modulo 2, i.e. only untwisted groups and groups with coefficients in $\mathcal{O}(1)$.

Our theorem says that we can compute all of these groups purely topologically. One of the main computational tools in topological K-theory is the Atiyah-Hirzebruch spectral sequence, which relates the K- or KO-groups of a space to its singular cohomology. Applying this sequence to a CW complex X which has cells only in even dimensions, we see that

- $\mathrm{KO}^0 X = \mathrm{KO}^4 X = \mathbb{Z}^e \oplus$ 2-torsion, where e is the number of cells of X of even complex dimension,
- $\mathrm{KO}^2 X = \mathrm{KO}^6 X = \mathbb{Z}^o \oplus$ 2-torsion, where o is the number of cells of X of odd complex dimension, and
- $\mathrm{KO}^{2i+1} X =$ 2-torsion of $\mathrm{KO}^{2i} X$. [28, 2.1 and 2.2]

This gives the free parts of the usual groups $\mathrm{KO}^i(\mathbb{C}P^n)$ straightaway. The complete computations of $\mathrm{KO}^i(\mathbb{C}P^n)$ were first published in a 1967 paper by Fujii [26], in which he also determines most of the multiplicative structure. It is not difficult to deduce the values of the twisted groups $\mathrm{KO}^i(\mathbb{C}P^n; \mathcal{O}(1))$ from these results: the Thom space $\mathrm{Th}(O_{\mathbb{C}P^n}(1))$ is homeomorphic to $\mathbb{C}P^{n+1}$, so

$$\begin{aligned}
\mathrm{KO}^i(\mathbb{C}P^n; \mathcal{O}(1)) &= \widetilde{\mathrm{KO}}^{i+2}(\mathrm{Th}(\mathcal{O}(1))) \\
&= \widetilde{\mathrm{KO}}^{i+2}(\mathbb{C}P^{n+1}).
\end{aligned}$$

We summarise the additive structure in a table.

i	$\mathrm{KO}^i(\mathbb{C}P^n)$			$\mathrm{KO}^i(\mathbb{C}P^n;\mathcal{O}(1))$		
	n even	$n \equiv 1$	$n \equiv 3 \pmod 4$	n odd	$n \equiv 0$	$n \equiv 2 \pmod 4$
0	$\mathbb{Z}^{s+1}$	$\mathbb{Z}^{s+1} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{s+1}$	$\mathbb{Z}^t$	$\mathbb{Z}^{t+1}$	$\mathbb{Z}^{t+1}$
1	0	$\mathbb{Z}_2$	0	0	0	0
2	$\mathbb{Z}^s$	$\mathbb{Z}^{s+1}$	$\mathbb{Z}^{s+1}$	$\mathbb{Z}^t$	$\mathbb{Z}^t$	$\mathbb{Z}^t \oplus \mathbb{Z}_2$
3	0	0	0	0	0	$\mathbb{Z}_2$
4	$\mathbb{Z}^{s+1}$	$\mathbb{Z}^{s+1}$	$\mathbb{Z}^{s+1} \oplus \mathbb{Z}_2$	$\mathbb{Z}^t$	$\mathbb{Z}^{t+1}$	$\mathbb{Z}^{t+1}$
5	0	0	$\mathbb{Z}_2$	0	0	0
6	$\mathbb{Z}^s \oplus \mathbb{Z}_2$	$\mathbb{Z}^{s+1} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{s+1} \oplus \mathbb{Z}_2$	$\mathbb{Z}^t$	$\mathbb{Z}^t \oplus \mathbb{Z}_2$	$\mathbb{Z}^t$
7	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0

The KO-groups of $\mathbb{C}P^n$. Here, $s = \lfloor \frac{n}{2} \rfloor$ and $t = \lceil \frac{n}{2} \rceil$.

In the classical theory of Witt groups, Arason showed in 1980 that the Witt group $\mathrm{W}^0(\mathbb{P}^n)$ of a projective space $\mathbb{P}^n$ over a field k agrees with the Witt group of the field [3], so for $\mathbb{C}P^n$ we have $\mathrm{W}^0(\mathbb{C}P^n) = \mathbb{Z}_2$. Higher Witt and Grothendieck-Witt groups have only been computed recently by Walter [10]. Under our identifications, his result agrees with the one displayed above.

References

[1] Daniel Quillen, *Higher algebraic K-theory. I*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), Springer, Berlin, 1973, pp. 85–147. Lecture Notes in Math., Vol. 341.

[2] John Milnor and Dale Husemoller, *Symmetric bilinear forms*, Springer-Verlag, New York, 1973. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 73.

[3] Jón Kr. Arason, *Der Wittring projektiver Räume*, Math. Ann. **253** (1980), no. 3, 205–212 (German).

[4] Paul Balmer, *Triangular Witt groups. I. The 12-term localization exact sequence*, K-Theory **19** (2000), no. 4, 311–363.

[5] ———, *Triangular Witt groups. II. From usual to derived*, Math. Z. **236** (2001), no. 2, 351–382.

[6] Paul Balmer and Stefan Gille, *Koszul complexes and symmetric forms over the punctured affine space*, Proc. London Math. Soc. (3) **91** (2005), no. 2, 273–299.

[7] I. Panin, *Oriented cohomology theories of algebraic varieties*, K-Theory **30** (2003), no. 3, 265–314. Special issue in honor of Hyman Bass on his seventieth birthday. Part III.

[8] Stefan Gille and Alexander Nenashev, *Pairings in triangular Witt theory*, J. Algebra **261** (2003), no. 2, 292–309.

[9] Alexander Nenashev, *Gysin maps in Balmer-Witt theory*, J. Pure Appl. Algebra **211** (2007), no. 1, 203–221.

[10] Charles Walter, *Grothendieck-Witt groups of projective bundles* (2003), available at `http://www.math.uiuc.edu/K-theory/0644/`. Preprint.

[11] ———, *Grothendieck-Witt groups of triangulated categories* (2003), available at `http://www.math.uiuc.edu/K-theory/0643/`. Preprint.

[12] Marco Schlichting, *Hermitian K-theory of exact categories* (2008), available at `http://www.math.lsu.edu/~mschlich/research.html`. Accepted for publication in Journal of K-theory.

[13] ———, *The Mayer-Vietoris principle for Grothendieck-Witt groups of schemes* (2008), available at `http://www.math.lsu.edu/~mschlich/research.html`. Preprint.

[14] ———, *Higher algebraic K-theory* (2007), available at `http://www.math.lsu.edu/~mschlich/research/sedano.pdf`. Notes.

[15] ———, *Hermitian K-theory, derived equivalences and Karoubi's fundamental theorem* (2007), available at `http://www.math.lsu.edu/~mschlich/research/draftGWsch2.pdf`. Draft.

[16] Jens Hornbostel and Marco Schlichting, *Localization in Hermitian K-theory of rings*, J. London Math. Soc. (2) **70** (2004), no. 1, 77–124.

[17] Jens Hornbostel, *A^1-representability of Hermitian K-theory and Witt groups*, Topology **44** (2005), no. 3, 661–687.

[18] William Fulton, *Intersection theory*, 2nd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 2, Springer-Verlag, Berlin, 1998.

[19] Heisuke Hironaka, *Triangulations of algebraic sets*, Algebraic geometry (Proc. Sympos. Pure Math., Vol. 29, Humboldt State Univ., Arcata, Calif., 1974), Amer. Math. Soc., Providence, R.I., 1975, pp. 165–185.

[20] Masayoshi Nagata, *Imbedding of an abstract variety in a complete variety*, J. Math. Kyoto Univ. **2** (1962), 1–10.

[21] W. Lütkebohmert, *On compactification of schemes*, Manuscripta Math. **80** (1993), no. 1, 95–111.

[22] M. F. Atiyah, *K-theory*, Lecture notes by D. W. Anderson, W. A. Benjamin, Inc., New York-Amsterdam, 1967.

[23] Raoul Bott, *Lectures on $K(X)$*, Mathematics Lecture Note Series, W. A. Benjamin, Inc., New York-Amsterdam, 1969.

[24] M. F. Atiyah, R. Bott, and A. Shapiro, *Clifford modules*, Topology **3** (1964), no. suppl. 1, 3–38.

[25] M. F. Atiyah and E. Rees, *Vector bundles on projective 3-space*, Invent. Math. **35** (1976), 131–153.

[26] Michikazu Fujii, *K_0-groups of projective spaces*, Osaka J. Math. **4** (1967), 141–149. MR 0219060 (36 #2143)

[27] Akira Kono and Shin-ichiro Hara, *KO-theory of complex Grassmannians*, J. Math. Kyoto Univ. **31** (1991), no. 3, 827–833.

[28] S. G. Hoggar, *On* KO *theory of Grassmannians*, Quart. J. Math. Oxford Ser. (2) **20** (1969), 447–463.

[29] Graeme Segal, *Equivariant K-theory*, Inst. Hautes Études Sci. Publ. Math. (1968), no. 34, 129–151.

[30] J. F. Adams, *Stable homotopy and generalised homology*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1995. Reprint of the 1974 original.

[31] Colin Patrick Rourke and Brian Joseph Sanderson, *Introduction to piecewise-linear topology*, Springer Study Edition, Springer-Verlag, Berlin, 1982. Reprint.