The stable converse soul question for positively curved homogeneous spaces

David González-Álvaro & Marcus Zibrowius

Abstract

The stable converse soul question (SCSQ) asks whether, given a real vector bundle E over a compact manifold, some stabilization $E \times \mathbb{R}^k$ admits a metric with non-negative (sectional) curvature. We extend previous results to show that the SCSQ has an affirmative answer for all real vector bundles over any simply connected homogeneous manifold with positive curvature, except possibly for the Berger space B^{13}. Along the way, we show that the same is true for all simply connected homogeneous spaces of dimension at most seven, for arbitrary products of simply connected compact rank one symmetric spaces of dimensions multiples of four, and for certain products of spheres. Moreover, we observe that the SCSQ is "stable under tangential homotopy equivalence": if it has an affirmative answer for all vector bundles over a certain manifold M, then the same is true for any manifold tangentially homotopy equivalent to M. Our main tool is topological K-theory. Over B^{13}, there is essentially one stable class of real vector bundles for which our method fails.

1 Introduction and statement of results

The *Soul Theorem* by Cheeger and Gromoll [CG72] determines the structure of complete open manifolds with non-negative sectional curvature: given such a manifold M, there exists a totally geodesic, totally convex compact submanifold S (called the soul) such that M is diffeomorphic to the normal bundle of S. The *Converse Soul Question* asks, conversely, which vector bundles over a compact non-negatively curved manifold S admit a non-negatively curved metric. In general, this question is still widely open. The only such vector bundles that are known *not* to admit a non-negatively curved metric occur over manifolds S with infinite fundamental group [BK03, ÖW94]. On the other hand, the only simply connected manifolds over which *all* vector bundles are known to admit a non-negatively curved metric are the spheres $\mathbb{S}^n$ with $n \leq 5$ [GZ00]. Other than that, there are only partial results, and a weaker question has been studied, which we formulate as follows. From now on, M will denote a (smooth) compact manifold.

Question (Stable Converse Soul Question, SCSQ)**.** *Let E be the total space of a real vector bundle over a compact manifold M. Is there an integer $k \geq 0$ such that the manifold $E \times \mathbb{R}^k$ admits a metric of non-negative sectional curvature?*

For simply connected M with non-negative sectional curvature, this is precisely Problem 5.4 of [APL]. In our formulation, we do not explicitly require any curvature related conditions on M. Note however that, as a consequence of the Soul Theorem, one can expect positive results for the SCSQ only when M is at least homotopy equivalent to a compact manifold with non-negative curvature. The existence of manifolds over which the SCSQ has a negative answer for *all* vector bundles is therefore immediate: simply consider compact manifolds whose Betti numbers violate Gromov's bound for the existence of metrics with non-negative curvature [Gro81], for example connected sums $\#^m \mathbb{CP}^n$ for sufficiently large m and arbitrary n.

On the positive side, the SCSQ is already known to have an affirmative answer for all real vector bundles over any sphere $\mathbb{S}^n$ [Rig78], and for all real vector bundles over $\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$ and $\mathbb{CP}^2 \#(-\mathbb{CP}^2)$ [GZ11]. In [Gon17], the first-named author extended these results to include all compact rank one symmetric spaces (CROSSes): the spheres $\mathbb{S}^n$, the projective spaces $\mathbb{RP}^n$, $\mathbb{CP}^n$ and $\mathbb{HP}^n$, and the Cayley plane.

In this article, we study the SCSQ over the larger class of compact simply connected homogeneous manifolds with an invariant metric of positive sectional

curvature. These manifolds are completely classified (see the recent classification [WZ]). In addition to the CROSSes they include the Wallach flag manifolds W^6, W^{12} and W^{24}, the Berger spaces B^7 and B^{13}, and the infinite family of Aloff-Wallach spaces $W_{p,q}^7$. Our main result is the following.

Main Theorem. *The SCSQ has a positive answer for all real vector bundles over any compact simply connected homogeneous manifold M with an invariant metric of positive sectional curvature, except possibly for $M \cong B^{13}$.*

In fact, we can show a more general statement. Recall that two manifolds M, N of the same dimension are defined to be tangentially homotopy equivalent if there exists a homotopy equivalence $f\colon M \to N$ such that the tangent bundle TM and f^*TN are stably isomorphic, i.e. such that $TM \times \mathbb{R}^k$ and $f^*TN \times \mathbb{R}^k$ are isomorphic as bundles over M for some integer $k \geq 0$.

Main Theorem (cont.). *More generally, the SCSQ has a positive answer for any real vector bundle over any compact manifold tangentially homotopy equivalent to one of the above manifolds $M \not\cong B^{13}$.*

Results of Wilking suggest that, in this form, the Main Theorem covers a fairly general class of manifolds with positive sectional curvature: any n-dimensional simply connected manifold M^n with positive sectional curvature is either isometric to a homogeneous space with positive sectional curvature or tangentially homotopy equivalent to a CROSS, provided its isometry group $I(M)$ satisfies one of the following [Wil06]:

- the dimension of $I(M)$ is at least $2n - 6$, or
- the cohomogeneity of the $I(M)$-action is $\ell \geq 1$ and $n \geq 18(\ell + 1)^2$.

On the other hand, the general form of the Main Theorem also applies to manifolds that do not admit a metric of non-negative sectional curvature:

Example 1.1. All homotopy spheres are tangentially homotopy equivalent since their tangent bundles are stably trivial [KM63, Theorem 3.1]. But there are homotopy spheres Σ^m of dimensions $m = 8n + 1$ and $8n + 2$ that do not admit a metric of non-negative sectional curvature. (See [DT11] for these and other pairs of homotopy equivalent manifolds of which only one admits a metric of non-negative curvature.)

We can use this example to illustrate the need for stabilization:

Example 1.2. Let Σ^m be one of the homotopy spheres of the previous example that does not admit a metric of non-negative sectional curvature. Then the total space $\Sigma^m \times \mathbb{R}$ of the trivial bundle likewise admits no metric of non-negative sectional curvature (see Lemma 4.4). Our Main Theorem implies that, nonetheless, some stabilization $\Sigma^m \times \mathbb{R}^k$ does admit such a metric. In this particular case, this can also be seen directly as follows: by the "Work Horse Theorem" of [TW15] (see Section 4.2 below), the product $\Sigma^m \times \mathbb{R}^{m+1}$ is diffeomorphic to $\mathbb{S}^m \times \mathbb{R}^{m+1}$, on which a metric of non-negative curvature is provided by the product metric. It would be interesting to find the minimum $2 \leq k \leq m + 1$ such that $\Sigma^m \times \mathbb{R}^k$ admits such a metric.

In the case of B^{13} our results are inconclusive. We can, however, make precise to what extent the general strategy used in this article fails for B^{13}: see Theorem 1.12 below.

Outline of the proof and related results

We proceed to sketch our proof of the Main Theorem and to describe some positive results for other families of homogeneous spaces that we have obtained along the way. First, we note that the second part of the theorem follows from the first:

Proposition 1.3. *If the SCSQ has a positive answer for all real (resp. all complex) vector bundles over M, then it also has a positive answer for all real (resp. all complex) vector bundles over any tangentially homotopy equivalent manifold N.*

Here, the SCSQ for a complex vector bundle is simply the SCSQ for the underlying real vector bundle. The proof of Proposition 1.3 given in Section 4.2 relies on the "Work Horse Theorem" already mentioned: the main point is that the total space of a vector bundle over N of sufficiently high rank is diffeomorphic to the total space of some vector bundle over M.

Given Proposition 1.3 and the results of [Gon17] on CROSSes, it remains to prove the Main Theorem for W^6, W^{12}, W^{24}, B^7 and $W^7_{p,q}$. Each of these is a compact homogeneous space, i.e. an orbit space G/H of a subgroup H acting on a compact Lie group G. Our basic strategy is the same as in [Gon17]: we prove that every vector bundle over one of these spaces is stably isomorphic to a homogeneous vector bundle $G \times_H \mathbb{R}^m$ for some representation of H. Such a homogeneous vector bundle always admits a metric of non-negative sectional curvature by O'Neill's theorem on Riemannian submersions. In fact, for each homogeneous metric $\langle,\rangle$ of non-negative curvature on G/H, there is a metric on $G \times_H \mathbb{R}^m$ with non-negative curvature and soul isometric to $(G/H, \langle,\rangle)$.

Remark 1.4. Given a stably homogeneous vector bundle E over G/H, one can estimate the minimal k needed to obtain non-negative curvature on $E \times \mathbb{R}^k$ as follows: Let ρ be a representation of H of rank $\rho \geq \dim G/H + 1$ and minimal among all representations such that the associated homogeneous bundle is stably isomorphic to E. Then the minimum k needed satisfies $k \leq \operatorname{rank} \rho - \operatorname{rank} E$. See [Gon17] for concrete bounds in the case $G/H = \mathbb{S}^n$.

The K-rings of real or complex vector bundles over a manifold M, denoted $KO(M)$ and $K(M)$, respectively, provide the natural framework to study stable classes of vector bundles. Let $RO(H)$ and $R(H)$ likewise denote the real and complex representation rings of a Lie group H. Sending a real or complex representation of H to the corresponding real or complex homogeneous vector bundle over M defines ring homomorphisms:

$$\alpha_O \colon RO(H) \to KO(G/H)$$
$$\alpha \colon R(H) \to K(G/H)$$

As observed in [Gon17], every real vector bundle over G/H can be stabilized to a homogeneous real vector bundle if and only if α_O is surjective. The same equivalence holds for complex vector bundles and α. What is more, the complex version $\alpha \colon R(H) \to K(G/H)$ is indeed surjective in many situations. In particular, it is well-known that α is surjective whenever G is compact simply connected and $H \subset G$ is a closed connected subgroup of maximal rank, i.e. when

rank $H = \operatorname{rank} G$. (The rank of a Lie group is the dimension of a maximal torus.) As we explain in Theorem 3.6 below, this is true more generally, though less well-known, for subgroups H with $\operatorname{rank} G - \operatorname{rank} H \leq 1$. All compact homogeneous positively curved spaces G/H satisfy this inequality, with equality if and only if the dimension of the space is odd. As an immediate corollary, we obtain the following result for complex vector bundles over such spaces.

Theorem 1.5. *Let G be a compact connected Lie group with $\pi_1(G)$ torsion-free, and let $H \subset G$ be a closed connected subgroup such that $\operatorname{rank} G - \operatorname{rank} H \leq 1$. Then the SCSQ has a positive answer for all complex vector bundles over any compact manifold tangentially homotopy equivalent to G/H.*

The homomorphism α_O, on the other hand, is surjective only in very special situations. For example, the results of the second-named author in [Zib15] imply that α_O is almost never surjective when G/H is a full flag manifold, i.e. when $H \subset G$ is a maximal torus T. The few positive cases are:

Theorem 1.6. *The SCSQ has a positive answer for all real vector bundles over the full flag manifolds $SU(m)/T$ with $m \in \{2, 3, 4, 5, 7\}$, over $Spin(7)/T$, over G_2/T and over the products of these specified in Proposition 5.6 below.*

It therefore came as a surprise that, nonetheless, α_O *is* surjective for all compact simply connected manifolds admitting a homogeneous metric of positive sectional curvature, except for B^{13}. For any given homogeneous space M, the surjectivity of α_O may of course depend on the precise presentation of M as G/H, but for all spaces considered here, it turns out that the "standard" presentation with simply connected G works. We will in fact show that α_O is surjective for all homogeneous spaces in certain families that include the positively curved ones, except for B^{13}. In particular, in each of our positive cases $W^6, W^{12}, W^{24}, B^7, W_{p,q}^7$, the surjectivity of α_O fits into one of the following two more general results.

Firstly, the spaces W^6, B^7 and $W_{p,q}^7$ are covered by a result in low dimensions: α_O is surjective for any simply connected compact homogeneous space G/H of dimension at most seven, at least if we choose, as we may, a presentation with simply connected G (Proposition 5.3). The dimensional bound is perhaps not surprising—real K-theory generally becomes more complicated starting from dimension eight—and indeed, for $\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$ the result is false (Remark 5.4). In any case, as a consequence we obtain:

Theorem 1.7. *The SCSQ has a positive answer for all real vector bundles over any simply connected compact homogeneous space of dimension at most seven.*

Secondly, the Wallach manifolds $W^{12} = Sp(3)/Sp(1)^3$ and $W^{24} = F_4/Spin(8)$ are covered by the following result (see Proposition 3.12):

Theorem 1.8. *Let G be a compact connected Lie group with $\pi_1(G)$ torsion-free, and let $H \subset G$ be a closed connected subgroup with $\operatorname{rank} H = \operatorname{rank} G$, and such that complex conjugation acts on the complex representation ring $R(H)$ as the identity map. Then the SCSQ has a positive answer for all real vector bundles over G/H.*

The maximal rank closed subgroups of compact Lie groups are described in [BDS49]. Conjugation acts trivially on $R(H)$ for all compact connected Lie groups H that have no circle factor and no simple factors of types A_n with $n \geq 2$, D_{2n+1} with $n \geq 4$ or E_6 (Remark 3.13).

The SCSQ for products

The class of spaces satisfying Theorem 1.8 is closed under taking products. In particular, it includes arbitrary products of $4n$-dimensional spheres, quaternionic projective spaces and the Cayley plane (for the usual homogeneous presentations). It turns out that, in good situations, surjectivity of α_O is also preserved when taking products with $\mathbb{CP}^{2n}$ (Proposition 3.14). Altogether, this gives the following result.

Theorem 1.9. *Let M be a finite product of simply connected CROSSes whose dimensions are multiples of four. Then the SCSQ has a positive answer for all real vector bundles over M.*

Besides the products of spheres included in Theorem 1.9, our K-theoretic computations yield positive results for other products of spheres.

Theorem 1.10. *Consider a product of spheres $S = \mathbb{S}^{n_1} \times \mathbb{S}^{n_2} \times \cdots \times \mathbb{S}^{n_\ell}$ Assume that the unordered tuple $\{n_1, \ldots, n_\ell\}$ is congruent modulo eight to one of the following, where* even, even$'$ *denote arbitrary even numbers and* odd *denotes an arbitrary odd number:*

$\ell = 1:$ $\{n\}$ *arbitrary*
$\ell = 2:$ $\{3,3\}$, $\{7,7\}$, $\{$even, odd$\}$ *or* $\{$even, even$'\} \not\equiv \{2,6\}$
$\ell = 3:$ $\{2,2,2\}$, $\{6,6,6\}$, $\{7,7,7\}$ *or* $\{$even, even$'$, odd$\} \notin \{\{2,6,\text{odd}\}, \{4,4,\text{odd}\}\}$
$\ell = 4:$ $\{2,2,2,\text{odd}\}$ *or* $\{6,6,6,\text{odd}\}$
$\ell \in \mathbb{N}:$ $\{$even$, n_2, \ldots, n_\ell\}$, *where $n_i \equiv 0$ or 4 for all i*
$\ell \in \mathbb{N}:$ $A \cup \{n_k, \ldots, n_\ell\}$, *where $n_i \equiv 0$ for all i and A is any of the tuples above.*

Then the SCSQ has a positive answer for all real vector bundles over S.

Finally, all of the above results that rely on the surjectivity of α_O can be improved using Bott periodicity, in the sense that we can allow products with an arbitrary number of spheres whose dimensions are multiples of eight. In combination with Proposition 1.3 we obtain:

Theorem 1.11. *Let M be any of the homogeneous spaces satisfying the hypotheses in Theorems 1.6, 1.7, 1.8, 1.9 or 1.10, and let S be a product of arbitrarily many spheres whose dimensions are multiples of eight. The SCSQ has a positive answer for all real vector bundles over any compact manifold tangentially homotopy equivalent to $M \times S$.*

The Berger space B^{13}

For the standard presentation of the Berger space B^{13} as $SU(5)/Sp(2) \times_{\mathbb{Z}_2} S^1$, in turns out that α_O is *not* surjective. As is shown in [FZ09, Proof of Theorem A], any compact homogeneous space homotopy equivalent to B^{13} is already

diffeomorphic to B^{13}, and the standard presentation is its only presentation as an orbit space of a simply connected Lie group. It follows that α_O cannot be surjective for *any* presentation of B^{13} as a homogeneous space (Lemma 3.5). In other words, the SCSQ over B^{13} cannot be addressed by the techniques used here. However, surjectivity of α_O fails only by a small margin:

Theorem 1.12. *The map α_O is not surjective for any presentation of B^{13} as a homogeneous space. Rather, its image is contained in a fixed subring $S \subset KO(B^{13})$ of index two. For the standard presentation of B^{13} as $SU(5)/Sp(2)\times_{\mathbb{Z}_2} S^1$, the image of α_O is equal to S.*

Let $\widetilde{KO}(B^{13}) \subset KO(B^{13})$ and $\widetilde{S} \subset S$ denote the ideals consisting of all virtual vector bundles of rank zero. Then $\widetilde{KO}(B^{13})$ splits, as a non-unitary ring, as a product

$$\widetilde{KO}(B^{13}) \cong S \times \mathbb{Z}_2 w$$

with $w^2 = 0$. The generator w is the unique nonzero class in $\widetilde{KO}(B^{13})$ with trivial Pontryagin classes.

In particular, there exists a real vector bundle E (of rank ≤ 13) over B^{13} such that $E \times \mathbb{R}^k$ is not a homogeneous vector bundle for any integer k. There are, in fact, countably infinitely many such bundles – all representatives of all elements in the w-coset of S – but there is a unique stable class of vector bundles with the additional property that the Pontryagin classes are trivial.

Acknowledgements

2 K-theory

In this section we review mostly well-known facts on the real and complex K-theory of a manifold. Classical references include [Ati67] and [Hus94]. We place particular emphasis on relations between the real and the complex theory. In particular, we discuss cohomological conditions that ensure the surjectivity of the realification map from complex to real K-theory for manifolds of dimension at most seven.

2.1 K-theory classifies vector bundles

Throughout this section, M will denote a compact Hausdorff space. We denote by $\text{Vect}_{\mathbb{F}}(M)$ the set of isomorphism classes of (continuous) $\mathbb{F}$-vector bundles over M, where $\mathbb{F}$ is either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. (In the last case, each fiber is considered as a right module over $\mathbb{H}$.) Over a manifold, any continuous vector bundle can be equipped with an essentially unique smooth structure, i.e. $\text{Vect}_{\mathbb{F}}(M)$ agrees with the set of isomorphism classes of *smooth* $\mathbb{F}$-bundles. The Whitney sum $\oplus$ endows $\text{Vect}_{\mathbb{F}}(M)$ with a semigroup structure, and we define $K_{\mathbb{F}}(M)$ as its group completion. The elements of $K_{\mathbb{F}}(M)$ can be described as formal differences of

elements in $\text{Vect}_{\mathbb{F}}(M)$. As an example, for a point we have $K_{\mathbb{F}}(pt) \cong \mathbb{Z}$. As usual in the literature, we denote $K_{\mathbb{R}}$, $K_{\mathbb{C}}$ and $K_{\mathbb{H}}$ by KO, K and KSp, respectively.

Products

When $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, the $\mathbb{F}$-tensor product gives $\text{Vect}_{\mathbb{F}}(M)$ a semiring structure. Thus $K_{\mathbb{F}}(M)$ is the ring completion of $\text{Vect}_{\mathbb{F}}(M)$ and $K(M)$ and $KO(M)$ are referred to as the complex and real K-rings of M, respectively. Moreover, recall that there is a bijective correspondence between $\text{Vect}_{\mathbb{R}}(M)$ (resp. $\text{Vect}_{\mathbb{H}}(M)$) and the set of isomorphism classes of $\mathbb{C}$-vector bundles endowed with a conjugate-linear automorphism J such that $J^2 = \text{id}$ (resp. $J^2 = -\text{id}$) [Ati67, §1.5]. In this way, the $\mathbb{C}$-tensor product induces product maps $KO(M) \otimes KSp(M) \to KSp(M)$ and $KSp(M) \otimes KSp(M) \to KO(M)$ giving the direct sum $KO(M) \oplus KSp(M)$ the structure of a $\mathbb{Z}_2$-graded ring. Similarly, the i-th exterior power of vector spaces induces the corresponding exterior power operations λ^i on vector bundles, making $K(M)$ and $KO(M)$ λ-rings and $KO(M) \oplus KSp(M)$ a $\mathbb{Z}_2$-graded λ-ring [Bot59, All74]. For two spaces M_1 and M_2 we also have *external* products defined in the usual way in terms of the (internal) products discussed above and the canonical projections $\pi_i \colon M_1 \times M_2 \to M_i$:

$$\begin{array}{rcl} \mu \colon K_{\mathbb{F}_1}(M_1) \otimes K_{\mathbb{F}_2}(M_2) & \to & K_{\mathbb{F}_{12}}(M_1 \times M_2) \\ \alpha \otimes \beta & \mapsto & \pi_1^*(\alpha) \cdot \pi_2^*(\beta) \end{array}$$

Here, $(\mathbb{F}_1, \mathbb{F}_2, \mathbb{F}_{12})$ is one of the triples $(\mathbb{C}, \mathbb{C}, \mathbb{C})$, $(\mathbb{R}, \mathbb{R}, \mathbb{R})$, $(\mathbb{R}, \mathbb{H}, \mathbb{H})$, $(\mathbb{H}, \mathbb{R}, \mathbb{H})$ or $(\mathbb{H}, \mathbb{H}, \mathbb{R})$. Bott periodicity implies that the external product for K- and KO-theory is an isomorphism when one of the two spaces is an even-dimensional sphere or a sphere of dimension a multiple of eight, respectively [Ati67, Cor. 2.2.3; Hus94, Thm 11.1.2]:

$$\begin{array}{rcl} K(M) \otimes K(\mathbb{S}^{2k}) & \xrightarrow{\cong} & K(M \times \mathbb{S}^{2k}) \\ KO(M) \otimes KO(\mathbb{S}^{8k}) & \xrightarrow{\cong} & KO(M \times \mathbb{S}^{8k}) \end{array} \qquad (1)$$

(Most references only state the case $k = 1$. The more general statement follows from [Hus94, Prop. 11.1.1].) See also [Ati62] for a general Künneth formula for complex K-theory.

Reduced K-groups

For a pointed space (M, pt), the reduced $K_{\mathbb{F}}$-theory is defined as the kernel of the restriction map $K_{\mathbb{F}}(M) \to K_{\mathbb{F}}(pt)$. When M is connected, this subgroup is independent of the chosen basepoint and we denote it by $\widetilde{K}_{\mathbb{F}}(M)$. It may in this case be described explicitly as follows. Denote by $n_{\mathbb{F}}$ the trivial $\mathbb{F}$-vector bundle over M of rank $n \in \mathbb{N}$. Then

$$\widetilde{K}_{\mathbb{F}}(M) = \{E - n_{\mathbb{F}} \in K_F(M) \mid \text{rank } E = n\}.$$

Alternatively, $\widetilde{K}_{\mathbb{F}}(M)$ can be described as the abelian group of stable classes of $\mathbb{F}$-vector bundles over M. Two vector bundles $E, F \in \text{Vect}_{\mathbb{F}}(M)$ are stably equivalent if there exist trivial bundles $n_{\mathbb{F}}, m_{\mathbb{F}}$ such that $E \oplus n_{\mathbb{F}}$ is isomorphic to $F \oplus m_{\mathbb{F}}$. In any case, we have a natural isomorphism of abelian groups:

$$K_{\mathbb{F}}(M) = \mathbb{Z} \oplus \widetilde{K}_{\mathbb{F}}(M). \qquad (2)$$

As we will ultimately be interested in the reduced group $\widetilde{K}_{\mathbb{F}}(M)$, we will often write the constant $\mathbb{Z}$-term in (2) in parentheses, i.e. $K_{\mathbb{F}}(M) = [\mathbb{Z}] \oplus \widetilde{K}_{\mathbb{F}}(M)$.

2.2 K-theory as a generalized cohomology theory

Atiyah and Hirzebruch constructed generalized cohomology theories $K^*(-)$ and $KO^*(-)$ on topological spaces such that $K^0(M) \cong K(M)$ and $KO^0(M) \cong KO(M)$ for any compact Hausdorff M. For negative indices i, the groups $K^i(M)$ and $KO^i(M)$ can be defined geometrically in terms of vector bundles over suspensions of M. An equivalent way to state Bott periodicity (1) is that these cohomology theories are periodic in the sense that there are natural isomorphisms $K^{i+2}(-) \cong K^i(-)$ and $KO^{i+8}(-) \cong KO^i(-)$. Their coefficient groups are as follows:

$i \bmod 8$	0	1	2	3	4	5	6	7
$K^{-i}(pt)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
$KO^{-i}(pt)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0

Both cohomology theories are multiplicative. Their coefficient rings may be written as

$$K^*(pt) = \mathbb{Z}[\beta^{\pm 1}]$$
$$KO^*(pt) = \mathbb{Z}[\eta, \alpha, \beta_O^{\pm 1}]/(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta_O)$$

with $\deg \beta = -2$, $\deg \eta = -1$, $\deg \alpha = -4$ and $\deg \beta_O = -8$. The graded rings $K^*(M) := \bigoplus_{i \in \mathbb{Z}} K^i(M)$ and $KO^*(M) := \bigoplus_{i \in \mathbb{Z}} KO^i(M)$ are graded $K^*(pt)$- and $KO^*(pt)$-modules, respectively, and the periodicity isomorphisms are given by multiplication with the so-called Bott elements β and β_O. In the quaternionic case, the same construction yields a graded abelian group $KSp^*(M) := \bigoplus_{i \in \mathbb{Z}} KSp^i(M)$. The periodicity in this case is again of period eight, as there are natural group isomorphisms:

$$KSp^i(M) \cong KO^{i-4}(M), \text{ for all } i \in \mathbb{Z}. \tag{3}$$

For connected M, we define higher *reduced* groups $\widetilde{K}^i(M)$ as in the case $i = 0$, so that we have natural splittings:

$$K^i(M) = [K^i(pt)] \oplus \widetilde{K}^i(M), \qquad KO^i(M) = [KO^i(pt)] \oplus \widetilde{KO}^i(M).$$

2.3 Relations between complex and real K-theory

We have the following natural maps between the K-groups of a space, referred to as complexification (c, c′), realification (r), quaternionification (q) and conjugation (t):

$$\mathsf{c}\colon KO(M) \to K(M) \qquad \mathsf{r}\colon K(M) \to KO(M) \qquad \mathsf{t}\colon K(M) \to K(M)$$
$$\mathsf{c}'\colon KSp(M) \to K(M) \qquad \mathsf{q}\colon K(M) \to KSp(M)$$

They satisfy the following identities, where 2 denotes multiplication by two:

$$\begin{array}{lll} \mathsf{r} \circ \mathsf{c} = 2 & \mathsf{c} \circ \mathsf{r} = \mathrm{id} + \mathsf{t} & \mathsf{q} \circ \mathsf{c}' = 2 \\ \mathsf{c}' \circ \mathsf{q} = \mathrm{id} + \mathsf{t} & \mathsf{t} \circ \mathsf{c} = \mathsf{c} & \mathsf{r} \circ \mathsf{t} = \mathsf{r} \\ \mathsf{t} \circ \mathsf{c}' = \mathsf{c}' & \mathsf{q} \circ \mathsf{t} = \mathsf{q} & \mathsf{t}^2 = \mathrm{id} \end{array}$$

The maps c and t are ring homomorphisms, while c', r and q are only group homomorphisms. More precisely, $(c, c') \colon KO(M) \oplus KSp(M) \to K(M)$ is a ring homomorphism. This gives $K(M)$ the structure of a $KO(M) \oplus KSp(M)$-module. The map $\binom{r}{q} \colon K(M) \to KO(M) \oplus KSp(M)$ is a $KO(M) \oplus KSp(M)$-module homomorphism with respect to this module structure on $K(M)$. For example, $r(c' x \cdot y) = x \cdot qy$ for all $x \in KSp(M)$, $y \in K(M)$.

All of the above maps extend to natural transformations of cohomology theories. Under the identification (3), $q \colon K(M) \to KSp(M)$ corresponds to $r \colon K^{-4}(M) \to KO^{-4}(M)$ and c' corresponds to c [Bot59]. The values of c, r and t on the coefficient rings are determined by:

$$ t\beta = -\beta \qquad c\beta_O = \beta^4 \qquad c\alpha = 2\beta^2 \qquad r\beta = \eta^2 \qquad r(\beta^2) = \alpha \qquad (4) $$

[Bou90, §1.1]. The realification and complexification maps in different degrees can be assembled into a periodic exact sequence known as the *Bott sequence*, of the following form:

$$ \cdots \to KO^i(M) \xrightarrow{\;c\;} K^i(M) \xrightarrow{\;r\beta^{-1}\;} KO^{i+2}(M) \xrightarrow{\;\eta\;} KO^{i+1}(M) \to \cdots \quad (5) $$

A lemma of Bousfield

We can view the conjugation $t \colon K(M) \to K(M)$ as an involution on the K-ring. In general, given a ring $A = (A, t)$ with an involution, i.e. with a ring automorphism t such that $t^2 = \mathrm{id}$, one can define the associated Tate ring as the $\mathbb{Z}_2$-graded ring $h^*(A, t) := h^+(A, t) \oplus h^-(A, t)$ with homogeneous components:

$$ h^+(A, t) = \frac{\ker(\mathrm{id} - t)}{\mathrm{im}(\mathrm{id} + t)} $$

$$ h^-(A, t) = \frac{\ker(\mathrm{id} + t)}{\mathrm{im}(\mathrm{id} - t)} $$

Lemma 2.1 (Bousfield). *For any space M with $K^1(M) = 0$, the complexification maps induce isomorphisms:*

$$ KO(M)/r \oplus KSp(M)/q \xrightarrow[c + c']{\cong} h^+(K(M), t) $$

$$ KO^2(M)/r \oplus KO^6(M)/r \xrightarrow[c + c]{\cong} h^-(K(M), t) $$

Proof. This is essentially [Bou05, Lemma 4.7]. See also [Zib15, §1.2, §2.1]. □

Remark 2.2. Likewise, for any compact Lie group G, complex conjugation defines an involution t on the complex representation ring RG, and the complexification map from the real/quaternionic representation ring $RO(G) \oplus RSp(G)$ to the complex representation ring induces a ring isomorphism:

$$ RO(G)/r \oplus RSp(G)/q \xrightarrow[c + c']{\cong} h^+(RG, t) $$

The odd Tate groups $h^-(RG, t)$ vanish in this case (see section Section 3.2).

Lemma 2.3. *Suppose (A, t) and (B, t) are commutative rings with involutions. Consider the tensor product ring $A \otimes B$ with the induced involution $\mathsf{t} \otimes \mathsf{t}$. If A is free as an abelian group, then the canonical ring homomorphism of $\mathbb{Z}_2$-graded rings $h^*(A) \otimes h^*(B) \to h^*(A \otimes B)$ is an isomorphism.*

Proof. As an abelian group, we can decompose A into a direct sum $A_+ \oplus A_- \oplus A' \oplus A'$ such that t acts on A_+ as the identity, on A_- as minus the identity and on $A' \oplus A'$ by interchanging the two factors [Bou90, Prop. 3.7]. It suffices to verify the claim separately for each of these three factors. $\qquad\square$

2.4 The Atiyah-Hirzebruch spectral sequences

In [AH61], Atiyah and Hirzebruch constructed a spectral sequence for calculating generalized cohomology theories like K^* and KO^* of a space from its ordinary cohomology. It is commonly known today as the Atiyah-Hirzebruch spectral sequence (AHSS).

A spectral sequence is a sequence of pages $\{E_k^{p,q}\}_{k \in \mathbb{Z}}$, each consisting of a grid of abelian groups together with certain homomorphisms $d_k^{p,q}$ between them, called differentials, that determine the next page. See [McC01] for a comprehensive introduction to this machinery. A spectral sequence converges to some graded group G^* if there exists a limit page $E_\infty^{p,q}$ such that each diagonal $p + q = c$ is isomorphic to the graded group associated with some bounded exhaustive filtration on G^c. We refer to this situation with the notation $E_2^{p,q} \Rightarrow G^{p+q}$.

The AHSS for the generalized cohomology theories $K^*(M)$ and $KO^*(M)$ have the form:

$$E_2^{p,q} = H^p(M, K^q(pt)) \Rightarrow K^{p+q}(M)$$
$$E_2^{p,q} = H^p(M, KO^q(pt)) \Rightarrow KO^{p+q}(M)$$

Alternatively, using reduced cohomology, we obtain a spectral sequence converging to the reduced K- and KO-theory of M. It differs from the unreduced AHSS only in that the first column is zero. We will repeatedly use the following facts:

(a) The images of all differentials are torsion. In particular, if $H^*(M, \mathbb{Z})$ is torsion-free, then there are no nontrivial differentials in the AHSS for $K^*(M)$, and consequently $E_2^{p,q} = E_\infty^{p,q}$. (See [AH61, §2.4] for the complex case. The real case follows by comparing the two spectral sequence for $K^*(-) \otimes \mathbb{Q}$ and $KO^*(-) \otimes \mathbb{Q}$.)

(b) In the AHSS for $KO^*(M)$, the differential $d_2 \colon E_2^{p,q} \to E_2^{p+2,q-1}$ is as follows:

$$d_2^{p,q} = \begin{cases} Sq^2 \circ \mathsf{red} & \text{for } q \equiv 0 \mod 8 \\ Sq^2 & \text{for } q \equiv -1 \mod 8 \\ 0 & \text{otherwise,} \end{cases}$$

Here, Sq^2 denotes the second Steenrod square and red is reduction of coefficients modulo two. [Fuj67, §1]

(c) The spectral sequences are multiplicative. That is, for each page E_r of the spectral sequence, we have a pairing of bigraded abelian groups $E_r \otimes E_r \to E_r$,

and with respect to this pairing, the differential d_r satisfies the Leibniz rule $d_r(x \cdot y) = d_r(x) \cdot y + (-1)^p x \cdot d_r(y)$ for $x \in E_r^{p,q}$ and $y \in E_r^{s,t}$. Up to a sign, the pairing on the E_2-page can be identified with the usual cup product [Dug03, Thm 3.4 and Rem. 3.6].

(d) The first few columns of the spectral sequences are closely related to certain characteristic classes. Write $K(M) = F^0 K(M) \supset F^1 K(M) \supset F^2 K(M) \supset \cdots$ for the filtration on $K(M)$ whose associated graded ring is isomorphic to the zero diagonal of the E_∞-page of the AHSS converging to $K(M)$. As all odd rows vanish, $F^{2i-1} K(M) = F^{2i} K(M)$ for all i. Moreover, $F^2 K(M) = \tilde{K}(M)$, the morphism $F^2 K(M) \to H^2(M, \mathbb{Z})$ coming from the spectral sequence can, up to a sign, be identified with the first Chern class, and the morphism $F^4 K(M) \to H^4(X, \mathbb{Z})$ can be identified with the restriction of the second Chern class, again up to a possible sign. Similarly, for the corresponding filtration on $KO(M)$, we have $F^1 KO(M) = \widetilde{KO}(M)$, the map $F^1 KO(M) \to H^1(M, \mathbb{Z}_2)$ is the first Stiefel-Whitney class, and the map $F^2 KO(M) \to H^2(M, \mathbb{Z}_2)$ is the second Stiefel-Whitney class [Zib14, Lem. 3.13].

Here's one quick application:

Lemma 2.4. *A product of spheres $S = \mathbb{S}^{n_1} \times \mathbb{S}^{n_2} \times \cdots \times \mathbb{S}^{n_\ell}$ satisfies $\widetilde{KO}(S) = 0$ only in the following cases:*

$\ell = 1$ *and* $n_1 \equiv 3, 5, 6, 7$ *modulo 8.*
$\ell = 2$ *and the unordered pair* $\{n_1, n_2\}$ *is congruent to one of the pairs* $\{3, 3\}$, $\{5, 6\}$, $\{6, 7\}$ *or* $\{7, 7\}$ *modulo 8.*
$\ell = 3$ *and the triple* $\{n_1, n_2, n_3\}$ *is congruent to* $\{7, 7, 7\}$ *modulo 8.*

Proof. The differentials in the AHSS for KO are trivial for any sphere. It follows from the multiplicativity of the spectral sequence that they are also trivial for S. Thus, $\widetilde{KO}(S)$ vanishes if and only if all terms on the diagonal $E_2^{p,-p}$ are zero for $p \geq 1$. As the cohomology of S is finitely generated free, this happens if and only if $H^k(S, \mathbb{Z})$ vanishes in all positive degrees $k \equiv 0, 1, 2, 4 \mod 8$. This condition is equivalent to the numerical conditions stated. $\qquad \square$

The structure maps r, c and t induce morphisms of spectral sequences between the AHSS computing the K- and KO-theory of a space. For example, r induces a sequence of maps $\mathsf{r}_k^{p,q} \colon E_k^{p,q}(K(M)) \to E_k^{p,q}(KO(M))$ that commute with the differentials and converge to $\mathsf{r} \colon K^*(M) \to KO^*(M)$. On the E_2-pages, these morphisms are simply the change-of-coefficient maps induced by the corresponding maps between the K- and KO-theory of a point. Using (4), we can easily describe them explicitly:

Conjugation $\mathsf{t}_2^{p,q} \colon E_2^{p,q}(K(M)) \to E_2^{p,q}(K(M))$

For $q \equiv 0 \mod 4$, $\mathsf{t}_2^{p,q} \colon H^p(M, \mathbb{Z}) \to H^p(M, \mathbb{Z})$ is the identity.
For $q \equiv 2 \mod 4$, $\mathsf{t}_2^{p,q} \colon H^p(M, \mathbb{Z}) \to H^p(M, \mathbb{Z})$ is multiplication by -1.
In all other cases, $\mathsf{t}_2^{p,q}$ vanishes.

Complexification $\mathsf{c}_2^{p,q} \colon E_2^{p,q}(KO(M)) \to E_2^{p,q}(K(M))$

For $q \equiv 0 \mod 8$, $\mathsf{c}_2^{p,q} \colon H^p(M, \mathbb{Z}) \to H^p(M, \mathbb{Z})$ is the identity.

For $q \equiv 4 \bmod 8$, $\mathsf{c}_2^{p,q} \colon H^p(M, \mathbb{Z}) \to H^p(M, \mathbb{Z})$ is multiplication by 2.
In all other cases, $\mathsf{c}_2^{p,q}$ vanishes.

Realification $\mathsf{r}_2^{p,q} \colon E_2^{p,q}(K(M)) \to E_2^{p,q}(KO(M))$

For $q \equiv 0 \bmod 8$, $\mathsf{r}_2^{p,q} \colon H^p(M, \mathbb{Z}) \to H^p(M, \mathbb{Z})$ is multiplication by 2.
For $q \equiv 4 \bmod 8$, $\mathsf{r}_2^{p,q} \colon H^p(M, \mathbb{Z}) \to H^p(M, \mathbb{Z})$ is the identity.
For $q \equiv 6 \bmod 8$, $\mathsf{r}_2^{p,q} \colon H^p(M, \mathbb{Z}) \to H^p(M, \mathbb{Z}_2)$ is reduction of coefficients.
In all other cases, $\mathsf{r}_2^{p,q}$ vanishes.

For example, the description of r_2 implies:

Lemma 2.5. *Suppose all differentials in the AHSS's affecting elements in the diagonals computing $K^0(M)$ and $KO^0(M)$ vanish. If*

- $H^k(M) = 0$ *for positive $k \equiv 0 \bmod 8$, and*
- *reduction of coefficients $H^k(M, \mathbb{Z}) \to H^k(M, \mathbb{Z}_2)$ is surjective for $k \equiv 2 \bmod 8$*

then the map $\mathsf{r} \colon \widetilde{K}(M) \to \widetilde{KO}(M)$ is surjective.

A similar argument to that in the proof of Lemma 2.4 yields the following corollary:

Corollary 2.6. *Consider a product of spheres $S = \mathbb{S}^{n_1} \times \mathbb{S}^{n_2} \times \cdots \times \mathbb{S}^{n_\ell}$. If, among all possible sums $n_{i_1} + \cdots + n_{i_k}$ $(1 \le i_1 < \cdots < i_k \le \ell)$, only the empty sum is congruent to zero modulo eight, then the realification map $\mathsf{r} \colon \widetilde{K}(S) \to \widetilde{KO}(S)$ is surjective.*

Observe that of course the products in Lemma 2.4 are included in Corollary 2.6.

2.5 Manifolds of dimension at most seven

Here, we apply the preceding discussion to simply connected compact manifolds M^n of small dimensions. By the reduced realification, we will mean the restriction of the realification map to reduced K-groups: $\mathsf{r} \colon \widetilde{K}(M) \to \widetilde{KO}(M)$.

Proposition 2.7. *Let M^n be a simply connected compact manifold of dimension $n \le 7$. If $H^{n-2}(M, \mathbb{Z})$ contains no two-torsion, then the reduced realification is surjective. If $H^5(M, \mathbb{Z})$ is torsion-free, then the absence of two-torsion in $H^{n-2}(M, \mathbb{Z})$ is equivalent to the surjectivity of the reduced realification.*

The proof below will moreover show that, when both cohomological assumptions are satisfied, the group $\widetilde{KO}(M)$ can be described as an extension of $H^2(M, \mathbb{Z}_2)$ by $H^4(M, \mathbb{Z})$. If $H^4(M, \mathbb{Z})$ is a torsion group of odd order, this extension splits, so

$$KO(M) \cong [\mathbb{Z}] \oplus H^2(M, \mathbb{Z}_2) \oplus H^4(M, \mathbb{Z}). \qquad (6)$$

Examples 2.8. Proposition 2.7 easily implies surjectivity of the reduced realification in the following cases:

- M compact simply connected of dim $M \le 4$.
- M compact simply connected and *homogeneous* of dim $M \le 7$, except for two cases (see Lemma 5.1).

- M is an Eschenburg biquotient with positive curvature [Esc82, Proposition 36].
- M is the total space of an $\mathbb{S}^3$-bundle over $\mathbb{S}^4$. Grove and Ziller proved in [GZ00] that they all admit non-negatively curved metrics.

Proof of Proposition 2.7. Let $H^i := H^i(M, \mathbb{Z})$ and $h^i := H^i(M, \mathbb{Z}_2)$. We note first that for a simply connected compact n-manifold M, our hypothesis that H^{n-2} contains no two-torsion is equivalent to the assumption that the reduction map $H^2 \to h^2$ is surjective. Indeed, as H_1 vanishes, the Universal Coefficient Theorem identifies H^2 and h^2 with $\mathrm{Hom}(H_2, \mathbb{Z})$ and $\mathrm{Hom}(H_2, \mathbb{Z}_2)$, respectively. The long exact sequence obtained by applying $\mathrm{Hom}(H_2, -)$ to the short exact sequence $\mathbb{Z} \rightarrowtail \mathbb{Z} \twoheadrightarrow \mathbb{Z}_2$ therefore yields an isomorphism between the cokernel of the reduction map $H^2 \to h^2$ and the kernel of multiplication by two on $\mathrm{Ext}(H_2, \mathbb{Z})$. As M is a compact manifold, H_2 is finitely generated, and $\mathrm{Ext}(H_2, \mathbb{Z})$ can be identified with the torsion subgroup of H_2. Moreover, by Poincaré duality, $H_2 \cong H^{n-2}$. So altogether the cokernel of $H^2 \to h^2$ is isomorphic to the kernel of multiplication by two on H^{n-2}.

Now assume in addition that $n \leq 7$, and consider the realification map from the AHSS computing $\widetilde{K}(M)$ to the AHSS computing $\widetilde{KO}(M)$. The surjectivity of $H^2 \to h^2$ implies that, on the E_2-page, realification is surjective on the zero diagonal computing $\widetilde{K}(M)$ and $\widetilde{KO}(M)$.

The only possibly nontrivial differentials on these two diagonals are the two differentials d_3 defined on $E_2^{2,-2}(K) = H^2$ and $E_2^{2,-2}(KO) = h^2$, respectively. Both take values in H^5 and thus vanish when H^5 is torsion-free. In that case, $\widetilde{K}(M)$ is an extension of H^2 by H^4, $\widetilde{KO}(M)$ is an extension of h^2 by H^4, and the reduced realification r defines a morphism of extensions that restricts to an isomorphism on the subgroups identified with H^4 (see above Lemma 2.5). So r is surjective if and only if $H^2 \to h^2$ is.

However, even when the differentials d_3 are nontrivial, the realification remains surjective on the zero diagonal of the E_3-page. Indeed, we have the following commutative diagram:

$$
\begin{array}{ccccccc}
0 \longrightarrow & E_3^{2,-2}(\widetilde{K}(M)) & \longrightarrow & H^2 = H^2(M, K^{-2}(pt)) & \xrightarrow{d_3} & H^5 = H^5(M, K^{-4}(pt)) \\
& \downarrow{\mathsf{r}_3^{2,-2}} & & \downarrow{\mathsf{r}_2^{2,-2}} & & \downarrow{\mathsf{r}_2^{5,-4}} \\
0 \longrightarrow & E_3^{2,-2}(\widetilde{KO}(M)) & \longrightarrow & h^2 = H^2(M, KO^{-2}(pt)) & \xrightarrow{d_3} & H^5 = H^5(M, KO^{-4}(pt))
\end{array}
$$

We are assuming that the second vertical arrow is surjective, and we know that the third vertical arrow is an isomorphism (again, see above Lemma 2.5). So it follows that the first vertical arrow is also surjective. Thus, again, realification is surjective on the zero diagonal of the E_∞-pages, and hence as a morphism $\widetilde{K}(M) \to \widetilde{KO}(M)$. $\qquad\square$

3 Homogeneous spaces and bundles

We outline the classical construction of a vector bundle over a homogeneous space from a representation. Moreover, we recall and extend existing results on how many stable classes of vector bundles arise in this way. These considerations will be key to the proof of the Main Theorem.

3.1 Homogeneous spaces

A homogeneous space is a manifold M that admits a smooth transitive action by a Lie group G. It may be identified with the orbit space G/H for a closed subgroup $H \subset G$, the stabilizer of a point under the given action. In general, there can be many different groups G acting transitively on M, so the presentation of M as G/H is not unique. However, the difference $\operatorname{rank} G - \operatorname{rank} H$ is a homotopy invariant of M (see [GOV97, II § 2.2, remark following Cor. 2]). Moreover, when G/H is compact and simply connected, we may assume the same for G:

Lemma 3.1. *Given any presentation of a simply connected compact homogeneous space as G/H, there is a simply connected compact Lie group G', a closed connected subgroup $H' \subset G$ and a homomorphism of Lie groups $f \colon G' \to G$ with $f(H') \subset H$ inducing a diffeomorphism $G'/H' \cong G/H$.*

Proof. Suppose G/H is some presentation of M. Let $H_1 \subset H$ and $G_1 \subset G$ be the identity components. As M is simply connected, we find that $H/H_1 \to G/G_1$ is an isomorphism, and hence that G/H is diffeomorphic to G_1/H_1. Next, by Montgomery's Theorem [GOV97, II S 2.3, Cor. 3], we have $G_1/H_1 \cong G^c/H^c$ for maximal compact subgroups $H^c \subset H_1$ and $G^c \subset G_1$. Let G' denote the compact factor of the universal cover of G^c, and let H' denote the preimage of H under the canonical homomorphism $j \colon G' \to G^c$. This homomorphism induces a diffeomorphism $G'/H' \cong G^c/H^c$. $\qquad\square$

From now on, all homogeneous spaces and all Lie groups considered in this article will be compact.

3.2 Representations and homogeneous bundles

As in Section 2, let $\mathbb{F}$ denote either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Given a compact Lie group G and a subgroup $i \colon H \to G$ as above, a representation ρ of H on $\mathbb{F}^m$ defines a right H-action on $G \times \mathbb{F}^m$ by the rule $((g,v),h) \mapsto (gh, \rho(h^{-1})v)$. The orbit space

$$E_\rho := G \times_H \mathbb{F}^m := (G \times \mathbb{F}^m)/H$$

is the total space of an $\mathbb{F}^m$-vector bundle over G/H. Vector bundles constructed in this way are called homogeneous.

Example 3.2. The tangent bundle of a homogeneous space G/H is always a homogeneous vector bundle [GOV97, II § 3.3]. The representation of H inducing the tangent bundle is called the isotropy representation for the presentation G/H.

The real and complex representation rings $RO(H)$ and $R(H) = RH$, and the quaternionic representation group $RSp(H)$, are constructed in a manner completely analogous to the construction of K-groups of a space: one starts with the set $\operatorname{Rep}_{\mathbb{F}}(H)$ of isomorphism classes of finite-dimensional $\mathbb{F}$-representations of H, endows it with a semigroup structure via the direct sum, and passes to the group completion $R_{\mathbb{F}}(H)$. Unsurprisingly, the properties of $R(H)$, $RO(H)$ and $RSp(H)$ are also analogous to those described in Section 2.1 for $K(M)$, $KO(M)$ and $KSp(M)$; see [Ada69, Ch. 3] for details. In particular, $R(H)$ and $RO(H)$

are rings via the tensor product, and $RO(H) \oplus RSp(H)$ is a $\mathbb{Z}_2$-graded ring. However, in some ways representation rings are simpler than K-rings:

- The representation ring RH is always free as an abelian group. A basis is given by the complex irreducible representations of H.

- The complexification maps $\mathsf{c} \colon RO(H) \to R(H)$ and $\mathsf{c}' \colon RSp(H) \to R(H)$ are always injective [Ada69, Prop. 3.22].

- The negative Tate cohomology $h^-(RH, \mathsf{t})$ always vanishes, as is immediate from the following description of t in terms of the given basis: Let us say that an irreducible complex representation is of real or quaternionic type if it is the complexification of a real or a quaternionic representation, respectively; otherwise we say that it is of complex type. On the basis elements of RH corresponding to irreducible representations of real or quaternionic type, t acts as the identity. The irreducible complex representations of complex type, on the other hand, come in pairs that are interchanged by t.

The construction of homogeneous vector bundles above yields well-defined morphisms of semigroups:

$$\begin{array}{ccc} \mathrm{Rep}_{\mathbb{F}}(H) & \longrightarrow & \mathrm{Vect}_{\mathbb{F}}(G/H) \\ \rho & \longmapsto & E_\rho \end{array}$$

These extend to group homomorphisms $\alpha_{\mathbb{F}} \colon R_{\mathbb{F}}(H) \to K_{\mathbb{F}}(G/H)$, which we also denote by α, α_O and α_{Sp}, respectively. The following lemmas are well known and can be checked directly from the definitions.

Lemma 3.3. *The maps α, α_O and α_{Sp} commute with the maps c, r , t, c' and q defined on representation and K-groups. They also commute with tensor products and exterior powers, i. e. they define λ-ring homomorphisms:*

$$\begin{aligned} \alpha \colon R(H) &\to K(G/H) \\ \alpha_O \colon RO(H) &\to KO(G/H) \\ \alpha_O \oplus \alpha_{Sp} \colon RO(H) \oplus RSp(H) &\to KO(G/H) \oplus KSp(G/H) \end{aligned}$$

The third of these is of course a graded homomorphism of $\mathbb{Z}_2$-graded λ-rings.

The product of two homogeneous spaces G_1/H_1 and G_2/H_2 is again a homogeneous space: $(G_1/H_1) \times (G_2/H_2) \cong (G_1 \times G_2)/(H_1 \times H_2)$. The maps $\alpha_{\mathbb{F}}$ also commute with the external products defined on representation and K-rings, i. e. we have commutative diagrams of the form:

$$\begin{array}{ccc} R_{\mathbb{F}_1}(H_1) \otimes R_{\mathbb{F}_2}(H_2) & \xrightarrow{\;\;\mu\;\;} & R_{\mathbb{F}_{12}}(H_1 \times H_2) \\ \Big\downarrow{\scriptstyle \alpha_{\mathbb{F}_1} \otimes \alpha_{\mathbb{F}_2}} & & \Big\downarrow{\scriptstyle \alpha_{\mathbb{F}_{12}}} \\ K_{\mathbb{F}_1}(G_1/H_1) \otimes K_{\mathbb{F}_2}(G_2/H_2) & \xrightarrow{\;\mu\;} & K_{\mathbb{F}_{12}}(G_1/H_1 \times G_2/H_2) \end{array}$$

Here, $(\mathbb{F}_1, \mathbb{F}_2, \mathbb{F}_{12})$ may be any of the triples $(\mathbb{C}, \mathbb{C}, \mathbb{C})$, $(\mathbb{R}, \mathbb{R}, \mathbb{R})$, $(\mathbb{R}, \mathbb{H}, \mathbb{H})$, $(\mathbb{H}, \mathbb{R}, \mathbb{H})$ or $(\mathbb{H}, \mathbb{H}, \mathbb{R})$. The commutativity of the square follows from the previous Lemma and the fact that $\alpha_{\mathbb{F}}$ commutes with the pullback maps along the projections to each factor. More generally, the maps $\alpha_{\mathbb{F}}$ are natural in the following sense:

Lemma 3.4 (Naturality)**.** *Let G and G' be compact Lie groups with closed subgroups H and H'. Let $f\colon G' \to G$ be a homomorphism of Lie groups such that $f(H') \subset H$, and let $\bar{f}\colon G'/H' \to G/H$ be the induced map. Then we have commutative diagrams as follows:*

$$
\begin{array}{ccc}
R_{\mathbb{F}}(H) & \xrightarrow{\ \alpha_{\mathbb{F}}^{H'}\ } & K_{\mathbb{F}}(G/H) \\
\downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle \bar{f}^*} \\
R_{\mathbb{F}}(H') & \xrightarrow{\ \alpha_{\mathbb{F}}^{H}\ } & K_{\mathbb{F}}(G'/H')
\end{array}
$$

In the next two subsections, we discuss some general situations in which $\alpha_{\mathbb{F}}$ is known to be surjective. Of course, given a homogeneous space M, the surjectivity of $\alpha_{\mathbb{F}}$ may depend on the chosen presentation of M as G/H. For simply connected M, we may restrict our attention to presentations with simply connected G:

Lemma 3.5. *Let M be a simply connected compact homogeneous space. If there is any presentation of M as G/H such that $\alpha\colon R(H) \to K(M)$ is surjective, then there is also such presentation with simply connected G and connected H.*

Indeed, this is immediate from Lemma 3.1 and naturality (Lemma 3.4).

3.3 Complex bundles: Pittie and Steinberg's Theorem

It turns out that the complex version of α is surjective in many interesting cases:

Theorem 3.6. *Let G be a compact connected Lie group with $\pi_1(G)$ torsion-free, and let $H \subset G$ be a closed connected subgroup such that $\operatorname{rank} G - \operatorname{rank} H \leq 1$. Then the natural map $\alpha\colon RH \to K^0(G/H)$ is surjective and induces an isomorphism*

$$
\mathbb{Z} \otimes_{RG} RH \cong K^0(G/H).
$$

If $\operatorname{rank} H = \operatorname{rank} G$, then moreover $K^0(G/H)$ is a finitely generated free abelian group, and $K^1(G/H) = 0$.

Remark 3.7. The rank assumptions here are optimal:
– When $\operatorname{rank} G - \operatorname{rank} H = 1$, the higher K-group $K^1(G/H)$ may no longer vanish, and $K^0(G/H)$ may not be free. For example, this happens for the Berger space B^{13} (Proposition 6.5).
– When $\operatorname{rank} G - \operatorname{rank} H = 2$, even the first part of the result no longer holds. Consider for example $G = Spin(4) = \mathbb{S}^3 \times \mathbb{S}^3$ and H the trivial subgroup. The map α is clearly trivial, whereas $K(\mathbb{S}^3 \times \mathbb{S}^3) = [\mathbb{Z}] \oplus \mathbb{Z}$ is not trivial.

When $\operatorname{rank} H = \operatorname{rank} G$, Theorem 3.6 is well-known. In many cases, it was already verified by Atiyah and Hirzebruch in the paper that founded topological K-theory [AH61, Thm 5.8]. In general, it can be obtained from a spectral sequence due to Hodgkin and a theorem of Pittie and Steinberg. We recall both these ingredients here, derive a corollary of the latter result, and explain how to obtain Theorem 3.6 under the stated weaker hypothesis that $\operatorname{rank} G - \operatorname{rank} H \leq 1$.

Hodgkin's spectral sequence is as follows:

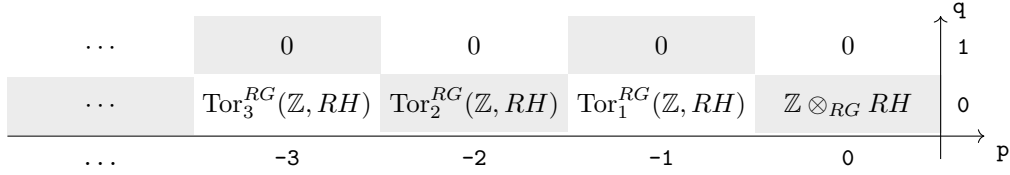| ...	0	0	0	0	1
...	$\mathrm{Tor}_3^{RG}(\mathbb{Z}, RH)$	$\mathrm{Tor}_2^{RG}(\mathbb{Z}, RH)$	$\mathrm{Tor}_1^{RG}(\mathbb{Z}, RH)$	$\mathbb{Z} \otimes_{RG} RH$	0
...	−3	−2	−1	0	p

Figure 1: The E_2-page of Hodgkin's spectral sequence converging to $K^*(G/H)$. The shaded entries compute $K^0(G/H)$, the remaining entries $K^1(G/H)$.

Theorem 3.8 ([Hod75; McL79, Thm 4.1])**.** *Suppose G is a compact, connected Lie group with torsion-free fundamental group. Then for G-spaces X and Y we have a strongly convergent spectral sequence*

$$E_2^{p,q} = \{\mathrm{Tor}_{-p}^{RG}(K_G^*(X), K_G^*(Y))\}^q \quad \Rightarrow \quad K_G^{p+q}(X \times Y).$$

(The G-spaces are assumed to be compactly generated and have the homotopy type of a CW complex, see [Hod75, §I.1]*.)*

In this theorem, the complex representation ring RG is to be viewed as a $\mathbb{Z}_2$-graded ring concentrated in degree zero, the equivariant K-groups $K_G^*(X)$ and $K_G^*(Y)$ are graded modules over this graded ring, and the Tor functors Tor_{-p}^{RG} are to be taken in the category of such graded modules. The q-coordinate refers to the internal grading of these Tor modules. To prove Theorem 3.6, one applies the spectral sequence to $X := G$ and $Y := G/H$. Then $K_G^*(X) \cong \mathbb{Z}$, $K_G^*(Y) \cong RH$ and $K_G^*(X \times Y) \cong K^*(G/H)$. In particular, all rings and modules appearing on the E_2-page are concentrated in degree zero, so that we can replace the graded Tor functor by the usual Tor functor. The spectral sequence thus takes the simple form displayed in Fig. 1. Moreover, the edge homomorphism $E_2^0 = \mathbb{Z} \otimes_{RG} RH \to K^0(G/H)$ is induced by α [Hod75, Lemma 9.1]. The contribution of Pittie and Steinberg is that, when rank H = rank G, all the other groups on the E_2-page vanish:

Theorem 3.9 ([Pit72, Ste75])**.** *Let G be as in Theorem 3.6. For every closed connected subgroup $H \subset G$ of maximal rank, RH is a finitely generated free RG-module.*

This implies Theorem 3.6 in the maximal rank case.

Note that the representation ring of Lie group G as above is isomorphic to the tensor product of a polynomial ring with a ring of Laurent polynomials. Therefore, a finitely generated RG-module is free if and only if it is projective if and only if it is flat: the first two notions coincide by a generalization of the Quillen-Suslin theorem [Swa78, Swa92], the second two coincide for finitely generated modules over arbitrary Noetherian rings. In particular, the notions of "projective dimension" and "weak/flat dimension" are equivalent over such RG. Below, we always write "projective dimension", but we use Tor rather than Ext in all arguments because the Tor-functors are the ones we are ultimately interested in.

Starting from Pittie and Steinberg's result, we prove:

18

Corollary 3.10. *Let G be as in Theorem 3.6, and let $H \subset G$ be an arbitrary closed connected subgroup. Then the projective dimension of RH as an RG-module is at most $\operatorname{rank} G - \operatorname{rank} H$.*

The corollary implies, in particular, that for G and H as stated we have:

$$\operatorname{Tor}_p^{RG}(RH, \mathbb{Z}) = 0 \text{ for } p > \operatorname{rank} G - \operatorname{rank} H \tag{7}$$

Together with Hodgkin's spectral sequence (Fig. 1), this implies the general form of Theorem 3.6.

The vanishing result (7) was observed even before Pittie announced his result by Snaith [Sna71, Thm 4.2]. His strategy for passing from full rank to arbitrary rank is similar to the strategy used in the proof below. However, in the absence of Steinberg and Pittie's result, Snaith had to work with completions.

Proof of Corollary 3.10. We use the usual notation $\operatorname{pd}_R M$ to denote the projective dimension of a module M over a ring R. Consider first the case that $H \subset G$ is an inclusion of tori $T' \subset T$. Then T' is a direct factor of T [HN12, Lem. 15.3.2], and $R(T) \cong R(T')[x_1^{\pm 1}, \ldots, x_m^{\pm 1}]$, where m is the difference of ranks between T and T'. In this case, the projective dimension $\operatorname{pd}_{RT} RT'$ is exactly m: we need only apply the First Change of Rings Theorem [Wei94, Thm 4.3.3] inductively to the elements $(x_1 - 1), (x_2 - 1), \ldots, (x_m - 1)$.

Next, consider the case of a torus T' contained in an arbitrary compact Lie group G with torsion-free fundamental group. Choose a maximal torus T of G such that $T' \subset T$, so that we may again view RT' as an RT-module. Applying the General Change of Rings Theorem [Wei94, Thm 4.3.1] to the restriction map $RG \to RT$, we obtain the inequality $\operatorname{pd}_{RG} RT' \leq \operatorname{pd}_{RT} RT' + \operatorname{pd}_{RG} RT$. By Theorem 3.9, the second summand on the right side vanishes, and by the previous step $\operatorname{pd}_{RT}(RT') \leq \operatorname{rank} G - \operatorname{rank} T'$. So altogether we obtain the required inequality.

Finally, for an arbitrary closed connected subgroup $H \subset G$, consider the inclusion of a maximal torus $i \colon T' \hookrightarrow H$. By [Ati68, Rem. 1 after Prop. 4.9], there exists a morphism of RH-modules $i_* \colon RT' \to RH$ providing a left inverse to the restriction morphism i^*. In the situation at hand, we can view i_* and i^* as morphisms of RG-modules. This shows that the vanishing of $\operatorname{Tor}_p^{RG}(RT', -)$ implies the vanishing of $\operatorname{Tor}_p^{RG}(RH, -)$, and hence that $\operatorname{pd}_{RG} RH \leq \operatorname{pd}_{RG} RT'$. $\square$

Remark 3.11. The results on the projective dimensions can be extended to a slightly larger class of groups. Let us say that a Lie group is *good* if it is compact and connected and if its complex representation ring RG is isomorphic to a polynomial ring tensored with a Laurent ring. As mentioned above, all compact connected Lie groups whose fundamental group is torsion-free are good. More precisely, a compact connected Lie group is good if and only if each simple factor of its semisimple part is either simply connected or of the form $SO(2n + 1)$ [Ste75]. Steinberg's theorem in fact says:

> Let G be a compact connected Lie group. Then RH is free over RG for every closed connected subgroup $H \subset G$ of maximal rank if and only if G is good.

Corollary 3.10 and its proof above likewise hold more generally for good G. However, in Theorem 3.6, the assumption that $\pi_1(G)$ is torsion-free *cannot* be weakened to "G is good". For example, consider $SO(2n+1)/SO(2n) \cong Spin(2n+1)/Spin(2n) \cong \mathbb{S}^{2n}$. Both $SO(2n+1)$ and $Spin(2n+1)$ are good, but the natural map $RH \to K(\mathbb{S}^{2n})$ referred to in the theorem is only surjective for the presentation with $G = Spin(2n+1)$. Indeed, by Lemma 3.4 the pullback along the covering homomorphism $\pi\colon Spin(2n) \twoheadrightarrow SO(2n)$ fits into a commutative triangle:

$$
\begin{array}{ccc}
R(Spin(2n)) & \xrightarrow{\ \alpha(Spin)\ } & \\
\pi^* \big\uparrow & \searrow & K(\mathbb{S}^{2n}) \\
R(SO(2n)) & \xrightarrow{\ \alpha(SO)\ } &
\end{array}
$$

The theorem implies that $\alpha(Spin)$ is an epimorphism that identifies $K(\mathbb{S}^{2n})$ with $\mathbb{Z}[\tilde{\Delta}_+]/(\tilde{\Delta}_+^2)$, where $\tilde{\Delta}_+ = \Delta_+ - \operatorname{rank}(\Delta_+)$ is one of the reduced half-spin representations. This generator does not lie in the image of $\alpha(SO)$.

3.4 Real bundles

Suppose that G and H are as above: G is a compact connected Lie group with torsion-free fundamental group, and H is a closed connected subgroup. The real and quaternionic versions of α are surjective only under far more restrictive conditions than those of Theorem 3.6. In [Sey73, Thm 5.24], Seymour describes how to compute the cokernel of $\alpha_O\colon RO(H) \to KO(G/H)$ based on a detailed understanding of how the complex version of α interacts with the involution on $K(G/H)$ and the decomposition of $R(H)$ into representations of different types. The precise result is both difficult to state and to work with, but it has the following simple corollary, of which we here provide a short and independent proof:

Proposition 3.12 ([Sey73, Cor. 5.25]). *Let G and H be as in Theorem 3.6, with $\operatorname{rank} G = \operatorname{rank} H$. If the conjugation on $R(H)$ is the identity map, then both $\alpha_O\colon RO(H) \to KO(G/H)$ and $\alpha_{Sp}\colon RSp(H) \to KSp(G/H)$ are surjective.*

Remark 3.13. The condition on $R(H)$ is satisfied for a large class of Lie groups: conjugation acts trivially on $R(H)$ for any compact connected Lie group H that has no circle factor and no simple factors of types A_n with $n \geq 2$, D_{2n+1} with $n \geq 4$ or E_6. For simply connected H, this follows from [Zib15, Table 1]. In general, H will be a quotient of a simply connected group $\tilde{H}$ by a finite central subgroup, and then $R(H)$ will embed into $R(\tilde{H})$; see the proof of Lemma 6.1.

On the other hand, the condition on $R(H)$ is by no means a necessary one. For example, α_O and α_{Sp} are both surjective for even-dimensional projective spaces $\mathbb{CP}^{2n} = U(2n+1)/(U(2n) \times U(1))$, even though the conjugation acts nontrivially on $R(U(2n) \times U(1))$.

Proof. As $K^1(\frac{G}{H})$ vanishes, we can apply Lemma 2.1. Thus, the different versions of α fit into the following exact diagram with exact rows (compare [Zib15, Prop. 2.2]):

$$
\begin{array}{ccccccc}
R(H) \oplus R(H) & \longrightarrow & RO(H) \oplus RSp(H) & \longrightarrow & h^+(RH) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \alpha\oplus\alpha} & & \downarrow{\scriptstyle \alpha_O\oplus\alpha_{Sp}} & & \downarrow{\scriptstyle h^+(\alpha)} & & \\
K(\frac{G}{H}) \oplus K(\frac{G}{H}) & \longrightarrow & KO(\frac{G}{H}) \oplus KSp(\frac{G}{H}) & \longrightarrow & h^+(K(\frac{G}{H})) & \longrightarrow & 0
\end{array}
$$

We already know that α is surjective. As the involution on $R(H)$ is trivial, so is the involution on $K(\frac{G}{H})$. Therefore, $h^+(\alpha)$ is obtained from α by tensoring with $\mathbb{Z}_2$. In particular, $h^+(\alpha)$ is also surjective. The surjectivity of α_O and α_{Sp} now follows from the diagram. $\qquad\square$

Note that the class of spaces satisfying the hypotheses in Proposition 3.12 is closed under products: if the conjugation acts as the identity both on $R(H_1)$ and $R(H_2)$, then it also acts as the identity on $R(H_1 \times H_2) \cong R(H_1) \otimes R(H_2)$. In fact, we have the slightly more general "product theorem":

Proposition 3.14. *Let G_1, H_1 and G_2, H_2 be as in Theorem 3.6, with* $\operatorname{rank} G_i = \operatorname{rank} H_i$. *Then α_O and α_{Sp} are surjective for the product $G_1/H_1 \times G_2/H_2$ if and only if they are surjective for both G_1/H_1 and G_2/H_2 and one of $h^-(K(G_1/H_1))$, $h^-(K(G_2/H_2))$ is zero.*

Remark 3.15. The condition $h^-(K(M)) = 0$ is satisfied in each of the following cases:

– When M is any of the homogeneous spaces included in Proposition 3.12, since for these conjugation acts trivially on $K(M)$ and $K(M)$ is torsion-free by Theorem 3.6.
– When $M = \mathbb{CP}^{2n}$. This can be verified directly, or using Lemma 2.1 and the known results for $KO^{2i}(\mathbb{CP}^{2n})/\mathsf{r}$ [Zib15, §4.2].
– When the cohomology of M is concentrated in degrees multiples of 4. This can be seen using the AHSS: we find that the conjugation acts trivially on each homogeneous summand of the graded ring associated with the cellular filtration on $K(M)$, and hence that h^- vanishes on each summand. It then follows inductively that h^- vanishes for the whole K-ring.

In contrast, when M is $\mathbb{S}^{8n+2}$, $\mathbb{S}^{8n+6}$ or $\mathbb{CP}^{2n+1}$, we have $h^-(K(M)) = \mathbb{Z}_2$.

Proof of Proposition 3.14. Consider the diagram used in the previous proof, with $H_1 \times H_2$ in place of H and $G_1 \times G_2$ in place of G. Using the Künneth theorems for complex representation rings [Ada69, Thm 3.65], for K-theory [Ati62] and for Tate cohomology (see Lemma 2.3 and Remark 2.2), we obtain:

$$R(H_1 \times H_2) \cong R(H_1) \otimes R(H_2) \qquad h^+(R(H_1 \times H_2)) \cong h^+(RH_1) \otimes h^+(RH_2)$$
$$K\big(\tfrac{G_1}{H_1} \times \tfrac{G_2}{H_2}\big) \cong K\big(\tfrac{G_1}{H_1}\big) \otimes K\big(\tfrac{G_2}{H_2}\big) \quad h^+(K\big(\tfrac{G_1}{H_1} \times \tfrac{G_2}{H_2}\big)) \cong h^+(K\big(\tfrac{G_1}{H_1}\big)) \otimes h^+(K\big(\tfrac{G_2}{H_2}\big))$$
$$\oplus\, h^-(K\big(\tfrac{G_1}{H_1}\big)) \otimes h^-(K\big(\tfrac{G_2}{H_2}\big))$$

The vertical map $\alpha_O \oplus \alpha_{Sp}$ is surjective if and only if the vertical map $h^+(\alpha)$ on the right is surjective, which is the case if and only if it is surjective for both $\frac{G_1}{H_1}$ and $\frac{G_2}{H_2}$ and if the additional summand $h^-(K\big(\tfrac{G_1}{H_1}\big)) \otimes h^-(K\big(\tfrac{G_2}{H_2}\big))$ vanishes. $\quad\square$

When one of the factors in the product is a sphere whose dimension is a multiple of eight, the other factor may be arbitrary — the isomorphism (1) in Section 2.1 and the surjectivity of α_O for $\mathbb{S}^{8n}$ imply:

Proposition 3.16. *If $M = G/H$ is a compact homogeneous space such that α_O is surjective, then*

$$\alpha_O \colon RO(H \times Spin(8n)) \to KO(M \times \mathbb{S}^{8n})$$

is also surjective.

4 Curvature on vector bundles

4.1 Curvature on homogeneous bundles

Any compact Lie group G admits a biinvariant metric, which in particular has non-negative sectional curvature. Moreover, isometric quotients of non-negatively curved manifolds also have non-negative sectional curvature by O'Neill's theorem on Riemannian submersions. It easily follows that compact homogeneous spaces G/H and homogeneous vector bundles $E_\rho = G \times_H \mathbb{F}^m$ over compact homogeneous spaces admit metrics with non-negative sectional curvature (where $\mathbb{F}^m$ is endowed with the Euclidean flat metric).

We will use the following result that relates the K and KO-theory of a homogeneous space to the SCSQ.

Proposition 4.1 ([Gon17])**.** *Let $M = G/H$ be a compact homogeneous space, where G is a compact Lie group, and assume that the map $\alpha_O \colon RO(H) \to KO(M)$ (respectively $\alpha \colon R(H) \to K(M)$) is surjective. Then for every real (resp. complex) vector bundle E over M, the product $E \times \mathbb{R}^k$ carries a metric of non-negative sectional curvature for some integer k.*

We give here the idea of the proof, which consists in showing that every vector bundle E is stably isomorphic to a homogeneous vector bundle (see [Gon17] for the details). By assumption, the virtual class $E - 0 \in K_{\mathbb{F}}(G/H)$ equals $E_{\rho_1} - E_{\rho_2}$ for some representations ρ_1, ρ_2 of H. It is well known that one can choose a representation ρ_3 such that $E_{\rho_2} \oplus E_{\rho_3}$ is a trivial bundle m. Thus

$$E = E_{\rho_1} - E_{\rho_2} + E_{\rho_3} - E_{\rho_3} = E_{\rho_1 \oplus \rho_3} - m$$

in $K_{\mathbb{F}}(G/H)$, and it follows that $E \times \mathbb{R}^k \cong E_{\rho_1 \oplus \rho_3 \oplus (k-m)}$ for some k.

4.2 Vector bundles over tangentially homotopy equivalent spaces

In this section we prove Proposition 1.3. It is a consequence of the "Work Horse Theorem" 4.2 below, which follows from classical results by Haefliger [Hae61] and Siebenmann [Sie69].

"Work Horse Theorem" 4.2 ([TW15, Theorem 10.1.6]). *Let $E_i \to M_i^n$ be vector bundles over compact manifolds of the same rank l, for $i = 1, 2$. Suppose that $f\colon E_1 \to E_2$ is a tangential homotopy equivalence, where $l \geq 3$ and $l > n$. Then f is homotopic to a diffeomorphism.*

Proof of Proposition 1.3. Let $f\colon M \to N$ be a tangential homotopy equivalence and $\pi\colon E \to N$ an arbitrary smooth real (resp. complex) vector bundle. We shall prove that the manifold $E \times \mathbb{R}^k$ carries a metric with non-negative sectional curvature for some k.

Consider the pullback of E along f:

$$
\begin{array}{ccc}
f^*E & \xrightarrow{\ h\ } & E \\
\downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\
M & \xrightarrow{\ f\ } & N
\end{array}
$$

When E is a complex vector bundle, the pullback f^*E is a also a complex vector bundle. But let us first ignore any possible complex structure on E and concentrate on the (underlying) real bundle/the total space of E. The map f is a homotopy equivalence, and the same holds for the bundle projections π and π'. Hence h is also a homotopy equivalence. By our assumption on f, we have $f^*(TN) = TM$ in $KO(M)$. Using the isomorphisms

$$TE \cong \pi^*TN \oplus \pi^*E$$
$$T(f^*E) \cong (\pi')^*(TM) \oplus (\pi')^*f^*E$$

[tD91, Satz IX.6.9] together with the commutativity of the above square we obtain that $h^*(TE) = T(f^*E)$ in $KO(f^*E)$. Thus h is also a tangential homotopy equivalence.

If necessary, replace E by $E \times \mathbb{R}^k$, where k is chosen large enough to satisfy the assumptions of the "Work Horse Theorem" 4.2. Possibly increasing k once more, we may by our assumption on M equip $f^*(E \times \mathbb{R}^k) = f^*E \times \mathbb{R}^k$ with a metric of non-negative sectional curvature. Then the "Work Horse Theorem" 4.2 implies that h defines a diffeomorphism $f^*E \times \mathbb{R}^k \to E \times \mathbb{R}^k$, and hence $E \times \mathbb{R}^k$ can be equipped with a metric of non-negative sectional curvature. $\qquad\square$

4.3   Products with the real line

Here we provide a proof of the claim in Example 1.2. The starting point is the following consequence of Smale's h-cobordism theorem.

Lemma 4.3 ([Bro65]). *Let M^n, N^n be compact simply connected manifolds of dimension $n \geq 5$. If $M \times \mathbb{R}$ is diffeomorphic to $N \times \mathbb{R}$, then M is diffeomorphic to N.*

We obtain the following equivalence for the existence of non-negatively curved metrics.

Lemma 4.4. *Let M^n be a compact simply connected manifold of dimension $n \geq 5$. Then $M \times \mathbb{R}$ admits a metric with non-negative sectional curvature if and only if so does M.*

Proof. One implication is trivial: if M admits a non-negatively curved metric, take the product metric on $M \times \mathbb{R}$.

Suppose conversely that $M \times \mathbb{R}$ admits a metric with non-negative sectional curvature. Then by the Soul Theorem, $M \times \mathbb{R}$ is diffeomorphic to the total space of a vector bundle over a soul S of $M \times \mathbb{R}$. Since any vector bundle projection is a homotopy equivalence, it follows that S is homotopy equivalent to M, and in particular of the same dimension. Hence $M \times \mathbb{R}$ is diffeomorphic to the total space of a real line bundle over S, which must be $S \times \mathbb{R}$ as S is simply connected. Using Lemma 4.3 we see that M is diffeomorphic to S. This completes the proof: recall that a soul S of $M \times \mathbb{R}$ inherits a metric of non-negative sectional curvature, as it is a totally geodesic submanifold. $\square$

5   The positive cases

We are now ready to apply the tools developed in the previous sections to the Stable Converse Soul Question (SCSQ). Almost all statements made in Section 1 are direct consequences of the results obtained in Propositions 5.3, 5.5 and 5.6 below:

Proof of Theorem 1.5. By Theorem 3.6, the map $\alpha \colon R(H) \to K(G/H)$ is surjective for all homogeneous spaces G/H in Theorem 1.5. Now apply the complex versions of Propositions 4.1 and 1.3. $\square$

Proofs of Theorems 1.6 to 1.11. For all the homogeneous spaces contained in each theorem, there is a presentation G/H for which the map $\alpha_O \colon RO(H) \to KO(G/H)$ is surjective: the summary below provides a reference for surjectivity of α_O in each case. Therefore Proposition 4.1 gives the result.

Theorem 1.6	$\Leftarrow$	Proposition 5.6
Theorem 1.7	$\Leftarrow$	Proposition 5.3
Theorem 1.8	$\Leftarrow$	Proposition 3.12
Theorem 1.9	$\Leftarrow$	Proposition 3.14, Prop. 5.1 in [Gon17] and Remark 3.15
Theorem 1.10	$\Leftarrow$	Proposition 5.5
Theorem 1.11	$\Leftarrow$	Proposition 3.16

(Theorem 1.11 also relies on Proposition 1.3.) $\square$

Proof of the Main Theorem. Given Proposition 1.3 and the classification of homogeneous spaces with positive curvature [WZ], we have to prove the statement for the CROSSes and for the spaces W^6, B^7, $W_{p,q}^7$, W^{12} and W^{24}. For any of the CROSSes, it is shown in [Gon17] that there is a presentation G/H for which $\alpha_O \colon RO(H) \to KO(G/H)$ is surjective, so Proposition 4.1 gives the result. For the spaces W^6, B^7 and $W_{p,q}^7$ the result follows from Theorem 1.7. Finally, recall that the usual presentations of W^{12} and W^{24} are $Sp(3)/Sp(1)^3$ and $F_4/Spin(8)$, respectively, and observe that conjugation acts trivially on both $Sp(1)^3$ and $Spin(8)$. Thus Theorem 1.8 completes the proof. $\square$

space	H^0	H^1	H^2	H^3	H^4	H^5	H^6	H^7	reference
Wu	$\mathbb{Z}$	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}$			[Bar65, pp. 9–10]
W^6	$\mathbb{Z}$	0	$\mathbb{Z}^2$	0	$\mathbb{Z}^2$	0	$\mathbb{Z}$		[AZ16, p. 1005]
$\widetilde{Gr}_{2,5}$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$		[Van99, Thm 11]
B^7	$\mathbb{Z}$	0	0	0	$\mathbb{Z}_{10}$	0	0	$\mathbb{Z}$	[CE03, p. 365]
$T_1\mathbb{S}^4$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}_2$	0	0	$\mathbb{Z}$	[CE03, p. 379]
$W^7_{p,q}$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}_{|p^2+q^2+pq|}$	$\mathbb{Z}$	0	$\mathbb{Z}$	[Esc82, Prop. 36]
$N^7_{p,q}$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}_{q^2}$	$\mathbb{Z}$	0	$\mathbb{Z}$	[KS88, p. 375]
$N^7_{p,q,r}$	$\mathbb{Z}$	0	$\mathbb{Z}^2$	0	$\mathbb{Z}_{|2pqr|}$	$\mathbb{Z}^2$	0	$\mathbb{Z}$	[Ber91, p. 85]

Table 1: Cohomologies of homogeneous spaces of dimension at most seven

5.1 Homogeneous spaces of dimension at most seven

Compact simply connected homogeneous spaces of dimension ≤ 7 are classified (see [Gor12, Kla88]). Such a space is diffeomorphic to one of the following:

- $\mathbb{S}^n = Spin(n+1)/Spin(n)$, with $2 \leq n \leq 7$

- $\mathbb{CP}^n = U(n+1)/(U(n) \times U(1))$ with $n = 2$ or $n = 3$

- a product of spheres, $\mathbb{CP}^2 \times \mathbb{S}^2$, or $\mathbb{CP}^2 \times \mathbb{S}^3$

- one of the following spaces appearing in the corresponding dimensions:

  $n = 5 :$ $Wu = SU(3)/SO(3)$, the Wu-manifold

  $n = 6 :$ $W^6 = SU(3)/T^2$, the Wallach flag manifold

  $n = 6 :$ $\widetilde{Gr}_{2,5} = SO(5)/(SO(2) \times SO(3))$, the oriented real Grassmannian

  $n = 7 :$ $T_1\mathbb{S}^4 = Spin(5)/Spin(3)$, the unit tangent bundle of $\mathbb{S}^4$
  (Here, the inclusion $Spin(3) \subset Spin(5)$ is the obvious one.)

  $n = 7 :$ $B^7 = Spin(5)/Spin(3)$, the Berger space
  (Here, the inclusion of $Spin(3)$ is defined by a nontrivial 5-dimensional real representation.)

  $n = 7 :$ $W^7_{p,q} = SU(3)/T^1_{p,q}$, the Aloff-Wallach spaces

  $n = 7 :$ $N^7_{p,q} = (SU(2) \times SU(3))/(U(1) \times SU(2))_{p,q}$, the Witten manifolds

  $n = 7 :$ $N^7_{p,q,r} = SU(2)^3/T^2_{p,q,r}$

  $n = 7 :$ $\mathbb{S}^2 \times Wu$

T^1 and T^2 denote the circle and the torus groups respectively. The subscripts in the quotient groups denote different inclusions parametrized by coprime integers p, q, r.

Table 1 displays the integer cohomology groups $H^k(-, \mathbb{Z})$ of the spaces described above, together with the corresponding references. Using Proposition 2.7, we easily deduce:

Lemma 5.1. *The reduced realification map* $\mathsf{r}\colon \widetilde{K}(M) \to \widetilde{KO}(M)$ *is surjective for all compact simply connected homogeneous spaces of dimension at most* 7, *except for* Wu *and* $\mathbb{S}^2 \times Wu$.

Proof. As the integer cohomology of $\mathbb{S}^n$ and $\mathbb{CP}^m$ has no torsion, the same holds for any possible product $M = \mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_i} \times \mathbb{CP}^{m_1} \times \cdots \times \mathbb{CP}^{m_j}$. In particular, they all satisfy the hypothesis on H^{n-2} in Proposition 2.7. All the spaces in Table 1 except for Wu likewise satisfy the hypothesis on H^{n-2}. So for all of these spaces, the reduced realification is surjective. For Wu, the same Proposition shows that the reduced realification is *not* surjective. For the only remaining space, $\mathbb{S}^2 \times Wu$, we will see non-surjectivity in the proof of Proposition 5.3 below. $\qquad\square$

The ranks of the Lie groups involved in the spaces listed above are:

$$\mathrm{rank}\,(U(n)) = \mathrm{rank}\,(SU(n)) + 1 = n$$
$$\mathrm{rank}\,(Spin(2n+1)) = \mathrm{rank}\,(Spin(2n)) = n$$

It is immediate to check which spaces satisfy $\mathrm{rank}\,G - \mathrm{rank}\,H \le 1$, for which Theorem 3.6 gives the following result.

Lemma 5.2. *Let* G/H *be a compact simply connected homogeneous space of dimension at most* 7 *and nondiffeomorphic to* $\mathbb{S}^3 \times \mathbb{S}^3$, *with* G *compact and simply connected. Then the map* $\alpha\colon R(H) \to K(G/H)$ *is surjective.*

Moreover, we show that the real version α_O is surjective for all homogeneous spaces of dimension at most 7.

Proposition 5.3. *Let* G/H *be a compact simply connected homogeneous space of dimension at most* 7 *with* G *compact and simply connected. Then the map* $\alpha_O\colon RO(H) \to KO(G/H)$ *is surjective.*

Remark 5.4. Proposition 5.3 is optimal in the sense that in dimension $n = 8$ it is no longer true: by Proposition 5.6 below, surjectivity of α_O fails for the presentation $SU(2)^{\times 4}/T$ of the product $\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$.

Proof of Proposition 5.3. When G/H is nondiffeomorphic to $\mathbb{S}^3 \times \mathbb{S}^3, Wu$ or $\mathbb{S}^2 \times Wu$, the result follows immediately from Lemmas 5.1 and 5.2.

For $G/H = \mathbb{S}^3 \times \mathbb{S}^3$, recall that $KO(\mathbb{S}^3 \times \mathbb{S}^3) = [\mathbb{Z}]$ (see Lemma 2.4), hence α_O is surjective for trivial reasons.

It remains to consider the cases when G/H is diffeomorphic to Wu or $\mathbb{S}^2 \times Wu$.

For $G/H = Wu$, the AHSS's and the description of the cohomology in Table 1 show that $\widetilde{K}(Wu)$ vanishes while $\widetilde{KO}(Wu) \cong \mathbb{Z}_2$, so clearly realification is not surjective in this case. An explicit generator of $\widetilde{KO}(Wu)$ is given by the reduced stable class of the tangent bundle of Wu. Indeed, by (d) in Section 2.4, the second Stiefel-Whitney class defines an isomorphism $\widetilde{KO}(Wu) \to H^2(Wu; \mathbb{Z}_2)$, and it is known that the second Stiefel-Whitney class of Wu is nontrivial (see for example [Bar65, Lem. 1.1], where the Wu-manifold is denoted by X_{-1}). On the other hand, the tangent bundle of a homogeneous space is a homogeneous vector bundle (see Section 3.2). This implies the surjectivity of $\alpha_O\colon RO(H) \to KO(Wu)$.

Essentially the same argument works for the product $M = \mathbb{S}^2 \times Wu = G/H$. Its cohomology is easily computed using the Künneth theorem. The AHSS's identify $\widetilde{K}(M)$ and $\widetilde{KO}(M)$ with the kernels of the corresponding differentials $d_3 \colon E_2^{2,-2} \to E_2^{5,-4}$, both of which vanish according to the Leibniz rule. Using above the description of the realification map above Lemma 2.5, we can thus identify the cokernel of $\mathsf{r} \colon \widetilde{K}(M) \to \widetilde{KO}(M)$ with the cokernel of the reduction map $\mathsf{red} \colon H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{Z}_2)$. This cokernel in turn can be identified with the $\mathbb{Z}_2$-term of $H^2(M, \mathbb{Z}_2)$ corresponding to $H^2(Wu, \mathbb{Z}_2)$ under the Künneth formula. Since the second Stiefel-Whitney class of $\mathbb{S}^2$ vanishes, it follows from the Whitney product formula that the cokernel of red is generated by the second Stiefel-Whitney class of M. So again, the reduced tangent bundle generates the cokernel of r, and hence $\alpha_O \colon RO(H) \to KO(M)$ is surjective. $\qquad\square$

5.2   Some products of spheres

A product of spheres is a homogeneous space G/H and the difference $\mathrm{rank}\, G - \mathrm{rank}\, H$ equals the number of odd-dimensional factors.

Proposition 5.5. *Each of the products of spheres in Theorem 1.10 has a presentation G/H for which the map $\alpha_O \colon RO(H) \to KO(G/H)$ is surjective.*

Proof. The case $\ell = 1$ has already been discussed in [Gon17]. Here, the most delicate case is the case of spheres of dimensions $n \equiv 1$: it is a result of Rigas [Rig78] that in this case α_O is surjective for some presentation of $\mathbb{S}^n$, though it is not apparent which presentation this is. In all other dimensions, and for all products of spheres considered below, the usual presentation of (products) of $\mathbb{S}^n$ as (products of) $Spin(n+1)/Spin(n)$ does the job, and we implicitly use this presentation without further specification.

For the tuples $\{3,3\}, \{7,7\}, \{7,7,7\}$, the product G/H satisfies $KO(G/H) = [\mathbb{Z}]$ by Lemma 2.4, hence $\alpha_O \colon RO(H) \to KO(G/H)$ is trivially surjective. For the remaining tuples in the cases $\ell = 2, 3, 4$, we have $\mathrm{rank}\, G - \mathrm{rank}\, H \leq 1$, so that $\alpha \colon R(H) \to K(G/H)$ is surjective by Theorem 3.6. All of these remaining tuples (except for those of the form $\{0, n\}$ and $\{0, \mathrm{even}, \mathrm{odd}\}$, which are covered by the last item) also satisfy the numerical condition of Corollary 2.6, hence realification $\mathsf{r} \colon K(G/H) \to KO(G/H)$ is surjective. Thus we get the surjectivity of $\alpha_O \colon RO(H) \to KO(G/H)$.

Now let us consider the tuples $\{\mathrm{even}, n_2, \ldots, n_\ell\}$ where $n_i \equiv 0$ or 4 for all i. For the product G_2/H_2 corresponding to the tuple $\{n_2, \ldots, n_\ell\}$, Proposition 3.12 implies that the map $\alpha_0 \colon RO(H_2) \to KO(G_2/H_2)$ is surjective and that $h^-(K(G_2/H_2)) = 0$ (see Remark 3.15). Observe that for the usual presentation of an even dimensional sphere G_1/H_1 we have that $\mathrm{rank}\, G_1 = \mathrm{rank}\, H_1$ and $\alpha_O \colon RO(H_1) \to KO(G_1/H_1)$ is surjective, hence Proposition 3.14 applies to the product $G/H = G_1/H_1 \times G_2/H_2$ and shows that $\alpha_O \colon RO(H) \to KO(G/H)$ is surjective.

Finally, consider $A \cup \{n_k, \ldots, n_\ell\}$, where $n_i \equiv 0$ for all i and A is any of the tuples above. We have already shown that α_O is surjective for the appropriate presentation of the product corresponding to the tuple A, so the claim follows directly from Proposition 3.16. $\qquad\square$

5.3 Few full flag manifolds

When $T \subset G$ is a maximal torus, the homogeneous space G/T is referred to as a full flag manifold. As always, we may and will assume that G is simply connected. The KO-theory such full flag manifolds is investigated in [Zib15]. The results there easily imply:

Proposition 5.6. *The only full flag varieties for which $\alpha_O \colon R(T) \to KO(G/T)$ is surjective are $SU(m)/T$ with $m \in \{2, 3, 4, 5, 7\}$, $Spin(7)/T$ and G_2/T, and the following products:*

$$SU(2)^{\times 2}/T \quad SU(3)^{\times 2}/T \quad (SU(3) \times SU(4))/T \quad (SU(3) \times G_2)/T$$
$$SU(2)^{\times 3}/T \quad SU(3)^{\times 3}/T \quad (SU(3) \times SU(5))/T$$

Proof. Let $b_{\mathbb{C}}$, $b_{\mathbb{R}}$ and $b_{\mathbb{H}}$ denote the number of basic representations of G of complex, real and quaternionic type, respectively. By the main result of [Zib15], the $\mathbb{Z}_4$-graded "Witt ring" $\bigoplus_i KO^{2i}(G/T)/\mathsf{r}$ is an exterior algebra over $\mathbb{Z}_2$ on $b_{\mathbb{H}}$ generators of degree 1 and $(b_{\mathbb{C}}/2) + b_{\mathbb{R}}$ generators of degree 3. By [Zib15, Example 2.3], α_O is surjective if and only if $KO^0(G/T)/\mathsf{r} = \mathbb{Z}_2$, i.e. if the only additive generator of the Witt ring in degree zero is the multiplicative unit. This happens if and only if all of the following conditions are met:

- There are at most 3 generators in degree 1, i.e. $b_{\mathbb{H}} \leq 3$.
- There are at most 3 generators in degree 3, i.e. $(b_{\mathbb{C}}/2) + b_{\mathbb{R}} \leq 3$.
- We do not simultaneously have generators in degree 1 and in degree 3, i.e. $b_{\mathbb{H}} = 0$ or $b_{\mathbb{C}} + b_{\mathbb{R}} = 0$.

The claim now follows from [Zib15, Table 1]. $\qquad\square$

6 The 13-dimensional Berger manifold

In this section we prove Theorem 1.12 — see Propositions 6.6, 6.9 and 6.10 below.

The space under consideration can be constructed as follows. Consider the standard embedding $j \colon Sp(2) \to SU(4)$, and define:

$$
\begin{aligned}
i \colon Sp(2) \times S^1 &\longrightarrow \quad SU(5) \\
(A, z) &\mapsto \mathrm{diag}(j(A)z, \overline{z}^4)
\end{aligned}
\tag{8}
$$

The homomorphism i has nontrivial kernel given by $\{(Id, 1), (-Id, -1)\} \cong \mathbb{Z}_2$. So i defines a two-fold covering from $Sp(2) \times S^1$ onto its image, which is usually denoted $Sp(2) \times_{\mathbb{Z}_2} S^1$ or simply $Sp(2) \cdot S^1$. The 13-dimensional **Berger space** B^{13} is the quotient

$$B^{13} := \frac{SU(5)}{Sp(2) \times_{\mathbb{Z}_2} S^1}$$

It is the total space of a differentiable $\mathbb{RP}^5$-bundle over $\mathbb{CP}^4$ [Zil07]:

$$\mathbb{RP}^5 \to B^{13} \to \mathbb{CP}^4 \tag{9}$$

6.1 Representation rings

We collect here some information on the representation rings of the groups used to construct B^{13}. Let v denote the standard five-dimensional complex representation of $SU(5)$, and let x be the standard one-dimensional representation of the circle group S^1. Let u denote the restriction of the standard four-dimensional representation of $SU(4)$ to $Sp(2)$, viewed as a quaternionic representation. The complex representation rings of $SU(5)$ and of $Sp(2) \times S^1$ can be written as follows:

$$R(SU(5)) \qquad \cong \mathbb{Z}[v, \lambda^2 v, \lambda^3 v, \lambda^4 v]$$
$$R(Sp(2) \times S^1) \cong R(Sp(2)) \otimes R(S^1)$$
$$\cong \mathbb{Z}[\mathsf{c}'u, \lambda^2(\mathsf{c}'u), x^{\pm 1}]$$

There are no hidden relations: the first is a polynomial ring, while the second is a tensor product of a polynomial ring and the ring of Laurent polynomials $\mathbb{Z}[x^{\pm 1}]$.

Lemma 6.1. *The representation ring $R(Sp(2) \times_{\mathbb{Z}_2} S^1)$ is the subring of $R(Sp(2) \times S^1)$ generated by $(\mathsf{c}'u)^2$, $\lambda^2(\mathsf{c}'u)$, $(\mathsf{c}'u)x^{\pm 1}$ and $x^{\pm 2}$. We may sometimes write this ring as*

$$R(Sp(2) \times_{\mathbb{Z}_2} S^1) \cong \mathbb{Z}[(\mathsf{c}'u)^2, \lambda^2(\mathsf{c}'u), (\mathsf{c}'u)x^{\pm 1}, x^{\pm 2}],$$

but note that there are relations among the generators.

Proof. In general, given a finite central subgroup of a compact Lie group, $\Gamma \subset G$, the representation ring of the quotient G/Γ can be computed as the ring of fixed points under the action of Γ on $R(G)$. More precisely, we have an action of Γ on $R(G) \otimes_{\mathbb{Z}} \mathbb{C}$, as follows. Take some $\gamma \in \Gamma$ and an irreducible representation ρ. As γ is central, Schur's Lemma implies that $\rho(\gamma)$ acts as multiplication by some complex scalar. Identify $\rho(\gamma)$ with this scalar. Define $\gamma.\rho$ to be the element $\rho \otimes \rho(\gamma) \in R(G) \otimes_{\mathbb{Z}} \mathbb{C}$, for any irreducible ρ, and extend this action $\mathbb{C}$-linearly. A representation of G descends to a representation of G/Γ if and only if is fixed by this action, so:
$$R(G/\Gamma) \cong (R(G) \otimes \mathbb{C})^{\Gamma} \cap R(G)$$
[Ste75, §3]. In our case, where $\Gamma = \{(I, 1), (-I, -1)\}$ and $G = Sp(2) \times S^1$, the action of Γ on the ring generators of $R(G)$ is easily computed: $(-I, -1)$ acts as multiplication by -1 on $\mathsf{c}'u$ and $x^{\pm 1}$, but as the identity on $\lambda^2(\mathsf{c}'u)$. It follows that the monomials in these generators that are fixed by the action of Γ are those monomials $(\mathsf{c}'u)^i (x)^j (\lambda^2 \mathsf{c}'u)^k$ with $i + j$ even. All these monomials are linearly independent, so the above result follows. $\square$

Lemma 6.2. *Let $\iota \colon Sp(2) \times_{\mathbb{Z}_2} S^1 \hookrightarrow SU(5)$ denote the inclusion of the image of i (8). The restriction $\iota^* \colon R(SU(5)) \to R(Sp(2) \times_{\mathbb{Z}_2} S^1)$ is determined by:*

$$\iota^*(v) = (\mathsf{c}'u)x^1 + x^{-4} \qquad\qquad \iota^*(\lambda^2 v) = \lambda^2(\mathsf{c}'u)x^2 + (\mathsf{c}'u)x^{-3}$$
$$\iota^*(\lambda^3 v) = \lambda^2(\mathsf{c}'u)x^{-2} + (\mathsf{c}'u)x^3 \qquad \iota^*(\lambda^4 v) = (\mathsf{c}'u)x^{-1} + x^4$$

Proof. The restriction of v can be read off the definition of i. For the other ring generators, we observe that restriction commutes with exterior powers and use the usual formulas for manipulating exterior powers (e. g. [FL85, p. 5]). Note that $\lambda^3(u) = u$: as a representation of $SU(4)$, the standard representation u is only conjugate to $\lambda^3 u$, but its restriction to $Sp(2)$ is of quaternionic type, hence self-conjugate. $\square$

Lemma 6.3. *$RO(Sp(2) \times_{\mathbb{Z}_2} S^1)$ is generated as a ring by 1, u^2, $\lambda^2 u$ and the image of the realification map.*

Proof. We first consider the real representation ring of the product $\widetilde{H} := Sp(2) \times S^1$. As u is a quaternionic representation, both u^2 and $\lambda^2 u$ are real representations. We claim that $RO(\widetilde{H})$ is generated as a ring by 1, u^2, $\lambda^2 u$ and the image of the realification map. To see this, recall from Remark 2.2 that c and c' define an isomorphism $RO(\widetilde{H})/\mathsf{r} \oplus RSp(\widetilde{H})/\mathsf{q} \cong h^+(R\widetilde{H})$ while $h^-(R\widetilde{H}) = 0$. Using Lemma 2.3, we find that:

$$
\begin{aligned}
h^+(R(\widetilde{H})) &= h^+(R(Sp(2))) \otimes h^+(R(S^1)) \\
&= \mathbb{Z}_2[\mathsf{c}'u, \lambda^2(\mathsf{c}'u)] \otimes \mathbb{Z}_2 \\
&= \mathbb{Z}_2[\mathsf{c}'u, \lambda^2(\mathsf{c}'u)]
\end{aligned}
$$

As we know which monomials in $\mathsf{c}'u$ and $\lambda^2(\mathsf{c}'u)$ are real and quaternionic, respectively, we find that $RO(\widetilde{H})/\mathsf{r} = \mathbb{Z}_2[u^2, \lambda^2 u]$. This implies the claim.

Now consider the quotient $H := Sp(2) \times_{\mathbb{Z}_2} S^1$. We will show that the inclusion $RH \hookrightarrow R\widetilde{H}$ induces an isomorphism $h^*(RH) \cong h^*(R\widetilde{H})$. To simplify the notation, we introduce the polynomial ring $A := \mathbb{Z}[\lambda^2(\mathsf{c}'u)]$, which we view as a subring of $R(H)$. Lemma 6.1 implies that, additively, we can decompose $R(H)$ and $R(\widetilde{H})$ as follows:

$$
\begin{aligned}
R(\widetilde{H}) &\cong \bigoplus\nolimits_{(i,j) \in \widetilde{J}} A \cdot (\mathsf{c}'u)^i x^j \\
R(H) &\cong \bigoplus\nolimits_{(i,j) \in J} A \cdot (\mathsf{c}'u)^i x^j
\end{aligned}
$$

The index set $\widetilde{J}$ ranges over all pairs (i,j) with $i \in \mathbb{N}_0$ and $j \in \mathbb{Z}$, while the subset $J \subset \widetilde{J}$ ranges only over those pairs with $i + j$ even. The involution t is trivial on A and on $\mathsf{c}'u$, while $\mathsf{t}(x) = x^{-1}$. Thus:

$$
\begin{aligned}
h^+(RH) &\cong h^+\big(\bigoplus\nolimits_{i+j \text{ even}} A \cdot (\mathsf{c}'u)^i x^j\big) \\
&= h^+\big(\bigoplus\nolimits_{i \text{ even}} A \cdot (\mathsf{c}'u)^i\big) \oplus \bigoplus\nolimits_{\substack{j>0 \\ i+j \text{ even}}} \underbrace{h^+\big(A \cdot (\mathsf{c}'u)^i x^j \oplus \mathsf{t}(A \cdot (\mathsf{c}'u)^i x^j)\big)}_{0} \\
&= h^+(\mathbb{Z}[\lambda^2(\mathsf{c}'u), (\mathsf{c}'u)^2]) \\
&\cong h^+(R\widetilde{H})
\end{aligned}
$$

The claim of the Lemma concerning $RO(H)$ now follows as it did above for $RO(\widetilde{H})$. $\square$

6.2 Cohomology

The cohomology of the Bazaikin spaces is known. For the Berger space B^{13}, we have $H^*(B^q, \mathbb{Z}) \cong \mathbb{Z}[\beta, \gamma]/(5\beta^3, \beta^5, \gamma^2, \gamma\beta^3)$ with $\deg \beta = 2$ and $\deg \gamma = 9$

[FZ09, Prop. 2.1]. Thus, the Atiyah-Hirzebruch spectral sequences (AHSS) computing K- and KO-theory will have the following lines:

$$
\begin{array}{lccccccccc}
& ^{(0)} & ^{(2)} & ^{(4)} & ^{(6)} & ^{(8)} & ^{(9)} & ^{(11)} & ^{(13)} \\
H^*(B^{13},\mathbb{Z}): & \mathbb{Z}\cdot 1 & \mathbb{Z}\cdot\beta & \mathbb{Z}\cdot\beta^2 & \mathbb{Z}_5\cdot\beta^3 & \mathbb{Z}_5\cdot\beta^4 & \mathbb{Z}\cdot\gamma & \mathbb{Z}\cdot\gamma\beta & \mathbb{Z}\cdot\gamma\beta^2 \\
H^*(B^{13},\mathbb{Z}_2): & \mathbb{Z}_2\cdot 1 & \mathbb{Z}_2\cdot\beta & \mathbb{Z}_2\cdot\beta^2 & & & \mathbb{Z}_2\cdot\gamma & \mathbb{Z}_2\cdot\gamma\beta & \mathbb{Z}_2\cdot\gamma\beta^2
\end{array}
$$

Lemma 6.4. *The total Stiefel-Whitney class of B^{13} is $w(B^{13}) = 1+\beta+\beta^2$ and its total Wu class is $\nu(B^{13}) = 1+\beta$. The second Steenrod square Sq^2 operates on $H^*(B^{13},\mathbb{Z}_2)$ as follows:*

$$Sq^2(\beta) = \beta^2 \qquad Sq^2(\beta^2) = 0 \qquad Sq^2(\gamma) = 0 \qquad Sq^2(\gamma\beta) = \gamma\beta^2$$

Proof. We first compute the low degree Stiefel-Whitney classes using the fibre bundle (9). In general, given a differentiable fibre bundle $F \xrightarrow{i} E \xrightarrow{\pi} X$, we have $i^*w(E) = w(F)$, c.f. [BH60, p. 5.1]. In our situation, $F = \mathbb{RP}^5$ with $H^*(\mathbb{RP}^5,\mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/\alpha^6$ and $w(\mathbb{RP}^5) = 1+\alpha^2+\alpha^4$ [MS74, Cor. 11.15 (p. 133)]. So we find

$$w(B^{13}) = 1 + \beta + \beta^2 + \text{ terms of degrees} \geq 9.$$

The Wu classes ν_k are determined by the Steenrod squares by the formula $Sq^k(x) = \nu_k \cup x$ for $x \in H^{13-k}(B^{13},\mathbb{Z}_2)$ [MS74, top of p. 132]. As Sq^k vanishes on elements of degree less than k, we necessarily have $\nu_k = 0$ for $k > 13-k$, i.e. for $k \geq 7$. So the total Wu class reduces to

$$\nu(B^{13}) = 1 + \nu_2 + \nu_4.$$

Both ν_2 and ν_4 and the higher Stiefel-Whitney classes are now determined by the relation between the Wu classes and the Stiefel-Whitney classes [MS74, Thm 11.14]:

$$w_k = \nu_k + Sq^1(\nu_{k-1}) + Sq^2(\nu_{k-2}) + \cdots$$

Keeping in mind that Sq^i vanishes on classes of degree less than i, we find that $\nu_2 = \beta$, $\nu_4 = 0$ and $w_9 = w_{11} = w_{13} = 0$.

Finally, we compute all Steenrod squares. For β and β^2 the claims are clear. For $\gamma\beta$, we use the formula $Sq^2(\gamma\beta) = \nu_2 \cup \gamma\beta$ defining ν_2 and the result $\nu_2 = \beta$ from above. The second Steenrod square of γ is now determined by the Cartan formula $Sq^2(ab) = Sq^2(a)b + Sq^1(a)Sq^1(b) + aSq^2(b)$. $\qquad\square$

6.3 K-theory: additive structure

Proposition 6.5. *The complex K-groups of the Berger space B^{13} are as follows:*

$$K^0(B^{13}) = [\mathbb{Z}] \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_5$$
$$K^1(B^{13}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

The (reduced) real K-groups $\widetilde{KO}^*(B^{13})$ and the maps η, c and r are as displayed in Fig. 2. In particular, the quaternionification $\mathsf{q}\colon \widetilde{K}^0(B^{13}) \to \widetilde{KO}^{-4}(B^{13})$ is surjective, while the realification $\mathsf{r}\colon \widetilde{K}^0(B^{13}) \to \widetilde{KO}^0(B^{13})$ has cokernel isomorphic to $\mathbb{Z}_2$.

$$
\begin{array}{cccc}
\widetilde{K}^0 = \mathbb{Z}^2 \oplus \mathbb{Z}_5^2 & \widetilde{K}^0 = \mathbb{Z}^2 \oplus \mathbb{Z}_5^2 & \widetilde{K}^0 = \mathbb{Z}^2 \oplus \mathbb{Z}_5^2 & \widetilde{K}^0 = \mathbb{Z}^2 \oplus \mathbb{Z}_5^2 \\
\end{array}
$$

Figure 2: The Bott sequence for B^{13}. Missing arrows indicate the zero map.

Proof. The AHSS for complex K-theory collapses: the torsion elements in $\mathbb{Z}_5$ cannot be reached by any nontrivial differential. This immediately determines $K^1(B^{13})$, and it determines $K^0(B^{13})$ up to an extension problem: there is a subgroup $F_5 \subset K^0(B^{13})$ that fits into a short exact sequence $0 \to \mathbb{Z}_5 \to F_5 \to \mathbb{Z}_5 \to 0$. In order to determine whether F_5 is $\mathbb{Z}_5 \oplus \mathbb{Z}_5$ or $\mathbb{Z}_{25}$, we consider the involution t on K^0 and the induced morphism of spectral sequences, as described in Section 2.4. We obtain, in particular, the following commutative diagram, which implies $F_5 = \mathbb{Z}_5 \oplus \mathbb{Z}_5$.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}_5 & \longrightarrow & F_5 & \longrightarrow & \mathbb{Z}_5 & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle\mathrm{id}} & & \downarrow{\scriptstyle\mathsf{t}} & & \downarrow{\scriptstyle-\mathrm{id}} & & \\
0 & \longrightarrow & \mathbb{Z}_5 & \longrightarrow & F_5 & \longrightarrow & \mathbb{Z}_5 & \longrightarrow & 0 \\
\end{array}
$$

We now turn to KO-theory. A portion of the E_2-page of the AHSS is displayed in Fig. 3. The differentials d_2 on the E_2-page are determined by Sq^2, which we computed in Lemma 6.4. Figure 3 displays those d_2's which are zero in gray and those which are not in red. Once we have passed to the E_3-page, there are only three further differentials which could be nonzero, two d_7's and one d_9. However, a comparison with the spectral sequence for K-theory shows that they all vanish. Thus, the non-circled entries in Fig. 3 constitute the E_∞-page.

The groups $\widetilde{KO}^i(B^{13})$ with $i = -2, \ldots, -6$ can be read off the spectral sequence directly. To compute $KO^{-1}(B^{13})$, consider the morphism of spectral sequences induced by realification, and concentrate first on the diagonals comput-

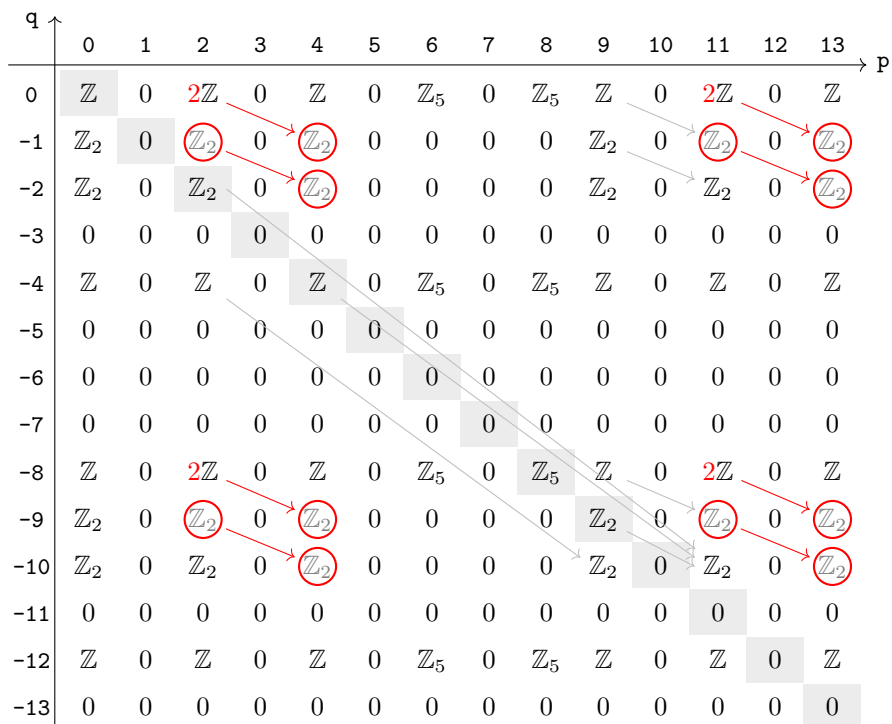| q \ p	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	$\mathbb{Z}$	0	$2\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}_5$	0	$\mathbb{Z}_5$	$\mathbb{Z}$	0	$2\mathbb{Z}$	0	$\mathbb{Z}$
-1	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	0	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
-2	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	0	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
-3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-4	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}_5$	0	$\mathbb{Z}_5$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
-5	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-6	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-7	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-8	$\mathbb{Z}$	0	$2\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}_5$	0	$\mathbb{Z}_5$	$\mathbb{Z}$	0	$2\mathbb{Z}$	0	$\mathbb{Z}$
-9	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	0	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
-10	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	0	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$
-11	0	0	0	0	0	0	0	0	0	0	0	0	0	0
-12	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}_5$	0	$\mathbb{Z}_5$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
-13	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Figure 3: The AHSS computing $KO^*(B^{13})$, with shaded zero diagonal. The red arrows are the nonzero differentials, and entries circled in red are those that vanish when passing to the E_3-page. The red twos indicate that $\mathbb{Z}$ is replaced by the subgroup $2\mathbb{Z}$ when passing to the E_3-page.

ing $K^0(B^{13})$ and $KO^0(B^{13})$. Compare the filtrations and the exact sequences that we obtain from the spectral sequences:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F^{n+1}K & \longrightarrow & F^n K & \longrightarrow & E_\infty^{n,-n}(K) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r_\infty} & & \\
0 & \longrightarrow & F^{n+1}KO & \longrightarrow & F^n KO & \longrightarrow & E_\infty^{n,-n}(KO) & \longrightarrow & 0
\end{array}
$$

The vertical arrow on the right comes from the change-of-coefficients morphism $H^n(-, K^{-n}) \to H^n(-, KO^{-n})$ induced by the realification $\mathsf{r}\colon K^{-n}(pt) \to KO^{-n}(pt)$, described in detail in Section 2.4. So we can compute cokernel and kernel of the realification map step by step. This gives us the following exact sequences:

$$
\begin{aligned}
0 \to 0 \to F^8 K \to F^8 KO \to \mathbb{Z}_2 \to 0 \\
0 \to \mathbb{Z}_5 \to F^6 K \to F^6 KO \to \mathbb{Z}_2 \to 0 \\
0 \to \mathbb{Z}_5 \to F^4 K \to F^4 KO \to \mathbb{Z}_2 \to 0 \\
0 \to \mathbb{Z} \oplus \mathbb{Z}_5 \to F^2 K \to F^2 KO \to (\mathbb{Z}_2 \text{ or } 0) \to 0
\end{aligned}
\tag{10}
$$

Note that $F^2 K = \widetilde{K}$ and $F^2 KO = \widetilde{KO}$, so that we find:

The realification map $\widetilde{K}(B^{13}) \to \widetilde{KO}(B^{13})$ has cokernel at most $\mathbb{Z}_2$.

Next, we analyse the map $\eta\colon \widetilde{KO}^{-1}(B^{13}) \to \widetilde{KO}^0(B^{13})$, playing the same game with the filtrations as above. There is only one step:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & F^9 KO^0 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow{\scriptstyle \cong} & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & F^9 KO^{-1} & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 0
\end{array}
$$

Thus, we find that $\widetilde{KO}^{-1}(B^{13}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. On the other hand, we know from above that $\widetilde{K}^{-1}(B^{13})$ is free. So the Bott sequence $\widetilde{K} \to \widetilde{KO} \to \widetilde{KO}^{-1} \to \widetilde{K}^{-1}$ implies:

The realification map $\widetilde{K}(B^{13}) \to \widetilde{KO}(B^{13})$ has cokernel exactly $\mathbb{Z}_2$.

Next, to compute $\widetilde{KO}^{-7}(B^{13})$, we consider the realification $\widetilde{K}^{-6}(B^{13}) \to \widetilde{KO}^{-6}(B^{13})$. Note that the entries $H^2(B^{13}, \mathbb{Z}) \cong \mathbb{Z}$ in the rows $q = 0$ of the spectral sequence for KO-theory get replaced by the subgroup $2\mathbb{Z}$ when passing to the E_3-page, so that $\mathsf{r}_\infty^{2,0}$ is surjective. Again arguing filtration-step-by-filtration-step, we find that $\widetilde{K}^{-6}(B^{13}) \to \widetilde{KO}^{-6}(B^{13})$ is surjective. The Bott sequence therefore implies that $\widetilde{KO}^{-7}(B^{13})$ injects into $\widetilde{K}^{-1}(B^{13})$, so $\widetilde{KO}^{-7}(B^{13})$ must be torsion-free. Thus, we find that $\widetilde{KO}^{-7}(B^{13})$ is as displayed.

Finally, for $KO^0(B^{13})$, the spectral sequence initially only tells us that $\widetilde{KO}^0(B^{13}) \cong \mathbb{Z} \oplus \mathbb{Z}_5 \oplus X$ with $X = \mathbb{Z}_2$ or $X = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $X = \mathbb{Z}_4$. But we also find that $\mathsf{r}\colon \widetilde{K}^1(B^{13}) \to \widetilde{KO}^1(B^{13})$ has cokernel $\mathbb{Z}_2$. As $\widetilde{K}^0(B^{13})$ contains no two-torsion, the Bott sequence implies that $\widetilde{KO}^0(B^{13})$ can contain at most one element of order a power of two. Thus, $\widetilde{KO}^0(B^{13})$ is as displayed.

The remaining maps in the Bott sequence are easily computed. As $\eta \in KO^{-1}(*)$ is two-torsion, the fact that most of the maps η are zero is clear. For the two

nonzero maps, we again use the spectral sequence and argue filtration-step-by-filtration-step. The additional partial information concerning the maps r and c displayed in Fig. 2 follows from the exactness of the sequence. $\qquad\square$

6.4 K-theory: multiplicative structure

Proposition 6.6. *The real and complex K-rings of B^{13} are as follows:*[1]

$$K(B^{13}) \cong \mathbb{Z}[u,y]/(5u^3, 5u^4, u^5, y^3, uy^2, u^3 y, y^2 - u^4,$$
$$y - u^2 + u^3 - u^4, uy - u^3 + u^4, u^2 y - u^4)$$
$$\cong \mathbb{Z}\cdot 1 \oplus \mathbb{Z}u \oplus \mathbb{Z}y \oplus \mathbb{Z}_5 u^3 \oplus \mathbb{Z}_5 u^4$$
$$KO(B^{13}) = \mathbb{Z}[y', w]/(5y'^2, 2w, w^2, wy', y'^3)$$
$$= \mathbb{Z}\cdot 1 \oplus \mathbb{Z}y' \oplus \mathbb{Z}_5 y'^2 \oplus \mathbb{Z}_2 w$$

The ring homomorphisms $\mathsf{t}\colon K(B^{13}) \to K(B^{13})$ *and* $\mathsf{c}\colon KO(B^{13}) \to K(B^{13})$ *and the group homomorphism* $\mathsf{r}\colon K(B^{13}) \to KO(B^{13})$ *are determined by:*

$\mathsf{t}(y) = y$	$\mathsf{c}(y') = y$	$\mathsf{r}(u) = y'$	$\mathsf{r}(u^3) = 3y'^2$
$\mathsf{t}(u) = y - u$	$\mathsf{c}(w) = 0$	$\mathsf{r}(y) = 2y'$	$\mathsf{r}(u^4) = 2y'^2$

Proof. Let $G = SU(5)$ and $H = Sp(2) \times_{\mathbb{Z}_2} S^1$, so that $B^{13} = G/H$. We first compute the complex K-ring. By Theorem 3.6, $K(B^{13})$ is isomorphic to $R(H)$ modulo the ideal generated by the restrictions of reduced generators of $R(G)$. Let us write

$$a := (\mathsf{c}'u)^2 \qquad b := x^2 \qquad c := x^{-2} \qquad d := (\mathsf{c}'u)x \qquad e := \lambda^2(\mathsf{c}'u)$$

for the generators of $R(H)$ determined in Lemma 6.1. Then in $R(H)$ we have the relations displayed in line (11) below. Using Lemma 6.2, we find that the restriction of reduced generators from $R(G)$ to $R(H)$ yield lines (12) and (13) as additional relations in $K(B^{13})$:

$$bc = 1 \qquad\qquad ab = d^2 \qquad\qquad (11)$$
$$d + c^2 = 5 \qquad\qquad eb + dc^2 = 10 \qquad\qquad (12)$$
$$ec + db = 10 \qquad\qquad dc + b^2 = 5 \qquad\qquad (13)$$

Thus, $K(B^{13})$ is isomorphic to a quotient of the polynomial ring $\mathbb{Z}[a,b,c,d,e]$ by all relations above. In particular, we see that a possible set of additive generators is given by $\{b^2, b, 1, c, c^2\}$. Equivalently, a set of additive generators is given by $\{b'^2, b', 1, c', c'^2\}$, where now $b' := b-1$ and $c' := c-1$ both lie $\widetilde{K}(B^{13})$. However, for our purposes, the following set of additive generators seems most suitable:

1	$u := b'$	$u^3 = 2b'^2 + c'^2 - 3b' - 3c'$
	$y := b' + c'$	$u^4 = b'^2 + c'^2 - 2b' - 2c'$

A tedious but straightforward calculation shows that these satisfy the relations and have the properties indicated above.

---

[1]The set of relations displayed for the complex K-ring is not minimal.

For our computation of the KO-ring, we will also need to understand in terms of our chosen generators the filtration $F^i K := F^i K(B^{13})$ that appears in the AHSS. We claim that this filtration is as follows:

$$
\begin{aligned}
F^1 K &= F^2 K = (u, y) \\
F^3 K &= F^4 K = (u^2, y) \\
F^5 K &= F^6 K = (u^3, uy, y^2) \\
F^7 K &= F^8 K = (u^4) \\
&\quad F^9 K = 0
\end{aligned}
\tag{14}
$$

For $F^1 K$, the kernel of the rank homomorphism, the claim is clear. As the filtration is multiplicative, we moreover know "$\supset$" in each of the indicated equalities:

$F^3 K = F^4 K$ must contain $(F^2 K)^2$, in particular u^2 and $uy = u^2 - y$, so u^2 and y.

$F^5 K = F^6 K$ must contain $F^1 K \cdot F^2 K$, in particular u^3, uy and y^2.

$F^7 K = F^8 K$ must contain $(F^1 K)^4$, in particular u^4.

In our specific situation, equality in (14) will follow if we can show that, assuming equality, each quotient $F^i K / F^{i+1} K$ is isomorphic to the corresponding entry of the $E_2 = E_\infty$-page of the AHSS. This is indeed the case:

$F^2 K / F^4 K \cong \mathbb{Z} \cdot u$

$F^4 K / F^6 K \cong \mathbb{Z} \cdot \{u^2, y\} / F^6 K \cong \mathbb{Z} \cdot u^2$ $\qquad$ (since $u^2 = y$ modulo uy)

$F^6 K / F^8 K \cong \mathbb{Z} \cdot \{u^3, uy, u^2 y, u^3 y\} / F^8 K \cong \mathbb{Z}_5 \cdot u^3$

$\qquad F^8 K \cong \mathbb{Z}_5 \cdot u^4$

For $F^6 K / F^8 K$ note that $uy = u^3 \mod F^8 K$: We know that $u^2 = uy + y$, so multiplying by u we obtain $u^3 = u^2 y + uy$. Using that $u^2 y = y^2$, we can rewrite this as: $u^3 = y^2 + uy$. The other possible generators of $F^6 K / F^8 K$ vanish: $u^2 y = y^2$ vanishes modulo $F^8 K$, and $u^3 y = y^3 = 0$ anyway.

We now turn to the KO-ring. The associated graded ring that the AHSS converges to has no nontrivial products, so this ring is not very helpful. We therefore run through the additive computation again, choosing names for all generators and taking a note of where they map under complexification.

First, we have $F^9 KO = \mathbb{Z}_2 w$ for some mysterious element w mapping to zero under complexification. Next we know that $F^8 KO \cong \mathbb{Z}_5 \oplus \mathbb{Z}_2$, and that $F^8 K \to F^8 KO$ has cokernel $\mathbb{Z}_2$. So the five-torsion in $F^8 KO$ must be generated by some element of the form $\mathsf{r}(y^2)$. The image of $\mathsf{r}(y^2)$ under c is $2y^2$, since y^2 is self-conjugate. Equivalently:

$$
F^8 KO = \mathbb{Z}_5 y'' \oplus \mathbb{Z}_2 w
$$

for $y'' := 3r(y^2)$, and $\mathsf{c}(y'') = y^2$. The next filtration step for $\mathsf{c} \colon KO \to K$ is:

$$
\begin{array}{ccccc}
\mathbb{Z}_2 w \oplus \mathbb{Z}_5 y'' & \longrightarrow & F^4 KO & \longrightarrow & \mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow 2 \\
\mathbb{Z}_5 u^3 \oplus \mathbb{Z}_5 y^2 & \longrightarrow & \mathbb{Z} y \oplus \mathbb{Z}_5 u^3 \oplus \mathbb{Z}_5 y^2 & \longrightarrow & \mathbb{Z} y
\end{array}
$$

The vertical map on the right is $c\colon KSp(*) \to K(*)$, which is multiplication by 2. So F^4KO contains an element mapping to $2y + \alpha u^3 + \beta y^2$ for some α, β. As y'' maps to y^2, there must also be an element mapping to $2y + \alpha u^3$. As the image of any element under complexification is self-conjugate, we must have $\alpha = 0$. We conclude that there must be some element $v \in F^4KO$ such that $c(v) = 2y$. So $F^4KO = \mathbb{Z}v \oplus \mathbb{Z}_5 y'' \oplus \mathbb{Z}_2 w$.

Finally, $F^2KO = \widetilde{KO}$, $F^2K = \widetilde{K}$, and we have seen that the cokernel of $c\colon \widetilde{KO} \to \widetilde{K}$ has no two-torsion. This implies that there is an element $y' \in F^2KO$ such that $c(y') = y$. Consider the two short exact sequences corresponding to this last filtration step:

$$
\begin{array}{ccccc}
\mathbb{Z}u \quad \oplus \mathbb{Z}_5 y'' \oplus \mathbb{Z}_2 w & \longrightarrow & F^2KO & \longrightarrow & \mathbb{Z}_2 y' \\
\downarrow & & \downarrow & & \downarrow 0 \\
\mathbb{Z}y \oplus \mathbb{Z}u^3 \oplus \mathbb{Z}_5 y^2 & \longrightarrow & \mathbb{Z}u \oplus \mathbb{Z}y \oplus \mathbb{Z}_5 u^3 \oplus \mathbb{Z}_5 y^2 & \longrightarrow & \mathbb{Z}u
\end{array}
$$

The short exact sequence along the top gives us:

$$F^2KO = \mathbb{Z}y' \oplus \mathbb{Z}_5 y'' \oplus \mathbb{Z}_2 w.$$

As $c(2y') = c(v)$, we have $v = 2y' + \alpha w$ for some α. As w is in the smallest piece of the filtration, we can replace the generator v by $2y'$ everywhere above. Thus, altogether, the filtration on $KO(B^{13})$ may be written as follows:

$$
\begin{aligned}
F^1KO = F^2KO = (y', w) \;\; &= \mathbb{Z} \cdot y' \;\; \oplus \mathbb{Z}_5 \cdot y'^2 \oplus \mathbb{Z}_2 \cdot w \\
F^3KO = F^4KO = (2y', w) &= \mathbb{Z} \cdot 2y' \oplus \mathbb{Z}_5 \cdot y'^2 \oplus \mathbb{Z}_2 \cdot w \\
F^5KO = F^8KO = (y'^2, w) &= \qquad\quad\; \mathbb{Z}_5 \cdot y'^2 \oplus \mathbb{Z}_2 \cdot w \\
F^9KO = (w) \quad &= \qquad\qquad\qquad\qquad \mathbb{Z}_2 \cdot w \\
F^{10}KO = 0 &
\end{aligned}
$$

It remains to determine the products of y', y'' and w. For filtration reasons, we have $(y')^3 = (y'')^2 = w^2 = y'w = y''w = 0$. As $c((y')^2) = c(y'')$, we moreover obtain the relation $(y')^2 = y'' + \delta w$ for some $\delta \in \{0, 1\}$. We will return to the value of δ at the end of the proof.

The behaviour of c has already been determined. It follows that $r(y) = r(cy') = 2y'$ and likewise $r(y^2) = r(cy'') = 2y''$. From $cr(u) = u + tu = y = c(y')$, we deduce $r(u) = y' + \alpha w$ for some $\alpha \in \mathbb{Z}_2$. As we already know that w does not lie in the image of r (c.f. (10) in the additive computation), α must be zero. Similarly, $cr(u^3) = u^3 + t(u^3) = u^3 + (y - u)^3 = \cdots = 3y^2$ implies $r(u^3) = 3y''$. Now $(y')^2 = r(u)y' = r(u \cdot cy')$ and $y'' = r(3y^2)$ are both in the image of r, so their difference δw must also be in the image of r. But again, we already know that w is not in the image of r. So $\delta = 0$, i.e. $(y')^2 = y''$. $\qquad\square$

6.5 The multiplicative map ϕ

For any Lie group G, there is a multiplicative map

$$\phi\colon R(G) \to RO(G)$$

such that $c\phi(x) = x(tx)$ for any element x [Ada69, Lemma 7.2]. Likewise, we have such map $\phi\colon K(X) \to KO(X)$. The idea is that for any complex vector

bundle E, there is a canonical conjugate linear automorphism J on $E \otimes \mathsf{t}E$ such that $J^2 = \text{id}$. More precisely, we have the following:

Proposition 6.7 (Bousfield). *For any finite CW complex X, there is a natural map*

$$\phi \colon K(X) \to KO(X)$$

such that for all $x, y \in K(X)$:

$$\phi(xy) = \phi(x)\phi(y)$$
$$\phi(x + y) = \phi(x) + \phi(y) + \mathsf{r}((\mathsf{t}x)y)$$
$$\mathsf{c}\phi(x) = x(\mathsf{t}x)$$
$$\phi(\mathsf{t}x) = \phi(x)$$
$$\phi(x) = \lambda^2(\mathsf{r}x) - \mathsf{r}(\lambda^2 x)$$

Reference. See [Bou05]: the existence of ϕ and the first three properties are asserted at the beginning of 6.11 and in Thm 6.5. The expression for $\phi(x)$ for $x \in K^0(X)$ in terms of exterior powers is implicit in Thm 6.7, see 6.2 (iv). The relation $\phi(\mathsf{t}x) = \phi(x)$ follows. $\qquad\square$

Note that the above properties allow us to easily compute ϕ *modulo the kernel of* c. For example, in the case of $X = B^{13}$, we must have $\phi(y) = y'^2 + \delta w$ for some $\delta \in \{0, 1\}$. In fact, $\delta = 0$ in this case:

Lemma 6.8. *The map $\phi \colon K(B^{13}) \to KO(B^{13})$ is determined by*

$$\phi(u) = -y'$$
$$\phi(y) = y'^2$$

and the above properties. In particular, the image of ϕ is contained in the image of r, and hence does not *contain the two-torsion summand of $KO(B^{13})$.*

Proof. $\phi(u)$ can be computed as follows. It is easy to compute ϕ on representation rings because there c is injective, so ϕ is completely determined by $\mathsf{c}\phi(x) = x(\mathsf{t}x)$. For the generator $x \in R(S^1)$, we find $\phi(x) = 1$, so $\phi(x^2) = 1$ for the element $x^2 \in R(H)$. Moreover, ϕ is compatible with the maps α, i.e. $\phi \circ \alpha = \alpha_O \circ \phi$. Thus, we find:

$$\phi(u) = \phi(\alpha(x^2 - 1)) = \alpha_O(\phi(x^2 - 1)) = \alpha_O(2 - \mathsf{r}(x^2))$$
$$= \alpha_O(\mathsf{r}(1 - x^2)) = \mathsf{r}(\alpha(1 - x^2)) = \mathsf{r}(-u)$$
$$= -y'$$

The value of $\phi(y)$ can be deduced from this:

$$\phi(y) = \phi(u + \mathsf{t}u) = \phi(u) + \phi(\mathsf{t}u) + \mathsf{r}(u^2)$$
$$= 2\phi(u) + \mathsf{r}(y + u^3 - u^4) = -2y' + 2y' + y'^2$$
$$= y'^2$$

Finally, note that $K(B^{13})$ is generated by u and y as a ring. Both $\phi(u)$ and $\phi(y)$ lie in the image of r. The properties of ϕ therefore imply that the whole image of ϕ is contained in the image of r. The image of r does not contain the two-torsion. $\qquad\square$

6.6 Non-surjectivity of α_O

Proposition 6.9. *In the notation of Proposition 6.6, the image of the ring homomorphism* $\alpha_O \colon RO(Sp(2) \times_{\mathbb{Z}_2} S^1) \to KO(B^{13})$ *is the subring*

$$S := \mathbb{Z} \cdot 1 \oplus \mathbb{Z}y' \oplus \mathbb{Z}_5 y'^2$$

generated by y'. *As a subgroup, S is of index two.*

Note that the product of y' with the additional generator w of $KO(B^{13})$ is trivial. Thus, if we take $\widetilde{S} := \mathbb{Z}y' \oplus \mathbb{Z}_5 y'^2$ to denote the rank zero ideal of S, we obtain the splitting $\widetilde{KO}(B^{13}) \cong \widetilde{S} \times \mathbb{Z}_2 w$ alluded to in Theorem 1.12.

Proof. By Proposition 6.6, the subring S is additively generated by the multiplicative unit 1 and by the image $\mathrm{im}(\mathsf{r})$ of the realification map $\mathsf{r} \colon K(B^{13}) \to KO(B^{13})$. For the inclusion $S \subset \mathrm{im}(\alpha_O)$, it therefore suffices to note that $1 \in \mathrm{im}(\alpha_O)$ (as α_O is a ring homomorphism) and $\mathrm{im}(\mathsf{r}) \subset \mathrm{im}(\alpha_O)$ (as the complex version α is surjective by Theorem 3.6 and $\alpha_O \mathsf{r} = \mathsf{r}\alpha$ by Lemma 3.3). It remains to show the converse inclusion: the image of α_O is contained in the subgroup of $KO(B^{13})$ generated by 1 and $\mathrm{im}(\mathsf{r})$.

By Lemma 6.3, $RO(H)$ is generated as a ring by 1, u^2, $\lambda^2 u$ and the image of the realification map. Clearly, α_O maps the image of the realification map to $\mathrm{im}(\mathsf{r})$, and it maps 1 to 1. So it suffices to show that α_O also sends the two additional generators u^2 and $\lambda^2 u$ to $\mathrm{im}(\mathsf{r})$. As noted in Lemma 3.3, the different flavours of α are compatible with multiplication and exterior powers. In particular, $\alpha_O(u^2) = \alpha_{Sp}(u)^2$ and $\alpha_O(\lambda^2 u) = \lambda^2(\alpha_{Sp} u)$. It therefore suffices to show that:

$$KSp(B^{13}) \cdot KSp(B^{13}) \subset \mathrm{im}(\mathsf{r}) \quad \text{in } KO(B^{13})$$
$$\lambda^2(KSp(B^{13})) \subset \mathrm{im}(\mathsf{r}) \quad \text{in } KO(B^{13})$$

As the quaternionification $\mathsf{q} \colon K(B^{13}) \to KSp(B^{13})$ is surjective (Proposition 6.5), we only need to show that $\mathrm{im}\,\mathsf{q} \cdot \mathrm{im}\,\mathsf{q} \subset \mathrm{im}\,\mathsf{r}$ and $\lambda^2(\mathrm{im}\,\mathsf{q}) \subset \mathrm{im}\,\mathsf{r}$. The first inclusion is clear from the usual multiplication rules for r: explicitly, $\mathsf{q}x \cdot \mathsf{q}y = \mathsf{r}(x \cdot \mathsf{c}'\mathsf{q}y)$. For the second inclusion, note that, for any X and any $z \in K(X)$, we have $\lambda^2(\mathsf{q}z) = \lambda^2(\mathsf{r}z) = \mathsf{r}(\lambda^2 z) + \phi(z)$ (Proposition 6.7). For $X = B^{13}$, Lemma 6.8 shows that $\mathrm{im}\,\phi \subset \mathrm{im}\,\mathsf{r}$, so the claim follows. $\qquad\square$

6.7 Pontryagin classes

Proposition 6.10. *In the notation of Proposition 6.6, the generator* $w \in \widetilde{KO}(B^{13})$ *is the unique nonzero element with trivial Pontryagin classes.*

Proof. We first compute the total Chern class $c = 1 + c_1 + c_2 + \ldots$ of the generators u and y of $K(B^{13})$. We know that $b = u + 1$ represents a line bundle, and the spectral sequence computations imply $c_1(b) = c_1(u) = \pm\beta$. So $c(u) = 1 \pm \beta$, and we deduce:

$$c(y) = c(u + \mathsf{t}u) = 1 - \beta^2$$
$$c(y^2) = c(b^2 + \mathsf{t}b^2 - 4b - 4\mathsf{t}b + 6) = (1 \pm 2\beta)(1 \mp 2\beta)/\big((1 \pm \beta)^4(1 \mp \beta)^4\big)$$
$$= (1 - 4\beta^2)/(1 - \beta^2)^4 = 1 - 6\beta^4 \ = 1 - \beta^4$$

The only nontrivial Pontryagin classes that the cohomology of B^{13} admits are p_1, with values in $\mathbb{Z}\beta^2$, and p_2, with values in $\mathbb{Z}_5\beta^4$. For the total Pontryagin classes $p = 1 + p_1 + p_2$ we find:

$$
\begin{aligned}
p(y') &= 1 - c_2(y) + c_4(y) &&= 1 + \beta^2 \\
p(y'^2) &= 1 - c_2(y^2) + c_4(y^2) &&= 1 - \beta^4 \\
p(w) &= 1 - c_2(\mathsf{c}(w)) + c_4(\mathsf{c}(w)) = 1 && (\text{as } \mathsf{c}(w) = 0)
\end{aligned}
$$

So for an arbitrary element $\mu y' + \nu y'^2 + \delta w \in \widetilde{KO}(B^{13})$, we find:

$$
\begin{aligned}
p(\mu y' + \nu y'^2 + \delta w) &= (1 + \beta^2)^\mu (1 - \beta^4)^\nu \\
&= (1 + \mu\beta^2 + \tbinom{\mu}{2}\beta^4)(1 - \nu\beta^4) \\
&= 1 + \mu\beta^2 + (\tbinom{\mu}{2} - \nu)\beta^4 \in 1 + \mathbb{Z}\beta^2 + \mathbb{Z}_5\beta^4
\end{aligned}
$$

This is equal to 1 if and only if $\mu = 0$ in $\mathbb{Z}$ and $\nu = 0$ in $\mathbb{Z}_5$. $\qquad\square$

References

[Ada69] J. F. Adams, *Lectures on Lie groups*, W. A. Benjamin, Inc., New York-Amsterdam, 1969. [MR0252560] ↑15, 16, 21, 37

[APL] AIM Problem Lists, *AimPL: Manifolds with non-negative sectional curvature*. Available at http://aimpl.org/nnsectcurvature. ↑2

[All74] J. Allard, *Adams operations in $KO(X) \oplus KSp(X)$*, Bol. Soc. Brasil. Mat. **5** (1974), no. 1, 85–96. [MR0372857] ↑8

[AZ16] M. Amann and W. Ziller, *Geometrically formal homogeneous metrics of positive curvature*, J. Geom. Anal. **26** (2016), no. 2, 996–1010. [MR3472826] ↑25

[Ati62] M. F. Atiyah, *Vector bundles and the Künneth formula*, Topology **1** (1962), 245–248. [MR0150780] ↑8, 21

[Ati67] _____, *K-theory*, Lecture notes by D. W. Anderson, W. A. Benjamin, Inc., New York-Amsterdam, 1967. [MR0224083] ↑7, 8

[Ati68] _____, *Bott periodicity and the index of elliptic operators*, Quart. J. Math. Oxford Ser. (2) **19** (1968), 113–140. [MR0228000] ↑19

[AH61] M. F. Atiyah and F. Hirzebruch, *Vector bundles and homogeneous spaces*, Proc. Sympos. Pure Math., Vol. III, American Mathematical Society, Providence, R.I., 1961, pp. 7–38. [MR0139181] ↑11, 17

[Bar65] D. Barden, *Simply connected five-manifolds*, Ann. of Math. (2) **82** (1965), 365–385. [MR0184241] ↑25, 26

[BK03] I. Belegradek and V. Kapovitch, *Obstructions to nonnegative curvature and rational homotopy theory*, J. Amer. Math. Soc. **16** (2003), no. 2, 259–284. [MR1949160] ↑2

[Ber91] F. Bermbach, *On Simply-Connected 7-Manifolds* (1991). Ph.D. Thesis, Johannes Gutenberg-Universität Mainz. ↑25

[BDS49] A. Borel and J. De Siebenthal, *Les sous-groupes fermés de rang maximum des groupes de Lie clos*, Comment. Math. Helv. **23** (1949), 200–221. [MR0032659] ↑6

[BH60] A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces. III*, Amer. J. Math. **82** (1960), 491–504. [MR0120664] ↑31

[Bou90] A. K. Bousfield, *A classification of K-local spectra*, J. Pure Appl. Algebra **66** (1990), no. 2, 121–163. [MR1075335] ↑10, 11

[Bou05] _____, *On the 2-primary v_1-periodic homotopy groups of spaces*, Topology **44** (2005), no. 2, 381–413. ↑10, 38

[Bot59] R. Bott, *Quelques remarques sur les théorèmes de périodicité*, Bull. Soc. Math. France **87** (1959), 293–310. [MR0126281] ↑8, 10

[Bro65] W. Browder, *Structures on $M \times R$*, Proc. Cambridge Philos. Soc. **61** (1965), 337–345. [MR0175136] ↑23

[CG72] J. Cheeger and D. Gromoll, *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math. (2) **96** (1972), 413–443. [MR0309010] ↑2

[CE03] D. Crowley and C. M. Escher, *A classification of S^3-bundles over S^4*, Differential Geom. Appl. **18** (2003), no. 3, 363–380. [MR1975035] ↑25

[DT11] A. Dessai and W. Tuschmann, *A note on nonnegative curvature and (rational) homotopy type* (2011). arXiv:1103.4514. ↑3

[tD91] T. tom Dieck, *Topologie*, de Gruyter Lehrbuch. [de Gruyter Textbook], Walter de Gruyter & Co., Berlin, 1991. [MR1150244] ↑23

[Dug03] D. Dugger, *Multiplicative structures on homotopy spectral sequences II* (2003). arXiv:math/0305187. ↑12

[Esc82] J.-H. Eschenburg, *New examples of manifolds with strictly positive curvature*, Invent. Math. **66** (1982), no. 3, 469–480. [MR662603] ↑14, 25

[FZ09] L. A. Florit and W. Ziller, *On the topology of positively curved Bazaikin spaces*, J. Eur. Math. Soc. (JEMS) **11** (2009), no. 1, 189–205. [MR2471136] ↑6, 31

[Fuj67] M. Fujii, *K_O-groups of projective spaces*, Osaka J. Math. **4** (1967), 141–149. [MR0219060] ↑11

[FL85] W. Fulton and S. Lang, *Riemann-Roch algebra*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 277, Springer-Verlag, New York, 1985. [MR801033] ↑30

[Gon17] D. González-Álvaro, *Nonnegative curvature on stable bundles over compact rank one symmetric spaces*, Adv. Math. **307** (2017), 53–71. [MR3590513] ↑2, 4, 22, 24, 27

[Gor12] V. V. Gorbatsevich, *Compact homogeneous manifolds of dimension at most 7 up to a finite-sheeted covering*, Izv. Ross. Akad. Nauk Ser. Mat. **76** (2012), no. 4, 27–40; English transl., Izv. Math. **76** (2012), no. 4, 669–680. [MR3013269] ↑25

[GOV97] V. V. Gorbatsevich, A. L. Onishchik, and E. B. Vinberg, *Foundations of Lie theory and Lie transformation groups*, Springer-Verlag, Berlin, 1997. [MR1631937] ↑15

[Gro81] M. Gromov, *Curvature, diameter and Betti numbers*, Comment. Math. Helv. **56** (1981), no. 2, 179–195. [MR630949] ↑2

[GZ11] K. Grove and W. Ziller, *Lifting group actions and nonnegative curvature*, Trans. Amer. Math. Soc. **363** (2011), no. 6, 2865–2890. [MR2775790] ↑2

[GZ00] ———, *Curvature and symmetry of Milnor spheres*, Ann. of Math. (2) **152** (2000), no. 1, 331–367. [MR1792298] ↑2, 14

[Hae61] A. Haefliger, *Plongements différentiables de variétés dans variétés*, Comment. Math. Helv. **36** (1961), 47–82. [MR0145538] ↑22

[HN12] J. Hilgert and K.-H. Neeb, *Structure and geometry of Lie groups*, Springer Monographs in Mathematics, Springer, New York, 2012. [MR3025417] ↑19

[Hod75] L. Hodgkin, *The equivariant Künneth theorem in K-theory*, Topics in K-theory. Two independent contributions, Springer, Berlin, 1975, pp. 1–101. Lecture Notes in Math., Vol. 496. [MR0478156] ↑18

[Hus94] D. Husemoller, *Fibre bundles*, 3rd ed., Graduate Texts in Mathematics, vol. 20, Springer-Verlag, New York, 1994. [MR1249482] ↑7, 8

[KM63] M. A. Kervaire and J. W. Milnor, *Groups of homotopy spheres. I*, Ann. of Math. (2) **77** (1963), 504–537. [MR0148075] ↑3

[Kla88] S. Klaus, *Einfach-zusammenhängende kompakte homogene Räume bis zur Dimension 9* (1988). Diploma thesis (in German), University of Mainz. ↑25

[KS88] M. Kreck and S. Stolz, *A diffeomorphism classification of 7-dimensional homogeneous Einstein manifolds with $SU(3) \times SU(2) \times U(1)$-symmetry*, Ann. of Math. (2) **127** (1988), no. 2, 373–388. [MR932303] ↑25

[McC01] J. McCleary, *A user's guide to spectral sequences*, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 58, Cambridge University Press, Cambridge, 2001. [MR1793722] ↑11

[McL79] J. McLeod, *The Kunneth formula in equivariant K-theory*, Algebraic topology, Waterloo, 1978 (Proc. Conf., Univ. Waterloo, Waterloo, Ont., 1978), Lecture Notes in Math., vol. 741, Springer, Berlin, 1979, pp. 316–333. [MR557175] ↑18

[MS74] J. W. Milnor and J. D. Stasheff, *Characteristic classes*, Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974. Annals of Mathematics Studies, No. 76. [MR0440554] ↑31

[ÖW94] M. Özaydin and G. Walschap, *Vector bundles with no soul*, Proc. Amer. Math. Soc. **120** (1994), no. 2, 565–567, DOI 10.2307/2159896. [MR1162091] ↑2

[Pit72] H. V. Pittie, *Homogeneous vector bundles on homogeneous spaces*, Topology **11** (1972), 199–203. [MR0290402] ↑18

[Rig78] A. Rigas, *Geodesic spheres as generators of the homotopy groups of* O*, B*O, J. Differential Geom. **13** (1978), no. 4, 527–545 (1979). [MR570216] ↑2, 27

[Sey73] R. M. Seymour, *The real K-theory of Lie groups and homogeneous spaces*, Quart. J. Math. Oxford Ser. (2) **24** (1973), 7–30. [MR0319193] ↑20

[Sie69] L. C. Siebenmann, *On detecting open collars*, Trans. Amer. Math. Soc. **142** (1969), 201–227. [MR0246301] ↑22

[Sna71] V. P. Snaith, *On the K-theory of homogeneous spaces and conjugate bundles of Lie groups*, Proc. London Math. Soc. (3) **22** (1971), 562–584. [MR0290403] ↑19

[Ste75] R. Steinberg, *On a theorem of Pittie*, Topology **14** (1975), 173–177. [MR0372897] ↑18, 19, 29

[Swa78] R. G. Swan, *Projective modules over Laurent polynomial rings*, Trans. Amer. Math. Soc. **237** (1978), 111–120. [MR0469906] ↑18

[Swa92] ———, *Gubeladze's proof of Anderson's conjecture*, Azumaya algebras, actions, and modules (Bloomington, IN, 1990), Contemp. Math., vol. 124, Amer. Math. Soc., Providence, RI, 1992, pp. 215–250. [MR1144038] ↑18

[TW15] W. Tuschmann and D. J. Wraith, *Moduli spaces of Riemannian metrics*, Oberwolfach Seminars, vol. 46, Birkhäuser Verlag, Basel, 2015. Second corrected printing. [MR3445334] ↑3, 23

[Van99] J. Vanžura, *The cohomology of $\widetilde{G}_{m,2}$ with integer coefficients*, Rend. Circ. Mat. Palermo (2) Suppl. **59** (1999), 201–208. The 18th Winter School "Geometry and Physics" (Srní, 1998). [MR1692271] ↑25

[Wei94] C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. [MR1269324] ↑19

[Wil06] B. Wilking, *Positively curved manifolds with symmetry*, Ann. of Math. (2) **163** (2006), no. 2, 607–668. [MR2199227] ↑3

[WZ] B. Wilking and W. Ziller, *Revisiting homogeneous spaces with positive curvature*, Journal für die reine und angewandte Mathematik (Crelles Journal). Published online 2015-10-21. doi:10.1515/crelle-2015-0053. ↑3, 24

[Zib15] M. Zibrowius, *KO-rings of full flag varieties*, Trans. Amer. Math. Soc. **367** (2015), no. 4, 2997–3016. [MR3301890] ↑5, 10, 20, 21, 28

[Zib14] ———, *Witt groups of curves and surfaces*, Math. Z. **278** (2014), no. 1-2, 191–227. [MR3267576] ↑12

[Zil07] W. Ziller, *Examples of Riemannian manifolds with non-negative sectional curvature*, Surveys in differential geometry. Vol. XI, Surv. Differ. Geom., vol. 11, Int. Press, Somerville, MA, 2007, pp. 63–102. [MR2408264] ↑28